Pricing European down-and-out call options: an application to Vietnamese financial derivatives market

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Abstract

In this paper, we present an application of Merton [2]'s pricing formula of European down-and-out call options under the Black-Scholes framework. We first derive the formula using a probabilistic approach. We then show in detail how to apply the formula to price European down-and-out call options written on Vietnamese stocks, say FPT stock. Our case-study research can serve as a good start for future researches on Vietnamese option market, which is going to develop soon.

1 Introduction

In Vietnam, derivatives market has just started officially very recently, in August, 2017, with its first products: futures contracts on VN30 index. Although the derivatives market is very new to Vietnamese investors, it is expected to strongly develop soon and will be one of the main pillars of Vietnamese financial market. In fact, trading volume on futures contracts on VN30 index is increasing significantly over last few months, and is expected to increase with an even faster rate. This fact shows great interest of investors to financial derivatives market.

One important type of financial derivatives products is options. In Vietnam, covered call (a type of call options) will be traded soon, as planed by the Vietnamese government. Options will bring greater leverage to speculators and bring more risk management tools for hedgers. Understanding clearly the pricing formulas for options is thus very urgent for investors investing in Vietnamese financial market.

Among options, barrier options are one of the most common ones used in foreign exchange, interest rate and equity option markets in the world. One of the reasons for the popularity of barrier options is that they provide a more flexible and cheaper way for hedging and speculating than their vanilla option counterparts. For instance, speculators can reduce costs with a down-and-out call option compared with the corresponding vanilla call option if the price of the underlying asset remains above a certain price level during the option life.

The pricing formula for European down-and-out call options under the Black-Scholes framework was first derived by Merton [2], using the heat equation approach. This study derives the pricing formula using a probabilistic approach. The formula is then used to calculate the call option price written on FPT stock as an illustration for application.

2 The formula's derivation

A down-and-out call option entitles the holder the right to buy the underlying with the exercise price K at the expiry time T if the underlying price has not touched a given price level B, called the barrier, during the option life time $[0, T]$. At any time t, if $S_t \leq B$ then the option is knocked out, i.e., immediately becomes worthless. We thus only consider the case $S_t > B$. We also assume $B \leq K$, as the call option holder is more likely to cease the exercise right when the asset falls below the strike price K.

Under the Black-Scholes framework, the value of a European down-and-out call options depends on the asset price (paying no dividend) with risk-neutral dynamics given by a Geometric Browninan motion:

$$dS_t = rS_t dt + \sigma S_t dZ_t, \tag{2.1}$$

where $\{S_t : 0 \leq t \leq T\}$ is the stock price process, $\{Z_t : 0 \leq t \leq T\}$ is a standard Brownian motion with respect to the risk-neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$. Here T, r, σ, which are positive constants, represent for the expiry time, the risk-free interest rate and the volatility rate, respectively. The price of a European down-and-out-call option at time t, denoted by $C_{d/o}(S_t, t)$, satisfies the Black-Scholes equation [2]:

$$\frac{\partial C_{d/o}}{\partial t}(S_t, t) + rS_t \frac{\partial C_{d/o}}{\partial S_t}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_{d/o}}{\partial S^2}(S_t, t) - rC_{d/o}(S_t, t) = 0.$$

From the Feynman-Kac theorem [3, p. 268], $C_{d/o}(S_t, t)$ can be computed from the formula:

$$C_{d/o}(S_t, t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[C_{d/o}(S_T, T)|\mathcal{F}_t],$$

where $\mathbb{E}^{\mathbb{Q}}$ represents the expectation under the risk neutral measure $\mathbb{Q}$, and $C_{d/o}(S_T, T)$ is the option pay-off:

$$C_{d/o}(S_T, T) = \max\{S_T - K, 0\}\mathcal{I}_{\min\limits_{t\leq u\leq T} S_u > B},$$

with $\mathcal{I}_{\min\limits_{t\leq u\leq T} S_u > B}$ is the indicator of the set $\{S_u > B\}$, i.e.,

$$\mathcal{I}_{\min\limits_{t\leq u\leq T} S_u > B} = \begin{cases} 1, & \text{if} \quad \min\limits_{t\leq u\leq T} S_u > B \\ 0, & \text{if} \quad \min\limits_{t\leq u\leq T} S_u \leq B \end{cases}.$$

In other words, the option holder will receive at expiry T the positive difference of S_T and K if $S_T > K$ and the barrier has not been hit up to time T.

Solving the stochastic differential equation (2.1), we obtain:

$$S_T = S_t e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma W^{\mathbb{Q}}_{T-t}} = S_t e^{\sigma \widehat{W}_{T-t}} \tag{2.2}$$

where $\widehat{W}_{T-t} = \nu(T-t) + W^{\mathbb{Q}}_{T-t}$ and $\nu = \dfrac{1}{\sigma}(r - \dfrac{1}{2}\sigma^2)$. By defining $m_{T-t} = \min\limits_{t\leq u\leq T} \widehat{W}_{u-t}$ we can express

$$\min_{t\leq u\leq T} S_u = \min_{t\leq u\leq T} S_t e^{\sigma \widehat{W}_{u-t}} = S_t e^{\sigma \min\limits_{t\leq u\leq T} \widehat{W}_{u-t}} = S_t e^{\sigma m_{T-t}}.$$

As a result, the payoff can be expressed as:

$$C_{d/o}(S_T,T) = \max\{S_T-K,0\}\mathcal{I}_{\{\min\limits_{t\le u\le T} S_u>B\}} = \max\{S_t e^{\sigma \widehat{W}_{T-t}}-K,0\}\mathcal{I}_{\{S_t e^{\sigma m_{T-t}}>B,\}}$$

$$= (S_t e^{\sigma \widehat{W}_{T-t}}-K)\mathcal{I}_{\{S_t e^{\sigma m_{T-t}}>B, S_t e^{\sigma \widehat{W}_{T-t}}>K\}} = (S_t e^{\sigma \widehat{W}_{T-t}}-K)\mathcal{I}_{\{m_{T-t}>\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right), \widehat{W}_{T-t}>\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)\}}$$

The down-and-out call option price at time t is

$$\begin{aligned}
C_{d/o}(S_t,t;K,B,T) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[C_{d/o}(S_T,T)|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_t e^{\sigma \widehat{W}_{T-t}}-K)\mathcal{I}_{\{m_{T-t}>\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right), \widehat{W}_{T-t}>\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)\}}\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}\int_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\infty}(S_t e^{\sigma x}-K)f^{\mathbb{Q}}_{m_{T-t},\widehat{W}_{T-t}}(a,x)\,da\,dx
\end{aligned}$$

where $f^{\mathbb{Q}}_{m_u,\widehat{W}_u}(m,\omega)$ is the joint probability density function of $(m_u,\widehat{W}_u)$ as given as in Chin et al. [1, p. 212]:

$$f^{\mathbb{Q}}_{m_u,\widehat{W}_u}(a,x) = \begin{cases} \dfrac{2(x-2a)}{u\sqrt{2\pi u}}\exp(\nu x - \dfrac{1}{2}\nu^2 u - \dfrac{(2a-x)^2}{2u}), & a<0, x\ge a, \\ 0, \quad \text{otherwise.} \end{cases}$$

Thus, we have:

$$f^{\mathbb{Q}}_{m_{T-t},\widehat{W}_{T-t}}(a,x) = \begin{cases} \dfrac{2(x-2a)}{(T-t)\sqrt{2\pi(T-t)}}\exp(\nu x - \dfrac{1}{2}\nu^2(T-t) - \dfrac{(2a-x)^2}{2(T-t)}), & a<0, x\ge a, \\ 0, \quad \text{otherwise.} \end{cases}$$

For convenience, we recall that $\widehat{W}_{T-t} = \nu(T-t)+W^{\mathbb{Q}}_{T-t}$ and $\nu = \dfrac{1}{\sigma}(r-\dfrac{1}{2}\sigma^2)$. The option price can now be computed as:

$$\begin{aligned}
C_{d/o}(S_t,t;K,B,T &= e^{-r(T-t)}\int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}\int_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\min(0,x)}\frac{2(S_t e^{\sigma x}-K)(x-2a)}{(T-t)\sqrt{2\pi(T-t)}}e^{\nu x-\frac{1}{2}\nu^2(T-t)-\frac{(2a-x)^2}{2(T-t)}}\,da\,dx \\
&= e^{-r(T-t)}\int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}\int_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\min(0,x)}\frac{(S_t e^{\sigma x}-K)}{\sqrt{2\pi(T-t)}}d(e^{\nu x-\frac{1}{2}\nu^2(T-t)-\frac{(2a-x)^2}{2(T-t)}})dx \\
&= e^{-r(T-t)}\int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}\frac{(S_t e^{\sigma x}-K)}{\sqrt{2\pi(T-t)}}e^{\nu x-\frac{1}{2}\nu^2(T-t)-\frac{(2a-x)^2}{2(T-t)}}\Big|_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\min(0,x)}dx \\
&= e^{-r(T-t)}\int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}\frac{(S_t e^{\sigma x}-K)}{\sqrt{2\pi(T-t)}}e^{\nu x-\frac{1}{2}\nu^2(T-t)}\left[e^{-\frac{(2\min(0,x)-x)^2}{2(T-t)}}-e^{-\frac{(\frac{2}{\sigma}\log\left(\frac{B}{S_t}\right)-x)^2}{2(T-t)}}\right]dx \\
&= e^{-r(T-t)}\int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}\frac{(S_t e^{\sigma x}-K)}{\sqrt{2\pi(T-t)}}e^{\nu x-\frac{1}{2}\nu^2(T-t)}\left[e^{-\frac{x^2}{2(T-t)}}-e^{-\frac{(\frac{2}{\sigma}\log\left(\frac{B}{S_t}\right)-x)^2}{2(T-t)}}\right]dx \\
&= S_t I_1 - K I_2 - (S_t I_3 - K I_4),
\end{aligned}$$

where

$$I_1 = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{-r(T-t)+\sigma x+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2} dx$$

$$I_2 = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{-r(T-t)+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2} dx$$

$$I_3 = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{-r(T-t)+\sigma x+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{\frac{2}{\sigma}\log\frac{B}{S_t}-x}{\sqrt{T-t}}\right)^2} dx$$

$$I_4 = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{-r(T-t)+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{\frac{2}{\sigma}\log\frac{B}{S_t}-x}{\sqrt{T-t}}\right)^2} dx$$

The computation of I_1, I_2, I_3, I_4 will ultimately lead to the computation of integrals having the form: $\frac{1}{\sqrt{2\pi T}}\int_L^U e^{ax-\frac{1}{2}(\frac{x}{\sqrt{T}})^2}dx$. We thus now focus on the latter.

$$\frac{1}{\sqrt{2\pi T}}\int_L^U e^{ax-\frac{1}{2}(\frac{x}{\sqrt{T}})^2}dx = \frac{1}{\sqrt{2\pi T}} e^{\frac{1}{2}a^2T}\int_L^U e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}-a\sqrt{T}\right)^2}dx.$$

Let $y = \frac{x}{\sqrt{T}} - a\sqrt{T}$, we obtain:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi T}}\int_L^U e^{ax-\frac{1}{2}(\frac{x}{\sqrt{T}})^2}dx &= e^{\frac{1}{2}a^2T}\frac{1}{\sqrt{2\pi}}\int_{\frac{L-aT}{\sqrt{T}}}^{\frac{U-aT}{\sqrt{T}}} e^{-\frac{1}{2}y^2}dy \\
&= e^{\frac{1}{2}a^2T}\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\frac{U-aT}{\sqrt{T}}} e^{-\frac{1}{2}y^2}dy - \int_{-\infty}^{\frac{L-aT}{\sqrt{T}}} e^{-\frac{1}{2}y^2}dy\right] \\
&= e^{\frac{1}{2}a^2T}\left[\Phi\left(\frac{U-aT}{\sqrt{T}}\right) - \Phi\left(\frac{L-aT}{\sqrt{T}}\right)\right],
\end{aligned}$$

with $\Phi(x) = \int_{-\infty}^{x} e^{-y^2/2}dy$ is the cumulative distribution of a standard normal random variable. Using the above result, we can compute I_1 as follows:

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi(T-t)}} e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)} \int_{\frac{1}{\sigma}log\left(\frac{K}{S_t}\right)}^{\infty} e^{(\sigma+\nu)x-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2}dx \\
&= e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)+\frac{1}{2}(\nu+\sigma)^2(T-t)}\left[\Phi(\infty) - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t)-(\nu+\sigma)(T-t)}{\sqrt{T-t}}\right)\right] \\
&= e^{-r(T-t)-\frac{1}{2}\left(\frac{2r-\sigma^2}{2\sigma}\right)^2(T-t)+\frac{1}{2}\left(\frac{2r+\sigma^2}{2\sigma}\right)^2(T-t)}\left[1 - \Phi\left(-\frac{\log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \\
&= \Phi\left(\frac{\log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

Similarly we can deduce

$$I_2 = e^{-r(T-t)}\Phi\left(\frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

The computation of I_3 and I_4 is a bit more complicated as shown below:

$$\begin{aligned}
I_3 =& \frac{1}{\sqrt{2\pi(T-t)}} e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}\log^2\left(\frac{B}{S_t}\right)} \times \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{\left[\nu+\sigma+\frac{2}{\sigma(T-t)}\log\left(\frac{B}{S_t}\right)\right]x-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2} dx \\
=& e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}\log^2\left(\frac{B}{S_t}\right)+\frac{1}{2}\left[\nu+\sigma+\frac{2}{\sigma(T-t)}\log\left(\frac{B}{S_t}\right)\right]^2(T-t)} \\
& \times \left[\Phi(\infty) - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t) - \left[\nu+\sigma+\frac{2}{\sigma(T-t)}\log(B/S_t)\right](T-t)}{\sqrt{T-t}}\right)\right] \\
=& \left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}} \Phi\left(\frac{\log(B^2/(S_tK)) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

$$\begin{aligned}
I_4 =& \frac{1}{\sqrt{2\pi(T-t)}} e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}\left(\log\left(\frac{B}{S_t}\right)\right)^2} \times \int_{x=\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{\left[\nu+\frac{2}{\sigma(T-t)}\log\left(\frac{B}{S_t}\right)\right]x-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2} dx \\
=& e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}\left(\log\left(\frac{B}{S_t}\right)\right)^2+\frac{1}{2}\left[\nu+\frac{2}{\sigma(T-t)}\log\left(\frac{B}{S_t}\right)\right]^2(T-t)} \\
& \times \left[\Phi(\infty) - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t) - \left[\nu+\frac{2}{\sigma(T-t)}\log(B/S_t)\right](T-t)}{\sqrt{T-t}}\right)\right] \\
=& e^{-r(T-t)}\left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}}\left[1 - \Phi\left(\frac{\log(S_tK/B^2) - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \\
=& e^{-r(T-t)}\left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}} \Phi\left(\frac{\log(B^2/(S_tK)) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

Therefore,

$$C_{d/o}\left(S_t, t; K, B, T\right) = C_{bs}\left(S_t, t; K, T\right) - \left(\frac{S_t}{B}\right)^{\lambda} C_{bs}\left(\frac{B^2}{S_t}, t; K, T\right) \tag{2.3}$$

where $C_{bs}(S_t, t; K, T) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$, with $d_1 = \dfrac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$ and $\lambda = 1 - \dfrac{2r}{\sigma^2}$.

3 Application to Vietnamese financial market

In this section, we show how to apply the formula (2.3) to obtain the price of a European down-and-out call option written on Vietnamese stocks. In particular, we choose FPT stock as an example for illustration. More specifically, the daily adjusted closing prices of FPT stock from December 12, 2006 to November 16, 2018 are used. The data can be downloaded from the free source: https://www.cophieu68.vn/export.php.

We price a European call option written on FPT stock with the starting date on November 16, 2018 and expiring in the next 6 months. The parameters for the option are assumed to be given as in the following table: To apply the formula (2.3), we still need one more input: the

S_0	Stock price at inception	$42,750$ VND
K	Strike price	$45,000$ VND
B	Down and out barrier price	$38,000$ VND
r	Annual risk-free interest rate	7%
T	Option expiration (in years)	0.5

Table 1: Option parameters

volatility of the FPT stock. It should be noted that the volatility is the only unobservable input. We now estimate this input by using the historical volatility of the stock. More specifically, we calculate the annualized daily volatility of FPT returns over the data period. Note that to annualize the daily volatility, we multiply the standard deviation of daily returns by the square root of 252 (a year is assumed to have 252 trading days). The following R code is used to obtain the historical volatility:

```
df <- read.csv(file="excel_fpt.csv", head=T,stringsAsFactors = F)
df <- df[,c(2:7)]
date <- as.Date(as.character(df$X.DTYYYYMMDD.), format="%Y%m%d")
class(date)
df <- cbind(date,df)
df <- df[,-2]
df <- df[order(df$date),]
names(df) <- paste(c("date", "Open", "High", "Low", "Close", "Volume"))
library(xts)
df <- xts(df[,2:6],order.by = df[,1])
volatility<-df[,4]
volatility$Ret<-diff(log(volatility$Close))
volatility<-volatility[-1,]
hist.vol<-sd(volatility$Ret)*sqrt(252)
vol <- hist.vol
```

As a result of running the above R code, we obtain the annual volatility of FPT $\sigma = 32.5\%$. We are now ready to compute the value of the European down-and-out call option by using the following R code:

```
BS_call <- function (S0,K,T,r,sigma){
d1 <- (log(S0/K)+(r+0.5*sigma^2)*T)/(sigma*sqrt(T))
d2 <- d1- sigma*sqrt(T)
opt.val <- S0*pnorm(d1)-K*exp(-r*T)*pnorm(d2)
return(opt.val)}
```

```
down_out_call <- function (S0,K, T, r, sigma,B){
lamda <- 1-2*r/sigma^2
opt.val <- BS_call(S0,K,T,r,sigma)-(S0/B)^lamda*BS_call(B^2/S0,K,T,r,sigma)
return(opt.val)}
down_out_call(42750,45000, 0.5, 0.07, 0.325, 38000)
```

As a result, we obtain the European down-and-out call option price as $3,018.038$ (VND). Compare with the price of the corresponding vanilla call, 3601.607 (VND), the barrier option is cheaper 16.2%. This is because the barrier option holder is willing to cease the exercise right if the stock price falls below $38,000$ (VND) during the option life.

4 Conclusion

In this paper, the Merton [2]'s pricing formula of European down-and-out call options under the Black-Scholes framework is derived, using a probabilistic approach. The application of the formula to price European down-and-out call options written on FPT stock is presented in detail. The R code is given to show how to: manipulate data, obtain the historical volatility of the stock and compute the option price. Our future research will be on the pricing formulas of other exotic options.

References

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