

Parameterized Inapproximability

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November 17, 2025

Clique is inapproximable due to the PCP Theorem, and we know that Clique is $W[1]$ -hard, but can we combine these approaches to get a FPT approximation algorithm? We introduce some important conjectures in the area:

Conjecture 1 (ETH). *There is no $2^{o(n)}$ time algorithm for solving 3SAT*

This is only for exact versions, so we also introduce the stronger statement that it is hard to even approximate 3SAT:

Conjecture 2 (Gap-ETH). *Given a 3SAT instance ϕ and any $\varepsilon > 0$, the $(1 - \varepsilon, \varepsilon)$ -3SAT problem is NP-hard.*

[?] suggests that unless 2 fails, there is no $\omega(k)$ fpt-approximation for the k -clique problem.

I Notation

- $\binom{S}{k}$ is the set of all k -subsets of S
- $\omega(G)$ is the size of a maximum clique in G
- Parameterized reduction – Let $(Q_1, \kappa_1), (Q_2, \kappa_2)$ be two parameterized problems. An fpt-reduction from (Q_1, κ_1) to (Q_2, κ_2) is an fpt-algorithm R such that for every instance x of Q_1 the algorithm R computes an instances $R(x)$ of Q_2 satisfying
 - $x \in Q_1 \iff R(x) \in Q_2$
 - $\kappa_2(R(x)) \leq g(\kappa_1(x))$ for a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ (independent of x).

- $W[1]$ consists of all parameterized problems that can be reduced to k -Clique by an fpt-reduction.
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Definition 1. *Let $\rho : \mathbb{N} \rightarrow \mathbb{N}$. Then an algorithm \mathcal{A} is an fpt-approximation of MAX-CLIQUE of ratio ρ if*

- *On an input graph G the algorithm \mathcal{A} computes a clique in G of size at least*

$$\frac{\omega(G)}{\rho(\omega(G))}$$

- *the running time of \mathcal{A} can be bounded by $f(\omega(W))n^{O(1)}$ for some computable $f : \mathbb{N} \rightarrow \mathbb{N}$*

II Parameterized optimization and approximation

Based on 1, there is always a trivial k approximation (output a single vertex), and this can be improved to $\lceil k/c \rceil$ for any constant $c \geq 1$ (naively enumerate all $\binom{n}{c} = O(n^c)$ subsets and check if they are cliques, but the authors conjecture that this is the best they can do:

Conjecture 3. *For every $p : \mathbb{N} \rightarrow \mathbb{N}$ such that $p(k) = o(k)$, there is no p -approximation for MAX-CLIQUE.*

III Constraint Satisfaction Problem

The parameterized inapproximability hypothesis states that MAX2CSP is constant-inapproximable by an FPT algorithm:

Conjecture 4 (Parameterized Inapproximability Hypothesis). *There is a constant $0 < \varepsilon < 1$ such that it is $W[1]$ -hard to distinguish between*

- *Satisfiable 2CSP instances*
- *2CSP instances where any assignment cannot satisfy ε -fraction of the constraints*

This amounts to showing that there is an FPT-reduction R from k -clique to MAX2CSP with the following properties:

- If G has a k -clique, then $R(G)$ is satisfiable
- If G has no k -clique, then no assignment satisfies an ε fraction of the constraints of $R(G)$

The important result is that assuming 4, it is $W[1]$ -hard to approximate k -clique within a constant factor, with the analysis following the FLGSS graph [2]. Also, we note that even though 4 may be for a specific ε' , we can use parallel repetition to achieve inapproximability for *any* $\varepsilon > 0$, which implies for any $c > 1$, there is no c -approximation.

IV Coding Theory

Fix a finite field \mathbb{F} . Note: This section is confusing, and to me (someone who is not studied in coding theory), they introduce a parallelized hadamard code which is in reality just matrix multiplication? Idk tho I will come back to this one

Definition 2 (Sidon Sets). *Let \mathbb{F} be a finite field and $d \geq 1$. A subset $S \subseteq \mathbb{F}^d$ is a linear Sidon Set if for all $r, r' \in \mathbb{F}^*$ and $u, u', v, v' \in S$ with $u \neq u'$ and $v \neq v'$, we have*

$$ru + r'u' = rv + r'v' \Rightarrow \{u, u'\} = \{v, v'\}$$

With some lemma for finding Sidon sets, we assume that for an instance of k -clique, $V(G)$ can be represented by a linear Sidon set.

IV.1 A simple combinatorial proof for the super constant inapproximability of clique

There are proofs of lower bounds of $k^{o(1)}$ -ratio inapproximability of k -clique that rely on various coding theory techniques, but recently there was a purely combinatorial proof [1] that does not rely on coding theory and the authors will present it here as a 2-step reduction, with an intermediary stop at a CSP.

IV.1.1 A forgotten definition

We introduce something that I should've introduced earlier, a way to parameterize the clique problem by something that is not $\omega(G)$; consider the multi-colored k -clique problem:

Definition 3 (MC- k -CLIQUE). *Instance:* A graph G and $k \geq 1$ such that $V(G) = \bigsqcup_{i \in [k]} V_i$, where each V_i is an independent set.

Parameter: k

Problem: Decide whether G has a k -clique.

The FPT-reduction from k -clique to 3 is trivial, i.e. make k disjoint copies of $V(G)$ and add an edge between (u, i) and (v, j) if $\{u, v\} \in E(G)$ and $i \neq j$, and clearly the original has a k -clique if and only if the multi-colored version does too.

IV.1.2 k -clique to CSP

Assume that we are dealing with the multi-colored version, and the CSP variable set will be the following:

$$X := \{x_{\bar{r}} \mid \bar{r} = (r_1, \dots, r_k) \in \mathbb{F}^k\}$$

where each variable is supposed to take the value

$$x_{\bar{r}} := \mathcal{H}_k^d(v_1, \dots, v_k)(\bar{r}) = r_1 v_1 + \dots + r_k v_k \in \mathbb{F}^d$$

for a purported clique on vertices $v_1 \in V_1, \dots, v_k \in V_k$. NOTE: This means that each $v \in V$ is represented by some $y \in \mathbb{F}^d$

To check whether an assignment really gives us a clique in G , we have the following tests as constraints:

- **Vertex Test:** for every $i \in [k], \bar{r} \in \mathbb{F}^k, r \in \mathbb{F}^*$, test whether

$$x_{\bar{r}+re_i} - x_{\bar{r}} \in rV_i$$

where $rV_i := \{ra \mid a \in V_i\}$.

- **Edge Test:** For every $\bar{r} = (r_1, \dots, r_k) \in \mathbb{F}^k, 1 \leq i < i' \leq k$, and $r, r' \in \mathbb{F}^*$, test whether

$$x_{\bar{r}+re_i+r'e_{i'}} - x_{\bar{r}} = rv + rv'$$

So far, the CSP instance has the following properties:

- Variables: $X = \{x_{\bar{r}} \mid (r_1, \dots, r_k) \in \mathbb{F}^k\}$
- Values: For each $x \in \mathbb{F}^k$, it can take a value $y \in \mathbb{F}^d$
- Constraints:
 - Vertex Test: For every $i \in [k], \bar{r} \in \mathbb{F}^k$, and $r \in \mathbb{F}^*$, test whether

$$x_{\bar{r}+re_i} - x_{\bar{r}} \in rV_i$$

where $rV_i := \{ra \mid a \in V_i\}$

References

- [1] Yijia Chen, Yi Feng, Bundit Laekhanukit, and Yanlin Liu. Simple combinatorial construction of the $k^{o(1)}$ -lower bound for approximating the parameterized k -clique, 2024.
- [2] Uriel Feige, Shafi Goldwasser, Laszlo Lovász, Shmuel Safra, and Mario Szegedy. Interactive proofs and the hardness of approximating cliques. *J. ACM*, 43(2):268–292, March 1996.