A HYPOTHETICAL COLLECTIVE PROGRESSION FOR CONCEPTUALIZING MATRICES AS LINEAR TRANSFORMATIONS

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In this paper we develop the notion of a hypothetical collective progression (HCP). We offer this construct as an alternative to the construct of hypothetical learning trajectory in order to (a) foreground the mathematical development of the collective rather than that of individuals, and (b) highlight the integral role of the teacher within this development. We offer an abbreviated example of an HCP from introductory linear algebra based on the "Italicizing N" task sequence, in which students work to generate and combine matrices that correspond to geometric transformations specified within the problem context. In particular, we describe the ways in which the HCP supports students in developing and extending local "matrix acting on a vector" views of matrix multiplication (focused on individual mappings of input vectors to output vectors) to more global views in which matrices are conceptualized in terms of how they transform a space in a coordinated way.

Keywords: Linear algebra, collective, learning, progression, linear transformation

The construct of a hypothetical learning trajectory (HLT) was initially developed for and has primarily been used by both teachers and researchers for the purpose of describing individual student learning in particular content domains (e.g., Clements & Sarama, 2004; Simon, 1995; Simon & Tzur, 2004; Steffe, 2004). As reported by Clements and Sarama, the variety of uses and interpretations has expanded to include both individual cognitivist and collective analyses of mathematical development. The abundant use of the term, however, may lead to confusion regarding which unit of mathematical development is under discussion. We hold the view that, in a classroom setting, individual student thinking shapes and is shaped by the development of mathematical meaning at the collective level (Cobb & Yackel, 1996). Choosing to focus on the classroom as the unit of analysis, we adapt the notion of an HLT to articulate a new construct, which we refer to as a hypothetical collective progression (HCP), appropriate to guide the mathematical development at the collective level. In this paper we offer an abbreviated illustration of an HCP in the context of undergraduate linear algebra.

Student difficulties in learning fundamental concepts in linear algebra are well documented (e.g., Carlson, 1993; Dorier, Robert, Robinet, & Rogalski, 2000; Harel, 1989; Hillel, 2000; Sierpinska, 2000). Symbolization of algebraic ideas relies heavily on the use of variables and

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functions (Arcavi, 1994), and research shows that students at the undergraduate level continue to struggle in their interpretations of variables and functions (Oehrtman, Carlson, & Thompson, 2008; Jacobs & Trigueros, 2008). This difficulty is amplified in the realm of linear algebra, where students must reckon with symbolization in multidimensional contexts.

A result of our work that we present in this piece is an HCP designed to support students in developing their understanding and symbolization of linear transformations defined by matrix multiplication. The main learning goals of this HCP are (a) interpreting a matrix as a mathematical object that transforms input vectors to output vectors, (b) interpreting matrix multiplication as the composition of linear transformations, (c) developing the imagery of an inverse as "undoing" the original transformation**,** and (d) coming to view matrices as objects that geometrically transform a space. These learning goals include a student transition from a localized view wherein matrices are interpreted as transforming one vector at a time to a more global view of a matrix transforming an entire space.

Theoretical Framework and Literature Review

This work draws on three instructional design heuristics of Realistic Mathematics Education (RME) (Freudenthal, 1983; summarized by Cobb, 2011). First, an instructional sequence should be based on experientially real starting points. In other words, tasks that comprise an instructional sequence should be set in a context that is sufficiently meaningful to students that they have a set of experiences through which to meaningfully engage in, interpret, and make some initial mathematical progress. Second, the task sequence should be designed to support students in making progress toward a set of mathematical learning goals associated with the instructional sequence. Third, classroom activity should be structured so as to support students in developing models-of their mathematical activity that can then be used as models-for subsequent mathematical activity. In other words, the process of students' reasoning on a task becomes reified so that the outcome of that process of reasoning can serve as a meaningful basis and starting point for students' reasoning on subsequent tasks.

In order to operationalize these RME heuristics into content-specific deliverables that are more explicitly related to instruction, a number of researchers have used the construct of hypothetical learning trajectory. Simon (1995) coined the term to describe the work teachers do in anticipating the path(s) of their students' learning in planning for classroom instruction, and he defined an HLT as consisting of "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133). In addition to its wide spread use in describing the hypothetical learning of individual students, the construct has been adapted by some to conjecture about the development of mathematical meaning at the collective level (e.g., Cobb, Stephan, McClain, & Gravemeijer, 2001; Gravemeijer, Bowers, & Stephan, 2003; Larson, Zandieh, & Rasmussen, 2008). Indeed, Cobb et al. (2001) describe viewing an HLT as "consisting of conjectures about the collective mathematical development of the classroom community" (p. 117), and Gravemeijer et al. (2003) describe it as a "possible taken-as-shared learning route for the classroom community" (p. 52). We follow this adaptation of the construct for the social perspective, adding here the explicit consideration of the role of the teacher as an integral aspect in the collective sense making that takes place in the classroom enactment.

In order to distinguish this differing collective perspective on HLTs it is necessary to put forth an alternative construct, namely that of a *hypothetical collective progression* (HCP). We define an HCP to be a storyline about teaching and learning that occurs over an extended period of time. The storyline includes four interrelated aspects:

1. Learning goals about student reasoning;
2. Evolution of students' mathematical activity;
3. The role of the teacher; and
4. A sequence of instructional tasks in which students engage.

The second aspect, the evolution of students' mathematical activity, is described in terms of both common difficulties and problematic conceptions that arise, as well as in ways of reasoning that potentially function as if shared. By detailing possible normative ways of reasoning (in the sense of Stephan & Rasmussen, 2002), an HCP emphasizes the potential mathematical development of the collective. It further pays homage to the reflexive relationship between individual thinking and collective development by noting common difficulties and problematic conceptions that arise within students' engagement in mathematical activity.

This construct further differs from than that of an HLT in its explicit inclusion of the role of the teacher as integral in the anticipated progression of mathematical activity in the classroom. The teacher is a unique and essential member of the classroom community with the role of not only pushing forward the mathematical development of the classroom but also fostering productive social and sociomathematical norms within that classroom. Thus, this framing highlights the multi-dimensional structure of classroom activity. As the first and second aspects highlight, a teacher must consider the learning goals she has for her classroom, as well as envision the evolution of students' mathematical development as these goals are actualized. The third and fourth aspects of an HCP—the role of a teacher and the sequence of instructional tasks in which the students engage—speak to how these could be carried out within a given classroom.

Toward Conceptualizing Matrices as Linear Transformations

Research on the learning of linear algebra identifies three common student interpretations of a matrix times a vector: matrix acting on a vector view (MAOV), vector acting on a matrix view (VAOM), and systems views (Larson, 2011). The MAOV view is based on the idea that the matrix acts on or transforms the input vector, thus turning it into the output vector. The VAOM view is based on the idea that the vector acts on the matrix by weighting the column vectors of the matrix, whose sum results in the output vector. A systems view of matrix multiplication is typified by an effort to reinterpret matrix multiplication by thinking of it as corresponding to a system of equations. The HCP we detail offers a means by which instructors can support students in developing and extending the MAOV view of a matrix times a vector to a more global view of how a linear transformation defined by a matrix affects an entire space and how transformations can be composed.

A transformation is a broad mathematical concept that can be represented in a number of ways. For example, a matrix is one specific way in which certain types of transformations (e.g., linear transformations) can be represented. A transformation (function) T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$. A linear transformation $T\colon \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a map that satisfies the following properties: (a) $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for every $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^n$, and (b) $T(a\mathbf{v}) = aT(\mathbf{v})$ for every scalar a and every $\mathbf{v}$ in $\mathbf{R}^n$. It can be shown that every transformation given in terms of matrix multiplication is a linear transformation when defining $T(\mathbf{v})=A\mathbf{v}$ for a given n x m matrix A. For instance, one may consider the transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ that rotates the plane ninety degrees counterclockwise; this transformation can be defined by the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. It is this conceptualization, which we refer to as conceptualizing matrices as linear transformations, that is the subject of this paper.

Data Sources and Methods

This research-based HCP grows out of a larger design research project that explores ways of building on students' current ways reasoning to help them develop more formal and conventional ways of reasoning, particularly in linear algebra. The instructional sequence described in this paper was developed and iteratively refined over the course of four semester-long classroom teaching experiments following the methodology described by Cobb (2000) that took place in inquiry-oriented introductory linear algebra classes at public universities in the southwestern United States. We use the term inquiry-oriented in a dual sense, where the term inquiry refers to the activity of the students as well as the teacher (Rasmussen & Kwon, 2007). Students engage in discussions of mathematical ideas, questions, and problems with which they are unfamiliar and do not yet have ways of approaching; thus, evaluating arguments and considering alternative explanations are central aspects of student activity. Teacher activity includes facilitating these discussions, which demands that the teacher constantly inquire into students' thinking. Students in these courses were generally sophomores or juniors in college, majoring in math, engineering, or computer science, and were required to have successfully completed two semesters of calculus prior to enrollment in the course.

During each classroom teaching experiment, we videotaped every class period using 3-4 video cameras that focused on both whole class discussion and small group work. We also collected student written work from each class day. As a research team, we met approximately three times a week in order to debrief after class, discuss impressions of student work and mathematical development, and plan the following class. We also used these meetings retrospectively to inform decisions regarding the following iteration of the classroom teaching experiment, as what we analyzed one semester became refined and informed the next iteration of the curriculum. One of our goals was to produce an empirically grounded instructional theory, and doing so involves a number of stages. Over the four years, we have refined not only our instructional tasks, but we also have deepened what we know about student thinking in linear algebra, refined the learning goals for our course, and increased our awareness of the role of the teacher. One of the results from this extensive iterative work is the notion of an HCP. Examples presented in this paper were taken from the fourth and latest classroom teaching experiment.

The HCP presented in this paper is the result of a retrospective analysis of the development of the instructional sequence and the associated set of learning goals, as well as the way in which the instructor used this instructional sequence to support students' mathematical activity in its classroom enactment. Instructor and student notes were used to reconstruct the broad progression of classroom activity across the set of tasks; these were used to identify relevant segments of classroom video from the classroom teaching experiment. Our research team generated memos to document students' mathematical activity and the role of the teacher in progressing through this particular enactment of the instructional sequence, paying particular attention to the variety of student interpretations elicited by the task, issues that were problematic for students, and the role of the instructor in negotiating sense-making around student generated notation and connecting to more conventional notation used by the broader mathematical community.

Prior to the instructional sequence driven by this HCP, the class had engaged in an RME-inspired instructional sequence focused on helping students develop a conceptual understanding of linear combinations, span, and linear independence (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, in press). The class also spent time developing solution techniques for linear systems to help answer questions regarding span and linear independence of sets of vectors. This led to the definition and exploration of equivalent systems, elementary row operations, matrices as an array of column vectors, augmented matrices, Gaussian elimination, row-reduced echelon form, pivots, and existence and uniqueness of solutions. This broad set of ideas was unified by developing and proving conjectures regarding how these ideas fit together for both square and non-square matrices.

Results

The HCP developed in this report encompasses four learning goals: (a) Interpreting a matrix as a mathematical object that transform input vectors to output vectors; (b) Interpreting matrix multiplication as the composition of linear transformations; (c) Developing the idea of an inverse as “undoing” the original transformation; and (d) Coming to view matrices as objects that geometrically transform a space. These learning goals are not intended to be achieved sequentially. Rather, these four learning goals interweave and support students in developing a robust conceptual understanding of matrices as linear transformations. For instance, one may see learning goal (a) as a local view of linear transformation, whereas learning goal (d) may be interpreted as a more global view. The global view is not meant to replace the local view; rather, it elaborates it. We want students to be able to draw on and coordinate both views, moving flexibly between them as need be. In fact, coordination of local and global views is an aspect of students’ mathematical activity that cuts across all tasks in the instructional sequence in a way that we argue is crucial to the development of productive normative ways of reasoning.

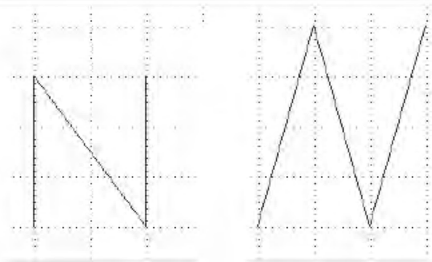
Our construct of HCP has four components, and we organize the results section in terms of the fourth: the sequence of instructional tasks in which students engage. We choose to do this because it is the aspect of the HCP in which the students do engage sequentially, so this allows for a more natural parallel to how the HCP may unfold in an actual enactment. For each task, we discuss what the students are being asked to do and how this is significant in terms of our learning goals. Using data from the fourth classroom teaching experiment in linear algebra, we describe students’ mathematical activity as they engage in each task, sources of difficulty, and the role of the teacher in supporting students to work through these difficulties. We focus especially on the role of the teacher in negotiating the use of mathematical notation to support sense making and to connect to symbolic and definitional conventions of the mathematical community.

Following an introduction to transformation view of $A\mathbf{x} = \mathbf{b}$, this particular HPC regarding linear transformations has three main tasks: (a) the Italicizing N task; (b) the Pat and Jamie task; and (c) the Getting Back to the N task.

Introduction to a Transformation View of $A\mathbf{x} = \mathbf{b}$

The first learning goal of this HCP is conceptualizing matrices as mathematical objects that transform input vectors to output vectors. That is, in contrast to interpreting $A\mathbf{x} = \mathbf{b}$ in terms of a vector equation or a system of equations, the goal is to encourage conceiving of $A\mathbf{x} = \mathbf{b}$ as a matrix A acting on the vector $\mathbf{x}$ to produce the vector $\mathbf{b}$. This goal involves a major interpretive shift for students, but their prior experiences working with functions serve as a good starting point for this new conceptualization of matrices. One way in which the teacher can support this shift is by introducing terminology that will support students’ work in the upcoming sequence of tasks by helping them analogize their work with matrices to their prior knowledge of functions. For instance, the teacher may introduce the terms like transformation, domain, and codomain, and discuss how $A\mathbf{x} = \mathbf{b}$ can be interpreted as an example of a transformation by defining $T(\mathbf{x}) =$

$A\mathbf{x}$. These introductory whole-class discussions offer students the opportunity to begin to lay a foundation for thinking of input-output pairs of vectors that are related through a matrix transformation. Rather than provide further specifics of this introductory aspect, we shift out focus to the main task of this HCP, students' mathematical development through interaction with this task, and the role of the teacher.

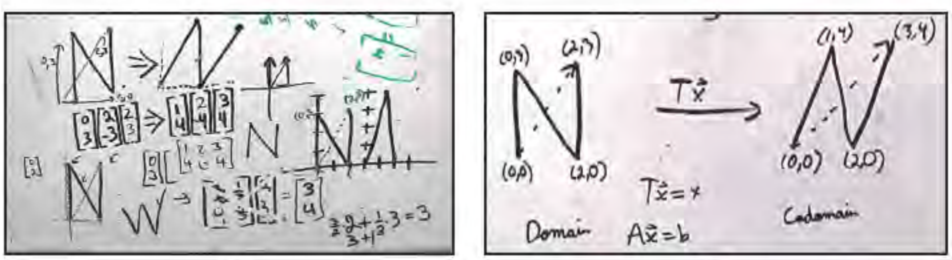


Suppose the "N" on the left is written in regular 12-point font. Find a matrix A that will transform N into the letter on the right, which is written in italics in 16-point font.

Figure 1. The Italicizing N Task

The Italicizing N Task

The Italicizing N task (see Figure 1) is the first task in our HCP, and it is through this task that students embark on their initial exploration of matrices as linear transformations. In this task, the students' goal is to determine a matrix A that represents the requested transformation of the N described in the problem statement. The teacher plays a crucial role in setting up this task by supporting students in developing a common interpretation of the setting and goals of the task, as well as in interpreting matrix multiplication as a transformation. Specifically, assumptions about the context and aspects of the mathematical goals need to be negotiated (e.g., how to represent the N mathematically in each image, as well as how to determine a matrix that maps the image on the left to the one on the right).



(a) (b)

Figure 2. Student work on the Italicizing N task.

It is nontrivial for students to determine that both the inputs and outputs for the transformation lie in $\mathbf{R}^2$ and that the mapping could be represented by a 2 x 2 matrix. Furthermore, students grapple with how to interpret and symbolize the representations of the N. Examination of past student work has revealed two common strategies: using vectors in $\mathbf{R}^2$ or using points in the x-y plane. For example, within the student work shown in Figure 2a, the N is represented with vectors whose tip and tail lay on the letter with tips originating from the same point on the letter (corresponding to a fixed origin). Other students represent the N with vectors whose tip and tail lay on the letter but with tips originating from different points on the letter (corresponding to a "floating" origin). On the other hand, within the student work shown in Figure 2b, locations on the N are labeled as points on the x-y plane with an origin anchored at the lower left vertex of each N.

Regardless of the way in which students represent the letter N, a potential normative way of reasoning is setting up a system of matrix equations – one matrix equation for each input-output pair – in order to determine the component values of A. An example of this approach is shown in Figure 3. The instructor is able to use the variety of representations students generate for the letter N as a starting point for a class discussion about the relationships among choices of representation (points versus vectors), the significance of where one chooses to place the origin, and whether those choices affect the matrix that transforms the letter in the desired way. This allows the teacher to push students to make connections among various approaches and bring out key mathematical ideas. The teacher, as a member of the mathematical community, is in a position to raise questions, such as why anchoring the origin would be advantageous, that the students are not necessarily in the position to make on their own. It is this interaction between the role of the teacher and students' mathematical progress on an instructional task that helps promote a climate of sense making, fosters social norms such as listening to others' reasoning and providing explanation of your own, and moves forward the mathematical goals.

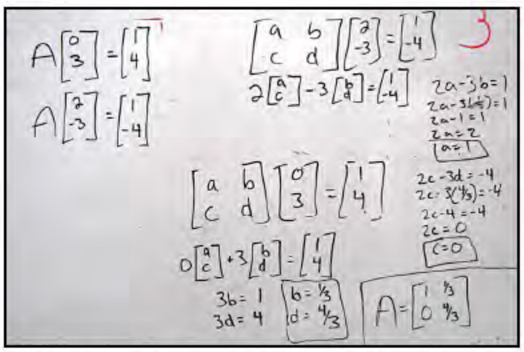


Figure 3. Students set up matrix equations to solve for the values of matrix A.

Other important aspects of this task that are not immediately obvious to students include how to select input-output pairs, how many input-output pairs are needed to determine the matrix, and whether the matrix will be unique. As students share their approaches for finding the matrix A, the teacher has the opportunity to ask students about these aspects. For instance, the teacher can facilitate a discussion about whether A is a unique matrix representation of the transformation (according to the standard basis, which has remained implicit at this point). Often, at least one group of students selects a linearly dependent set of vectors to generate their system of matrix equations – and these students often argue that A is not unique. With the instructor's facilitation, this disagreement over A provides an opportunity for students to discuss what it means for A to be unique, or under what criteria is A unique, and if so, what the criteria are for selecting sets of input-output pairs that uniquely determine A.

In our classroom teaching experiments, we follow the Italicizing N Task with activities that ask students to investigate other transformations of the plane, such as stretching, rotating, etc. The emphasis here begins to shift away from only considering particular input-output pairs to how the transformation defined by A affects the entire plane, without needing to go through the motions of plotting particular pairs. While still working in $\mathbf{R}^2$, the teacher suggests other transformations (such as stretching and skewing images in Quadrant 3) to develop a connection to geometric interpretations of the standard 2 x 2 transformation matrices. This leads into and is not disjoint from the fourth learning goal of coming to view matrices as objects that geometrically transform a space.

The Pat and Jamie Task

The Pat and Jamie task (see Figure 4) was the first task introduced associated with the learning goal of interpreting matrix multiplication as a composition of linear transformations. This is a follow-up to the Italicizing N Task, and it sets up a scenario in which the students must first decide if the approach of two "fellow students," Pat and Jamie, is valid, and then determine the matrices that represent the transformation via their approach. Note that in the problem statement, the order in which Pat and Jamie transform the N is not vague (they make it taller first and then italicize it), but the way in which to computationally accomplish this is purposefully left vague. Students are meant to struggle with how to combine and symbolize one transformation followed by another and why that is sensible.

> Last semester, two linear algebra students—Pat and Jamie—described their approach to the Italicizing N Task in the following way:
>
> > "In order to find the matrix A, we are going to find a matrix that makes the "N" taller (from 12-point to 16-point), find a matrix that italicizes the taller "N," and the combination of those will be the desired matrix A."
>
> 1. Does their approach seem sensible to you? Why or who not?
> 2. Do you think their approach allowed them to find a matrix A? If so, do you think it was the same matrix A we found this semester?
> 3. Try Pat and Jamie's approach. You should either: a) come up with a matrix A by using their approach, or b) be able to explicitly explain why this approach does not work.

Figure 4. The Pat and Jamie Task

The Pat and Jamie task sets the stage for a shift in students' mathematical activities and goals; students are asked to *combine* matrices that define transformations in addition to determining what those transformations matrices are. This is a shift from the goal of constructing a single transformation matrix based on inputs and outputs, such as in the Italicizing N task. This shift is significant because matrices are beginning to be positioned as objects of students' mathematical activity rather than solely the result of a mathematical process (Sfard, 1991). Students often are successful in constructing matrices for the individual transformations (which is a natural continuation of their work on the Italicizing N task) but struggle more with what a sensible way to "combine" these matrices would be. For instance, Figure 5 shows a student's correct matrix A for making the N taller, but the student's matrix (also labeled A) for italicizing the taller N is incorrect (but rather would italicize the shorter, original N correctly). Thus, this student did not attend to the chain of transformations, in which the output for one transformation (making taller) serves as the input for the following one (italicizing). Furthermore, the student writes, "How do we combine these?" on his/her paper, which further indicates the student's struggle with this problem. The teacher needs to be aware of common problematic conceptions such as this, and because of that, it is the role of the teacher to ask questions, make comments, and use notation that direct the class towards to the desired mathematical progress.

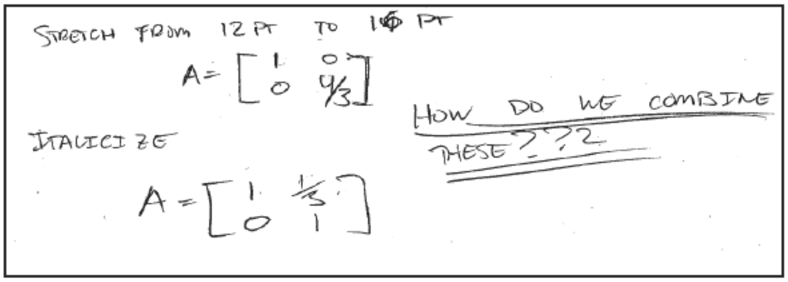


Figure 5. Student expresses uncertainty of how to combine the separate transformations to first change from 12 to 16-pt font and then italicize.

The class begins the Pat and Jamie task by positing that this approach should give the same final matrix as in the initial Italicizing N task. Thus, students often experiment with ways of creating and combining the two intermediary matrices until the combination yielded the desired matrix A (which they had found in the previous activity), mostly trying addition or multiplication. However, students often have difficulty knowing why their operation choice and order is logical. For instance, after working for a while, every small group may have the correct matrix to make the 12-pt N the taller 16-pt N. The class negotiates to name this matrix B because T is used for other things (namely, to refer to generic linear transformations). However, students struggle to find the matrix that would yield the desired 'lean.' The teacher plays a role in working through this struggle in whole class discussion by having various students explain their thoughts and approaches. This is peppered throughout with the teacher asking clarifying questions and revoicing the students' approaches in both words and symbols. For instance, some groups may (correctly) use the vectors from the middle N as the inputs for the lean transformation to correctly determine L. The teacher may choose to summarize and revoice this type of explanation with mathematical symbolism on the board by restating the explanation and explicitly discussing how the output of the first transformation becomes the input for the second and illustrate this with particular input-output pairs (see Figure 6). This provides a way to move the mathematical agenda forward but still allow students to reason for themselves why the correct order of matrix multiplication in sensible according to Pat and Jamie's approach. This also promotes the sociomathematical norms of developing mathematical justifications for computational choices and explaining them.

To summarize, two main choices often surface through the students' work on the task. First, many groups determine, at least initially, that the matrix for the "lean" transformation is $L = \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$ (this is consistent with the matrices in Figure 5). Students who remain with this (incorrect) choice discover that $BL = A$. Other groups determine (as described above), that the "lean" transformation is $L = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}$ and that the (correct) matrix multiplication of LB yields the desired matrix A. Note that the two approaches have the same matrix B but two different matrices for L. The teacher has the opportunity to write both matrix equations on the board, highlight how students got the correct A in two different ways, and ask which approach is what Pat and Jamie did and how they could be certain.

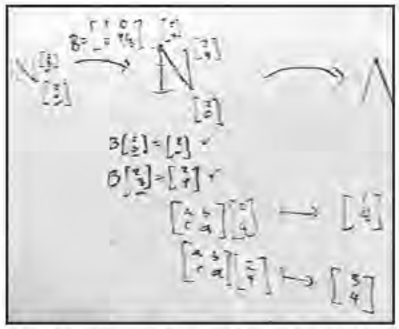
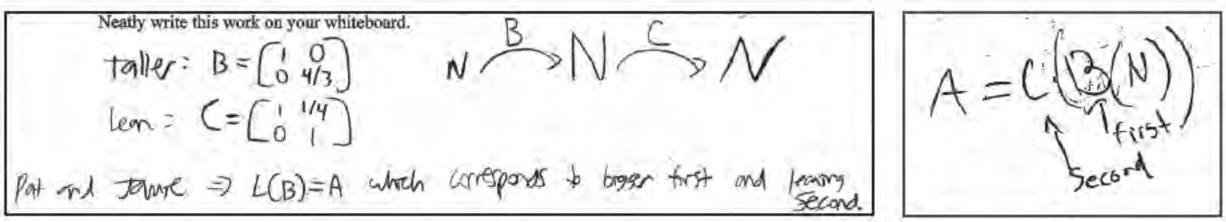


Figure 6. The teacher's symbolic revoicing of students' approaches to the Pat & Jamie Task.

We highlight this episode as significant because it illustrates (a) probable student difficulties with developing an intuitive notion of function composition in the context of linear transformations, and (b) the role of the teacher in connecting to student thinking as she moves her mathematical agenda forward. We posit that the teacher's purposeful use of symbolic notation on the whiteboard during whole class discussion serves as an example of the pedagogical content tool of transformational record (Rasmussen & Marrongelle, 2006). She recorded student thinking regarding input-output pairs for the various transformations and added notation—such as the three N's, the arrows and corresponding transformation matrices between the N's, and the vector and matrix equation representations of the input-output pairs—that later served as tools in students' reasoning about which order of matrix multiplication correctly matched Pat and Jamie's approach.



(a) (b)

Figure 7. A snapshot of one student's work to explain which approach is correct and why.

For instance, the images in Figure 7a and 7b serve as examples of common ways of reasoning after a discussion such as the one highlighted in Figure 6. Figure 7a highlights how a student found the two transformations and the correct order of multiplication; the student also wrote "$CB = A$ which corresponds to bigger first and leaning second." The student work in 7b highlights a bit of the compositional nature of the task, where the "$(B(N))$" seems similar to the notion of "$f(x)$," with the matrix C then acting on the output of $B(N)$. Note that both of these responses only begin to hint at the compositional nature of the two transformations, with the output of B becoming the input for C. It is the role of the teacher to facilitate discussion of this. These two examples of student work also illustrate the potential to shift away from only focusing on how individual vectors are transformed to how a space is transformed. As such, this task, in addition to developing the notion of matrix multiplication as the composition of functions, further fosters the first and fourth learning goals of the local and global aspects of matrices as transformations.

Finally, it should be mentioned how the Pat and Jamie task is built upon in subsequent classroom discussion. Along the way in determining which matrix equation describes Pat and

Jamie's approach, the students work to determine that, in this case, the order in which the matrices are multiplied affect the answer. The teacher has the opportunity and responsibility, as a member of the mathematics community, to introduce and connect the term *commutativity* to the students' work. The teacher also works to connect this to the notion of composition of functions as an interpretation of matrix multiplication. Again, as a member of the mathematics community, the teacher serves as a broker (Rasmussen, Zandieh, & Wawro, 2009) between students' authentic mathematical activity and the terminology and notation commonly used in the mathematics community.

The Getting Back to N Task

The last main task in the instructional sequence within this HCP is the Getting Back to N task (see Figure 8), and it is mainly associated with the learning goal of developing the idea of an inverse as "undoing" the original transformation. It is also intended to further the learning goal of reasoning about matrix multiplication as a composition of linear transformations. This task asks students to determine a matrix C that will transform the letter on the right back into the letter on the left; that is, from the 16-pt italicized N to the original N. Furthermore, this task asks students to determine the matrix C in two ways: through one direct transformation and through Pat and Jamie's method (i.e., in two steps). To the expert, this task introduces the notion of inverse transformations. This naturally follows from students' work on the previous tasks, although it is by no means trivial for students, because investigating the Pat and Jamie approach to "getting back to the N" reiterates their work regarding function composition and matrix multiplication. A rationale behind our development of this task is consistent with Oehrtman et al.'s (2008) observation that more sophisticated, process-oriented views of an inverse function coincide with conceiving of it as the function that undoes the action of the original function, versus a less sophisticated, action-oriented view that associates the concept of inverse function with a surface action such as determining the associated matrix inverse via a memorized formula. It additionally requires students to consider the importance of order in function composition and matrix multiplication when one or more of the functions under consideration has the action of "undoing" the action of a previous transformation.

Regarding the Italicizing N Task, complete the following:

Find a matrix C the will transform the letter on the right back into the letter on the left.

1. Find C using either your method or one of your classmate's method for finding A
2. Find C using Pat and Jamie's method for finding A.

Figure 8. The Getting Back to N task.

Initial student work on the first prompt of finding the matrix C in one step is often unproblematic. Given the students' engagement with the previous two tasks, a potential normative way of reasoning is to determine C by creating a matrix equation that coordinates particular input-output pairs from the tall, italicized N to the original N. Part of the role of the teacher, however, is to call attention to the relationship between these input-output pairs and those from the original Italicizing N task; namely, that the inputs in the Getting Back to N task served as the outputs in the Italicizing N task and vice versa. This emphasis on the role of the various parts of matrix equations such as $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{b} = \mathbf{x}$ lays the groundwork for subsequently labeling the matrix C as the inverse of matrix A. Given that students are not, at this stage, formally aware of C as the inverse of A, the symbol used to notate this relationship is tied to students' ways of thinking and symbolizing. For instance, based on students' normative way of reasoning that the transformation defined by C "undoes" the effect of the transformation defined by A, the teacher leads the class in defining $C = U_A$, where U_A stands for "undoes A."

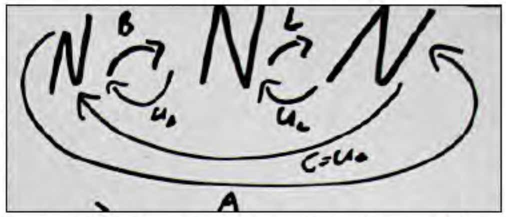
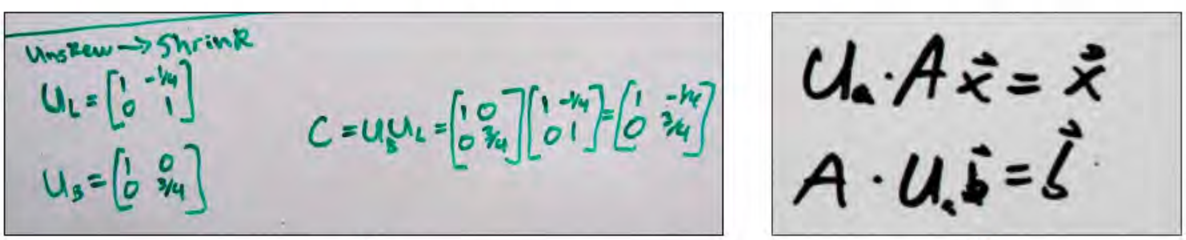


Figure 9. Negotiation of meaning for the matrices U_B, U_L, and U_A, which are the matrices that "undo" the actions of B, L, and A, respectively.

When working on the second prompt of determining the matrix C via Pat and Jamie's method, the students are faced again with not only again determining the matrix transformations for the constituent parts but also how to combine those matrices in a sensible manner. As seen in Figure 9, the class's work often begins by again creating names for each matrix associated with a transformation. Here student work illustrates a naming scheme developed during the class; students chose to use the letter B to label the matrix that made the matrix bigger, L for the matrix that made the N lean, and A is the original transformation matrix from the Italicizing N task (note, however, that what was labeled C in Figure 8 is labeled L in Figure 10). Also, the arrows going the opposite directions correspond to the transformations U_L, U_B, and U_A, which "undo" the original transformations defined by L, B, and A, respectively. Also note that the notation in Figure 9 is layered upon the symbolic representations developed in the original Pat and Jamie task (see Figures 6 and 7a), but that the standard symbolism for inverses is not used here. The teacher can use this development of notation to further foster sociomathematical norms of explicitly defining symbolic notation and providing justification for these choices. This also serves to connect to students' current ways of reasoning that the teacher could then leverage into formal notation and definitions used by the mathematics community.



(a) (b)

Figure 10. Student work on the Getting Back to N task.

The notation developed in Figure 9 can subsequently be used by students to calculate the numerical values of U_L, U_B, and U_A. The example of student work in Figure 10a illustrates a potential normative way of reasoning that matrix multiplication for a composition of functions is constructed right to left, with the matrix for the first transformation being multiplied on the left by matrices for the subsequent transformations. The example of student work shows the correct order of computation to find the inverses for Pat and Jamie's approach. The student wrote, "unskew → shrink" to indicate that the first action is to undo the lean and the second is to undo making it bigger (notated by the matrices U_L and U_B, respectively). The second example of student work (see Figure 10b) shows a student's notation illustrating that A composed with its "undoing" matrix U_A in either order leaves the input vector unchanged. This connects back to the matrix equation $A\mathbf{x} = \mathbf{b}$ interpretation of A acting on the vector $\mathbf{x}$ to produce the vector $\mathbf{b}$. In the first line of Figure 10b, matrix A acts on the vector $\mathbf{x}$, and matrix U_A acts on the resultant vector $A\mathbf{x}$ to produce vector $\mathbf{x}$. Similarly, the second line of Figure 10b connects to the aforementioned matrix equation $C\mathbf{b} = \mathbf{x}$ in that the matrix U_A acts on the vector $\mathbf{b}$, and matrix A acts on the resultant vector $U_A\mathbf{b}$ to produce vector $\mathbf{b}$. This sophisticated way of symbolizing gives the teacher a launching point from student reasoning within the task setting to connect to the definitions and notation of the mathematics community. For instance, she can facilitate a conversation about interpreting the matrix transformations (U_AA) and (AU_A) in Figure 10b as transformations that have the action of "doing nothing" to any given input vector, leading to a symbolic notation of these composition transformations as a "do nothing" transformation, notated by the letter I; that is, $(U_AA)\mathbf{x} = I\mathbf{x} = \mathbf{x}$ and $(AU_A)\mathbf{x} = I\mathbf{x} = \mathbf{x}$. Finally, as a member of the classroom community and the mathematics community, the teacher acts as a broker between them by connecting the class's work with the formal definitions of inverse for both linear transformation (i.e., a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible given that there exists a transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ such that $T(S(\mathbf{x}) = \mathbf{x}$ and $S(T(\mathbf{x})) = \mathbf{x}$) and matrix multiplication (i.e., a matrix A is invertible given that there exists an n x n matric C such that $AC = I$ and $CA = I$).

Conclusion

This abbreviated example of an HCP highlights several aspects of the interrelationships among the components in our definition of the construct. First, we highlight the fact that the learning goals are not sequential in nature, and we posit that the way in which these goals cut across tasks is important for supporting the development of productive normative ways of reasoning. For example, students repeatedly engage in the mathematical activity of coordinating input-output pairs to construct matrices, and this comes to function as the basis for later reasoning about how to conceptualize those matrices as mappings that can be combined (composed) and undone (inverted). Second, we highlight the fact that the variety of student approaches and sources of student difficulty function as a source of need for public negotiation of meaning – and that these conversations can and should contribute to the development of ideas that come to function as-if-shared in the classroom. This type of public negotiation of meaning is inextricably linked to the work the teacher does in using students' mathematical activity as a basis for group sense-making that moves forward the mathematical agenda. Finally, we highlighted the complexity of the role the teacher in negotiating meaning around student generated notation and in introducing more conventional notation in a way that honors student generated notation and connects to the broader mathematical community.

Looking back across the HCP, we see the first and fourth learning goals (interpreting a matrix as a mathematical object that transforms input vectors to output vectors, and coming to view matrices as objects that transform a space) cutting through the sequence of activities. More specifically, students repeatedly coordinate local and global views of matrix multiplication – including in the context of composing and inverting mappings. In addition, by repeatedly restructuring the original problem context in a variety of ways (as a two-step map for composing, and as a backwards map for inverting), students' activity shifts from an initial focusing solely on coordination of inputs and outputs to construct matrix representations for particular mappings, to a later focus on coordinating and combining those mappings (which also involves coordination of inputs and outputs but is in service of the goal of combining mappings).

We conclude by noting that this is consistent with Oehrtman, Carlson, and Thompson's (2008) recommendations for teaching ideas about functions. Namely, they recommend explicitly asking students to explain ideas about functions in terms of inputs and outputs, as well as asking students to explain function behavior on entire intervals (as opposed to just asking about function behavior in a pointwise fashion). Our HCP encourages students to coordinate inputs and outputs for the purpose of constructing matrices that yield desires mappings. In order to construct mappings that yield desired geometric transformations, students have a need to conceptualize the domain and codomain in a coordinated way for the purpose of selecting (linearly independent) sets of inputs and outputs in the domain and codomain, respectively.

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