Optimization and Numerical Methods Solutions

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Preface

This project has two purposes. First, it is an attempt to organize my solutions to the course Optimization and Numerical Methods in a structured way. Second, it provides a justification to try and learn Quarto.

1 Chapter 1

No exercises.

2 Chapter 2

Chapter 2 is the first chapter that actually entails exercises.

2.1 Exercises (2.7 in the notes)

1. Consider the multinomial likelihood in Equation 2.1 for a model (for a two-way contingency table) assuming independence. Can you simplify the likelihood?

$$\sum_{j=1}^{R}\sum_{k=1}^{C} n_{jk}\ln(\pi_{jk}) \qquad \sum_{j=1}^{R}\sum_{k=1}^{C} \pi_{jk} = 1 \tag{2.1}$$

Solution

$$\begin{aligned}
\ell(\pi) &= \sum_{j=1}^{R}\sum_{k=1}^{C} n_{jk}\ln(\pi_{jk}) \\
&= \sum_{j=1}^{R}\sum_{k=1}^{C} n_{jk}\ln(\pi_{j+}\cdot\pi_{+k}) \\
&= \sum_{j=1}^{R}\sum_{k=1}^{C} n_{jk}\ln\pi_{j+} + n_{jk}\ln\pi_{+k} \\
&= \sum_{j=1}^{R} n_{j+}\ln\pi_{j+} + \sum_{k=1}^{C} n_{+k}\ln\pi_{+k}
\end{aligned} \tag{2.2}$$

2. In a mixed model, optimization is carried out using the marginal likelihood (the likelihood with the random effects integrated out). Define the marginal likelihood for the one-way random effects ANOVA model.

One-way random effects ANOVA with group-specific effects $\mu_j \sim \mathcal{N}(0, \sigma_\mu^2)$, and

$$y_{ij} = \beta + \mu_j + \epsilon_{ij},$$

with $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$, with a groups indexed j, and n_j individuals in every group.

Solution

So, the likelihood consists of two components. For the individuals within each group, we have

$$\prod_{i=1}^{n_j} \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(- \frac{(y_{ij} - \beta - \mu_j)^2}{2\sigma_\epsilon^2} \right),$$

whereas for the groups themselves, we have

$$\prod_{j=1}^{a} \frac{1}{\sqrt{2\pi\sigma_\mu^2}} \exp\left(- \frac{\mu_j^2}{2\sigma_\mu^2} \right).$$

Combining these components, and integrating out the random effects, we obtain the marginal likelihood

$$\prod_{j=1}^{a} \int \prod_{i=1}^{n_j} \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} \exp\left(- \frac{(y_{ij} - \beta - \mu_j)^2}{2\sigma_\epsilon^2} \right) \frac{1}{\sqrt{2\pi\sigma_\mu^2}} \exp\left(- \frac{\mu_j^2}{2\sigma_\mu^2} \right) d\mu_j.$$

3. Suppose you do a simple linear regression analysis using a t_ν-distribution for the residuals (density: $f_\nu(y) = C\sqrt{\lambda}\left(1+\frac{\lambda(y-\mu)^2}{\nu}\right)^{-\frac{\nu+1}{2}}$ where μ is the mean (for $\nu > 1$), λ is a scale parameter and C is a normalizing constraint that does not depend on μ or λ). Define the (log-)likelihood for n observations (y_i, x_i), such that $\mu_i = \beta_0 + \beta_1 x_i$.

Solution

$$L(\beta) = \prod_{i=1}^{n} C\sqrt{\lambda}\left(1 + \frac{\lambda(y_i - \beta_0 - \beta_1 x_i)^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

$$\ell(\beta) = N \ln C + \frac{N}{2}\ln\lambda - \sum_{i=1}^{n} \frac{\nu+1}{2} \ln\left(1 + \frac{\lambda(y_i - \beta_0 - \beta_1 x_i)^2}{\nu}\right)$$

3 Chapter 3

References