Parameterized inapproximability: From PIH to Clique

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We are reviewing [1] in order to give a talk on it for my final pproject in the class I am fake auditing.

Contents

1 Introduction

There are strong inapproximability results for the k-clique problem. One gets straightaway from the PCP theorem and the FGLSS graph that there it is NP-hard to approximate within a constant factor, and it turns out for any $\varepsilon > 0$, it is NP-hard to approximate within a factor of $\frac{1}{n^{1-\varepsilon}}$. It is also known from the parameterization side that this problem is $\mathsf{W}[1]$-hard, and so an FPT-algorithm would imply that $\mathsf{P} = \mathsf{NP}$, with $\mathsf{W}[1]$ playing the analogue role to NP in parameterized complexity. One desires to rule out approximation algorithms that are based on approximation *and* parameterization, i.e. we want to be able to say that even in the parameterized setting, one cannot approximate k-Clique well.

It actually has been proven that a constant-factor approximation for k-Cliqueis $\mathsf{W}[1]$-hard, as was shown in [2]. This was quickly improved to show that it is $\mathsf{W}[1]$-hard to approximate within factor $k^{o(1)}$. Though they do not use a parameterized PCP theorem, the proof in [2] follows the usual PCP-based approximation formula. That is, they do the following

1. Reduce k-clique to an algebraic problem, k-vector sum
2. Using Hadamard codes with local testing and decoding, k-vector sum is reduced to 2CSP
3. The 2CSP instances are turned back into k-clique instances, but now with a gap

It has been suggested that the constant inapproximability of the parameterized binary CSP would serve as a parameterized PCP theorem, and it has thus been dubbed the *parameterized inapproximability hypothesis* (PIH henceforth.) The paper will serve as an explanation for these.

2 Definitions

We give a list of definitions here (only the unusual ones or ones I don't recognize):

Definition 2.0.1 — Algebra

Let q be a prime power. Then $\mathbb{F}_q$ is the unique finite field with q elements. If q itself is prime, then $\mathbb{F}_q$ is a *prime field*. Boldface letters $\mathbf{u}$ denote vectors in $\mathbb{F}_q^d$, and for every $v \in \mathbb{F}_q^d$ and $i \in [d]$ is denoted as $v[i]$.

Definition 2.0.2 — FPT Reduction

Let $P_1 = (Q_1, \kappa_1)$ and $P_2 = (Q_2, \kappa_2)$ be two parameterized problems. An FPT-reduction from P_1 to P_2 is an FPT-algorithm R such that for every instance x of P_1, the algorithm R computes an instance $R(x)$ of P_2 satisfying

- $x \in Q_1 \iff R(x) \in Q_2$
- $\kappa_2(R(x)) \leq g(\kappa_1(x))$ for a computable function $g : \mathbb{N} \to \mathbb{N}$ independent of x.

This means that $\mathsf{W}[1]$ is the set of all problems that can be reduced to k-Clique by an FPT-reduction.

3 Parameterized optimization and approximation

Instead of considering Max-Clique and the traditional k-clique problem, the paper will consider the multicolored k-clique problem:

Problem 3.0.1: Multi-colored k-clique
Input: A graph G such that $V(G) = V_1 \sqcup \cdots \sqcup V_k$ and each V_i is an independent set. Parameter: k Output: Yes if G contains a k-clique, no o.w.

If G is an instance of 3.0.1, then $\omega(G)$ (independence number of G) is at most k, since each V_i is an independent set. The reduction from k-clique is trivial; make k disjoint copies of $V(G)$ and add an edge between $\{u, i\}$ and $\{v, j\}$ if $\{u, v\} \in E(G)$ and $i \neq j$. This new graph G' has a k-clique if and only if G does, so Max-Clique is FPT iff 3.0.1 is.

References

[1] Yijia Chen and Bingkai Lin. Parameterized inapproximability: From clique to pih. *Computer Science Review*, 59:100834, 2026.

[2] Bingkai Lin. Constant approximating k-clique is w[1]-hard. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2021, page 1749–1756, New York, NY, USA, 2021. Association for Computing Machinery.