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PRICING EXTERNAL-CHAINED BARRIER OPTIONS WITH EXPONENTIAL BARRIERS

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Abstract. External barrier options are two-asset options with stochastic variables where the payoff depends on one underlying asset and the barrier depends on another state variable. The barrier state variable determines whether the option is knocked in or out when the value of the variable is above or below some prescribed barrier level. This paper derives the explicit analytic solution of the chained option with an external single or double barrier by utilizing the probabilistic methods - the reflection principle and the change of measure. Before we do this, we examine the closed-form solution of the external barrier option with a single or double-curved barrier using the methods of image and double Mellin transforms. The exact solution of the external barrier option price enables us to obtain the pricing formula of the chained option with the external barrier more easily.

1. Introduction

Barrier options are one of the most popular path-dependent derivatives in various markets, particularly in over-the-counter markets and FX markets. The main reasons barrier options have become so popular are flexibility and a lower price compared to vanilla options. Merton [13] first proposed the analytic pricing formula for a barrier option with a lower knock-out boundary. Reiner and Rubinstein [15] derived the closed-form formula for all eight types of barriers with a cumulative normal distribution function. Kunitomo and Ikeda [11] obtained double-barrier options with two curved (exponential) barriers.

An external barrier option is a contract whereby the payoff of an underlying asset is created by the crossing of a barrier determined by a state variable. In other words, in an external barrier option, one underlying asset depends on the actual option payoff, and the other asset depends on whether the option is knocked in or out. Heynen and Kat [6] derived closed-form formulas for

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European-style barrier options with a single external barrier, where the external variable is not the underlying asset price. Carr [3] obtained the analytical valuation for the up-and-in call option with an external single barrier. Kwok et al. [12] studied analytic formulas for multi-asset option prices with an external single barrier so that the barrier level is exponential, and Chen et al. [4] derived the closed-form formulas for Parisian external single-barrier options. Additionally, Wong and Kwok [16] used the splitting direction technique for the pricing of multi-asset options with an external double-barrier. Kim and Kim et al. [10] obtained the Laplace transforms of the prices for the external single and double-barrier options so that the underlying asset prices follow a regime-switching model with finite regimes and then calculated the option prices through numerical inversion of the Laplace transforms.

This paper examines the pricing of external-chained barrier options with curved barriers based on the external barrier option pricing formula. Jun and Ku [8] found the exact formula of chained barrier options where the monitoring of the barrier begins at a random time as the underlying asset first crosses two barrier levels in a specified order. The authors also addressed the pricing formula of chained barrier options with exponential barriers [9].

Jun and Ku [8] derived the pricing formula of chained barrier options using the reflection property and the change of measure and computing the given expectation directly under a risk-neutral probability. However, if we are attempting to calculate the expectation directly after implementing the change of measure under an equivalent martingale measure in case of the external-chained barrier option, then the computation of the barrier option price may be more complicated than the case of the chained barrier option mentioned in Jun and Ku [8] because the dynamics of the assets of the external-chained barrier option are those of a two-factor model. Hence, to compute the closed-form formula of the external-chained barrier option with exponential barriers effectively and easily, we utilize the external European barrier option price described in Section 2. After using the reflection principle method and the change of measure under risk-neutral probability, we change the expectation representation of the external-chained barrier option into the form of the expectation of the external European barrier option. Then, because the pricing formula of the external European barrier option is expressed by the bivariate cumulative normal distribution function, we obtain the closed solution of the external-chained barrier option directly without any calculation of the expectation. Similarly, by using the external double-barrier option price mentioned in Section 3 immediately, we obtain the semi-analytic formula of the external-chained double-barrier option.

The pricing formulas of the external European call option price proposed by Heynen and Kat [6] were derived by the properties of reflection and probabilistic approaches. However, the derivation of the option pricing formula using these methods requires the complexity of the calculation. To resolve this problem, we use the double Mellin transform methods and the method of images to derive the valuation formula of the given option price. The Mellin transform, which is considered the integral transform of the multiplicative version of the two-sided Laplace transform, is a useful instrument for the transformation of partial differential equations. Yoon [17] used Mellin transforms to obtain a closed formula for European options under a Hull-White stochastic interest rate. Yoon and Kim [18] utilized double Mellin transforms to derive the exact-form solution for European vulnerable options under a constant interest rate and a stochastic interest rate. Additionally, Jeon, Kang, and Yoon [7] considered the pricing of path-dependent options using double Mellin transforms for an explicit-form pricing formula of path-dependent options.

The method of images was first proposed by Buchen [1] and is closely related to the reflection principle of the expectations solution. Buchen [1] obtained the closed formula of barrier options more easily using the PDE method of images compared to the existing probabilistic method. Therefore, using the double Mellin transforms and the method of images for the derivation of the option pricing formula solves the pricing formula of the external European call option presented in Section 2 effectively and easily. To find the semi-analytic valuation formula of the external double-barrier option price in Section 3, the double Mellin transform and the image operators of the method of images are tools that find the solution simply.

This paper is organized as follows. Section 2 uses the Mellin transform technique and the method of images to derive the external European barrier option with one or two curved barriers. Section 3 considers the derivation of the price of the external double-barrier option with the double exponential barrier. Section 4 shows an explicit analytic solution for an external-chained single barrier option using the reflection principle and the formula of the given external European barrier option directly. Section 5 addresses the semi-analytic solution for the external double-barrier option, and Section 6 presents concluding remarks.

2. External European barrier option with a single curved barrier

2.1. Model formulation

Assume that S_t is the value of an asset underlying the option with the constant drift rate μ and the volatility σ, and Z_t is the barrier state variable with the constant drift rate μ_* and the volatility σ_*, where both σ and σ_* are positive constants. Then, the dynamics of S_t and Z_t have the following stochastic differential equations (SDEs)

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t^s, \\ dZ_t &= \mu_* Z_t dt + \sigma_* Z_t dB_t^z, \end{aligned} \tag{2.1}$$

and B_t^s and B_t^z are standard Brownian motions satisfying $d\langle B_t^s, B_t^z\rangle_t = \rho dt$. Using the Girsanov theorem in Oksendal [14], in the risk neutral world P^*, the equation (2.1) is changed into

$$dS_t = rS_t dt + \sigma S_t dB_t^{s*},$$
$$dZ_t = rZ_t dt + \sigma_* Z_t dB_t^{z*}, \tag{2.2}$$

where r is a constant interest rate, and B_t^{s*} and B_t^{z*} are the transformed Brownian motions of B_t^s and B_t^z, respectively, with $d\langle B_t^{s*}, B_t^{z*}\rangle_t = \rho dt$.

Based upon the SDEs (2.2), we consider European up-and-out call options where the terminal payoff depends on the underlying asset S_t, and the knock-out occurs when the barrier state variable Z_t breaches the upstream exponential barrier $\beta(t) = Ue^{b(T-t)}$. If K is the strike price of the option, then, under the risk-neutral measure P^*, the price of the up-and-out call with the exponential barrier is given by

$$P(t,s,z) := E^{P^*}\left[e^{-r(T-t)}\tilde{h}(S_T, Z_T)|S_t = s, Z_t = z\right], \tag{2.3}$$

where the payoff function $\tilde{h}$ is expressed by

$$\tilde{h}(S_T, Z_T) = (S_T - K)^+ \mathbf{1}_{\left\{\max_{0\le\gamma\le T}(Z_\gamma - \beta(\gamma))<0\right\}}.$$

Using the Feynman-Kac formula in Oksendal [14], $P(t,s,z)$ leads to the following PDE problem

$$\begin{aligned} &\mathcal{L}P(t,s,z) = 0, \quad t < T, \\ &P(T,s,z) = h(s) = (s-K)^+, \quad \text{on} \quad z < U, \\ &P(t,s,\beta(t)) = 0\,, \\ &\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}{\sigma_*}^2 z^2 \frac{\partial^2}{\partial z^2} + \rho\sigma\sigma_* sz \frac{\partial^2}{\partial s \partial z} \\ &\qquad + r\left(s\frac{\partial}{\partial s} + z\frac{\partial}{\partial z}\right) - rI \end{aligned} \tag{2.4}$$

on the domain $\{(t,s,z) : 0 \le t < T,\ 0 \le x < \infty,\ 0 \le z < \beta(t)\}$, where I is the identity operator.

If we apply the change of variables $V(t,x,y) = \frac{P(t,s,z)}{e^{b(T-t)}}$, $x = \frac{s}{e^{b(T-t)}}$, and $y = \frac{z}{e^{b(T-t)}}$ to (2.4), then, the PDE (2.4) yields

$$\begin{aligned} &\tilde{\mathcal{L}}V(t,x,y) = 0, \quad t < T, \\ &V(T,x,y) = h(x) = (x-K)^+, \quad \text{on} \quad y < U \\ &V(t,x,U) = 0\,, \\ &\tilde{\mathcal{L}} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}{\sigma_*}^2 y^2 \frac{\partial^2}{\partial y^2} + \rho\sigma\sigma_* xy \frac{\partial^2}{\partial x \partial y} \\ &\qquad + (r+b)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - (r+b)I \end{aligned} \tag{2.5}$$

on the domain $\{(t,x,y) : 0 \le t < T,\ 0 \le x < \infty,\ 0 \le y < U\}$.

2.2. Derivation of the external European barrier option price with a single exponential barrier: Double Mellin transform approach

This subsection investigates the price of an external European option with a single exponential barrier. To find an explicit closed-form solution, we use the method of images mentioned in Buchen [1] and a double Mellin transform consistent with Hassan and Adem [5].

2.2.1. *The review of the method of images.* Let $f(t,x,y)$ be a differentiable function with respect to (t,x,y). Then, the image function of $f(t,x,y)$ described by $f^*(t,x,y) = \mathcal{I}_B f(t,x,y)$ to the barrier B has the following properties

(a) ${\mathcal{I}_B}^2 = I$, where I is the identity operator.
(b) If $\mathcal{L}\, f(t,x,y) = 0$, then $\mathcal{L}\,\mathcal{I}_B[f(t,x,y)] = 0$.
(c) When $x = B$ or $y = B$, $\mathcal{I}_B[f] = f$, that is, $(I - \mathcal{I}_B)[F] = 0$.
(d) If $x > B$ or $y > B$ ($x < B$ or $y < B$) is the active domain of $f(t,x,y)$, then $x < B$ or $y < B$ ($x > B$ or $y > B$) is the active domain of $f^*(t,x,y) = \mathcal{I}_B f(t,x,y)$,

where $\mathcal{I}_B$ is the image operator with respect to the barrier level B, and $\mathcal{L}$ is a parabolic differential operator.

Then, for a differentiable function $\tilde{f}(t,x,y)$, we consider the following PDE

$$\begin{aligned} &\mathcal{L}\tilde{f}(t,x,y) = 0; \\ &\tilde{f}(t,B,y) = 0, \qquad x < B\ ,\ t < T \\ &\tilde{f}(T,x,y) = g(x), \end{aligned} \tag{2.6}$$

where $g(x)$ is the terminal payoff function of $\tilde{f}(t,x,y)$. To solve this PDE (2.6), the following related PDE with an unrestricted domain with respect to x should be considered.

$$\begin{aligned} &\mathcal{L}f(t,x,y) = 0; \\ &f(T,x,y) = g(x)\mathbf{1}_{\{x<B\}}, \qquad x > 0\ ,\ t < T. \end{aligned}$$

Then, we obtain the solution of the PDE (2.6), which is expressed by

$$\tilde{f}(t,x,y) = f(t,x,y) - f^*(t,x,y),$$

where $f^*(t,x,y)$ is the image solution of $f(t,x,y)$.

2.2.2. *Review of the double Mellin transform.* Referring to Hassan and Adem [5], we review the double Mellin transform to solve the PDE with the given final and boundary conditions. For a locally Lebesgue integrable function $f(x_1,y_1),\ x_1,\ y_1 \in \mathbb{R}^+$, the double Mellin transform is given by

$$\mathcal{M}_{x_1 y_1}(f(x_1,y_1),w_1,w_2) := \hat{f}(w_1,w_2) = \int_0^\infty \int_0^\infty f(x_1,y_1) x_1^{w_1-1} y_1^{w_2-1} dx_1 dy_1,$$

where w_1 and w_2 are complex numbers. If $a < \mathrm{Re}(w_1)$, $\mathrm{Re}(w_2) < b$, and if c_1 and c_2 such that $a < c_1 < b$ and $a < c_2 < b$ exist, the inverse of the double Mellin transform is the complex integral function satisfying

$$\begin{aligned} f(x_1, y_1) &= \mathcal{M}^{-1}_{x_1 y_1}(\hat{f}(w_1, w_2)) \\ &= \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} \hat{f}(w_1, w_2) x_1^{-w_1} y_1^{-w_2} dw_1 dw_2. \end{aligned}$$

In PDE (2.5), letting $W(t, x, y) = V(t, x, U\, y)$ and $\tilde{r} = r + b$, (2.5) is transformed into the following PDE:

$$\begin{aligned} &\tilde{\mathcal{L}} W(t, x, y) = 0, \quad t < T, \\ &W(T, x, y) = h(x) = (x - K)^+, \quad \text{on} \quad y < 1 \\ &W(t, x, 1) = 0\,, \\ &\tilde{\mathcal{L}} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}{\sigma_*}^2 y^2 \frac{\partial^2}{\partial y^2} + \rho\sigma\sigma_* xy \frac{\partial^2}{\partial x \partial y} \\ &\qquad + \tilde{r}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) - \tilde{r} I \end{aligned} \tag{2.7}$$

in the region $\{(t, x, y) : 0 \le t < T,\ 0 \le x < \infty,\ 0 \le y < 1\}$.

To solve the problem (2.7), we must consider $\bar{w}(t, x, y)$ satisfying the following PDE

$$\begin{aligned} &\tilde{\mathcal{L}} \bar{w}(t, x, y) = 0, \\ &\bar{w}(T, x, y) = h(x) \mathbf{1}_{\{y<1\}}. \end{aligned} \tag{2.8}$$

with the unrestricted domain $\{(t, x, y) : 0 \le t < T,\ 0 \le x < \infty,\ 0 \le y < \infty\}$. Using the PDE method of images stated in Buchen [1], if we obtain the image solution of the above $\bar{w}(t, x, y)$, then we can find the solution of $W(t, x, y)$ in (2.7) by integrating $\bar{w}(t, x, y)$ and the image solution of $\bar{w}(t, x, y)$.

If we define $\bar{w}(t, x, y)$ as the inverse double Mellin transform of $\hat{w}(t, x^*, y^*)$ satisfying

$$\bar{w}(t, x, y) = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} \hat{w}(t, x^*, y^*) x^{-x^*} y^{-y^*} dx^* dy^*,$$

the PDE (2.7) is transformed into

$$\begin{aligned} &\frac{d\hat{w}}{dt} + A(x^*, y^*)\hat{w} = 0, \\ &A(x^*, y^*) := \frac{\sigma^2}{2} {x^*}^2 + \rho\sigma\sigma_* x^* y^* + \frac{{\sigma_*}^2}{2} {y^*}^2 - \left(\tilde{r} - \frac{\sigma^2}{2} \right) x^* \\ &\qquad - \left(\tilde{r} - \frac{{\sigma_*}^2}{2} \right) y^* - \tilde{r}, \end{aligned} \tag{2.9}$$

and from the solution $\hat{w}(t,x^*,y^*)$ of the ordinary differential equation (2.9), we obtain

(2.10)
$$\bar{w}(t,x,y) = \frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} \hat{h}(x^*,y^*)e^{A(x^*,y^*)(T-t)}x^{-x^*}y^{-y^*}dx^*dy^*,$$

where $\hat{h}(x^*,y^*)$ is the double Mellin transform of $\bar{w}(T,x,y) = h(x)\mathbf{1}_{\{y<1\}}$.

Hence, to compute (2.10), let us consider

$$\tilde{B}(t,x,y) = \frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} e^{A(x^*,y^*)(T-t)}x^{-x^*}y^{-y^*}dx^*dy^*. \tag{2.11}$$

Because $e^{A(x^*,y^*)(T-t)}$ and $\hat{h}(x^*,y^*)$ are the double Mellin transforms of $\tilde{B}(t,x,y)$ and $h(x)\mathbf{1}_{\{y<1\}}$, respectively, by using the Mellin convolution property stated in Hassan and Adem [5], we have

$$\bar{w}(t,x,y) = \int_0^\infty\int_0^\infty h(u_1)\mathbf{1}_{\{u_2<1\}}\tilde{B}\left(\tau,\frac{x}{u_1},\frac{y}{u_2}\right)u_1^{-1}u_2^{-1}du_1du_2. \tag{2.12}$$

To find the value of the double integral in (2.12), we calculate $\tilde{B}(t,x,y)$ in (2.11). The computation of the $\tilde{B}(t,x,y)$ is given by Yoon and Kim [18] and

(2.13)
$$\begin{aligned}
&\tilde{B}(\tau,x,y) \qquad (\tau = T-t)\\
&= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\exp\left[\left\{\frac{1}{2}(1-\rho^2)\sigma^2x^{*2} + \left(\frac{\rho\sigma\sigma_*}{2}(k_*-1) - \left(\tilde{r}-\frac{\sigma^2}{2}\right)\right)x^*\right.\right.\\
&\qquad \left.\left.-\frac{{\sigma_*}^2(k_*-1)^2}{8} - \tilde{r}\right\}\tau\right]x^{-x^*}b(\tau,x^*,y)dx^*,\\
&b(\tau,x^*,y) := \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\exp\left[\frac{{\sigma_*}^2}{2}\tau\left\{y^* + \left(\frac{\rho\sigma}{\sigma_*}x^* - \frac{k_*-1}{2}\right)\right\}^2\right]y^{-y^*}dy^*,\\
&k_* := \frac{2\tilde{r}}{{\sigma_*}^2}.
\end{aligned}$$

As mentioned earlier, $W(t,x,y)$ of the PDE problem in (2.7) is expressed as the sum of $\bar{w}(t,x,y)$ and the image solution of $\bar{w}(t,x,y)$. To find the image solution of $\bar{w}(\tau,x,y)$, the following lemma is useful.

Lemma 2.1. *If we define* $R(x^*,y^*) := \left\{y^* + \left(\frac{\rho\sigma}{\sigma_*}x^* - \frac{k_*-1}{2}\right)\right\}^2$ *in the equation* $b(\tau,x^*,y)$ *of* (2.13)*, then*

$$\tilde{B}(\tau,x,y) = y^{-(k_*-1)}\tilde{B}\left(\tau, y^{-\frac{2\rho\sigma}{\sigma_*}}x, \frac{1}{y}\right).$$

Proof. By the definition of $R(x^*,y^*)$, we obtain

$$R(x^*,y^*) = R\left(x^*, -y^* - 2\left(\frac{2\rho\sigma}{\sigma_*}x^* - \frac{k_*-1}{2}\right)\right).$$

Then,

$$\begin{aligned} b(\tau, x^*, y) &= \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \exp\left[\frac{{\sigma_*}^2}{2}\tau R\left(x^*, y^*\right)\right] y^{-y^*} dy^* \\ &= y^{\frac{2\rho\sigma}{\sigma_*}x^*} y^{-(k_*-1)} \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \exp\left[\frac{{\sigma_*}^2}{2}\tau R\left(x^*, y^{**}\right)\right] y^{y^{**}} dy^{**}, \\ &= y^{\frac{2\rho\sigma}{\sigma_*}x^*} y^{-(k_*-1)} b\left(\tau, x^*, \tfrac{1}{y}\right). \end{aligned}$$

where $y^{**} = -y^* - 2\left(\frac{2\rho\sigma}{\sigma_*}x^* - \frac{k_*-1}{2}\right)$.

In (2.13), if we let

$$l(x^*) = \left\{\frac{1}{2}(1-\rho^2)\sigma^2 {x^*}^2 + \left(\frac{\rho\sigma\sigma_*}{2}(k_*-1) - \left(\tilde{r} - \frac{\sigma^2}{2}\right)\right) x^* - \frac{{\sigma_*}^2(k_*-1)^2}{8} - \tilde{r}\right\},$$

then $\tilde{B}(\tau, x, y)$ yields

(2.14)

$$\tilde{B}(\tau, x, y) = \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \exp\left[\tau l(x^*)\right] y^{\frac{2\rho\sigma}{\sigma_*}x^*} y^{-(k_*-1)} b\left(\tau, x^*, \tfrac{1}{y}\right) x^{-x^*} dx^*.$$

$$\Longrightarrow y^{(k_*-1)}\tilde{B}(\tau, x, y) = \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \exp\left[\tau l(x^*)\right] y^{\frac{2\rho\sigma}{\sigma_*}x^*} b\left(\tau, x^*, \tfrac{1}{y}\right) x^{-x^*} dx^*.$$

Replacing $\frac{1}{y}$ with y again, and using

$$\tilde{B}(\tau, x, y) = \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \exp\left[\tau l(x^*)\right] l\left(\tau, x^*, y\right) x^{-x^*} dx^*$$

in (2.11), we obtain

$$\begin{aligned} y^{-(k_*-1)}\tilde{B}\left(\tau, x, \frac{1}{y}\right) &= \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \exp\left[\tau l(x^*)\right] y^{-\frac{2\rho\sigma}{\sigma_*}x^*} b\left(\tau, x^*, y\right) x^{-x^*} dx^* \\ &= \tilde{B}\left(\tau, y^{\frac{2\rho\sigma}{\sigma_*}} x, y\right). \end{aligned} \tag{2.15}$$

Hence, $\tilde{B}(\tau, x, y) = y^{-(k_*-1)}\tilde{B}\left(\tau, y^{-\frac{2\rho\sigma}{\sigma_*}} x, \frac{1}{y}\right)$. □

Therefore, from Lemma 2.1, $\bar{w}(t, x, y)$ of (2.12) is described by

$$\begin{aligned} \bar{w}(t, x, y) = \int_0^\infty \int_0^\infty & h(u_1) \mathbf{1}_{\{u_2<1\}} \left(\frac{y}{u_2}\right)^{-(k_*-1)} \\ & \tilde{B}\left(\tau, \left(\frac{y}{u_2}\right)^{-\frac{2\rho\sigma}{\sigma_*}} \frac{x}{u_1}, \frac{u_2}{y}\right) u_1^{-1} u_2^{-1} du_1 du_2, \end{aligned} \tag{2.16}$$

and yields

$$y^{-(k_*-1)}\bar{w}\left(t, y^{-\frac{2\rho\sigma}{\sigma_*}} x, \frac{1}{y}\right) = \int_0^\infty \int_0^\infty h(u_2^{-\frac{2\rho\sigma}{\sigma_*}} u_1) u_2^{-(k_*-1)} \mathbf{1}_{\{u_2>1\}} \tag{2.17}$$

$$\tilde{B}\left(\tau, \frac{x}{u_1}, \frac{y}{u_2}\right) u_1^{-1} u_2^{-1} du_1 du_2.$$

Lemma 2.2. *Let* $\bar{w}^*(t,x,y) = y^{-(k_*-1)}\bar{w}\left(t, y^{-\frac{2\rho\sigma}{\sigma_*}}x, \frac{1}{y}\right)$*. Then,* $\bar{w}^*(t,x,y)$ *is an image function of* $\bar{w}(t,x,y)$ *satisfying* $\tilde{\mathcal{L}}\bar{w}^*(t,x,y) = 0$*. Moreover, the solution of* (2.5) *is given by*

$$V(t,x,y) = \bar{V}(t,x,y) - \left(\frac{y}{U}\right)^{-(k_*-1)} \bar{V}\left(t, \left(\frac{y}{U}\right)^{-\frac{2\rho\sigma}{\sigma_*}} x, \frac{U^2}{y}\right), \tag{2.18}$$

where $\bar{V}(t,x,y)$ *is a solution of* $\tilde{\mathcal{L}}\bar{V}(t,x,y)=0$ *with the final condition* $\bar{V}(T,x,y) = h(x)\mathbf{1}_{\{y<U\}}$.

Proof. As described above by the property of the double Mellin transform convolution, $\bar{w}(t,x,y)$ in (2.12) is a solution of the PDE problem of (2.7). Hence, by the relationship between (2.7) (PDE problem) and (2.12) (Mellin convolution), we note that $\bar{w}^*(t,x,y) = y^{-(k_*-1)}\bar{w}\left(t, y^{-\frac{2\rho\sigma}{\sigma_*}}x, \frac{1}{y}\right)$ given in (2.17) is the solution of the PDE $\tilde{\mathcal{L}}\bar{w}^*(t,x,y) = 0$ with the terminal condition $\bar{w}^*(T,x,y) = h(y^{-\frac{2\rho\sigma}{\sigma_*}}x)y^{-(k_*-1)}\mathbf{1}_{\{y>1\}}$. Additionally, to prove that $\bar{w}^*(t,x,y)$ is an image function of $\bar{w}(t,x,y)$, we define

$$W(t,x,y) = \bar{w}(t,x,y) - \bar{w}^*(t,x,y). \tag{2.19}$$

Then,

$$\tilde{\mathcal{L}}\bar{w}(t,x,y) = 0 \ \text{ and } \tilde{\mathcal{L}}\bar{w}^*(t,x,y) = 0, \tag{2.20}$$

$$W(t,x,1) = \bar{w}(t,x,1) - \bar{w}^*(t,x,1) = 0 \tag{2.21}$$

at $y = 1$, and

$$\begin{aligned} W(T,x,y) &= \bar{w}(T,x,y) - \bar{w}^*(T,x,y) \\ &= h(x)\mathbf{1}_{\{y<1\}} - h(y^{-\frac{2\rho\sigma}{\sigma_*}}x)y^{-(k_*-1)}\mathbf{1}_{\{y>1\}} \\ &= \begin{cases} h(x) & \text{if } y < 1 \\ -h(y^{-\frac{2\rho\sigma}{\sigma_*}}x)y^{-(k_*-1)} & \text{if } y > 1 \end{cases} \end{aligned} \tag{2.22}$$

are satisfied. Hence, by the properties of the image function mentioned in Buchen [1], $\bar{w}^*(t,x,y)$ is the image function of $\bar{w}(t,x,y)$.

Moreover, (2.19) and (2.20) lead to $\tilde{\mathcal{L}}W(t,x,y) = 0$, and by combining (2.19) to (2.22), $W(t,x,y) = \bar{w}(t,x,y) - \bar{w}^*(t,x,y)$ becomes the solution of the PDE (2.7).

Therefore, the solution of (2.7) is expressed by

$$W(t,x,y) = \bar{w}(t,x,y) - y^{-(k_*-1)}\bar{w}\left(t, y^{-\frac{2\rho\sigma}{\sigma_*}}x, \frac{1}{y}\right), \tag{2.23}$$

and by replacing y with $\frac{y}{U}$ in (2.23), we obtain

$$W(t,x,\frac{y}{U}) = \bar{w}(t,x,\frac{y}{U}) - \left(\frac{y}{U}\right)^{-(k_*-1)} \bar{w}\left(t, \left(\frac{y}{U}\right)^{-\frac{2\rho\sigma}{\sigma_*}} x, \frac{U}{y}\right), \tag{2.24}$$

and from $W(t,x,y) = V(t,x,Uy)$, $V(t,x,y)$ is given by

$$V(t,x,y) = \bar{V}(t,x,y) - \left(\frac{y}{U}\right)^{-(k_*-1)} \bar{V}\left(t, \left(\frac{y}{U}\right)^{-\frac{2\rho\sigma}{\sigma_*}} x, \frac{U^2}{y}\right),$$

where $\bar{V}(t,x,y)$ is a solution of $\tilde{\mathcal{L}}\bar{V}(t,x,y)=0$ with the final condition $\bar{V}(T,x,y) = h(x)\mathbf{1}_{\{y<U\}}$ in the unrestricted region $\{(t,x,y) : 0 \le t < T,\ 0 \le x < \infty,\ 0 \le y < \infty\}$. $\square$

Theorem 2.1. *The price of the external European call option price with the single exponential barrier, defined by* (2.4)*, is expressed by*

(2.25)

$$\begin{aligned} &P(t,s,z) \\ &= s\mathcal{N}_2\left(d_1^r\left(\tau, \tfrac{s}{K}\right), -d_2^r\left(\tau, \tfrac{z}{\beta(t)}\right), -\rho\right) - e^{-r\tau}K\mathcal{N}_2\left(d_3^r\left(\tau, \tfrac{s}{K}\right), -d_4^r\left(\tau, \tfrac{z}{\beta(t)}\right), -\rho\right) \\ &\quad + \left(\tfrac{z}{\beta(t)}\right)^{-\frac{2\rho\sigma}{\sigma_*}-(k_*-1)} s\mathcal{N}_2\left(d_1^r\left(\tau, \left(\tfrac{z}{\beta(t)}\right)^{-\frac{2\rho\sigma}{\sigma_*}} \tfrac{s}{K}\right), d_2^r\left(\tau, \tfrac{\beta(t)}{z}\right), -\rho\right) \\ &\quad - e^{-r\tau} K\left(\tfrac{z}{\beta(t)}\right)^{-(k_*-1)} \mathcal{N}_2\left(d_3^r\left(\tau, \left(\tfrac{z}{\beta(t)}\right)^{-\frac{2\rho\sigma}{\sigma_*}} \tfrac{s}{K}\right), -d_4^r\left(\tau, \tfrac{\beta(t)}{z}\right), -\rho\right), \end{aligned}$$

where $\mathcal{N}_2$ is the bivariate normal cumulative distribution function defined by

$$\mathcal{N}_2(n_1, n_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{n_1} \int_{-\infty}^{n_2} e^{-\frac{1}{2(1-\rho^2)}\left(p^2 - 2\rho pq + q^2\right)} dpdq$$

and $d_1^r(\tau, \varsigma_1) = \frac{\ln(\varsigma_1) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$, $d_2^r(\tau, \varsigma_2) = \frac{\ln(\varsigma_2) + (r - \frac{{\sigma_*}^2}{2} + \rho\sigma\sigma_*)\tau}{\sigma_*\sqrt{\tau}}$, $d_3^r(\tau, \varsigma_3) = \frac{\ln(\varsigma_3) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$ *and* $d_4^r(\tau, \varsigma_2) = \frac{\ln(\varsigma_4) + (r - \frac{{\sigma_*}^2}{2})\tau}{\sigma_*\sqrt{\tau}}$.

Proof. First, Lemma 2.2 shows that the $\bar{V}(t,x,y)$ is the solution satisfying

$$\begin{aligned} &\tilde{\mathcal{L}}\bar{V} = 0, \\ &\bar{V}(T,x,y) = h(x)\mathbf{1}_{\{y<U\}} = (x-K)^+\mathbf{1}_{\{y<U\}}, \end{aligned} \tag{2.26}$$

where $\tilde{\mathcal{L}}$ is the operator given by (2.5).

To solve the PDE (2.5), we utilize the double Mellin transform technique. First, we solve the PDE (2.26). However, since the payoff function h of the European call option is not bounded, the double Mellin transforms may not exist. Therefore, we modify h by defining a sequence of functions h_n such that h_n is given by

$$\begin{aligned} &h_n(x,y) = h_n^1(x)h_n^2(y), \\ &h_n^1(x) := \begin{cases} x-K, & \text{if } K \le x < n \\ 0, & \text{otherwise,} \end{cases} \quad h_n^2(y) := \begin{cases} 1, & \text{if } y \le U \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{2.27}$$

and each h_n is bounded and $\lim_{n\to\infty} h_n(x,y) = h(x)1_{\{y<U\}}$.

If we define sequential functions $\bar{V}_n(t,x,y)$ satisfying the following PDE

$$\begin{aligned} &\tilde{\mathcal{L}}\,\bar{V}_n(t,x,y) = 0 \\ &\bar{V}_n(T,x,y) = h_n(x,y) \end{aligned} \tag{2.28}$$

on domain $\{(t,x,y) : 0 \le t < T,\ 0 \le x < \infty,\ 0 \le y < \infty\}$, where the operator $\tilde{\mathcal{L}}$ is described by (2.5), then the limit $V(t,x,y) = \lim_{n\to\infty} V_n(t,x,y)$ should be the solution of the PDE (2.5).

By using the Mellin convolution property, we have

$$\bar{V}_n(t,x,y) = \int_0^\infty \int_0^\infty h_n(u_1,u_2)\,\tilde{B}\left(\tau, \frac{x}{u_1}, \frac{y}{u_2}\right) u_1^{-1} u_2^{-1} du_1 du_2.$$

Here, more detail on the computation of the $\tilde{B}(t,x,y)$ is given by the equation (2.12) in Yoon and Kim [18].

Therefore, from $h_n(s,v) = h_n^1(s)h_n^2(v)$ in (2.26),

$$\bar{V}_n(t,x,y) = \int_0^U \int_K^n e^{\theta_1(u)}(u-K)\theta_2(u,w)\frac{1}{u}\frac{1}{w}dudw,$$

where

$$\theta_1 = -\frac{\tau}{8(1-\rho^2)}\left(\sigma^2(k-1)^2 + {\sigma_*}^2(k_*-1)^2 - 2\rho\sigma\sigma_*(k-1)(k_*-1)\right) - \tilde{r}\tau,$$

$$\theta_2(u,w) = \left(\frac{x}{u}\right)^{\frac{\frac{\rho\sigma\sigma_*\tau}{2}(k_*-1) - \left(\tilde{r} - \frac{\sigma^2}{2}\right)\tau + \frac{\rho\sigma}{\sigma_*}\ln(y/w)}{(1-\rho^2)\sigma^2\tau}} \left(\frac{y}{w}\right)^{\frac{\frac{\rho\sigma\sigma_*\tau}{2}(k-1) - \left(\tilde{r} - \frac{{\sigma_*}^2}{2}\right)\tau + \frac{\rho\sigma_*}{\sigma}\ln(x/u)}{(1-\rho^2){\sigma_*}^2\tau}}$$
$$\frac{e^{-\frac{1}{2}\left(\frac{\ln(x/u)}{\sigma\sqrt{(1-\rho^2)\tau}}\right)^2}}{\sigma\sqrt{2\pi\tau}} \frac{e^{-\frac{1}{2}\left(\frac{\ln(y/w)}{\sigma_*\sqrt{(1-\rho^2)\tau}}\right)^2}}{\sigma_*\sqrt{2\pi(1-\rho^2)\tau}},$$

$$k = \frac{2\tilde{r}}{\sigma^2}, \quad \text{and} \quad k_* = \frac{2\tilde{r}}{{\sigma_*}^2}.$$

Then, taking the limit $n \to \infty$ yields

$$\begin{aligned} \bar{V}(t,x,y) &= \lim_{n\to\infty} \bar{V}_n(t,x,y) \\ &= \int_0^U \int_K^\infty e^{\theta_1(u)} u\,\theta_2(u,w)\frac{1}{u}\frac{1}{w}dudw \\ &\quad - \int_0^U \int_K^\infty e^{\theta_1(u)} K\,\theta_2(u,w)\frac{1}{u}\frac{1}{w}dudw \\ &:= V^1(t,x,y) - V^2(t,x,y). \end{aligned} \tag{2.29}$$

Using a similar method described in Yoon and Kim [18], to find the value of $V_1(t,x,y)$ in (2.28), if we let $p = \frac{\ln(x/u)}{\sigma\sqrt{\tau}}$ and $q = \frac{\ln(y/w)}{\sigma_*\sqrt{\tau}}$ and use the method of undetermined coefficients, then

$$V^1(t,x,y) = \frac{x}{2\pi\sqrt{1-\rho^2}} \int_\infty^{\frac{\ln(y/U)}{\sigma_*\sqrt{\tau}}} \int_{\frac{\ln(x/K)}{\sigma\sqrt{\tau}}}^{-\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left((p+a)^2\right.\right. \tag{2.30}$$

$$+(q+b)^2\Big)+\frac{\rho}{1-\rho^2}(p+a)(q+b)\Bigg\}\,dpdq.$$

where $a=\frac{\left(\tilde{r}+\frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sigma}$, $b=\frac{\left(\rho\sigma\sigma_*+\left(\tilde{r}-\frac{\sigma_*{}^2}{2}\right)\right)\sqrt{\tau}}{\sigma_*}$.

Also, $V^2(t,x,v)$ in (2.28) is expressed by

(2.31)

$$V^2(t,x,y)=\frac{K}{2\pi\sqrt{1-\rho^2}}\int_{\infty}^{\frac{\ln(y/U)}{\sigma_*\sqrt{\tau}}}\int_{\frac{\ln(x/K)}{\sigma\sqrt{\tau}}}^{-\infty}\exp\Bigg\{D_2-\frac{1}{2(1-\rho^2)}\Big((p+c)^2$$
$$+(q+d)^2\Big)+\frac{\rho}{1-\rho^2}(p+c)(q+d)\Bigg\}\,dpdq,$$

where $c=\frac{\left(\tilde{r}-\frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sigma}$, $d=\frac{\left(\tilde{r}-\frac{\sigma_*{}^2}{2}\right)\sqrt{\tau}}{\sigma_*}$ and $D_2=-\tilde{r}\tau$.

Therefore, by setting $p_1=p+\frac{\left(\tilde{r}+\frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sigma}$ and $q_1=q+\frac{\left(\rho\sigma\sigma_*+\left(\tilde{r}-\frac{\sigma_*{}^2}{2}\right)\right)\sqrt{\tau}}{\sigma_*}$ in (2.29), and $p_2=p+\frac{\left(\tilde{r}-\frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sigma}$ and $q_2=q+\frac{\left(\tilde{r}-\frac{\sigma_*{}^2}{2}\right)\sqrt{\tau}}{\sigma_*}$ in (2.30), $\bar{V}(t,x,y)$ yields the following formula

$$\begin{aligned}
&\bar{V}(t,x,y)\\
&=x\left[\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{\infty}^{\frac{\ln(y/w)}{\sigma_*\sqrt{\tau}}+\frac{(\rho\sigma\sigma_*+(\tilde{r}-\frac{\sigma_*{}^2}{2}))\sqrt{\tau}}{\sigma_*}}\int_{\frac{\ln(x/u)}{\sigma\sqrt{\tau}}+\frac{(\tilde{r}+\frac{\sigma^2}{2})\sqrt{\tau}}{\sigma}}^{-\infty}e^{-\frac{1}{2(1-\rho^2)}\left(p_1^2-2\rho p_1q_1+y_1^2\right)}dp_1dq_1\right]\\
&\quad-Ke^{-\tilde{r}\tau}\left[\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{\infty}^{\frac{\ln(y/w)}{\sigma_*\sqrt{\tau}}+\frac{(\tilde{r}-\frac{\sigma_*{}^2}{2})\sqrt{\tau}}{\sigma_*}}\int_{\frac{\ln(x/u)}{\sigma\sqrt{\tau}}+\frac{(\tilde{r}-\frac{\sigma^2}{2})\sqrt{\tau}}{\sigma}}^{-\infty}e^{-\frac{1}{2(1-\rho^2)}\left(p_2^2-2\rho p_2q_2+q_2^2\right)}dp_2dq_2\right] \qquad (2.32)\\
&=x\mathcal{N}_2\left(d_1^{\tilde{r}}\left(\tau,\tfrac{x}{K}\right),-d_2^{\tilde{r}}\left(\tau,\tfrac{y}{U}\right),-\rho\right)\\
&\quad-e^{-\tilde{r}\tau}K\mathcal{N}_2\left(d_3^{\tilde{r}}\left(\tau,\tfrac{x}{K}\right),-d_4^{\tilde{r}}\left(\tau,\tfrac{y}{U}\right),-\rho\right). \qquad (2.33)
\end{aligned}$$

Therefore, from this closed-form solution of $\bar{V}(t,x,y)$ with respect to the bivariate normal cumulative distribution function, we find the image solution $\left(\frac{y}{U}\right)^{-(k_*-1)}\bar{V}\left(t,\left(\frac{y}{U}\right)^{-\frac{2\rho\sigma}{\sigma_*}}x,\frac{U^2}{y}\right)$ to $\bar{V}(t,x,y)$. We combine the two results to derive a closed-form formula for the option price $V(t,x,v)$ of the PDE (2.5).

Therefore,

$$\begin{aligned}
&V(t,x,y)\\
&=x\mathcal{N}_2\left(d_1^{\tilde{r}}\left(\tau,\tfrac{x}{K}\right),-d_2^{\tilde{r}}\left(\tau,\tfrac{y}{U}\right),-\rho\right)-e^{-\tilde{r}\tau}K\mathcal{N}_2\left(d_3^{\tilde{r}}\left(\tau,\tfrac{x}{K}\right),-d_4^{\tilde{r}}\left(\tau,\tfrac{y}{U}\right),-\rho\right)\\
&\quad+\left[\left(\tfrac{y}{U}\right)^{-\frac{2\rho\sigma}{\sigma_*}-(k_*-1)}x\mathcal{N}_2\left(d_1^{\tilde{r}}\left(\tau,\left(\tfrac{y}{U}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\frac{x}{K}\right),d_2^{\tilde{r}}\left(\tau,\frac{U}{y}\right),-\rho\right)\right.\\
&\quad\left.-e^{-\tilde{r}\tau}K\left(\tfrac{y}{U}\right)^{-(k_*-1)}\mathcal{N}_2\left(d_3^{\tilde{r}}\left(\tau,\left(\tfrac{y}{U}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\frac{x}{K}\right),-d_4^{\tilde{r}}\left(\tau,\frac{U}{y}\right),-\rho\right)\right].
\end{aligned}$$

Because $V(t,x,y) = \frac{P(t,s,z)}{e^{b(T-t)}}, x = \dfrac{s}{e^{b(T-t)}},\ y = \frac{z}{e^{b(T-t)}}$ and $d_j^{\tilde{r}}(\tau, \frac{s}{Ke^{b\tau}}) = d_j^r(\tau, \frac{s}{K})$ for $j = 1,2,3,4$,

(2.34)

$$
\begin{aligned}
&P(t,s,z)\\
&= s\mathcal{N}_2\left(d_1^r\left(\tau,\tfrac{s}{K}\right), -d_2^r\left(\tau,\tfrac{z}{\beta(t)}\right), -\rho\right) - e^{-r\tau}K\mathcal{N}_2\left(d_3^r\left(\tau,\tfrac{s}{K}\right), -d_4^r\left(\tau,\tfrac{z}{\beta(t)}\right), -\rho\right)\\
&\quad + \left(\tfrac{z}{\beta(t)}\right)^{-\frac{2\rho\sigma}{\sigma_*}-(k_*-1)} s\mathcal{N}_2\left(d_1^r\left(\tau, \left(\tfrac{z}{\beta(t)}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\tfrac{s}{K}\right), d_2^r\left(\tau,\tfrac{\beta(t)}{z}\right), -\rho\right)\\
&\quad - e^{-r\tau}K\left(\tfrac{z}{\beta(t)}\right)^{-(k_*-1)}\mathcal{N}_2\left(d_3^r\left(\tau, \left(\tfrac{z}{\beta(t)}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\tfrac{s}{K}\right), -d_4^r\left(\tau,\tfrac{\beta(t)}{z}\right), -\rho\right).
\end{aligned}
$$

□

3. External European barrier option with the double-curved barrier

This section addresses the price of an external double-barrier option using the image method mentioned by Buchen and Konstandatos [2] and the double Mellin transforms. First, we transform the PDE problem of the external double-barrier option with a boundary and final conditions with a final condition on the extended domain of a barrier state variable. From double Mellin transforms, we derive the pricing formula of the external double-barrier option in the unrestricted region. From the image operators of the method of images, combining the external double-barrier option price in the unrestricted region and the image solution of the option yields the semi-analytic solution of the external European barrier option with the double-curved barrier.

3.1. Model formulation

Let us consider the arbitrage-free pricing of double knock-out external barrier call options. A double-barrier option is an option with two barriers, which contains an upstream barrier at $\alpha(t) = Ae^{a(T-t)}$ and a downstream barrier at $\beta(t) = Be^{b(T-t)}$. If the barrier state variable Z_t reaches one of the two barriers at any time before the expiration, then the nullification of the option contract will be aroused. Then, the price of all double knock-out external barriers is described by

$$P(t,s,z) := E^{P^*}[e^{-r(T-t)}\hat{h}(S_T, Z_T) \mid S_t = s, Z_t = z] \tag{3.1}$$

under the risk-neutral measure, and the payoff function $\hat{h}$ is given by

$$\hat{h}(S_T, Z_T) = (S_T - K)^+ \mathbf{1}_{\{\{\max_{0\le\gamma\le T}(Z_\gamma - \alpha(\gamma))<0\}\cap\{\min_{0\le\gamma\le T}(Z_\gamma-\beta(\gamma))>0\}\}}. \tag{3.2}$$

Using the Feynman-Kac formula, the external double-barrier problem yields the following PDE

$$
\begin{aligned}
&\mathcal{L}P(t,s,z) = 0, \quad 0 \le t < T,\\
&P(T,s,z) = h(s) = (s-K)^+ \quad \text{on } \; B < z < A,\\
&P(t,s,\alpha(t)) = P(t,s,\beta(t)) = 0,\\
&\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}{\sigma_*}^2 z^2 \frac{\partial^2}{\partial z^2} + \rho\sigma\sigma_* sz \frac{\partial^2}{\partial s \partial z}\\
&\quad + r\left(s\frac{\partial}{\partial s} + z\frac{\partial}{\partial z}\right) - rI
\end{aligned} \tag{3.3}
$$

on the domain $\{(t,x,z) : 0 \le t < T,\ 0 \le s \le \infty,\ \beta(t) < z < \alpha(t)\}$, and I is the identity operator.

In a similar manner to the calculation in Section 2, to solve the PDE (3.3), we consider the following related PDE in an unrestricted domain.

$$
\begin{aligned}
&\mathcal{L}\phi(t,s,z) = 0, \quad 0 \le t < T,\\
&\phi(T,s,z) = h(s,z) = (s-K)^+ \mathbf{1}_{\{B<z<A\}}\\
&\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}{\sigma_*}^2 z^2 \frac{\partial^2}{\partial z^2} + \rho\sigma\sigma_* sz \frac{\partial^2}{\partial s \partial z}\\
&\quad + r\left(s\frac{\partial}{\partial s} + z\frac{\partial}{\partial z}\right) - rI
\end{aligned} \tag{3.4}
$$

in the region $\{(t,s,z) : 0 \le t < T,\ 0 \le s \le \infty,\ 0 \le z \le \infty\}$, and I is the identity operator.

Here, $\phi(t,s,z)$ is a solution of the unrestricted initial value (IV) problem (3.4), and the solution of the restricted initial and boundary value (IBV) problem (3.3) is given by

$$P(t,s,z) = \phi(t,s,z) - \text{image function of } \phi(t,s,z)$$

using the PDE method of images mentioned earlier.

To solve the closed-form solution of the PDE (3.4), we take advantage of the double Mellin transform method. The technique obtains the solution of the PDE (3.4) more easily, and the solution is expressed by

$$
\begin{aligned}
&\phi(t,s,z)\\
&= s\mathcal{N}_2\left(d_1^r(\tau,\frac{s}{K}), -d_2^r(\tau,\frac{z}{A}), -\rho\right) - e^{-r\tau}K\mathcal{N}_2\left(d_3^r(\tau,\frac{s}{K}), -d_4^r(\tau,\frac{z}{A}), -\rho\right)\\
&\quad - s\mathcal{N}_2\left(d_1^r(\tau,\frac{s}{K}), -d_2^r(\tau,\frac{z}{B}), -\rho\right) + e^{-r\tau}K\mathcal{N}_2\left(d_3^r(\tau,\frac{s}{K}), -d_4^r(\tau,\frac{z}{B}), -\rho\right),
\end{aligned} \tag{3.5}
$$

where d_1^r, d_2^r, d_3^r, and d_4^r are given by equation (2.25), and $\mathcal{N}_2$ is the bivariate cumulative normal distribution function.

3.2. The derivation of the price of the external double-barrier option with the double exponential barrier: Image operator of the double-barrier option

Buchen and Konstandatos [2] examined pricing formulas of double-barrier options with an arbitrary payoff using an image operator. Based on these results, we find the analytic solution of the external double-barrier option using the image operator of the method of images. Therefore, for double-barrier option pricing with exponential barriers, the following Lemma is useful. Additionally, Lemma 3.1 enables us to find the image operator of the external double-barrier option with curved (exponential) barriers in this section.

Lemma 3.1. *If $Q(t,s,z)$ is a solution satisfying the PDE $\mathcal{L}\,Q = 0$, where $\mathcal{L}$ is expressed by (3.3), then the image function of Q with respect to the exponential barrier $z = \beta(t) = Be^{b(T-t)}$ is given by*

$$(3.6)\quad Q^*(t,s,z) = \mathcal{I}_\beta Q(t,s,z) = \left(\tfrac{z}{\beta(t)}\right)^{-(k_{z,b}-1)} Q\left(t, \left(\tfrac{z}{\beta(t)}\right)^{-2\frac{\rho\sigma}{\sigma_*}} s, \tfrac{\beta(t)^2}{z}\right),$$

where $k_{z,b} = \frac{2(r+b)}{\sigma_*{}^2} - 1$ and B are constants mentioned before.

Proof. First, we define a function $Q_b(t,s,z)$ such that

$$(3.7)\qquad Q(t,s,z) := e^{b(T-t)} Q_b(t, se^{-b(T-t)}, ze^{-b(T-t)}).$$

For any b, if we define the following parabolic PDE operator

$$(3.8)\quad \mathcal{L}_b := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}\sigma_*{}^2 z^2 \frac{\partial^2}{\partial z^2} + \rho\sigma\sigma_* sz \frac{\partial^2}{\partial s \partial z} + b\left(s\frac{\partial}{\partial s} + z\frac{\partial}{\partial z}\right) - bI,$$

then $\mathcal{L}\,Q = \mathcal{L}_r\,Q = \mathcal{L}_{r+b}\,Q_b = 0$.

By defining Q_b^* as an image function of Q_b with respect to $v = B$ for the PDE operator $\mathcal{L}_{r+b}$, from Lemma 2.2, we obtain

$$(3.9)$$

$$\begin{aligned} &Q_b^*(t, e^{-b(T-t)}s, e^{-b(T-t)}z) \\ &= \left(\tfrac{z}{Be^{b(T-t)}}\right)^{-(k_{z,b}-1)} Q_b\left(t, \left(\tfrac{z}{Be^{b(T-t)}}\right)^{-2\frac{\rho\sigma}{\sigma_*}} se^{-b(T-t)}, \tfrac{(Be^{b(T-t)})^2}{z} e^{-b(T-t)}\right), \end{aligned}$$

where $\mathcal{L}_{r+b}\,Q_b^* = 0$.

Then, from (3.7), the image solution of $Q(t,s,z)$ is expressed by $Q^*(t,s,z) = e^{b(T-t)} Q_b^*(t, se^{-b(T-t)}, ze^{-b(T-t)})$.

Hence, the image function of Q with respect to $z = \beta(t)$ has

$$(3.10)$$

$$\begin{aligned} &Q^*(t,s,z) \\ &= e^{b(T-t)} Q_b^*(t, se^{-b(T-t)}s, ze^{-b(T-t)}) \end{aligned}$$

$$= e^{b(T-t)} \left(\tfrac{z}{Be^{b(T-t)}}\right)^{-(k_{z,b}-1)} Q_b \left(t, \left(\tfrac{z}{Be^{b(T-t)}}\right)^{-2\frac{\rho\sigma}{\sigma_*}} se^{-b(T-t)}, \tfrac{(Be^{b(T-t)})^2}{z} e^{-b(T-t)}\right)$$

$$= \left(\tfrac{z}{\beta(t)}\right)^{-(k_{z,b}-1)} Q\left(t, \left(\tfrac{z}{\beta(t)}\right)^{-2\frac{\rho\sigma}{\sigma_*}} s, \tfrac{\beta(t)^2}{z}\right).$$

□

Using this lemma, the infinite sequence sum of image operators $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta$ for exponential barrier $\alpha(t)$ and $\beta(t)$, respectively, which is defined by the following Lemma 3.2, allows us to obtain the semi-analytic solution of the PDE (3.3) in terms of $\phi(t,s,z)$. Contrary to the image operator of the external single-barrier option shown in Section 2, the image operator of the external double-barrier option is expressed by any sequence of continuative image operators, similar to Buchen and Konstandatos [2].

Lemma 3.2. *Let $\mathcal{J}_\alpha^\beta$ denote the doubly infinite sequence of image operators*

$$\begin{aligned} \mathcal{J}_\alpha^\beta = I &- \mathcal{I}_\alpha + \mathcal{I}_{\beta\alpha} - \mathcal{I}_{\alpha\beta\alpha} + \mathcal{I}_{\beta\alpha\beta\alpha} + \cdots \\ &-\mathcal{I}_\beta + \mathcal{I}_{\alpha\beta} - \mathcal{I}_{\beta\alpha\beta} + \mathcal{I}_{\alpha\beta\alpha\beta} + \cdots . \end{aligned} \tag{3.11}$$

Then, the solution of the PDE (3.3) *is given by*

$$P(t,s,z) = \mathcal{J}_\alpha^\beta[\phi(t,s,z)],$$

where $\phi(t,s,z)$ is described by (3.5).

Proof. First, because $\mathcal{J}_\alpha^\beta$ is the infinite sequence of the image operator, the property of the image operator leads to $\mathcal{L}\left(\mathcal{J}_\alpha^\beta[\phi(t,x,v)]\right) = 0$. Additionally, if we define

$$\mathcal{A}_{\beta\alpha} = I - \mathcal{I}_\beta + \mathcal{I}_{\beta\alpha} - \mathcal{I}_{\alpha\beta\alpha} + \cdots \quad \text{and} \quad \mathcal{A}_{\alpha\beta} = I - \mathcal{I}_\alpha + \mathcal{I}_{\alpha\beta} - \mathcal{I}_{\beta\alpha\beta} + \cdots$$

then, $\mathcal{J}_\alpha^\beta$ and $\mathcal{J}_\beta^\alpha$ have the following decompositions

$$\mathcal{J}_\alpha^\beta = (I - \mathcal{I}_\alpha)\mathcal{A}_{\beta\alpha} \quad \text{and} \quad \mathcal{J}_\beta^\alpha = (I - \mathcal{I}_\beta)\mathcal{A}_{\alpha\beta},$$

and, from this, we obtain $\mathcal{J}_\alpha^\beta[\phi(t,s,\alpha)] = \mathcal{J}_\alpha^\beta[\phi(t,s,\beta)] = 0$.

Next, from (3.11), if we consider $\mathcal{J}_\alpha^\beta[\phi(T,s,z)] = \phi(T,s,z)+$ image sequence of $\phi(T,s,z)$, then $\phi(T,s,z) = h(s,z)$ for $B < z < A$ in (3.4), and the property of the image operator, imply that the image sequence of $\phi(T,s,z)$ extinguishes at an outside interval (B,A). Hence, $\mathcal{J}_\alpha^\beta[\phi(T,s,z)] = h(s,z)$.

Therefore, $\mathcal{J}_\alpha^\beta[\phi(t,s,z)]$ is the solution of the PDE (3.3), and $P(t,s,z) = \mathcal{J}_\alpha^\beta[\phi(t,s,z)]$. □

To calculate $\mathcal{J}_\alpha^\beta[\phi(t,s,z)]$, the following Lemma 3.3 and Lemma 3.4 are useful.

Lemma 3.3. *For $\alpha(t) = Ae^{a(T-t)}$, $\beta(t) = Be^{b(T-t)}$ defined in Section 3.1 and $\gamma(t) = Ce^{c(T-t)}$,*

$$\mathcal{I}_{\gamma\beta\alpha} := \mathcal{I}_\gamma\mathcal{I}_\beta\mathcal{I}_\alpha = A^{\frac{2(c-b)}{{\sigma_*}^2}} B^{\frac{2(a-c)}{{\sigma_*}^2}} C^{\frac{2(b-a)}{{\sigma_*}^2}} \mathcal{I}_{\frac{\alpha\gamma}{\beta}}. \tag{3.12}$$

Proof. Using Lemma 3.1, the direct calculations lead to,

(3.13)

$$
\begin{aligned}
&\mathcal{I}_{\gamma\beta\alpha}[Q(t,s,z)] \\
&= \mathcal{I}_{\gamma}\mathcal{I}_{\beta}\left[\left(\tfrac{z}{\alpha}\right)^{-(k_{z,a}-1)} Q\left(t, \left(\tfrac{z}{\alpha}\right)^{-2\frac{\rho\sigma}{\sigma_*}} s, \frac{\alpha^2}{z}\right)\right] \\
&= \mathcal{I}_{\gamma}\left[\left(\frac{z}{\beta}\right)^{-(k_{z,b}-1)} \left(\frac{\beta^2}{\alpha z}\right)^{-(k_{z,a}-1)} Q\left(t\left(\frac{\beta z}{\alpha}\right)^{-2\frac{\rho\sigma}{\sigma_*}} s, \frac{\alpha^2}{\beta^2} z\right)\right] \\
&= \left(\frac{z}{\gamma}\right)^{-(k_{z,c}-1)} \left(\frac{\gamma^2}{\beta z}\right)^{-(k_{z,b}-1)} \left(\frac{\beta^2\gamma^2 z}{\alpha}\right)^{-(k_{z,a}-1)} Q\left(t, \left(\frac{\beta z}{\alpha\gamma}\right)^{-2\frac{\rho\sigma}{\sigma_*}} s, \left(\frac{\alpha\gamma}{\beta}\right)^2 \frac{1}{z}\right) \\
&= \alpha^{(k_{z,c}-k_{z,b})}\beta^{(k_{z,a}-k_{z,c})}\gamma^{(k_{z,b}-k_{z,a})} \left(\frac{\beta z}{\alpha\gamma}\right)^{-(k_{z,a}+k_{z,c}-k_{z,b}-1)} Q\left(t, \left(\frac{\beta z}{\alpha\gamma}\right)^{-2\frac{\rho\sigma}{\sigma_*}} s, \left(\frac{\alpha\gamma}{\beta}\right)^2 \frac{1}{z}\right) \\
&= A^{\frac{2(c-b)}{{\sigma_*}^2}} B^{\frac{2(a-c)}{{\sigma_*}^2}} C^{\frac{2(b-a)}{{\sigma_*}^2}} \left(\frac{\beta z}{\alpha\gamma}\right)^{-(k_{z,a}+k_{z,c}-k_{z,b}-1)} Q\left(t, \left(\frac{\beta z}{\alpha\gamma}\right)^{-2\frac{\rho\sigma}{\sigma_*}} s, \left(\frac{\alpha\gamma}{\beta}\right)^2 \frac{1}{z}\right),
\end{aligned}
$$

where $k_{z,a} = \frac{2(r+a)}{{\sigma_*}^2}$, $k_{z,b} = \frac{2(r+b)}{{\sigma_*}^2}$ and $k_{z,c} = \frac{2(r+c)}{{\sigma_*}^2}$. □

For image operators $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta$, if we define the image operator $\mathcal{T}^n_{\alpha\beta}$ as a composition of image operators for all integers $n > 0$,

$$
\begin{aligned}
\mathcal{T}^n_{\alpha\beta} &= \mathcal{I}_{\alpha\beta}\mathcal{I}_{\alpha\beta}\cdots\mathcal{I}_{\alpha\beta} \quad \{n\,(\alpha\beta) - \text{pairs}\} \\
&= \mathcal{I}_{\alpha\beta\alpha\beta\cdots\alpha\beta},
\end{aligned}
\tag{3.14}
$$

we obtain the following lemma.

Lemma 3.4. (a) $\mathcal{T}^n_{\alpha\beta} = \mathcal{I}_\beta \cdot \mathcal{I}_{(\frac{\beta^{n+1}}{\alpha^n})}$. *Hence, for* $n > 0$, *we define* $\mathcal{T}^{-n}_{\alpha\beta} = \mathcal{I}_\beta \cdot \mathcal{I}_{(\frac{\beta^{-n+1}}{\alpha^{-n}})}$.

(b) *From* (a), *for* $n > 0$, $\mathcal{T}^n_{\alpha\beta} = \mathcal{T}^{-n}_{\beta\alpha}$.

(c) $\mathcal{J}^\beta_\alpha = (I - \mathcal{I}_\alpha)\sum_{n=-\infty}^{\infty} \mathcal{T}^n_{\alpha\beta} = (I - \mathcal{I}_\beta)\sum_{n=-\infty}^{\infty} \mathcal{T}^n_{\alpha\beta}$
$= (I - \mathcal{I}_\alpha)\sum_{n=-\infty}^{\infty} \mathcal{T}^n_{\beta\alpha} = (I - \mathcal{I}_\beta)\sum_{n=-\infty}^{\infty} \mathcal{T}^n_{\beta\alpha}$.

Proof. The proof of this lemma (a) follows from Lemma 3.3. Also, combining (a) and (b), and using the direct computation, we obtain the formula of $\mathcal{J}^\beta_\alpha$, (c). Refer to Lemma 3.7, Lemma 3.9, Corollary 3.8, and Corollary 3.10 in Buchen and Konstandatos [2]. □

Hence, from Lemma 3.4,

$$
\begin{aligned}
P(t,s,z) &= \mathcal{J}^\beta_\alpha[\phi(t,s,z)] \\
&= (I - \mathcal{I}_\alpha)\sum_{n=-\infty}^{\infty} \mathcal{T}^n_{\alpha\beta}[\phi(t,s,z)]
\end{aligned}
\tag{3.15}
$$

$$= \sum_{n=-\infty}^{\infty} \left[\mathcal{T}^n_{\alpha\beta} - \mathcal{I}_\alpha \mathcal{T}^n_{\alpha\beta}\right] \phi(t,s,z)$$
$$= \sum_{n=-\infty}^{\infty} \left[\mathcal{T}^n_{\beta\alpha} - \mathcal{I}_\alpha \mathcal{T}^n_{\alpha\beta}\right] \phi(t,s,z).$$

First, in (3.15), to find $\mathcal{T}^n_{\beta\alpha}[\phi(t,s,z)]$, Lemma 3.4(a) yields

$$\begin{aligned} \mathcal{T}^n_{\beta\alpha} &= \mathcal{I}_\alpha \, \mathcal{I}_{\frac{\alpha^{n+1}}{\beta^n}} \\ &= \mathcal{I}_\alpha \, \mathcal{I}_\delta, \end{aligned} \tag{3.16}$$

where $\delta(t) = \frac{\alpha^{n+1}(t)}{\beta^n(t)} = \frac{A^{n+1}}{B^n} e^{\{(n+1)a-nb\}(T-t)}$ and $k_{z,\delta} = \frac{2(r+(n+1)a-nb)}{{\sigma_*}^2}$.

Then, $\mathcal{T}^n_{\beta\alpha}[\phi(t,s,z)]$ has

$$\begin{aligned} &\mathcal{T}^n_{\beta\alpha}[\phi(t,s,z)] \\ &= \left(\frac{\alpha}{z}\right)^{(k_{z,a}-1)} \left(\frac{\delta\, z}{\alpha^2}\right)^{(k_{z,\delta}-1)} \phi\left(t, \left(\frac{\delta}{\alpha}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \frac{\delta^2}{\alpha^2} z\right) \\ &= \left(\frac{z}{\alpha}\right)^{\frac{2(a-b)n}{{\sigma_*}^2}} \left(\frac{\alpha}{\beta}\right)^{(k_{z,\delta}-1)n} \phi\left(t, \left(\frac{\alpha}{\beta}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \left(\frac{\alpha}{\beta}\right)^{2n} z\right). \end{aligned} \tag{3.17}$$

Next, in (3.15), to find $\mathcal{I}_\alpha \mathcal{T}^n_{\beta\alpha}[\phi(t,s,z)]$, we use Lemma 3.4(b) to obtain

$$\begin{aligned} &\mathcal{I}_\alpha \mathcal{T}^n_{\alpha\beta}[\phi(t,s,z)] \\ &= \mathcal{I}_\alpha \mathcal{T}^{-n}_{\beta\alpha}[\phi(t,s,z)] \\ &= \mathcal{I}_\alpha \left(\frac{z}{\alpha}\right)^{\frac{-2(a-b)n}{{\sigma_*}^2}} \left(\frac{\alpha}{\beta}\right)^{-(k_{z,\delta}-1)n} \phi\left(t, \left(\frac{\alpha}{\beta}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \left(\frac{\alpha}{\beta}\right)^{-2n} z\right) \\ &= \left(\frac{z}{\alpha}\right)^{(k_{z,a}-1)-\frac{2(a-b)n}{{\sigma_*}^2}} \left(\frac{\alpha}{\beta}\right)^{-(k_{z,\delta}-1)n} \phi\left(t, \left(\frac{\alpha}{\beta}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \left(\frac{\alpha}{\beta}\right)^{-2n} \frac{\alpha^2}{z}\right). \end{aligned} \tag{3.18}$$

Hence, by integrating (3.15), (3.17), and (3.18), we obtain following results.

Theorem 3.1. *The unique arbitrage-free price of the external double knock-out call given by PDE* (3.3) *can be expressed explicitly by the doubly infinite sum in terms of* $\phi(t,s,z)$ *mentioned in* (3.5)*. Then, we obtain the following formula*

$$\begin{aligned} P(t,x,z) = \sum_{n=-\infty}^{\infty} \Bigg[&\left(\tfrac{z}{\alpha}\right)^{\frac{2(a-b)n}{{\sigma_*}^2}} \left(\tfrac{\alpha}{\beta}\right)^{((\frac{2(r+a)}{{\sigma_*}^2}-1))n} \phi\left(t, \left(\tfrac{\alpha}{\beta}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \left(\tfrac{\alpha}{\beta}\right)^{2n} z\right) \\ &- \left(\tfrac{z}{\alpha}\right)^{(\frac{2(r+a)}{{\sigma_*}^2}-1)+\frac{2(a-b)n}{{\sigma_*}^2}} \left(\tfrac{\alpha}{\beta}\right)^{(k_{z,\delta}-1)n} \phi\left(t, \left(\tfrac{\alpha}{\beta}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \left(\tfrac{\alpha}{\beta}\right)^{2n} \tfrac{\alpha^2}{z}\right)\Bigg], \end{aligned} \tag{3.19}$$

where $k_{z,\delta} = \frac{2(r+(n+1)a-nb)}{{\sigma_*}^2}$, *and* a *and* b *are constants in* $\alpha(t) = Ae^{a(T-t)}$ *and* $\beta(t) = Be^{b(T-t)}$.

Proof. By combining (3.15), (3.17), (3.18), and $\phi(t,s,z)$ in (3.5), we obtain the semi-analytic solution (3.19) of the external double-barrier European call option. □

4. External-chained single barrier option with curved barriers

This section addresses barrier options where barrier monitoring begins at a random time when the barrier state variable first hits an exponential barrier or two exponential barriers in a specified order. To find the exact-form formula of the external-chained barrier option with the bivariate normal cumulative distribution function, we exploit the pricing formula of the external European call option in Section 2. From the method of reflection principle and the change of measure under risk-neutral probability, we replace the expectation of the external-chained barrier option with the expectation from the external European call option. Then, we derive the closed-form formula of the external-chained barrier option with exponential barriers immediately.

In SDE (2.2), S_t and Z_t have the following geometric Brownian motion.

$$\begin{aligned} S_t &= S_0 \exp(\tilde{\mu}\, t + \sigma W_t^{s*}), \qquad \tilde{\mu} = (r - \frac{\sigma^2}{2}) \\ Z_t &= Z_0 \exp(\tilde{\mu}_*\, t + \sigma_* W_t^{z*}), \qquad \tilde{\mu}_* = (r - \frac{{\sigma_*}^2}{2}) \end{aligned} \tag{4.1}$$

where r is the risk-free interest rate, σ is a positive constant, S_t and Z_t are underlying asset prices, and the barrier state variable, respectively, and W_t^{s*} and W_t^{z*} are standard Brownian motions under the risk-neutral measure P^*.

Let the upper and lower exponential barriers in the interval $[0,T]$ be $U(t) = Ae^{\xi_1 t}$ and $D(t) = Be^{\xi_2 t}$ $(A > Z(0) > B,\ \xi_1 \geq \xi_2)$, respectively, and

$$\begin{aligned} &G_t = \frac{1}{\sigma_*}\ln\left(\frac{Z_t}{Z_0}\right) = W_t^{z*} + \frac{\tilde{\mu}_*}{\sigma_*}t, \quad u(t) = \frac{1}{\sigma_*}\ln\left(\frac{U(t)}{Z_0}\right) = \frac{\xi_1}{\sigma_*}t + a, \\ &d(t) = \frac{1}{\sigma_*}\ln\left(\frac{D(t)}{Z_0}\right) = \frac{\xi_2}{\sigma_*}t + b, \end{aligned} \tag{4.2}$$

where $a = \frac{1}{\sigma_*}\ln\frac{A}{Z_0}$ and $b = \frac{1}{\sigma_*}\ln\frac{B}{Z_0}$.

4.1. The case of crossing a barrier

First, we derive the pricing formula for a down-and-out call option commencing at a random time when the barrier state variable reaches the upper exponential barrier $U(t)$. If the barrier state process rises above $U(t)$ and then goes below $D(t)$ before the time of expiry T, the payoff of the option holder becomes zero, and its payoff is a call otherwise.

Theorem 4.1. *Let us consider a down-and-out external barrier call option, which is activated at time $\tau = \min\{t > 0 : Z_t = U(t)\}$. For $t^* < \tau$, the price of the external-chained barrier option activated at time τ is described by*

(4.3) $DOC_u(t^*, s, z)$

$$
\begin{aligned}
&= e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} s\mathcal{N}_2\left(d_1^r(T-t^*,\tfrac{s}{K}),-d_2^r(T-t^*,\tfrac{D(t^*)z}{U(t^*)^2}),-\rho\right)\\
&\quad - e^{-r(T-t^*)}K\mathcal{N}_2\left(d_3^r(T-t^*,\tfrac{s}{K}),-d_4^r(T-t^*,\tfrac{D(t^*)z}{U(t^*)^2}),-\rho\right)\\
&\quad + e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}\Bigg[\left(\tfrac{D(t^*)z}{U(t^*)^2}\right)^{-\frac{2\rho\sigma}{\sigma_*}-(\tilde{k}_*-1)}\\
&\qquad s\mathcal{N}_2\left(d_1^r(T-t^*,\left(\tfrac{D(t^*)z}{U(t^*)^2}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\tfrac{s}{K}),d_2^r(T-t^*,\tfrac{U(t^*)^2}{D(t^*)z}),-\rho\right)\Bigg]\\
&\quad - e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}\Bigg[e^{-r(T-t^*)}\,K\left(\tfrac{D(t^*)z}{U(t^*)}\right)^{-(\tilde{k}_*-1)}\\
&\qquad \mathcal{N}_2\left(d_3^r(T-t^*,\left(\tfrac{D(t^*)z}{U(t^*)^2}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\tfrac{s}{K}),-d_4^r(T-t^*,\tfrac{U(t^*)^2}{D(t^*)z}),-\rho\right)\Bigg],
\end{aligned}
$$

where $\tilde{k}_* = \frac{2(-r+{\sigma_*}^2+\xi_2)}{{\sigma_*}^2}$ *and* d_1^r, d_2^r, d_3^r, *and* d_4^r *are given by equation* (2.25).

Proof. Under the risk-neutral measure, the expectation representation of the knock-out external barrier call option price at time $t^* < \tau$

(4.4)
$$
\begin{aligned}
&DOC_u(t^*,s,z)\\
&= e^{-r(T-t^*)}\\
&\quad E^{P^*}[(S(T)-K)^+\,\mathbf{1}_{\{\min_{\tau<\gamma<T}(Z_\gamma-D(\gamma))>0,\,\tau\le T,\,Z_\tau=U(\tau)\}}|\,S_{t^*}=s,\,Z_{t^*}=z]\\
&= e^{-r(T-t^*)}\\
&\quad E^{P^*}[(S(T)-K)^+\,\mathbf{1}_{\{\min_{\tau<\gamma<T}(G_\gamma-d(\gamma))>0,\,\tau\le T,\,G_\tau=u(\tau)\}}|\,S_{t^*}=s,\,Z_{t^*}=z].
\end{aligned}
$$

If we define a process $H_t = G_t - \frac{\xi_1}{\sigma_*}t = W_t^{z*} + \frac{1}{\sigma_*}(\tilde{\mu}_* - \xi_1)\,t$, then H_t is a standard Brownian motion under the measure Q, defined by

$$\frac{dQ}{dP^*} = \exp\left(-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,W_T^{z*} - \frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T\right). \tag{4.5}$$

Also, let us define a process $\tilde{H}_t,\ t\in[0,T]$, defined by the formula

$$\tilde{H}_t := \begin{cases} H_t & t\le\tau \\ 2a - H_t & t>\tau \end{cases}. \tag{4.6}$$

Using the reflection principle, the process $\tilde{H}_t$ also follows a standard Brownian motion under Q and from the definition of H_t, (4.5) and (4.6), (4.4) leads to

(4.7)
$$
\begin{aligned}
&DOC_u(t^*,s,z)\\
&= e^{-r(T-t^*)}E^{P^*}\big[(S(T)-K)^+
\end{aligned}
$$

$$\mathbf{1}_{\min_{\tau<\gamma<T}\left(H_\gamma+\frac{\xi_1}{\sigma_*}\gamma-d(\gamma)\right)>0,\,\tau\le T,\,H_\tau=a} \mid S_{t^*}=s, Z_{t^*}=z\Big]$$

$$= e^{-r(T-t^*)}E^Q\Big[e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,H_T-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T}$$

$$(S(T)-K)^+\,\mathbf{1}_{\{\min_{\tau<\gamma<T}\left(H_\gamma+\frac{\xi_1}{\sigma_*}\gamma-d(\gamma)\right)>0,\,\tau\le T,\,H_\tau=a\}} \mid S_{t^*}=s, Z_{t^*}=z\Big]$$

$$= e^{-r(T-t^*)}E^Q\Big[e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,(2a-\tilde{H}_T)-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T}$$

$$(S(T)-K)^+\,\mathbf{1}_{\{\min_{0<\gamma<T}\left(2a-\tilde{H}_\gamma+\frac{\xi_1}{\sigma_*}\gamma-d(\gamma)\right)>0\}} \mid S_{t^*}=s, Z_{t^*}=z\Big]$$

$$= e^{-r(T-t^*)}\,e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}E^Q\Big[e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\tilde{H}_T-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T}$$

$$(S(T)-K)^+\,\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{H}_\gamma-\frac{\xi_1}{\sigma_*}\gamma-(2a-d(\gamma))\right)<0\}} \mid S_{t^*}=s, Z_{t^*}=z\Big].$$

Let us again define an equivalent probability measure $\tilde{P}^*$ by setting

$$\frac{d\,\tilde{P}^*}{d\,Q}=\exp\left(-\frac{1}{\sigma_*}\,(\tilde{\mu}_*-\xi_1)\;\tilde{H}_T-\frac{1}{2{\sigma_*}^2}\,(\tilde{\mu}_*-\xi_1)^2\;T\right) \tag{4.8}$$

so that the process $\tilde{W}_t^{z*} := \tilde{H}_t+\frac{1}{\sigma_*}\,(\tilde{\mu}_*-\xi_1)\;t,\;\; t\in[0,T]$ follows a standard Brownian motion under $\tilde{P}^*$.

Using the definition of H_t and $\tilde{H}_t$ mentioned above, we define $\tilde{W}_t^{z*}$

$$\tilde{W}_t^{z*}=\begin{cases} W_t^{z*}+\dfrac{2(\tilde{\mu}_*-\xi_1)}{\sigma_*}t & t\le\tau \\ 2a-W_t^{z*} & t>\tau \end{cases}. \tag{4.9}$$

Then, (4.7) leads to

(4.10)

$$e^{-r(T-t^*)}\,e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}E^Q\Big[e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\tilde{H}_T-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T}$$

$$(S(T)-K)^+\,\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{H}_\gamma-\frac{\xi_1}{\sigma_*}\gamma-(2a-d(\gamma))\right)<0\}} \mid S_{t^*}=s, Z_{t^*}=z\Big]$$

$$= e^{-r(T-t^*)}\,e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}E^Q\Big[e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\tilde{H}_T-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T}$$

$$(S(T)-K)^+\,\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{G}_\gamma-\frac{\tilde{\mu}_*}{\sigma_*}\gamma-(2a-d(\gamma))\right)<0\}} \mid S_{t^*}=s, Z_{t^*}=z\Big]$$

$$= e^{-r(T-t^*)}\,e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}$$

$$E^{\tilde{P}^*}\Big[(S(T)-K)^+\,\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{H}_\gamma-\frac{\xi_1}{\sigma_*}\gamma-(2a-d(\gamma))\right)<0\}} \mid S_{t^*}=s, Z_{t^*}=z\Big]$$

$$= e^{-r(T-t^*)}\,e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}$$

$$E^{\tilde{P}^*}\Big[(S(T)-K)^+\,\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{W}_\gamma^{z*}-\frac{\tilde{\mu}_*}{\sigma_*}\gamma-(2a-d(\gamma))\right)<0\}} \mid S_{t^*}=s, Z_{t^*}=z\Big]$$

$$= e^{-r(T-t^*)}\,e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}$$

$$
\begin{aligned}
& E^{\tilde{P}^*}\left[(S(T)-K)^+\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{G}_\gamma-(2a-d(\gamma))\right)<0\}}\mid S_{t^*}=s, Z_{t^*}=z\right] \\
= {} & e^{-r(T-t^*)}\, e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} \\
& E^{\tilde{P}^*}\left[(S(T)-K)^+\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{G}_\gamma-\tilde{d}(\gamma)\right)<0\}}\mid S_{t^*}=s, Z_{t^*}=z\right],
\end{aligned}
$$

where $\tilde{G}_t=\tilde{H}_t-\frac{\xi_1}{\sigma_*}t$ and $\tilde{d}(t)=2a-d(t)$.

Now, let us define $\tilde{Z}_t$ satisfying following SDE:

$$
d\,\tilde{Z}_t=(-\tilde{\mu}_*+\frac{{\sigma_*}^2}{2})\,\tilde{Z}_t\,d\,t+\sigma_*\,\tilde{Z}_t\,d\,\tilde{W}_t^{z*}. \qquad \tilde{Z}_0=Z_0. \tag{4.11}
$$

Then, from the definition of G_t and $d(t)$ in (4.2), we obtain $\tilde{G}_t=\frac{1}{\sigma_*}\ln\left(\frac{\tilde{Z}_t}{Z_0}\right)$ and $\tilde{d}(t)=\frac{1}{\sigma_*}\ln\left(\frac{\tilde{D}(t)}{Z_0}\right)$. To compute the conditional expectation of (4.10), we review the external barrier call option mentioned in Section 2 where the terminal payoff depends on the payoff state variable S_t, and knock-out occurs when the barrier state variable $\tilde{Z}_t$ breaches the upstream exponential barrier $\tilde{D}(t)=\frac{A^2}{D(t)}$. From (4.9) and SDE (4.11), for $t<\tau$, we have

$$
\tilde{Z}_t=Z_t\,e^{-2\xi_1 t}, \tag{4.12}
$$

and the price of this up-and-out call option with an external barrier variable $\tilde{Z}_t$ and the upstream exponential barrier $\tilde{D}(t)=\frac{A^2}{D(t)}$ is expressed by

$$
\begin{aligned}
& \tilde{P}(t^*,s,\tilde{z}) \\
& = E^{\tilde{P}^*}\left[e^{-r(T-t^*)}\,(S_T-K)^+\mathbf{1}_{\left\{\max_{0\le\gamma\le t}(\tilde{Z}_\gamma-\tilde{D}(\gamma))<0\right\}}\,|S_{t^*}=s,\tilde{Z}_{t^*}=\tilde{z}\right],
\end{aligned} \tag{4.13}
$$

where the closed formula of $\tilde{P}(t^*,s,\tilde{z})$ is given by (2.25).

Finally, from the definition of $\tilde{G}$ and $\tilde{d}$ stated above, the equation (4.10) yields

$$
\begin{aligned}
& e^{-r(T-t^*)}\,e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} \\
& E^{\tilde{P}^*}\left[(S(T)-K)^+\mathbf{1}_{\{\max_{0<\gamma<T}\left(\tilde{Z}_\gamma-\tilde{D}(\gamma)\right)<0\}}\mid S_{t^*}=s,\tilde{Z}_{t^*}=ze^{-2\xi_1 t^*}\right]
\end{aligned} \tag{4.14}
$$

$$
= e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}}\,\tilde{P}(t^*,s,z\,e^{-2\xi_1 t^*}). \tag{4.15}
$$

However, because $\tilde{P}(t^*,s,z\,e^{-2\xi_1 t^*})$ is the up-and-out call option with an external barrier in (4.13), from (2.25), we obtain the desired closed-form formula of DOC_u given by (4.3). □

4.2. Case of crossing two barriers

This subsection considers barrier options activated in the event that the barrier state variable crosses two barrier levels successively in a specified order. Alternatively, we derive the pricing formula for an up-and-out call option created when the external barrier state variable hits the lower exponential barrier

$D(t)$ after hitting the upper exponential barrier $U(t)$. The payoff value of the option is zero if the barrier state process increases above $U(t)$, then falls below $D(t)$, and then rises above $U(t)$ before expiration time T, and its payoff is a call otherwise.

Theorem 4.2. *Let us consider an up-and-out external barrier call option that is activated at time* $\tau_2 = \min\{t > \tau_1 : Z_t = D(t), \tau_1 = \min\{t > 0 : Z_t = U(t)\}\}$. *Then, for* $t^* < \tau_1$, *the price of the external-chained barrier option activated at time* τ_2 *is given by*

(4.16)

$$\begin{aligned}
&UOC_{ud}(t^*, s, z) \\
&= e^{\theta} s\mathcal{N}_2\left(d_1^r(T-t^*, \frac{s}{K}), -d_2^r(T-t^*, \frac{D(t^*)^2\, z}{U(t^*)^3}), -\rho\right) \\
&\quad - e^{-r(T-t^*)} K\mathcal{N}_2\left(d_3^r(T-t^*, \frac{s}{K}), -d_4^r(T-t^*, \frac{D(t^*)^2\, z}{U(t^*)^3}), -\rho\right) \\
&\quad + e^{\theta}\left[\left(\frac{D(t^*)^2 z}{U(t^*)^3}\right)^{-\frac{2\rho\sigma}{\sigma_*}-(\hat{k}_*-1)}\right. \\
&\qquad\qquad \left. s\mathcal{N}_2\left(d_1^r(T-t^*, \left(\frac{D(t^*)^2\, z}{U(t^*)^3}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\frac{s}{K}), d_2^r(T-t^*, \frac{U(t^*)^3}{D(t^*)^2\, z}), -\rho\right)\right] \\
&\quad - e^{\theta}\left[e^{-r(T-t^*)}\, K\left(\frac{D(t^*)^2\, z}{U(t^*)^3}\right)^{-(\hat{k}_*-1)}\right. \\
&\qquad\qquad \left. \mathcal{N}_2\left(d_3^r(T-t^*, \left(\frac{D(t^*)^2\, z}{U(t^*)^3}\right)^{-\frac{2\rho\sigma}{\sigma_*}}\frac{s}{K}), -d_4^r(T-t^*, \frac{U(t^*)^3}{D(t^*)^2 z}), -\rho\right)\right],
\end{aligned}$$

where $\hat{k}_* = \frac{2(r-\xi_1)}{{\sigma_*}^2}$, $\theta := 2(b-a)\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*} - 2(b-2a)\frac{(\xi_2-\xi_1)}{\sigma_*}$, $a = \dfrac{1}{\sigma_*}\ln\frac{A}{Z_0}$, $b = \frac{1}{\sigma_*}\ln\frac{B}{Z_0}$, *and* $U(t) = Ae^{\xi_1 t}$ *and* $D(t) = Be^{\xi_2 t}$ *are the upper and lower exponential barriers, respectively, satisfying* $A > V(0) > B$, *where* $\xi_1 \geq \xi_2$.

Proof. First, under the risk-neutral measure, the price of the up-and-out call option at time $t < \tau_1$ is given by

(4.17)

$$\begin{aligned}
&UOC_{ud}(t^*, s, z) \\
&= e^{-r(T-t^*)} E^{P^*}[(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\max_{\tau_2<\gamma<T}(Z_\gamma - U(\gamma))<0,\, \tau_1<\tau_2\leq T,\, Z_{\tau_1}=U(\tau_1)\,, Z_{\tau_2}=D(\tau_2)\}} |\, S_{t^*}=s,\, Z_{t^*}=z] \\
&= e^{-r(T-t^*)} E^{P^*}[(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\max_{\tau_2<\gamma<T}(G_\gamma - u(\gamma))<0,\, \tau_1<\tau_2\leq T,\, G_{\tau_1}=u(\tau_1)\,, G_{\tau_2}=d(\tau_2)\}} |\, S_{t^*}=s,\, Z_{t^*}=z],
\end{aligned}$$

$$
\begin{aligned}
&= e^{-r(T-t^*)} E^{P^*}\Big[(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\max_{\tau_2<\gamma<T}\left(H_\gamma+\frac{\xi_1}{\sigma_*}-u(\gamma)\right)<0,\,\tau_1<\tau_2\le T,\,H_{\tau_1}=a,\,H_{\tau_2}=k(\tau_2)\}} \,|\, S_{t^*}=s,\, Z_{t^*}=z\Big] \\
&= e^{-r(T-t^*)} E^{Q}\Big[e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,H_T-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T}(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\max_{\tau_2<\gamma<T}\left(H_\gamma+\frac{\xi_1}{\sigma_*}\gamma-u(\gamma)\right)<0,\,\tau_1<\tau_2\le T,\,H_{\tau_1}=a,\,H_{\tau_2}=k(\tau_2)\}} \,|\, S_{t^*}=s, Z_{t^*}=z\Big],
\end{aligned}
$$

where $H_t = G_t - \frac{\xi_1}{\sigma_*} t = W_t^{z*} + \frac{1}{\sigma_*}(\tilde{\mu}_* - \xi_1)\, t$ is a standard Brownian motion under the measure Q so that Q is defined by

$$
\frac{dQ}{dP^*} = \exp\left(-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,W_T^{z*} - \frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T\right) \tag{4.18}
$$

and $k(t) = \frac{(\xi_2-\xi_1)}{\sigma_v} t + b$.

Similar to Section 4.1, we consider a process $\tilde{H}(t)$, $\in [0,T]$ defined by the formula

$$
\tilde{H}(t) = \begin{cases} H(t) & t \le \tau_1 \\ 2a - H(t) & t > \tau_1 \end{cases}, \tag{4.19}
$$

and the reflection principle implies that the process $\tilde{H}(t)$ is a Brownian motion under Q.

Hence, (4.17) leads to

(4.20)

$$
\begin{aligned}
&e^{-r(T-t^*)} E^{Q}\Big[e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,H_T-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T}(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\max_{\tau_2<\gamma<T}\left(H_\gamma+\frac{\xi_1}{\sigma_*}\gamma-u(\gamma)\right)<0,\,\tau_1<\tau_2\le T,\,H_{\tau_1}=a,\,H_{\tau_2}=k(\tau_2)\}} \,|\, S_{t^*}=s, Z_{t^*}=z\Big] \\
&= e^{-r(T-t^*)} E^{Q}\Big[e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,(2a-\tilde{H}_T)-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T}\,(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\max_{\tau_2<\gamma<T}\left(-\tilde{H}_\gamma+\frac{\xi_1}{\sigma_*}\gamma+(2a-u(\gamma))\right)<0,\,\tau_2\le T,\,\tilde{H}_{\tau_2}=2a-k(\tau_2)\}} \,|\, S_{t^*}=s, Z_{t^*}=z\Big] \\
&= e^{-r(T-t^*)+2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} E^{Q}\Big[e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\tilde{H}_T-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T}\,(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\min_{\tau_2<\gamma<T}\left(\tilde{H}_\gamma-\frac{\xi_1}{\sigma_*}\gamma-(2a-u(\gamma))\right)>0,\,\tau_2\le T,\,\tilde{H}_{\tau_2}=2a-k(\tau_2)\}} \,|\, S_{t^*}=s, Z_{t^*}=z\Big].
\end{aligned}
$$

Additionally, we introduce a process L_t so that $L_t = \tilde{H}_t + \frac{(\xi_2-\xi_1)}{\sigma_*} t$ is a standard Brownian motion under the measure R, defined by

$$
\frac{dR}{dQ} = \exp\left(-\frac{(\xi_2-\xi_1)}{\sigma_*}\,\tilde{H}_T - \frac{1}{2\sigma_*{}^2}(\xi_2-\xi_1)^2\,T\right). \tag{4.21}
$$

Then, (4.20) yields

$$
\begin{aligned}
&e^{-r(T-t^*)+2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} E^{Q}\Big[e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\tilde{H}_T-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T}\,(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\min_{\tau_2<\gamma<T}\left(\tilde{H}_\gamma-\frac{\xi_1}{\sigma_*}\gamma-(2a-u(\gamma))\right)>0,\,\tau_2\le T,\,\tilde{H}_{\tau_2}=2a-k(\tau_2)\}} \,|\, S_{t^*}=s, Z_{t^*}=z\Big]
\end{aligned}
$$

$$
\begin{aligned}
(4.22)\quad &= e^{-r(T-t^*)}\, e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} \\
&\quad E^R\Big[e^{\frac{(\xi_2-\xi_1)}{\sigma_*}\tilde{H}_T-\frac{1}{2\sigma_*{}^2}(\xi_2-\xi_1)^2\,T}\, e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\tilde{H}_T-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T} \\
&\qquad (S(T)-K)^+\,\mathbf{1}_{\Lambda_1}\,|\,S_{t^*}=s, Z_{t^*}=z\,\Big] \\
(4.23)\quad &= e^{-r(T-t^*)}\, e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} E^R\Big[e^{\frac{(\xi_2-\xi_1)}{\sigma_*}(L_T-\frac{(\xi_2-\xi_1)}{\sigma_*}T)+\frac{1}{2\sigma_*{}^2}(\xi_2-\xi_1)^2\,T} \\
&\quad e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,(L_T-\frac{(\xi_2-\xi_1)}{\sigma_*}T)-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T} \\
&\quad (S(T)-K)^+\,\mathbf{1}_{\Lambda_2}\,|\,S_{t^*}=s, Z_{t^*}=z\,\Big],
\end{aligned}
$$

where Λ_1 and Λ_2 in (4.22) and (4.23) are described by

$$
(4.24)\quad
\begin{aligned}
\Lambda_1 &= \left\{\min_{\tau_2<\gamma<T}\left(\tilde{H}_\gamma-\frac{\xi_1}{\sigma_*}\gamma-(2a-u(\gamma))\right)>0,\ \tau_2\le T,\ \tilde{H}_{\tau_2}=2a-k(\tau_2)\right\} \\
&\quad \text{and} \\
\Lambda_2 &= \left\{\min_{\tau_2<\gamma<T}\left(L_\gamma-\frac{\xi_2}{\sigma_*}\gamma-(2a-u(\gamma))\right)>0,\ \tau_2\le T,\ L_{\tau_2}=2a-b\right\},
\end{aligned}
$$

respectively.

To compute the expectation under the measure R in (4.23), we consider a process $\tilde{L}_t,\ t\in[0,T]$ defined by the formula

$$
(4.25)\qquad \tilde{L}_t := \begin{cases} L_t & t\le\tau_2 \\ 2(2a-b)-L_t & t>\tau_2 \end{cases}.
$$

Using the reflection principle, the process $\tilde{L}_t$ is a standard Brownian motion under R, and (4.23) leads to

$$
\begin{aligned}
&(4.26) \\
&e^{-r(T-t^*)}\, e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} E^R\Big[e^{\frac{(\xi_2-\xi_1)}{\sigma_*}(L_T-\frac{(\xi_2-\xi_1)}{\sigma_*}T)+\frac{1}{2\sigma_*{}^2}(\xi_2-\xi_1)^2\,T} \\
&e^{-\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,(L_T-\frac{(\xi_2-\xi_1)}{\sigma_*}T)-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T}\,(S(T)-K)^+\,\mathbf{1}_{\Lambda_3}\,|\,S_{t^*}=s, Z_{t^*}=z\Big] \\
&= e^{-r(T-t^*)}\, e^{\theta} E^R\Big[e^{-\frac{(\xi_2-\xi_1)}{\sigma_*}(\tilde{L}_T+\frac{(\xi_2-\xi_1)}{\sigma_*}T)+\frac{1}{2\sigma_*{}^2}(\xi_2-\xi_1)^2\,T} \\
&e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,(\tilde{L}_T+\frac{(\xi_2-\xi_1)}{\sigma_*}T)-\frac{1}{2\sigma_*{}^2}(\tilde{\mu}_*-\xi_1)^2\,T}\,(S(T)-K)^+\,\mathbf{1}_{\Lambda_3}\,|\,S_{t^*}=s, Z_{t^*}=z\Big],
\end{aligned}
$$

where θ is defined by $\theta := 2(b-a)\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}-2(b-2a)\frac{(\xi_2-\xi_1)}{\sigma_*}$, and Λ_3 in (4.26) is given by $\Lambda_3=\left\{\max_{0\le\gamma<T}\left(\tilde{L}_\gamma+\frac{\xi_2}{\sigma_*}\gamma-(u(\gamma)-2(a-b))\right)<0\right\}$.

Let us again introduce a process $\hat{H}_t$ so that $\hat{H}_t = \tilde{L}_t + \frac{(\xi_2-\xi_1)}{\sigma_*}\, t$ is a standard Brownian motion under the measure Q^*, defined by

$$\frac{d\,Q^*}{d\,R} = \exp\left(-\frac{(\xi_2-\xi_1)}{\sigma_*}\,\tilde{L}_T - \frac{1}{2{\sigma_*}^2}\,(\xi_2-\xi_1)^2\, T\right). \tag{4.27}$$

Hence, (4.26) is expressed by

(4.28)

$$\begin{aligned}
&e^{-r(T-t^*)}\, e^{\theta} E^R\Big[e^{-\frac{(\xi_2-\xi_1)}{\sigma_*}\,(\tilde{L}_T+\frac{(\xi_2-\xi_1)}{\sigma_*}T)+\frac{1}{2{\sigma_*}^2}(\xi_2-\xi_1)^2\,T} \\
&\quad e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,(\tilde{L}_T+\frac{(\xi_2-\xi_1)}{\sigma_*}T)-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T}\,(S(T)-K)^+\,\mathbf{1}_{\Lambda_3}\,|\,S_{t^*}=s, Z_{t^*}=z\Big] \\
&= e^{-r(T-t^*)}\, e^{\theta}\, E^{Q^*}\Big[e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\hat{H}_T-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T} \\
&\quad (S(T)-K)^+\,\mathbf{1}_{\{\max_{0\le\gamma<T}\left(\hat{H}_\gamma+\frac{\xi_1}{\sigma_*}\gamma-(u(\gamma)-2(b-a))\right)<0\}}\,|\,S_{t^*}=s, Z_{t^*}=z\Big].
\end{aligned}$$

Finally, let us define an equivalent probability measure $\hat{P}^*$ by setting

$$\frac{d\,\hat{P}^*}{d\,Q^*} = \exp\left(\frac{1}{\sigma_*}\,(\tilde{\mu}_*-\xi_1)\;\hat{H}_T - \frac{1}{2{\sigma_*}^2}\,(\tilde{\mu}_*-\xi_1)^2\;T\right) \tag{4.29}$$

so that the process $\hat{W}^{z*}_t := \hat{H}_t - \frac{1}{\sigma_*}\,(\tilde{\mu}_*-\xi_1)\;t,\ t\in[0,T]$ follows a standard Brownian motion under $\hat{P}^*$.

By the definition of H_t and $\hat{H}_t$ stated previously, we define the following $\hat{W}^{z*}_t$

$$\hat{W}^{z*}_t := \begin{cases} W^{z*}_t + \frac{2(\xi_2-\xi_1)}{\sigma_*}t & t\le\tau_1 \\ 2a - W^{z*}_t - \frac{2(\tilde{\mu}_*-\xi_2)}{\sigma_*}t & \tau_1<t\le\tau_2 \\ W^{z*}_t + 2(a-b) & \tau_2<t \end{cases}. \tag{4.30}$$

Then, (4.28) satisfies the following equations

(4.31)

$$\begin{aligned}
&e^{-r(T-t^*)}\, e^{\theta}\, E^{Q^*}\Big[e^{\frac{1}{\sigma_*}(\tilde{\mu}_*-\xi_1)\,\hat{H}_T-\frac{1}{2{\sigma_*}^2}(\tilde{\mu}_*-\xi_1)^2\,T} \\
&\quad (S(T)-K)^+\,\mathbf{1}_{\{\max_{0\le\gamma<T}\left(\hat{H}_\gamma+\frac{\xi_1}{\sigma_*}\gamma-(u(\gamma)-2(b-a))\right)<0\}}\,|\,S_{t^*}=s, Z_{t^*}=z\Big] \\
&= e^{-r(T-t^*)}\, e^{\theta}\, E^{\hat{P}^*}\Big[(S(T)-K)^+ \\
&\quad \mathbf{1}_{\{\max_{0\le\gamma<T}\left(\hat{W}^{z*}_\gamma+\frac{\tilde{\mu}_*}{\sigma_*}\gamma-(u(\gamma)-2(b-a))\right)<0\}}\,|\,S_{t^*}=s, Z_{t^*}=z\Big] \\
&= e^{-r(T-t^*)}\, e^{\theta}\, E^{\hat{P}^*}\Big[(S(T)-K)^+\,\mathbf{1}_{\{\max_{0\le\gamma<T}\left(\hat{G}_\gamma-\hat{u}(\gamma)\right)<0\}}\,|\,S_{t^*}=s, Z_{t^*}=z\Big],
\end{aligned}$$

where $\hat{G}_t = \hat{H}_t + \frac{\xi_1}{\sigma_*}$ and $\hat{u}(t) = u(t) - 2(b-a)$.

Similar to Section 4.1, to compute the expectation of (4.31), we consider $\hat{Z}_t$ satisfying the following SDE:

$$d\,\hat{Z}_t = (\tilde{\mu}_* + \frac{{\sigma_*}^2}{2})\,\hat{Z}_t\,dt + \sigma_*\,\hat{Z}_t\,d\,\hat{W}_t^{z*}, \qquad \hat{Z}_0 = Z_0. \tag{4.32}$$

From the definition of G_t and $u(t)$ in (4.2), we have $\hat{G}_t = \frac{1}{\sigma_*}\ln\left(\frac{\hat{Z}_t}{Z_0}\right)$ and $\hat{u}(t) = \frac{1}{\sigma_*}\ln\left(\frac{\hat{U}(t)}{Z_0}\right)$. Then, we review the price of the external up-and-out barrier call option given by

$$\begin{aligned}(4.33)\quad & \hat{P}(t^*, s, \hat{z}) \\ &= E^{\hat{P}^*}\left[e^{-r(T-t^*)}\,(S_T-K)^+\mathbf{1}_{\left\{\max_{0\le\gamma\le t}(\hat{Z}_\gamma-\hat{U}(\gamma))<0\right\}}\,|S_{t^*}=s, \hat{Z}_{t^*}=\hat{z}\right],\end{aligned}$$

where the terminal payoff depends on the payoff state variable S_t, and knock-out occurs when the barrier state variable $\tilde{Z}_t$ breaches the upstream exponential barrier $\hat{U}(t) = \frac{A^2}{B^2}U(t)$. Then, the closed-form solution of $\hat{P}(t, s, \hat{z})$ is described by (2.25).

For $t < \tau_1$, because $\hat{Z}_t = Z_0\,\exp\left(\hat{\mu}_*\,t + \sigma_*\,\hat{W}_t^{z*}\right)$,

$$\hat{Z}_t = Z_t\,e^{2(\xi_2-\xi_1)t} \tag{4.34}$$

and (4.31) leads to

$$\begin{aligned}(4.35)\quad & \\ & e^{-r(T-t^*)}\,e^\theta\,E^{\hat{P}^*}\left[\,(S(T)-K)^+\,\mathbf{1}_{\{\max_{0\le\gamma<T}\left(\hat{G}_\gamma-\hat{u}(\gamma)\right)<0\}}\,|\,S_{t^*}=s, Z_{t^*}=z\right] \\ &= e^{-r(T-t^*)}\,e^\theta E^{\hat{P}^*}\left[\,(S(T)-K)^+\right. \\ &\quad \left.\mathbf{1}_{\{\max_{0<\gamma<T}\left(\hat{Z}_\gamma-\hat{U}(\gamma)\right)<0\}}\,|\,S_{t^*}=s, \hat{Z}_{t^*}=z\,e^{2(\xi_2-\xi_1)t^*}\,\right] \\ &= e^\theta\,\hat{P}(t^*, s, z\,e^{2(\xi_2-\xi_1)t^*}).\end{aligned}$$

Therefore, because $\hat{P}(t^*, s, z\,e^{2(\xi_2-\xi_1)t^*})$ is the up-and-out call option with the external barrier given by (2.25), the desired closed formula of UOC_{ud} given by (4.16) is derived. □

5. External-chained double-barrier options with curved barriers

This section derives the semi-analytic solution of the double-barrier options where monitoring of a double-barrier starts at the time when the barrier state variable first crosses one exponential barrier or two exponential barriers in a specified order. To derive the pricing formula of the external-chained double-barrier option, as in the case in Section 4, we use the closed formula of the external double-barrier call option mentioned in Section 3. In a similar way to Section 4, we substitute the expectation representation of the external-chained double-barrier option for the expectation form of the external double-barrier call option and then find the semi-analytic form formula of the external-chained double-barrier option price with exponential barriers directly.

5.1. Case of crossing an exponential barrier: External double-barrier option

This subsection considers the pricing formula for an external double knock-out barrier option starting at a random time when the barrier state variable reaches the upper exponential barrier $U(t)$. Let us consider an external double knock-out barrier call option ($EDKC_u$) activated at time $\tau = \min\{t > 0 : Z_t = U(t)\}$. Then, the expectation representation of the external double knock-out barrier call option is described by

(5.1)

$$
\begin{aligned}
&EDKC_u(t,s,z)\\
&= e^{-r(T-t)} E^{P^*}[(S(T)-K)^+\\
&\quad \mathbf{1}_{\{\max_{\tau\le\gamma\le T}(Z_\gamma - U(\gamma))<0\}\cap\{\min_{\tau\le\gamma\le T}(Z_\gamma - D(\gamma))>0,\, \tau\le T,\, Z_\tau=U(\tau)\,\}} \mid S_t=s, Z_t=z].
\end{aligned}
$$

Theorem 5.1. *For $t^* < \tau$, the price of the external-chained double knock-out external option activated at time τ, $EDKC_u$ defined by* (5.1)*, is given by*

$$
\begin{aligned}
&EDKC_u(t^*,s,z)\\
&= e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} \sum_{n=-\infty}^{\infty} \Bigg[z^{\frac{2(\xi_2-\xi_1)n}{{\sigma_*}^2}} \left(\frac{D(t^*)}{U(t^*)}\right)^{(\frac{2(-r+{\sigma_*}^2+\xi_2)}{{\sigma_*}^2}-1)n}\\
&\quad J\left(t^*, \left(\frac{D(t^*)}{U(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, A^2 \left(\frac{D(t^*)}{U(t^*)}\right)^{2n} \frac{z}{U(t^*)^2}\right)\\
&\quad - z^{(\frac{2(-r+{\sigma_*}^2+\xi_2)}{{\sigma_*}^2}-1)+\frac{2(\xi_2-\xi_1)n}{{\sigma_*}^2}} \left(\frac{D(t^*)}{U(t^*)}\right)^{(\tilde{k}_{z,\delta}-1)n}\\
&\quad J\left(t^*, \left(\frac{D(t^*)}{U(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, A^2 \left(\frac{D(t^*)}{U(t^*)}\right)^{2n} \frac{U(t^*)^2}{D(t^*)^2\, z}\right)\Bigg],
\end{aligned}
$$

where $J(t,s,z)$ is the solution of the unrestricted PDE defined by (3.4)*, and the closed solution is described by* (3.5) *and* $\tilde{k}_{z,\delta} = \frac{2(-r+{\sigma_*}^2+(n+1)\xi_2-n\xi_1)}{{\sigma_*}^2}$.

Proof. Using the same procedure as in Section 4.1, under a probability measure $\tilde{P}^*$,

(5.2)

$$
\begin{aligned}
&EDKC_u(t^*,s,z)\\
&= e^{-r(T-t^*)} E^{P^*}[(S(T)-K)^+\\
&\quad \mathbf{1}_{\{\max_{\tau\le\gamma\le T}(Z_\gamma - U(\gamma))<0\}\cap\{\min_{\tau\le\gamma\le T}(Z_\gamma - D(\gamma))>0,\, \tau\le T,\, Z_\tau=U(\tau)\,\}} \mid S_{t^*}=s, Z_{t^*}=z]\\
&= e^{2a\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*}} e^{-r(T-t^*)} E^{\tilde{P}^*}\big[(S(T)-K)^+\\
&\quad \mathbf{1}_{\{\min_{0\le\gamma\le T}(\tilde{Z}_\gamma - \tilde{U}(\gamma))>0\}\cap\{\max_{0\le\gamma\le T}(\tilde{Z}_\gamma - \tilde{D}(\gamma))<0\}} \mid S_{t^*}=s, \tilde{Z}_{t^*} = z\, e^{-2\xi_1 t^*}\big],
\end{aligned}
$$

where $\tilde{D}(t) := \frac{A^2}{D(t)}$ (the upstream exponential barrier), $\tilde{U}(t) := \frac{A^2}{U(t)}$ (the downstream exponential barrier) and $\tilde{Z}_t$ are defined by Section 4.1.

If we call

$$e^{-r(T-t^*)}E^{\tilde{P}^*}\Big[(S(T)-K)^+ \\ \mathbf{1}_{\{\min_{0\le\gamma\le T}(\tilde{Z}_\gamma-\tilde{U}(\gamma))>0\}\cap\{\max_{0\le\gamma\le T}(\tilde{Z}_\gamma-\tilde{D}(\gamma))<0\}}\,|\,S_{t^*}=s, \tilde{Z}_{t^*}=z\,e^{-2\xi_1 t^*}\Big]$$

in (5.2) $H^*(t^*,s,z)$, then $H^*(t^*,s,z)$ is the double knock-out external barrier option with the upstream exponential barrier $\tilde{D}(t)$ and with the downstream exponential barrier $\tilde{U}(t)$ from Section 3. Then, by Theorem 3.1,

(5.3)

$$\begin{aligned} &H^*(t^*,s,z)\\ &= e^{-r(T-t^*)}E^{\tilde{P}^*}\Big[(S(T)-K)^+ \\ &\quad \mathbf{1}_{\{\min_{0\le\gamma\le T}(\tilde{Z}_\gamma-\tilde{U}(\gamma))>0\}\cap\{\max_{0\le\gamma\le T}(\tilde{Z}_\gamma-\tilde{D}(\gamma))<0\}}\,|\,S_{t^*}=s, \tilde{Z}_{t^*}=z\,e^{-2\xi_1 t^*}\Big]\end{aligned}$$

in (5.2) is expressed by

$$\begin{aligned} H^*(t^*,s,z) = \sum_{n=-\infty}^{\infty}\Bigg[& z^{\frac{2(\xi_2-\xi_1)n}{{\sigma_*}^2}}\left(\frac{D(t^*)}{U(t^*)}\right)^{(\frac{2(-r+{\sigma_*}^2+\xi_2)}{{\sigma_*}^2}-1)n}\\ & J\left(t^*, \left(\frac{D(t^*)}{U(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, A^2\left(\frac{D(t^*)}{U(t^*)}\right)^{2n}\frac{z}{U(t^*)^2}\right)\\ &- z^{(\frac{2(-r+{\sigma_*}^2+\xi_2)}{{\sigma_*}^2}-1)+\frac{2(\xi_2-\xi_1)n}{{\sigma_*}^2}}\left(\frac{D(t^*)}{U(t^*)}\right)^{(\tilde{k}_{z,\delta}-1)n}\\ & J\left(t^*, \left(\frac{D(t^*)}{U(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, A^2\left(\frac{D(t^*)}{U(t^*)}\right)^{2n}\frac{U(t^*)^2}{D(t^*)^2\,z}\right)\Bigg], \end{aligned} \tag{5.4}$$

where $J(t,s,z)$ is a solution of the following unrestricted domain PDE

$$\begin{aligned} &\mathcal{L}J(t,s,z)=0, \quad 0\le t<T,\\ &J(T,s,z)=h(s,z)=(s-K)^+\mathbf{1}_{\{\tilde{B}<z<\tilde{A}\}}\\ &J(t,s,\tilde{A})=J(t,s,\tilde{B})=0,\\ &\mathcal{L}:=\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2}{\partial s^2}+\frac{1}{2}{\sigma_*}^2 z^2\frac{\partial^2}{\partial z^2}+\rho\sigma\sigma_* sz\frac{\partial^2}{\partial s\partial z}\\ &\qquad +r\left(s\frac{\partial}{\partial s}+z\frac{\partial}{\partial z}\right)-rI \end{aligned} \tag{5.5}$$

with the domain $\{(t,s,z): 0\le t<T, 0\le s\le\infty, 0\le z\le\infty\}$, and $\tilde{A}$ and $\tilde{B}$ are two constants satisfying $\tilde{U}(t) := \frac{A^2}{U(t)} = \tilde{B}e^{\tilde{\xi}_1(T-t)}$ and $\tilde{D}(t) := \frac{A^2}{D(t)} = \tilde{A}e^{\tilde{\xi}_2(T-t)}$, respectively.

Then, as seen in (3.5) of Section 3, the closed-form solution $J(t,s,z)$ of the PDE problem in (5.5) is given by

(5.6)

$$
\begin{aligned}
&J(t,s,z)\\
&= s\mathcal{N}_2\left(d_1^r(\tau,\frac{s}{K}),-d_2^r(\tau,\frac{z}{\tilde{A}}),-\rho\right)-e^{-r\tau}K\mathcal{N}_2\left(d_3^r(\tau,\frac{s}{K}),-d_4^r(\tau,\frac{z}{\tilde{A}}),-\rho\right)\\
&\quad - s\mathcal{N}_2\left(d_1^r(\tau,\frac{s}{K}),-d_2^r(\tau,\frac{z}{\tilde{B}}),-\rho\right)+e^{-r\tau}K\mathcal{N}_2\left(d_3^r(\tau,\frac{s}{K}),-d_4^r(\tau,\frac{z}{\tilde{B}}),-\rho\right).
\end{aligned}
$$

Hence, by combining (5.2), (5.3) and (5.4), we obtain the desired result of Theorem 5.1. □

5.2. Case of crossing two exponential barriers: External double-barrier option

This subsection addresses an external double knock-out barrier option activated in the event that the barrier state variable hits the upstream barrier $U(t)$ followed by reaching the downstream barrier $D(t)$, or vice versa.

Theorem 5.2. *For $t^* < \tau_1$, if we consider the external double knock-out barrier call option ($EDKC_{ud}$) activated at a random time τ_2 defined by*

$$\tau_2 = \min\{\, t > \tau_1 \,:\, Z_t = D(t),\ \tau_1 = \min\{\, t > 0 \,:\, Z_t = U(t)\}\}, \tag{5.7}$$

then, the price of the option, $EDKC_{ud}$ is given by

(5.8)

$$
\begin{aligned}
EDKC_{ud}(t^*,s,z) = e^{\theta}\sum_{n=-\infty}^{\infty}\Bigg[&\left(\frac{D(t^*)^2 z}{U(t^*)^2}\right)^{\frac{2(\xi_2-\xi_1)n}{\sigma_*^2}}\left(\frac{U(t^*)}{D(t^*)}\right)^{(\frac{2(r-\xi_1)}{\sigma_*^2}-1)n}\\
&\hat{J}\left(t^*,\left(\frac{U(t^*)}{D(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}}s,\frac{A^2}{B^2}\left(\frac{U(t^*)}{D(t^*)}\right)^{2n}\frac{D(t^*)^2\, z}{U(t^*)^2}\right)\\
&- z^{(\frac{2(r-\xi_1)}{\sigma_*^2}-1)+\frac{2(\xi_2-\xi_1)n}{\sigma_*^2}}\left(\frac{U(t^*)}{D(t^*)}\right)^{(\hat{k}_{z,\delta}-1)n}\\
&\hat{J}\left(t^*,\left(\frac{U(t^*)}{D(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}}s,\frac{A^2}{B^2}\left(\frac{U(t^*)}{D(t^*)}\right)^{2n}\frac{U(t^*)^4}{D(t^*)^2\, z}\right)\Bigg],
\end{aligned}
$$

where $\theta := 2(b-a)\frac{(\tilde{\mu}_*-\xi_1)}{\sigma_*} - 2(b-2a)\frac{(\xi_2-\xi_1)}{\sigma_*}$, and $\hat{J}(t,s,z)$ is the solution of the unrestricted PDE defined by (3.4), and the closed solution is expressed by (3.5) and $\hat{k}_{z,\delta} = \frac{2(r-(n+1)\xi_1+n\xi_2)}{\sigma_*^2}$.

Proof. Using a similar method as in Section 4.2, under a probability measure P^*,

(5.9)

$$
\begin{aligned}
&EDKC_{ud}(t^*, s, z)\\
&= e^{-r(T-t^*)} E^{P^*}[(S(T)-K)^+\\
&\quad \mathbf{1}_{\{\{\max_{\tau_2 \le \gamma \le T}(Z_\gamma - U(\gamma)) < 0\} \cap \{\min_{\tau_2 \le \gamma \le T}(Z_\gamma - D(\gamma)) > 0,\ \tau_1 < \tau_2 \le T,\ Z_{\tau_1} = U(\tau_1),\ Z_{\tau_2} = U(\tau_2)\}\}} \mid S_{t^*} = s, Z_{t^*} = z]\\
&= e^{\theta} e^{-r(T-t^*)} E^{\hat{P}^*}[(S(T)-K)^+\\
&\quad \mathbf{1}_{\{\{\max_{0 \le \gamma \le T}(\hat{V}_\gamma - \hat{U}(\gamma)) < 0\} \cap \{\min_{0 \le \gamma \le T}(\hat{V}_\gamma - \hat{D}(\gamma)) > 0,\ \}\}} \mid S_{t^*} = s, \hat{Z}_{t^*} = z e^{2(\xi_2 - \xi_1)t^*}],
\end{aligned}
$$

where $\hat{U}(t) := \frac{A^2\, U(t)}{B^2}$ (upstream exponential barrier), $\hat{D}(t) := \frac{A^2\, D(t)}{B^2}$ (downstream exponential barrier), and $\hat{Z}_t$ are defined by Section 4.2.

Similar to Section 5.1,

(5.10)

$$
\begin{aligned}
&e^{-r(T-t^*)} E^{\hat{P}^*}[(S(T)-K)^+\\
&\mathbf{1}_{\{\{\max_{0 \le \gamma \le T}(\hat{Z}_\gamma - \hat{U}(\gamma)) < 0\} \cap \{\min_{0 \le \gamma \le T}(\hat{Z}_\gamma - \hat{D}(\gamma)) > 0,\ \}\}} \mid S_{t^*} = s, \hat{Z}_{t^*} = z e^{2(\xi_2 - \xi_1)t^*}]
\end{aligned}
$$

in (5.9) becomes the price of the external double knock-out barrier option with the upstream exponential barrier $\hat{U}(t)$ and the downstream exponential barrier $\hat{D}(t)$ mentioned in Section 3. Let us define (5.10) as $\hat{H}(t^*, s, z)$. By Theorem 3.1, $\hat{H}$ is expressed by

$$
\begin{aligned}
&\hat{H}(t^*, s, z)\\
&= \sum_{n=-\infty}^{\infty} \Bigg[\left(\frac{D(t^*)^2 z}{U(t^*)^2}\right)^{\frac{2(\xi_2-\xi_1)n}{\sigma_*{}^2}} \left(\frac{U(t^*)}{D(t^*)}\right)^{(\frac{2(r-\xi_1)}{\sigma_*{}^2}-1)n}\\
&\quad \hat{J}\left(t^*, \left(\frac{U(t^*)}{D(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \frac{A^2}{B^2}\left(\frac{U(t^*)}{D(t^*)}\right)^{2n} \frac{D(t^*)^2\, z}{U(t^*)^2}\right)\\
&\quad - z^{(\frac{2(r-\xi_1)}{\sigma_*{}^2}-1)+\frac{2(\xi_2-\xi_1)n}{\sigma_*{}^2}} \left(\frac{U(t^*)}{D(t^*)}\right)^{(\hat{k}_{z,\delta}-1)n}\\
&\quad \hat{J}\left(t^*, \left(\frac{U(t^*)}{D(t^*)}\right)^{\frac{2\rho\sigma}{\sigma_*}} s, \frac{A^2}{B^2}\left(\frac{U(t^*)}{D(t^*)}\right)^{2n} \frac{U(t^*)^4}{D(t^*)^2\, z}\right)\Bigg],
\end{aligned} \tag{5.11}
$$

where $\hat{J}(t, s, z)$ is a solution of an unrestricted domain PDE satisfying

$$
\begin{aligned}
&\mathcal{L}\hat{J}(t, s, z) = 0, \quad 0 \le t < T,\\
&\hat{J}(T, s, z) = h(s, z) = (s-K)^+ \mathbf{1}_{\{\hat{B} < z < \hat{A}\}}\\
&\hat{J}(t, s, \hat{A}) = \hat{J}(t, s, \hat{B}) = 0,
\end{aligned} \tag{5.12}
$$

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}{\sigma_*}^2 z^2 \frac{\partial^2}{\partial z^2} + \rho\sigma\sigma_* sz \frac{\partial^2}{\partial s \partial z} + r\left(s\frac{\partial}{\partial s} + z\frac{\partial}{\partial z}\right) - rI,$$

where the domain $\{(t, s, z) : 0 \leq t < T, 0 \leq s \leq \infty, 0 \leq z \leq \infty\}$ and $\hat{A}$ and $\hat{B}$ are two constant values so that $\hat{U}(t) = \frac{A^2\, U(t)}{B^2} = \hat{A}e^{\hat{\xi}_1(T-t)}$ and $\hat{D}(t) := \frac{A^2\, D(t)}{B^2} = \hat{B}e^{\hat{\xi}_2(T-t)}$, respectively.

Then, using the double Mellin transform as seen in Section 3, we obtain the closed-form formula of $\hat{J}(t, s, z)$ given by

(5.13)

$$\begin{aligned}&\hat{J}(t, s, z)\\ &= s\mathcal{N}_2\left(d_1^r(\tau, \frac{s}{K}), -d_2^r(\tau, \frac{z}{\hat{A}}), -\rho\right) - e^{-r\tau}K\mathcal{N}_2\left(d_3^r(\tau, \frac{s}{K}), -d_4^r(\tau, \frac{z}{\hat{A}}), -\rho\right)\\ &\quad - s\mathcal{N}_2\left(d_1^r(\tau, \frac{s}{K}), -d_2^r(\tau, \frac{z}{\hat{B}}), -\rho\right) + e^{-r\tau}K\mathcal{N}_2\left(d_3^r(\tau, \frac{s}{K}), -d_4^r(\tau, \frac{z}{\hat{B}}), -\rho\right).\end{aligned}$$

Finally, by integrating (5.9), (5.10) and (5.11), the desired result of Theorem 5.2 is obtained. □

6. Concluding remarks

This paper examined the valuation formulas for external-chained barrier options or external-chained double-barrier options with curved barriers so that barrier options are activated when the external barrier state hits a specified barrier level. First, we derived explicit valuation formulas for the external-chained barrier options using the reflection principle method and the change of measure and by changing the expectation representation of the external-chained barrier options into the external European barrier option price with a closed formula, which is described by the bivariate cumulative normal distribution function. Similarly, using the semi-analytic formula of the external double-barrier options, we obtained the pricing valuation formula for the external-chained double-barrier options with infinite sum in terms of the bivariate cumulative normal distribution function. A significant contribution of our methodology is the resolution of the complicated calculation of the pricing of the external-chained barrier (double-barrier) option with two underlying assets by using the closed-form formula (semi-analytic formula) of the well-known external barrier (double-barrier) option directly. Then, the pricing formulas of the external barrier (double-barrier) option using the double Mellin transform method and the method of images are derived more easily and effectively compared to existing probabilistic approaches.

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