NANYANG TECHNOLOGICAL UNIVERSITY
SINGAPORE

SPMS21081 – Cluster Algebras and Skein theory

Presented by Yu Xiaoming

Supervised by Assoc Prof Andrew James Kricker, Co-supervisor Mr. Wang Zhihao

1. Basic Concepts

(1). Skein algebra $\mathrm{Sk}_q(\Sigma)$ of a surface Σ with marked points is defined as

$$\mathrm{Sk}_q(\Sigma) \coloneqq \mathbb{Z}_q^{\text{links}}/I$$

Here $\mathbb{Z}_q = \mathbb{Z}\left[q^{\pm\frac{1}{2}}\right]$, q is indeterminant, $\mathbb{Z}_q^{\text{links}}$ is the free $\mathbb{Z}_q$ module with equivalent class of links in Σ as basis, and I stands for the following four relations:

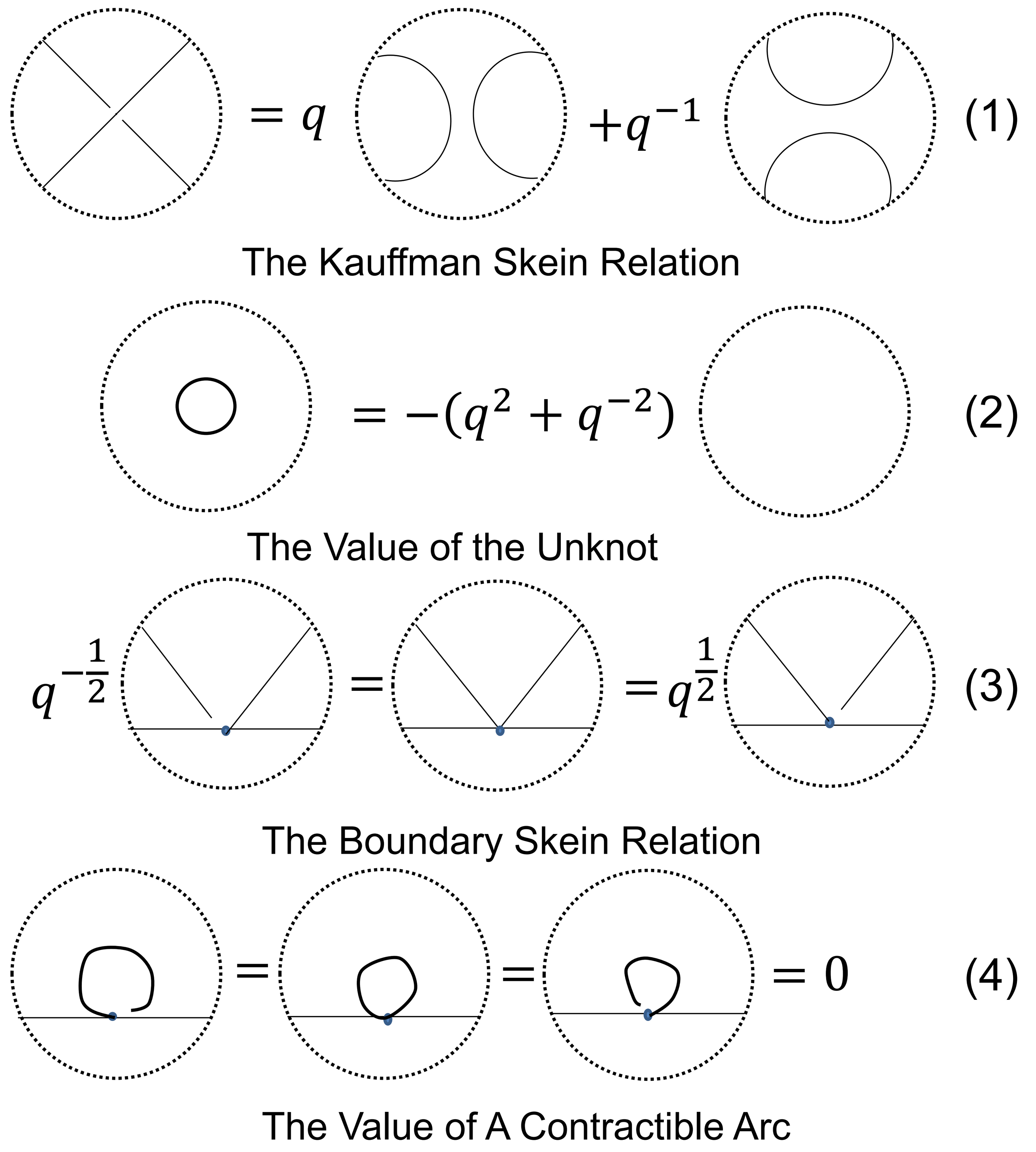

Figure 1. The Relation Set I

(2). Quantum cluster algebra consists of a quantum seed and mutation rule. It is the quantized version of cluster algebra ($q = 1$).

A quantum seed over a skew-field $\mathcal{F}$ is a triple (B, Λ, M), with B is extended skew-symmeterizable $N{\times}\mathrm{ex}$ matrix ($\mathrm{ex} \subset \{1, \dots, N\}$), $\Lambda B = D\iota$ (D is a $N{\times}N$ diagonal matrix with $D_{ii} > 0$), and M satisfies

$$M\colon \mathbb{Z}^N \to \mathcal{F}\backslash\{0\} \qquad M(\alpha)M(\beta) = q^{\frac{1}{2}\Lambda(\alpha,\beta)}M(\alpha+\beta)$$

If we mutate the seed (B, Λ, M) at $i \in \mathrm{ex}$ to (B', Λ', M') that are both in $\mathcal{F}$, there are 3 conditions for B', M' are as follows

$$B'_{jk} = \begin{cases} -B_{jk} & i = j \text{ or } i = k \\ B_{jk} + \frac{1}{2}\left(|B_{ji}|B_{ik} + B_{ji}|B_{ik}|\right) & \text{otherwise} \end{cases}$$

$$M'(e_i) = M\left(-e_i + \textstyle\sum_{B_{ji}>0} B_{ji}e_j\right) + M\left(-e_i - \textstyle\sum_{B_{ji}<0} B_{ji}e_j\right)$$

$$\alpha \in \mathbb{Z}^N, \alpha_i = 0 \Longrightarrow M(\alpha) = M'(\alpha)$$

2. Quantum Trace Map

(1). Skein coordinate map with a given triangulation:

$$\varphi_\Delta\colon \mathrm{Sk}_q(\Sigma) \hookrightarrow \mathrm{Sk}_q(\Sigma)[\Delta^{-1}] \cong \mathfrak{X}(\Delta)$$

Here $\mathrm{Sk}_q(\Sigma)[\Delta^{-1}]$ is the localization of skein algebra with monomials in the triangulation as fractions, $\mathfrak{X}(\Delta)$ is called Muller Algebra, a quantum torus that satisfies $\mathbb{T}(P, q, X)$, where P is the vertex matrix. In the classical case, the image are called shear coordinates.

(2). Shear-to-skein map with a given triangulation:

$$\psi\colon \mathcal{Y}(\Delta) \to \mathfrak{X}(\Delta)$$

Here $\mathcal{Y}(\Delta) \coloneqq \mathbb{T}(Q^\circ, q^{-1}, Y)$ is called Chekhov-Fock Algebra, Q is the face matrix, Q° is the submatrix with inner edges. In the classical case, the image are the Penner coordinates.

(3). Quantum trace map with a given triangulation:

$$\kappa\colon \mathrm{Sk}_q^\circ(\Sigma) \to \mathcal{Y}^{\mathrm{bl}}$$

Here $\mathcal{Y}^{\mathrm{bl}}$ is a submodule of $\mathcal{Y}(\Delta)$ spanned by y^k, with $k \in \mathbb{Z}^{\Delta^\circ}$ such that $k(\tau) \coloneqq k(a) + k(b) + k(c)$ is even for all triangles in the triangulation, $\mathrm{Sk}_q^\circ(\Sigma)$ is the skein algebra of a surface with no punctures. Explicitly, $\kappa = \psi_\Delta^{-1} \circ \varphi_\Delta$.

3. The Case $q = -1$ and Representation Theory

Assumption: $\partial\Sigma = \emptyset$. Then the diagram below commutes

$$\begin{tikzcd} \mathrm{Sk}_{-1}(\Sigma) \arrow[r, "T^\omega"] \arrow[d, "\cong"'] & Z(\mathrm{Sk}_\omega(\Sigma)) \arrow[r, "\alpha_\rho"] & \mathbb{C} \\ \mathfrak{X}_\Sigma(\mathrm{SL}_2(\mathbb{C})) \arrow[urr] & & \end{tikzcd}$$

Here ω^2 is a root of unity such that $\omega^{2N} = 1$ for some odd integer N. $T^\omega[k] = T_N[k]$ is called Chebyshev homomorphism, T_N is the N^{th} Chebyshev polynomial.

For irreducible representation $\rho\colon \mathrm{Sk}_\omega(\Sigma) \to \mathrm{End}(V)$, by Schur's lemma, $\rho(x) = \alpha_\rho(x) \cdot \mathrm{Id}_V$.

$\mathfrak{X}_\Sigma(\mathrm{SL}_2(\mathbb{C}))$ is the $\mathrm{SL}_2(\mathbb{C})$ character variety of Σ.

4. References:

1. G. Muller, Skein algebras and cluster algebra of marked surfaces, Preprint arXiv:1204.0020
2. T.T.Q.Le, Quantum Teichmuller Spaces and Quantum Trace Map, Preprint arXiv: 1511.06054
3. C. Frohman, J.K.Bartoszynska, T.T.Q.Le, Unicity for Representations of the Kauffman Bracket Skein Relation, Preprint arXiv: 1707.09234