Introduction to the Italicizing N Task Sequence (Back to Top)

The first learning goal of the instructional sequence is interpreting matrices as mathematical objects that transform input vectors to output vectors. Thus, a goal of this introductory whole-class discussion is to help students conceive of input-output pairs of vectors that are related through a matrix transformation.

It is appropriate to introduce students to this through a mini-lecture with a few examples. This will provide the pre-requisite information students need to engage in the Italicizing N task sequence.

Time Required: Approximately 20 minutes.

Assumed prior knowledge

- Linear combinations, span, and linear (in)dependence
- Methods for determining solutions to a linear system such as Gaussian elimination; existence and uniqueness of solutions
- The interpretation of $A\mathbf{x} = \mathbf{b}$ as a vector equation both algebraically and geometrically
- The interpretation of $A\mathbf{x} = \mathbf{b}$ as a system of equations both algebraically and geometrically

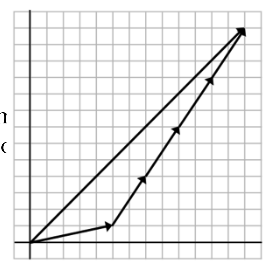
Mini-lecture on transformations

The instructor could begin by reminding the class that they have already worked with two interpretations of the equation Ax = b (vector equation & system of linear equations).

> **Example Discussion Topic:**
>
> Let's review the interpretations of $\boldsymbol{Ax} = \boldsymbol{b}$ we've seen by considering the example $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \end{bmatrix}$.

As a vector equation:

- $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1\begin{bmatrix} 5 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \end{bmatrix}$.
- The solution (1, 4) is the number of the first colun vector and the number of the second column vecto needs to become the linear combination $\begin{bmatrix} 13 \\ 13 \end{bmatrix}$.

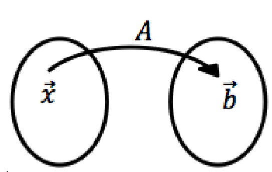


As a systems of equations:

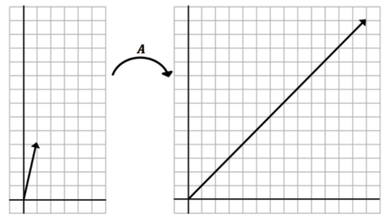
- $5(1) + 2(4) = 13$
  $1(1) + 3(4) = 13$
- The solution (1, 4) is the location in the Cartesian plane in which the equations $5x + 2y = 13$ and $x + 3y = 13$ intersect.

As a linear transformation:

- One can also consider $\boldsymbol{Ax} = \boldsymbol{b}$ **is** as A transforming the vector **x** into the vector **b**. That is, we can think of A as "acting on **x**" to turn it into a vector **b**.



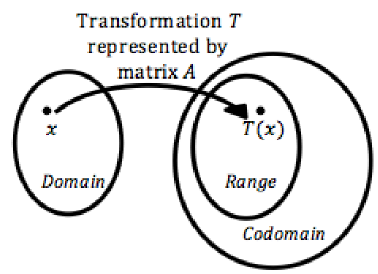
- Considering the same example $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \end{bmatrix}$, the transformation defined by $A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$ transforms the vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ into the vector $\begin{bmatrix} 13 \\ 13 \end{bmatrix}$.

- Graphically: 

Definitions:
The instructor might give the following definitions and briefly relate them to how students are familiar with them from high school and calculus (functions from R to R).

> **NOTE:**
> Depending on the students' backgrounds and/or the rigor of the linear algebra course, the instructor could choose to wait until after students have worked on Task 1 to introduce these terms, connecting them to the work students will have done in Task 1.

- A **transformation** (function) $T: \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns to each $x \in \mathbb{R}^n$ a vector $T(x) \in \mathbb{R}^m$.
- The **domain** is the set of all possible input vectors x. Here, the domain is $\mathbb{R}^n$
- The output $T(x)$ is the **image** of x under the transformation T
- The **range** is the set of all images under the transformation T
- **Codomain**: The (vector) space that contains the range of the transformation T



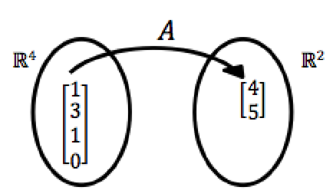
Examples (Use these if the above definitions have been given):

Familiar examples:

1. $T: \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^2$
   - The domain and codomain are both $\mathbb{R}$, whereas the range is only $[0, \infty)$.
2. $T: \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 5$
   - The domain and codomain are both $\mathbb{R}$, and the range is also $\mathbb{R}$.

New Examples:

1. Let $T: \mathbb{R}^4 \to \mathbb{R}^2$ be a transformation defined by $A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 5 \end{bmatrix}$. That is, $T(x_1, x_2, x_3, x_4)$ $=(x_1 + 3x_3 + 2x_4, 2x_1 + x_2 + 5x_4)$. Note that A is transforming vectors from $\mathbb{R}^4$ into vectors in $\mathbb{R}^2$. For examples, let $\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$. Then $A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. A graphical interpretation is hard to do for this, but we can think of it set theoretically:

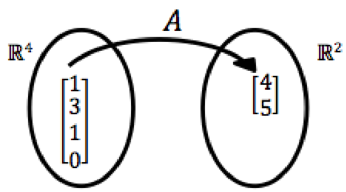


2. Let T be a transformation defined by A. Let $A = \begin{bmatrix} 1 & -2 \\ 5 & -9 \\ -3 & 6 \end{bmatrix}$, let $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the image of u under the transformation T.

3. If the domain for a transformation T is $\mathbb{R}^5$ and the codomain for T is $\mathbb{R}^3$, and T is defined by a matrix A, what would the dimensions of A have to be? (i.e., how many rows and how many columns does A have and why?)

Examples (use the versions below if the above definitions have NOT been given at this time)

1. Let $A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 5 \end{bmatrix}$. Note that A transforms vectors from $\mathbb{R}^4$ to vectors in $\mathbb{R}^2$. For example, let $\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$. Then $A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. A graphical interpretation is hard to do for this, but we can think of it set theoretically:



2. Let $A = \begin{bmatrix} 1 & -2 \\ 5 & -9 \\ -3 & 6 \end{bmatrix}$, let $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the image of u under multiplication by A.

3. What is the size of a matrix that sends vectors in $\mathbb{R}^5$ to vectors in $\mathbb{R}^3$?