NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Supervisor: Assoc Prof. Andrew James Kricker
Mentor: Mr. Wang Zhihao

Skein Algebra and Quantum Invarients of Surface Diffeomorphisms

Yu Xiaoming (U2040367D)

Part 1: Basic Concepts, Assumptions and Notations

1. Throughout this poster, we assume S is a compact, orientable surface with punctures and $\varphi: S \to S$ is a psudo-Anosov map.

2. We assume n is a positive odd integer and q is a primitative n^{th} root of unity.

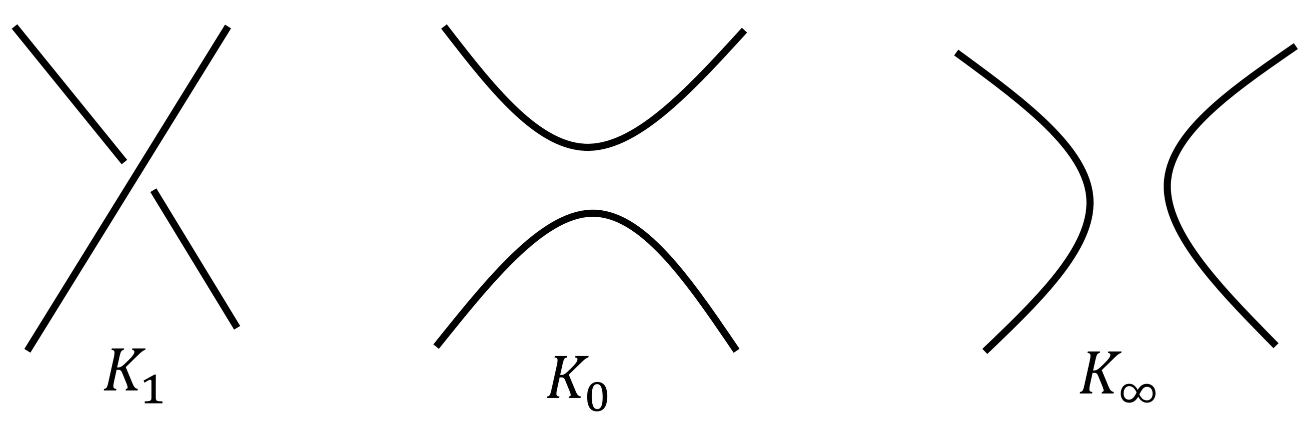
3. $SL_2(\mathbb{C})$-character variety of S is

$$\mathcal{X}_{SL_2(\mathbb{C})}(S) \coloneqq \mathrm{Hom}\big(\pi_1(S), SL_2(\mathbb{C})\big)//SL_2(\mathbb{C})$$

Here '//' means the element is considered up to conjugation action by $SL_2(\mathbb{C})$ elements. $\mathcal{X}_{PSL_2(\mathbb{C})}(S)$ is defined similarly.

4. Kauffman bracket skein algebra $\mathcal{K}^q(S)$ is the algebra concerning framed links in $S\times[0,1]$, modulo the relation

$$K_1 = q^{\frac{1}{2}}K_0 + q^{-\frac{1}{2}}K_\infty$$



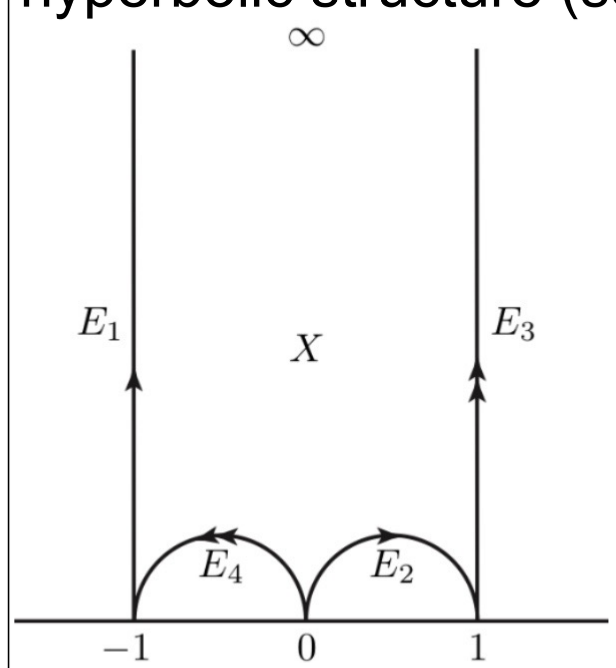
5. Mapping torus $M_{\varphi,r}$ of S is defined as identifying $(x,1)$ with $(\varphi(x),0)$ of $S\times[0,1]$. It is a 3-manifold.

6. φ acts on $\mathcal{K}^q(S)$ by $\varphi_*: \mathcal{K}^q(S) \to \mathcal{K}^q(S)$ in the following way: $\varphi_*([K]) = \big[(\varphi\times\mathrm{Id}_{[0,1]})(K)\big]$. A φ-invarient character is a character r satisfying $[r] = [r \circ \varphi_*]$

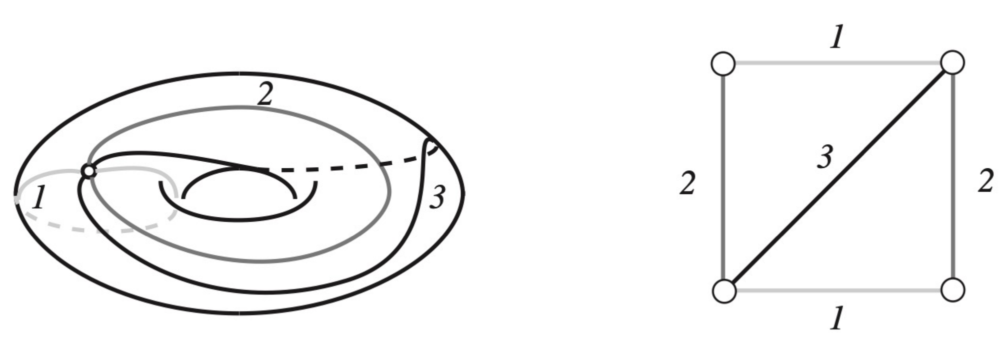
7. Ideal triangulation sweep means either a re-indexing of edges or a diagonal flip of a given triangulation.

8. Chekhov-Fock algebra is $\mathcal{T}_\tau^q(S) = \mathbb{C}[X_1^{\pm1}, \dots, X_e^{\pm1}]$, e is number of edges, satisfying $X_iX_j = q^{\sigma_{ij}}X_jX_i$, (σ_{ij}) is the face matrix.

9. A once-punctured torus $S_{1,1}$ is a torus with one point being removed. It admits a triangulation with 3 edges and a complete hyperbolic structure (so it tesselate $\mathbb{H}^2$), as shown below.



Once-Punctured torus in $\mathbb{H}^2$



Ideal Triangulation of once-punctured torus

Part 2: Main Conjecture

We assume that $[r] \in \mathcal{X}_{SL_2(\mathbb{C})}(S)$ is φ-invarient and $\Lambda_{\varphi,r}^q: V \to V$ is an intertwiner. We choose φ-invarient puncture weights $\theta_v \in \mathbb{C}$ such that $Tr(\alpha_v) = -e^{\theta_v} - e^{-\theta_v}, \forall \alpha_v \in \pi_1(S)$ around v, the puncture weight is $p_v = e^{\frac{\theta_v}{n}} + e^{-\frac{\theta_v}{n}}$. Then the conjecture is

$$\lim_{n\to\infty} \frac{\ln\left|\mathrm{Tr}\left(\Lambda_{\varphi,r}^q\right)\right|}{n} = \frac{\mathrm{vol}_{\mathrm{hyp}}\left(M_{\varphi,r}\right)}{4\pi}$$

The literatures proves the case of once-punctured torus.

Part 3: Representation Theory of $\mathcal{K}^q(S)$

1. Suppose v is a puncture. We define a puncture weight $p_v \in \mathbb{C}$ for each puncture such that $p_v = p_{\varphi(v)}$, then we say they are φ-invarient.

2. Suppose $\rho: \mathcal{K}^q(S) \to \mathrm{End}(V)$ is a representation of $\mathcal{K}^q(S)$. By uniqueness property, we know that $\rho \circ \varphi_*$ and ρ are unnique up to isomorphism. That is, $(\rho \circ \varphi_*)(X) = \Lambda_{\varphi,r}^q \circ \rho(S) \circ \left(\Lambda_{\varphi,r}^q\right)^{-1}, \forall X \in \mathcal{K}^q(S)$. Then $\Lambda_{\varphi,r}^q$ is the intertwiner in Part 2. WLOG, assume $\left|\det \Lambda_{\varphi,r}^q\right| = 1$.

Part 4: Chekhov-Fock Intertwiner $\overline{\Lambda_{\varphi,r}^q}$

Chekhov-Fock intertwiner is defined similarly as that of intertwiner, but this time we consider representation of Chekhov-Fock algebras, instead of skein algebra.

By results in [1], an intertwiner is isomorphic to a Chekhov-Fock intertwiner and thus has the same trace.

Part 5: Chekhov-Fock Intertwiner of $S_{1,1}$

Suppose $(a_0, b_0, c_0), \dots, (a_{k_0}, b_{k_0}, c_{k_0}) = (a_0, b_0, c_0)$ is an edge weight system (shear-band parameters corresponds to triangulations of sequence of flips). Suppose also that $e^{\theta_v} = a_0b_0c_0$, then we have

$$Tr\left(\Lambda_{\varphi,r}^q\right) = \frac{\sum_{i_1,i_2,\dots,i_{k_0}=1}^{n} \prod_{k=1}^{k_0} QDL^q(u_k, v_k|2i_k)q^B}{n^{\frac{k_0}{2}} \prod_{k=1}^{k_0} |D^q(u_k)|^{\frac{1}{n}}}$$

$$B = \left(\sum_{k=1}^{k_0} i_k^2(\epsilon_k + \epsilon_{k+1} + 2) - 4\epsilon_{k+1}i_ki_{k+1}\right) + \epsilon_1\widehat{l_0}i_1 + \frac{-\epsilon_1\widehat{l_0} - \widehat{m_0} + \widehat{n_0}}{2}i_{k_0}$$

Here ϵ_k indicates whether the k^{th} elementary intertwiner is Left or Right, where $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Specifically,

$$\epsilon_k = \begin{cases} 1, \varphi_k = L \\ -1, \varphi_k = R \end{cases}$$

QDL is Discrete Quantum Dilogrithm function, each u_k, v_k are chosen such that we can apply results of Part 4 and that $v_k^n = 1 + u_k^n, \forall k$.

Part 6: Hyperbolic volume of $M_{\varphi,r}$ when $\varphi = LR$

In this case, $M_{\varphi,r}$ is diffeomorphic to Figure-8 knot complement, and is therefore isometric to $\mathbb{H}^3/\widehat{\Gamma}_8$, where $\widehat{\Gamma}_8 = \langle\varphi_1, \varphi_3, \tau\rangle$, $\varphi_1 = \frac{z+1}{z+\omega^{-1}}, \varphi_3 = \frac{z-1}{-z+\omega^{-1}}, \tau = z + \omega, \omega = e^{\frac{\pi i}{3}}$. Thus

$$\mathrm{v}_{\mathrm{hyp}}\left(M_{\varphi,r}\right) = 6\int_0^{\frac{1}{2}}\int_0^{\sqrt{3}x}\int_{\sqrt{1-x^2-y^2}}^{\infty} \frac{dx\,dy\,dz}{z^3} = 6\Lambda\left(\frac{\pi}{3}\right)$$

where $\Lambda(x)$ is Lobachevsky function. The proof of the conjecture requires asymptotic analysis of results in Part 5.

References:
1. Asymptotics of Quantum Invariants of Surface Diffeomorphisms 1: Conjecture And Algebraic Computations; Francis Bonahon, Helen Wong, Tian Yang; arxiv: 2112.12852
2. Asymptotics of Quantum Invariants of Surface Diffeomorphisms 2: The Figure-Eight Knot Complement; F. Bonahon, H. Wong, T. Yang; arxiv: 2203.05730
3. Quantum Hyperbolic Invariants for Diffeomorphisms of Small Surfaces; Liu Xiaobo; arxiv: 0603467
4. Low-Dimensional Geometry: From Euclidean Surfaces to Hyperbolic Knots; Francis Bonahon; American Mathematical Society; 2009.