MAFS5130: Time series analysis

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1 Exercises

1. Find the ACF and PACF and plot the ACF ρ_k for $k = 0, 1, 2, 3, 4$ and 5 for each of the following models:
   (a) $r_t - 0.5r_{t-1} = a_t$;
   (b) $r_t + 0.98r_{t-1} = a_t$;
   (c) $r_t - 1.3r_{t-1} + 0.4r_{t-2} = a_t$.

   Solution:

   - ACF:
   $$E(r_t * r_{t-k})$$
   - PACF:
   $$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

2. For each of the following models,
   (a) $(1 - 0.9B)(r_t - 10) = a_t$;
   (b) $r_t = 10 - 0.9a_{t-1} + a_t$;
   (c) $(1 - 0.5B)(r_t - 10) = a_t - 0.9a_{t-1}$.
   where $\sigma_a^2 = 2$. Given $r_1 = 1.2$ and $r_2 = 0.1$, find the l-step ahead forecast values and forecast variances for $l = 1, 2, 3, 4$.

   Solution:

   - l-step ahead forecast:
   $$\hat{r}_t(l) = E(r_{t+l}|\mathcal{F}_t)$$
   - l-step forecast variance:
   $$e_t(l) = r_{t+l} - \hat{r}_t(l)$$
   $$\text{Var}\left[e_t(l)\right] = \sigma_a^2\left(1 + \phi^2 + \phi^4 + \cdots\right)$$

3. Find the ACF and PACF for $k = 0, 1, 2, 3$ and 4 for each of the following models:
   (a) $r_t = (1 - 0.8B)a_t$;
   (b) $r_t = \left(1 - 1.2B + 0.5B^2\right)a_t$.

   Solution:

   - ACF

- PACF

4. Verify whether or not the following models are stationary and / or invertible:
(a) $(1-B)r_t=(1-1.5B)a_t$;
(b) $(1-0.8B)r_t=(1-0.5B)a_t$;
(c) $\left(1-1.1B+0.8B^2\right)r_t=\left(1-1.7B+0.72B^2\right)a_t$
(d) $(1-0.6B)r_t=\left(1-1.2B+0.2B^2\right)a_t$
Solution:
For AR(1), MA(1), ARMA(1,1),
    - Invertibility condition: $|\theta|<1$
    - Stationarity condition: $|\phi|<1$

    For other model, solve equation like $1-1.1z+0.8z^2=0$ in (c), if the roots lies outside the unit circle, then it's stationary or invertible; otherwise it's not.

5. Consider the two models:
(a) $(1-0.43B)(1-B)r_t=a_t$;
(b) $(1-B)r_t=(1-0.43B)a_t$
where a_t is i.i.d. $N(0,1)$. Given the observations $r_{49}=33.4$ and $r_{50}=33.9$, compute their forecasts $r_{50}(l)$, for $l=1,2,3,4$, and the corresponding 90% forecast intervals.
Solution:
    - Forecast values: $\hat{r_{50}}(1)=E(r_{51}|\mathcal{F}_t)=34.115$
    - Forecast variance: $\operatorname{Var}\left[e_n(l)\right]=\sigma_a^2\sum_{j=0}^{l-1}\phi^{2j}$
    - Forecast intervals: $\left[\widehat{r}_n(l)-N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\phi^{2j}},\widehat{r}_n(l)+N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\phi^{2j}}\right]$
    where $\alpha=0.1$ in this case and $N_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ -quantile of the standard normal distribution.

6. Find the ACF for the following seasonal models:
(a) $r_t=\left(1-\theta_1B\right)\left(1-\Theta_1B\right)a_t$;
(b) $\left(1-\Phi_1B^s\right)r_t=\left(1-\theta_1B\right)a_t$;
(c) $\left(1-\Phi_1B^s\right)\left(1-\phi_1B\right)r_t=a_t$
where a_t is i.i.d. $N(0,1)$.
Solution:

7. Consider the ARCH model:
$$a_t=\eta_t\sigma_t,\sigma_t^2=\alpha_0+\alpha_1a_{t-1}^2$$
Show that the unconditional variance of a_t is $\operatorname{Var}\left(a_t\right)=\alpha_0/\left(1-\alpha_1\right)$,
where $\alpha_0>0,0\leq\alpha_1<1$ and η_t is i.i.d. $N(0,1)$.
Solution: Use conditional expectation to solve the problem.

8. Give the stationarity condition and its representation in terms of $\{\eta_t\}$ for the GARCH model:
$$a_t=\eta_t\sigma_t,\sigma_t^2=\alpha_0+\alpha_1a_{t-1}^2+\beta_1\sigma_{t-1}^2$$
where $\alpha_0>0,\alpha_1,\beta_1\geq0$, and η_t is i.i.d. $N(0,1)$. Futhermore, give Ea_t^4 and the prediction of the conditional variances σ_{t+s}^2
Solution:
    - Stationarity condition: $0\leq\alpha_1,\beta_1\leq1,\left(\alpha_1+\beta_1\right)<1$; [$\sum_{i=1}^{\max(m,s)}\left(\alpha_i+\beta_i\right)<1$ for GARCH(m,s)]
    - Representation in terms of $\{\eta_t\}$: $\sigma_t^2=\alpha_0\left[1+\sum_{j=1}^{\infty}\prod_{i=1}^{j}\left(\alpha_1\eta_{t-i}^2+\beta_1\right)\right]$
    - 1-step ahead forecast: $\sigma_t^2(1)=E\left(\sigma_{t+1}^2|\mathcal{F}_t\right)=\sigma_{t+1}^2=\alpha_0+\alpha_1Z_t^2+\beta_1\sigma_t^2$
    - 2-step ahead forecast: $\sigma_t^2(2)=E\left(\sigma_{t+2}^2|\mathcal{F}_t\right)=\alpha_0+E\left(\alpha_1\epsilon_{t+1}^2+\beta_1|\mathcal{F}_t\right)\sigma_{t+1}^2=\alpha_0+\left(\alpha_1+\beta_1\right)\sigma_t^2(1)$

- In general: $\sigma_t^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1)\,\sigma_t^2(\ell - 1)$
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9. Give the stationarity and invertibility conditions, MA and AR representation and ACFs of the seasonal ARMA models:
(a) $y_t = \phi y_{t-s} + a_t$;
(b) $y_t = \theta a_{t-s} + a_t$
where $\{a_t\}$ is white noise and variance σ_a^2.
Solution:
   - MA representation: $y_t = \frac{1}{1-\phi B^s} a_t = \sum_{j=0}^{\infty} (\phi B^s)^j\, a_t$
   - AR representation: $at = \frac{y_t}{1+\theta B^s} = \sum_{j=0}^{\infty} (-\theta B^s)^j\, y_t$
   - ACFs

10. Consider the following EGARCH(1,1) model
$$a_t = \sigma_t \epsilon_t, \quad (1 - \beta B) \ln\left(\sigma_t^2\right) = \alpha_0 + \alpha g\left(\epsilon_{t-1}\right)$$
where $\epsilon_t \sim N(0,1)$ and $E\left(|\epsilon_t|\right) = \sqrt{2/\pi}$ and
$$g\left(\epsilon_t\right) = \theta \epsilon_t + \left[|\epsilon_t| - E\left(|\epsilon_t|\right)\right]$$
Show the representation of $\ln(\sigma_t^2)$ in terms of ϵ_t and give its mean and variance.
Solution:
   - Expectation:
$$\begin{aligned} E(\ln(\sigma_t^2)) &= E\left[\tfrac{1}{1-\beta B}(\alpha_0 + \alpha g(\epsilon_{t-1}))\right] \\ &= E\left[\tfrac{\alpha_0}{1-\beta} + \alpha \textstyle\sum_{i=0}^{\infty} \beta^i B^i g(\epsilon_{t-1})\right] \\ &= E\left[\tfrac{\alpha_0}{1-\beta} + \alpha \textstyle\sum_{i=0}^{\infty} \beta^i g(\epsilon_{t-i-1})\right] \\ &= \tfrac{\alpha_0}{1-\beta} + \tfrac{\alpha}{1-\beta} E\left[g(\epsilon_{t-i-1})\right] \\ &= \tfrac{\alpha_0}{1-\beta} \end{aligned}$$
   - Variance:
$$\mathrm{Var}(\ln(\sigma_t^2)) = \mathrm{Var}\left(\frac{1}{1-\beta B}[\alpha_0 + \alpha g(\epsilon_{t-1})]\right) = \mathrm{Var}\left(\frac{\alpha}{1-\beta B} g(\epsilon_{t-1})\right) = \mathrm{Var}\left(\frac{\alpha}{1-\beta} g(\epsilon_{t-i-1})\right)$$
Since
$$\begin{aligned} \mathrm{Var}[g(\epsilon_{t-1})] &= E[g^2(\epsilon_{t-1})] \\ &= E\left[\left(\theta \epsilon_t + |\epsilon_t| - E\left(|\epsilon_t|\right)\right)^2\right] \\ &= E(\theta^2 \epsilon_t^2) + E(|\epsilon_t|^2) + \tfrac{2}{\pi} + 2E\left(\theta \epsilon_t |\epsilon_t|\right) - 2E\left(|\epsilon_t| \sqrt{\tfrac{2}{\pi}}\right) - 2E\left(\theta \epsilon_t \sqrt{\tfrac{2}{\pi}}\right) \\ &= (\theta^2 + 1) + \tfrac{2}{\pi} - \tfrac{4}{\pi} = (\theta^2 + 1) - \tfrac{2}{\pi} \end{aligned}$$
where
$$E\left(\epsilon_t |\epsilon_t|\right) = 0$$
Then
$$\mathrm{Var}(\ln(\sigma_t^2)) = \frac{\alpha^2((\theta^2+1) - \frac{2}{\pi})}{(1-\beta)^2}$$

11. Consider the following bivariate VAR model:
$$\begin{aligned} y_{1t} &= 0.3 y_{1,t-1} + 0.8 y_{2,t-1} + a_{1t} \\ y_{2t} &= 0.9 y_{1,t-1} + 0.4 y_{2,t-1} + a_{2t} \end{aligned}$$

with $E(a_{1t}a_{1\tau}) = 1$ if $t = \tau$ and 0 otherwise, $E(a_{2t}a_{2\tau}) = 2$ if $t = \tau$ and 0 otherwise, and $E(a_{1t}a_{2\tau}) = 0$ for all t and τ.

(a) Is this system stationary?
(b) Calculate the two-step ahead forecast variance for variable $y_{1,t+2}$, that is

$$E\left[y_{1,t+2} - E\left(y_{1,t+2}|Y_t, Y_{t-1}, \cdots\right)\right]^2$$

where $Y_t = (y_{1t}, y_{2t})'$

Solution:

- Stationarity or invertibility condition:

$$\lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \Phi_1$$

- VAR(p) model:
  - $\hat{y_{1,t}}(2) = E\left(y_{1,t+2}|Y_t, Y_{t-1}, \cdots\right)$
  - $\text{Var}\left(e_t(1)\right) = \Sigma$
  - $\text{Var}\left(e_t(l)\right) = \Sigma + \sum_{j=1}^{l-1} \Psi_j \Sigma \Psi_j^T$
  - $\Sigma = E\left[a_t a_{t+k}'\right]$, if $k = 0$.

12. Write down the bivariate system into an VAR model and show that it is not stationary:

$$y_{1t} = \gamma y_{2t} + \varepsilon_{1t}$$
$$y_{2t} = y_{2,t-1} + \varepsilon_{2t}$$

where $\gamma \neq 0, \varepsilon_{1t}$ and ε_{2t} being uncorrelated white noise processes.

Solution:
Stationarity or invertibility condition:

$$\lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \Phi_1$$

13. Show that the following VAR model

$$\mathbf{y}_\mathrm{t} = \sum_{\mathrm{i}=1}^{\mathrm{p}} \Phi_\mathrm{i} \mathrm{y}_{\mathrm{t}-\mathrm{i}} + \varepsilon_\mathrm{t}$$

can be written as following VCE model:

$$\Phi(B)\mathbf{y_t} = \mathbf{\Phi}^*(\mathbf{B})(1-\mathbf{B})\mathbf{y_t} + \mathbf{\Phi}(1)\mathbf{B}\mathbf{y_t}$$

where $\Phi^*(B) = \mathrm{I_m} - \sum_{\mathrm{i}=1}^{\mathrm{p}-1} \Phi_\mathrm{i}^* \mathrm{B}^\mathrm{i}$ with $\Phi_i^* = -\sum_{j=i+1}^{p} \Phi_j$.

Solution:
Assume $|\Phi(z)| = \left|I_m - \sum_{i=1}^{p} \Phi_i z^i\right| = 0$ has $d < m$ unit root and the remaining roots outside the unit circle.
The rank of $\Phi(1) = I_m - \sum_{i=1}^{p} \Phi_i$ is r and $r = m - d$.
We can decompose $\Phi(1)$ as

$$\alpha\beta' = -\Phi(1) = -\mathbf{I}_m + \Phi_1 + \ldots + \Phi_p$$

$\Phi(B)$ can be re-expressed as

$$\Phi(B) = \Phi^*(B)(1-B) + \Phi(1)B$$

where $\Phi^*(B) = \mathbf{I}_m - \sum_{i=1}^{p-1} \Phi_i^* B^i$ with $\Phi_i^* = -\sum_{j=i+1}^{p} \Phi_j$.

14. Consider the two dimensional vector AR(2) model:

$$\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} -0.2 & 0.1 \\ 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} + \begin{bmatrix} 0.8 & 0.7 \\ -0.4 & 0.6 \end{bmatrix} \begin{bmatrix} Z_{1,t-2} \\ Z_{2,t-2} \end{bmatrix}$$

where $\{(a_{1t}, a_{2t})'\}$ i sa sequence of i.i.d. standard normal random vectors. Show that it is a partially non-stationary AR model and give its cointegration vector.

Solution:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -0.2 & 0.1 \\ 0.5 & 0.2 \end{bmatrix} - \begin{bmatrix} 0.8 & 0.7 \\ -0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.8 \\ -0.1 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.1 \end{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix}$$

Since $|1 + \alpha_1 + \alpha_2\beta_1| = |1 + 0.4 + 0.2| > 1$, so it's not stationary.
And the cointegrating vector is $\beta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

15. Determine the stationarity and invertibility of the following two dimensional vector models and find their correlation matrix function, ρ_k, for $k = \pm1, \pm2, \pm3$:
(a) $(I - \Phi_1 B)\, Z_t = a_t$, where $\Phi_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.6 \end{bmatrix}$ and $\Sigma = I$;
(b) $(I - \Phi_1 B)\, Z_t = a_t$, where $\Phi_1 = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.8 \end{bmatrix}$ and $\Sigma = I$;
(c) $Z_t = (I - \Theta_1 B)\, a_t$, where $\Theta_1 = \begin{bmatrix} 0.6 & 1.2 \\ 0.4 & 0.8 \end{bmatrix}$ and $\Sigma = I$;

Solution:

- Stationarity or invertibility condition:

$$\lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \Phi_1$$

- VAR(1) model:

$$\Sigma = \Gamma(0) - \Phi_1 \Gamma(0) \Phi_1'$$

$$\Gamma(k) = \begin{cases} \Gamma(-1)\Phi_1' + \Sigma, & \text{if } k = 0 \\ \Gamma(k-1)\Phi_1' = \Gamma(0)\left(\Phi_1'\right)^k, & \text{if } k \geq 1 \end{cases}$$

$$\Gamma(k) = \Gamma'(-k) \geq 0$$

$$\rho(k) = D^{-1}\Gamma(k)D^{-1}$$

where $D = \begin{bmatrix} \sigma_{11}(k) & 0 & \cdots & 0 \\ 0 & \sigma_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{mm}(k) \end{bmatrix}$.

- solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \Gamma(0) - \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.6 \end{bmatrix} \Gamma(0) \begin{bmatrix} 0.8 & 0.1 \\ 0.3 & 0.6 \end{bmatrix}$$

$$\rho(k) = \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix} \Gamma(k) \begin{bmatrix} \sigma^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix} = \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix} \Gamma(0) \begin{bmatrix} 0.8^k 0.1^k \\ 0.3^k 0.6^k \end{bmatrix} \begin{bmatrix} \sigma & -1 \\ 0 & \sigma_2^{-1} \end{bmatrix}$$

16. Consider the process

$$\begin{aligned} Z_{1t} &= Z_{1,t-1} + a_{1t} + \theta a_{1,t-1} \\ Z_{2t} &= \phi Z_{1t} + a_{2t} \end{aligned}$$

where $|\phi| < 1, |\theta| < 1$ and $a_t = [a_{1t}, a_{2t}]' \sim N(0, \Sigma)$. (a) Write the process in a vector form (b) Is the process $[Z_{1t}, Z_{2t}]'$ stationary and invertible? (c) Write down the model for the vector of the first differences $(I - B)Z_t$, where $Z_t = [Z_{1t}, Z_{2t}]'$. Is the resulting model stationary and invertible?

Solution:

- (a)

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi & 0 \end{bmatrix} \begin{bmatrix} z_{1,t-1} \\ z_{2,t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \phi & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} \theta & 0 \\ \phi & 0 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

- (c)

$$
\begin{aligned}
\Delta z_{1t} &= z_{1t} - z_{1,t-1} = a_1 + \theta a_{1+1} \\
\Delta z_{2t} &= \phi \Delta z_{1t} + a_{2t} - a_{2,t-1} = \phi a_{1t} + \phi\theta a_{1,t-1} + a_{2t} - a_{2,t-1}
\end{aligned}
$$

$$
\begin{bmatrix} \Delta z_{1t} \\ \Delta z_2 t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} \theta & 0 \\ \phi\theta & -1 \end{bmatrix} \cdot \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}
$$

$$
\begin{vmatrix} \lambda - \theta & 0 \\ \phi - \phi\theta & \lambda + 1 \end{vmatrix} = (\lambda - \theta)(\lambda + 1) = 0
$$

  Since $|\lambda| \geq 1$, so it's not invertible;
  And it's vector MA model, so it's stationary.

17. Show that the process $y_t = z_t - z_{t-1}$ is weakly stationary, where $z_t = 0.9z_{t-1} + a_t$ and $\{a_t\}$ is white noise series.

    Solution:

$$
\begin{aligned}
y_t &= 0.9z_{t-1} + a_t - 0.9z_{t-2} - a_{t-1} \\
&= 0.9y_{t-1} + a_t - a_{t-1} \\
\Rightarrow &(1 - 0.9B)y_t = (1 - B)a_t
\end{aligned}
$$

    Since $|\phi_1| = |0.9| < 1$, then $y_t = z_t - z_{t-1}$ is stationary.