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Developing instructor support materials for an inquiry-oriented curriculum

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ABSTRACT

The purpose of this paper is to describe the process of designing web-based instructor support materials for an inquiry oriented abstract algebra curriculum. First we discuss the ways in which the research literature influenced the design of the instructor support materials. Then we discuss the design-based research methods used to develop the instructor support materials, elaborating the ways in which the research phases of our work contributed to the design of the instructor support materials. This discussion includes specific examples of important insights from our research and precisely how these were incorporated into the support materials.

1. Introduction

In recent years reform-oriented mathematics curricula have become increasingly prevalent, gaining attention among policy-makers and enjoying, in some instances, widespread implementation. This has occurred primarily at the K-12 level, with school districts across the country using curricula such as Investigations (TERC, 1995) and the Connected Mathematics Project (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). There have also been innovations of this type in undergraduate mathematics instruction, including the Harvard Calculus program (Calculus Consortium at Harvard, 1994) and Henderson's 'Experiencing Geometry' text (Henderson & Taimina, 2004).

Recently, a number of researchers have conducted instructional design studies aimed at leveraging the theory of Realistic Mathematics Education (RME) for supporting the learning of undergraduate mathematics. There are ongoing RME-guided instructional design projects at the undergraduate level in the areas of differential equations (Rasmussen, 2007), abstract algebra (Larsen, Johnson, & Bartlo, 2013), and advanced calculus (Oehrtman & Swinyard, 2011). These studies are resulting in innovative student-centered instructional approaches that place significant demands on instructors in terms of making sense of and leveraging students' thinking to advance the instructional agenda (Johnson & Larsen, 2012; Speer & Wagner, 2009).

The process of scaling up these innovations to serve a wider audience is a significant challenge but is necessary if such innovations are to have a real impact on STEM (Science, Technology, Engineering, and Mathematics) education. The National Science Foundation calls for such work in its solicitation for the TUES (Transforming Undergraduate Education in Science, Technology, Engineering, and Mathematics) program:

> This solicitation especially encourages projects that have the potential to transform the conduct of undergraduate STEM education, for example, by bringing about widespread adoption of classroom practices that embody

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Table 1
Project activities featuring data collection.

Year	Stage	Activity	Units
2002	1	Sequence of Three Small-Scale Design Experiments (*7–8 90-minute sessions w/two students*)	Group & Isomorphism
2004–2005	2	Experimental Teaching in Group Theory Course w/Designer as Teacher (*approx. 30 students for 10-week course*)	Group & Isomorphism
2005	2–3	Teaching Experiment in Algebra Course for K-12 Teachers (*10 2-hour sessions with 4 sections of 15–24 teachers*)	Group & Isomorphism
2006	1	Small-Scale Design Experiment (*10 90-minute sessions with two students*)	Quotient Groups
2007	2–3	Whole Class Teaching Experiment w/Mathematician as Teacher (*approximately 35 students for 10-week course*)	Full Curriculum[a]
2008–2009	3	Implementation by Three Mathematicians w/Limited Support – Instructor Notes & Email Conversations (*each approximately 35 students for 10-week course*)	Full Curriculum
2011	3	Implementation by a Mathematician at a Different University w/Only Web-Based ISMs for Support[b] (*approx. 30 students for 15-week course*)	Full Curriculum

[a] Debriefing/planning video for full course + classroom video for *quotient groups* only.
[b] Data include smart board video capture and audio recordings of class sessions.

understanding of how students learn most effectively. Thus transferability and dissemination are critical aspects for projects developing instructional materials and methods and should be considered throughout the project's lifetime (National Science Foundation, 2011, p. 4).

The primary goal of the *Teaching Abstract Algebra for Understanding* (*TAAFU*) project has been the scaling up of an RME-based group theory curriculum, which we refer to as the TAAFU curriculum. This curriculum was developed through a series of design experiments (Design-Based Research Collective, 2003) and has been refined through several iterations of classroom trials. More recently, a number of mathematicians (who were not involved in the development of the curriculum) have implemented the curriculum. The experiences and backgrounds of three of these mathematicians are discussed in detail in Johnson, Caughman, Fredericks, and Gibson (2013).

The process of scaling up to include instructors not involved with the creation of the TAAFU curriculum has motivated us to design Instructor Support Materials (ISMs) to support teachers in successfully implementing the curriculum. These ISMs take the form of an interactive website that provides instructors with a number of resources to help them implement the curriculum effectively and faithfully.

The purpose of this paper is to describe the process of designing our ISMs. First we discuss the ways in which the research literature influenced the design of the ISMs. Then we discuss the design-based research methods used to develop the ISMs. Finally, we discuss the ways in which the research phases of our work contributed to the design of the ISMs. This discussion includes specific examples of important insights from our research and precisely how these were incorporated into the support materials. Before we describe our design process, we briefly describe the TAAFU curriculum and the ISMs that we have created.

2. Overview of the TAAFU curriculum and instructor support materials

The development of the TAAFU curriculum and the accompanying ISMs was a multi-year, multi-stage project. Table 1 provides a timeline of the primary project activities that featured data collection. All small-scale design experiments were video-recorded using one camera and (unless otherwise noted) all classroom activity was recorded using two cameras to capture video of small group and whole class activity.

2.1. The curriculum

The TAAFU curriculum is a research based, inquiry-oriented abstract algebra curriculum that actively engages students in developing the fundamental concepts of group theory. The *TAAFU* curriculum was primarily designed to be used in an upper-division, undergraduate abstract algebra course and is composed of three primary units: groups/subgroups, isomorphisms, and quotient groups. Each unit begins with a *reinvention phase* in which students develop concepts based on their intuition, informal strategies, and prior knowledge. The end product of the reinvention phase is a formal definition (or definitions) constructed by the students and a collection of conjectures. The deductive phase begins with the formal definitions that are relevant to the concept. During this phase, students work to prove various theorems (often based on conjectures arising during the reinvention phase) using the formal definitions and previously proved results. For a detailed description of the curriculum please see Larsen et al. (2013).

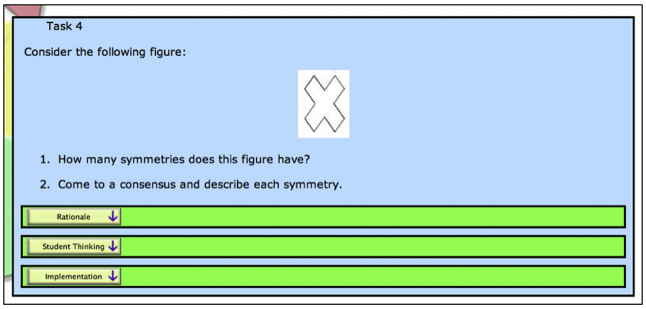


Fig. 1. Default lesson page.

2.2. The instructor support materials

The design of the website puts the instructor in control of the content. The goal is to avoid overwhelming the instructor with information while still making available any information the teacher might find useful. The default setting (Fig. 1) displays only the instructional tasks. For any task, the instructor can then select to expand any (or all) of three expandable text boxes (Fig. 2).

One of these text boxes contains an elaboration of the **rationale** for the task. At the top of each lesson page, the objectives of the lesson are stated. The information in the rational textbox explains the purpose of the task in supporting the learning objectives of the lesson. The goal is to address such questions as:

- *How does the task contribute to meeting instructional goals?*
- *What kinds of thinking is it meant to evoke, leverage, or challenge?*

The second textbox contains information about **student thinking**. This includes information about how students are likely to approach the task as well as what difficulties they might encounter. The goal is to address such questions as:

- *What are some surprising ways they might think about or approach the task?*
- *What kinds of answers/strategies will they likely come up with?*
- *What kind of difficulties are they likely to have?*

The third expandable textbox addresses the **implementation** of the task. It contains suggestions regarding facilitation as well as some discussion of known challenges involved in implementing the task. The goal is to address such questions as:

- *What kinds of discourse protocols might be most productive?*
- *What kinds of follow-up questions might be productive?*
- *What options are there for re-sequencing tasks?*

When appropriate, these textboxes contain relevant media including images of student solutions and classroom video. Further, selected expandable textboxes contain links to relevant published research articles or to special pages dedicated to addressing particularly important issues. Each lesson page also features an option to expand or contract all textboxes of a

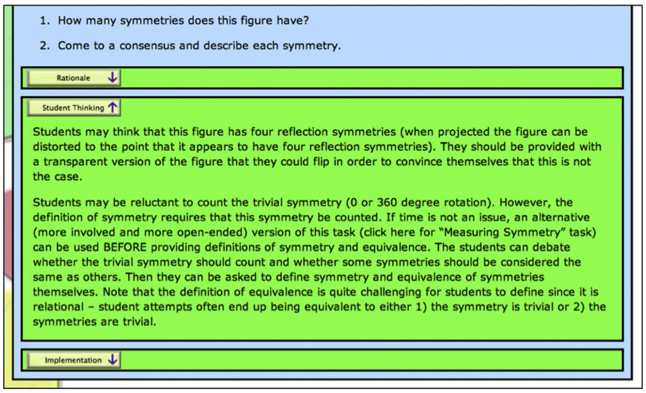


Fig. 2. Lesson page with expanded student thinking textbox.

certain kind as well as an option to print a lesson sheet containing all of the instructional tasks and the information in any expanded boxes.

3. Insights from the literature

In developing the ISMs, we considered recommendations from existing research on curricular materials as we made decisions about how to create and design our ISMs. In this section we discuss the literature on support materials, making connections both to research on teacher knowledge and our work in designing the ISMs for the TAAFU curriculum. We then discuss literature related to the Connected Mathematics Project (Lappan et al., 1998) and make connections between their design process and our own.

3.1. Literature related to instructor support materials (teacher knowledge)

The relationship between a teacher, a curriculum, and the teacher's implementation of that curriculum is a complex matter (Ball & Cohen, 1996; Ben-Perez, 1990; Shulman, 1986). In 1996, Ball and Cohen raised questions regarding what role curriculum materials might play, both in instructional reform and in teacher learning. They called for researchers to attend more carefully to the ways in which teachers might meaningfully and effectively use curriculum materials, and they suggested that teachers' guides could be potentially powerful avenues by which to support curriculum enactment.

Ball and Cohen (1996) note that, "teachers' guides could help teachers to learn how to listen to and interpret what students say, and to anticipate what learners may think about or do in response to instructional activities" (p. 7). Additionally, based on her research into teachers' interactions with curriculum materials, Remillard (2000) suggests that, "texts must prompt teachers to examine students' thinking with respect to the goals of the mathematical tasks. This sort of reading might be encouraged through descriptions of possible student responses or work and how they could be interpreted" (p. 345). Similarly, Davis and Krajcik (2005) discuss the educative nature of instructor support materials. They provide specific guidelines for the design of educative curriculum materials. One of these guidelines is that educative curriculum materials "could help teachers learn how to anticipate and interpret what learners may think about or do in response to instructional activities" (p. 5).

These papers emphasize the value of having curricular materials address student thinking. We see these recommendations as closely related to both *specialized content knowledge* and *knowledge of content and students*. Ball, Thames, and Phelps (2008) define specialized content knowledge as a category of Mathematical Knowledge for Teaching that is specialized because "it is not needed or used in settings other than mathematics teaching" (p. 397). This specialized content knowledge supports the mathematical work of teaching, such as "looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general" (p. 400). Ball et al. define *knowledge of content and students* as "knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing" (p. 402).

In the *student thinking* sections of our instructor support materials we share information about difficulties that students are likely to experience, as well as information about what kind of strategies they are likely to bring to bear on a task. We supplement this information with media in the form of written samples of student work and video clips of classroom activity. The goal is to support teachers in developing *knowledge of content and students*. At the same time, we offer mathematical analyses of likely student approaches in an effort to help teachers develop the "uncanny kind of unpacking of mathematics that is not needed – or even desirable – in settings other than teaching" (p. 400) that Ball et al. (2008) associate with *specialized content knowledge*.

The literature also suggests that curricular materials and ISMs could help teachers contextualize lessons and content across an entire unit, term, or even school year. Davis and Krajcik (2005) state that, "curriculum materials could help teachers consider ways to relate units during the year" (p. 5). They suggest that such activity could help the teacher reflect on how the current lesson fits into the larger curricular picture. Similarly, Ball and Cohen (1996) note that, "teachers' guides rarely help teachers to think about the temporal dimension of curriculum construction" (p. 7). They see the potential in teachers' guides to provide some broader context for teachers, making teachers more aware of how a given lesson fits into the entire year.

We see these recommendations as related to Ball et al.'s (2008) *knowledge of content and curriculum*. This category of Mathematical Knowledge for Teaching is defined in terms of Shulman's (1986) curriculum knowledge construct. Shulman defines one type of curriculum knowledge as "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Shulman, 1986, p. 10). In our ISMs, we attempt to address these recommendations in our *rationale* sections. In these sections, we explain the purpose of each instructional task and situate it within the larger instructional trajectory.

Researchers also argue that curriculum and support materials have the potential not only to help teachers learn content, but also to help them learn how to implement the content in the classroom setting. Remillard (2000) discusses the need to help teachers in their decision-making processes; while Davis and Krajcik (2005) argue that educative curriculum materials should serve to "promote a teacher's pedagogical design capacity, or his ability to . . . adapt curriculum to achieve productive instructional ends" (p. 5).

We see these recommendations as being related to *knowledge of content and teaching*, which Ball et al. (2008) define as knowledge that "combines knowing about teaching and knowing about mathematics" (p. 401). They point out that it is important for teachers to be able to coordinate their knowledge of mathematical content with how they might teach that content in the classroom. We attempt to support teachers' development of this kind of knowledge in the *implementation* sections of our ISMs. We relate our implementation recommendations to the task rationale and information about student thinking. The goal is to support teachers in drawing on their content knowledge to make pedagogical decisions.

In summary, our review of the literature related to ISMs suggests that such materials can support teachers in developing Mathematical Knowledge for Teaching. Specifically, we have identified recommendations that we interpret as calls for ISMs to address four of the domains of Mathematical Knowledge for Teaching described by Ball et al. (2008). In response, we have (1) designed the student thinking sections of our ISMs to support teachers' development of *specialized content knowledge* and *knowledge of content and students*, (2) designed the rationale sections of our ISMs to support teachers' development of *knowledge of content and curriculum*, and (3) designed the implementation sections of our ISMs to support teachers' development of *knowledge of content and teaching*.

3.2. Literature related to instructor support materials (practical suggestions)

The literature on instructor support materials also provides a number of practical suggestions regarding the design of ISMs. For example, Stylianides (2007) highlights language, coherence, and presentation as key issues for curriculum designers, and he offers examples of different formats through which teacher guidance is displayed. Davis and Krajcik (2005) note that it is imperative to consider "form and format, as well as content" (p. 9) when designing educative curriculum materials. They also argue for the potential benefit of presenting support materials online, highlighting the fact that such an online structure could facilitate the inclusion of audiovisual resources such as videos of teachers' lessons.

Our interactive website design specifically addresses the issues identified by Stylianides (2007) and Davis and Krajcik (2005). In particular, the online format allows teachers to engage with the provided information interactively. We include resources that contain audio and video elements to help teachers get a sense of what the implementation of the curriculum may actually look like. Indeed, in debriefing interviews with teachers, some noted that it was valuable to see what the classroom implementation actually looked like, and to get a feel for the classroom dynamics.

3.3. Literature related to instructor support materials (CMP)

Schneider and Krajcik (2002) note that there is little in the research literature on the development of ISMs. However, the Connected Mathematics Project (CMP) is one exception and serves as a significant example illustrating the process of designing instructor support materials. In this section we briefly outline the CMP development process from the designers' perspective and draw parallels between their design process and our own.

Lappan and Phillips (2009) indicate that a problem-centered curriculum "requires that the teacher possess a broad view of mathematics and mathematical goals. . .and a deeper knowledge of pedagogy to support 'inquiry"' (p. 13). In response to this need, Lappan and Phillips provided examples of possible classroom discourse. As noted in the previous session, the TAAFU instructor support materials include samples of actual classroom discussions.

Phillips, Lappan, Friel, and Fey (2001) note that the CMP authors also chose to provide mathematical overviews of the lessons in the teacher guides, allowing for teachers to have a better overall sense of the mathematical trajectory. In designing our ISMs, we similarly sought to provide teachers with resources that would enable them to have a sense of the broader context, so that they could be oriented to the overall mathematical trajectory of the curriculum. For each task, we provide a rationale explaining how the task supports the instructional goals of the curriculum.

The CMP curriculum and teacher guides were developed via an iterative field-testing and refinement process. Lappan and Phillips (2009) indicate that their curriculum underwent three years of field-testing, and they carefully incorporated feedback from teachers and students as they refined the curriculum and considered what should be incorporated into teacher guides. As a specific example of how this iterative field-testing approach affected the teacher guides, Lappan and Phillips point out, "scenarios of teacher and student interactions with the materials became the most compelling parts of the teacher guides" (p. 13). Thus, the designers' experiences with field-testing the curriculum directly supported the development of authentic classroom scenarios that could be shared with teachers in the ISMs. This has also been an important aspect of our design process. Our work collaborating with mathematicians who implemented the TAAFU curriculum has resulted in a library of classroom scenarios that we have incorporated into the ISMs in the form of classroom discussion excerpts and video clips. In the following section we describe our method for designing our ISMs and highlight the iterative nature of our design process.

4. Methods: the design process

Our development of the TAAFU curriculum and the ISMs is rooted in a design research methodology. The Design-Based Research Collective (2003) calls design-based research "an emerging paradigm for the study of learning in context through the systematic design and study of instructional strategies and tools" (p. 5). Central to the process of design-based research

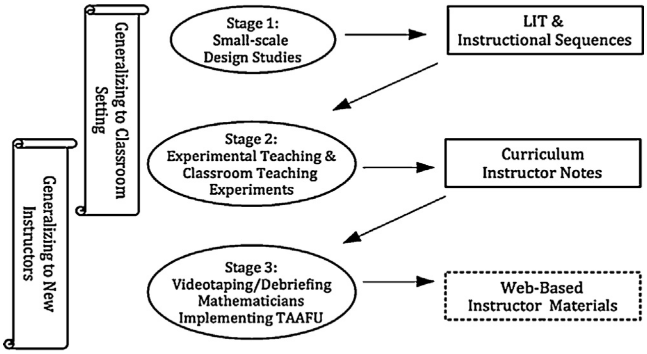


Fig. 3. TAAFU project design.

is its iterative nature, in which "development and research occur through iterative cycles of design, enactment, analysis, and redesign" (p. 5). Fig. 3 illustrates the overall design of the TAAFU project.

The first stage of the TAAFU project (supported by the RME instructional design theory) resulted in the initial design of the three core instructional units of the curriculum. These initial designs emerged along with local instructional theories from a set of small-scale design experiments conducted with pairs of students (see Larsen, 2013; Larsen & Lockwood, 2013). These instructional theories included a rationale for the instructional sequences. This rationale, and insights gained about student learning, provided a starting point for the design of our ISMs. In fact, the early written descriptions of these local instructional theories (Larsen, 2004; Larsen, Johnson, Bartlo, & Rutherford, 2009) can be considered to be the pilot version of the ISMs.

The primary goal of the second stage of the research and design process was to generalize from the initial design context (two students in a laboratory setting) to a more authentic context (a full classroom). Two significant products emerged from this second stage, a full group theory curriculum and a set of instructor notes. These notes included insights gained from adapting the initial instructional sequences for use in a regular classroom setting. An illustrative example appears in Fig. 4.

The third stage of the project featured a shift to instructors who had not been involved in the curriculum design process. Dr. James was the first mathematician to implement the full curriculum in a regular group theory course. The TAAFU team met with Dr. James once a week for video-recorded debriefing and planning meetings. Based on this trial, we redesigned the ISMs. These new written notes included three types of information for each task: (1) the goal of each task and how it fit within the larger unit, (2) implementation suggestions, and (3) descriptions of the kinds of student thinking the instructor should expect. An illustrative example appears in Fig. 5.

More extensive trials followed Dr. James' initial implementation of the curriculum. Over the next year, three mathematicians (including Dr. James again) implemented the curriculum, and every regular session of each of these courses was videotaped. Based on our analyses of this video data, we initiated a major redesign of the ISMs. We shifted to the current web-based interactive format for the materials and began to include a variety of media to support teachers including pictures of student work, transcripts of classroom discourse, classroom video clips, and links to relevant published research reports. On a global level, our goal was to improve the accessibility and quality of the information we found to be most important for supporting instructors. In the next section, we discuss in detail the ways in which our research activities have been informing our development of these online ISMs.

1. Prove or disprove these are groups:
   a. The set {-1, 0, 1} with regular addition.
   b. The set {-1, 0, 1} with multiplication.
   c. The set {-1, 1} with multiplication.

Notes:
- Have them do these individually.
- Then have them compare with their neighbor.
- Then have some of them yell out the "answers"

Tricky parts:
- Many students have some bad habits that should be addressed here. They tend to check the inverse property first (which of course makes no sense until the identity is identified (pun intended)).
- Tell them that they should really do it in the order: closure, identity, inverses, associativity
- If you don't address this issue here it can be done later on the Day 7 task with the 2x2 groups. The possible downside of doing it here is that they already "know" the identity in these cases.

Fig. 4. First version of instructor notes.

Activities/Prompts

1. Can a group have two identity elements?

2. State and prove a cancellation law for groups.

3. Prove or disprove these are groups:
   a. The set {-1, 0, 1} with regular addition
   b. The set {-1, 0, 1} with multiplication
   c. The set {-1, 1} with multiplication

Student Thinking

1. Students may not know where to start with this proof. It may be helpful to direct them to the definition of an identity element or back to their operation table of the symmetries of a triangle. In either case they should be able to form an algebraic expression about what it would mean for two elements to be the identity elements of a group. After finding an algebraic expression, they should be able to answer this question using the property of inverses. This would be a good place to discuss quantifiers.

2. The students might have a difficult time making sense of this prompt, however they have most likely been aware, and possibly using, this property while calculating the symmetries of a triangle. While working with inverses students often leave off the "for all" quantifier, or possibly include it without understanding how to work with it.

3. Students may have a difficult time switching between additive and multiplicative identities and inverses. Also, students may be looking for inverses before they look for the identity element. To help then with this, remind them of the definitions that they used as part of the definition of a group.

Students may want to say that they already know that addition of integers is associative. Which is correct, but it is up to you whether or not you want them to verify this for these small sets.

Fig. 5. Paper version of instructor support materials.

5. The impact of research phases on instructor support materials

In this section we describe specific ways in which the research phases of the design process had an impact on the ISMs. We highlight three examples that detail the ways in which findings from our analyses of teaching and learning in the context of the TAAFU curriculum influenced the development of the ISMs. The first example illustrates the role that the research phases played in informing our creation of the rationale and implementation sections of the ISMs. We see the rationale sections as promoting teachers' knowledge of content and curriculum and the implementation sections as promoting teachers' knowledge of content and teaching. The second example shows how we are leveraging our research efforts to promote teachers' development of knowledge of content and students. We do so by using our research findings to inform the student thinking sections of the ISMs. The third example demonstrates how we are using our research efforts to support teachers' development of specialized content knowledge (also via the student thinking sections).

5.1. *Influence of research: knowledge of content and curriculum & knowledge of content and teaching*

The first unit of the TAAFU curriculum is designed to support students in reinventing the group concept. The unit starts with an investigation in the context of geometric symmetry. First, students are asked to identify and symbolize the symmetries of an equilateral triangle. Later, they are to consider all combinations of two symmetries. For each combination, they are asked to figure out which symmetry is equivalent to it. The intent is for the group concept to emerge from this activity (see Larsen, 2013).

5.1.1. *Key insight*

The primary objective of the first design experiment was to identify informal student strategies that anticipated the group concept and figure out how consistently to evoke these strategies and leverage them to develop the formal concepts. Perhaps the most important student strategy that emerged during the first design experiment was the use of rule-based calculations to calculate combinations of symmetries. The first pair of students we worked with, Jessica and Sandra, quickly began to notice patterns and then formulated rules that they used to compute combinations. For example, Jessica noticed that the combination of a flip followed by a rotation and then another flip was equivalent to the same rotation in the opposite direction.

Jessica: So if a movement, a rotation, comes in between two flips then it does the reverse of what it did between them. Does that make sense? If there's something that happens between the two flips then it gives the reverse direction.
Sandra: Okay yeah. That's pretty cool.

The students then used these kinds of observations to compute combinations of symmetries. In the next excerpt, we see Sandra verbalizing such a calculation (note that *CC* represents a 120° counterclockwise rotation, *CL* represents a 120° clockwise rotation, and *F* represents a flip across the vertical axis of symmetry).

Sandra: *FCL* to *FCC* it gives clockwise. *FCL* and *F*, these two get counterclockwise, and counterclockwise and counterclockwise give clockwise.

We can see that in addition to using Jessica's observation about the interaction between flips and rotations, Sandra also implicitly uses the associative law when she groups *FCL* and *F*. After regrouping, she then substituted *CC* for this combination and then used a previous observation that the combination *CC CC* is equivalent to *CL*.

Rationale

This will be the first step toward establishing a common set of symbols for the class. The expressions that the students produce here will be considered when the set of symbols is selected. The students should be encouraged to generate multiple answers to each as this will result in a richer discussion and a more informed selection of final symbols.

This task is designed to move the students toward a set of symbols that are expressed in terms of two generators. This is crucial since the resulting composite symbols (e.g. *FR*) will promote the use of rule-based calculations to compute combinations. When symmetries expressed as compositions are combined the students will notice that they can regroup symbols to simplify using inverse and identity properties. For example, the expression *F*(*FR*) should suggest to students the strategy of grouping the *F*'s together and then cancelling them to get *R*.

Note that this task also will help students to develop rules for calculating combinations because the multiple expressions they produce for each symmetry will provide them with relations (rules). For example, they will produce the expressions *FR* and *RRF* (or something similar) for one of the flips across a diagonal axis establishing the dihedral relation *FR* = *RRF*.

Fig. 6. Expanded rationale textbox.

Jessica and Sandra performed a number of calculations of this type, many of which also used the identity property (e.g., the combination of the "do nothing" move, *N*, with any other symmetry was replaced by that symmetry) and the inverse property (e.g., the combination *FF* was replaced by *N*). Since these strategies relied on the properties that defined the group concept, it was clear that this student strategy could be a cornerstone of the local instructional theory for the group concept. Thus, during our ongoing and retrospective analyses associated with this first design experiment, we strove to learn how to evoke this strategy and to leverage it to support the reinvention of the formal group concept.

During our retrospective analysis it became clear that the students' use of "compound" symbols was key to the development of the rule-based computation strategy. The first pair of students used the term compound to refer to the symmetries they designated by *FCL* and *FCC*. These compound symbols, when concatenated with another symbol, evoked the strategy of regrouping to associate primitive symmetries that could then be cancelled or replaced with an equivalent primitive symmetry. Note that if the students had used six independent symbols for the symmetries, all combinations would be represented by a pair of symbols, which would clearly not promote the idea of regrouping (Larsen, 2013).

5.1.2. *Modification of ISMs in relation to key insight*

This observation from our analysis led to the design of a task in which students are asked to express (in multiple ways) each of the six symmetries in terms of two symmetries (a reflection across the vertical axis, *F*, and a clockwise rotation of 120°, *R*). In the ISMs we provided a rationale for this task (Fig. 6), explaining the role of the task in evoking the strategy of performing rule-based calculations.

As the project transitioned from the initial design studies to trials in regular classrooms, and with instructors who were not involved in the curriculum design process, our research efforts continued to provide insights that informed the design of the ISMs. Some of these insights, related to pedagogical decisions and content, were captured in the implementation sections. For instance, after the students create a number of expressions for each symmetry of an equilateral triangle in terms of *F* and *R*, a follow up task includes a whole class discussion in which students select the common set of symbols to be used by the class. The implementation section for this task (Fig. 7) discusses mathematical considerations and relates these to suggestions for facilitating this discussion.

5.1.3. *Discussion*

This first example illustrates the contribution of our research activities to the development of the rationale and implementation sections of the ISMs. The initial design experiments were particularly important as we developed the rationale sections of the ISMs because they produced the instructional theories that underlie the instructional sequences. We see these rationale sections as promoting the development of teachers' *knowledge of content and curriculum*. For each task we provide information about how this task fits into the larger instructional plan and make connections to mathematical ideas that will relate to subsequent lessons. This is exemplified in Fig. 6 by the statement that the development of compound symbols will help evoke the strategy of performing rule-based calculations. In this way, we attempt to contextualize the content for the teachers, clearly pointing out how a given concept fits within the curriculum as a whole.

As we moved to classroom trials, we gained insight into how the content was related to the pedagogical decision-making that is necessary to implement the curriculum. In our implementation sections we share these insights with instructors. This is exemplified by the suggestion in Fig. 7 to discuss the need to have a consistent notational system rather than one that mixes additive and multiplicative notation. In this way, we see our implementation sections addressing the need identified by Remillard (2000) for support materials to aid teachers in their decision-making processes. Specifically, we see these sections as promoting the development of the *teachers' knowledge of content and teaching*.

Implementation

Have some students share their results from Task 3 and record these on the board or on the overhead projector. You will need a way to articulate which symmetry is being expressed that is independent of the symbols. Diagrams illustrating the movement of the vertices work well. Click here for a table using this approach that could be used as an overhead transparency for recording answers. In this link we can see a what one teacher recorded during this classroom discussion.

After multiple expressions are recorded for each symmetry lead a discussion to choose one of these (for each symmetry) as a common symbol for the class. Debate the merits of the various expressions and eliminate those that are not likely to be useful as symbols (e.g. *FFF* as an expression for *F*).

Be explicit about the use of additive and multiplicative notation (as we see an instructor do in this clip) and discuss the merits of each. Have the students give arguments in support of both and debate them. Finally have a vote or make a decision. Be sure to discuss the need to have a consistent system - not one that mixes additive and multiplicative. Finally have the class make fine-tuning decisions (like whether to use RR or R^2).

Fig. 7. Expanded implementation textbox.

5.2. *Influence of research: knowledge of content and students*

The initial design experiments generated insights into how students would think about the operation of combining symmetries. In particular, we found that students tended to think about the operation in terms of a left to right sequential procedure. For example, the combination *FR* would be seen as sequence in which one first flipped a triangle along the vertical axis of symmetry and then rotated the triangle 120° clockwise. This is a very natural way of thinking about this operation, and it is a powerful perspective in many ways. However, it does present some challenges as students engage in some of the TAAFU instructional tasks. For example, Larsen (2010) reported that this perspective influenced how students thought about the associative and commutative properties. However, this insight gained from our initial design work did not find its way into our ISMs until we conducted research into mathematicians' experiences implementing the curriculum (Johnson & Larsen, 2012).

5.2.1. *Key insight*

As we began conducting research on teachers' implementations of the TAAFU curriculum, we were able to gain new insight into challenges that instructors might face. Further, recognizing and analyzing instances in which teachers struggled sometimes led us to discover new insights regarding student learning. As a result we were able to incorporate these insights into the ISMs in order to provide more support to teachers in responding to the corresponding instructional challenges. A significant illustrative example is provided by an episode that came to our attention as we were researching Dr. Bond's experience teaching with the TAAFU curriculum. We learned that, while students were typically comfortable multiplying both sides of a symmetry equivalence on the right, they seemed to have doubts about the validity of multiplying both sides by something on the left. After describing how this issue arose in the classroom and our analysis of this episode, we will detail how we incorporated the results of our analysis into the ISMs.

This issue emerged when Dr. Bond's class was trying to develop a minimal set of rules for calculating combinations of symmetries, and in so doing they were discussing whether or not two rules were equivalent. Specifically, the class was working on proving that $FR = R^2F$ follows from other rules (specifically the group axioms and another version of the dihedral relation, in this case $FRFR = I$). A student presented the proof shown in Fig. 8 (when read top to bottom, this proof contains a common error in that it actually proves the converse of the desired statement).

Notice that the first step of this proof involves multiplying both sides of the equivalence $FR = R^2F$ on the left by the symmetry *FR*. During the class discussion a student explained that, given a symmetry equation, it made sense that right multiplication preserves equivalence; however, he was unsure as to why left multiplication did. The transcript below highlights this in the third line, as Adam clarifies his initial statement. We see in the transcript that the instructor, Dr. Bond, struggled to make sense of this distinction between left and right multiplication, even after Mark tried to explain Adam's concern.

$$FR = R^2F$$
$$FRFR = FRR^2F$$
$$FRFR = F(RR^2)F$$
$$FRFR = FIF$$
$$FRFR = FF$$
$$FRFR = I$$

Fig. 8. Proof using left multiplication.

Adam: Well, I wanted to note that everyone agreed that if you're at two equivalent equations, like if the left side equals the right side, then you are saying that both of the shapes are in the exact same position. So obviously if you do the same operation to the end of them, without question that has to work. They are in the same spot and you are doing the same thing to both sides. But this is different from that.

Dr. Bond: So reading down you feel comfortable with –

Adam: No, I'm saying that if on the left side it's some string of things and on the right side it's some string of things that we are saying are equal, and if we add two equivalent operations to the end of both sides, then it's gonna not change anything, so you should be able to do that. But adding two things to the beginning that are equivalent seems a little bit more confusing.

Dr. Bond: I don't understand the difference between your beginning and your end.

Mark: He's saying we are doing the *FR* operation before the operations we already had.

Dr. Bond: So opposed to putting the *FR* on this side?

Mark: Yes, if we had the triangle in the equivalent state, then whatever we did from there would clearly not change anything.

Adam: Exactly.

Dr. Bond: If you start with something that's balanced... oh if you start with $FR = FR$ and then add another piece?

Mark: No, doing *FR* before equivalent operations is different in his mind than after.

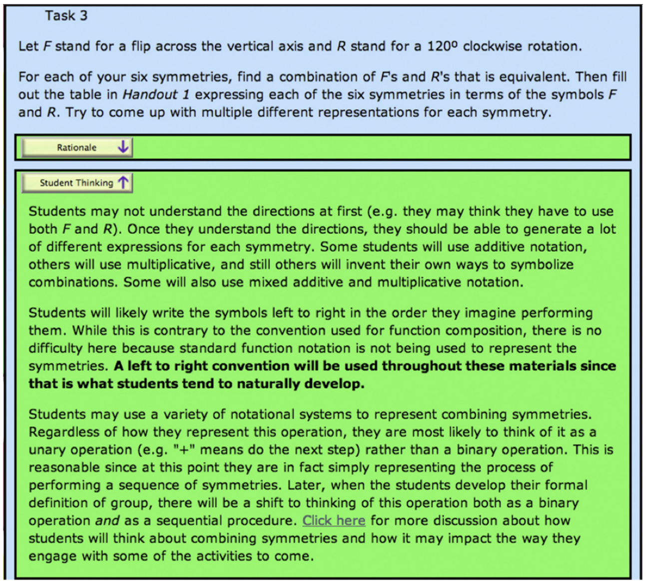
Dr. Bond: I'm not sure if I'm 100% clear on that, but that's okay.

As a research team we analyzed this episode in order to understand Adam's thinking and to determine whether there was a key piece of knowledge that could have helped Dr. Bond make sense of, and address, Adam's concern. In doing so, we were able to make a connection between Adam's concern and some findings from our initial design experiments.

Larsen (2004, 2010) found that students in his small-scale design experiments struggled to differentiate between the ability to regroup symbols (associativity) and the ability to reorder symbols (commutativity). In an effort to make sense of the students' difficulties, Larsen turned to research on children's use of bracketing (Kieran, 1979) and research on teachers' understanding of the associative and commutative properties (Zaslavsky & Peled, 1996). By connecting these research findings to his own data, Larsen concluded that his students' "difficulties with the associative and commutative properties may stem from (1) a tendency to think about expressions involving binary operations in terms of a sequential procedure and (2) a lack of preciseness in the informal language used in association with these properties" (p. 42).

When one conceptualizes the operation of combining symmetries as a left to right sequential procedure, Adam's comments in Dr. Bond's class become clear. For Adam, left multiplication and right multiplication were not conceptually the same. From his perspective, a symmetry equation tells us that two sequences of symmetries represent same overall transformation. Therefore, with right multiplication, an additional symmetry is applied from this equivalent ending point. This would certainly maintain equivalence since, for both expressions, this new symmetry is applied to the same ending point. With left multiplication, it is still known that the two original strings of symmetries result in the same overall transformation; however, we are now changing the start of these two transformations. It takes more thought to convince oneself that equivalence is preserved in this case.

Having made this connection between how students may think of the operation and how students may conceive of left and right multiplication, we began to think about how we might appropriately revise our ISMs. It is important to note Adam's perspective is completely in alignment with the TAAFU curriculum's approach to the operation of combining symmetries. The TAAFU curriculum initiates the investigation by encouraging students to physically manipulate triangles and the symbols emerge in an effort to denote the process of performing consecutive symmetries on a triangle. As a result, early in the instructional sequence, this sequential view of the operation is likely to remain at the forefront of the students' thinking. As Larsen (2010) explained:


Task 3

Let *F* stand for a flip across the vertical axis and *R* stand for a 120º clockwise rotation.

For each of your six symmetries, find a combination of *F*'s and *R*'s that is equivalent. Then fill out the table in *Handout 1* expressing each of the six symmetries in terms of the symbols *F* and *R*. Try to come up with multiple different representations for each symmetry.

Rationale

Student Thinking

Students may not understand the directions at first (e.g. they may think they have to use both *F* and *R*). Once they understand the directions, they should be able to generate a lot of different expressions for each symmetry. Some students will use additive notation, others will use multiplicative, and still others will invent their own ways to symbolize combinations. Some will also use mixed additive and multiplicative notation.

Students will likely write the symbols left to right in the order they imagine performing them. While this is contrary to the convention used for function composition, there is no difficulty here because standard function notation is not being used to represent the symmetries. **A left to right convention will be used throughout these materials since that is what students tend to naturally develop.**

Students may use a variety of notational systems to represent combining symmetries. Regardless of how they represent this operation, they are most likely to think of it as a unary operation (e.g. "+" means do the next step) rather than a binary operation. This is reasonable since at this point they are in fact simply representing the process of performing a sequence of symmetries. Later, when the students develop their formal definition of group, there will be a shift to thinking of this operation both as a binary operation *and* as a sequential procedure. Click here for more discussion about how students will think about combining symmetries and how it may impact the way they engage with some of the activities to come.

Fig. 9. Expanded student thinking textbox.

> The students were working with a step-by-step procedure that they symbolized by writing the symmetry symbols from left to right in the order they were to be performed. The binary operation concept began to emerge as the students started to develop rules for manipulating the symbols and was only crystallized when they formulated the definition of operation as they wrote their definition of a group (p. 46).

Therefore, the impact of our research findings on the ISMs was not to inform teachers how to avoid or correct this view of operation – but rather to explain this perspective, inform teachers that this operation conception was likely, and make teachers aware of potential consequences of this conception as students engaged with instructional tasks.

5.2.2. Modification of ISMs in relation to the key insight

Our analysis of the events in Dr. Bond's class gave us insight into how the sequential view of the operation of combining symmetries could have an impact on the ways that students and teachers engaged with the TAAFU curriculum. As a result, we incorporated information about how students were likely conceiving of combining symmetries into the ISMs. Fig. 9 illustrates this in the student thinking section for a task in which students are asked to name each of the six symmetries in terms of a vertical flip and a 120° clockwise rotation.

Additionally, a new page was added to the website that dealt specifically with how students would likely be thinking of combining symmetries as they worked through the reinvention of the group concept. This new page (which can be retrieved by clicking the link seen in Fig. 10) includes an expanded discussion of the ways students conceive of the operation, and

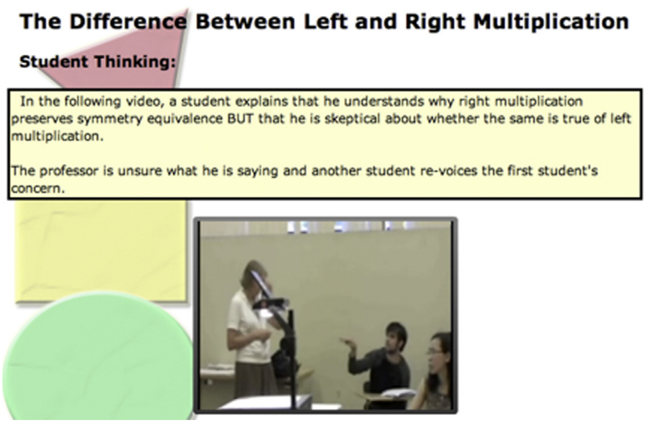


Fig. 10. ISMs page with video clip.

how this could have an impact on the way that students engage with various instructional task. This page also has a link to Larsen's (2010) paper in which the complexities surrounding associativity and commutativity are discussed in this context.

We also made refinements to the ISMs associated with the task in which students are asked to reduce their list of rules. We have included a link to a page that discusses issues around proof that are likely to come up for students. These proof related issues include assuming the conclusion, using cancelation (which, at that point in the TAAFU curriculum, is not yet established as valid), and multiplying both sides of an equation by a symmetry. As part of the discussion on multiplying both sides of an equation by a symmetry, we inform instructors that students may view left multiplication as different than right multiplication. We also provide a link to a video from Dr. Bond's class in which this issue arises (see Fig. 10) and explain the connection to the sequential view of the operation. Finally, we provide a link to the page created to discuss how students view operations during the reinvention process.

5.2.3. Discussion

Notice that both research on student learning and research on teaching informed these revisions to the ISMs. Through our research on teachers' implementation of the TAAFU curriculum we were able to identify challenges for teachers and new insights into student learning that we had not anticipated. In this case, the key insight is that students' sequential view of the operation of combining symmetries may influence how they think about left and right multiplication of symmetries. By analyzing this episode, in light of our experiences developing the TAAFU curriculum, we were able to make sense of this student conception by reflecting on how students may be thinking of operation at this point in the curriculum. The results of this research were then incorporated into the ISMs in the form of written explanations, samples of student work, classroom videos, and links to research articles.

Our approach to sharing this information is consistent with recommendations from the literature. Ball and Cohen (1996) suggest that "the sort of curriculum guidance for teachers that we imagine. . .could offer concrete examples of what student work might look like, what reasoning might underlie students' work, and what other teachers have done in similar situations" (p. 8). Remillard (2000) also argues for the potential value of providing "samples of discourse that might occur as a class pursues a task" (p. 345). Such *knowledge of content and students* can help teachers better anticipate what might arise in a given lesson, and support them in making sense of their students' thinking and making decisions about which student ideas they should pursue.

5.3. Influence of research: specialized content knowledge

We draw another example of the impact of our research on the ISMs from the quotient group unit (see Larsen & Lockwood, 2013). The first task in the quotient group unit asks the students, "Can you find anything like the evens and odds in D_8?" where D_8 is the group of symmetries of a square. This activity is later generalized as students are asked to find ways to partition D_8 into four subsets that form a group under set multiplication. By generating and analyzing partitions students discover that some partitions actually work (form a group) while most do not. In doing so, students make and prove conjectures regarding necessary conditions for forming groups out of subsets. The first such condition is that one of the subsets must be a subgroup (the one that acts as the identity element). After this condition has been established, the students are asked to determine how the remaining subsets should be formed. It is in the context of this task that we find our next example of the impact of our research efforts on the development of the ISMs.

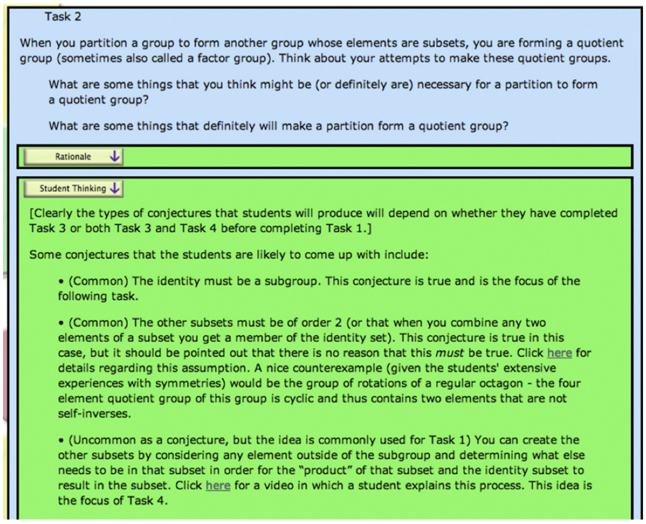
5.3.1. Key insight

As teachers' implemented the TAAFU curriculum, it became evident that some classes struggled to make adequate progress given only the prompts and tasks provided. For instance, although Dr. Bond's class was successful in identifying all of the quotient groups of D_8 of order 2 and of order 4, they struggled to develop viable conjectures as to why these partitions worked while others did not. Instead of considering the structure of a group and what was known about identity elements, the students became focused on characteristics of their working examples. This fixation on characterizing the properties of their working examples was evidenced by the types of conjectures and observations made by the class. The students proposed a number of observations related to the fact that in each of their working examples every element is of order two. For example, Mark observed that, "the one that we worked, in the beginning, the combination of the two elements always gives you an element of the identity set".

The students in Dr. Bond's class developed a number of conjectures of this type including: that each subset must be of order two, that the combination of any two elements in the same set must result in an element of the identity subset, and that for each element of a given subset the inverse of that element must also be in that subset (see Johnson & Larsen, 2012). While the research team had been aware that such conjectures were likely, or new research into mathematicians' experiences implementing the curriculum produced new insight into how students' fixation on this property of the quotient groups they had constructed could derail the conjecturing process.

5.3.2. Modification of ISMs in relation to key insight

Based on the insights gained by our investigation of Dr. Bond's implementation of the quotient group unit, we refined the ISMs to include information about the conjectures students were likely to make regarding necessary conditions for partitions of D_8 to form a quotient group. For example, in the student thinking section shown in Fig. 11 we make teachers aware that


Task 2

When you partition a group to form another group whose elements are subsets, you are forming a quotient group (sometimes also called a factor group). Think about your attempts to make these quotient groups.

What are some things that you think might be (or definitely are) necessary for a partition to form a quotient group?

What are some things that definitely will make a partition form a quotient group?

Rationale

Student Thinking

[Clearly the types of conjectures that students will produce will depend on whether they have completed Task 3 or both Task 3 and Task 4 before completing Task 1.]

Some conjectures that the students are likely to come up with include:

- (Common) The identity must be a subgroup. This conjecture is true and is the focus of the following task.

- (Common) The other subsets must be of order 2 (or that when you combine any two elements of a subset you get a member of the identity set). This conjecture is true in this case, but it should be pointed out that there is no reason that this *must* be true. Click here for details regarding this assumption. A nice counterexample (given the students' extensive experiences with symmetries) would be the group of rotations of a regular octagon - the four element quotient group of this group is cyclic and thus contains two elements that are not self-inverses.

- (Uncommon as a conjecture, but the idea is commonly used for Task 1) You can create the other subsets by considering any element outside of the subgroup and determining what else needs to be in that subset in order for the "product" of that subset and the identity subset to result in the subset. Click here for a video in which a student explains this process. This idea is the focus of Task 4.

Fig. 11. Student thinking textbox with quotient group conjectures.

students are likely to overgeneralize from the quotient groups they construct and propose (for example) that the subsets should always be self-inverses. This expandable textbox also suggests a counterexample that would likely be accessible to the students given their previous experiences with groups of symmetries.

Our analysis of Dr. Bond's implementation of the quotient group unit also made us aware that students may struggle to determine how to partition a group after selecting a subgroup to act as the identity. This observation motivated us to design a new instructional task in the form of a handout (Fig. 12). This handout was inspired by the work of a student, Rick, who participated in the initial design experiment, and it is designed to focus students on the structure of a quotient group and specifically on the identity property of the identity subset.

Our analysis of the events in Dr. Bond's class encouraged us to engage in more design work based on our knowledge of Rick's approach from the initial design experiment. This design work resulted in changes to both ISMs and the TAAFU curriculum materials. This illustrates the interconnectedness of our research on teaching and learning in the context of the TAAFU curriculum and our ongoing design efforts dedicated to improving the curriculum and the ISMs.

5.3.3. *Discussion*

Because the mathematical ideas are developed via guided reinvention, an instructor using the TAAFU curriculum may have to make sense of rather idiosyncratic mathematical contributions on the part of the students. It is unlikely that a mathematician would have considered a conjecture like the one observed in Dr. Bond's class because a mathematician would have access to a great number of examples of quotient groups that contain elements of order greater than two. To support teachers in making sense of TAAFU-specific student contributions, the development team provided both predictive and descriptive information on contributions that were likely to arise. We see this information as supporting teachers'

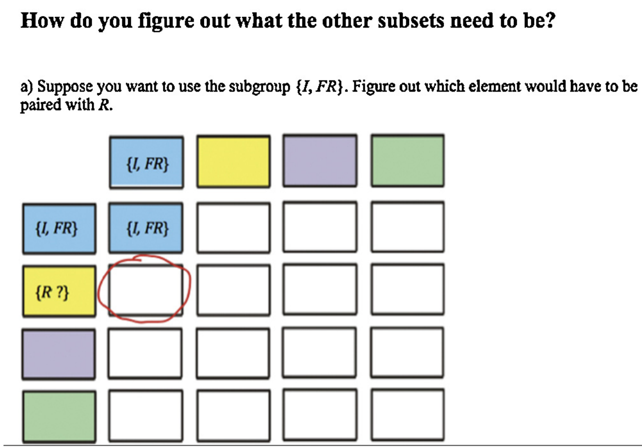


Fig. 12. Coset formation handout.

development of *specialized content knowledge* because it is unlikely to be needed in a setting other than teaching. As part of the student thinking section in the ISMs, we inform teachers that the conjecture that emerged in Dr. Bond's class is likely to occur and we provide examples of equivalent formulations of this idea. We also explain that this conjecture does in fact hold for each quotient group example the students have experienced thus far, and we provide a counterexample in a context that would likely make sense to the students (see Fig. 11). All of this information is designed to "help teachers learn how to anticipate and interpret what learners may think about or do in response to instructional activities" (Davis & Krajcik, 2005, p. 5).

6. Discussion and conclusions

As the TAAFU research team began to address scaling up considerations, it became imperative to develop materials to support teachers (who were not members of the research team) as they worked to implement the TAAFU curriculum. By conducting research on both student learning and the challenges and opportunities faced by mathematicians as they implemented the curriculum, we were able to progress through a series of iterative design phases corresponding to each round of curriculum field-testing. The result of these iterative refinements is a set of research-based ISMs designed to support teachers in successfully implementing the TAAFU curriculum without the direct support of the design team. These ISMs include for each task: the rationale for the task in terms of the learning objectives of the lesson, information about how students are likely to approach the task and what difficulties they might encounter, and suggestions regarding facilitation as well as some discussion of known challenges involved in implementing the task.

We see this report on the development of our ISMs as making two contributions to the research literature. First, as noted by Schneider and Krajcik (2002), there is little in the research literature on the development of ISMs. Here we illustrate how our research into teaching and learning has served to inform the development of our ISMs. Central to the iterative development of our ISMs was the fact that our research, from the initial design experiments to the field-testing, was focused on developing not only the TAAFU curriculum but also a set of support materials designed to help teachers implement the curriculum effectively.

By presenting and illustrating our development process we hope to address a second need in the research literature – models for scaling up design-based research. As other research teams work to develop similar reform-oriented curriculum (e.g., Rasmussen, 2007; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, in press; Oehrtman & Swinyard, 2011) it is likely they will also have to address scaling-up considerations. Thus, we argue that there is an audience that could gain from considering the development of our ISMs and how such materials can play a role in the process of scaling up an innovation.

Finally, the development of our ISMs opens new avenues for research, including investigating into the ways in which the ISMs support teachers. For instance, one reviewer of the ISMs (and field-tester) stated, "Generally, I find the all information here to be useful. The lesson does not take that long to read through as it is, and I find that all three categories of information are helpful for me." However, at this time is not well understood *how* the information provided was helpful. Specifically, one factor worth investigating is the extent to which the descriptive, and predictive, nature of ISMs supports teachers in implementing curricula. As one teacher commented, "it was spooky the way my students would express their thinking about the tasks in almost exactly the way the student thinking sections said that they would." While we suspect such information is useful for teachers, further research is needed to understand whether and how such knowledge could support successful implementation of inquiry-oriented curriculum. We are also interested in exploring other methods of supporting instructors. Next steps could include the design of workshops and an online learning community for mathematicians who are interested in implementing the curriculum. As with our web-based ISMs, our research into teaching and learning in the context of the TAAFU curriculum will be crucial to such design efforts.

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