MA4271 Notes

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1 Part A Theory of curves in $\mathbb{R}^3$

A.1 Parameterize the curves, regular curves, arc length

(This part corresponds to 1.2, 1.3 in Do Carmo's book)

Definition A.1.1 A *smooth curve* is a smooth map $\alpha : (a, b) \to \mathbb{R}^3$, where a, b can both be infinite. In terms of analysis, write $\alpha(t) = (x(t), y(t), z(t))$ (position vector), then the derivative of α can be written as $\alpha'(t) = (x'(t), y'(t), z'(t))$ (velocity vector), and is known as the tangent vector.

Example A.1.1 The image of a curve is defined as the trace of the curve. Then notice that even if $\alpha(t) = (\cos t, \sin t, 0)$ and $\beta(t) = (\cos 2t, \sin 2t, 0)$ have the same trace, the tangent vector of them are different.

Example A.1.2 The curve $\alpha(t) = (a \cos t, a \sin t, bt), \beta(t) = (a \cos t, -a \sin t, bt)$ are all possible ways to define helix, but they are always opposite (check the y components).

Example A.1.3 The curve $\alpha(t) = (t^3, t^2, 0)$ has a cusp at the origin.

To avoid cases like those in Example A.1.3, the definition of regular curves are required.

Definition A.1.2 Suppose $\alpha : I \to \mathbb{R}^3$, then α is a *regular smooth curve* if $\alpha'(t) \neq 0, \forall t \in I$.

Parameterization of tangent line of α at the point t_0 can be given by $\alpha(t_0) + t(\alpha'(t_0))$, notice that the locus of this line corresponds to the first two terms in Taylor expansion of α centred at t_0.

Theorem A.1.1 Arc length of a regular curve always exists and can be written as

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$$

, and hence that $\frac{ds}{dt} = |\alpha(t)|$ (by Fundamental theorem of calculus). Further, arc length is independent of parameters (hence canonical parameter).

Given a curve $\alpha(t)$, where t is a general parameter, the method of finding an arc length parameter is as follows: First solve the ODE $s'(t) = |\alpha'(t)|$, so we can have an expression of $s(t)$. Since arc length is monotone, the inverse exists, by inverse function theorem, $t(s)$ can be obtained and is the arc length parameter. Notice that **if $|\alpha'(t)| = 1, \ \forall t \in I$, then t is an arc length parameter**.

All arc length parameter has $|\alpha'(s)| = 1$ by the following reasoning: $|\alpha'(s)| = |\alpha'(t)|t'(s)| = |s'(t)||t'(s)| = 1$, where the first equality holds by chain rule (regarding t as a function of s), and the second holds by differentiating $s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$.

A.2 Local theory of curves

(This part corresponds to 1.5 in Do Carmo's book)

Definition A.2.1 Suppose $\alpha : I \to \mathbb{R}^3$ is a curve parameterized by arc length, then we define $k(s) := |\alpha''(s)|$ as the curvature of α at point s. (Note that the curvature might not be a constant.)

Notice that $\alpha'(s) \cdot \alpha''(s) = 0$ since $|\alpha'(s)| = 1 \iff \alpha'(s) \cdot \alpha'(s) = 1$. Taking derivative on both sides gives the result.

Write $n(s)$ to be the unit vector in the direction of $\alpha''(s)$. Then obviously $\alpha''(s) = k(s)n(s)$. The curvature describes how sharp the curve bends at a given point. And we write $\alpha'(s) = t(s)$ to emphasise the idea of tangent vector.

Example A.2.1 A line in $\mathbb{R}^3$ has zero curvature.

Example A.2.2 If $\alpha(s) = (a \cos \frac{s}{a}, a \sin \frac{s}{a}, 0)$ has curvature $\frac{1}{a}$. It is easy to verify that the above parameter is arc length parameter.

Example A.2.3 Let $\alpha(s) = (a \cos s, a \sin s, bs), \ a^2 + b^2 = 1$, and $\beta(s) = (\frac{1}{a} \cos as, \frac{1}{a} \sin as, 0)$, then both of them have the same curvature a, but the former is a space curve, whereas the latter is a planer curve.

Definition A.2.2 s is called *singular point of order 1* of a curve α if $\alpha''(s) = 0$.

Definition A.2.3 If a curve α does not have a singular point of order 1, then the *osculating plane* associated to α at point s is spanned by $\{t(s), n(s)\}$. The normal vector associated with this plane is known as the *binormal vector* and is defined as $b(s) := t(s) \times n(s)$.

Notice that if a curve is planar, then the binormal vector is a constant.

Definition A.2.4 The *torsion* of a curve α, $\tau(s)$, is defined as $\tau(s) := |b'(s)|$.

By definition of $b(s)$, we have $b'(s) = t'(s) \times n(s) + t(s) \times n'(s)$, but since $t'(s)$ and $n(s)$ are coplaner, the first term is 0. Hence we write $b'(s) = t(s) \times n'(s)$. Then we know that $b'(s)$ is orthogonal to $t(s), n'(s)$ and to $b(s)$ by definition of derivative. We know that $b'(s)$ is parallel to $n(s)$ (the reasoning behind is, since $b(s)$ is orthogonal to $b'(s)$, $b'(s)$ lies in the osculating plane, but since that $b'(s)$ is orthogonal to $t(s)$, and that $n(s)$ is orthogonal to $t(s)$, we know that $b'(s)$ must have the same direction as $n(s)$ does). And thus we write $b'(s) = \tau(s)n(s)$.

There are two important properties of torsion: (1). $\tau(s) = b'(s) \cdot n(s)$. (2). $\tau(s) = \frac{-(\alpha'(s) \times \alpha''(s)) \cdot \alpha'''(s)}{(k(s))^2}$.

The first property is easy to explain: $b(s) \cdot n(s) = \tau n(s) \cdot n(s) = \tau(s)$ since $n(s)$ is a unit vector.

Example A.2.4 A curve is planar iff its torsion is 0 at any point in its domain.

Example A.2.5 As in Example A.2.3, the torsion of α is $-b$.

Notice that $\{t(s), n(s), b(s)\}$ is a basis on $\mathbb{R}^3$.

Definition A.2.5 The *Frenet formula* is given by

$$\begin{aligned} t'(s) &= k(s)n(s) \\ n'(s) &= -\tau(s)b(s) - k(s)t(s) \\ b'(s) &= \tau(s)n(s) \end{aligned}$$

Notice that the plane spanned by $\{n, b\}$ is called normal plane, and that spanned by $\{t, b\}$ is called rectifying plane.

The next theorem is called Fundamental theorem of local theory of curves. The rough idea is to prove that given expressions of curvature and torsion, a unique curve, up to rigid motion, will be determined. Rigid motion means rotation and /or translation.

Theorem A.2.1 Given $k(s) > 0, \tau(s), s \in I$, there exists a unique, up to rigid motion, regular, parameterized curve with these curvature and torsion. i.e. $\beta = \rho\alpha + c$, where $\rho \in \mathrm{SO}(3), c \in \mathbb{R}^3$.

Proof. We first require two lemmas.

Lemma A.2.1 Consider the system of ODE $\{\frac{dx_i}{ds} = f_i(s, x_i(s), \cdots, x_k(s))\},\ 1 \le i \le k, s \in I$, and each f_i are smooth functions on $I \times \mathbb{R}^k$, subject to initial conditions $x_i(s_0) = a_i,\ 1 \le i \le k$, then there exists an open interval $J \subset I$ containing s_0 and a unique solution of this system on J.

Lemma A.2.2 Suppose further that the system of ODE in the previous lemma is linear, then $J = I$.

We first prove existence. Then suppose the coordinates of t, n, b are $t = (x_1, x_2, x_3), n = (x_4, x_5, x_6), b = (x_7, x_8, x_9)$. Then the system of Frenet formula becomes a linear differential system with these 9 variables. The matrix associates to the 9 variables are given by

$$\begin{bmatrix} 0 & 0 & 0 & k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k & 0 & 0 & 0 \\ -k & 0 & 0 & 0 & 0 & 0 & -\tau & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 & 0 & -\tau & 0 \\ 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 & -\tau \\ 0 & 0 & 0 & \tau & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau & 0 & 0 & 0 \end{bmatrix}$$

Given $\{t_0, n_0, b_0\}$ as initial conditions, by the previous two lemmas, there exists a unique solution $t(s), n(s), b(s)$ on I such that $t(s_0) = t_0, n(s_0) = n_0, b(s_0) = b_0$.

To conclude the proof of existence, we need to show that $\{t(s), n(s), b(s)\}$ are everywhere orthonormal. This is done by analyzing the system of equations

$$\begin{aligned} \frac{d(t,n)}{ds} &= k(n,n) - k(t,t) - \tau(t,b) \\ \frac{d(t,b)}{ds} &= k(n,b) + \tau(t,n) \\ \frac{d(n,b)}{ds} &= -k(t,b) - \tau(b,b) + \tau(n,n) \\ \frac{d(n,n)}{ds} &= 2k(t,n) \\ \frac{d(b,b)}{ds} &= 2\tau(b,n) \end{aligned}$$

where the bracket denotes Euclidean inner products. By inspecting that $(0, 0, 0, 1, 1, 1)$ is a solution, together with uniqueness of solution lemma, we know that this is the unique solution required.

Now we prove that the solution constructed above is unique up to rigid motion. We choose a point p_0 and denote the solution above as $\alpha(s) = p_0 + \int_{s_0}^{s} t(v)dv$. We suppose that $\alpha(s)$ is parameterized by arc length, and that $k_\alpha(s) = k(s), \tau_\alpha(s) = \tau(s)$.

Given another such curve, $\beta(s) = q_0 + \int_{s_0}^{s} \overline{t(v)}dv$, where $\begin{bmatrix} t_0 \\ n_0 \\ b_0 \end{bmatrix} = A \begin{bmatrix} \overline{t_0} \\ \overline{n_0} \\ \overline{b_0} \end{bmatrix}$, and $A \in \mathrm{SO}(3)$. then by the lemma for uniqueness, we know that $\begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} = A \begin{bmatrix} \overline{t(s)} \\ \overline{n(s)} \\ \overline{b(s)} \end{bmatrix}$

So in this case, we re-write $\alpha(s) = Aq_0 + A\int_{s_0}^{s} \overline{t(v)}dv + (p_0 - Aq_0)$. This concludes uniqueness. $\square$

A.3 The local canonical form

This corresponds to 1.6 in Do Carmo's book.

Suppose that at a point s_0, the Frenet frame are t_0, n_0, b_0 with $k(s_0) = k, \tau(s_0) = \tau$. Consider the Taylor expansion of α at 0 (because it does not matter for which s_0 we choose, so WLOG we assume it is 0), we can easily deduce

$$\begin{aligned} \alpha'(0) &= t_0 \\ \alpha''(0) &= kn_0 \\ \alpha'''(0) &= k'n_0 - k^2t_0 - k\tau b_0 \end{aligned}$$

Then we have $\alpha(s) - \alpha(0) = (s - \frac{k^2s^3}{6})t_0 + (\frac{ks^2}{2} + \frac{k's^3}{6})n_0 - \frac{s^3k\tau}{6}b_0 + R$, where $R = o(s^3)$. Then the *local canonical form* of α near 0 is given by

$$\begin{aligned} x(s) &= s - \frac{k^2s^3}{6} + R_x \\ y(s) &= \frac{ks^2}{2} + \frac{k's^3}{6} + R_y \\ z(s) &= -\frac{s^3k\tau}{6} + R_z \end{aligned}$$

where $R = R_x + R_y + R_z$ (Basically what we have done here is to resolve R in three directions).

Remark This would be quite useful to analyze some problems regarding limits and local behaviours that seems unsolvable. When doing such question, we generally initialize the point to be the origin and apply the local canonical form at the origin to ease any other unnecessary computations.

Then there are two applications of this canonical form. One is, since $k > 0, y \geq 0$. The equality holds iff $s = 0$. So The curve is above or on the rectifying plane. Another is how to interpret the sign of torsion. If a curve pass through the osculating plane from -ve to +ve, the torsion is negative, vice versa.

A.4 Global properties of closed curves *NOT TESTED*

This part corresponds to 1.7 in Do Carmo's book

(a) Simple closed curves

Definition A.4.1 A *closed plane curve* is a regular parameterized curve $\alpha : [a, b] \to \mathbb{R}^2$ such that $\alpha^n(a) = \alpha^n(b)\ \forall n \geq 0$. A *simple curve* means there is no self-intersections except the endpoints.

Theorem A.4.1 *Jordan Curve Theorem* Only simple closed curve γ in $\mathbb{R}^2$ has an interior and exterior. Further, the interior is bounded and the exterior is unbounded. Both interior and exterior themselves are connected, but they are disconnected.

(b) Isoparametric inequality

Theorem A.4.2 Suppose that γ is simple closed curve with length l. Then

1. $l^2 \geq 4\pi A(\mathrm{int}\gamma)$
2. The equality holds iff γ is a circle

To avoid defining measures when finding the area, we use Green theorem instead. So one of the parameterization of area is given by $A = \frac{1}{2}\int_C xdy - ydx$. Note that integrating this 1-form is well-defined since we assume that the curve is smooth.

(c) Rotation index of simple closed curves

If we write $\alpha(s) = (x(s), y(s))$, then $N(s) := (-y'(s), x'(s))$, and we can rewrite $t'(s) = k_r(s)N(s)$. Then the function $k_r(s)$ is known as *signed curvature*.

Here I will include two formulas concerning curvature, both of which can be easily applied in case of the parameter might not be arc length.

$$k(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha(t)|^3} \text{ for curvature}$$
$$k_r(t) = \frac{x'y'' - x''y'}{((x')^2 + (y'^2))^{\frac{3}{2}}} \text{ for } \textit{signed} \text{ curvature}$$

Remark: For signed curvature, we write $\alpha(t) = (x(t), y(t))$. So when the parameter given in the question is not arc length, there is absolutely no need to convert it to arc length since it would be very time-consuming and error-prone, just apply the formula above would suffice.

Let $\theta(s)$ be the angle between $t(s)$ and the *positive* x-axis, notice that this is a multivalued function (because each of the value is unique up to an addition of multiple of 2π). But this suggests that its derivative is well-defined everywhere.

Then notice that $k_r(s) = \theta'(s)$. This can be easily verified by chain rule.

Definition A.4.2 *Rotation index* is defined as $\frac{1}{2\pi}\int_\gamma k_r(s)ds$.

Theorem A.4.3 If γ is a simple closed curve, then the rotation index is 1.

(d) Convex curves and four-vertex theorem

Definition A.4.3 A regular curve (not necessarily closed) $\alpha : [a, b] \to \mathbb{R}^2$ is called convex if $\forall s \in [a, b]$, the trace of α lies entirely on the side of the closed half plane determined by the tangent line.

Theorem A.4.4 Suppose γ is a simple closed curve, then it is convex iff it fixes an orientation, $k_r(s)$ does not change sign.

If $k_r(s) = 0$, we call s as a *vertex* of γ.

Theorem A.4.5 *Four Vertex Theorem* If γ is a simple closed curve, then γ has at least four vertices.

2 Part B The theory of surfaces in $\mathbb{R}^3$

B.1 Regular surfaces

Definition B.1.1 $S \subset \mathbb{R}^3$ is a *regular surface* if $\forall p \in S, \exists V_\epsilon(p) \subset \mathbb{R}^3$ and a map $P : U \to V \cap S$, where U is an open set in $\mathbb{R}^2$ such that

1. P is diffeomorphism
2. dP_p has full rank (the map is injective)

Here the word *diffeomorphism* means differentiable and homeomorphism. Then the map $P : (u, v) \to (x(u, v), y(u, v), z(u, v))$, call the resulting function as $P(u, v)$ to stress on the idea that this function has two variables.

Recall that we have that $dP_p = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$, and thus that $dP_p(e_1) = P_u, dP_p(e_2) = P_v$, where e_1, e_2 are standard bases in $\mathbb{R}^2$.

Example B.1.1 Here are two methods to cover S^2. One is to consider $P_N : U \to S^2 \cap \{z > 0\}$ given by $(x, y) \to (x, y, \sqrt{1 - x^2 - y^2})$. Then we repeat this with other five cases, so together we have 6 coordinate patches to cover S^2. Here $U := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$

Or we consider $S^2 - \{x^2 + z^2 = 1\}$. Define $P_1 : U \to S^2 - \{x^2 + z^2 = 1\}$ given by $(\theta, \phi) \to (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $U := \{(\theta, \phi) | 0 < \theta < \pi, 0 < \phi < 2\pi\}$.

Then there are two ways to generate regular surfaces.

Theorem B.1.1 The graph of differentiable function is a regular surface. That is, suppose $D \subset \mathbb{R}^2$ to be open, let $f : D \to \mathbb{R}, P : D \to (D, f(D))$. Then P is diffeomorphism.

Do note that the method above generate a piece of surface only.

Definition B.1.2 A *regular value* of f is a real number such that $\nabla f|_\lambda \neq 0$.

Theorem B.1.2 Suppose $f : U \to \mathbb{R}$ is a differentiable map, and $U \subset \mathbb{R}^3$, and $\lambda \in \mathbb{R}$ is a regular value of f. Then the fibre $f^{-1}(\lambda)$ is a regular surface.

Example B.1.2 $f : \mathbb{R}^3 \to \mathbb{R}$ given by $(x, y, z) \to \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ has 1 as regular value and is known as ellipsoid. Similarly, if it maps to $-x^2 - y^2 + z^2$, 1 is again a regular value, the resulting surface is called hyperbolid.

Example B.1.3 *Standard torus* Here is a way to cover the torus. Define the map $P : (0, 2\pi) \times (0, 2\pi) \to T - (C_1 \cup C_2)$, where T is the torus and C_1 is the boundary lines $\{u = 0\} \cup \{u = 2\pi\}$. C_2 is defined for v in the similar way. The map has explicit expression $P(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$.

Here r is the radius of the inner circle, $a + r$ is the radius of the outer circle. It is obvious that this map is a diffeomorphism, and P_v, P_u are linearly independent.

B.2 The tangent plane and normal line of regular surfaces

This part corresponds to section 2.4 in Do Carmo's book.

Tangent Plane

Definition B.2.1 Suppose p is a point on a regular surface S, a *tangent vector* to S at p is the tangent vector of a parameterized curve on S. The curve can be described by, $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$, and we have $v = \alpha'(0)$.

We write $T_p(S) := \{\text{all tangent vectors at } p\}$, and this is known as *tangent space* at p.

Theorem B.2.1 $T_p(S)$ is a 2 dimensional vector space.

Proof. In this proof, we claim that $\text{Span}\{dP_p(e_1), dP_p(e_2)\}$ is $T_p(S)$. Because this proves that the tangent space is 2 dimensional, and the remaining parts are easy to verify.

Choose a coordinate patch U around p. Suppose V is a tangent vector of S at p, then by Definition B.2.1, there exists such a curve α.

Then in U, we have $\alpha(t) = P(u(t), v(t))$, where P is the map from the manifold to $\mathbb{R}^2$. The next step is to calculate $\alpha'(0)$ in two ways.

By chain rule, we know that $\alpha'(0) = P_u u'(0) + P_v v'(0)$. And by definition, $\alpha'(0) = v$. Since our choice of α is arbitrary, the result holds at once. $\square$

Theorem B.2.2 Change of parameterization results in change of basis on the tangent space.

Proof. It follows at once from the compatibility condition of charts and altas, here is a sketch. Suppose that U, V are two coordinate patches that have nonempty intersections. Let $P : U \to S$ and $h : V \to U$, so the map $P \circ h : V \to S$ is well-defined. Then by chain rule, $\begin{bmatrix}(P\circ h)_s & (P \circ h)_t\end{bmatrix} = \begin{bmatrix}P_u & P_v\end{bmatrix} \begin{bmatrix} u_s & u_t \\ v_s & v_t \end{bmatrix}$ Notice that the matrix is simply the Jacobian (change of basis) $\square$

Example B.2.1 Consider the unit sphere $x^2 + y^2 + z^2 = 1$, let $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$. To find $T_p(S^2)$, we have two methods. One is the function and graph method, that is, we parameterize the sphere by $(u, v, \sqrt{1 - u^2 - v^2})$. Notice that $P_u(p) = (1, 0, -\frac{1}{\sqrt{2}}), P_v(p) = (0, 1, -\frac{1}{\sqrt{2}})$, then we know immediately that $T_p(S) = p + tP_u(p) + \mu P_v(p)$, where $t, \mu \in \mathbb{R}$ are variables. Alternatively, take cross product could get the alternative form.

The other method is to use spherical coordinate. Let $P(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. In this case, we take $\theta = \phi = \frac{\pi}{4}$. To complete the computation, we know that $P_\theta(p) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}), P_\phi(p) = (\frac{-1}{2}, \frac{1}{2}, 0)$.

Normal line

Definition B.2.2 Given $p \in S$, the straight line passing through p orthogonal to the tangent space is the *normal line* of S at p.

Choose a coordinate patch $P : U \to S$, in the neighborhood of p, consider the normal vector $n_{p,P} := \frac{P_u \times P_v}{|P_u \times P_v|}(p)$, then the normal line is $p + tn_{p,P}$. Then, for example, the equation of normal line in previous example is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}) + t(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$. Further, the tangent plane equation can be written as $((x, y, z) - p) \cdot n_{p,P} = 0$.

Theorem B.2.3 Change of parameterization will remain $n_{p,P}$ up to a change of sign.

Proof. We only give an outline here. Write $(P \circ h)_s, (P \circ h)_t$ explicitly by multiplying the matrices. Then notice that cross product is bilinear, skew-symmetric. $\square$

If the sign is positive, then we say the two parameterizations are orientation-preserving, otherwise we say they are orientation-reversing.

Definition B.2.3 Under parameterization P, we have $\{P_u, P_v, n_{p,P}\}$. The set is known as *frame field* or Gauss field.

But the question arises when we have a smooth map $F : \mathbb{R}^3 \to \mathbb{R}$, and $a \in \mathbb{R}$ is a regular value and consider the regular surface $F^{-1}(a)$. What can we say about the normal line? The next theorem solves this issue.

Theorem B.2.4 Suppose $p \in F^{-1}(a), p = (x_0, y_0, z_0)$, then the gradient vector at p is a normal vector.

Proof. Suppose w is a tangent vector at $p \in F^{-1}(a)$. Then by definition, $w = \alpha'(0)$ for some curve α with $\alpha(0) = p$. Notice that $F \circ \alpha = a$ as α is the preimage of a. Then by chain rule, $\frac{d}{dt}F\alpha = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} = 0$. This is the same as $\nabla F \cdot \alpha'(t)$. Evaluate this at $t = 0$ shows that $\nabla F \cdot w = 0$, which is the same as saying ∇F is a normal vector. $\square$

Example B.2.2 Let $F : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \to x^2 + y^2 + z^2$, then we know that $F^{-1}(1) = S^2$, the gradient vector $(2x, 2y, 2z)$ is a normal to the unit sphere.

To conclude this section, we discuss an application of normal vectors. Suppose two surfaces S_1, S_2 intersects. To find the angle between the intersection, we first compute the normal vectors n_1, n_2 of the surface S_1, S_2 respectively. Then the angle between n_1, n_2 is what we want. (Here it would be nice if we normalize it since we do not need to worry about denominators.)

B.3 The first fundamental form

This corresponds to section 2.5 in Do Carmo's book

Definition B.3.1 The *first fundamental form*: Suppose S is a regular surface, and $p \in S$. On $T_p(S)$, the quadratic form $I_p(w) = |w|^2 \geq 0$ is called the first fundamental form of S at p.

Since there is a one-to-one correspondence between quadratic forms and bilinear forms, we could define the corresponding bilinear form as $I_p(w_1, w_2) = \frac{1}{2}(I_p(w_1 + w_2) - I_p(w_1) - I_p(w_2))$.

Choose a parameterization P near p, choose an open set U containing p, consider $P(u, v)$. Let $w = \alpha'(0)$ for some curve α in the definition. By some computations, we know that $I_p(w) = P_u \cdot P_u u'(0)^2 + 2P_u \cdot P_v u'(0)v'(0) + P_v \cdot P_v v'(0)^2$. So denote $E = P_u \cdot P_u, F = P_u \cdot P_v, G = P_v \cdot P_v$, so we have $du : T_p(S) \to \mathbb{R}, w \to u'(0)$ and define dv similarly, so the first fundamental form is $I(w) = Edu^2 + 2Fdudv + Gdv^2$.

In some less obvious sense, but clear from definition of a quadratic form, the first fundamental form can be written as $\begin{bmatrix} du & dv \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$.

The next step is to consider a change of parameter. Suppose $Q(s, t)$ is a new parameterization comparing to the original one $P(u, v)$. By noticing that $Q_s = P_u u_s + P_v v_s$ and $Q_t = P_u u_t + P_v + v_t$, and apply the definition of the first fundamental form, we immediately have $\begin{bmatrix} E(s,t) & F(s,t) \\ F(s,t) & G(s,t) \end{bmatrix} = J \begin{bmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{bmatrix} J^T$, where $J = \begin{bmatrix} u_s & v_s \\ u_t & v_t \end{bmatrix}$ is the Jacobian. More explicitly, we have $E(s, t) = E(u, v)u_s^2 + 2F(u, v)u_s v_s + G(u, v)v_s^2, F(s, t) = E(u, v)u_s u_t + F(u, v)(u_s v_t + u_t v_s) + G(u, v)v_s v_t, G(s, t) = E(u, v)u_t^2 + 2F(u, v)u_t v_t + G(u, v)v_t^2$.

Example B.3.1 We write a plane equation as $P(u, v) = p + uw_1 + vw_2$, then we know that $E = |w_1|^2, F = w_1 \cdot w_2, G = |w_2|^2$ are all constants.

Example B.3.2 If we use the spherical coordinate to parameterize the unit sphere, i.e. $P(\theta, \phi) = (\cos\theta \cos\phi, \cos\theta \sin\phi, \sin\theta)$, then clearly we have $E = 1, F = 0, G = \cos\theta$.

Example B.3.3 Consider the curve $\alpha(v) = (f(v), g(v))$ lying on the $x - z$ plane, then if we revolve the curve around the z axis, then we have the obvious parameterization for the surface of revolution: $P(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$, where u is the angle between a point on the surface and the $x - z$ axis.

Then we discuss the application of the fundamental form:

(1). We could have the arc length in surfaces $\alpha : I \to S$, where I here is an interval. Then by definition, we have $s = \int_a^b |\alpha'(t)| dt = \int_a^b \sqrt{I(\alpha'(t))} dt = \int_a^b \sqrt{Eu'(t)^2 + 2Fu'(t) + Gv'(t)^2} dt$. This sometimes ease the computation.

Example B.3.4 Given $P(u, v) = (u \cos v, u \sin v, \ln \cos v + u)$, where $-\frac{\pi}{2} < v < \frac{\pi}{2}$, and $u \in \mathbb{R}$. Show that $P(u = a, v), P(u = b, v)$ determines segments of equal lengths of all curves of the form $P(u, v = c)$. By the arc length formula above, $s = \int_a^b \sqrt{Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)^2} du = \int_a^b \sqrt{E(u, v_0)} du$, notice that $E(u, v) = P_u \cdot P_u = 2$, and hence the arc length is $\sqrt{2}(b - a)$.

(2). The second application is to compute angle between curves on a surface. Suppose that α, β are all such curves. In a chart, it is parameterized by $\alpha(t) = P(u_1(t), v_1(t)), \beta(t) = P(u_2(t), v_2(t))$, we assume that the point of intersection is when $t = 0$ we denote it by p. So the angle between the curve is

$$\begin{aligned} \cos\theta &= \frac{\alpha'(0) \cdot \beta'(0)}{|\alpha'(0)||\beta'(0)|} \\ &= \frac{(P_u u_1'(0) + P_v v_1'(0)) \cdot (P_u u_2'(0) + P_v v_2'(0))}{\sqrt{I(\alpha'(0)) I(\beta'(0)}} \end{aligned}$$

where the I in the second line is the first fundamental form. It could be expanded though, but it is super complicated. The next theorem is much neater. This can be easily proven by brute force.

Theorem B.3.1 For coordinate curves, $P(a, v), P(b, v)$, the angle is $\cos\theta = \frac{F}{\sqrt{EG}}|_p$.

Example B.3.5 Find the curve such that it equally divides the angle of two coordinates. Consider the curve $\gamma : I \to S$ and that $\gamma(t) = P(u(t), v(t))$. Obviously that $\gamma'(t) = P_u u'(t) + P_v v'(t)$. Now we compute angles between the curve γ' and P_u, P_v respectively.

$$\cos(\gamma', P_u) = \frac{Eu'(t) + F'v(t)}{\sqrt{EI}}$$
$$\cos(\gamma', P_v) = \frac{Fu'(t)_G v'(t)}{\sqrt{GI}}$$

where the I above is the first fundamental form. Since we are given that the two angles above are the same, we immediately get, by clearing denominators, that

$$\sqrt{G}(Eu'(t) + Fv'(t)) = \sqrt{E}(Fu'(t) + Gv'(t))$$
$$\implies (EG - F^2)(Eu'(t)^2 - Gv'(t)^2) = 0$$
$$\implies Eu'(t)^2 = Gv'(t)^2$$

where the second line is obtained by squaring the first line, and the third line is obtained by dividing $EG - F^2$, since it is never zero (more rigorously, it is always positive by Cauchy inequality). Take the square root of the last line, we get two possible cases, and they are just ODEs with t as independent variable. Solving it gives the solution curve we want.

(3) The last application that will be discussed here is the area of a region. A region of S is a union of a domain with boundary. A regular domain is an open, connected subset of S. The boundary is diffeomorphic to a circle. Here we allow punctures.

Suppose a region $R \subset U$ is parameterized by chart $U, P(u,v)$, then $A(R) = \iint_{P^{-1}(R)} |P_u \times P_v| dudv$ is defined to be the area of such region.

Remark This expression is well-defined. That is, it does not depend on the choice of parameterization. This is easy to prove, we simply need to recall change of parameter formula and the definition of Jacobian.

Theorem B.3.2 $A(R) = \iint_{P^{-1}(R)} \sqrt{EG - F^2} dudv$

Proof. Notice that $|P_u \times P_v|^2 + (P_u \cdot P_v)^2 = |P_u|^2 |P_v|^2$. Then everything else is trivial. □

Example B.3.6 Surface area of a torus. $P(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$. Then we calculate that $\sqrt{EG - F^2} = r(r\cos u + a)$ and hence that $A = \int_0^{2\pi}\int_0^{2\pi} r(r\cos u + a) dudv = 4\pi^2 ra$.

To conclude B.3, we have two appendix, the first one is not tested.

Appendix B.3.1

We discuss what is meant by $\int_S f dS$, where $f : S \to \mathbb{R}$. We already know that $\int_S dS$ is the surface area. Then suppose that $S = \bigcup_\alpha \{U_\alpha\}$, so there exists $\phi_\alpha : S \to \mathbb{R}^+ \cup \{0\}$, for every choice of α such that $\overline{\text{supp}\phi_\alpha} \subset U\alpha$ (this basically means that the nonzero images of $\phi_\alpha \in U_\alpha$), and that $\forall x \in S, \sum_\alpha \phi_\alpha(x) = 1$. This is basically known as *partition of unity*. Then $\int_S f dS$ is defined as

$$\int_S f dS = \sum_\alpha \int_{U_\alpha} f\phi_\alpha dS$$
$$= \sum_\alpha \iint_{P^{-1}(U_\alpha)} f\phi_\alpha(\sqrt{EG - F^2})|_{U\alpha} dudv$$

Do notice that this is well defined.

Appendix B.3.2 We discuss smoothness and differential of $f : S \to \mathbb{R}$.

f is smooth if $\forall x \in S, \exists U$ such that $f|_U$ is smooth about 2 parameters.

There are two ways to define the differential here.

This one is geometric. Suppose that α is a curve on the surface such that $\alpha(0) = p, \alpha'(0) = w$. Then the differential at this point is given by $df_p(w) = \frac{d}{dt}(f \circ \alpha(t))|_{t=0}$. Notice that it is obvious that the differential is well-defined.

The other one is algebraic. Choose a chart $U, P(u,v)$ around the point p. We know $T_p(S) = \text{span}\{P_u, P_v\}$. The differential is a linear map such that $df_p(P_u) = f_u, df_p(P_v) = f_v$.

The differential in this case is a number.

B.4 The definition of Gauss map and its fundamental properties

This corresponds to section 3.2 in Do Carmo's book.

(1) Defining Gauss map and its differential.

The orientation of a surface S is defined as **EITHER** differnentiable field of unit normal vectors on S **OR** there exists consistent cover of charts, where consistent means that sign of determinant of the Jacobian is positive. An example is the surface $F^{-1}(a)$ for a regular value a, then $\frac{\nabla F}{|\nabla F|}$ is a differentiable field of unit normal vectors.

Definition B.4.1 Let S be an orientiable surface, then the Gauss map is $N : S \to S^2, p \to N(p)$.

Example B.4.1 For a plane $ax + by + cz = d$, $N(p) = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$ for any point in the plane.

For a sphere, $(x, y, z) \to (x, y, z)$ or $-(x, y, z)$.

For a cylinder, $x^2 + y^2 = 1$, then the $N(x, y, z) = (x, y, 0)$ or $(-x, -y, 0)$.

In this case, $T_p(S)$ and $T_{N(p)}(S^2)$ can be identified. Suppose that $\alpha(t)$ is a curve on S, then it is mapped to $N \circ \alpha(t)$, where $N \circ \alpha(0) = N_p$. So the differential of Gauss map is $dN_p : w \to (N \circ \alpha(t))'|_{t=0}$, where $w = \alpha'(0)$.

The Gauss map is coordinate-free, that is, no charts and altas are chosen.

Remark: Suppose $f : \Sigma_1 \to \Sigma_2$ be a map between two surfaces, then the differential map at point p is given by $df_p : T_p\Sigma_1 \to T_p\Sigma_2$. The geometric interpretation is that the differential map is a linearization of f, in which we could apply linear algebra in the tangent space.

Now we re-interpret the example Gauss maps in the previous example.

As for the plane $P : ax + by + cz = d$, the differential of Gauss map $dN_p : T_pP \to T_qS^2 \cong T_pP$, where we identify the space T_qS^2 with T_pP. Since the Gauss map here is a constant map, $dN_p(w) = 0, \forall w \in T_pP$ is the zero map. As for the sphere, if we take the + sign, it is just the identity map.

The case of a cylinder is different. Suppose we take the + sign. Then consider the differential $dN_p : T_pS \to T_pS$. Let $w \in T_pS$, so $w = \alpha'(t)|_{t=0}$ for some curve α on the surface. In this case, we could write $\alpha(t) = (x(t), y(t), z(t))$ and hence that $w = (x'(0), y'(0), z'(0))$. So we have $dN_p(w) = (x'(0), y'(0), 0)$. In more concrete settings, consider the resulting vectors given by $dN_p(0, 0, 1) = (0, 0, 0), dN_p(1, 0, 0) = (1, 0, 0)$. We can easily deduce that the eigenvalues of the tangent space are 0,1.

Example B.4.2 Consider the hyperbolid $z = y^2 - x^2$. We may parameterize the surface in the following way, let $P(u, v) = (u, v, v^2 - u^2)$ hence we have $P_u = (1, 0, -2u), P_v = (0, 1, 2v)$. By some painful calculations, one can eventually get the unit normal vector is

$$(\frac{u}{\sqrt{u^2+v^2+1/4}}, \frac{-v}{\sqrt{u^2+v^2+1/4}}, \frac{1}{2\sqrt{u^2+v^2+1/4}})$$

Consider the Gauss map $N : S \to S^2$. Take $p = (0, 0, 0)$, we investigate T_pS. It is easy to see that $N_p(p) = (0, 0, 1)$. Notice that at p, x_u, x_v are unit normal vectors along the x, y axis, respectively. Suppose $\alpha(t)$ is a curve with $\alpha(0) = p$, so by the above reasoning, $\alpha'(0) = (u'(0), v'(0), 0)$. Thus $N'(0) = (N \circ \alpha(t))'|_{t=0} = (2u'(0), -2v'(0), 0)$ is the derivative of unit normal vector. Hence $dN_p(u'(0), v'(0), 0) = (2u'(0), -2v'(0), 0)$. Then it is trivial that $(1, 0, 0), (0, 1, 0)$ are eigenvectors of this map, with eigenvalues $2, -2$, respectively. In this case, we could say the differential map is equivalent to the matrix $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

Choose a parameterization around p, $P(u, v)$, then $dN_p(P_u) = N_u, dN_p(P_v) = N_v$ and hence that $dN_p(\alpha'(0)) = N_uu'(0) + N_vv'(0)$.

Theorem B.4.1 The differential of Gauss map is a self-adjoint operator with respect to I_p, the first fundamental form.

Proof. We need to prove for any $w_1, w_2 \in T_pS$, we have $I_p(dN_p(w_1), w_2) = I_p(dN_p(w_2), w_1)$

It suffices to show that $N_u \cdot P_v = P_u \cdot N_v$. But this holds trivially true since $N \cdot P_u = N \cdot P_v = 0$, N is normal vector. □

(II) The second fundamental form and third fundamental form

Definition B.4.2 The second fundamental form is given by $II_p(v, w) = -dN_p(v) \cdot w$ and $II_p(v) = -dN_p(v) \cdot v$. The third fundamental form is given by $III_p(v) = |dN_p(v)|$. Note that by the self-adjoint property, we could write the second fundamental form as $-v \cdot dN_p(w)$.

Definition B.4.3 Let c be a regular curve in the surface passing through $p \in S$, and n is the normal vector of the curve at p. Denote N to be the normal vector of the surface at p, then we define the *normal curvature* as $k_n = k \cos\theta$, where k is the curvature of C at p, and $\cos\theta = n \cdot N$.

Theorem B.4.2 Suppose $\alpha(s)$ is a regular curve in the surface parameterized by the arc length parameter with $\alpha(0) = p$. Then $II_p(\alpha'(0)) = k_n(p)$.

Proof. This is direct calculation in the following manner.

$$\begin{aligned} II_p(\alpha'(0)) &= -dN_p(\alpha'(0)) \cdot \alpha'(0) \\ &= -N'(0) \cdot \alpha'(0) \\ &= N(0) \cdot \alpha''(0) \\ &= N \cdot kn \\ &= k(N \cdot n) \\ &= k_n(p) \end{aligned}$$

where the third equality holds by the fact $N(s) \cdot \alpha'(s) = 0$

This leads to a corollary

Corollary B.4.1 (Memsmer) All curves lying on the surface and having a given point $p \in S$. These curves have the same tangent line at this point and they have the same curvature. (In this case, we only consider the tangent vectors, instead of curves, since they are all equivalent) □

Remark: Suppose we have $N_u = a_{11}P_u + a_{12}P_v, N_v = a_{21}P_u + a_{22}P_v$, then the differential of the Gauss map is equivalent to the matrix (a_{ij}).

The next problem follows is that when given $p \in S$, v is a tangent vector and N is the normal vector, how to explicitly find a curve with tangent vector v and normal vector N.

Definition B.4.4 In the set of all curves passing through p, with a given tangent vector v, there exists a special one obtained from the intersection of S with the plane spanned by N, v, then the curve is defined to be the normal section of S at p along v.

Denote the curve by $C_{p,v}$, then the normal vector of it at p is either N or $-N$ (depends on orientation). Hence we have $II_p(v) = k(C_{p,v})\cos 0$ (or we could replace 0 by π, depends on orientation). Other curves along v have curvature no less than $k(C_{p,v})$.

Example B.4.3 Consider the surface of revolution obtained from $z = y^4$. By some computation, we could easily get $II_0(v) = 0$. So the differential $dN_0 = 0 \forall v$. Similarly, $II_p(v) = 0$ for all planes.

Example B.4.4 Another example here is sphere, and p is the north pole. At any direction, we could easily get $II_p(v) = 1$, so the differential of Gauss map is equivalent to I or $-I$, where I is the 2×2 identity matrix.

Example B.4.5 Consider the cylinder $x^2 + y^2 = 1$. At any point p, we could find two directions e_1 (horizontal) and e_2 (vertical), and that $II_p(e_1) = 1, II_p(e_2) = 0$. Hence we have $0 \leq II_p(v) \leq 1$ for any directions.

(III) Principle curvatures and principle directions

The fact that dN_p is self-adjoint operator implies that it has two orthonormal eigenvectors. Denotes the two eigenvectors by $e_1, e_2 \in T_p(S)$. WLOG, we could assume that $dN_p(e_1) = -k_1e_1, dN_p(e_2) = -k_2e_2, k_1 \geq k_2$. Thus any vector lying on the tangent plane can be written, uniquely, as $v = e_1\cos\theta + e_2\sin\theta$. Hence, the second fundamental form $II_p(v) = -dN_p \cdot v = k_1\cos^2\theta + k_2\sin^2\theta$. Obviously, we have the following inequality holds: $k_1 \geq II_p(v) \geq k_2$.

Definition B.4.5 k_1, k_2 are called the principle curvature of S at p, and e_1, e_2 are called principle directions.

Definition B.4.6 Let $p \in S, dN_p : T_p(s) \to T_p(S)$ be the differential of Gauss map, then we have the following curvatures: $K = \det(dN_p)$ is the Gauss curvature, $H = -\frac{\operatorname{tr}(dN_p)}{2}$ is the mean curvature.

(IV) Special points on the surface

Definition B.4.7 A point on the surface S is called

(1). Elliptic if $K > 0$

(2). Hyperbolic if $K < 0$

(3). Parabolic if only one eigenvalue of dN_p is 0.

(4). Planar if all eigenvalues are 0.

So these are local models, Gauss curvature is continuous on S. If a point has positive K, and another point with negative K, then there exists a point in between with $K = 0$.

Definition B.4.8 A point p with $k_1 = k_2$ is called umbilical point. (It means we cannot distinguish directions)

Theorem B.4.3 If all points of a connected surface S are umbilical, then S is either contained in a sphere or a plane.

Proof. Choose a parameterization $P(u, v)$ around p. Let $w \in T_p(S)$, then $w = a_1P_u + a_2P_v$. Since p is umbilical, we have $dN_p(w) = \lambda(p)w$ (since umbilical point has only one distinct eigenvalue, and it is a function depending on points on the surface).

By definition of dN_p, we have $dN_p(P_u) = N_u = \lambda(p)P_u$, similarly we have $N_v = \lambda(p)P_v$. So λ is smooth near p. We calculate N_{uv} in two different ways and hence that $\lambda_vP_u = \lambda_uP_v$. By our assumption, we have $\lambda_u = \lambda_v = 0$ (by linear independence). So we have $\lambda = c$ for some real c.

Case 1: $\lambda \equiv 0$. Then $N = N_0$ is a constant vector, this shows the surface is part of a plane.

Case 2: $\lambda \neq 0$. Differentiate $P(u, v) - \frac{N(u,v)}{\lambda}$ with respect to u, v respectively, we have that $P - \frac{N}{\lambda} = y_0$ is a constant. Hence by taking norm on both sides gives the sphere equation. □

(V) Special curves on the surface (1) Line of curvature

Definition B.4.9 Suppose S is a regular surface, if $C \subset S$ is a curve such that $\forall p \in C$, the tangent of C at p is a principle direction at p. Then C is the line of curvature of S.

Theorem B.4.4 C is line of curvature iff $N'(t) = \lambda(t)\alpha'(t)$. (Clear from definition)

Definition B.4.10 Suppose $p \in S$. An asymptotic direction of S at p is a direction of $T_p(S)$ for which the normal curvature is 0. $C \subset S$ is an asymptotic curve if at each point $p \in C$, the tangent direction is an asymptotic direction.

Example B.4.6 (i) At elliptic points, there are no asymptotic direction. (ii) At hyperbolic points, there are 2 asymptotic directions.

B.5 Gauss map and the second fundamental form in local coordinates

This part corresponds to Section 3.3 in Do Carmo's book.

(I) Gauss and Mean curvature

Suppose $\alpha : I \to S$ with $\alpha(0) = p, \alpha(t) = P(u(t), v(t))$. Hence, $\alpha'(t) = P_u u'(0) + P_v v'(0)$.

In this case, we have $II_p(\alpha'(0)) = -dN_p(\alpha'(0) \cdot \alpha'(0)$. By some simple algebra, we have

$$\alpha'(0) = -N_u \cdot P_u u'(0)^2 - (N_u \cdot P_v + N_v \cdot P_u)u'(0)v'(0) - N_v \cdot P_v v'(0)^2$$

But because $N \cdot P_v = 0 = N \cdot P_u$, we could have $-N_u \cdot P_u = N \cdot P_{uu}, -N_u \cdot P_v = -N_v \cdot P_u = N \cdot P_{uv}, -N_v \cdot P_v = N \cdot P_{vv}$. Denote the three expression by e, f, g respectively. Hence we derive the formula

$$II_p = edu^2 + 2fdudv + gdv^2$$

By some brute forces, we derive

$$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

By notice that $a_{21} = a_{12}$ (because of the self-adjoint property), the variables in the equation above are all smooth, local functions in the neighborhood around p. So this equation gives a way to explicitly write down the differential of Gauss map and hence we could explicitly deduce the eigenvalues.

Again, by brute forces, we get the following theorem

Theorem B.5.1 We can write our elements explicitly as

$$a_{11} = \frac{fF - eG}{EG - F^2}, a_{12} = \frac{gF - fG}{EG - F^2}, a_{21} = \frac{eF - fE}{EG - F^2}, a_{22} = \frac{fF - gE}{EG - F^2}$$

Then we have some useful alternative expression for mean and Gauss curvature:

$$K = \frac{eg - f^2}{EG - F^2}, H = \frac{eG + gE - 2fF}{2(EG - F^2)}$$

Also, the principle curvatures can be written as $H + \sqrt{H^2 - K}$ and $H - \sqrt{H^2 - K}$, when we are only known the Gauss and mean curvature.

Alternatively, solve the equation

$$\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - k_i I \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0, i = 1, 2$$

gives $xP_u + yP_v$ as the principle direction.

Example B.5.1 In case of the unit sphere, it is easy to check that $K = H = k_1 = k_2 = 1$, by repeating the procedure above.

Example B.5.2 (Graph of functions) Consider the map $h : \mathbb{R}^2 \to \mathbb{R}$, and the surface $z = h(x, y)$. Parameterize the surface as $P(u, v) = (u, v, h(u, v))$. Notice that $P_u = (1, 0, h_u), P_v = (0, 1, h_v)$. Hence the first fundamental form is given by $E = 1 + h_u^2, F = 0, G = 1 + h_v^2$, and the normal vector is given by $N = \frac{(-h_u, -h_v, 1)}{\sqrt{1+h_u^2+h_v^2}}$. By routine calculations, one can obtain the following coefficients for the second fundamental form as follows:

$$e = \frac{h_{uu}}{\sqrt{1 + h_u^2 + h_v^2}}, f = \frac{h_{uv}}{\sqrt{1 + h_u^2 + h_v^2}}, g = \frac{h_{vv}}{\sqrt{1 + h_u^2 + h_v^2}}$$

And hence the Gauss and mean curvature, by the explicit formulae above, are

$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{\left(1 + h_u^2 + h_v^2\right)^2}, H = \frac{\left(1 + h_u^2\right) h_{vv} - 2h_u h_v h_{uv} + \left(1 + h_v^2\right) h_{uu}}{2\left(1 + h_u^2 + h_v^2\right)^{\frac{3}{2}}}$$

respectively.

Example B.5.3 (Standard Torus) $P(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$ is a parameterization of the standard torus. By some calculations, we know that $K = \frac{\cos u}{r(a+r\cos u)}$. Here is a remark of this Gauss curvature. When $K = 0$, we have $u = \frac{\pi}{2}$ and $u = \frac{3\pi}{2}$, which corresponds to the top and bottom line of the torus, respectively.

(II) Application of the second fundamental form

Recall that $II_p(v) = k(C_{pv}), K = k_1 k_2$, where k_1, k_2 stands for principle curvatures.

Theorem B.5.2 Suppose $p \in S$ is elliptic point, then there exists a neighborhood $V_p \subset S$ such that $\forall x \in V_p$ belong to one side of $T_p(S)$. If p is a hyperbolic point, then $\forall x \in V_p$ belong to two sides of $T_p(S)$.

Proof. Suppose at p, the parameterization $P(u,v)$ has $P(0,0)=p$. Consider the distance function $d(u,v) := (P(u,v)-P(0,0))\cdot N(p)$. This function measures the signed distance from a point on the surface to the tangent plane $T_p(S)$. Then consider the Taylor expansion of $P(u,v)$,

$$P(u,v) = P(0,0) + P_u\cdot u + P_v\cdot v + \frac{1}{2}(P_{uu}u^2 + 2P_{uv}uv + P_{vv}v^2) + \overline{R}$$

where we assume $\overline{R} = o(u^2+v^2)$. Then plug in this expansion to the distance function, hence

$$d(u,v) = \frac{1}{2}(N\cdot P_{uu}u^2 + 2N\cdot P_{uv}uv + N\cdot P_{vv}v^2) + \overline{R}\cdot N$$

This is clear in the sense that N is orthogonal to $T_p(S)$, hence the P_u, P_v terms vanish.

Then in case $P(u,v)$ is sufficiently close to p, we have $d(u,v) = \frac{1}{2}II_p(P_u u + P_v v)$.

This leads to two cases:

Case 1: $K>0$, we know p is elliptic, so $II_p(P_u u + P_v v)$ has fixed sign, then all points lies on one side of the tangent plane.

Case 2: $K<0$, we know p is hyperbolic, so $II_p(P_u u + P_v v)$ can take either positive or negative values, depending on the relative position of the point, so in a small neighborhood, some of the points are on the one side of the tangent plane, while other points are on the other side of the tangent plane. This concludes the proof. □

The above theorem can be extended in the following way to give a shortcut for finding asymptotic curves in certain special cases.

Theorem B.5.3 (i)Suppose $p\in S$ is hyperbolic. Then the coordinate curves of $P(u,v)$ are asymptotic curves iff $e=g=0$.

(ii) Suppose p is not umbilical, then the coordinate curves of P are lines of curvature iff $F=f=0$.

To add on some points for this part, we consider some practical computation techniques.

To find asymptotic curves, we solve the equation $eu'^2 + 2fu'v' + gv'^2 = 0$. To find lines of curvature, we solve the following differential equation: $(fE-eF)u'^2 + (gE-eG)u'v' + (gF-fG)v'^2 = 0$.

(III) Another geometric interpretation of II_p

Theorem B.5.4 Let $p\in S$, $K(p)\neq 0$, consider $V_p\subset S$ such that $K(V)$ have the same sign. Then we could find a set B such that $p\in B\subset V$, such that $\lim_{A(B)\to 0}\frac{A(N(B))}{A(B)} = |K(p)|$ (Here $N(B)$ stands for the image of B in S^2 under the Gauss map). (Remark: This theorem is actually the original intuition of Gauss curvature)

Proof. We apply the formula of surface area directly in the following way:

$$\begin{aligned} A(B) &= \iint_R \sqrt{EG-F^2}dudv \\ A(N(B)) &= \iint_R |N_u\times N_v|dudv \\ &= \iint_R |dN_p(P_u)\times dN_p(P_v)|dudv \\ &= \iint_R |dN_p||P_u\times P_v|dudv \end{aligned}$$

Hence that

$$\lim_{A(B)\to 0}\frac{A(N(B))}{A(B)} = \lim_{A(B)\to 0}\frac{\frac{A(N(B))}{A(R)}}{\frac{A(B)}{A(R)}} = \frac{|K|\cdot|P_u\times P_v|}{|P_u\times P_v|} = |K|$$

□

B.6 Isometry and conformal maps

This part corresponds to Section 4.2 in Do Carmo's book.

Definition B.6.1 A diffeomorphism $\phi: S\to\tilde{S}$ is an isometry if $\forall p\in S, \forall w_1, w_2\in T_p(S)$, we have $(w_1\cdot w_2)_p = (d\phi_p(w_1)\cdot d\phi_p(w_2))_{\phi(p)}$. (In other words, isometry map preserves inner product structure.) So two surfaces are isometric if there exists an isometry map between them.

Remark: One can also define isometry by requiring that $\forall p\in S, \forall w\in T_p(S)$, we have $I_p(w) = I_{d\phi(p)(w)}$. This is clear from the definition of the first fundamental form. More generally, if a quadratic form satisfying $w_1\cdot w_2 = \frac{1}{2}(I_p(w_1+w_2) - I_p(w_1) - I_p(w_2))$, then we say it is a quadratic refinement.

The idea of local isometry can be defined by the following theorem:

Theorem B.6.1 Suppose $V\subset S, \tilde{V}\subset\tilde{S}, U\subset\mathbb{R}^2$,such that $P(U)=V, P(U)=\tilde{V}$. Then define $\phi = \tilde{P}\circ P^{-1}$ to establish a map between $V,\tilde{V}$ via U. Then ϕ is a local isometry iff $E=\tilde{E}, F=\tilde{F}, G=\tilde{G}$.

Proof. Suppose $\alpha(t) = P(u(t), v(t)), \alpha'(0) = w$, then compute the differential of ϕ at p gives

$$\begin{aligned} d\phi_p(w) =&(\phi \circ \alpha)|_{t=0} \\ =&\frac{d}{dt}(\tilde{P} \circ P^{-1}) \circ (P(u(t), v(t)))|_{t=0} \\ =&\frac{d}{dt}\tilde{P}(u(t), v(t))|_{t=0} \\ =&\tilde{P}_u u'(0) + \tilde{P}_v v'(0) \end{aligned}$$

Then this result holds by definition of isometry map and first fundamental form. □

The following parts provide some examples and non-examples of isometry.

Example B.6.1 Let $U = (0, \pi) \times (0, 2\pi) \subset \mathbb{R}^2$, consider $P(\theta, \phi) = (\theta, \phi, 0)$ as the zero extension map, and $\tilde{P}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ as the map to a unit sphere with the longitude removed. This is not an isometry because $G = 1, \tilde{G} = \sin^2\theta$.

Example B.6.2 Let $U = (0, 2\pi) \times \mathbb{R} \subset \mathbb{R}^2$. Consider $P(u, v) = (\cos u, \sin u, v), \tilde{P}(u, v) = (u, v, 0)$. By checking the first fundamental forms, the map is an isometry. (For reference, $E = G = 1, F = 0$.)

Example B.6.3 Let $U = (0, \infty) \times (0, 2\pi\sin\alpha)$, where α is a constant. Define $P(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, 0)$ and $\tilde{P}(\rho, \theta) = (\rho\sin\alpha\cos\frac{\theta}{\sin\alpha}, \rho\sin\alpha\sin\frac{\theta}{\sin\alpha}, \rho\cos\alpha)$. Then the map ϕ is an isometry. (For reference, $E = 1, F = 0, G = \rho^2$.)

Example B.6.4 Consider the catenoid $P(u, v) = (c\cosh v\cos u, c\cosh v\sin u, c)$, where $(u, v) \in U := (0, 2\pi) \times \mathbb{R}$, and the helicoid $Q(\tilde{u}, \tilde{v}) = (\tilde{v}\cos 2\tilde{u}, \tilde{v}\sin 2\tilde{u}, 2c\tilde{u})$, where $(\tilde{u}, \tilde{v}) \in \tilde{U} := (0, \pi) \times \mathbb{R}$, subject to relation $\tilde{v} = c\sinh v, \tilde{u} = \frac{1}{2}u$. This is an isometry, and $E = c^2\cosh^2 v, F = 0, G = c^2\sinh^2 v$.

To conclude the isometry part, we introduce the distance between two points on the surface.

Definition B.6.2 If $p, q \in S$, then $d(p, q) := \inf(L(\gamma))$, where L is the arc length function, and $\gamma : [0, 1] \to S$ is a piecewise smooth curve such that $\gamma \subset S, \gamma(0) = p, \gamma(1) = q$.

Theorem B.6.2 Suppose $\phi : S \to \tilde{S}$ is an isometry, then the distance in the above definition is preserved.

Proof. We apply the first fundamental form version of arc length integral directly:

$$L(\gamma) = \int_0^1 \sqrt{I_S(\gamma'(t))}dt$$

And that

$$\begin{aligned} L(\phi \circ \gamma) &= \int_0^1 \sqrt{I_{\tilde{S}}(d\phi(\gamma(t))}dt \\ &= \int_0^1 \sqrt{I_S(\gamma'(t))}dt \end{aligned}$$

as required. The second equality holds by the definition of differential. □

Remark: Length, angle, area and Gauss curvature are all preserved under isometry.

(II) Conformal map

Definition B.6.3 A diffeomorphism $\phi : S \to \tilde{S}$ is conformal map if $\forall p \in S, \forall w_1, w_2 \in T_p(S)$, we have $(d\phi_p(w_1) \cdot d\phi_p(w_2))_{\phi(p)} = \lambda^2(p)(w_1 \cdot w_2)_p$, where λ^2 is a nonzero differentiable function on S.

Consider $\alpha(t), \beta(t) \subset S$ satisfying $\alpha(0) = \beta(0)$, denote the angle of intersection by θ, then if we have a locally conformal map, then we have

$$\begin{aligned} \cos\tilde{\theta} =&\frac{d\phi_p(\alpha'(0) \cdot d\phi_p(\beta'(0)}{|d\phi_p(\alpha'(0)||d\phi_p(\beta'(0)|} \\ =&\frac{\lambda^2(\alpha'(0) \cdot \beta'(0)}{\lambda^2|\alpha'(0)||\beta'(0)|} \\ =&\cos\theta \end{aligned}$$

As a quick example, Example B.6.1 is not an isometry but is locally conformal.

Theorem B.6.3 Any two regular surfaces are locally conformal.

B.7 The Gauss theorem, Compatibility equation, and The first fundamental form of local theory of surfaces

This part corresponds to Section 4.3 in Do Carmo's book.

Under this section, we assume that $\alpha(s) \subset S$ is a curve on the surface parameterized by arc length. Thus, we have $\{t, n, b\}$ as the Frenet frame. Let $U \subset S$, $P(u, v)$ be a local parameterization of a surface, thus, $\{P_u, P_v, N\}$ is the Gauss frame.

This part aims to differentiate Gauss frame, similar to what we have done in differentiating the Frenet frame. We obtain the following system of equations by differentiating the Gauss frame. Here Γ_{ij}^k is called Christoffel symbol.

$$\begin{aligned}
P_{uu} &= \Gamma_{11}^1 P_u + \Gamma_{11}^2 P_v + L_1 N \\
P_{uv} &= \Gamma_{12}^1 P_u + \Gamma_{12}^2 P_v + L_2 N \\
P_{vu} &= \Gamma_{21}^1 P_u + \Gamma_{21}^2 P_v + \overline{L_2} N \\
P_{vv} &= \Gamma_{22}^1 P_u + \Gamma_{22}^2 P_v + L_3 N \\
N_u &= a_{11} P_u + a_{21} P_v \\
N_v &= a_{12} P_u + a_{22} P_v
\end{aligned}$$

The next step is to compute all the unknowns. Notice that a_{ij} have explicit formulas in Theorem B.5.1. Also, under the assumption that P is smooth, we have $\Gamma_{12}^1 = \Gamma_{21}^1, \Gamma_{12}^2 = \Gamma_{21}^2$. Then observe that if we take inner products with N on both sides, for the first four equations, we obtain that $L_1 = e, L_2 = \overline{L_2} = f, L_3 = g$. The only problem remaining is to compute all Christoffel symbols.

Then it suffices to consider, for example, equation 1,2,4 above. Taking inner product on both sides with P_u, P_v, respectively, we have

$$\Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2} E_u \tag{1}$$

$$\Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \frac{1}{2} E_v \tag{2}$$

$$\Gamma_{12}^1 E + \Gamma_{12}^2 F = \frac{1}{2} E_v \tag{3}$$

$$\Gamma_{12}^1 F + \Gamma_{12}^2 G = \frac{1}{2} G_u \tag{4}$$

$$\Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \frac{1}{2} G_u \tag{5}$$

$$\Gamma_{22}^1 F + \Gamma_{22}^2 G = \frac{1}{2} G_v \tag{6}$$

Note that N is orthogonal to P_u, P_v.

Now these symbols are easy to compute in the sense that equation 1,2; equation 3,4; and equation 5,6 are all simultaneous equations.

Remark All geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isomteries. The Christoffel symbols depend only on the first fundamental form.

The next concept is the compatibility conditions.

Theorem B.7.1 We have the following Gauss equations:

$$\begin{aligned}
KF &= (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 \\
-KE &= (\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{11}^1 \Gamma_{12}^2
\end{aligned}$$

Together with Mainardi-Codazzi equations:

$$\begin{aligned}
e_v - f_u &= e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2 \\
f_v - g_u &= e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2
\end{aligned}$$

Proof. We notice the following:

$$\begin{aligned}
(P_{uu})_v - (P_{uv})_u &= 0 \\
(P_{vv})_u - (P_{vu})_v &= 0 \\
N_{uv} - N_{vu} &= 0
\end{aligned}$$

Working under the Gauss frame, the system of equations above can be written in the following way:

$$\begin{aligned}
A_1 P_u + B_1 P_v + C_1 N &= 0 \\
A_2 P_u + B_2 P_v + C_2 N &= 0 \\
A_3 P_u + B_3 P_v + C_3 N &= 0
\end{aligned}$$

So we need all of the A_i, B_i, C_i to be 0. To obtain the Gauss equations, consider the first row, i.e. consider $A_1 = B_1 = C_1 = 0$. Equating P_u gives the first Gauss equation and equating P_v gives the second Gauss equation. The Mainardi-Codazzi equations are obtained by considering $C_1 = 0, C_2 = 0$, respectively. $\square$

Then these formulae lead to an important observation.

Theorem B.7.2 (*Gauss Theorema Egregium*) K is invarient by local isometries.

Alternatively, Gauss Theorema Egregium can be phrased as 'Gauss curvature is completely determined by the first fundamental form'. The phrase 'Theorema Egregium' is a Latin term meaning 'fantastic theorem'. This theorem is fantastic because originally Gauss curvature is defined by the first and second fundamental form, but this theorem shows that the second fundamental form is actually not necessary for finding K.

Similar to the fundamental theorem of Frenet frames, we have the Theorem of Bonnet for surfaces.

Theorem B.7.3 Given an open set $W \subset \mathbb{R}^2$, smooth functions E, F, G, e, f, g, such that $E > 0, G > 0, EG - F^2 > 0$, and they satisfy compatibility conditions. Then for any $q \in W, \exists U \subset W$ containing q and a diffeomorphism $P : U \to P(U)$ such that $P(U)$ is a regular surface with the given I and II. Further, if $\tilde{P}$ is another such surface, then $\tilde{P} = \rho \circ P + c, \rho \in SO(3), c \in \mathbb{R}^3$.

To conclude B.7, note the following remarks. In case the coordinate neighborhood has no umbilical points and $F = f = 0$, the Mainardi-Codazzi equation can be simplified as

$$e_v = \frac{E_v}{2}\left(\frac{e}{E} + \frac{g}{G}\right)$$
$$g_u = \frac{G_u}{2}\left(\frac{e}{E} + \frac{g}{G}\right)$$

In case when $F = 0$, the Gauss curvature can be written as

$$K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

Do note that these formulas need proof if used in exams.

B.8 A global property of surface

This part corresponds to Section 2.6 in Do Carmo's book.

(I) Orientation

We recall the followings from previous lectures

Definition B.8.1 A surface S is orientable if there exists a smooth unit vector field N on S.

Theorem B.8.1 S is orientable iff there exists a compatible cover of coordinate patches. Recall that compatible means the Jacobian for the intersection of two coordinate patches is positive.

Example B.8.1 S^2 is orientable.

Example B.8.2 If a surface S can be covered by two coordinate patches such that the intersection is connected, then S is orientable.

Example B.8.3 Graph of function f is orientable. i.e. $P(u, v) = (u, v, f(u, v))$ for some f.

Example B.8.4 $S = F^{-1}(a)$ for some function F, and a is not a critical value, also, we require that $\nabla F|_S \neq 0$. Then S is orientable.

Example B.8.5 Möbius band is not orientable.

Consider two coordinate patches, $P(u, v) = ((2 - v\sin\frac{u}{2})\sin u, (2 - v\sin\frac{u}{2})\cos u, v\cos\frac{u}{2})$, where $(u, v) \in (0, 2\pi) \times (-1, 1)$, and $\tilde{P}(\tilde{u}, \tilde{v}) = ((2 - \tilde{v}\sin(\frac{\tilde{u}+\pi}{2}))\sin(\frac{\tilde{u}+\pi}{2}), (2 - \tilde{v}\sin(\frac{\tilde{u}+\pi}{2}))\cos(\frac{\tilde{u}+\pi}{2}), \tilde{v}\cos(\frac{\tilde{u}+\pi}{2}))$, the domain is the same as the previous map. Consider two sets $W_1 = \{P(u, v) | \pi < u < 2\pi\}, W_2 = \{P(u, v) | 0 < u < \pi\}$. Then on W_1, we have $\tilde{u} = u - \pi, \tilde{v} = v$. And on $W_2, \tilde{u} = u + \pi, \tilde{v} = -v$. Then we can easily compute the Jacobian: $J_{12}|_{W_1} = 1, J_{12}|_{W_2} = -1$. Now we prove by contradiction. Suppose the Möbius band is orientable. Then there exists a unit normal vector field $N : M \to \mathbb{R}^3$ such that $N|_{\text{Patch 1}} = \frac{P_u \times P_v}{|P_u \times P_v|}, N|_{\text{Patch 2}} = \frac{P_{\tilde{u}} \times P_{\tilde{v}}}{|P_{\tilde{u}} \times P_{\tilde{v}}|}$ and one can switch $u, v, \tilde{u}, \tilde{v}$ such that the Jacobian on W_1, W_2 are both 1. But clearly this is impossible from the previous calculation. This completes the proof.

(II) Classification of closed surfaces

This is covered in topology. Basically the result is as follows: an orientable surface is homeomorphic to a sphere with g handles, and an unorientable surface is homeomorphic to a connected sum of a sphere with g handles and either a projective plane or a Klein bottle.

Part C Curves in Surfaces

C.1 Parallel Transport and Geodesics

This part corresponds to Section 4.4 in Do Carmo's book.

The intuition behind C.1 is to generalize the idea 'directional derivative' to an arbitrary manifold.

Definition C.1.1 A (tangent) vector field is an open set $V \subset S$ defining a correspondence w that assigns to each $p \in V$ a vector $w(p) \in T_p(S)$.

w is differentiable if for some $P : U \subset \mathbb{R}^2 \to V$, we could write $w = f(u,v)P_u + g(u,v)P_v$, where f, g are smooth. A quick example is that in $\mathbb{R}^2, w = f(x,y)e_1 + g(x,y)e_2$ is a vector field.

Definition C.1.2 Directional derivative. Suppose $p \in S, f : S \to \mathbb{R}, v \in T_p(S)$, then the directional derivative is defined as $D_v f|_p := \frac{d}{dt} f(\alpha(t))|_{t=0}$, where $\alpha : I \to S, \alpha(0) = p, \alpha'(0) = v$. (Notice that this coincides with the differential of f evaluated at v, $df(v)$. Also notice that the definition is independent of choice of α, directional derivative is uniquely defined up to equivalence class of curves.)

In $\mathbb{R}^2$, we have the directional derivative for a vector field w in the following sense: if $w = f(x,y)e_1 + g(x,y)e_2$, then $D_v(w) = D_v f e_1 + D_v g e_2$. To generalize directional derivative to surfaces, we replace e_1, e_2 in the expression above to P_u, P_v, for a parameterization P. Thus if we have $w = gP_u + hP_v$, $D_v(w) = (D_v g)P_u + g(D_v P_u) + (D_v h)P_v + h(D_v P_v)$.

We also define $D_y w|_p = \text{proj}_{T_p(S)} \frac{dw(\alpha(t))}{dt}|_{t=0}$, that is, we project the vector above to the tangent plane. In other words, we remove the elements on the normal direction to the surface.

The next part is devoted to compute $D_y w|_p$ locally. First we write $w(t) = a(u(t),v(t))P_u + b(u(t),v(t))P_v, \alpha(t) = P(u(t),v(t)), P(u(0),v(0)) = p, y = P_u u'(0) + P_v v'(0)$. Then we compute $\frac{dw}{dt}$.

$$\begin{aligned}
\frac{dw}{dt} &= a(P_{uu}u' + P_{uv}v') + a'P_u + b(P_{vu}u' + P_{vv}v') + b'P_v \\
&= a[(\Gamma^1_{11}P_u u' + \Gamma^2_{11}P_v u' + eNu') + (\Gamma^1_{12}P_u v' + \Gamma^2_{12}P_v v' + fNv')] + a'P_u \\
&\quad + b[(\Gamma^1_{21}P_u u' + \Gamma^2_{21}P_v u' + fNu') + (\Gamma^1_{22}P_u v' + \Gamma^2_{22}P_v v' + gNv')] + b'P_v
\end{aligned}$$

where the first equality holds by chain rule, and the second equality holds by applying the Christoffel symbols. Project $\frac{dw}{dt}$ to $T_p(S)$ gives

$$\begin{aligned}
\text{proj}_{T_p(S)} \frac{dw}{dt} =& (a'(0)P_u + b'(0)P_v) \\
&+ P_u(a(0)\Gamma^1_{11}u'(0) + a\Gamma^1_{12}v'(0) + b(0)\Gamma^1_{21}u'(0) + b(0)\Gamma^1_{22}v'(0)) \\
&+ P_v(a(0)\Gamma^2_{11}u'(0) + a(0)\Gamma^2_{12}v'(0) + b(0)\Gamma^2_{21}u'(0) + b(0)\Gamma^2_{22}v'(0))
\end{aligned}$$

Example C.1.1 We parameterize a plane P as $P(u,v) = (u,v,0)$. Write $w = ae_1 + be_2$, with $y = (m,n)$, then $D_y w(0,0) = (a_u(0,0)m + a_v(0,0_n)e_1 + (b_u(0,0)m + b_v(0,0)n)e_2$ because all Christoffel symbols are 0.

The next idea is a vector along a curve α in a surface S.

Definition C.1.3 Let $\alpha : I \to S$ be a parameterized curve in S. A vector field w along α is an assignment that assigns each $t \in I$ a vector $w(t) \in T_{\alpha(t)}S$. We say w is differentiable if for some parameterization $P, w(t) = aP_u + bP_v$, where a, b are both differentiable functions of t.

And similarly, the directional derivative is defined in this case. $D_{\alpha'(t)}w := \text{proj}_{T_{\alpha(t)}S} \frac{dw}{dt}$

Definition C.1.4 A vector field w along a parameterized curve $\alpha \subset S$ is parallel if $\frac{Dw}{dt} := D_{\alpha'(t)}w = 0, \forall t \in I$. (In this case, for a plane, the idea of parallel vector field is the same as our expectation, but for an arbitrary surface, a parallel vector field may not look parallel)

Example C.1.2 Consider S^2, let α be a longitude parameterized by arc length. Then $w = \alpha'(s)$ is a vector field along $\alpha(s)$. We have $\frac{Dw}{ds} = 0$ here because $\frac{dw}{ds} = \alpha''(s)$ which is parallel to $N(s)$, and hence when projecting into the tangent space, the resulting vector is zero vector.

Theorem C.1.1 Suppose w, v are parallel vector fields along α, then $w(t) \cdot v(t)$ is a constant.

Proof. By direct computation

$$\begin{aligned}
\frac{d}{dt}(w(t) \cdot v(t)) &= \left(\frac{dw}{dt} \cdot v(t)\right) + \left(w(t) \cdot \frac{dv}{dt}\right) \\
&= \left(\frac{Dw}{dt} \cdot v(t)\right) + \left(w(t) \cdot \frac{Dv}{dt}\right) \\
&= 0
\end{aligned}$$

where the second equality holds by noticing that v, w are linear combinations of P_u, P_v, and that the dot product of N component of $\frac{dw}{dt}, \frac{dv}{dt}$ with v, w is 0; and the last equality holds by definition of parallel vector fields. ☐

Theorem C.1.2 If w is a parallel vector field along $\alpha(t)$, and $\sigma(u) = t$, then $w(\sigma(u)) = 0$.

Proof. By chain rule,

$$\frac{Dw(\sigma(u))}{du} = \frac{Dw(t)}{dt}\frac{dt}{du} = 0$$

□

Theorem C.1.3 Let $w_0 \in T_{\alpha(t)}S$, then there exists uniquely a parallel vector field $w(t)$ along α such that $w(t_0) = w_0$.

Proof. (Sketch of proof) This is an initial problem of ODE. We need to solve $\frac{Dw}{dt} = 0$ provided the initial condition $w(t_0) = w_0$. The existence and uniqueness follows from ODE theory. □

Definition C.1.5 Under same assumption as Let w be the parallel vector field along α such that $w(t_0) = w_0$, then $w(t_1), t_1 \in I$ is called the parallel transport of w_0 along α at t_1. (To be concise, we start from a vector in the tangent plane $T_{\alpha(t_0)}S$, we aim to find a vector field such that at every point $t \in I$, we have the vector $w(t)$ in the tangent plane at t has the same direction as w_0 does).

So we could regard the parallel transport as a linear map from $T_p(S)$ to $T_q(S)$. It is linear because the ODE associated is linear. The map is isometry.

Definition C.1.6 A curve $\gamma : I \to S$ is geodesic if $\frac{D\gamma'(t)}{dt} = 0$.

So $|\gamma'(t)| = c$, thus the arc length is $s = ct$ for some $c \in \mathbb{R}^+$.

Definition C.1.7 A geometric regular curve C is said to be a geodesic if it is a parameterized geodesic with arc length parameter.

Example C.1.3 Consider plane $P(u, v) = (u, v, 0)$, $\alpha(t) = (u(t), v(t), 0)$ is a geodesic iff $u''(t)e_1+v''(t)e_2 = 0$, which happens iff $u'' = 0 = v''$, so $\alpha(t) = (a, b, 0)+t(c, d, 0)t$ is a necessary and sufficient condition for a geodesic in this case.

Example C.1.4 Geodesics of S^2 are great arcs.

The next step is to consdier geodesic equation in local coordinates. Let $\alpha(t) : I \to S$ be a parameterized geodesic, we need to have $\frac{D\alpha'(t)}{dt} = 0$. Let $P(u, v)$ be a local parameterization and $\alpha(t) = P(u(t), v(t))$, so $\alpha'(t) = P_u u'(t) + P_v v'(t)$. By applying the definition of directional derivatives and Christoffel symbols, we obtain that

$$\begin{aligned}\frac{D\alpha'(t)}{dt} =& P_u(u''(t) + \Gamma^1_{11}(u'(t))^2 + 2\Gamma^1_{12}u'(t)v'(t) + \Gamma^1_{22}(v'(t))^2)\\ =& P_v(v''(t) + \Gamma^2_{11}(u'(t))^2 + 2\Gamma^2_{12}u'(t)v'(t) + \Gamma^2_{22}(v'(t))^2)\end{aligned}$$

To satisfy the condition of a geodesic, one must have

$$\begin{aligned}u''(t) + \Gamma^1_{11}(u'(t))^2 + 2\Gamma^1_{12}u'(t)v'(t) + \Gamma^1_{22}(v'(t))^2 = 0\\ v''(t) + \Gamma^2_{11}(u'(t))^2 + 2\Gamma^2_{12}u'(t)v'(t) + \Gamma^2_{22}(v'(t))^2 = 0\end{aligned}$$

The system of ODE above is called differential equations of the geodesics of S.

For reference, a plane has first fundamental form $I = du^2 + dv^2$. All Christoffel symbols degenerate, so a straight line on the plane satisfy the ODE above in the obvious sense.

Back to Example C.3.3 above, we know that a straight line in the plane is a geodesic. Also, we know that there is an isometry between $P(u, v) = (u, v, 0), \tilde{P} = (\cos u, \sin u, v)$ where the Euclidean open set is $E = (0, 2\pi)$. Essentially, geodesics are preserved via isometry. (In other words, geodesics stand for the curve with shortest distance with respect to a given metric, so as long as the metric does not change, isometry maps a geodesic to a geodesic.) Notice that $\tilde{P}$ is a cylinder. So geodesics of a cylinder are of the form $(\cos as, \sin as, bs)$. Notice that it is possible to connect two points on a cylinder via infinite number of geodesics.

Back to Example B.6.3, we know that geodesics of P are straight lines, thus the geodesics of a cone are 'oblique ellipses'.

Example C.1.5 Consider surface of revolution $P(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$. By some routine computation, we obtain $E = f^2, F = 0, G = (f')^2 + (g')^2, \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0, \Gamma^2_{11} = \frac{-ff'}{f'^2+g'^2}, \Gamma^1_{12} = \frac{f'}{f}, \Gamma^2_{22} = \frac{ff'+gg'}{f'^2+g'^2}$. Then we obtain the ODE of geodesics as

$$\begin{aligned}u''(s) + \frac{2f'}{f}u'(s)v'(s) =& 0\\ v''(s) + \frac{-ff'}{f'^2 + g'^2}u'(s)^2 + \frac{ff' + gg'}{f'^2 + g'^2}v'(s)^2 =& 0\end{aligned}$$

Obviously $u = \text{constant}, v = v$ is a solution.

Theorem C.1.4 Given $S, p \in S, w \in T_p(S)$, then there exists $\epsilon > 0$ and unique parameterized geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p, \gamma'(0) = w$. (This can be proven using theory of ODE)

Remark: In particular, $D_{\gamma'(t)}\gamma'(t) = 0$. Suppose $P(p,q) = \{\gamma \in S, \gamma(0) = p, \gamma(1) = q\}$, then $L : P(p,q) \to \mathbb{R}$ given by $L(\gamma) = \int_0^1 |\gamma| dt$ determines the length. Geodesics give the smallest length.

The next part devotes to study Darboux Frame.

Definition C.1.8 $\{t, N \times t, N\}$ is called the Darboux frame of a curve in a surface. More conventionally, we write $B = N \times t$. So the Darboux frame is denoted by $\{t, B, N\}$.

Definition C.1.9 Geodesic curvature of a curve α: $k_g := (\frac{Dt(s)}{ds} \cdot B(s))$.

So by definition of a geodesic, we know that if α is a geodesic, then it has $k_g = 0$.

Remark: We have $\frac{D\alpha'(s)}{ds}$ is parallel to $B(s)$ (this can be seen, for example, $\frac{D\alpha'(s)}{ds}$ is orthogonal to $t(s)$ (by differentiating the identity $(t(s) \cdot t(s)) = 1$) and N(by definition)). So sometimes geodesic curvature is defined as the unique number k_g satisfying the equation $\frac{Dt(s)}{ds} = k_g B(s)$. Another remark is that when the surface is a plane, $k_s = k_g$, recall that k_s is the signed curvature of a curve.

Definition C.1.10 Geodesic torsion of a curve α: $\tau_g := -(\frac{dB}{ds} \cdot N)$.

Now we have Darboux theorem. Note that it is analogous to Frenet theorem

Theorem C.1.5 *Darboux Theorem*

$$\frac{d}{ds}\begin{bmatrix} t \\ B \\ N \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & -\tau_g \\ -k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} t \\ B \\ N \end{bmatrix}; \begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} t \\ B \\ N \end{bmatrix}$$

Recall that the angle θ is the angle between n, N.

Also recall that $k_n = k\cos\theta$. We have $k_g = (\alpha''(s) \cdot N \times t) = (kn \cdot B) = k(n \cdot B) = k \sin\theta$. Hence

$$k^2 = k_n^2 + k_g^2$$

For plane curve, $k_n = 0$, So $k_g = k_s$.

Theorem C.1.6 $\tau_g = \tau - \frac{d\theta}{ds}$, alternatively, $\tau = \tau_g + \frac{d}{ds} \arctan \frac{k_g}{k_n}$.

Proof. By Darboux Theorem, $b = -\cos\theta B + \sin\theta N$, so $b' = \sin\theta \frac{d\theta}{ds} B - \cos\theta \frac{dB}{ds} + \cos\theta \frac{d\theta}{ds} N + \sin\theta \frac{dN}{ds}$

By noticing that $n = \cos\theta N + \sin\theta B$ and that $\tau = b' \cdot n$, the formula holds at once. □

The rest of C.1 is about Liouville theorem

Definition C.1.11 Algebraic value of the covarient derivative of w at t is a function λ satisfying $\frac{Dw}{dt} = \lambda(N \times w)$.

Now suppose v, w are two vector fields along α. The purpose is to understand the angle function from v to w at each point. Define $\overline{v}(t) := N(t) \times v(t)$, and we define $\overline{w}(t)$ in the similar manner. So we could write $w(t) = a(t)v(t) + b(t)\overline{v}(t)$, where a, b are smooth functions of t, with $a^2 + b^2 = 1$. Because of this equation, we may assume that at a point t_0, $a(t_0) = \cos\phi_0, b(t_0) = \sin\phi_0$. But this is only the case at point t_0. The question now is how to generalize this ϕ to a smooth function ϕ on I such that $a = \cos\phi(t), b = \sin\phi(t), \phi(t_0) = \phi_0$.

Lemma C.1.1 There exists a solution ϕ that satisfy the given property. Explicitly, it can be written as $\phi = \phi_0 + \int_{t_0}^t ab' - ba' dt$.

Proof. (Sketch of proof) It suffices to prove that $(a - \cos\phi)^2 + (b - \sin\phi)^2 = 0$ for all $t \in I$. This is equivalent to prove that $a\cos\phi + b\sin\phi = 1$. Then prove LHS has derivative 0. □

Lemma C.1.2 Suppose v, w are two vector fields along α, $\phi(t)$ is one of the differentiable determinations of the angle from v to w given by $\frac{d\phi}{dt} = [\frac{Dw}{dt}] - [\frac{Dv}{dt}]$, where the bracket means the algebraic value of covarient derivative.

Proof. First notice that $w = \cos\phi v + \sin\phi\overline{v}, \overline{w} = \cos\phi\overline{v} - \sin\phi v$. Compute w' and notice that $(v' \cdot \overline{v}) = -(v \cdot \overline{v}')$, thus

$$w' = -\sin\phi\frac{d\phi}{ds}v + \cos\phi v' + \cos\phi\frac{d\phi}{ds}\overline{v} + \sin\phi\overline{v}'$$

$$w' \cdot \overline{w} = \frac{d\phi}{ds} + \cos^2\phi(v' \cdot \overline{v}) - \sin^2\overline{v}' \cdot v = \frac{d\phi}{ds} + (v' \cdot \overline{v})$$

Notice that $(w' \cdot \overline{w}) = (\frac{Dw}{dt} \cdot \overline{w}) = [\frac{Dw}{dt}]$, the result follows at once. □

Corollary C.1.1 If $\alpha(s)$ is a geodesic, v is a parallel vector field along α and $w = \alpha'$, then $\frac{d\phi}{ds} = k_g$.

Theorem C.1.7 Parameterize the surface S such that $F = 0$, choose a curve and a unit vector field. Let ϕ be the angle between P_u and w. Then

$$\frac{d\phi}{ds} = \frac{-1}{2\sqrt{EG}}\left(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right) + \left[\frac{Dw}{dt}\right]$$

Proof. Normalize P_u, P_v so that we could apply Lemma C.1.2, so write $e_1 = \frac{P_u}{\sqrt{E}}, e_2 = \frac{P_v}{\sqrt{G}}$. Notice that $e_1 \times e_2 = N$, then by lemma 2, take $v = e_1$, we have $\frac{d\phi}{dt} = [\frac{Dw}{dt}] - [\frac{De_1}{dt}]$. Now we calculate $\frac{De_1}{dt}$ in another way.

$$\begin{aligned}\left[\frac{De_1}{dt}\right] &= \frac{de_1}{dt} \cdot (N \times e_1) \text{ (From definition)} \\ &= \frac{de_1}{dt} \cdot e_2 \\ &= ((e_1)_u \cdot e_2)\frac{du}{dt} + ((e_1)_v \cdot e_2)\frac{dv}{dt} \text{ (Chain rule)} \\ &= \left(\left(\frac{P_u}{\sqrt{E}}\right)_u \cdot \frac{P_v}{\sqrt{G}}\right)\frac{du}{dt} + \left(\left(\frac{P_u}{\sqrt{E}}\right)_v \cdot \frac{P_v}{\sqrt{G}}\right)\frac{dv}{dt}\end{aligned}$$

The rest of the proof is done by noticing that $(P_{uu} \cdot P_v) = \frac{-1}{2}E_v$, for example, then apply similar reasoning to other terms, the theorem is clear. ☐

Theorem C.1.8 (*Liouville Theorem*) Under the assumptions above, assume $F = 0$, then $k_g = k_{g1}\cos\phi + k_{g2}\sin\phi + \frac{d\phi}{ds}$, where k_{g1}, k_{g2} are geodesic curvature of coordinate curves.

Proof. By the previous theorem when $w = \alpha'(s)$, and notice that $[\frac{Dw}{dt}] = k_g$, we have

$$k_g = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{ds} - E_v\frac{du}{ds}\right) + \frac{d\phi}{ds}$$

Also notice the followings:

$$k_{g1} = \frac{-E_v}{2E\sqrt{G}}, k_{g2} = \frac{G_u}{2G\sqrt{E}}, \sqrt{E}\frac{du}{ds} = \alpha'(s) \cdot \frac{P_u}{\sqrt{E}} = \cos\phi$$

Then everything is clear. ☐

The rational for these theorems is to help prove Gauss-Bonnet Theorem in the next section.

C.2 The Gauss-Bonnet Theorem and Its Applications

This part corresponds to Section 4.5 in Do Carmo's book.

Part I: Local Gauss-Bonnet Theorem

Definition C.2.1 A curve $\alpha : [0, l] \to S$ is a simple closed piecewise regular curve if (i) $\alpha(0) = \alpha(l)$, (ii) $t_1 \neq t_2 \implies \alpha(t_1) \neq \alpha(t_2)$, (iii) there exists subdivision $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$ of the interval such that α is smooth and regular in each of $[t_i, t_{i+1}], 0 \leq i \leq k$. (The last condition is the same as saying α is a curved polygon.) We call $\alpha(t_i)$ as vertices.

Definition C.2.2 A region $R \subset S$ is called a simple region if R is homeomorphic to a disk and the boundary of R is a trace of a simple closed piecewise regular parameterized curve $\alpha : I \to S$.

We also have the following assumption: α is positively oriented. i.e. If we walk on the curve in the positive direction of α with head pointing to N. The region is our left.

Let $P : U \to S$ be a coordinate patch of S. We need $P_u \times P_v = \lambda N$ with $\lambda > 0$. (Positive orientation) Let $R \subset P(U)$ be a bounded region of S (possibly with holes). Also, we define $\iint_R f d\sigma := \iint_{P^{-1}(R)} f\sqrt{EG - F^2}dudv$. Do notice that $\sqrt{EG - F^2}dudv$ is known as area form. It is a differential 2-form.

Theorem C.2.1 (*Local Gauss-Bonnet Theorem*)Suppose R is a simple region, under the previous assumptions,

$$\sum_{i=0}^{k}\theta_i = 2\pi - \sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} k_g(s)ds - \iint_R K d\sigma$$

where each θ_i are external angles of each vertex.

Corollary C.2.1 If R is a geodesic polygon in $P(U)$, then

$$\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} k_g(s)ds = 0$$

In particular, if R is a geodesic triangle, and A, B, C are all internal angles, then we have $A + B + C = \pi + \iint_R K d\sigma$.

Example C.2.1 Spherical triangles. If A is the north pole of the unit sphere, AB, AC are half arc of two great circles with the angle between them to be α. Then by noticing that $K = 1$ here, the sum of angle is thus $\pi + [ABC] = \pi + 2\pi \times \frac{\alpha}{2\pi} = \pi + \alpha$, where $[ABC]$ stands for the area of spherical triangle ABC.

Remark: If $K = 0$, then the sum of internal angles s π. This is equivalent to parallel postulate. $K > 0, K < 0$ are not Euclidean geometry.

Part II: Global Gauss-Bonnet Theorem

This part assumes the knowledge of triangulation and Euler characteristic.

Theorem C.2.2 (*Global Gauss-Bonnet Theorem*) Suppose $R \subset S$ is a regular region of an orientable surface S. Let $C_1, \cdots, C_n$ be the closed, simple, piecewise regular curve which form the boundary of R. Suppose each C_i is positively oriented. Let $\theta_1, \cdots, \theta_p$ be the set of all external angles of $C_1, \cdots, C_n$. Then

$$\sum_{i=1}^{p} \theta_i = 2\pi\chi(R) - \sum_i \int_{C_i} k_g - \iint_R K d\sigma$$

Then we present two corollaries **Corollary C.2.2** If R is simple, then $\chi(R) = 1$, we have local Gauss-Bonnet.

Corollary C.2.3 If there is no boundary component, then

$$2 - 2g = \chi(S) = \frac{1}{2\pi} \iint_S K d\sigma$$

This is the end of Part II.

Part III: Applications

Theorem C.2.3 A closed surface of positive curvature must be homeomorphic to S^2.

Proof. (Outline) Prove by contradiction, check signs on both side of Corollary C.2.3. □

Theorem C.2.4 Let S be with $K \leq 0$. Then two geodesics starting from a point $p \in S$ cannot meet again at q in such a way that they consists of the boundary of a simple region $R \subset S$.

Theorem C.2.5 Suppose S is homeomorphic to a cylinder and $K < 0$. Then S has at most 1 simple closed geodesic.

Proof. (Outline) By contradiction, if C_1, C_2 are such geodesics, then apply Gauss-Bonnet for the part of S enclosed by the two geodesics. □

Theorem C.2.6 If S is closed, $K > 0$, suppose γ_1, γ_2 are two simple geodesics, then they must intersect.

Theorem C.2.7 Let $\alpha : I \to \mathbb{R}^3$ be a closed regular curve, with $k \neq 0$. Assume $n(I)$ (translate the unit normal vector to the origin) is simple on S^2, it divide the unti sphere to 2 parts with equal area.

Proof. (Outline) By Gauss-Bonnet theorem, we have

$$0 = 2\pi - \iint_R K d\sigma - \int_{\partial R} k_g ds$$

Then notice that the double integral represents area, and the line integral is 0. □

The final part of C.2 is to prove Gauss-Bonnet theorem

Part IV: Proof of Gauss-Bonnet Theorem

We first begin proving the local Gauss-Bonnet theorem.

Proof.

$$\begin{aligned}
\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} k_g ds &= \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \left(\frac{G_u}{2\sqrt{EG}} \frac{dv_i}{ds} - \frac{E_v}{2\sqrt{EG}} \frac{du_i}{ds} \right) ds + \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \frac{d\phi_i}{ds} ds \\
&= \iint_{P^{-1}(R)} \left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u dudv + \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} d\phi_i \\
&= - \iint_{P^{-1}(R)} K\sqrt{EG} dudv + \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} d\phi_i \\
&= - \iint_R K d\sigma + \left(2\pi - \sum_{i=0}^{k} \theta_i \right)
\end{aligned}$$

Notice that the second equality holds because of Green theorem as the sum of line integrals forms a loop. The last equality holds by theorem of turning tangents. The intuition behind it is that on edges we gradually turn the tangents but on vertices we suddenly turn tangents. □

Here an outline of proof of the Global Gauss-Bonnet theorem is provided. We first triangulate a region such that each triangle has a parameterization with $F = 0$, apply local Gauss-Bonnet theorem for each of the triangles, counting the contributions of v, e, f we get the Euler characteristics.

That is the end of this module.