

# MAFS5130: Time series analysis

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## 1 Exercises

1. Find the ACF and PACF and plot the ACF  $\rho_k$  for  $k = 0, 1, 2, 3, 4$  and 5 for each of the following models:

- (a)  $r_t - 0.5r_{t-1} = a_t$ ;  
 (b)  $r_t + 0.98r_{t-1} = a_t$ ;  
 (c)  $r_t - 1.3r_{t-1} + 0.4r_{t-2} = a_t$ .

**Solution:**

- ACF:

$$E(r_t * r_{t-k})$$

- PACF:

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

2. For each of the following models,

- (a)  $(1 - 0.9B)(r_t - 10) = a_t$ ;  
 (b)  $r_t = 10 - 0.9a_{t-1} + a_t$ ;  
 (c)  $(1 - 0.5B)(r_t - 10) = a_t - 0.9a_{t-1}$ .

where  $\sigma_a^2 = 2$ . Given  $r_1 = 1.2$  and  $r_2 = 0.1$ , find the  $l$ -step ahead forecast values and forecast variances for  $l = 1, 2, 3, 4$ .

**Solution:**

For AR(1) model:

- 1-step ahead forecast:

$$\hat{r}_t(l) = E(r_{t+l} | \mathcal{F}_t)$$

- 1-step forecast variance:

$$e_t(l) = r_{t+l} - \hat{r}_t(l)$$

$$\text{Var}[e_t(l)] = \sigma_a^2 (1 + \phi^2 + \phi^4 + \cdots)$$

3. Find the ACF and PACF for  $k = 0, 1, 2, 3$  and 4 for each of the following models:

- (a)  $r_t = (1 - 0.8B)a_t$ ;  
 (b)  $r_t = (1 - 1.2B + 0.5B^2)a_t$ .

**Solution:**

- ACF
- PACF

4. Verify whether or not the following models are stationary and / or invertible:

- (a)  $(1 - B)r_t = (1 - 1.5B)a_t$ ;
- (b)  $(1 - 0.8B)r_t = (1 - 0.5B)a_t$ ;
- (c)  $(1 - 1.1B + 0.8B^2)r_t = (1 - 1.7B + 0.72B^2)a_t$
- (d)  $(1 - 0.6B)r_t = (1 - 1.2B + 0.2B^2)a_t$

**Solution:**

For AR(1), MA(1), ARMA(1,1),

- Invertibility condition:  $|\theta| < 1$
- Stationarity condition:  $|\phi| < 1$

For other model, solve equation like  $1 - 1.1z + 0.8z^2 = 0$  in (c), if the roots lies outside the unit circle, then it's stationary or invertible; otherwise it's not.

5. Consider the two models:

- (a)  $(1 - 0.43B)(1 - B)r_t = a_t$ ;
- (b)  $(1 - B)r_t = (1 - 0.43B)a_t$

where  $a_t$  is i.i.d.  $N(0, 1)$ . Given the observations  $r_{49} = 33.4$  and  $r_{50} = 33.9$ , compute their forecasts  $r_{50}(l)$ , for  $l = 1, 2, 3, 4$ , and the corresponding 90% forecast intervals.

**Solution:**

- Forecast values:  $r_{50}(1) = E(r_{51}|\mathcal{F}_t) = 34.115$
- Forecast variance:  $\text{Var}[e_n(l)] = \sigma_a^2 \sum_{j=0}^{l-1} \phi^{2j}$
- Forecast intervals:  $\left[ \hat{r}_n(l) - N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \phi^{2j}}, \hat{r}_n(l) + N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \phi^{2j}} \right]$   
where  $\alpha = 0.1$  in this case and  $N_{\frac{\alpha}{2}}$  is the  $\frac{\alpha}{2}$ -quantile of the standard normal distribution.

6. Find the ACF for the following seasonal models:

- (a)  $r_t = (1 - \theta_1 B)(1 - \Theta_1 B)a_t$ ;
- (b)  $(1 - \Phi_1 B^s)r_t = (1 - \theta_1 B)a_t$ ;
- (c)  $(1 - \Phi_1 B^s)(1 - \phi_1 B)r_t = a_t$

where  $a_t$  is i.i.d.  $N(0, 1)$ .

**Solution:**

7. Consider the ARCH model:

$$a_t = \eta_t \sigma_t, \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2$$

Show that the unconditional variance of  $a_t$  is  $\text{Var}(a_t) = \alpha_0 / (1 - \alpha_1)$ , where  $\alpha_0 > 0, 0 \leq \alpha_1 < 1$  and  $\eta_t$  is i.i.d.  $N(0, 1)$ .

**Solution:** Use conditional expectation to solve the problem.

8. Give the stationarity condition and its representation in terms of  $\{\eta_t\}$  for the GARCH model:

$$a_t = \eta_t \sigma_t, \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where  $\alpha_0 > 0, \alpha_1, \beta_1 \geq 0$ , and  $\eta_t$  is i.i.d.  $N(0, 1)$ . Furthermore, give  $Ea_t^4$  and the prediction of the conditional variances  $\sigma_{t+s}^2$

**Solution:**

- Stationarity condition:  $0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1$ ;  $[\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1 \text{ for GARCH}(m,s)]$
- Representation in terms of  $\{\eta_t\}$ :  $\sigma_t^2 = \alpha_0 \left[ 1 + \sum_{j=1}^{\infty} \prod_{i=1}^j (\alpha_1 \eta_{t-i}^2 + \beta_1) \right]$
- 1-step ahead forecast:  $\sigma_t^2(1) = E(\sigma_{t+1}^2 | \mathcal{F}_t) = \sigma_{t+1}^2 = \alpha_0 + \alpha_1 Z_t^2 + \beta_1 \sigma_t^2$

- 2-step ahead forecast:  $\sigma_t^2(2) = E(\sigma_{t+2}^2 | \mathcal{F}_t) = \alpha_0 + E(\alpha_1 \epsilon_{t+1}^2 + \beta_1 | \mathcal{F}_t) \sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2(1)$
- In general:  $\sigma_t^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2(\ell - 1)$

9. Give the stationarity and invertibility conditions, MA and AR representation and ACFs of the seasonal ARMA models:

(a)  $y_t = \phi y_{t-s} + a_t$ ;

(b)  $y_t = \theta a_{t-s} + a_t$

where  $\{a_t\}$  is white noise and variance  $\sigma_a^2$ .

**Solution:**

- MA representation:  $y_t = \frac{1}{1-\phi B^s} a_t = \sum_{j=0}^{\infty} (\phi B^s)^j a_t$

- AR representation:  $a_t = \frac{y_t}{1+\theta B^s} = \sum_{j=0}^{\infty} (-\theta B^s)^j y_t$

- ACFs

10. Consider the following EGARCH(1,1) model

$$a_t = \sigma_t \epsilon_t, \quad (1 - \beta B) \ln(\sigma_t^2) = \alpha_0 + \alpha g(\epsilon_{t-1})$$

where  $\epsilon_t \sim N(0, 1)$  and  $E(|\epsilon_t|) = \sqrt{2/\pi}$  and

$$g(\epsilon_t) = \theta \epsilon_t + [|\epsilon_t| - E(|\epsilon_t|)]$$

Show the representation of  $\ln(\sigma_t^2)$  in terms of  $\epsilon_t$  and give its mean and variance.

**Solution:**

- Expectation:

$$\begin{aligned} E(\ln(\sigma_t^2)) &= E\left[\frac{1}{1-\beta B}(\alpha_0 + \alpha g(\epsilon_{t-1}))\right] \\ &= E\left[\frac{\alpha_0}{1-\beta} + \alpha \sum_{i=0}^{\infty} \beta^i B^i g(\epsilon_{t-1})\right] \\ &= E\left[\frac{\alpha_0}{1-\beta} + \alpha \sum_{i=0}^{\infty} \beta^i g(\epsilon_{t-i-1})\right] \\ &= \frac{\alpha_0}{1-\beta} + \frac{\alpha}{1-\beta} E[g(\epsilon_{t-i-1})] \\ &= \frac{\alpha_0}{1-\beta} \end{aligned}$$

- Variance:

$$\text{Var}(\ln(\sigma_t^2)) = \text{Var}\left(\frac{1}{1-\beta B}[\alpha_0 + \alpha g(\epsilon_{t-1})]\right) = \text{Var}\left(\frac{\alpha}{1-\beta B}g(\epsilon_{t-1})\right) = \text{Var}\left(\frac{\alpha}{1-\beta}g(\epsilon_{t-i-1})\right)$$

Since

$$\begin{aligned} \text{Var}[g(\epsilon_{t-1})] &= E[g^2(\epsilon_{t-1})] \\ &= E[(\theta \epsilon_t + |\epsilon_t| - E(|\epsilon_t|))^2] \\ &= E(\theta^2 \epsilon_t^2) + E(|\epsilon_t|^2) + \frac{2}{\pi} + 2E(\theta \epsilon_t |\epsilon_t|) - 2E(|\epsilon_t| \sqrt{\frac{2}{\pi}}) - 2E\left(\theta \epsilon_t \sqrt{\frac{2}{\pi}}\right) \\ &= (\theta^2 + 1) + \frac{2}{\pi} - \frac{4}{\pi} = (\theta^2 + 1) - \frac{2}{\pi} \end{aligned}$$

where

$$E(\epsilon_t |\epsilon_t|) = 0$$

Then

$$\text{Var}(\ln(\sigma_t^2)) = \frac{\alpha^2((\theta^2 + 1) - \frac{2}{\pi})}{(1 - \beta)^2}$$

11. Consider the following bivariate VAR model:

$$y_{1t} = 0.3y_{1,t-1} + 0.8y_{2,t-1} + a_{1t}$$

$$y_{2t} = 0.9y_{1,t-1} + 0.4y_{2,t-1} + a_{2t}$$

with  $E(a_{1t}a_{1\tau}) = 1$  if  $t = \tau$  and 0 otherwise,  $E(a_{2t}a_{2\tau}) = 2$  if  $t = \tau$  and 0 otherwise, and  $E(a_{1t}a_{2\tau}) = 0$  for all  $t$  and  $\tau$ .

(a) Is this system stationary?

(b) Calculate the two-step ahead forecast variance for variable  $y_{1,t+2}$ , that is

$$E[y_{1,t+2} - E(y_{1,t+2}|Y_t, Y_{t-1}, \dots)]^2$$

where  $Y_t = (y_{1t}, y_{2t})'$

**Solution:**

- Stationarity or invertibility condition:

$$\lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \Phi_1$$

- VAR(p) model:

- $y_{1,t}(2) = E(y_{1,t+2}|Y_t, Y_{t-1}, \dots)$
- $\text{Var}(e_t(1)) = \Sigma$
- $\text{Var}(e_t(l)) = \Sigma + \sum_{j=1}^{l-1} \Psi_j \Sigma \Psi_j^T$
- $\Sigma = E[a_t a_{t+k}']$ , if  $k = 0$ .

12. Write down the bivariate system into an VAR model and show that it is not stationary:

$$\begin{aligned} y_{1t} &= \gamma y_{2t} + \varepsilon_{1t} \\ y_{2t} &= y_{2,t-1} + \varepsilon_{2t} \end{aligned}$$

where  $\gamma \neq 0$ ,  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  being uncorrelated white noise processes.

**Solution:**

Stationarity or invertibility condition:

$$\lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \Phi_1$$

13. Show that the following VAR model

$$\mathbf{y}_t = \sum_{i=1}^p \Phi_i \mathbf{y}_{t-i} + \varepsilon_t$$

can be written as following VCE model:

$$\Phi(B)\mathbf{y}_t = \Phi^*(B)(1 - B)\mathbf{y}_t + \Phi(1)\mathbf{y}_t$$

where  $\Phi^*(B) = \mathbf{I}_m - \sum_{i=1}^{p-1} \Phi_i^* B^i$  with  $\Phi_i^* = -\sum_{j=i+1}^p \Phi_j$ .

**Solution:**

Assume  $|\Phi(z)| = |I_m - \sum_{i=1}^p \Phi_i z^i| = 0$  has  $d < m$  unit root and the remaining roots outside the unit circle.

The rank of  $\Phi(1) = I_m - \sum_{i=1}^p \Phi_i$  is  $r$  and  $r = m - d$ .

We can decompose  $\Phi(1)$  as

$$\alpha\beta' = -\Phi(1) = -\mathbf{I}_m + \Phi_1 + \dots + \Phi_p$$

$\Phi(B)$  can be re-expressed as

$$\Phi(B) = \Phi^*(B)(1 - B) + \Phi(1)B$$

where  $\Phi^*(B) = \mathbf{I}_m - \sum_{i=1}^{p-1} \Phi_i^* B^i$  with  $\Phi_i^* = -\sum_{j=i+1}^p \Phi_j$ .

14. Consider the two dimensional vector AR(2) model:

$$\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} -0.2 & 0.1 \\ 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} + \begin{bmatrix} 0.8 & 0.7 \\ -0.4 & 0.6 \end{bmatrix} \begin{bmatrix} Z_{1,t-2} \\ Z_{2,t-2} \end{bmatrix}$$

where  $\{(a_{1t}, a_{2t})'\}$  is a sequence of i.i.d. standard normal random vectors. Show that it is a partially non-stationary AR model and give its cointegration vector.

**Solution:**

$$-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -0.2 & 0.1 \\ 0.5 & 0.2 \end{bmatrix} + \begin{bmatrix} 0.8 & 0.7 \\ -0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} -0.4 & 0.8 \\ 0.1 & -0.2 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.1 \end{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix}$$

Since  $|1 + \alpha_1 + \alpha_2\beta_1| = |1 - 0.4 - 0.2| < 1$ , so ...

And the cointegrating vector is  $\beta = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ .

15. Determine the stationarity and invertibility of the following two dimensional vector models and find their correlation matrix function,  $\rho_k$ , for  $k = \pm 1, \pm 2, \pm 3$ :

(a)  $(I - \Phi_1 B) Z_t = a_t$ , where  $\Phi_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.6 \end{bmatrix}$  and  $\Sigma = I$ ;

(b)  $(I - \Phi_1 B) Z_t = a_t$ , where  $\Phi_1 = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.8 \end{bmatrix}$  and  $\Sigma = I$ ;

(c)  $Z_t = (I - \Theta_1 B) a_t$ , where  $\Theta_1 = \begin{bmatrix} 0.6 & 1.2 \\ 0.4 & 0.8 \end{bmatrix}$  and  $\Sigma = I$ ;

**Solution:**

- Stationarity or invertibility condition:

$$\lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \Phi_1$$

- VAR(1) model:

$$\Sigma = \Gamma(0) - \Phi_1 \Gamma(0) \Phi_1'$$

$$\Gamma(k) = \begin{cases} \Gamma(-1)\Phi_1' + \Sigma, & \text{if } k = 0 \\ \Gamma(k-1)\Phi_1' = \Gamma(0)(\Phi_1')^k, & \text{if } k \geq 1 \end{cases}$$

$$\Gamma(k) = \Gamma'(-k) \geq 0$$

$$\rho(k) = D^{-1} \Gamma(k) D^{-1}$$

$$\text{where } D = \begin{bmatrix} \sigma_{11}(k) & 0 & \cdots & 0 \\ 0 & \sigma_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{mm}(k) \end{bmatrix}.$$

- solution

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \Gamma(0) - \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.6 \end{bmatrix} \Gamma(0) \begin{bmatrix} 0.8 & 0.1 \\ 0.3 & 0.6 \end{bmatrix} \\ \rho(k) &= \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix} \Gamma(k) \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix} = \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix} \Gamma(0) \begin{bmatrix} 0.8^k 0.1^k \\ 0.3^k 0.6^k \end{bmatrix} \begin{bmatrix} \sigma & -1 \\ 0 & \sigma_2^{-1} \end{bmatrix} \end{aligned}$$

16. Consider the process

$$\begin{aligned} Z_{1t} &= Z_{1,t-1} + a_{1t} + \theta a_{1,t-1} \\ Z_{2t} &= \phi Z_{1t} + a_{2t} \end{aligned}$$

where  $|\phi| < 1, |\theta| < 1$  and  $a_t = [a_{1t}, a_{2t}]' \sim N(0, \Sigma)$ . (a) Write the process in a vector form (b) Is the process  $[Z_{1t}, Z_{2t}]'$  stationary and invertible? (c) Write down the model for the vector of the first differences  $(I - B)Z_t$ , where  $Z_t = [Z_{1t}, Z_{2t}]'$ . Is the resulting model stationary and invertible?

**Solution:**

- (a)

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi & 0 \end{bmatrix} \begin{bmatrix} z_{1,t-1} \\ z_{2,t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \phi & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} \theta & 0 \\ \phi & 0 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

- (c)

$$\Delta z_{1t} = z_{1t} - z_{1,t-1} = a_1 + \theta a_{1+1}$$

$$\Delta z_{2t} = \phi \Delta z_{1t} + a_{2t} - a_{2,t-1} = \phi a_{1t} + \phi \theta a_{1,t-1} + a_{2t} - a_{2,t-1}$$

$$\begin{bmatrix} \Delta z_{1t} \\ \Delta z_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} \theta & 0 \\ \phi\theta & -1 \end{bmatrix} \cdot \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

$$\begin{vmatrix} \lambda - \theta & 0 \\ \phi - \phi\theta & \lambda + 1 \end{vmatrix} = (\lambda - \theta)(\lambda + 1) = 0$$

Since  $|\lambda| \geq 1$ , so it's not invertible;

And it's vector MA model, so it's stationary.

17. Show that the process  $y_t = z_t - z_{t-1}$  is weakly stationary, where  $z_t = 0.9z_{t-1} + a_t$  and  $\{a_t\}$  is white noise series.

**Solution:**

$$y_t = 0.9z_{t-1} + a_t - 0.9z_{t-2} - a_{t-1}$$

$$= 0.9y_{t-1} + a_t - a_{t-1}$$

$$\Rightarrow (1 - 0.9B)y_t = (1 - B)a_t$$

Since  $|\phi_1| = |0.9| < 1$ , then  $y_t = z_t - z_{t-1}$  is stationary.