

# Parameterized Inapproximability

Whitaker Thompson

November 17, 2025

Clique is inapproximable due to the PCP Theorem, and we know that Clique is  $W[1]$ -hard, but can we combine these approaches to get a FPT approximation algorithm? We introduce some important conjectures in the area:

**Conjecture 1** (ETH). *There is no  $2^{o(n)}$  time algorithm for solving 3SAT*

This is only for exact versions, so we also introduce the stronger statement that it is hard to even approximate 3SAT:

**Conjecture 2** (Gap-ETH). *Given a 3SAT instance  $\phi$  and any  $\varepsilon > 0$ , the  $(1 - \varepsilon, \varepsilon)$ -3SAT problem is NP-hard.*

[?] suggests that unless 2 fails, there is no  $\omega(k)$  fpt-approximation for the  $k$ -clique problem.

## I Notation

- $\binom{S}{k}$  is the set of all  $k$ -subsets of  $S$
- $\omega(G)$  is the size of a maximum clique in  $G$
- Parameterized reduction – Let  $(Q_1, \kappa_1), (Q_2, \kappa_2)$  be two parameterized problems. An fpt-reduction from  $(Q_1, \kappa_1)$  to  $(Q_2, \kappa_2)$  is an fpt-algorithm  $R$  such that for every instance  $x$  of  $Q_1$  the algorithm  $R$  computes an instances  $R(x)$  of  $Q_2$  satisfying
  - $x \in Q_1 \iff R(x) \in Q_2$
  - $\kappa_2(R(x)) \leq g(\kappa_1(x))$  for a computable function  $g : \mathbb{N} \rightarrow \mathbb{N}$  (independent of  $x$ ).
- $W[1]$  consists of all parameterized problems that can be reduced to  $k$ -Clique by an fpt-reduction.
- 

**Definition 1.** *Let  $\rho : \mathbb{N} \rightarrow \mathbb{N}$ . Then an algorithm  $\mathcal{A}$  is an fpt-approximation of MAX-CLIQUE of ratio  $\rho$  if*

- On an input graph  $G$  the algorithm  $A$  computes a clique in  $G$  of size at least

$$\frac{\omega(G)}{\rho(\omega(G))}$$

- the running time of  $\mathcal{A}$  can be bounded by  $f(\omega(W))n^{O(1)}$  for some computable  $f : \mathbb{N} \rightarrow \mathbb{N}$

## II Parameterized optimization and approximation

Based on 1, there is always a trivial  $k$  approximation (output a single vertex), and this can be improved to  $\lceil k/c \rceil$  for any constant  $c \geq 1$  (naively enumerate all  $\binom{n}{c} = O(n^c)$  subsets and check if they are cliques, but the authors conjecture that this is the best they can do:

**Conjecture 3.** *For every  $p : \mathbb{N} \rightarrow \mathbb{N}$  such that  $p(k) = o(k)$ , there is no  $p$ -approximation for MAX-CLIQUE.*

## III Constraint Satisfaction Problem

The parameterized inapproximability hypothesis states that MAX2CSP is constant-inapproximable by an FPT algorithm:

**Conjecture 4** (Parameterized Inapproximability Hypothesis). *There is a constant  $0 < \varepsilon < 1$  such that it is  $W[1]$ -hard to distinguish between*

- Satisfiable 2CSP instances
- 2CSP instances where any assignment cannot satisfy  $\varepsilon$ -fraction of the constraints

This amounts to showing that there is an FPT-reduction  $R$  from  $k$ -clique to MAX2CSP with the following properties:

- If  $G$  has a  $k$ -clique, then  $R(G)$  is satisfiable
- If  $G$  has no  $k$ -clique, then no assignment satisfies an  $\varepsilon$  fraction of the constraints of  $R(G)$

The important result is that assuming 4, it is  $W[1]$ -hard to approximate  $k$ -clique within a constant factor, with the analysis following the FLGSS graph [2]. Also, we note that even though 4 may be for a specific  $\varepsilon'$ , we can use parallel repetition to achieve inapproximability for *any*  $\varepsilon > 0$ , which implies for any  $c > 1$ , there is no  $c$ -approximation.

## IV Coding Theory

Fix a finite field  $\mathbb{F}$ . Note: This section is confusing, and to me (someone who is not studied in coding theory), they introduce a parallelized hadamard code which is in reality just matrix multiplication? Idk tho I will come back to this one

**Definition 2** (Sidon Sets). *Let  $\mathbb{F}$  be a finite field and  $d \geq 1$ . A subset  $S \subseteq \mathbb{F}^d$  is a linear Sidon Set if for all  $r, r' \in \mathbb{F}^*$  and  $u, u', v, v' \in S$  with  $u \neq u'$  and  $v \neq v'$ , we have*

$$ru + r'u' = rv + r'v' \Rightarrow \{u, u'\} = \{v, v'\}$$

With some lemma for finding Sidon sets, we assume that for an instance of  $k$ -clique,  $V(G)$  can be represented by a linear Sidon set.

### IV.1 A simple combinatorial proof for the super constant inapproximability of clique

There are proofs of lower bounds of  $k^{o(1)}$ -ratio inapproximability of  $k$ -clique that rely on various coding theory techniques, but recently there was a purely combinatorial proof [1] that does not rely on coding theory and the authors will present it here as a 2-step reduction, with an intermediary stop at a CSP.

### IV.1.1 A forgotten definition

We introduce something that I should've introduced earlier, a way to parameterize the clique problem by something that is not  $\omega(G)$ ; consider the multi-colored  $k$ -clique problem:

**Definition 3** (MC- $k$ -CLIQUE). *Instance: A graph  $G$  and  $k \geq 1$  such that  $V(G) = \bigsqcup_{i \in [k]} V_i$ , where each  $V_i$  is an independent set.*

*Parameter:  $k$*

*Problem: Decide whether  $G$  has a  $k$ -clique.*

The FPT-reduction from  $k$ -clique to 3 is trivial, i.e. make  $k$  disjoint copies of  $V(G)$  and add an edge between  $(u, i)$  and  $(v, j)$  if  $\{u, v\} \in E(G)$  and  $i \neq j$ , and clearly the original has a  $k$ -clique if and only if the multi-colored version does too.

### IV.1.2 $k$ -clique to CSP

Assume that we are dealing with the multi-colored version, and the CSP variable set will be the following:

$$X := \{x_{\bar{r}} \mid \bar{r} = (r_1, \dots, r_k) \in \mathbb{F}^k\}$$

where each variable is supposed to take the value

$$x_{\bar{r}} := \mathcal{H}_k^d(v_1, \dots, v_k)(\bar{r}) = r_1 v_1 + \dots + r_k v_k \in \mathbb{F}^d$$

for a purported clique on vertices  $v_1 \in V_1, \dots, v_k \in V_k$ . NOTE: This means that each  $v \in V$  is represented by some  $y \in \mathbb{F}^d$

To check whether an assignment really gives us a clique in  $G$ , we have the following tests as constraints:

- **Vertex Test:** for every  $i \in [k], \bar{r} \in \mathbb{F}^k, r \in \mathbb{F}^*$ , test whether

$$x_{\bar{r}+re_i} - x_{\bar{r}} \in rV_i$$

where  $rV_i := \{ra \mid a \in V_i\}$ .

- **Edge Test:** For every  $\bar{r} = (r_1, \dots, r_k) \in \mathbb{F}^k, 1 \leq i < i' \leq k$ , and  $r, r' \in \mathbb{F}^*$ , test whether

$$x_{\bar{r}+re_i+r'e_{i'}} - x_{\bar{r}} = rv + rv'$$

So far, the CSP instance has the following properties:

- Variables:  $X = \{x_{\bar{r}} \mid (r_1, \dots, r_k) \in \mathbb{F}^k\}$
- Values: For each  $x \in \mathbb{F}^k$ , it can take a value  $y \in \mathbb{F}^d$
- Constraints:
  - Vertex Test: For every  $i \in [k], \bar{r} \in \mathbb{F}^k$ , and  $r \in \mathbb{F}^*$ , test whether

$$x_{\bar{r}+re_i} - x_{\bar{r}} \in rV_i$$

where  $rV_i := \{ra \mid a \in V_i\}$

## References

- [1] Yijia Chen, Yi Feng, Bundit Laekhanukit, and Yanlin Liu. Simple combinatorial construction of the  $k^{o(1)}$ -lower bound for approximating the parameterized  $k$ -clique, 2024.
- [2] Uriel Feige, Shafi Goldwasser, Laszlo Lovász, Shmuel Safra, and Mario Szegedy. Interactive proofs and the hardness of approximating cliques. *J. ACM*, 43(2):268–292, March 1996.