University of Missouri
Department of Chemical and Biomedical Engineering

BIOL_EN/BME 8970

Nuclear Magnetic Resonance and Magnetic Resonance Imaging

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All Exercise Sets, Summer 2024
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Github: `https://github.com/thomenr/NMR8000`

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Problem Set #1: Foundational Math

Problem 1. Given the following vectors

$$\vec{v}_1 = \cos\theta\cos t\hat{x} + \sin t\hat{y} + \sin\theta\cos t\hat{z}$$
$$\vec{v}_2 = \cos\theta\sin t\hat{x} - \cos t\hat{y} + \sin\theta\sin t\hat{z}$$

evaluate $\vec{v}_1 \cdot \vec{v}_2$ and $\vec{v}_1 \times \vec{v}_2$.

Be sure to simplify all expressions as much as possible[1].

Problem 3 (25 points). Evaluate the following and express them in both additive ($a + ib$) and exponential ($re^{i\theta}$) forms. Plot each result in the Argand plane.

(a) $e^{i\pi/2} \cdot (1 + i)$

(b) $e^{i\pi/2} + (1 + i)$

(c) $(-1 - i) \cdot (-1 + i) \cdot e^{1+2i}$

(d) $(e^{i\pi} + i) \cdot e^{-i3\pi/4} - 1$

(e) $i^i \cdot \sqrt[i]{i}$

Problem 4 (10 points). The following matrix $\hat{T}$ describes an active transformation[2]:

$$\hat{T} = \begin{bmatrix} 3i & 1-2i \\ e^{-i\pi} & \frac{2}{\sqrt{2}}e^{i3\pi/4} \end{bmatrix}$$

Calculate the transformation of the following vector $\vec{v} = 2e^{i2\pi}\hat{x} + (1 + 3i)\hat{y}$.

Problem 5. Plot the following vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Then, replot your vectors following this transformation

$$\hat{T} = \begin{bmatrix} 0.143 & -1.4 \\ -.3 & 1.1 \end{bmatrix}$$

BONUS: Animate this tranformation (Check the lecture notes for some coding hints!)

[1] Can you visualize these vectors or their products? Hint: what do they look like if $\theta = 0$?
[2] Can you visualize this transformation?

Problem Set #2: Fourier Theory

Problem 1. Find the Fourier Transforms of the following 1D functions of t:

(a) $f(t) = \begin{cases} e^{i\omega_0 t} \cdot e^{-t/T_2} & t \geq 0 \\ 0 & t < 0 \end{cases}$

(b) $f(t) = \delta(t - i\pi/4)$

(c) $f(t) = \begin{cases} \cos(t\pi/a), & -a \leq t \leq a \\ 0, & \text{else} \end{cases}$

Problem 2. Find the inverse Fourier Transforms of the following 1D functions of ω:

(a) $F(\omega) = \delta(\omega - \omega_0)$

(b) $F(\omega) = \frac{1}{1+i\omega}$

Problem 3. Let's code some Fourier Transforms using your favorite programming language! First, create a 512-element vector in time starting at $t = 0$ and with a dwell time of $t_d = 1.95 \times 10^{-5}$ seconds (what would the bandwidth be?). Next create a signal which follows:

$$S(t) = \sum_k A_k e^{i\phi_k} e^{i\omega_k t} e^{-t/T_{2k}}, \quad \text{where} A_k = \begin{bmatrix} 1 \\ 0.4 \\ 0.05 \end{bmatrix}, \phi_k = \begin{bmatrix} 0 \\ -\pi/2 \\ \pi/4 \end{bmatrix}, \omega_k = 2\pi \begin{bmatrix} 0 \\ 700 \\ 7000 \end{bmatrix}, T_2 = \begin{bmatrix} 0.001 \\ 0.002 \\ 0.5 \end{bmatrix}$$

Plot the real part of this signal.
Fourier transform your signal and plot both the real and imaginary parts of the transform on the same plot. Remember to create an accurate labeling for the ω-axis.
Since there were 3 frequencies added together, you should see 3 peaks. The peak at 7000 Hz is unwanted in our signal - remove it in frequency space, then plot the new time domain signal.

Problem 4. Clone the homework github if you haven't already: `https://github.com/thomenr/NMR8000` In the HW2 folder is a 2-column csv file. The first column is a vector of time in seconds, the second is acquired signal (like an oscilloscope reading). Plot this data. Plot the Fourier transform of this data (modulus only). What frequencies do you see and what are their relative amplitudes?

Problem Set #3: Precession and Excitation

Problem 1a (5 points). A particular spin-$\frac{1}{2}$ nucleus has magnetic moment $\mu_a = 1.41 \times 10^{-26}$ [J/T]. What is the gyromagnetic ratio of this nucleus in units of rad s^{-1} T^{-1}?

Problem 1b (5 points). What is the NMR Larmor frequency of μ_a nuclei in a $B_0 = 1.5$T magnetic field?

Problem 1c (5 points). What flip angle will these μ_a nuclei experience if we deliver an on-resonance excitation box pulse of $B_1 = 50$ μT for 100 μs.

Problem 1d (10 points). A second nucleus has magnetic moment $\mu_b = 1.411 \times 10^{-26}$ [J/T]. What is the angle B_{eff} makes with respect to the B_0 axis for these nuclei given the excitation pulse described in part (c)?

Problem 1e (20 points). Plot the excitation pulse $B_1(t)$ from part (c) and its Fourier Transform $B_1(\omega)$ - be sure to quantitatively label your axes. You may draw your plots in either the laboratory or rotating frame (be sure to indicate which).

Problem 1f (5 points). Draw a line on the Fourier Transform plot at the resonant frequency of the μ_b nuclei in the B_0 field. What is the Fourier amplitude $B_1(\omega_b)$ at the resonant frequency of the μ_b nuclei?

Problem 1g (10 points). Imagine we deliver a sinc pulse instead of a box pulse on resonance with our μ_a nuclei in a $B_0 = 0.5$T magnetic field. What is the minimum duration of the center lobe of this sinc pulse which ensures the μ_b nuclei experience **no** excitation?

Problem 1h (10 points). The homework github contains a csv file which describes an excitation pulse - column 1 is time in milliseconds, and column 2 is the B_1 pulse in μT. Plot this pulse. Plot the Fourier transform of this pulse. What is the excitation frequency and the transmit bandwidth of this pulse?

Exercise Set #4: Relaxation and Detection

Problem 1 . Determine the fraction of protons aligned with an external magnetic field of $B =$ 3.0T if the protons are at room temperature ($T = 300$K).

Problem 2 . If protons are in a $B = 3.0$ T magnetic field, what temperature T must they be for 50.1% of them to be aligned with the field?

Problem 3 . If protons are in a $B = 3.0$ T magnetic field, what temperature T must they be to have polarization $P = 50.1\%$?

Problem 4 . We have a sample of room-temperature water in a 1 mm × 1 mm × 1 mm cube. This cube is placed in a $B = 1.0$ T magnetic field. Calculate the Boltzmann Magnetism, M_0, of this sample.

Problem 5 . Use the following data for all parts of problem 4:

	Tissue A	Tissue B	Tissue C
M_0	1.0	1.5	2.0
T_1 [ms]	500	300	600
T_2 [ms]	90	50	30

(a) Which tissue will appear brightest in a spin-density image?

(b) Which tissue will appear brightest in a T_2-weighted image with TE = 30 ms?

(c) Which tissue will appear brightest in a T_1-weighted image with TR = 400 ms?

(d) Which tissue will appear brightest in an NMR experiment where TR = 400 ms and TE = 30 ms?

(e) What TE should we use if we want Tissues A and C to have the same brightness in a T_2-weighted image?

(f) Using your answer for the previous part, what will the signals be for all tissues?

Problem 5 . The github contains 2 csv files which are 100×100 arrays of weighted image data. 'T1w.csv' contains a T_1-weighted image of a circular phantom with a TR of 500 ms, and 'T2w.csv' contains a T_2-weighted image acquired with TE = 15 ms. Create T_1 and T_2 maps of this phantom. (You may assume M_0 is constant within the phantom).

Exercise Set #5: Hardware

Problem 1a (5 points). A particular superconducting magnet is ramped to exactly $B_0 = 3.0$ T. We deliver on-resonance $B_1 = 30\mu$T box pulses to excite protons to a flip angle of 90°. What is the duration of these pulses?

Problem 1b (5 points). The magnet has resistance of 1 nΩ and inductance of 100 H. What will the magnetic field strength be in 5 years?

Problem 1c (5 points). By how much will the larmor frequency of protons have changed in Hz in 5 years?

Problem 1d (10 points). Although the magnetic field will have changed in 5 years, the B_1 excitation pulses have not. What is $B_{0,\text{eff}}$ for protons in the new magnetic field?

Problem 2 (30 points). Let's build an NMR spectrometer! We will place a 3mm-diameter NMR tube full of water in a $B = 3.0$ T magnetic field. We will wrap 28 AWG wire around a 10 mm section of the tube to behave as our NMR coil. We will use 30 AWG wire and wrap it in a single layer with no spaces between windings[3].

(a) How many windings of 28 AWG wire will we fit in a single layer around the tube assuming no gaps in the windings?

(b) What is the resistance of the wire coil?

(c) What is the inductance of the wire coil?

(d) What capacitance will we need to resonate hydrogen in the $B = 3.0$ T field?

(e) What is the Q of this LRC circuit?

(f) How far off-resonance can an excitation pulse be from coil resonance before the coil attenuates the signal by half?

[3] You may need to Google some properties of 28 AWG wire for this problem...

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Problem Set #6: 1D NMR

Problem 1a (15 points). We deliver a 90° pulse to a spin sample of protons and find that the resultant Lorentzian lineshape in frequency space shows a peak at the spins' Larmor frequency of $\omega_0 = 100$ MHz. At $\omega_1 = 100.001$ MHz the Lorentzian has amplitude of 20% of the peak amplitude.

(a) What is the T_2^* of the FID?

(b) What is the FWHM of the Lorentzian?

(c) What is the maximum achievable T_2^* for a proton sample in this field?

Problem 1b (15 points). Following the FID decay from the previous problem, we apply a 180° pulse at time $\tau = 80$ ms and find that the peak of the resultant echo reaches 50% of the FID's initial signal value.

(a) Illustrate the full NMR experiment with a signal diagram drawn to scale.

(b) What is the T_2 of this sample?

(c) What is the field inhomogeneity ΔB?

Problem 2 (20 points). A particular tissue has a $T_1 = 200$ ms. You would like to perform NMR on this tissue with a flip angle of $\theta = 15°$.

(a) What TR should be prescribed to ensure optimal signal?

(b) For the same tissue, we would like to use a TR of 50ms. In this case what should our prescribed flip angle be for optimum signal?

(c) How long will each scenario take to reach steady state[4]?

(d) How many excitations will each scenario take to reach steady state?

Problem 3. The github contains a csv file containing NMR data from 2 spin-echo experiments. The second experiment was performed exactly 500 ms after the first - that is, the excitation pulses for the experiments were separated in tiem by TR = 500 ms. At what time was the 180° refocusing pulse delivered? What is the T_2^* of the sample? What is the T_2 of the sample? What is the T_1 of the sample?

Problem 4 (10 points). The gyromagnetic ratio of ^{1}H to full precision is $\gamma_{^1H} = 42.577478$ MHz. In a 1.5T field, we detect resonances at 0 ppm, 1, ppm and 4.5 ppm. What frequencies did we detect?[5]

[4] Use the definition that steady state is achieved once the difference in signal between 2 consecutive excitations is less than 1%

[5] Here, 0ppm is not the DSS signal, just the resonant signal of un-shifted protons.

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Problem Set #7: Thermodynamics and Quantum Mechanics

Problem 1 (10 points). In quantum mechanics the fundamental equation which describes the nature of a particle's spin is the cannonical commutator:

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k$$

where ϵ_{ijk} is the Levi-Civita symbol (i, j, k in this context represent any permutation of x, y, z) - note that the i in front of the Levi-Civita is the imaginary unit which satisfies $i^2 = -1$. So for example $[\hat{S}_z, \hat{S}_y] = -i\hat{S}_x$. It turns out that the equations of motion for a spin in a magnetic field can be derived directly from this commutator. In order to do this we apply the Heisenberg equation:

$$\frac{d\hat{A}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}]$$

which says that the rate of change of an operator variable $\hat{A}$ is proportional to its commutator with the system Hamiltonian $\hat{H}$. The Hamiltonian of a spin in a magnetic field is as follows:

$$\hat{H} = \gamma\hbar B_0\hat{S}_z$$

Find the equations of motion for $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ for a spin in a magnetic field.[6]

BONUS (10 points): *Solve* the equations of motion for $\hat{S}_x(t)$, $\hat{S}_y(t)$, and $\hat{S}_z(t)$. This might look a bit familiar to an excercise from Module 3.

Problem 2a (10 points). In class we discussed how the theory of blackbody radiation was developed by modeling the system as being composed of many tiny oscillators. The average energy per oscillator follows from the Boltzmann distribution and was calculated by Rayleigh and Jeans according to

$$\bar{E} = \frac{\int\limits_0^\infty E e^{-E/kT} dE}{\int\limits_0^\infty e^{-E/kT} dE}$$

Using this expression calculate $\bar{E}$.

Problem 2b (10 points). The quantum revolution began when Plank made the same calculation, but instead of allowing $dE \to 0$, he postulated that energy emission is in fact discrete ($E = n\epsilon$,

[6] The following property may be handy: $[a\hat{A}, \hat{B}] = a[\hat{A}, \hat{B}]$. This says that scaling an operator will necessarily result in an equally scaled commutator.

where n is an integer and ϵ is the size of an individual energy packet). The average energy then becomes a summation rather than an integral:

$$\bar{E} = \frac{\sum\limits_{n=0}^{\infty} E e^{-E/kT}}{\sum\limits_{n=0}^{\infty} e^{-E/kT}}$$

Using this expression calculate $\bar{E}$.[7]

[7]**Hint:** Recall the identity $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$

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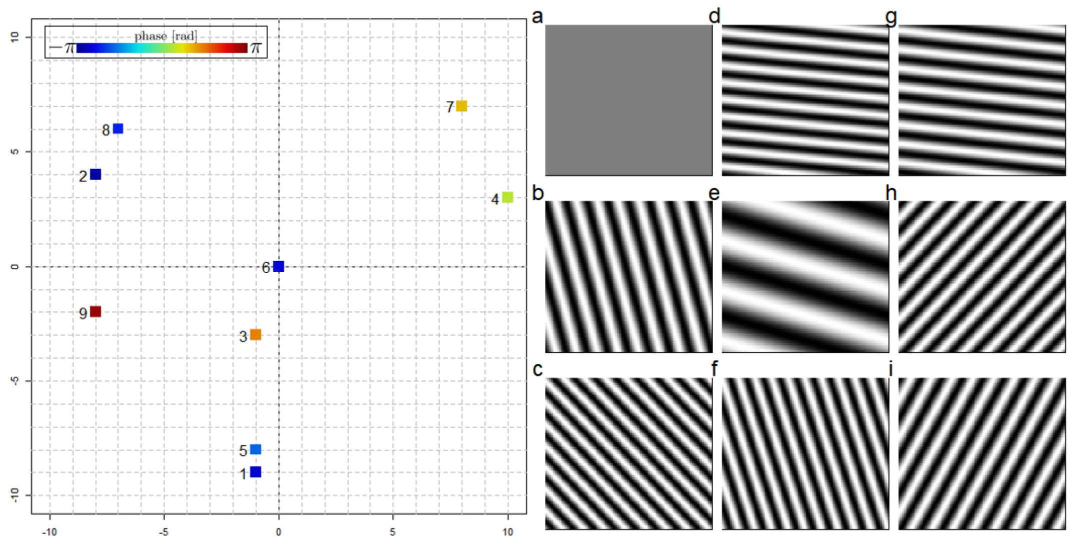
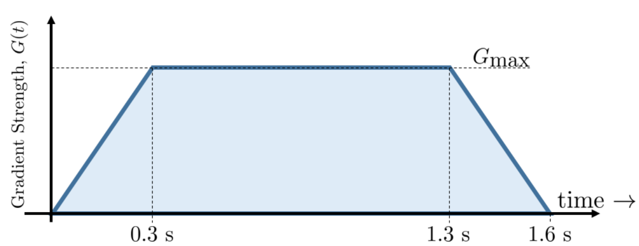
Problem Set #8: Magnetic Resonance Imaging

Problem 1 (20 points). Using the images below, match the numbered locations in k-space with their corresponding images.



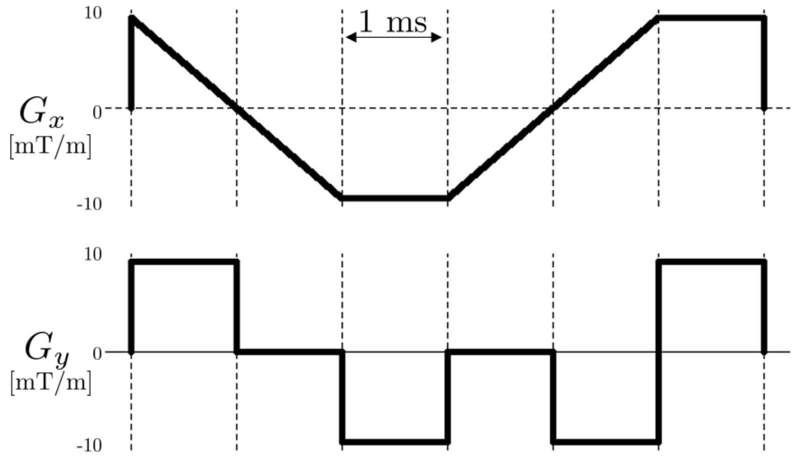
Problem 2 (25 points). We with to acquire an image of the brain which has fields of view of $\text{FOV}_x = 16$ cm and $\text{FOV}_y = 20$ cm. Our radiologists require a resolution of 2-mm in-plane to accurately assess pathology. Find the k-Space step sizes Δk_x and Δk_y, and the Nyquist frequencies in $k_{x,N}$ and $k_{y,N}$. Also determine the k-space matrix size.

Problem 2 (15 points). Gradients often take time to 'ramp' up to their maximum strength. This is called the gradients' *slew rate* and is often specified in units of mT/m/s (milliTesla per meter per second). Following is a plot of gradient strength as a function of time, $G(t)$ with a maximum gradient strength of $G_{\text{max}} = 60$ mT/m.



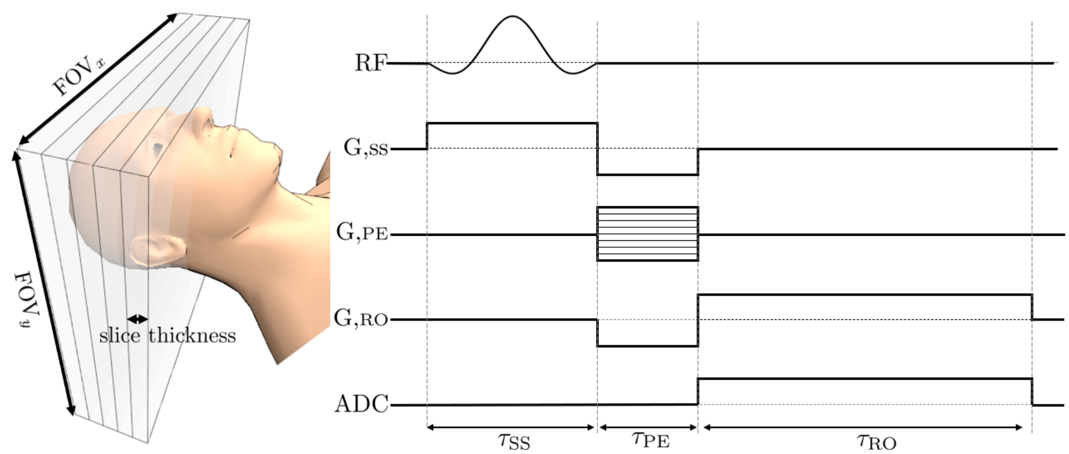
If the gradient in the plot is ramped as quickly as possible to its maximum strength, what is its slew rate? What spatial frequency is delivered to protons following this gradient?

Problem 4 (30 points). The following gradients G_x and G_y are delivered to a sample of protons; the dashed spacings indicate 1 ms separations in time (so the entire sequence lasts 6 ms). Plot $k_x(t)$ and $k_y(t)$ for this spin sample quantitatively indicating each spatial frequency amplitude. Also, draw a plot of the k-space trajectory; again, be quantitative.



Problem Set #9: Pulse Sequences

Problem 1 (50 points). You want to perform head MRI on an individual using a Gradient Echo sequence as shown in the figure below: 180 mm × 200 mm Field of View in x and y respectively, 3 mm slice thickness, 5 slices. We would like 1 mm in-plane resolution (voxel dimension = 1×1×4 mm^3). The scanner has a maximum gradient strength of 30 mT/m (you may assume an infinite gradient slew rate). We will use a 3-lobe sinc pulse for excitation as shown. The sequence must be as short as possible, but TR = 50 ms and TE = 10 ms.



Completely describe all parameters of this Gradient Echo Sequence:

(a) What are the matrix dimensions and which dimension should be the phase-encoding dimension?

(b) What are k-space Δk_x and Δk_y?

(c) What are the Nyquist frequencies in $k_{x,N}$ and $k_{y,N}$?

(d) What is duration of the slice selection gradient/pulse τ_{SS}?

(e) What is the duration of the phase encode/slice rewinder gradient, τ_{PE}?[8]

(f) By how much will the phase encode gradient change for each TR: ΔG_{PE}?

(g) What is the readout time τ_{RO}

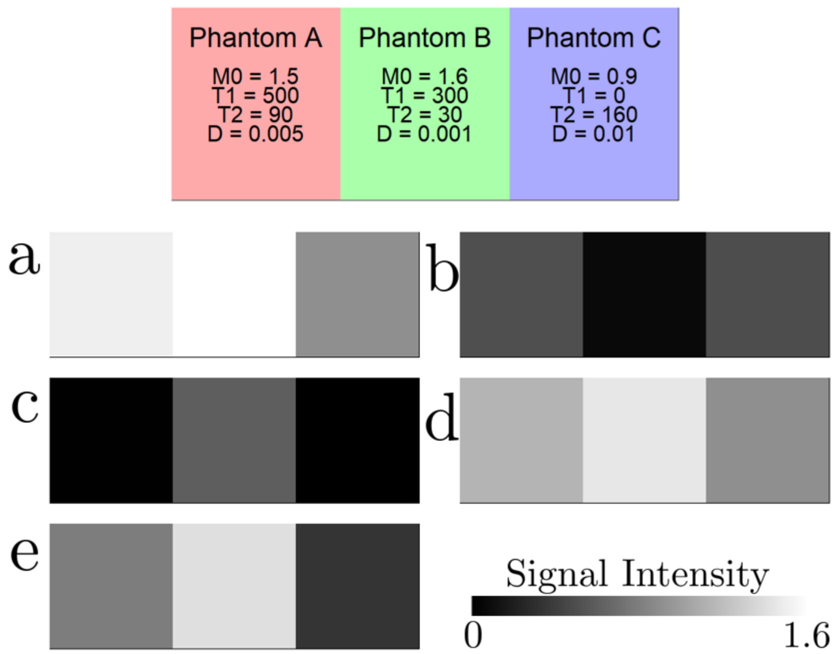
(h) How long will this sequence take acquire all 5 slices?

(i) What is the bandwidth of the readout?

(j) If the field homogeneity is 0.5 ppm within the imaging volume, how much has the signal decayed by time we detect the echo peak?

[8] Note that the shortest possible slice rewinder may not be the same duration as the shortest possible Nyquist phase encode gradient!

Problem 2 (20 points). The phantom in the Figure below consists of 3 wells of different samples: A, B, and C. The samples' respective M_0's, T_1's, T_2's, and diffusion coefficients D's are given in the figure for each (T_1 and T_2 in units of ms; D in units of mm^2/s). Five NMR experiments are performed with the TR, TE, and b-values given in the Table below. The image results of the five experiments comprise the displayed grayscale images the figure (image a-e). Match the images a-e with their respective experiment number. Of course, justify your choices with appropriate calculations.



Experiment	TR [ms]	TE [ms]	b-value [s/mm^2]
1	∞	0	0
2	700	0	0
3	∞	100	0
4	∞	0	1,000
5	1000	0	100

Table 1: Table of experimental values used in each NMR experiment in problem 3.

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Problem Set #10: Advanced Topics

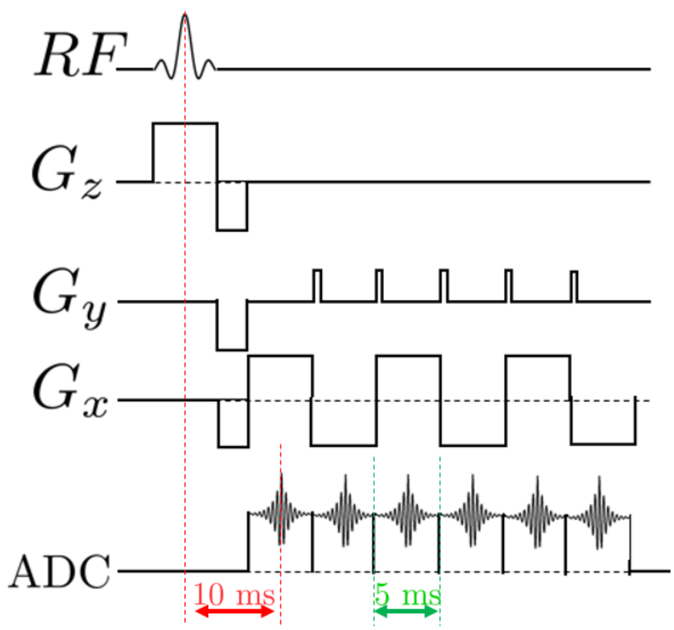
Problem 1. We wish to image a tissue sample of $T_2 = 60$ ms using the following EPI sequence. The volume in which the sample resides has a field homogeneity of $\Delta B = .7$ ppm.

(a) What will the signal intensity of the final echo be (6th echo in image) expressed as a fraction of the magnetization of the first echo?

(b) We must not allow our signal to decay to less than 20% of the magnitude of the first echo.

(c) How many echoes can we acquire in a single excitation? (You may assume the time in between echoes is negligible.)

(d) The image matrix size is 100 PE ×128 RO. What is the bandwidth of the acquired readouts?



Problem 2a. A 1-liter volume of ^{129}Xe is hyperpolarized to 35% and has a $T_1 = 20$ min in a magnetic field. What will the polarization be after 10 minutes?

Problem 2b. We image the xenon using a 8° flip angle and a 128×128 matrix GRE. The excitations are performed in rapid succession (i.e. TR is near 0). How much polarization is left at the end of the scan?

Problem 2c. Once the polarization of the ^{129}Xe drops below 5% it cannot provide enough signal for high quality images. Thus we want to ensure that at the end of the scan we are left with exactly 5% polarization. What flip angle should we use to ensure this is the case for a 128×128 image? What if we used an EPI with acceleration factor of 4?