Topics in mathematical olympiad

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Abstract

This is an on going note about high school mathematical olympiad. The sources of problems are various: I created most problems in chapter 1 and some other problems in chapter 2, and the rest of the problems are from Putnam competitions, IMO, USMO, and other national mathematical olympiad. It may contain errors or typos. My ultimate goal is to write a book on high school mathematical olympiad.

Contents

Chapter 1

Some applications of polynomials to functional equations

1.1 Basic problems

Problem 1.1.1. Find all function $f : (0, \infty) \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) + A \tag{1.1.1}$$

$$f(x^n) = f(x)^n + B \tag{1.1.2}$$

for all $x, y \in (0, \infty)$, and A, B are real constant.
Solution
Assume that f is a function satisfying the hypothesis.
Let $p \in \mathbb{N}$ and $p > 0$, then for all $m \in \mathbb{Z}^+$, we have

$$f(mp) = mf(p) + (m-1)A$$

Now for all $x > 0$ and $m \in \mathbb{Z}^+$, from 1.1.2 we have

$$f((x+mp)^n) = f(x+mp)^n + B$$

$$\begin{aligned} f((x+pm)^n) &= f(\sum_{k=0}^{n} \binom{n}{k} (pm)^k x^{n-k}) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (pm)^k f(x^{n-k}) + m^n p^{n-1} f(p) + (\sum_{k=0}^{n-1} \binom{n}{k} (pm)^k - 1)A + (m^n p^{n-1} - 1)A \end{aligned} \tag{1.1.3}$$

$$
\begin{aligned}
f(x+mp)^n &= (f(x)+m(f(p)+A))^n + B \\
&= \sum_{k=0}^{n} \binom{n}{k} (m(f(p)+A))^k f(x)^{n-k} + B
\end{aligned} \tag{1.1.4}
$$

Case 1: $A \neq 0$.
The coefficient of m^{n-1} in $f((x+pm)^m)$ is $np^{n-1}f(x) + np^{n-1}A$.
The coefficient of m^{n-1} in $f(x+mp)^n$ is $n(f(p)+A)^{n-1}f(x)$.
Therefore,

$$np^{n-1}f(x) + np^{n-1}A = n(f(p)+A)^{n-1}f(x)$$

or

$$((f(p)+A)^{n-1} - p^{n-1})f(x) = p^{n-1}A$$

Because $A \neq 0$, we have $f(x)$ is a constant function, hence $f(x) = -A$ and $-A = (-A)^n + B$.
Case 2: $A = 0$. The coefficient of m^{n-2} in $f((x+mp)^n)$ is $\binom{n}{2}p^{n-2}f(x^2)$.
The coefficient of m^{n-2} in $f(x+mp)^n$ is $\binom{n}{2}f(p)^{n-2}f(x)^2$.
Thus

$$p^{n-2}f(x^2) = f(p)^{n-2}f(x)^2$$

So for all $x > a$

$$f(x^2) = cf(x)^2$$

where $c = (\frac{f(p)}{p})^{n-2}$.
f is also additive, so $f = ax$.
If $a \neq 0$ then $B = A = 0$ and $f(x) = ax$ with $a^n = a$.

Problem 1.1.2. Let $a > 0$, $A, B \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) + A \tag{1.1.5}$$

$$f(x^n) = f(x)^n + B \tag{1.1.6}$$

Solution Similar to Problem 1, just fix $p \in \mathbb{N}$ and $p > a$. Then we have 2 cases:
Case 1: $A \neq 0$ then the only solution is a constant function.
Case 2: $A = 0$ then we will have $f(x^2) = cf(x)^2$ where c is a constant for all $x > a$. This implies $f(x) \geq 0$ for all $x > a^2$ or $f(x \leq 0)$ for all $x > a^2$.
Fix $b > a^2$ then $f(rb) = rf(b)$ for all $\frac{a^2}{b} < r \in \mathbb{Q}$.
If $c > 0$ then $f(x) \geq 0$ for all $x > a^2$.
For $x > a$, let $x = ub$.

Pick a sequence $\{u_k\}_{k=1,2,..}$ of rational numbers increasing to u with $u_1 b > a$. Fix k and choose $M_k \in \mathbb{N}$ large enough such that $M_k(u-u_k)b > a^2$ then

$$M_k(f(ub) - f(u_k b)) = f(M_k(u-u_k)b) \geq 0$$

So $f(x) = f(ub) \geq f(u_k b) = u_k f(b)$
Taking the limit we get $f(x) \geq uf(b)$.
Similarly, by picking a sequence of rational numbers decreasing to u, we have $f(x) \leq uf(b)$.
Therefore, $f(x) = uf(b) = \frac{f(b)}{b}x$ for all $x > a$.
So $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.
This forces $B = 0$ and $\alpha^n = \alpha$ If $c < 0$, we also get $f(x) = \beta x$ for some $\beta \in \mathbb{R}$.

Problem 1.1.3. Let $a > 0$, $n \in \mathbb{Z}$, $n > 1$. $A, B \in \mathbb{R}$, $A > 0$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x+y) - f(x) - f(y) = \text{constant}$$

$$f(Ax^n) = Bf(x)^n + \text{constant}$$

Solution Let $f(x+y) - f(x) - f(y) = \alpha$ and $f(Ax^n) - Bf(x)^n = \beta$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.
Fix $p \in \mathbb{N}$ and $p > a$. Then again, for all $m \in \mathbb{Z}^+$, we have

$$Bf(x+mp)^n = B(f(x) + mf(p) + m\alpha)^n = B(f(x) + mb)^n$$

and

$$\begin{aligned} f(A(x+mp)^n) &= f(A(\sum_{k=0}^{n-1}\binom{n}{k}(mp)^k x^{n-k}) + (mp)^n) \\ &= \sum_{k=0}^{n-1}\binom{n}{k}(mp)^k f(Ax^{n-k}) + m^n p^{n-1} f(Ap) + \\ &+ (\sum_{k=0}^{n-1}\binom{n}{k}(mp)^k + m^n p^{n-1} - 1)\alpha \end{aligned}$$

Comparing the coefficients of m^n, we have

$$Bb^n = p^{n-1} f(Ap)$$

Comparing the coefficients of m^{n-1}, we have

$$b^{n-1}Bf(x) = p^{n-1}(f(Ax) + \alpha)$$

Comparing the coefficients of m^{n-2}, we have

$$b^{n-2}Bf(x)^2 = p^{n-2}f(Ax^2) + p^{n-2}\alpha$$

If $b = f(p) + \alpha = 0$, then

$$p^{n-2}(f(Ax^2) + \alpha) = 0$$

for all $x > a$, thus $f(x) = -\alpha$ for all $x > Aa^2, x > a$.
Now fix $y > a$, then pick x large enough, we will have

$$f(x + y) = -\alpha$$

and $f(x) = -\alpha$ hence $f(y) = -\alpha$ Therefore $f(x) = -\alpha$ for all $x > a$.
So $-\alpha = B(-\alpha)^n + \beta$.
If $b = f(p) + \alpha \neq 0$, then

$$p^{n-1}f(Ap) = Bb^n \neq 0$$

hence $f(Ap) \neq 0$.
We have

$$p^{n-1}f(Ap)f(x) = Bb^n f(x) = bp^{n-1}(f(Ax) + \alpha)$$

Hence $f(Ap)f(x) = f(Ax) + \alpha$ Now again, for all $y > a$ then

$$f(x + my) = f(x) + m(f(y) + \alpha)$$

$$f(A(x + my)) = f(Ax) + m(f(Ay) + \alpha)$$

Therefore, we have

$$f(Ap)(f(x) + m(f(y) + \alpha)) = f(Ax) + m(f(Ay) + \alpha) + \alpha$$

This holds for every $m \in \mathbb{Z}^+$, so

$$f(Ap)(f(y) + \alpha) = f(Ay) + \alpha$$

But

$$f(Ap)f(y) = f(Ay) + \alpha$$

so

$$f(Ap)\alpha = 0$$

Because $f(Ap) \neq 0$, we have $\alpha = 0$.
Therefore

$$f(Ax^2) = B(\frac{b}{p})^{n-2} f(x)^2$$

for all $x > a$.
This shows that $f(x) = 0$ for all $x > a$ or $f(x) \leq 0$ for all $x > a$ or $f(x) \geq 0$ for all $x > a$.
By similar argument to the previous problem, we have $f(x) = cx$ for some real constant c.

Problem 1.1.4. Let $a > 0$, $m, n \in \mathbb{N}^*$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) \tag{1.1.7}$$

$$f(x^{n+m}) = x^n f(x^m) \tag{1.1.8}$$

for all $x \in (a, \infty)$.
Solution
Let $k = n + m$ Let f be a function satisfying the problem. Fix $p \in \mathbb{N}$ and $p > a$. For all $r \in \mathbb{N}^*$, we have

$$f(rp) = rf(p)$$

Now

$$\begin{aligned} f((x+rp)^k) &= f((rp)^k + \sum_{i=0}^{k-1} \binom{k}{i} (rp)^i x^{k-i}) \\ &= r^k p^{k-1} f(p) + \sum_{i=0}^{k-1} \binom{k}{i} (rp)^i f(x^{k-i}) \end{aligned} \tag{1.1.9}$$

Regard $f((x + rpk)^k)$ as a polynomial in r, then the coefficient of r^{k-1} is $kp^{k-1}f(x)$

$$(x+rp)^n f((x+rp)^m) = (\sum_{i=0}^{n} \binom{n}{i} (rp)^i x^{n-i})(\sum_{i=0}^{m-1} \binom{m}{i} (rp)^i f(x^{m-i}) + r^m p^{m-1} f(p)) \tag{1.1.10}$$

Regard $(x+rp)^n f((x+rp)^m)$ as a polynomial in r, then the coefficient of $r^{n+m-1} = r^{k-1}$ is $p^{k-1}mf(x) + p^{k-2}nxf(p)$. Now from

$$f((x+rp)^k) = (x+rp)^n f((x+rp)^m)$$

for all $r \in \mathbb{N}^*$, we must have

$$kp^{k-1}f(x) = p^{k-1}mf(x) + p^{k-2}nxf(p)$$

for all $x > a$. So $f(x) = cx$ for all $x > a$ and c is a constant.

Problem 1.1.5. Let $a > 0$, $m, n \in \mathbb{N}^*$, find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) \tag{1.1.11}$$

$$f(x^{n+m}) = x^n f(x)^m \tag{1.1.12}$$

for all $x \in (a, \infty)$.

Solution

Let $k = m + n$.

Similar the Problem 2, just fix $p > a$ and $p \in \mathbb{N}^*$. Regard $f((x + rpk)^k)$ as a polynomial in r, then the coefficient of r^{k-1} is $kp^{k-1}f(x)$.

Regard $(x+rp)^n f((x+rp))^m$ as a polynomial in r, then the coefficient of $r^{n+m-1} = r^{k-1}$ is $p^n f(p)^{m-1} mf(x) + p^{n-1}nxf(p)^n$. So we have

$$(m+n)p^{m+n-1}f(x) = mp^n f(p)^{m-1}f(x) + np^{n-1}f(p)^n x$$

or

$$(m+n)p^m f(x) = pmf(p)^{m-1}f(x) + nf(p)^n x$$

or

$$p((m+n)p^{m-1} - mf(p)^{m-1})f(x) = nf(p)^n x$$

If $f(p) = 0$ then $f(x) = 0$ for all $x > a$.

If $f(p) \neq 0$ then we have $f(x) = cx$ for all $x > a$ and some constant $c \in \mathbb{R}$ satisfying $c = c^m$.

So $f(x) = 0$ or $f(x) = cx$ with $c^{m-1} = 1$.

Problem 1.1.6. Let $a > 0$, $m, n \in \mathbb{N}^*$,$A > 0$, $B \in \mathbb{R}$, find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x + y) - f(x) - f(y) = \text{constant}$$

and

$$f(Ax^n) - Bx^n f(x^m) = \text{constant}$$

for all $x, y > a$.

Solution Solution uses the same method.

Problem 1.1.7. Let $p, q \in \mathbb{Z}^+$, $a > 0$. $A, B \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x + y) = f(x) + f(y) + A \tag{1.1.13}$$

$$f(x^p) = \frac{f(x)^{p+q}}{x^q} + B \tag{1.1.14}$$

for all $x, y > a$.

Solution Let $D = (a, \infty)$.

As before, fix $r > a$, $r \in \mathbb{N}$.

We have $f(mr) = mf(r)$ for all $m \in \mathbb{Z}^+$.

We have

$$f(x)^{p+q} = x^q f(x^p) - Bx^q$$

For all $m \in \mathbb{Z}^+$ then

$$f(x + mr) = f(x) + m(f(r) + A) = f(x) + mb$$

with $b = f(r) + A$.

Thus

$$f(x + mr)^{p+q} = (f(x) + mb)^{p+q} = \sum_{k=0}^{p+q} \binom{p+q}{k} m^k (b^k f(x)^{p+q-k})$$

as a polynomial in m, then the coefficient of m^{p+q-1} is $(p+q)b^{p+q-1}f(x)$ On the other hand,

$$\begin{aligned} f((x + mr)^p) &= f(\sum_{k=0}^{p-1} \binom{p}{k})(mr)^k x^{p-k} + (mr)^p) \\ &= \sum_{k=0}^{p-1} \binom{p}{k} (mr)^k f(x^{p-k}) + m^p r^{p-1} f(r) + (m^p r^{p-1} + \sum_{k=0}^{p-1} \binom{p}{k} (mr)^k - 1)A \end{aligned}$$

Thus

$$(x+mr)^q(f((x+mr)^p)-B) = (x+mr)^q(\sum_{k=0}^{p-1}\binom{p}{k}(mr)^k f(x^{p-k}) + m^p r^{p-1} f(r) + (m^p r^{p-1}$$
$$+\sum_{k=0}^{p-1}\binom{p}{k}(mr)^k - 1)A)$$

As a polynomial in m, the coefficient of m^{p+q-1} is

$$r^q p r^{p-1}(f(x)+A) + qxr^{p+q-2}f(r)$$

So we have

$$(p+q)(f(r)+A)^{p+q-1}f(x) = pr^{p+q-1}(f(x)+A) + qxr^{p+q-2}f(r)$$

Thus

$$\alpha f(x) = \beta x$$

where $\alpha = (p+q)(A+f(r))^{p+q-1} - pr^{p+q-1}$ and $\beta = qxr^{p+q-2}f(r)$.
If $\exists r \in \mathbb{N}, r > a$ with $f(r) \neq 0$ then we have $f(x) = cx$ for all $x > a$ for some real constant c, this forces $A = B = 0$.
If $f(r) = 0$ for all $r \in \mathbb{N}$ with $r > a$, then we have

$$\alpha = (p+q)A^{p+q-1}$$

and

$$\beta = 0$$

so $(p+q)A^{p+q-1}f(x) = 0$ for all $x > a$. If $A \neq 0$ then $f(x) = 0$ for all $x > a$, so $B = 0$.
If $A = 0$ then we have

$$f(x+y) = f(x) + f(y)$$

for all $x, y > a$.
then

$$f(x+mr) = f(x) + mf(r) = f(x)$$

Thus

$$f(x+mr)^{p+q} = f(x)^{p+q}$$

$$\begin{aligned} f((x+mr)^p) &= f(\sum_{k=0}^{p}\binom{p}{k}(mr)^k x^{p-k}) \\ &= \sum_{k=0}^{p-1}\binom{p}{k}(mr)^k f(x^{p-k}) + f((mr)^p) \\ &= \sum_{k=0}^{p-1}\binom{p}{k}(mr)^k f(x^{p-k}) \end{aligned}$$

So we have

$$(x+mr)^q(\sum_{k=0}^{p-1}\binom{p}{k}(mr)^k f(x^{p-k}) - B) = f(x)^{p+q}$$

for all $x > a$. The right hand side if a polynomial in m, so the leading coefficient is 0, hence $r^q\binom{p}{p-1}r^{p-1}f(x) = 0$, hence $f(x) = 0$.
In conclusion, we have the following cases
Case 1: $f(x) = cx$ for some constant $c \in R$, this forces $A = B = 0$.
Case 2: $f(x) = c$ for some constant $c \in \mathbb{R}$, this forces $c = B = 0$ and hence $A = 0$.

Problem 1.1.8. Let $a > 0$, $m, n \in \mathbb{N}^*$,$A > 0$, $B \in \mathbb{R}$, find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x+y) - f(x) - f(y) = \text{constant}$$

and

$$f(Ax^n) - B\frac{f(x^{m+n})}{x^m} = \text{constant}$$

for all $x, y > a$.

1.2 Some applications

Problem 1.2.1. (Romanian 2009) Find all function $f : \mathbb{R} \to R$ such that

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)$$

for all $x, y \in \mathbb{R}$.
This is a special case of the following problem.

Problem 1.2.2. Let $a > 0$, $m, n \in \mathbb{Z}^+$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that
$$f(x^{n+m} + y^{n+m}) = x^n f(x^m) + y^n f(y^m)$$

Problem 1.2.3. (China TST ??) Let $n \in \mathbb{Z}$, $n > 1$. Find all function $f : \mathbb{R} \to R$ such that
$$f(x^n + f(y)) = f(x)^n + y$$
for all $x, y \in \mathbb{R}$.

Problem 1.2.4. Let $a > 0$, $A \in \mathbb{R}$, $n \in \mathbb{Z}$ and $n \geq 2$. Find all function $f : (a, \infty) \to (a - a^n, \infty)$ such that
$$f(x^n + f(y)) = f(x)^n + y + A$$

Problem 1.2.5. Let $a > 0$, $m, n \in \mathbb{Z}^+$, $A \in \mathbb{R}$. Find all function $f : (a, \infty) \to (a - a^n, \infty)$ such that
$$f(x^{n+m} + f(y)) = x^n f(x^m) + y + A$$

Problem 1.2.6. Let $a > 0$, $m, n \in \mathbb{Z}^+$. $S = \{n \in \mathbb{N} : s > a\}$. Find all function $f : S \to \mathbb{R}$ such that
$$f(x^{n+m} + y^{n+m}) = x^n f(x^m) + y^n f(y^m)$$

Problem 1.2.7. Let $a > 0$, $m, n \in \mathbb{Z}^+$. $S = \{n \in \mathbb{N} : s > a\}$. Find all function $f : S \to \mathbb{R}$ such that
$$f(x^{n+m} + f(y)) = x^n f(x^m) + y$$

Problem 1.2.8. Let $a > 0$, $m, n \in \mathbb{Z}^+$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that
$$f(x^m + y^m + z^m) = \frac{f(x^{n+m})}{x^m} + \frac{f(y^{n+m})}{y^m} + \frac{f(z^{m+n})}{z^m}$$

Problem 1.2.9. Let $a > 0$, $m, n \in \mathbb{Z}^+$. $S = \{n \in \mathbb{N} : s > a\}$. Find all function $f : S \to \mathbb{Z}^+$ such that
$$f(x^n + f(y)) = \frac{f(x^{n+m})}{x^m} + y$$

Problem 1.2.10. Let $m, n \in \mathbb{Z}^+$. Find all function $f : (0, \infty) \to \mathbb{R}$ such that

$$f(\frac{1}{x^m} + \frac{1}{y^m} + \frac{1}{z^m}) = \frac{f(x^m)}{x^{n+m}} + \frac{f(y^n)}{y^{n+m}} + \frac{f(z^n)}{z^{n+m}}$$

Problem 1.2.11. Let $p, q, r \in \mathbb{Z}^+$ and $a \in \mathbb{R}^+$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x^p + y^q + z^r) = \frac{f(x)^{p+q}}{x^q} + \frac{f(y)^{q+r}}{y^r} + \frac{f(z)^{r+p}}{z^p}$$

Solution Assume that f is such a function. Fix $b > a$.
Then let $y = z = b$, we have

$$f(x^p + A_1) = \frac{f(x)^{p+q}}{x^q} + A$$

where $A_1 = b^q + b^r$ and $A = \frac{f(b)^{q+r}}{b^r} + \frac{f(b)^{r+p}}{b^p}$.
Similarly, let $B_1 = b^r + b^p, C_1 = b^p + b^q$ and $B = \frac{f(b)^{r+p}}{b^p} + \frac{f(b)^{p+q}}{b^q}$, $C = \frac{f(b)^{p+q}}{b^q} + \frac{f(b)^{q+r}}{b^r}$. Then we have

$$f(y^q + B_1) = \frac{f(y)^{q+r}}{y^r} + B$$

$$f(z^r + C_1) = \frac{f(z)^{r+p}}{z^p} + C$$

So we have

$$f(x^p + y^q + z^r) = f(x^p + A_1) + f(y^q + B_1) + f(z^r + C_1) + D$$

where $D = -A - B - C$ for all $x, y, z > a$.
Let $T > a^p, a^q, a^r$ then

$$f(x + y + z) = f(x + A_1) + f(y + B_1) + f(z + C_1)$$

for all $x, y, z > T$.
Fix $z > T$, then for all $x > T + A1$ and $y > T$, we have

$$f(x+A_1)+f(y+B_1)+f(z+C_1) = f(x+y+z) = f(x-A1+y+A_1+z) = f(x)+f(y+A_1+B_1)+f(z+C_1$$

Therefore

$$f(x) - f(x - A_1) = f(y + A_1 + B_1) - f(y + B_1)$$

for all $x > A_1 + T$ and $y > T$. So for all $x, y > T + A_1, T + B_1$ then $f(x + A_1) - f(x) = f(y + A_1) - f(y)$ so for all $x > T + A_1$ then $f(x + A_1) = f(x) + D_1$ for some $C_1 \in \mathbb{R}$.
Now fix $\alpha \in R$ with $\alpha > T + A_1, T + B_1, T + C_1, 1$ then by a similar argument as above, we have for all $y, z > \alpha$ then

$$f(y + B_1) = f(y) + D_2$$

$$f(z + C_1) = f(z) + D_3$$

for some real constant D_2, D_3.
Then we will have

$$f(x + y + z) = f(x) + f(y) + f(z) + E$$

for all $x, y, z > \alpha$ and $E = D + D_1 + D_2 + D_3$.
Also, for $\alpha > 1$ then if $x > \alpha$ then $x^p\alpha$, hence

$$f(x^p + A_1) = f(x^p) + D_1$$

Therefore,

$$f(x^p) = \frac{f(x)^{p+q}}{x^q} + F$$

with $F = A - D_1$
Now we go back to a basic problem: find all function $f : (\alpha, \infty) \to \mathbb{R}$ such that

$$f(x + y + z) = f(x) + f(y) + f(z) + E$$

and

$$f(x^p) = \frac{f(x)^{p+q}}{x^q} + F$$

Now replace z by $z + t$ in $f(x + y + z + t)$ we have

$$f(x + y + z + t) = f(x + y) + f(z) + f(t) + E = f(x) + f(y) + f(z + t) + E$$

for all $x, y, z, t > \alpha$.
Therefore

$$f(x + y) - f(x) - f(y) = f(z + t) - f(z) - f(t)$$

for all $x, y, z, t > \alpha$.
so

$$f(x+y) - f(x) - f(y) = G$$

for all $x, y > \alpha$ and G is a real constant.
Then we have

$$f(x+y) - f(x) - f(y) = \text{constant}$$

and

$$f(x^p) - \frac{f(x)^{p+q}}{x^q} = \text{constant}$$

for all $x, y > \alpha$. So the only possible functions are

$$f(x) = cx$$

for some $c \in \mathbb{R}$.
Plug in the original equation

$$f(x^p + y^q + z^r) = \frac{f(x)^{p+q}}{x^q} + \frac{f(y)^{q+r}}{y^r} + \frac{f(z)^{r+p}}{z^p}$$

we have $cx^p + cy^q + cz^r = c^{p+q}x^q + c^{q+r}y^q + c^{r+p}z^r$ for all $x, y, z > a$.
Hence $c = c^{p+q} = c^{q+r} = c^{r+p}$.
If $c \neq 0$ then $c^p = c^q = c^r = 1$,so $c = 1$.
So the only possible functions are $f(x) = 0$ for all $x > a$ or $f(x) = x$ for all $x > a$.

Problem 1.2.12. Let $p, q, r \in \mathbb{Z}^+$ and $a \in \mathbb{R}^+$, $A \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x^p + y^q + z^r) = \frac{f(x)^{p+q}}{x^q} + \frac{f(y)^{q+r}}{y^r} + \frac{f(z)^{r+p}}{z^p} + A$$

Solution Solve this problem in the same way as the previous problem, then
If $A \neq 0$, there is no function.
If $A = 0$ then $f(x) = 0$ for all $x > a$ or $f(x) = x$ for all $x > a$.

Problem 1.2.13. Let $A, B, C \in \mathbb{Z}^+$, $a, b, c, u \in \mathbb{R}^+$, $\alpha, p, q, r \in \mathbb{R}$. Find all function $f : (u, \infty) \to \mathbb{R}$ such that

$$f(ax^A + by^B + cz^C) = pf(x)^A + qf(y)^B + rf(z)^C + \alpha$$

Solution As before, just fix some $v > u$. Then let $x = y = z = v$ respectively, we will have for all $x, y, z > u$ then

$$f(ax^A + D_1) = pf(x)^A + E_1$$

$$f(by^B + D_2) = qf(y)^B + E_2$$

$$f(cz^C + D_3) = rf(z)^C + E_3$$

where $D_1, D_2, D_3 > 0$ and $E_2, E_2, E_3 \in \mathbb{R}$.
so from the original equation, we will have

$$f(ax^A + by^B + cz^C) = f(ax^A + D_1) + f(bx^B + D_2) + f(cx^C + D_3) + E_4$$

where $E_4 = -E_1 - E_2 - E_3$ (the values of E_4 does not matter much).
Just take $w > 0$ large enough then from the above equation, we have

$$f(x + y + z) = f(x + D_1) + f(y + D_2) + f(z + D_3)$$

for all $x, y, z > w$.
Now using the standard trick by writing

$$\begin{aligned} x + y + z + D_1 &= x + (y + D_1) + z \\ &= (x + D_1) + y \end{aligned}$$

we have

$$f(x+D_1)+f(y+D_1+D_2)+f(z+D_3) = f(x+2D_1)+f(y+D_2)+f(z+D_3)$$

for all $x, y, z > w$.
Hence

$$f(x + 2D_1) - f(x + D_1) = f(y + D_1 + D_2) - f(y + D_2)$$

for all $x, y > w$.
Now just replace w by

$$w_1 = w + D_1 + D_2 + D_2$$

, then

$$f(x + D_1) - f(x) = f(y + D_1) - f(y)$$

for all $x, y > w_1$.
Therefore,

$$f(x + D_1) - f(x) = \text{constant}$$

for all $x > w_1$.
Similarly,

$$f(y + D_2) - f(y) = \text{constant}$$

$$f(z + D_3) - f(z) = \text{constant}$$

for all $y, z > w_1$.
Thus

$$f(x + y + z) = f(x) + f(y) + f(z) + E$$

for all $x, y, z > w_1$ and E is a real constant.
Now just replace $z = x_1 + y_1$ for $x_1, y_1 > w_1$ then

$$\begin{aligned} f(x + y + x_1 + y_1) &= f(x) + f(y) + f(x_1 + y_1)) + E \\ &= f(x + y) + f(x_1) + f(y_1) + E \end{aligned}$$

for all $x, y, x_1, y_1 > w_1$.
Therefore,

$$f(x + y) - f(x) - f(y) = \text{constant}$$

for all $x, y > w_1$.
Now be enlarging w_1, we will have $ax^A > w_1$ for all $x > w_1$, thus we will have

$$f(ax^A + D_1) = f(ax^A) + \text{constant}$$

for all $x > w_1$.
Therefore,

$$f(ax^A) = pf(x)^A + \text{constant}$$

We go back a basic problem: Let $a, u > 0, A \in \mathbb{Z}^+$, $p \in \mathbb{R}$. Find all function $f : (u, \infty) \to \mathbb{R}$ such that

$$f(x + y) - f(x) - f(y) = \text{constant}$$

and

$$f(ax^A) = pf(x)^A + \text{constant}$$

Problem 1.2.14. Let $p, q, r \in \mathbb{Z}^+$ and $a, A, B, C \in \mathbb{R}^+$, and $A_1, B_2, C_2\alpha \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(Ax^p + By^q + Cz^r) = A_1 f(x)^p + B_1 f(y)^q + C_1 f(z)^r + \alpha$$

for all $x, y, z > a$.
Special case:
1,Find all function $f : (2019, \infty) \to \mathbb{R}$ such that

$$f(x^3 + 2y^5 + 3z^7) = f(x)^3 - 2f(y)^5 + 3f(z)^7$$

Problem 1.2.15. Let $p, q, r \in \mathbb{Z}^+$ and $a, A, B, C \in \mathbb{R}^+$, and $A_1, B_2, C_2\alpha \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(Ax^p + By^q - Cz^r) = A_1 f(x)^p + B_1 f(y)^q + C_1 f(z)^r + \alpha$$

for all $x, y, z > a$ such that $Ax^B y^q - Cz^r > a$
Solution Just fix $z_0 > a$, and we can assume that $x, y > M > a$ where M is large enough.
We will have

$$f(Ax^p + a_1) = A_1 f(x)^p + b_1$$
$$f(By^q + a_2) = B_1 f(y)^q + b_2$$

then we will have $f(x + y - Cz_0^r) = f(x + a_1) + f(y + b_1) + D$ with D is a constant (notice that z_0 is a constant).
Then by the usual trick, write $x+y$ as $x+y = x-a_1+y+a_1 = x+b_1+y-b_1$ then we will have

$$f(x + a_1) = f(x) + \text{constant}$$

and

$$f(y + b_1) = f(y) + \text{constant}$$

So then we have

$$f(x + y - E) = f(x) + f(y) + \text{constant}$$

Replace x by $x + E$ and y by $y + E$ respectively, we will have

$$f(x + E) + f(y) = f(x) + f(y + E)$$

hence

$$f(x + E) = f(x) + \text{constant}$$

Therefore, we again have

$$f(x+y) = f(x) + f(y) + \text{constant}$$

and

$$f(Ax^p) = A_1 f(x)^p + \text{constant}$$

From our basic problem, the only possible solution is $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.

Example 1.2.16. *Find all function $f : (2019, \infty) \to \mathbb{R}$ such that*

$$f(2x^3 - 3y^4 + 4z^5) = 2f(x)^3 - 3f(y)^4 + 4f(z)^5$$

for all $x, y, z > 2019$ and $2x^3 - 3y^4 + 4y^5 > 2019$.
Solution *Just fix $y = a > 2019$ then we have*

$$f(2x^3 + 4z^5 - A) = 2f(x)^3 + 4f(z)^5 - B$$

for all $x, y > b$ with b large enough, then

$$f(2x^3 + \mathit{constant}) = 2f(x^3) + \mathit{constant}$$

and

$$f(4z^5 + \mathit{constant}) = 4f(z)^5 + \mathit{constant}$$

Hence

$$f(2x^3 + 4z^5 - A) = f(2x^3 + \mathit{constant}) + f(4z^5 + \mathit{constant}) + \mathit{constant}$$

So we go back to a basis problem:

$$f(x + z - A) = f(x + B) + f(z + C) + D$$

and

$$f(2x^3 + B) = 2f(x)^3 + E$$

where A, B, C, D, E are constant.
just replace z by $z + A$, we have

$$f(x + z) = f(x + B) + f(z + A + C) + D$$

so we only have to look at

$$f(x+z) = f(x+B) + f(z+C) + D$$

and

$$f(2x^3+B) = 2f(x)^3 + E$$

Using the usual trick,

$$f(x) + f(z+B+C) + D = f(x-B+z+B) = f(x+B) + f(z+C) + D$$

hence

$$f(x+B) - f(x) = f(z+B+C) - f(z+C)$$

so

$$f(x+B) - f(x) = \textit{constant}$$

hence

$$f(2x^3+B) = f(2x^3) + \textit{constant}$$

Similar

$$f(z+C) - f(z) = \textit{constant}$$

This show that

$$f(x+z) = f(x) + f(z) + \textit{constant}$$

and

$$f(2x^3) = 2f(x)^3 + \textit{constant}$$

Thus

$$f(x) = cx + d$$

so

$$c(2x^3) + d = 2(cx+d)^3 + \textit{constant}$$

$c = 0$ *then* $f(x)$ *is a constant function and*

$$d = 2d^3 - 3d^4 + 4d^5$$

$c \neq 0$ *then comparing the coefficients of* x^2 *forces* $d = 0$ *and* $c = \pm 1$.
Plug in the original equation, we get $c = 1$,*so* $f(x) = x$ *or* $f(x) = \textit{constant}$ *for* x *large enough.*
Now just fix $x > 2019$, *and pick* y, z *large enough then* $2x^3 - 3y^4 + 4z^5$ *is large enough, hence we can compute*

$$2f(x)^3 = f(2x^3 - 3y^4 + 4z^5) + 3f(y)^4 - 4f(z)^5$$

So $f(x) = x$ *for all* $x > 2019$ *or* $f(x) = \textit{constant}$ *for all* $x > 2019$.

Example 1.2.17. *Find all function $f : (2019, \infty) \to \mathbb{R}$ such that*

$$f(2x^2 - 3y^3 + 6z^4) = 2f(x^2) - 3f(y)^3 + 6f(z)^4$$

for all $x, y, z > 2019$ such that $2x^2 - 3y^3 + 6z^4 > 2019$.
Solution *$f(x) = x$ or $f(x) = c$ with $c = 2c^2 - 3c^3 + 6c^4$ so $c = 0$ or $c = \frac{1}{2}$*

Example 1.2.18. *Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that*

$$f(2x^2 - 3y^3 + 6z^4) = 2f(x)^2 + g(y) + 6zf(z)^3$$

Solution *Just fix y and use the usual method, we will get $f(x) = x$ or $f(x) = 0$.*
Then $g(y) = -3y^3$.

Example 1.2.19. *Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that*

$$f(2x^2 - 3y^3 + 6z^4) = g(x) + h(y) + 6zf(z)^3$$

Solution*Fix $y = a > 2019$. Then we go back to our basis problem*

$$f(2x^2 + 6z^4 - A) = g(x) + 6zf(z)^3 + B$$

where A, B are constant. So fix $x, z = b$ $2b^2 - A, 6b^4 - A > 2019, 4$, then

$$g(x) = f(2x^2 + 6b^4 - A) - 6bf(b)^3 - B$$

and

$$6zf(z)^3 = f(6z^4 + 2b^2 - A) - g(b) - B$$

Then now

$$f(2x^2 + 6z^4 - A) = f(2x^2 + C) + f(6z^4 + D) + E$$

and

$$f(6z^4 + D) = 6zf(z)^3 + F$$

This forces f is a linear function on x. Hence we can compute value of g, h.

Example 1.2.20. *Let P is a polynomial with real coefficients. Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that*

$$f(2x^2 + P(y) + 6z^4) = 2f(x)^2 + g(y) + 6zf(z)^3$$

for all $x, y, z > 2019$ such that $2x^2 + 6z^4 + P(y) > 2019$.
Solution *Just fix y as usual, then we have $f(x) = 0$ or $f(x) = x$. Therefore $g(y) = P(y)$.*

Example 1.2.21. *Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that*

$$f(2x^2 - \frac{y^3 + 2}{y} + 6z^4) = 2f(x)^2 + g(y) + 6zf(z)^3$$

for all $x, y, z > 2019$ such that $2x^2 + 6z^4 - \frac{y^3+2}{y} > 2019$.

Example 1.2.22. *Find all function $f : (2019, \infty) \to \mathbb{R}$ such that*

$$f(x^2 - y^3 + z^4) = \frac{f(x)^4}{x^2 + 1} - \frac{f(y)^6}{y^3 + 1} + \frac{f(z)^8}{z^4 + 1}$$

Example 1.2.23. *Find all function $f : (2018, \infty) \to \mathbb{R}$ such that*

$$f(x^2 + y^3 + z^4) = \frac{f(x)^4}{x^2 + 1} + \frac{f(y)^6}{y^3 + 1} + \frac{f(z)^8}{z^4 + 1}$$

Example 1.2.24. *Find all function $f : (2019, \infty) \to \mathbb{R}$ such that*

$$f(x^3 + y^3 + z^3 + 3w^3) = \frac{f(x)^5}{x^2 + 1} + \frac{f(y)^5}{y^2 + 1} + \frac{f(z)^5}{z^2 + 1} + 3\frac{f(w)^5}{w^2 + 1}$$

Example 1.2.25. *Find all $a \in R$ such that there exists a function $f : (2019, \infty) \to \mathbb{R}^*$ such that*

$$f(x^3 + y^3 + z^3)(\frac{x^2 + a}{f(x)} + \frac{y^2 + a}{f(y)} + \frac{z^2 + a}{f(z)}) = 1$$

Solution *Assume $a \in \mathbb{R}$ and $f : (2019, \infty) \to \mathbb{R}$ such that*

$$f(x^3 + y^3 + z^3)(\frac{x^2 + a}{f(x)} + \frac{y^2 + a}{f(y)} + \frac{z^2 + a}{f(z)}) = 1$$

then for all $x, y, z > 2019$.
Let $g(x) = \frac{1}{f(x)}$, then we have

$$g(x^3 + y^3 + z^3) = (x^2 + a)g(x) + (y^2 + a)g(y) + (z^2 + a)g(z)$$

This equation is familiar and we know that $g(x)$ is a linear function on x, hence $g(x) = cx + d$ with $c, d \in \mathbb{R}$.
Then $a = 0$ and hence $g(x) = cx$, so $f(x) = \frac{1}{cx}$ with $c \neq 0$.

Example 1.2.26. *Find all $a, b \in \mathbb{R}$ such that there exists a function $f : (2019, \infty) \to \mathbb{R}^*$ such that*

$$f(x^3 + y^3 + z^3)\left(\frac{x^2 + ax + 1}{f(x)} + \frac{y^2 + ax + 1}{f(y)} + \frac{z^2 + ax + 1}{f(z)}\right) = 1$$

Solution *Let $g(x) = \frac{1}{f(x)}$ then g is linear function on x hence $g(x) = \alpha x + \beta$. Plug in we have*

$$\alpha(x^3 + y^3 + z^3) + \beta = \sum (x^2 + ax + b)(\alpha x + \beta)$$

Comparing the coefficients of x^2 forces $a = 0$, and comparing the coefficients of x forces $b = 0$.

Chapter 2

Some polynomial problems

Problem 2.0.1. Find all polynomials f,g in $\mathbb{C}[X]$ such that $f^3 - g^2$ is a nonzero constant.
Solution
If f or g is a constant then both f, g are constant.
Now assume that $\deg(f), \deg(g) > 0$ and $f^3 - g^2 = a \in \mathbb{C}^*$.
Let $b = a^{1/3}$ then we have $g^2 = f^3 - b^3 = (f-b)(f^2 + bf + b^2)$.
If $f - b$ and $f^2 + bf + b^2$ have a common root x_0 then we have

$$0 = f(x_0)^2 + bf(x_0) + b^2 = b^2 + b^2 + b^2 = 3b^2$$

, thus $b = 0$ and hence $a = 0$. So $f - b$ and $f^2 + bf + b^2$ are relatively prime. Therefore both $f - b$ and $f^2 + bf + b^2$ are squares of polynomials in $\mathbb{C}[X]$. Now let $f - b = A(x)^2$ and $f^2 + bf + b^2 = B(x)^2$ then

$$B^2 = (A^2 + b)^2 + b(A^2 + b) + b^2 = A^4 + 3bA^2 + 3b^2$$

so

$$B^2 = (A^2 + c)(A^2 + d)$$

where $c = \frac{b(-3+i\sqrt{3})}{2}$ and $d = \frac{b(-3-3i\sqrt{3})}{2}$.
Again, if $A(x)^2 + c$ and $A(x)^2 + d$ have a common root x_1 in $\mathbb{C}$ then $c - d = A(x_1)^2 + c - A(x_1)^2 - d = 0$, so $c = d$, which is not possible. So $A^2 + c$ and $B^2 + c$ are relatively prime in $\mathbb{C}[X]$.
Therefore $A(x)^2 + c = h(x)^2$.
But then $c = (h - A)(h + A)$, thus both $h - A, h + A$ are constant, and hence h, A are also constant.

Therefore $f = b + A^2$ is also a constant polynomial, which contradicts to $\deg(f) > 0$.
So f, g are constant polynomials.

Problem 2.0.2. Find all pairs of polynomials $P(x)$ and $Q(x)$ with real coefficients for which

$$P(x)Q(x+1) - P(x+1)Q(x) = 1$$

for all $x \in R$.
Solution
Suppose P, Q satisfy

$$P(x)Q(x+1) - P(x+1)Q(x) = 1$$

Then none of P, Q can be 0 and P, Q have no common non constant factors. We have

$$P(x+1)Q(x) - P(x+1)Q(x) = P(x-1)Q(x) - P(x)Q(x-1) = 1$$

Thus

$$P(x)(Q(x+1) - Q(x-1)) = Q(x)(P(x+1) + P(x-1)$$

Because P, Q have no non constant factors, we have $P(x)|P(x+1)+P(x-1)$.
This implies that $P(x+1) + P(x-1) = 2P(x)$.
So

$$P(x+1) - P(x) = P(x) - P(x-1)$$

Let $H(x) = P(x+1) - P(x)$ then $H(x) = H(x-1)$, thus H is a constant polynomial.
Therefore $P(x) - P(x-1) = a \in \mathbb{R}$. Therefore $P(x) = ax + b$.
Similarly, $Q(x) = cx + d$.
Then

$$P(x)Q(x+1) - P(x+1)Q(x) = bc - ad$$

So $1 = bc - ad$.
Therefore $P(x) = ax + b$ and $Q(x) = cx + d$ with $bc - ad = 1$.

Problem 2.0.3. Prove that every prime number is a divisor of the polynomial

$$x^6 - 11x^4 + 36x^2 - 36$$

which does not have rational roots.
Solution Let $P(x) = x^6 - 11x^4 + 36x^2 - 36$ then $P(x) = (x^2-2)(x^2-3)(x^2-6)$
So P has no rational roots.
Let p be a prime number greater than 3.
If $\left(\frac{2}{p}\right) = 1$ or $\left(\frac{3}{p}\right) = 1$ then $p|P(x)$, else we have $\left(\frac{6}{p}\right) = 1$.

Problem 2.0.4. Find the number of pairs of polynomials $P(x), Q(x) \in \mathbb{R}[X]$ such that

$$P(x)^2 + Q(x)^2 = x^{2n} + 1$$

and $\deg(P) > \deg(Q)$.
Solution Because $\deg(P) > \deg(Q)$, we have the leading coefficient of P is ± 1.
Now assume that the leading coefficient of P is 1, then we have

$$(P + iQ)(P - iQ) = \prod_{k=0}^{n-1}(x + e^{\frac{i(2k+1)\pi}{2n}}).\prod_{k=0}^{n-1}(x - e^{\frac{i(2k+1)\pi}{2n}})$$

$P + iQ$ is a polynomial of degree n, in order for P, Q to have real coefficients, then for each k in $\{0, 1, ..., n-1\}$, exactly one of $x + e^{\frac{i(2k+1)\pi}{2n}}$ or $x - e^{\frac{i(2k+1)\pi}{2n}}$ is a factor of $P + iQ$. Thus there are 2^n pairs of P, Q with the leading coefficient of P is 1.
If the leading coefficient of P is -1, we have 2^n pairs of P, Q. Therefore, there are 2^{n+1} pairs of P, Q satisfying the hypothesis.

Problem 2.0.5. Let f be a non constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.
Solution Let $f(x) = a_0 + a_1x + ... + a_dx^d$ with $a_0, ..., a_d \in \mathbb{Z}^+$. We have

$$f(f(n)+1) = a_0 + a_1(1+f(n)) + ... + a_d(1+f(n))^d \equiv a_0 + ... + a_d \equiv f(1) \mod f(n)$$

If $n = 1$ then of course $f(f(n) + 1) \equiv 0 \mod f(n)$.
If $n > 1$ then $f(1) < f(n)$ because $a_0, ..., a_d \in \mathbb{Z}^+$, therefore $f(f(n) + 1) \not\equiv 0 \mod f(n)$.

Problem 2.0.6. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), ..., h(p^2-1)$ are distinct modulo p^2. Prove that $h(0), h(1), ..., h(p^3-1)$ are distinct modulo p^3.
Solution We use Taylor's theorem:

$$h(x+y) = h(x) + h^{'}(x)y + \frac{h^{(2)}(x)}{2!}y^2 + ... + \frac{h^{(n)}(x)}{n!}y^n$$

Here $h^{'}(x), ..., \frac{h^{(n)}(x)}{n!} \in \mathbb{Z}[X]$.
For $x = 0, 1, ..., p-1$, we have

$$h(x+p) \equiv h(x) + ph^{'}(x) \mod p^2$$

Since $h^{'}(x) \in \mathbb{Z}[x]$, we have $h^{'}(x+mp) \equiv h^{'}(x) \mod p$ for every $m \in \mathbb{Z}$. Therefore, $h^{'}(x) \not\equiv 0 \mod p$ for all $x \in \mathbb{Z}$.
Now for $x = 0, 1, ..., p^2-1$ and $y = 0, 1, ..., p-1$ we have

$$h(x+yp^2) \equiv h(x) + p^2 yh^{'}(x) \mod p^3$$

Thus $h(x), h(x+p^2), ..., h(x+(p-1)p^2)$ run over all of the residue classes modulo p^3 congruent to $h(x)$ modudo p^2. Because $h(x)$ covers all the residue classes modulo p^2, $h(0), ..., h(p^3-1)$ are distinct modulo p^3.

Problem 2.0.7. Let

$$P_n(x) = 1 + 2x + 3x^2 + ... + nx^{n-1}$$

Prove that P_n and P_m are relatively prime for every $n \neq m$.
Solution
Lemma 1 Let

$$f(x) = a_0 + a_1 x + ... + a_n x^n$$

be a polynomial with $0 < a_0 \leq a_1 ... \leq a_n$ then let z be a complex root of f then $|z| \leq 1$.
Proof Let $f(z) = 0$ then from $(z-1)f(z) = 0$, we have

$$a_n z^{n+1} = (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$$

If $|z| > 1$ then $|a_n z^{n+1}| \leq (|a_n - a_{n-1}| + ... + |a_1 - a_0| + |a_0|)|z|^n = |a_n||z|^n$, contradiction.

Therefore, $|z| \leq 1$.

Lemma 2

Let $f(x) = a_0 + a_1x + ... + a_nx^n$ be a polynomial with positive coefficients then for very root $z \in \mathbb{C}$ of f satisfies $r \leq |z| \leq R$, for

$$r = \min\{\frac{a_0}{a_1}, ..., \frac{a_{n-1}}{a_n}\}$$

$$R = \max\{\frac{a_0}{a_1}, ..., \frac{a_{n-1}}{a_n}\}$$

Proof Apply lemma 1 to polynomial $f(\frac{x}{R})$, we have $|z| \leq R$. Apply lemma 1 to the reverse of polynomial $f(\frac{x}{r})$, we have $|z| \geq r$.

Suppose that $P_m(z) = P_n(z)$ for some $z \in \mathbb{C}$ and $0 < m < n \in \mathbb{Z}^+$.

we cannot have $n = m + 1$ because $P_{m+1} - P_m = (m + 1)z^{m+1}$.

Thus $n - m \geq 2$.

Apply lemma 2 to P_m we have $|z| \leq 1 - \frac{1}{m}$.

Apply lemma 2 to $\frac{P_n(x) - P_m(x)}{x^{n-m}}$, we have $|z| \geq 1 - \frac{1}{m+2}$.

So $1 - \frac{1}{m} \geq 1 - \frac{1}{m+2}$, contradiction.

Chapter 3

Some other problems

Problem 3.0.1. Let S be a set of n elements. Prove that for every $N \in \mathbb{N}$ such that $0 \leq N \leq 2^n$, we can color the set of subsets of S by two colors red or blue such that the union of any two red subsets is a red subset and the union of any two blue subsets is a blue subset.
Solution: We prove by induction on n.
If $n = 1$, then let $S = \{x\}$. We can easily see that the statement is true for $n = 1$.
Assume that the statement is true for n. Consider the set S with $n + 1$ elements.
Assume that $S = \{x_1, ..., x_n, x_{n+1}\}$ and let $T = \{x_1, ..., x_n\}$.
Let $N \in \mathbb{N}$ such that $0 \leq N \leq 2^{n+1}$.
If $N \leq 2^n$, then by induction hypothesis, we can color N subsets of T with the red color and all other subsets of T with the blue color such that the union of any 2 red subsets is a red subset and the union of any 2 blue subsets is a blue subset.
Now let $X \subset T$, if X is colored red, we color $X \cup \{x_{n+1}\}$ red too, and if X is colored blue then $X \cup \{x_{n+1}\}$ is colored blue too. Then of course union of any two red subsets is a red subset and the union of any two blue subsets is a blue subset.
If $2^n < N \leq 2^{n+1}$.
Let $N = 2^n + a$ with $0 < a \leq 2^n$.
If $a = 2^n$ then we just color all subsets of S red.
If $a < 2^n$, we color a subsets of T red and $2^n - a$ subsets of T blue such that the union of any two red subsets is red and the union of any two blue subsets is blue. Now if $X \subset T$ is colored red then $X \cup \{x_{n+1}\}$ is also colored red, if

$Y \subset T$ is colored blue then $Y \cup \{x_{n+1}\}$ is also colored blue.
Now, we color all subsets of T with red, then we have $2^n + a$ red subsets. And the union of any two red subsets is red, the union of any two blue subsets is blue.

Problem 3.0.2. Let S be an arithmetic progression of length 2^{n+1}. Prove that we can divide S into 2 subsets A, B, each has 2^n elements such that

$$\sum_{x\in A} x^k = \sum_{y\in B} y^k \tag{3.0.1}$$

for all $k \in \{0, 1..., n\}$.
Solution: We prove by induction on n.
For $n = 1$, we can assume $S = \{a, a + d, a + 2d, a + 3d\}$.
$A = \{a, a + 2d\}$ and $B = \{a + d, a + 3d\}$ satisfy (0.0.1).
Assume that the statement is true for n.
Let $S(n + 1) = \{a, a + d, ..., a + (2^{n+2} - 1)d\}$.
By induction, we can partition $S(n) = \{a, a + d, ..., a + (2^{n+1} - 1)d\}$ into 2 subsets A, B, each has 2^n elements such that

$$\sum_{x\in A} x^k = \sum_{y\in B} y^k$$

for all $k \in \{0, 1, ..., n\}$.
Now let $\alpha = 2^{n+1}d$, then we have $A+\alpha\cup B+\alpha = \{a+2^{n+1}d, ..., a+(2^{n+2}-1)d\}$.
Let $A_1 = A \cup B + \alpha$ and $B_1 = B \cup A + \alpha$.
We show that

$$\sum_{x\in A_1} x^k = \sum_{y\in B_1} y^k$$

for all $k \in \{0, 1, ..., n + 1\}$.
We have

$$\sum_{x\in A_1} x^k = \sum_{x\in A} x^k + \sum_{x\in B} (x + \alpha)^k$$
$$\sum_{y\in B_1} y^k = \sum_{y\in B} y^k + \sum_{y\in A} (y + \alpha)^k$$

Because

$$\sum_{x\in A} x^k = \sum_{y\in B} y^k$$

is true for all $k \in \{0, 1, ..., n\}$, we have

$$\sum_{x \in A_1} x^k = \sum_{y \in B_1} y^k$$

for all $k \in \{0, 1.., n\}$.
For $k = n + 1$, we have

$$\begin{aligned}
\sum_{x \in A_1} x^{n+1} &= \sum_{x \in A} x^{n+1} + \sum_{y \in B} (y + \alpha)^{n+1} \\
&= \sum_{x \in A \cup B} x^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} \alpha^{n+1-k} (\sum_{y \in B} y^k)
\end{aligned}$$

$$\begin{aligned}
\sum_{y \in B_1} y^{n+1} &= \sum_{y \in B} y^{n+1} + \sum_{x \in A} (x + \alpha)^{n+1} \\
&= \sum_{y \in B \cup A} y^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} \alpha^{n+1-k} (\sum_{x \in A} x^k)
\end{aligned}$$

For $0 \leq k \leq n$ then

$$\sum_{x \in A} x^k = \sum_{y \in B} y^k$$

so

$$\sum_{x \in A_1} x^{n+1} = \sum_{y \in B_1} y^{n+1}$$

So the statement is true for $n + 1$.

Problem 3.0.3. Let S be a set of $m(n + 1) - 1$ elements. Partition the set of all n element subsets of S into two classes. Prove that there are at least m pairwise disjoint sets in the same class.
Solution We prove by induction on m.
For $m = 1$, the problem is obvious.
Assume that the statement is true for $m > 0$.
We prove that the statement is also true for $m + 1$.
Let S be a set with $(m + 1)(n + 1) - 1$ elements.
If all n element subsets of S are in the same class, then there is nothing to prove.
Assume that not all n element subsets of S are in the same class. Let A and

B be two n element subsets such that $|A \cap B|$ has the maximum size. Then we claim that

$$|A \cap B| = n - 1$$

Indeed, assume $|A \cap B| = k < n - 1$. Let $x \in A - B$ and $y \in B - A$.
Let $C = (B - \{y\}) \cup \{x\}$ then $|A \cap C| = k + 1$ and $|C \cap B| = n - 1$.
A and C are in different classes or B and C are in different classes, a contradiction.
Now let $T = S - (A \cup B)$, then $|T| = (m+1)(n+1) - 1 - (n+1) = m(n+1) - 1$, so by induction, there are m pairwise disjoint n element subsets in the same class, A or B is in that class too.

Problem 3.0.4. Prove that for every n then $7^{7^n} + 1$ has at least (not necessarily distinct) $2n + 3$ prime factors.
Solution We prove by induction on n.
For $n = 0$, then $7^{7^n} + 1 = 8$ has 3 prime factors.
Assume that $7^{7^n} + 1$ has at least $2n + 3$ prime factors.
Let $x = 7^{7^n}$ then

$$\begin{aligned}
\frac{x^7+1}{x+1} &= \frac{(x+1)^7 - ((x+1)^7 - x^7 - 1}{x+1} \\
&= (x+1)^6 - \frac{7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x}{x+1} \\
&= (x+1)^6 - \frac{7x(x+1)(x^2+x+1)^2}{x+1} \\
&= (x+1)^6 - 7^{7^n+1}(x^2+x+1)^2 \\
&= ((x+1)^3 - 7^m(x^2+x+1))((x+1)^3 + 7^m(x^2+x+1))
\end{aligned}$$

where $m = \frac{1+7^n}{2} \in \mathbb{Z}^+$.
Now

$$\begin{aligned}
(x+1)^3 - 7^m(x^2+x+1) &= x^2(x - 7^m) + x(3x - 7^m) + 3x + 1 - 7^m \\
&> x^2 + x > 2
\end{aligned}$$

So $(x+1)^3 - 7^m(x^2+x+1)$ has at least 1 prime factor and $(x+1)^3 + 7^m(x^2 + x + 1)$ has at least 1 prime factor.
By induction hypothesis then $x+1$ has at least $2n+3$ prime factors. Therefore, $7^{7^{n+1}} + 1$ has at least $2n + 3 + 2 = 2(n + 1) + 3$ prime factors.

Problem 3.0.5. Find all pairs of positive integers (m, n) such that

$$(x^2 + x + 1)^m | (x + 1)^n - x^n - 1$$

Solution
Lemma $x^2 + x + 1 | (x + 1)^k - x^k$ if and only if $k \equiv 0 \mod 6$

Proof. If $k \equiv 0 \mod 6$ then let $k = 6m$ with $m \in \mathbb{N}$

$$\begin{aligned}(x + 1)^{6m} - x^{6m} &= (x^2 + 2x + 1)^{3m} - (x^3)^{2m} \\ &\equiv x^{3m} - 1^{2m} \mod x^2 + x + 1 \\ &\equiv 1 - 1 \equiv 0 \mod x^2 + x + 1\end{aligned}$$

If $k \not\equiv 0 \mod 6$ then just consider $k = 6m + r$ with $r = 1, 2, ..., 5$ we see that $x^2 + x + 1 \nmid (x + 1)^{6m+r} - x^{6m+r}$. $\square$

Now let $f(x) = (x + 1)^n - x^n - 1$.
If $m = 1$, then $x^2 + x + 1 | (x + 1)^n - x^n - 1$, this is equivalent to $6 | n \pm 1$.
If $m = 2$, then $(x^2 + x + 1)^2 | f(x)$, thus $x^2 + x + 1 | f^{'}(x) = n((x + 1)^{n-1} - x^{n-1})$.
By the lemma then $6 | n - 1$. When $6 | n - 1$ then $x^2 + x + 1 | f(x)$ by the case $m = 1$.
If $m > 2$ then $(x^2 + x + 1)^3 | f(x)$, thus $x^2 + x + 1 | f^{''}(x) = n(n - 1)((x + 1)^{n-2} - x^{n-2})$ and $x^2 + x + 1 | f^{'}(x) = n((x + 1)^{n-1} - x^{n-1})$.
By the lemma then $6 | n - 2$ and $6 | n - 1$, a contradiction.
Therefore, all pairs of (m, n) such that $(x^2 + x + 1)^m | (x + 1)^n - x^n - 1$ are $(1, 6k \pm 1), (2, 6k)$ with $k \in \mathbb{Z}^+$.

Problem 3.0.6. Let a be a perfect square. Assume that $7 + a$ has d prime divisors (not necessarily distinct). Show that $7^{7^n} + a^{7^n}$ has at least $2n + d$ prime divisors(not necessarily distinct).
Solution Just prove by induction on n.
Let $7^{7^n} = x$ and $a^{7^n} = y$ then

$$\begin{aligned}\frac{x^7 + y^7}{x + y} &= \frac{(x + y)^7 - ((x + y)^7 - x^7 - y^7)}{x + y} \\ &= (x + y)^6 - 7xy(x^2 + xy + y^2)^2\end{aligned}$$

Let $a = m^2$ then $y = b^2$ with $b = m^{7^n}$ then $7xy = (7^{\frac{7^n+1}{2}} b)^2 = c^2$.
So

$$x^7 + y^7 = (x + y)((x + y)^3 - c(x^2 + xy + y^2))((x + y)^3 + c(x^2 + xy + y^2))$$

We have

$$(x+y)^3+c(x^2+xy+y^2) > (x+y)^3-c(x^2+xy+y^2) = (x+y)^3-\sqrt{7xy}(x^2+xy+y^2) > 1$$

Indeed, we show that

$$(x+y)^3 - \sqrt{7xy}(x^2+xy+y^2) > 1$$

Problem 3.0.7. Find the minimum of the function

$$F(m,n) = (m+n)^3 - \sqrt{7mn}(m^2+mn+n^2)$$

where $m, n \in \mathbb{Z}^+$ and $m \neq n$.
Solution Assume $m > n$ then $F(m,n) \geq F(n+1,n) \geq F(2,1) = 27 - 5\sqrt{14}$

Problem 3.0.8. Find all monic polynomial $P(x)$ with integer coefficients such that there exists a positive integer n satisfying

$$P(x)^2 | (x+1)^n - x^n - 1$$

Solution
If $P(x)$ is a constant then $P(x) = 1$ because $P(x)$ is monic.
Assume now that $P(x)$ has a positive degree.
Let $f(x) = (x+1)^n - x^n - 1$.
Let α be a complex root of $P(x)$ then α is a common root of $f(x)$ and $f^{'}(x)$.
Thus

$$(\alpha+1)^{n-1} - \alpha^{n-1} = (\alpha+1)^n - \alpha^n - 1 = 0$$

But then

$$\begin{aligned}(\alpha+1)^n - \alpha^n &= (\alpha+1)(\alpha+1)^{n-1} - \alpha^n - 1\\ &= (\alpha^n + \alpha^{n-1}) - \alpha^n - 1\\ &= \alpha^{n-1} - 1\end{aligned}$$

So

$$(\alpha+1)^{n-1} = \alpha^{n-1} = 1$$

Thus

$$|\alpha| = |\alpha+1| = 1$$

Let $\alpha = a + ib$ with $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$.
Then from $|\alpha + 1| = 1$, we have

$$(a+1)^2 + b^2 = 1 = a^2 + b^2$$

so $a = -\frac{1}{2}$ and $b = \frac{\pm\sqrt{3}}{2}$.
Therefore if α is a root of $P(x)$ then $\alpha = \frac{-1\pm\sqrt{3}}{2}$.
The minimal polynomial of $\frac{-1+\pm\sqrt{3}}{2}$ over $\mathbb{Q}$ is x^2+x+1, and $P(x)$ is a monic polynomial, we have

$$P(x) = (x^2+x+1)^m$$

for some $m \in \mathbb{Z}^+$.
Now from α is a root of $x^3 - 1 = (x-1)(x^2+x+1)$ and $\alpha^{n-1} = 1$, we have

$$x^3 - 1 | x^{n-1} - 1$$

.
Hence $3|n-1$.
If $n = 6m+1$ then $x^2+x+1|(x+1)^n - x^n - 1$ and $x^2+x+1|(x+1)^{n-1} - x^{n-1}$.
If $n = 6m+4$ then $x^2+x+1 \nmid (x+1)^{n-1} - x^{n-1}$.
Therefore, $6|n-1$.
In this case, if $m > 2$ then $x^2+x+1|f^{(3)}(x) = n(n-1)((x+1)^{n-2} - x^{n-2})$, which is not possible when $6|n-1$.
Therefore, $P(x) = x^2+x+1$.

Problem 3.0.9. Find all pairs of positive integers (m, n) such that there exists a monic polynomial $P(x)$ with integer coefficients satisfying

$$P(x)^m | (x+1)^n - x^n - 1$$

Solution For $m = 1$, just take $P(x) = (x+1)^n - x^n - 1$ for every $n \in \mathbb{Z}^+$.
For $m > 1$, using arguing as the Problem 8, we have $m = 2$, $P(x) = x^2+x+1$ and $n = 6k+1$ with $k \in \mathbb{N}$.

Problem 3.0.10. Let $n \in \mathbb{Z}^+$, find $\gcd((x+1)^n - x^n, (x+1)^{n^2+1} - x^{n^2+1} - 1)$ in $\mathbb{Q}[x]$
Solution.
Let $P(x) = \gcd((x+1)^n - x^n, (x+1)^{n^2+1} - x^{n^2+1} - 1)$.
If $P(x)$ is not a constant polynomial, then let α be a root of $P(x)$.
Then

$$(\alpha+1)^n = \alpha^n$$

and

$$(\alpha+1)^{n^2+1} - \alpha^{n^2+1} - 1 = 0$$

But then

$$\begin{aligned}(\alpha+1)^{n^2+1}-\alpha^{n^2+1}-1 &= (\alpha+1)((\alpha+1)^n)^n-\alpha^{n^2+1}-1\\ &= (\alpha+1)(\alpha^n)^n-\alpha^{n^2+1}-1\\ &= \alpha^{n^2}-1\end{aligned}$$

So

$$\alpha^{n^2}=1$$

Therefore $|\alpha|=1$, hence $|\alpha+1|^n=|\alpha|^n=1$.
So

$$|\alpha+1|=|\alpha|=1$$

Write $\alpha=a+ib$ with $a,b\in\mathbb{R}$ then

$$\alpha=\frac{-1\pm\sqrt{3}}{2}$$

So $P(x)=(x^2+x+1)^k$ for $k\in\mathbb{Z}^+$.
We know that

$$x^2+x+1|(x+1)^n-x^n \iff 6|n$$

and $x^2+x+1\nmid(x+1)^{n-1}-x^{n-1}$ if $x^2+x+1|(x+1)^n-x^n$.
Therefore

$$\begin{aligned}\gcd((x+1)^n-x^n,(x+1)^{n^2+1}-x^{n^2}-1) &= 1 \text{ if } 6\nmid n\\ &= x^2+x+1 \text{ if } 6|n\end{aligned}$$

Problem 3.0.11. Let $m,n\in\mathbb{Z}^+$ such that $n|m-1$. Find $\gcd((x+1)^n-x^n,(x+1)^m-x^m-1)$.
Solution As above, if $6|n$ then x^2+x+1 is the gcd.
If $6\nmid n$ then the gcd is 1.

Problem 3.0.12. Let $n\geq 3\in\mathbb{Z}$. Let $a_1,a_2,...,a_n$ be real numbers such that $|a_i-a_j|\geq 1$ for all $1\leq i<j\leq n$. Let F be a concave up increasing function on $[1,\infty)$. Find the minimum value of

$$\sum_{i=1}^{n}F(|a_i|)$$

Solution We have

$$|a_i|+|a_{n+1-i}|\geq|a_{n+1-i}-a_i|\geq n+1-2i$$

for all $i = 1, ..., n$.
Therefore

$$F(|a_i|)+F(|a_{n+1-i}|) \geq 2F(\frac{|a_i| + |a_{n+1-i}|}{2}) \geq 2F(\frac{n+1-2i}{2}) = 2F(\frac{n+1}{2}-i)$$

Now, if $n = 2m$ then

$$\begin{aligned} S &= F(|a_1|) + F(|a_{2m}|) + ... + F(|a_{m-1}|) + F(|a_{m+1}|) \\ &\geq 2F(m-\frac{1}{2}) + 2F(m-\frac{3}{2}) + ... + 2F(\frac{1}{2}) \end{aligned}$$

Equality holds for example $a_1 = -m + \frac{1}{2}$, $a_2 = -m + \frac{3}{2}$,...,$a_{m-1} = -\frac{1}{2}$, $a_{m+1} = \frac{1}{2}$, $a_{m+2} = 1 + \frac{1}{2}$,...,$a_{2m} = m - \frac{1}{2}$.
If $n = 2m + 1$ then

$$\begin{aligned} S &= F(|a_1|) + F(|a_2|) + ... + F(|a_{2m+1}|) \\ &= F(|a_1|) + F(|a_{2m+1}|) + ... + F(|a_m|) + F(|a_{m+2}|) + F(|a_{m+1}|) \\ &\geq 2F(m) + 2F(m-1) + ... + 2F(1) + F(0) \end{aligned}$$

Equality holds for example $a_1 = -m$,$a_2 = -(m-1)$,...,$a_m = -1$, $a_{m+1} = 0$, $a_1 = 1$, $a_2 = 2$,...,$a_{2m+1} = m$.

Problem 3.0.13. Let $n \geq 3 \in \mathbb{Z}$, find the minimum value of

$$\sum_{i=1}^{n} |a_i|^3$$

where $a_1, ..., a_n$ are real numbers satisfying $|a_i - a_j| \geq 1$ for all $1 \leq i < j \leq n$.
Solution $f(x) = x^3$, then f is increasing and concave up on $\mathbb{R}^+$.

Problem 3.0.14. Let $n \geq 3 \in \mathbb{Z}$, find the minimum value of

$$\sum_{i=1}^{n} \frac{a_i^4}{a_i^2+1}$$

where $a_1, ..., a_n$ are real numbers satisfying $|a_i - a_j| \geq 1$ for all $1 \leq i < j \leq n$.
Solution Let

$$f(x) = \frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$$

Then

$$f^{'}(x) = 2x - 1 - \frac{2x}{(x^2+1)^2} \geq 0$$

for $x \geq 1$.

$$f^{"}(x) = 2 + \frac{2(x^2-1)}{(x^2+1)^3} > 0$$

for $x \geq 1$.
Therefore on $[1, \infty)$ then $f(x)$ is increasing and concave up.

Problem 3.0.15. Find the maximum value of

$$\sum_{1 \leq i < j \leq n} |x_i - x_j|$$

where $x_i \in [0, 1]$ for all $i = 0, 1, ..., n$.
Solution
Lemma Let $a_1, ..., a_n \in \mathbb{R}$ then the function

$$F(x) = \sum_{i=1}^{n} |x - a_i|$$

where $x \in [0, 1]$ satisfying

$$F(x) \leq \max\{F(0), F(1)\}$$

Proof. Just assume that $a_1 \leq a_2 \leq a_3 ... \leq a_n$.
We prove by induction on n.
For $n = 1$, this is obvious.
Assume that this is true for n.
Consider

$$F(x) = \sum_{i=1}^{n+1} |x - a_i|$$

where $a_1 \leq a_2 \leq ... \leq a_{n+1}$.
If $x < a_1$ then

$$F(x) = \sum_{i=1}^{n+1} (a_i - x) \leq F(0)$$

If $a_i \le x \le a_{i+1}$ then $F(x)$ has the maximum at $x = a_i$ or $x = a_{i+1}$.
By induction then

$$F(a_i), F(a_{i+1}) \le \max\{F(0), F(1)\}$$

So

$$F(x) \le \max\{F(0), F(1)\}$$

If $x > a_{n+1}$ then

$$F(x) = \sum_{i=1}^{n+1}(x - a_i) \le F(1)$$

So by induction, the lemma is true. □

Now, use the lemma then

$$\sum_{1 \le i < j \le n} |x_i - x_j| \le \sum_{1 \le i < j \le n, x_i, x_j \in \{0,1\}} |x_i - x_j|$$

Now assume that there are s values of $0's$ and $n-s$ values of $1's$ in $\{x_1, ..., x_n\}$.
Then

$$F = s(n-s) \le \lfloor \frac{n^2}{4} \rfloor$$

So

$$\max F = \lfloor \frac{n^2}{4} \rfloor$$

Problem 3.0.16. Find the maximum value of

$$\sum_{1 \le i < j \le n} \frac{|x_i - x_j|}{1 + |x_i - x_j|}$$

Solution
Lemma Let

$$F(x) = \sum_{i=1}^{n} \frac{1}{1 + |x - a_i|}$$

where $x \in [0, 1]$ and $a_1, ..., a_n$ are real constant.
Then

$$F(x) \le \max\{F(0), F(1)\}$$

Now apply the lemma then the sum attains the maximum when $x_i \in \{0, 1\}$.

Problem 3.0.17. Assume that n is a given positive integer $n > 1$, consider positive integers a, b, c, d that satisfy $\frac{b}{a} + \frac{d}{c} < 1$, $b + d \leq n$. Find the maximum value of $\frac{b}{a} + \frac{d}{c}$.
Solution

Problem 3.0.18. Let A be a subset of the set $\mathbb{N}^*$ of positive integers. For any $x, y \in A$, $x \neq y$, we have $|x - y| \geq \frac{xy}{36}$. Find the maximum value of $|A|$.
Solution Let $A = \{a_1 < a_2 < ... < a_n\}$ then $a_k \geq k$ for $k = 1, 2, ..., n$.
We have

$$\frac{1}{a_i} - \frac{1}{a_{i+1}} \geq \frac{1}{36}$$

Therefore

$$\frac{1}{a_k} - \frac{1}{a_n} \geq \frac{n-k}{36}$$

Thus

$$n - k < \frac{36}{a_k} \leq \frac{36}{k}$$

hence

$$n < k + \frac{36}{k}$$

for all $k < n$.
Let $k = 6$ then $n < 12$, so $n \leq 11$.
Let $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $a_5 = 5$, $a_6 = 6$, $a_7 = 8$, $a_8 = 11$, $a_9 = 16$, $a_{10} = 29$, $a_1 1 = 29 \cdot 30$ then this sequence satisfies the problem.
So $\max|A| = 11$.

Problem 3.0.19. Let $\alpha \in \mathbb{R}$. Find the minimum of $G(x, y, z) = (2 - x)^\alpha + (2 - y)^\alpha + (2 - z)^\alpha$ where x, y, z are non negative real numbers satisfying $x + y + z = 1$.
Solution If $\alpha \geq 1$ then we have

$$\frac{G(x, y, z)}{3} \geq (\frac{(2 - x + 2 - y + 2 - z)}{3})^\alpha = (\frac{5}{3})^\alpha$$

Equality holds when $x = y = z = \frac{1}{3}$.
If $\alpha < 0$ then

$$G(x, y, z) \geq 3\sqrt[3]{\prod(2 - x)}^\alpha \geq 3(\frac{2 - x + 2 - y + 2 - z}{3})^\alpha = 3(\frac{5}{3})^\alpha$$

Equality holds when $x = y = z = \frac{1}{3}$.
If $\alpha \in (0,1)$ then consider the function $F(t) = (a+t)^\alpha + (a-t)^\alpha$ where $t \in [0,b]$ and $a > b > 0$.
We have

$$F^{'}(t) = \alpha((a+t)^{\alpha-1} - (a-t)^{\alpha-1} \leq 0$$

Therefore $F(t) \geq F(b)$.
From the result, we have $G(x,y,z) = (2-x)^\alpha + (2-y)^\alpha + (2-z)^\alpha \geq G(0, x+y, z) \geq G(0,0,x+y+z) = 1 + 2^{1+\alpha}$