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Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra

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ABSTRACT

The purpose of this paper is to describe the process by which a pair of undergraduate students, participating in a teaching experiment, reinvented (with guidance) the concepts of group and isomorphism beginning with an exploration of the symmetries of an equilateral triangle. The intent of this description is to highlight some important insights provided by an analysis of the students' mathematical activity. First, the analysis resulted in the identification of a number of informal student strategies that anticipated the formal concepts. Second, the analysis provided insight into how these strategies could be evoked. Third, the analysis provided insight into how these strategies could be leveraged to support the development of the formal concepts.

1. Introduction

Freudenthal (1973) argued against group theory instruction in which one begins by defining an abstract group and then proceeds by proving general theorems. He claimed that mathematics does not develop this way in the minds of individuals, but instead moves from the particular to the general. Freudenthal also noted that, historically, formal definitions only appear at the end of a period of exploration. For example, Kleiner (1986) traced the beginnings of group theory back to the work of Lagrange beginning in 1770, while an abstract definition first appeared in 1854 and did not appear its modern form (including the inverse axiom) until 1882.

Freudenthal (1973) argued that groups should be introduced as systems of automorphisms of structures under composition. He suggested that when groups are introduced in this way, the group axioms can be verified conceptually rather than algorithmically. For example, it is clear that if one combines two symmetries of an equilateral triangle this combination is also a symmetry of an equilateral triangle (so the set of symmetries is closed under composition). Similarly, Burn (1996) argued that the notions of permutation and symmetry should be regarded as the fundamental concepts of group theory.

However, Dubinsky, Dautermann, Leron, and Zazkis (1997) noted that although from a mathematician's perspective group concepts may be visible in specific examples, it may still be difficult for a student to abstract those group concepts from the specific examples. Indeed it has been well reported that students often have great difficulty in abstract algebra courses (e.g., Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Hart, 1994; Leron, Hazzan, & Zazkis, 1995; Selden & Selden, 1987, 2008; Weber, 2001)

Inspired by the ideas of Burn (1996) and Freudenthal (1973) and with Dubinsky et al.'s (1997) words of caution in mind, I set out to investigate how students might be able to come to understand the abstract theory of groups beginning with an

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exploration of the symmetries of a geometric figure. I began by conducting a series of three teaching experiments (Steffe, 1991) with pairs of undergraduate students. The goal of each of these teaching experiments was to learn how students might be supported in reinventing the concepts of group and isomorphism based on their work in the context of geometric symmetry. In particular, I wanted to[1]:

1. Discover student strategies and ways of thinking that anticipated the formal concepts.
2. Develop instructional activities that could be used to evoke these strategies and ways of thinking.
3. Develop instructional activities that could be used to leverage these strategies and ways of thinking to support the development of the formal concepts.

Here I will present the case of Jessica and Sandra, the participants of the first of the three teaching experiments, keeping these three goals central to the discussion. This presentation of Jessica and Sandra's reinvention of the concepts of group and isomorphism will be presented primarily in the form of five critical episodes, each of which represents an important stage in the reinvention process and in my learning as a researcher/instructional designer.

2. The teaching experiment: overview

As noted, the overall design of the research project consisted of a sequence of three teaching experiments, each based on the constructivist teaching experiment (Steffe, 1991; Steffe & Thompson, 2000). Each teaching experiment was conducted with two undergraduate students. I played the dual role of teacher/researcher and was also in many ways a student, learning the mathematics in a new way as the teaching experiments progressed. The goal of each teaching experiment was to learn how students might reinvent the concepts of group and isomorphism. By reinvention, I refer to the concept of guided reinvention from the instructional design theory of Realistic Mathematics Education (RME). Gravemeijer and Doorman (1999) explain that the idea of guided reinvention "is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible" (p. 116). It should be noted that the guided reinvention heuristic does not imply that the students must reinvent the ideas without the assistance of a teacher see Rasmussen and Marrongelle (2006) for a discussion of the role of the teacher in guided reinvention) and it will be clear that my role in the reinvention process described here was significant.

Jessica and Sandra participated in the first teaching experiment. Jessica was a double major in mathematics and mathematics education. She was a very strong student who had just received an "A" in a transition-to-proof course. Sandra was not seeking a degree but was studying advanced mathematics to support her interest in science. She had just received a "C" in the same transition-to-proof course. I met with Jessica and Sandra seven times, with each session lasting 90–120 min. All of the sessions of the teaching experiments were videotaped and the students' written work was collected. Additional sources of data included my notes and the learning activities themselves, which documented the ongoing development of the instructional approach. The primary instructional mode was problem-based pair work combined with discussion. In addition to posing questions and tasks, I occasionally interjected with interview type questions in order to learn more about an individual student's thinking.

Most of the instructional activities used during the teaching experiments were inspired by the instructional design heuristics of RME. While some details are provided, it is beyond the scope of this paper to fully describe the initial design of the instructional sequence. The focus here will be on describing the students' mathematical activity and how it contributed to the subsequent reconstruction of the instructional approach. For more information on the initial design process see Larsen (2004).

The retrospective data analysis consisted of multiple phases of iterative analysis of the videotapes and the students' written work. This analysis was based upon techniques described in Cobb and Whitenack (1996) and Lesh and Lehrer (2000). For example, one phase of analysis was focused on identifying student strategies that seemed to anticipate aspects of the formal concepts of group and isomorphism, and then a subsequent phase of analysis was focused on identifying aspects of the tasks, or of the students' earlier mathematical activity, that could have been responsible for evoking these strategies.

Before describing Jessica and Sandra's reinvention of the concepts of group and isomorphism in detail, I should make a comment on the pedagogical implications of this research. I must acknowledge that it is in general not practical in a group theory course to provide nearly 14 hours of instruction, with a one-to-two teacher/student ratio, simply to arrive at the definitions of group and isomorphism. Nevertheless, this kind of in-depth work with a small number of students can support the development of an instructional theory that can in turn be used to design curriculum that can be practically implemented in an undergraduate classroom. The research presented here provided (in part) the foundation for the development of such an instructional theory (Larsen, 2004; Larsen, Johnson, Bartlo, & Rutherford, 2009) and subsequently a 10-week group theory course that has been implemented multiple times at a large urban public university (Bartlo, Larsen, & Lockwood, 2008).

[1] These three objectives are drawn from Gravemeijer's (1998) description of the ingredients of a local instructional theory.

3. The reinvention of the concepts of group and isomorphism

The reinvention of the concepts of group and isomorphism will be presented here in the form of five episodes. Each episode represented a significant step in the reinvention process. The first episode included the students' invention and refinement of a set of symbols for the six symmetries of an equilateral triangle. This episode was important primarily because it provided the foundation for the mathematical activity that followed. The second episode featured the emergence of an unexpected student strategy that anticipated important aspects of the group concept and was a key step in the reinvention process. The third episode took place as the students were formulating the definition of group and provided significant insight into how students could come to see a need for the inverse axiom in the definition of group. The fourth episode provided important insight into the weaknesses of what seemed to be a reasonable task with which to initiate the reinvention of the isomorphism concept. At the same time this episode provided insight into the complexity of the isomorphism concept and, despite the weaknesses of the task, also shed some light on how students might begin to reinvent the isomorphism concept. The fifth episode featured a struggle to make explicit a crucial aspect of the isomorphism concept and provided insight into how students' might be supported in accomplishing this. These five episodes do not tell the entire story of the reinvention process, but they do represent some of the most important aspects of the process. Whenever it was possible without excessively lengthening the narrative, I have attempted to include additional details (including some from the second and third teaching experiments) to fill out the story.

3.1. *Episode 1: Identifying and developing symbols for the symmetries of an equilateral triangle*

3.1.1. *Episode introduction*

It is not uncommon for instructors to use the context of symmetry to help students make sense of abstract algebra. However, when this is done these symmetries tend to be presented as a ready-made system. In particular, students are *given* the set of symbols used to represent these symmetries as well as the representation system for the operation (usually the function composition sign and a Cayley table). A crucial aspect of the reinvention process described here is the genesis of the symbols used to represent the symmetries of an equilateral triangle. The students developed these symbols, so for them the symbols were strongly connected to the physical movements they represented. Additionally, the students created their own system for representing the operation of combining symmetries, eventually creating an operation table.

3.1.2. *Episode description*

I began the teaching experiment by giving Jessica and Sandra a sheet of paper with an equilateral triangle drawn on it (called the outline of the triangle) and then placing a cardstock cutout of an equilateral triangle (of the same size) on top of the outline. I asked the students to determine all the ways that the triangle could be moved and land back on the outline. Sandra pointed out that there were "infinite ways you can pick it up and put it back down and have it be back on the outline." This initiated a negotiation of the meaning of equivalence for symmetries. To facilitate this discussion, I labeled the vertices of the triangle (with the numbers 1, 2, and 3) to draw the students' attention to the effect that each motion had on these vertices. Eventually the students agreed to consider symmetries to be the same if, in Sandra's words, "the triangle ends up in the exact same place in position due to these numbers." The students then identified the six possible configurations of the vertices and then identified six motions that would result in these configurations (assuming a standard starting configuration). They developed a flow chart to illustrate these motions and their effect on the triangle (Fig. 1).

After Jessica and Sandra had identified six different symmetries, I asked them what would happen if two of these were performed consecutively. They said that this would be the same as one of the six motions they had already identified because,

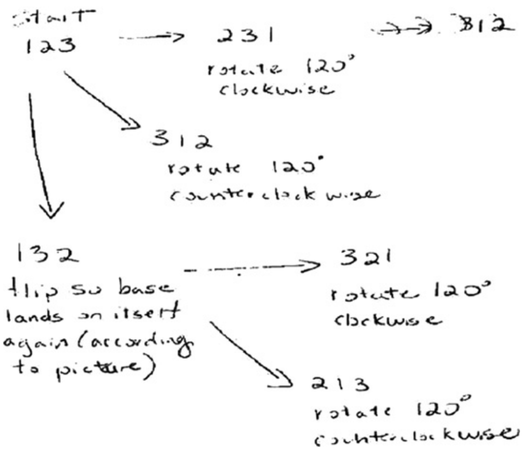

Fig. 1. Flow chart describing the six different symmetries of an equilateral triangle.

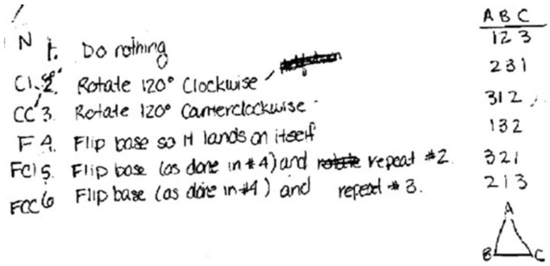

Fig. 2. A table describing the symmetries of an equilateral triangle.

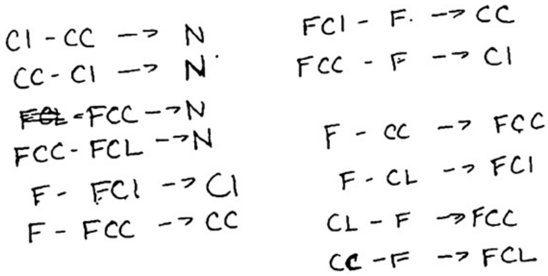

Fig. 3. A loosely structured list of symmetry combination results.

according to Sandra, "you end up with the results of another one of the moves." At this point, I presented the students with the central task of the first phase of the instructional sequence: I asked them to determine to which of the original six motions each combination of two motions was equivalent. I encouraged them to first develop a set of symbols for the motions that were short enough to make repeated use convenient while still carrying information about the nature of the motions they represented.

Jessica: "Half-turn" right here. Half-turn clockwise? Half-turn, no that's still, no that's not half-turn, that's 120 degrees.
Sandra: I would say um, 120 degrees clockwise and 120 degrees "CC" for counterclockwise. "N" for nothing. "Flip."
Jessica: I think he means like a name, like a [trails off and laughs].
Sandra: Well, it's shorter. Flip, flip and, oh, we don't even need to do 120 because they're both. "Clockwise" and "counterclockwise." We should have "Flip *CL*," "Flip *CC*" that's it.

After deciding on a set of symbols, these were added to the table shown in Fig. 2, which describes the six motions the symbols represent, and the effect each motion has on the vertices of the triangle.

Jessica and Sandra then proceeded to consider each combination of two symmetries to determine to which of the six symmetries it was equivalent. The means used to record these results became more structured over time. At first, the students recorded their results in a loosely structured list (Fig. 3).

Later, motivated by the need to determine whether they had considered all possible combinations, Jessica and Sandra added additional structure to their list (Fig. 4).

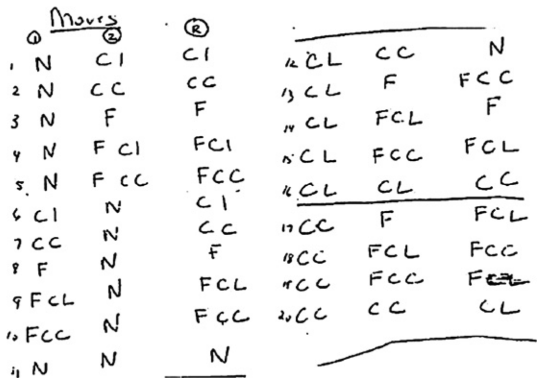

Fig. 4. A structured list of symmetry combination results.

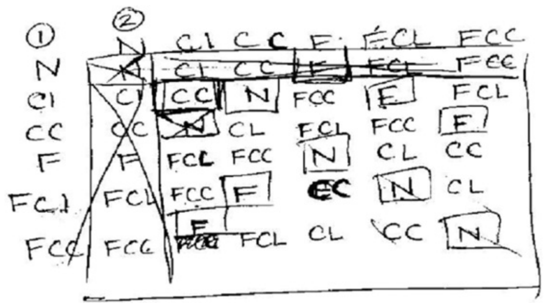

Fig. 5. An operation table for the symmetries of an equilateral triangle.

This more structured list grew to contain 26 combinations. At this point, the students again became concerned with whether they had considered all possibilities. I asked them how many times each symmetry would need to appear in the first column of their list. They decided that each symmetry would have to appear six times and concluded that there should be 36 total combinations. However, it was not obvious to them *which* 10 combinations were missing. I asked them whether there was a way to present their results that would make easier to tell if they had considered all possibilities. Jessica suggested that they could make a "chart" and the students eventually produced the operation table shown in Fig. 5.

3.1.3. Episode wrap-up

Later the students would modify their symbol set after completing a task in which they expressed each of the six symmetries using only the symmetries *F* and *CL* as building blocks (generators). I suggested that if all the symmetries were expressed this way, they would no longer need to specify the direction of the rotations. After some discussion, they decided to replace *CL* with *R* and *CC* with R^{-1}. In following sessions it would become apparent that the time the students used in structuring the system of symmetries of a triangle and developing their symbol system was well spent, and that the particular system they developed (one that includes composite symbols like *F CL*) was particularly well suited for building an algebraic system.

3.2. Episode 2: Creating a calculus for computing combinations of symmetries

3.2.1. Episode introduction

My expectation was that Jessica and Sandra would determine the result of combining two symmetries by physically manipulating the cardstock triangle I had provided. They did do some manipulation of the triangle, but they quickly developed a calculus for determining the results of combinations without performing the actions. This was perhaps the most significant episode in terms of my learning as an instructional designer. The system the students developed featured both general group axioms and important relations specific to this group. Thus, it anticipated the formal definition of group and would also turn out to be important for developing the notion of isomorphism. (Note that this episode overlaps chronologically with the previous episode as the students began developing their system for calculating symmetries even as they were developing their system for recording their results.)

3.2.2. Episode description

Jessica began noticing rules and relationships as the students computed various combinations of symmetries by manipulating the triangle. Sandra's participation in this activity was limited at first to making supportive comments, but later she was able to make more substantial contributions. The first of Jessica's observations concerned the nature of the "do nothing" move.

Jessica: So if we do "do nothing" and one of these other ones, it's gonna be the same thing.
Sandra: Right.
Jessica: So should we address that?
SL: Yeah, you could write that down.
Sandra: That's a very good observation. I like that.

Early on, Jessica made the first of several observations regarding the way that flips and rotations interacted. A specific example of the property Jessica describes below is the relation *FCC F* = *CL*.

Jessica: So if a movement, a rotation, comes in between two flips then it does the reverse of what it did between them. Does that make sense? If there's something that happens between the two flips then it gives the reverse direction. . .
Sandra: Okay yeah. That's pretty cool.

Fig. 6. Using regrouping to calculate a combination.

In addition to supporting Jessica, Sandra was also able to compare different observations and even extend some of the more basic observations.

Jessica: If you go *CL CC* that gives the same. . .
Sandra: Right because you're just doing the opposite. So if *CL* is the first term and *CC* is the second term then that's also reflexive because you can switch and make *CC* the first term and *CL* the second term they're going to be the same. And let's see. . .
Jessica: Because this is the reverse of the other thing.
Sandra: Right.
Jessica: [Thinking] Yeah.
Sandra: Okay.
Jessica: Because any time you go clockwise and then counterclockwise, you're just going back to your spot. The same spot that you started with.
Sandra: That's a good observation. That's different from the first example because if *N* is the first term and *CL* is the second term you're going to end up with *CL* but then in our second example here, you go from *CL* as your first term and *CC* as your second term you're not going to end up with *CC* you're going to end up with nothing. So they're all just canceling, these four moves are just basically canceling each other out. Because of the rotation.
Jessica: Yeah, all the clockwise and counterclockwise.

Later Sandra was able to explain one of Jessica's calculations to me. Specifically, she demonstrated understanding of the strategy of regrouping and substituting to simplify combinations.

SL: Which one were you doing just now?
Sandra: *FCL* to *FCC* it gives clockwise. . . *FCL* and *F*, these two get counterclockwise, and counterclockwise and counterclockwise give clockwise (see Fig. 6).

However, Sandra pointed out herself (in the excerpt below) that she was primarily operating on a procedural level (moving the triangle and then writing down the result). She makes this observation in reaction to Jessica's argument that a combination involving one flip will result in a move that contains a flip. Jessica's argument indicated that she was able to imagine the entire process (of performing the combination of symmetries) and *anticipate* the result without actually going through each step, while Sandra noted that she "can't quite see it yet like that." This suggests that Sandra was limited at this point to an *action* conception of the operation of combining symmetries, while Jessica seemed to be operating with a *process* understanding (see Brown, DeVries, Dubinsky, & Thomas, 1997).

Jessica: Wait, this isn't right.
Sandra: What isn't right?
Jessica: If you flip it you'll have *F* something. You should have *F* something, *F CC*. Does that make sense? If you only have one *F* here, you should have an *F* over here.
Sandra: You know what, I've just been looking at it from how we've been writing down our results, so I can't quite see it yet like that.

Sandra was less fluent at developing rules and performing rule-based calculations. However, her conviction that this activity was valuable not only encouraged Jessica, but also effectively established this kind of activity as normative for our mini-classroom.

Jessica: I'm just noticing these rules.
Sandra: Yeah, that's great. That's what we're supposed to be doing.

3.2.3. Episode wrap-up

Going into the teaching experiments, it had not occurred to me that students might spontaneously develop a calculus for computing combinations of symmetries. Instead, I expected that the students would notice the essential properties of the group only after reflecting on properties of various group operation tables. It was a welcome surprise to see the students using some of these properties (identity property, inverse elements, associativity) spontaneously to calculate combinations.

Rules

1. Associative Rule
   $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
   def'n: $A \cdot B \cdot C = (A \cdot B) \cdot C$
2. Identity
   $A \cdot I = I \cdot A = A$ for any move A.
3. ~~An inverse~~
   ~~If~~ For every move A,
   if ~~A•B=B•A then~~ $A \cdot B = B \cdot A = I$,
   then B is the inverse of A written A^{-1}.
   Every move has an inverse.
4. $F^2 = R^3 = I$ and $F \cdot R^{-1} = R \cdot F$

Fig. 7. Reduced set of rules for the symmetries of an equilateral triangle.

As a result, I did not challenge Sandra's assertion that this was what they were supposed to be doing. And ultimately, this particular mathematical activity would become a crucial part of the emerging local instructional theory and curriculum design.

However, during the other two teaching experiments, the students did not spontaneously begin to calculate combinations of symmetries. Instead they performed each combination using the paper triangle I provided. In those teaching experiments (and in whole-class implementations that have followed), I have intentionally encouraged this activity by asking the students to think about any shortcuts they may have used in filling out the table (typically students will at least use the identity property and some cancellation of inverse elements). I then ask the students to try to develop enough rules (or shortcuts) to fill in the operation table using calculations.

It is also worth noting that the emergence of this strategy was probably promoted by the symbol set the students developed. Specifically, it was important that some of the symmetries were expressed in terms of other symmetries. The symmetries *FCC* and *FCL* (Sandra called these "compound moves") when combined with other symmetries offered numerous opportunities for regrouping and canceling (e.g., $F(FCC) = (FF)\,CC = CC$).

In response to follow-up tasks, the students' list of rules was subject to further structuring. First, additional rules were added. For example, the associative law was added to the list after a surprisingly interesting discussion about the meaning of regrouping in this context, and the relationship between regrouping and commutativity (see Larsen, in press). Later, the list of rules was pared down as students proved that some of the rules could be deduced from others (following a suggestion from Jessica that they might not need all of their rules). See Larsen (2004) for more detail on this process. The reduced set of rules that resulted from this activity appears in Fig. 7. Note that this list of rules includes the three axioms featured in the formal definition of group as well as the relations found in the typical generator-and-relations representation of this particular dihedral group.

3.3. Episode 3: Is the identity element unique?

3.3.1. Episode introduction

Following the completion of the activities in the context of the equilateral triangle, I presented the students with a collection of similar tasks in different contexts. The first of these contexts was the symmetries of a square. Jessica and Sandra symbolized the eight symmetries using notation very much like what they had developed in the triangle context. The students were able to complete the entire operation table in about 45 min, using a short list of rules similar to those developed in the triangle context. While Jessica again took the lead in developing rules and was more fluent using these rules to perform calculations, Sandra was able to perform a number of the more complex calculations with little or no assistance. The second context I asked Jessica and Sandra to explore involved the SNAP game described by Huetinck (1996). This game features a 3×3 grid of golf tees and three rubber bands. From an expert perspective, the SNAP game is a representation of the group of permutations on three elements. I also developed a variation of this game that modeled the Klein 4-group. Of course, consistent with my overall approach to the project, I did not consider these games to represent the given groups. Rather, the groups emerged from the students' mathematical activity as they worked with the games.

After the students had explored these additional contexts, I introduced the term "group" and told Jessica and Sandra that the systems we had been working with were all examples of groups. I asked them to try to write a definition for group[2] based on properties that these systems shared. This task offered a number of challenges: It was a struggle for the students to articulate what constituted a group (a set and an operation) and then to formulate a definition for operation. These issues

[2] Recall that a group is a set with a binary operation for which (1) the associative property holds, (2) the set contains an identity element, and (3) each element has an inverse.

were resolved with my assistance. However, the most interesting aspect of this phase of the teaching experiment was initiated by Jessica's inclusion of an extra condition (uniqueness of the identity element) in her formulation of the definition of group.

3.3.2. Episode description

After initially writing that a group has an identity element, Jessica changed this aspect of her definition, writing that a group has a *unique* identity element. I asked Jessica if she could prove that the identity must be unique using the rest of her definition. After she worked for about 3 min, I asked her to tell me what she was doing.

SL: So what are you doing?
Jessica: My proof skills are horrible [laughs].
SL: So you have s times x equal to the identity and s times y equal to the identity?
Jessica: So then this equals this [$s \cdot x = s \cdot y$] so x must equal y. But am I allowed to say that then?

Note that a few moments later Jessica discovered her error and replaced the equations $s \cdot x = I$ and $s \cdot y = I$ with $s \cdot x = s$ and $s \cdot y = s$. I acknowledged that we do this sort of thing in arithmetic (conclude that if $s \cdot x = s \cdot y$ then $x = y$) and then asked Jessica why it works.

SL: Okay. So, but why is it okay? Why does it work?
Jessica: Well because... if you take one number right and any other like we took our chart [gestures as if pointing to two inputs on an imagined operation table] we took one number and here, any other number up here you would have different answers in every little slot here [moves a finger as if pointing to each entry in a row of the imagined operation table].
SL: Did we prove that?
Jessica: No, but it's there [laughs].

Jessica's comments reveal two important insights. First, she was able to express what it would mean to have two identity elements using algebraic symbols ($s \cdot x = s$ and $s \cdot y = s$) and use this to generate the equation $s \cdot x = s \cdot y$. She also knew that she wanted to be able to say that this equation implies that x is equal to y. This led to her second important insight, which is that this follows from the fact that, in the tables they had worked with, no element appeared twice in the same row. At this point Sandra returned from a short break, and I asked Jessica to explain what she had been doing.

SL: Okay that's an interesting point. Let's discuss that with Sandra.
Sandra: Uh oh.
SL: So she started on a proof. So do you want to explain what you tried to do?
Jessica: Okay. So I'm supposing that it's not unique right?
Sandra: Okay.
Jessica: So I'm gonna take two... two elements of S. Pretend that these are both the identity.
Sandra: Okay.
Jessica: So I'm saying that you can go just like the identity says you can go s dot x equals s or s dot y equals s [$s \cdot x = s$ or $s \cdot y = s$].
Sandra: Okay.
Jessica: So that means that s dot x equals s dot y [$s \cdot x = s \cdot y$]. And Sean asked me why... how could I just conclude this, so then x equals y. 'Cause this is what we want.' Cause if we could show that x equals y that would mean that it's unique.

Jessica then went on to consider the question of how to deduce $x = y$ from the equation $s \cdot x = s \cdot y$. As Jessica worked through this aloud, she began creating the proof shown in Fig. 8.

SL: So she thinks it would be fun if she could just say that x equals y, conclude that from that.
Jessica: And I'm saying that you can say it cause you have dut dut dut dut, dut dut dut dut dut [sketching the operation table in Fig. 8] and you take this little guy s here right, and you put x and y here well there's unique ones for each of these,

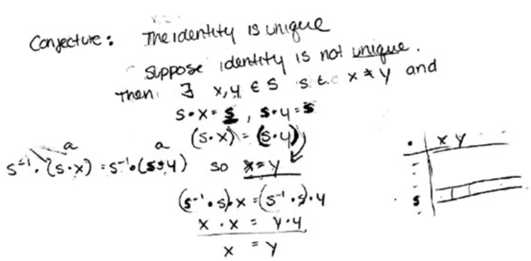

Fig. 8. Jessica's proof that the identity element is unique.

unique responses for each one of these so it must be in the same column is what I'm trying to say but I don't know what rule or what entitles me to say that.

SL: Do you know what she's talking about?

Sandra: Yeah I mean I'm looking at it. So you're saying that x has to equal y for it to work which is basically the same thing as saying that x and y aren't really separate they're not really two different things.

SL: Right that's what her overall proof is, but now she's trying to explain how she could go from this step to this step. That if s times something is the same as... if s times x is the same as s times y, she's trying to tell you why she can say that x has to be the same as y.

Sandra: Cause you can... you can cancel x out I mean s out.

After a brief discussion to establish that Sandra understood Jessica's argument that the uniqueness of the identity followed from the fact (not proven at this point) that each element occurred exactly once in each row of the table, I asked Sandra to explain her canceling idea.

SL: So what were you saying she could do to go from here to here?

Sandra: Uh I just said that you would just cancel out the s, so take the inverse maybe.

Jessica: Oh, both sides.

SL: So try it and see if it works.

3.3.3. Episode wrap-up

Freudenthal (1991) argues that, "knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others" (p. 47). Jessica and Sandra developed the symbols and notation systems that they used. This was intentional since, agreeing with Freudenthal, I felt that this would result in symbols and notation systems that were meaningful to the students. Further, since the formal notions were developed from the students' informal ideas, I felt that there would be a stronger connection between the students' informal understandings and the formal concepts. The research literature suggests that these kinds of connections can often be missing for students and that this can negatively impact their ability to construct proofs (Moore, 1994; Tall & Vinner, 1981; Weber & Alcock, 2004). Jessica's use of the operation table (and her observation that each element appeared only once in each row) is an encouraging example of a student using their informal understanding and a meaningful (to the student) notational system to construct a formal proof.

This episode was also particularly important to my learning as an instructional designer. I noticed during the three teaching experiments that the students did not need to include the existence of inverses in their list of rules used to calculate combinations of symmetries. Only Jessica and Sandra included this rule in their list. (They only did so because they happened to include in their extended list of rules all of the products that resulted in the identity.) The other two pairs of students did not include this property in their lists. Thus I was uncertain how I could motivate the need for this property as the students moved toward formulating a definition of group. Jessica's approach to her proof suggested an approach that has proven to be quite effective. While constructing their operation table, students inevitably make the observation that each element appears exactly once in each row. So, after the students have generated their reduced list of rules, I now ask them to treat this observation as a conjecture. I ask them to try to prove it using their rules and, if they cannot do this,[3] I ask them to develop an additional rule to add to their list that would allow them to prove this conjecture. As Jessica's proof illustrates, this conjecture can be proven if the existence of inverses is assumed along with the identity and associative properties.

3.4. Episode 4: Negotiating what it means for two groups to be essentially the same

3.4.1. Episode introduction

While working on the first SNAP activity, the students considered the possible consequences of using different symbols for the elements (configurations of rubber bands). At one point Jessica said, "I am wondering if we numbered them in a different way if that would make a difference," referring to the way the configurations were numbered and subsequently symbolized. Sandra responded by saying that if they changed the number of the identity element, it would still be the identity element. Jessica agreed, and Sandra went on to argue that the element they labeled 6, "because it's a cross would probably always be the one that comes up with this pattern [in the table]. So I mean they would still have the relations, they would just be labeled differently. I don't think it matters." This discussion anticipated the concept of isomorphism and I hoped that the students could build on this idea to eventually develop the formal notion of isomorphism.

I initiated the students' exploration of the isomorphism concept by asking them to determine how many different groups there were with four elements. Because this task requires the concept of equivalent groups (otherwise there would be infinitely many such groups), it seemed that it would be a good context for initiating the reinvention of the isomorphism concept. However, this was not the case. This task presented a number of obstacles that made it difficult for the students to build on their idea that the labels of the elements of a group do not matter. However, my efforts to make sense of the

[3] Actually this can be proven with just the dihedral relations, the identity property, and the associative property. For instance, these can be used to generate the entire table to prove the property. However, it can be much more efficiently obtained using the existence of inverses.

difficulties that emerged as the students engaged in the task raised my awareness of a number of complexities regarding the isomorphism concept that I had not anticipated. Additionally, this analysis informed the development of a new task sequence that has proven to be much more effective in building on students' informal notions of isomorphism.

3.4.2. Episode description

Jessica was immediately enthusiastic about the task saying, "This is a good one" and she began making operation tables with four elements (labeling the elements *I*, *a*, *b*, and *c*). I asked the students to predict how many groups there would be. Jessica predicted "three to six" and Sandra said there would be an infinite number. These predictions suggested that the two students had very different interpretations of the question. I briefly left the room to allow the students time to think. While I was gone, they had a spirited discussion about the meaning of the question.

Jessica: Well you can look at something with four elements like you said you take one is *I*. So you know this is going to be your chart. So basically how many ways can you fill this in so it would work and it would satisfy all the requirements of a group? How many different ways can you fill in these nine boxes so that it would have all the different ways to be a group?

Sandra: But that would just be with *a*, *b*, and *c* as the elements.

Jessica: There's four different elements, those are just random elements.

Sandra: But isn't that a different question? Isn't that how many different table configurations can you get with four specific elements?

Jessica: You're saying, you're looking at how many different elements can be in a group of four elements?

Sandra: Well that's what the question, I mean the question's written that way. I mean I guess it's just being, I mean it's a different question. Do you see? This is how many groups have four elements. Well there's an infinite number of groups that can have an infinite number of four elements. Because you can take any element from anywhere. But this question is um how many different configurations of a table can you get with four specific elements.

In retrospect it is not surprising that the students had different interpretations of the question. Since the notion of equivalent groups had not yet been established, Sandra's interpretation makes perfect sense. To Sandra, two groups would be different if they were formed from different sets. Furthermore, even given the notion of isomorphism, there are times when it makes sense to think of isomorphic groups as different. For example, there are a number of contexts in which one would think of the group of real numbers under addition as a different group than the group of positive real numbers under multiplication, even though these groups are isomorphic. On the other hand, Jessica's interpretation is also valid and can be seen as a generalization of the idea that two groups are the same if the only difference between them is the way the elements are labeled. By themselves, these competing interpretations are not really problematic. In fact one reason that I chose this task was because I thought it would motivate a discussion of what it would mean for two groups to be in some sense the same.

However, a significant underlying complexity here is the distinction between the labels (or symbols) of the elements of a group and the elements themselves. If one thinks of the four symbols that Jessica used (*I*, *a*, *b*, *c*) as simply labels, then it makes sense to think that one could identify all possible distinct groups by considering group tables formed using these four symbols. However, if one thinks of these symbols as the actual elements of the group, then Sandra's view (that groups using other symbols would be different) makes perfect sense.

Following this exchange between Jessica and Sandra, I engaged the students in a discussion of what it might mean for two groups to be the same. This discussion did not resolve the issue, but it did eventually result in the students making progress on the task. The students determined that they could construct four different Latin Squares (Fig. 9) that had an identity element and inverses.

Jessica and Sandra agreed that one of these was different from the others (because each element is its own inverse). However, when I asked about the three remaining tables, Jessica's answer was surprising given the way she seemed to be thinking about the task. She said, "I'd like to say that they are different, but I could see how you think that they are the same."

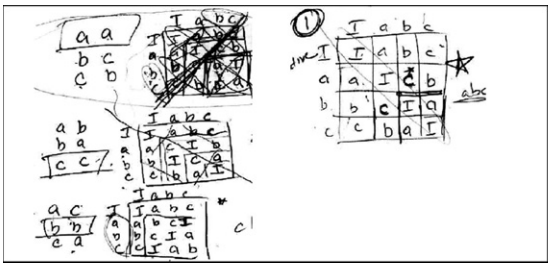

Fig. 9. Group table candidates with four elements.

The Same Or Different?

Consider the following two group tables and rules from last time:

Table #1

•	I	a	b	c
I	I	a	b	c
a	a	b	c	I
b	b	c	I	a
c	c	I	a	b

$c = a^{-1}$ $\quad a^4 = 1$
$b = b^{-1}$ $\quad b^2 = 1$
$a = c^{-1}$ $\quad c^4 = 1$

Table #2

•	I	a	b	c
I	I	a	b	c
a	a	c	I	b
b	b	I	c	a
c	c	b	a	I

$b = a^{-1}$ $\quad a^4 = 1$
$a = b^{-1}$ $\quad b^4 = 1$
$c = c^{-1}$ $\quad c^2 = 1$

We were wondering last time if these were really the same group with different names for the elements.

1. How should we rename I, a, b, and c in Table #1 if we want to end up with a table that is the same as Table #2? Why?

2. Make the changes, and compare the new Table #1 with Table #2? Are they the same? Why?

Fig. 10. "The Same Or Different" activity.

I asked if they could check to see if they were the same by changing the names of the elements in one table to try to get one of the other tables. Jessica suggested that she could change the elements (a, b, c) in one of the tables (bottom left in Fig. 9) to (c, b, a). However, when Jessica began to consider what would happen if she actually did this, she wondered whether she needed to change her inverses as well.

Jessica: Now, how am I going to do the inverse? Am I? I have to keep the same inverses.
SL: But aren't you already determining. . . you already have your whole table. I thought all you were doing was changing the names. You were making this into *c b a* instead right? But doesn't that mean the *a* has to be. . .
Jessica: Well you're changing the inverses too. Or I'm not changing the inverses?
SL: Well why don't you just forget about that, you just used that to come up with the table.
Jessica: But I think it's important.
SL: Okay.
Jessica: I think that's what makes these each different.

There are a couple of important points to make about this discussion excerpt. First, Jessica is clearly uncertain as to what it means to "change the names." At the time, I thought this could only mean that you replace each instance of a symbol with the new symbol. However, Jessica's question suggests that this is not at all obvious. In fact it was quite difficult for the students to determine what kinds of manipulations were permitted and what kinds were not. At one point, Jessica asked, "Is there any restrictions basically? Like you can't change the inverses or. . .?" The second related point is that to Jessica (and Sandra) the pattern of the inverses for a group was a very tangible quality of the group. If one takes the interpretation that the symbols I, a, b, c are the actual elements of the group in the bottom left of Fig. 9, then it would not be valid to change the names in a way that results in a being a self-inverse. As a result, while the students saw that the structure of the group whose table appears on the right side of Fig. 9 is clearly different from the other three groups (because the structure of the inverses is much different), they also saw the structures of the other three groups as somewhat different from each other because different elements appear as self-inverses.

It is worth noting that the students' working definition of equivalence of groups was limited at this point to the idea that two groups were the same if the only difference was the names of the elements. Given this, it is not surprising that the students were leaning toward thinking of all of the groups depicted in Fig. 9 as being different. In this case, the names of the elements were *not* different (from their perspective) but the structures, as seen in the inverse patterns, *were* different. Nevertheless, the fact that one of the groups was seen as being *more* different provided a starting point for expanding the notion of equivalence. With this in mind, I posed a new task (Fig. 10) in which I focused the students' attention on two of the groups that had been identified as being similar (in that they had the same number of self-inverses).

It immediately became clear that in the context of this task, Sandra did not think of changing the symbols as merely changing the names of the elements. As the following excerpt illustrates, to Sandra the two sets of elements were the same for each group. From this perspective, changing the names in this case would actually mean interchanging elements rather than re-labeling them.

Sandra: I'm still trying to figure out what you mean in the question. It's still not clear to me. [Reading directions] "We were wondering last time if these were really the same group with different names for the elements." Well the elements of the set are the same in both groups.

SL: That's true.
Sandra: So, [Reading directions] "How should we rename *I*, *a*, *b*, and *c* in Table Number 1 if we want to end up with a table that is the same as Table Number 2?" We don't need to rename *I*, *a*, *b*, and *c*. What you need to do is change the inverse relations. You don't need to rename *a*, *b*, and *c*, you just need to shuffle around positions of *a*, *b* and *c* in the table. . .

First, note that my agreement to Sandra's assertion that the elements of the set are the same in both groups indicates that I also had yet to tease out the complexities involved with changing the names of the elements when the two groups use the same set of symbols. From the perspective that the two sets are the same (not just the labels) one should not think in terms of changing the names but rather in terms of creating a correspondence between these two copies of the same set. This can only be seen as changing the names if one takes the perspective that while the symbol sets are the same in each group, the underlying elements might in fact be different.

The consequence of Sandra's perspective in the context of the task I posed was that the procedure I proposed (changing the names) seemed both arbitrary and irrelevant to the task. As she said, "We don't need to rename *I*, *a*, *b*, and *c*. What you need to do is change the inverse relations." Recall that Jessica also felt that the inverse relations were fundamental structural features of the two groups.

The students saw these groups as being different, and saw my name-changing procedure as actually changing the group on which it was performed. Given this, it makes sense that Sandra seemed to be interpreting the task as a request to change one group to obtain the other. So when Sandra actually began to "change the names" for one of the groups, she did so only on the inside of the table. This was perfectly sensible from her perspective, but certainly did not address the task I intended to pose.

In an effort to explain what I meant in the task instructions by "changing the names," I created an operation table for a two-element group (with elements *I* and *a*) and then changed the *a*'s to *b*'s everywhere on the table. Sandra agreed that this would not change the group, but then she pointed out that this situation was different than the task I had given them. In this case, the set of symbols used for the group elements was not the same for both groups; whereas in the task I had posed the symbol set was exactly the same for both groups.

SL: Is that different or the same as what I had before?
Sandra: It's the same kind of relationship, yeah. Except *b* is. . . If you're naming *b* different from *a*, it's probably a different element from *a* so I mean but it's kind of the same relationship. I mean it looks the same it's just a different element, it's a different name for a different element. So I mean if you say change the bottom table to look like the top table, well okay, change the *b*'s to *a*'s.
SL: So change them here [changes them only on the inside at the board]?
Sandra: Change them everywhere.
SL: Change them everywhere?
Sandra: Yeah.
SL: But that's not what you were saying you were going to do before.
Sandra: But you drew a different table than what's here, because here the rows out here are the same but the rows inside are different. So that's a different question than what you are asking on the paper. I mean, because these structures are similar, but they're not exactly the same. Like that one was pretty much exactly the same.

Note that in the context of two groups using *different* symbol sets, Sandra was comfortable with the idea of changing the names of the elements of one group in order to see if the two groups were really the same. To try to make a link between this idea and the task I had proposed, I drew up one of the 4×4 tables on the whiteboard.

SL: But what if I just changed the names though? For instance what if I change them to 1234, 1234 and then do the exact same thing in here. Would that be a significant change?
Sandra: I don't think so. . .
Jessica: I don't think it would because you're just changing the names.
Sandra: I mean I don't think it would. . . as long as you kept the symmetry between the answers in the first and the answers in the second.
SL: So like if I did *I* equal to 1, it should always be equal to 1?
Sandra: Right then you should have the same relationship.
SL: So you think that would be okay.
Sandra: Sure, it doesn't matter if you name it '*a*' or 'green slimy thing'.
SL: The problem is we are doing something a little bit trickier than this though.
Sandra: Okay
SL: The trickier thing is that we're changing the names, but we're using the same letters. In other words, *a* isn't turning into some, up till now not seen, symbol. It's gonna be one of these guys.

This seemed to be an important moment in the discussion perhaps because I finally acknowledged the difficulty of working with two groups that use the same symbol sets. I was then able to more successfully link this task to the students' idea that it does not really matter how you label the elements of a group.

SL: But it's a little more confusing I think because now we have the same letters already. What if the person who made this table, all they did was, they changed each of these names to one of the others and made their table. Then would they have really made a different table if they used all of the same answers?
Sandra: No.
SL: So that's what we're trying to discover.
Sandra: Okay
SL: Are these really the same but just the names are different?
Sandra: Oh I see... Okay... so you're saying it can be legal to look at both these tables and realize that *a* is a different thing, that *a* could be a different thing.

Sandra's realization that the symbol '*a*' could represent two different things in the two tables is a very important one. This realization makes it possible to imagine that the tables could be the result of two different individuals representing the same groups but having made different decisions when labeling the elements. From this perspective it would make sense to change the names of the elements of one group to see if it is really the same as the other group (even though the symbol sets are the same).

3.4.3. Episode wrap-up

When I asked Jessica and Sandra to determine how many groups there were with four elements, my hope was that they would begin trying to figure out how many different Latin Squares they could form with four elements (and Jessica did do this). I expected that they would then check pairs of tables to see if they could rename and reorder one table to obtain the other (with the idea that they represented the same group if this could be done). However, as seen in the episode description, a number of complexities emerged.

First, Sandra felt that this procedure would not really address the question because in this case only one set of four elements would be considered (and clearly there are infinitely many sets with four elements). Drawing on Sandra's earlier assertion that one could change the names of the elements of a group without changing the group, I was able to convince her to at least pursue Jessica's approach. However, it was difficult for both students to figure out exactly what would make two of these tables the same and what transformations of a table could be performed without changing the group. This was particularly challenging given the fact that the four tables the students produced all used the same set of four symbols (this seems to be natural given the task). Jessica's lists of inverses seemed to her to be capturing an essential difference between the groups. Meanwhile Sandra was uncomfortable because what I called "changing the names" looked more like moving elements around in the table. In fact Sandra seemed to interpret the task I proposed as a request to change one group in order to turn it into the other, rather than a request to see if the two were really the same.

In retrospect there were two problems with the symbol sets that emerged from the students' engagement in this activity. First, the symbols did not represent anything. By way of contrast, when Sandra initially noted that one could change the names of the elements of a group, she did so in the context of a specific group (the SNAP group) in which the elements could be identified independently of their symbol. Second, the group tables the students were comparing all used the same symbol set. Thus it appeared to the students that these were four different groups made up of the same set of elements. So it was not just the symbols that were shared, but the sets, making the differences in the operation tables seem more significant.

In response to the difficulties that emerged as the students worked on this task, I developed a new task for the following session. I introduced the term *isomorphism* as a way to distinguish between pairs of groups that were exactly the same, and pairs that were *essentially* the same (same structure but perhaps different labels for the elements). In this new task, I asked the students if the group of symmetries of an equilateral triangle and the SNAP group could be isomorphic. The students were much more successful in making sense of what this question was asking, probably because they were able to imagine that the symbols in the two tables stood for concrete objects.

For the third teaching experiment, I developed a new starting point for the isomorphism sequence based on these lessons. I presented the students with a 6×6 "mystery table" (Fig. 11) and asked if it could be an operation table for the group of symmetries of an equilateral triangle.

This task has proven to be an effective starting point for the reinvention of isomorphism, probably because it directly builds on an idea that students typically come up with themselves – namely that you would still get the same group if you chose different names for the symmetries of an equilateral triangle.

3.5. Episode 5: Steps toward a formal definition

3.5.1. Episode introduction

As described above, in spite of some complexities introduced by the task design, Jessica and Sandra were able to mathematize the notion of two groups being essentially the same in the form of a procedure involving changing the names of the elements of one group and then comparing its operation table to the operation table of the second group. Such a proce-

*	A	B	C	D	E	F
A	B	A	D	C	F	E
B	A	B	C	D	E	F
C	F	C	B	E	D	A
D	E	D	A	F	C	B
E	D	E	F	A	B	C
F	C	F	E	B	A	D

Fig. 11. The mystery table.

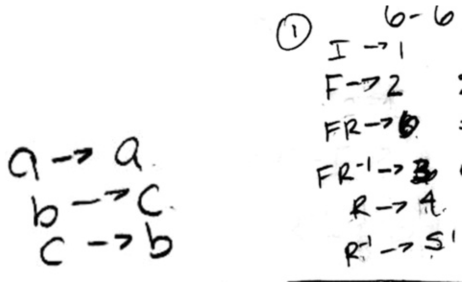

Fig. 12. Arrow diagrams for renaming schemes.

dure anticipates the formal definition of isomorphism since it involves a mapping from one set to the other that (when the groups are essentially the same) demonstrates that the operations are equivalent. However, there are a number of significant details to be worked out in moving from this kind of informal procedure to a formal definition.[4] The students were not able to work out all of these details without significant intervention on my part. However, they did develop a number of important insights that moved them toward the formal definition, and provided inspiration for the design of tasks that have subsequently proved successful in supporting students' reinvention of the isomorphism concept and its formal definition.

3.5.2. Episode description

The first insight that is needed to move from an informal renaming procedure toward the formal definition is the idea that this renaming can be seen as a function. The first appearance of the function idea in the students' work was in the way they represented their renaming schemes when comparing two groups to see if they were essentially the same (see Fig. 12).

It is unclear whether the students saw these diagrams as representing functions. It was not necessary to do so in this small finite case because the students could easily perform the renaming and then check to see if their renaming worked by rearranging the tables (or just by checking that all of the answers were the same across the two tables). With the goal of providing a context in which the students might think of a renaming scheme as a function, I posed the task of determining whether (and then proving that) the group of integers under addition is isomorphic to the group of integer multiples of 5 under addition. I expected that in this infinite case, the students might think of the renaming in terms of a rule and hence as a function.

When faced with this task, Jessica expressed uncertainty as to how she should proceed. She may have been uncomfortable because the students' informal procedure (changing the names and rewriting the table) could not be carried out in this infinite case, or she may have been uncomfortable because such a procedure did not feel like a proof to her. This uncertainty provided an opportunity for me to shift the mathematical activity toward formulating a definition of isomorphism.

Jessica: I don't understand what I need to do to show that they are isomorphic, to prove it.

SL: I think that you should hold that thought, because one reason why we can't prove it is because we don't have a definition.

[4] Recall that formally the groups $(G, \bullet)$ and $(H, \blacklozenge)$ are isomorphic if there is a bijective function $\varphi: G \to H$ with the property that for any two elements a, b in G, $\varphi(a \cdot b) = \varphi(a) \blacklozenge \varphi(b)$.

Are the triangle and SNAP groups the same?

We had two examples of groups that had 6 elements. One was the group of symmetries of an equilateral triangle. The second was the group from the SNAP game. The tables you formed are shown below. Are these the two groups actually the same? Please prove your assertion.

Triangle Group

$\bullet$	I	R	R^{-1}	F	FR	FR^{-1}
I	I	R	R^{-1}	F	FR	FR^{-1}
R	R	R^{-1}	I	FR^{-1}	F	FR
R^{-1}	R^{-1}	I	R	FR	FR^{-1}	F
F	F	FR	FR^{-1}	I	R	R^{-1}
FR	FR	FR^{-1}	F	R^{-1}	I	R
FR^{-1}	FR^{-1}	F	FR	R	R^{-1}	I

SNAP Group

S	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	5	6	3	4
3	3	4	1	2	6	5
4	4	3	6	5	1	2
5	5	6	2	1	4	3
6	6	5	4	3	2	1

Fig. 13. Isomorphism task – "Are the triangle and SNAP groups the same?"

I then suggested that we might be able to formulate a definition after figuring out how to prove these two infinite groups were isomorphic. However, Jessica immediately began to attempt a formulation. After a few moments, I asked her to explain her thinking to Sandra.

Jessica: I was just saying that... that's basically what I was saying that you have to multiply by five 'cause if you take everybody in here and you want to get something in $5\mathbb{Z}$, you just have to multiply by 5.
SL: That's right.
Jessica: And also I was thinking that it has to work the other way. And so I said that they are isomorphic if there's a function from *A* to *B* and then I was thinking about it, you have to be able to get somebody... it has to go both ways. Like you have to be able to say it goes from $\mathbb{Z}$ to $5\mathbb{Z}$ and then from $5\mathbb{Z}$ to $\mathbb{Z}$. 'Cause you have to be able to take one fifth. Given this function you have to be able to rename it over here.
SL: Right, right.
Jessica: So I said it has to be required that it has to be bijective.

As noted above, I did expect that because the groups were infinite, the students might see their renaming in terms of a function (in this case the renaming appears in the form of a rule for a linear function). Jessica's description of the renaming, "if you take everybody in here and you want to get something in $5\mathbb{Z}$, you just have to multiply by 5," certainly sounds like an informal description of a function. And she followed this immediately with an attempt to describe isomorphism in terms of a function.

However, the thinking that resulted in Jessica including the bijectivity condition was not what I expected. I had anticipated that if the students decided to include the condition that the function must be bijective, it would be because this condition is needed to ensure that the size of the groups were the same. Instead Jessica saw the bijectivity as necessary because one should be able to rename the groups in either direction. In retrospect, Jessica's idea that the renaming has to work in both directions makes perfect sense given that she was beginning with the idea that the two groups were equivalent, and that the purpose of the renaming is to associate the elements that correspond. From this perspective, it is clear that the concept of isomorphism is symmetric in the sense that a renaming must work in both directions.

The condition that the function must be bijective is not sufficient for distinguishing between functions that are isomorphisms and functions that are not. It is also necessary for the function to respect the operation in the sense that corresponding pairs of elements should yield products that correspond. This idea is formally captured in the definition with the equation $\varphi(ab) = \varphi(a)\varphi(b)$. When the students worked on the activity "Are the Triangle and SNAP Groups the same?" (Fig. 13), their first renaming scheme did not satisfy this property.

I asked why this renaming ($I \rightarrow 1$, $F \rightarrow 2$, $FR \rightarrow 3$, $FR^{-1} \rightarrow 6$, $R \rightarrow 4$, $R^{-1} \rightarrow 5$) could not work. Jessica eventually stated that "it has to do with its combinations with other things... yeah, its relationship with other elements." I then had them consider a specific example.

SL: So you gave *F* a name and you gave *R* a name. Right? You gave *F* 2 and you gave *R* 4. Isn't *F* times *R* already determined in this table. I mean *F* times *R* is given in this table right?
Sandra: That's the problem, the combinations.
Jessica: Mhm.
SL: So what is the problem? So tell me the problem Sandra.

Sandra: So you gave 2 a name of flip and you gave R a name of 4 but then FR is a name of 3, which is a problem.
SL: What do you think it should be?
Sandra: It should be um. . .
Jessica: 6.
Sandra: 6, yeah.
SL: Because?
Jessica: 2 dot 4 is 6.

After this discussion, Jessica generated a new renaming by choosing new names for F and R (based on the patterns of inverses) and then determining the new names for the remaining elements, implicitly using the this idea that the operations should be respected. However, she did not include this property as a condition when she drafted her first definition of isomorphism.

In order to help the students identify and articulate the operation preserving property as a necessary condition for an isomorphism, I called the students' attention to a bijective renaming of the symmetries of a triangle that did not work to show it was isomorphic to the SNAP group.

SL: Well you said all I needed was a bijective function right? And I'm saying if I put a 6 in there, I've got a bijective function.
Jessica: [Laughing] A bijective function that fits.
SL: That fits?
Jessica: [Laughing] How can you say it?
SL: So maybe you should tell me what it means for it to fit.
Jessica: [Laughing] I don't know.

At this point, Jessica realized that there was something else needed in her definition. I then recalled the earlier situation in which we were comparing groups with four elements. On the board I wrote the bijection, $I \to a$, $a \to b$, $b \to c$, and $c \to I$ and asked what needed to happen for this bijection to show that the groups are isomorphic.

SL: What has to be true for it to work out? For instance if a times b happened to be equal to c, what has to be true? So, the question is if a times b happened to be equal to c, what has to be true?
Jessica: Oh, then b dot c equals I.
SL: Do you buy that Sandra? So she says that if a dot b equals c, b dot c has to equal I.
Sandra: Right.
SL: Why is that the case?
Sandra: Because It's a bijective function.
SL: Right, but why. . .?
Jessica: You just renamed them.
SL: But why is this a necessary property. Why is it necessary that this be equal to I?
Jessica: 'Cause that's how you renamed them.
Sandra: Because c maps to I.
SL: Right, but who cares how they multiply? What does that have to do with anything? What am I trying to show about the groups?
Jessica: That they are isomorphic?
SL: How do I tell that they are isomorphic?
Sandra: You can rename them and. . .
SL: What has to happen?
Sandra: The operation has to work the same for both sets.
SL: The operation has to work the same.
Jessica: [Laughing] Yeah, so how do we say that?

3.5.3. *Episode wrap-up*

The idea of thinking of a renaming as a function and then formulating the definition of isomorphism in terms of a bijective function seemed quite natural to the students. It was interesting that Jessica realized the function needed to be bijective because she knew it should work in both directions (i.e., because she saw isomorphism as necessarily symmetric).

However, the additional property required of an isomorphism (operation preservation) did not emerge easily. Jessica and Sandra often seemed to be aware that the key was that the renaming had to respect the operation. In fact, Jessica had made fairly explicit use of operation preservation when she generated her renaming schemes to compare the group of symmetries of an equilateral triangle with the SNAP group. Jessica chose two numbers to associate with R and F (based on the patterns of inverses) and then multiplied to determine the appropriate correspondences for the remaining elements. Furthermore, when considering a specific renaming that did not work, Jessica and Sandra were able to explain that it failed because a pair

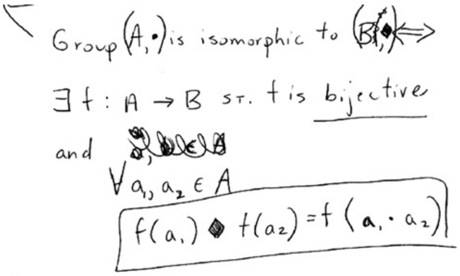

Fig. 14. A definition of isomorphism.

Fig. 15. Kathy's formulation of the operation preservation property.

of corresponding elements did not have products that corresponded. Eventually, building on this I was able to help them formulate the condition formally so that they could complete their definition (Fig. 14).

In the subsequent teaching experiments, I found that the other students also had difficulty pinning down and articulating this final condition needed for an isomorphism. However, when I retrospectively analyzed all three of the teaching experiments, I found (in different forms) the same two areas of success that I found in Jessica and Sandra's work.

First, in all three teaching experiments, the students were able to explain for specific examples what had to happen for the mapping to work. For example, during the second teaching experiment I asked the students to determine whether a given 6×6 operation table (the mystery table shown in Fig. 11) could be the operation table for the SNAP group. Erika explained that, "So our table is basically ordered pairs mapping to one thing. Right? Like (2, 3) goes to 4. So if we just renamed it using these letters we'd have like (*B*, *C*) goes to *E*. And then we check it on this table to see if it equals *E*, which it does. Then if you did that for like every ordered pair and they were the same then they would be equal." Here Erika notes that in the SNAP group, the product of 2 and 3 yields 4. Using the renaming scheme they had established, this would imply that the product of *B* and *C* is *E* in the mystery table. Since this is the case, the renaming works at least for this pair of elements. Note that Erika also was aware that this check needed to be performed for all of the pairs of elements, but at this point was unable to formulate an expression to represent this check for an arbitrary pair of elements.

Subsequently, these examples were helpful in terms of making the operation preservation property more explicit and in terms of supporting the formulation of a general expression of this property. For example, following Erika's explanation, her partner Kathy was able to formulate a general statement of the property (Fig. 15) that is equivalent to (but significantly different from) the statement usually included in the definition of isomorphism.

Second, in all three teaching experiments, the students used operation preservation implicitly to construct renamings that worked. For example, during the second and third teaching experiments, the students used partial operation tables to figure out how to map remaining elements after arbitrarily assigning some of the elements (to elements with the same order). This can be seen in Erika's work during the second teaching experiment (Fig. 16).

The work shown in Fig. 16 comes from Erika's work on the mystery table task (with a different table from the one shown in Fig. 11). She decided to let *E* from the mystery table correspond to the element 4 from the SNAP group (because neither of these were self-inverses). She let the element *F* correspond to the element 5 (these were the only remaining elements that were not self-inverses). She then drew the portion of the mystery table that displayed the products in which either *E* or *F* was multiplied by *B*, *C*, or *D* (Fig. 16 left). Then she drew the corresponding portion of the SNAP table – the part that displayed products in which either 4 or 5 was multiplied by 2, 3, or 6 (Fig. 16 center). Finally she arbitrarily let element *B* from the mystery table correspond to 2 from the SNAP table (these two non-identity elements are self-inverses) and then replaced all of the 2's in the partial SNAP table with *B*'s (Fig. 16 right). Then by looking up the products *EB* and *FB* in the partial mystery table, she was able to determine that *D* would have to correspond to 6 and *C* would have to correspond to 3. This completed the needed mapping (the two identity elements had already been paired together).

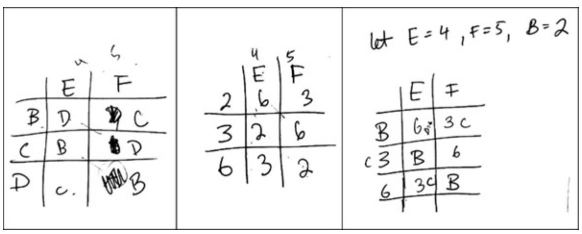

Fig. 16. Partial tables used by Erika to complete a mapping that respected the operations.

Building on these two ideas, I have been able to construct an instructional sequence that seems to effectively support students in completing the formulation of the definition of isomorphism. The sequence begins with a version of the mystery group task (in the new version they are asked to compare the mystery group to the group of symmetries of an equilateral triangle). The second task in the sequence presents the students with a renaming ($A \leftrightarrow F$, $B \leftrightarrow I$, $C \leftrightarrow FR^2$, $D \leftrightarrow R$, $E \leftrightarrow FR$, $F \leftrightarrow R^2$) that seems sensible (e.g., self-inverses map to self-inverses) but does not work. The students are asked to explain why it does not work, providing them with an opportunity to notice and articulate the fact that the renaming has to respect the operations. This task is followed by one in which I provide a partial renaming (the generators F and R are assigned to corresponding elements of the mystery table), and the students are asked to figure out how to rename the remaining elements. This task provides an opportunity for the students to *use* the idea that the operation must be preserved. The sequence has been effective in whole-class settings and seems to prepare the students for the (still nontrivial) task of formally expressing this property and formulating the definition.

4. Summary

The purpose for conducting the teaching experiment with Jessica and Sandra was to figure out *how* they could reinvent the concepts of group and isomorphism. Drawing on the ideas of Freudenthal (1973) and Burn (1996), I decided to launch the process by having the students explore the symmetries of an equilateral triangle and then mathematize this activity with a focus on combining symmetries.

As the teaching experiment unfolded, I was unsure how the group axioms would emerge. I quickly learned from Jessica and Sandra that at least some of these could emerge naturally from a shift toward *calculating* combinations rather than carrying out the sequence of motions physically. During my retrospective analysis of all three teaching experiments, I learned that the inverse axiom would not necessarily emerge this way, because it is not needed to perform calculations (although specific inverse pairs are commonly cancelled). Again, I learned from the students how to support the emergence of this axiom (by considering the conjecture that each element must appear exactly once in each row of the operation table).

I attempted to initiate the reinvention of the isomorphism concept by having the students figure out how many four-element groups existed. I expected that they would figure out that they could construct four different group tables and then figure out that three of them were really the same (by renaming the elements and re-ordering the tables). In this case, what I learned from the students was that this task presented a number of obstacles that inhibited them from drawing on their intuitive ideas of isomorphism to construct a procedure for determining whether two groups were essentially the same. I later found that they could successfully construct such a procedure in a context where the symbol sets for the two groups were *not* the same, and where one of the groups was a familiar group (e.g., the symmetries of an equilateral triangle).

After the students had constructed a procedure for determining whether two groups were isomorphic, the challenge was to support them in developing a definition of isomorphism based on this procedure. Again from the students I learned some effective strategies. First I learned that students could examine a (failed or successful) renaming and articulate in specific cases the need for the operations to be respected. Second I learned that students may implicitly use this property to construct renamings that work. These lessons led to the development of a task in which students *explicitly* use operation preservation to construct a renaming, setting the stage for formulating this property formally.

Freudenthal (1973) and Burn (1996) argued for the potential value of the context of geometric symmetry in supporting students' learning of group theory. Dubinsky et al. (1997) warned that although an expert might see the formal concepts in this geometric setting, it may still be quite difficult for a student to abstract the formal concepts from the examples. Assuming that the context of geometric symmetry can provide a rich and natural context for developing the concepts of group theory (and the results reported here suggest that it can), the challenge is to learn how to leverage this context to support students' learning. The five episodes described here exemplify how we can learn *from students* how to support students' learning. Working with Jessica and Sandra I discovered student strategies that anticipated the formal concepts of group theory that I wanted them to learn. By analyzing their mathematical activity, and the tasks with which the students engaged, I was able to learn how these strategies could be evoked and how they could then be leveraged to support the development of the formal concepts. In addition to these specific lessons regarding how one could support the reinvention of the concepts of group and isomorphism, these episodes emphasize the value of attending to student thinking not just during instruction, but also during the curriculum design process.

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