INTERNATIONAL UNIVERSITY
DEPARTMENT OF MATHEMATICS

THESIS REPORT

Pricing European barrier options with rebates

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Chapter 1

The urgency of the topic

The situation of Vietnamese derivatives market this year. Derivatives offer bright future for Vietnamese stocks. On August 10, 2017, Vietnam has opened a derivatives market to draw more investment to its capital markets with future contracts. The market will start with two main derivative products which are stock index and government bond futures and once fully operational, more instruments will be introduced. In addition, according to the predict of economic experts, option contract will be operated in Vietnam in December.

The discourse will respect to the barrier option option which offers the right to buy or sell securities but not the obligation. The special features here is that payoffs depends on whether or not the underlying asset price has reached or exceeded some barrier level during the option's life. The barrier options are popular and attractive thanks to benefits that they give investors more flexibility to express their view on the asset price movement in the option contract. There are some advantages of this option. Firstly, The barrier option depends on the predict of the investor about price of stock in the future, so it is suitable to beliefs about the future behavior of the market. Secondly, it match hedging needs more closely that it helps the investor reduce risk because of the fluctuation of prices. Lastly, Premiums are generally low by not paying a premium to cover scenarios he or she views as unlikely.

Chapter 2

Backgroud of Securities Market

2.1 Structure

In an economic system, we have found out two different sectors. They are household who saves money and business who needs money for the purpose of prodution or sale of goods and services.

Financial market acts as an intermediary between the savers and investors of money where traders buy and sell stocks, bonds, derivatives, foreign exchange and commodies.

There are three conditions to classifying market. Firstly, based on working capital, we have primary market where the new security issues sold to initial buyers. Typically involves an investment bank who underwrites the offering and second market where Securities previously issued are bought and sold. Involves both brokers and dealers. Secondly, based on operating approach, markets be assigned by exchange market that Stock trades conducted via centralized place. Buy/sell is conducted through the exchange; no direct contract between seller and buyer and OTC market (Over-The-Counter) where no centralized place. Trading is done directly between two parties, without the supervision of an exchange. Lastly, based on goods, there are stock market, bond market and derivative instrument. Let's look at stock market, a financial market that enables investors to buy and sell shares of publicly traded companies. There are common stock and preferred stock. Then bond market, a financial market where participants can issue

new debt or buy and sell debt securities. The form may be bonds, notes, bills, and so on. For derivative instrument, a financial market that trades securities that derive its value from its underlying asset including stock right, warrant, option.
Consider more about option contract which granting the owner the right to buy or sell shares of a security in the future at a given price involving call and put option. (Barrier, American, European, Asian, Lookback. . .).
The option is valid for the holder. The holder is guaranteed a nonnegative terminal payoff, so he must pay a premium get into the option.

2.2 The potential European barrier option

2.2.1 European option

European options are contracts that give the holder the right but not the obligation, to buy or sell the underlying security at a specific price (the strike price) on the option's expiration date. The holder must pay a premium at the initial time. This option is the path independent option. If it is exercised, time is at expiration.

$$\text{European call option payoff} = max((S - K), 0)$$
$$\text{European put option payoff} = max((K - S), 0)$$

Where S: the current price
K: the strike price

2.2.2 Barrier option

The barrier option is the most popular path dependent option. Barrier option is a type of option whose payoff depends on whether or not the underlying asset price has reached or exceeded some barrier level during the life of the option.

Classification:

- Knock-out option: the option can expire worthless if the underlying asset price touches the barrier.
  - Down and out barrier option: If the underlying asset's price falls below the barrier at any point in the option's life, the option will be worthless.
  - Up and out barrier option: if the underlying asset's price increases above the barrier at any point in the option's life, the option will be worthless.
- Knock-in option: the option has no value until the underlying asset price crosses the in-barrier.
  - Down and in barrier option: the underlying asset price moves below a barrier at any point in the option's life, the option comes into existence.
  - Up and in barrier option: if the price of the underlying asset rises above the barrier at any point in the option's life, the option comes into existence.

  Consider a portfolio of one European in-option and one European out-option: both have the same barrier, strike price and date of expiration. Then,

$$c_{ordinary} = c_{down-and-out} + c_{down-and-in}$$
$$p_{ordinary} = p_{up-and-out} + p_{up-and-in}$$

2.2.3 European barrier call option

It is a type of option including characters of both European option and barrier option which means the option will be exercised at expiration date unless the underlying asset price reached or exceeded a lower barrier during the option's life. Especially, the option cannot be activated again if the underlying asset price go up after down to a lower barrier.

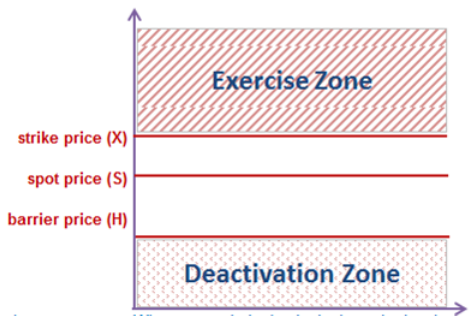

Figure 2.1: Down and Out option X >H

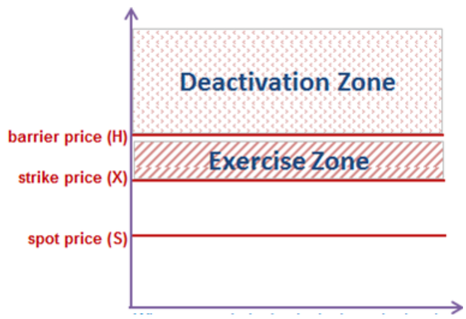

Figure 2.2: Up and Out option X <H

2.2.4 European barrier call option with rebates

This is a specified of European barrier call option which a knock-out occurs, the holder of the option will receive a partial rebate on the premium paid to the option writer. This reduces the writer's profit.

European barrier call option with rebate are cheaper than the respective standard European options because a zero payoff maybe occur before expiry time T. Lower premiums are usually offered for more exotic barrier option, which make them particularly attractive to hedgers in the financial market.

Chapter 3

Methodology

We may formulate the pricing models of barrier options using the probabilistic approach that includes the martingales pricing approach and derive the corresponding price formulas by computing the expectation of the discounted terminal payoff under the risk neutral measure Q. The price of the European down-and-out call option with rebates is given by

$$
\begin{aligned}
c(S,\tau) = & c_E(S,\tau) - \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S},\tau\right) \\
& + \int_0^\tau e^{-r\omega} \frac{ln\frac{S}{B}}{\sqrt{2\pi}\sigma} \frac{exp(\frac{-[ln\frac{S}{B}+(r-\frac{\sigma^2}{2})\omega]^2}{2\sigma^2\omega})}{\omega^{\frac{3}{2}}} R(\tau-\omega)d\omega
\end{aligned}
\tag{3.0.1}
$$

In particular,under the Black–Scholes pricing paradigm, when the martingale approach is used, we obtain the transition density function using the reflection principle in the Brownian process literature. To compute the expected present value of the rebate payment, we derive the density function of the first passage time to the barrier. Therefore, the Black-Sholes model is very necessary.

Chapter 4

The Black - Scholes - Merton model

4.1 Continuous Random Variables

X is a continuous random variable. The cumulative distribution of X is given by

$$P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx = 1 \tag{4.1.1}$$

$$P\{X \in B\} = \int_{B} f(x)dx \tag{4.1.2}$$

f : the probability density function, $f \geq 0$
$x \in (-\infty, \infty)$
Any set $B \in \mathbb{R}$

The relationship between the cumulative distribution F and the probability density function is expressed by

$$F(a) = P\{X \leq a\} = P\{X \in (-\infty, a]\} = \int_{-\infty}^{a} f(x)dx$$

The expected value of X,

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

The expected value of any real-valued function g,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

The variance

$$Var(X) = E[(X - \mu)^2] = E[X^2] - (E[X]^2)$$
$$Var(aX + b) = a^2 \sqrt{Var(X)}$$

μ : expected value of X

4.2 Normal Random Variables

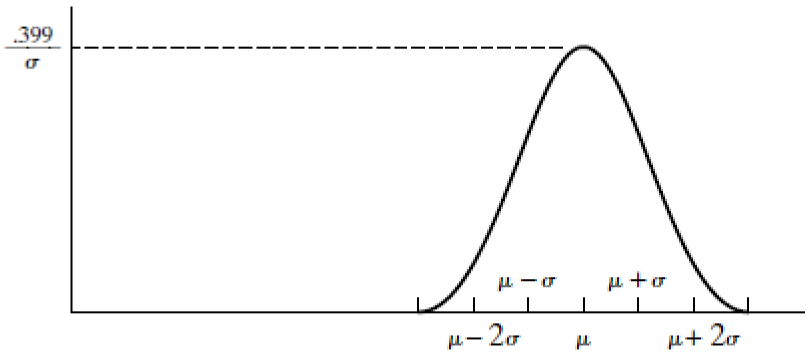

Figure 4.1: Arbitrary μ, σ^2

X is a normal random variable (normally distributed) with parameter μ and σ^2, the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

If $Y = aX + b$, then Y is normally distributed with parameter $a\mu + b$ and $a^2\sigma^2$.

- The cumulative distribution of Y

$$F_Y(x) = P\{Y \leq x\} = F_X\left(\frac{x-b}{a}\right)$$

- The density function of Y

$$f_Y(x) = \frac{1}{\sqrt{2\pi}a\sigma}e^{\frac{-(x-b-a\mu)^2}{2(a\mu)^2}}$$

The standard normal random variable

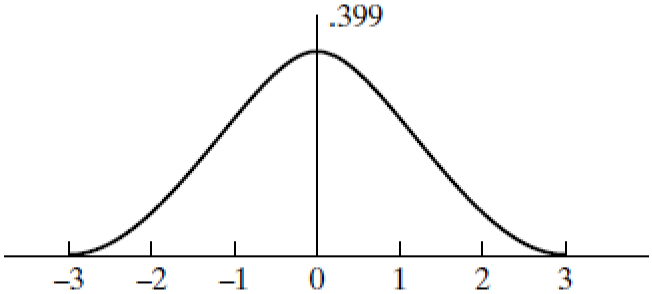

Figure 4.2: $\mu = 0$, $\sigma = 1$

$Z = \dfrac{X-\mu}{\sigma}$ is standard normally distributed with parameters 0 and 1.

Proof

$$
\begin{aligned}
E\left(\frac{X-\mu}{\sigma}\right) &= \frac{1}{\sigma}E(X-\mu) \\
&= \frac{1}{\sigma}[E(X)-E(\mu)] \\
&= \frac{1}{\sigma}(\mu-\mu) \\
&= 0. \\
Var\left(\frac{X-\mu}{\sigma}\right) &= \frac{1}{\sigma^2}Var(X-\mu) \\
&= \frac{1}{\sigma^2}[Var(X)-Var(\mu)] \\
&= \frac{\sigma^2}{\sigma^2} \\
&= 1.
\end{aligned}
$$

The cumulative distribution function of standard normal random variable

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-y^2}{2}}dy \tag{4.2.1}$$

$$\phi(-x) = 1-\phi(x) \tag{4.2.2}$$

Proof

Firstly, equation (4.2.1)
Let $Y = \dfrac{X-\mu}{\sigma}$, then

$$
\begin{aligned}
\phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(y-\mu)^2}{2\sigma^2}}dy \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\times 1}e^{\frac{-(y-0)^2}{2\times 1^2}}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-y^2}{2}}dy
\end{aligned}
$$

Secondly, equation (4.2.2)

$$
\begin{aligned}
\phi(-x) &= P\{X< -x\} \\
&= P\{X>x\} \\
&= P\{X \in (x,\infty)\} \\
&= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{\frac{-x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-x^2}{2}}dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{\frac{-x^2}{2}}dx \\
&= 1-\phi(x)
\end{aligned}
$$

4.3 Lognormal property of stock price

We have random variable X

$$X = e^Y \sim Log\text{-}N(\mu,\sigma^2) \text{ is distributed log-normally,}$$

$$\text{then } lnX = Y \sim N(\mu,\sigma^2) \text{ is normally distributed}$$

The lognormal distribution is bounded below by 0 and skewed to the right. It is extremely useful when analyzing stock prices which cannot fall below zero.

The mean value and variance of the log-normal distribution is given by

$$E(Y) = e^{\mu+\frac{1}{2}\sigma^2} \tag{4.3.1}$$

$$Var(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \tag{4.3.2}$$

- The probability density function of $LN(\mu,\sigma^2)$ is

$$f(x) = \frac{1}{x\sqrt{2\pi}\sigma}e^{\frac{-(lnx-\mu)^2}{2\sigma^2}}$$

- The cumulative distribution function of $LN(\mu,\sigma^2)$ is

$$\int_0^\infty f(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty e^{\frac{-(y-\mu)^2}{2\sigma^2}}dy = 1$$

Where

$$x = e^y$$
$$lnx = y$$
$$\frac{dx}{x} = dy$$

Consider the continuously compounded rate of return between times 0 and T, S_T is a lognormal distribution random variable. If the rate of return r is continuously compounded then the future stock price can be expressed as

$$S_T = S_0 e^r T \tag{4.3.3}$$

Then S_T is a lognormal distributed.
And

$$\frac{S_T}{S_0} = e^r T$$
$$ln\frac{S_T}{S_0} = rT$$

The continuously compounded rate of return between times 0 and T is normally distributed.
Consider $ln\frac{S_T}{S_0}$ is normally distributed with mean and standard deviation following

$$ln\frac{S_T}{S_0} \sim \phi[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T] \tag{4.3.4}$$

$$lnS_T \sim \phi[lnS_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T] \tag{4.3.5}$$

More explaination

$$X + a \sim \phi(\mu + a, \sigma)$$
$$E(X + a) = E[X] + E[a]$$
$$= \mu + a$$
$$Var(X + a) = Var(X) + Var(a)$$
$$= Var(X)$$

4.4 The distribution of the rate of return

We define the continuously compounded rate of return per annum realized between time 0 and T as μ (equation (4.3.3)), so that

$$
\begin{aligned}
S_T &= S_0 e^r T && (4.4.1)\\
r &= \frac{1}{T} ln \frac{S_T}{S_0} \quad \text{from (4.3.4), it leads to} \\
&\sim \frac{1}{T}\phi[(\mu - \frac{\sigma^2}{2}), \sigma^2 T] \\
&\sim \phi(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}) && (4.4.2)
\end{aligned}
$$

μ is the arithmetic of the returns realized in may short intervals of time.
$\mu - \dfrac{\sigma^2}{2}$ is a geometric average which is the expected continuously compound return realized over a larger period of time.

4.5 Volatility

The volatility, σ, of a stock is a measure of our uncertainty about the returns provided by the stock.

Estimating Volatility from Histical data.
The stock price is usually observed at fixed intervals of time. Define
n+1 is number of observation
S_i is the Stock price at and of ith interval, with $i = 0, 1, 2, \ldots, n$
τ is length of time interval in years
and let

$$u_i = ln\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for} \quad i = 1, 2, \ldots, n$$

The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i-1}^{n}(u_i - \bar{u})^2} \tag{4.5.1}$$

$$= \sqrt{\frac{1}{n-1}\sum_{i-1}^{n}u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2} \tag{4.5.2}$$

Where $\bar{u}$ is the mean of u_i.

From equation (4.3.4), the standard deviation of the u_i is $\sigma\sqrt{\tau}$. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$.

The foregoing analysis assumes that the stock pays no dividends, but it can be adapted to accommodate dividend-paying stocks. The return, u_i, during a time interval that includes an ex-dividend day is given by

$$u_i = ln\left(\frac{S_i + D}{S_{i-1}}\right)$$

Volaitily per annum = Volatility per trading day $\times\sqrt{\text{Number of trading days per annum}}$
The number of trading days in a year is usually assumed to be 252 for stocks.

4.6 Risk neutral valuation

Black-Scholes Merton differential equation would be independence of risk preference. The matter of this will be concentrated on the risk neutral.

Risk neutral investor only looks at the potential gains of each investment which is expected return of his investment, and ignores the potential downside risk.

Risk-neutral valuation neans that you can value option in terms of their expected payoffs, discounted from expiration to the present, assuming that they grow on averge at the risk free rate.

4.7 The Black-Scholes-Merton pricing formulas

The Black–Scholes–Merton formulas for the prices of European call option is given by

$$c = S_0N(d_1) - Ke^{-rT}N(d_2) \tag{4.7.1}$$

Where

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}T\right)}{\sigma\sqrt{T}} \tag{4.7.2}$$

$$d_2 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}T\right)}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{4.7.3}$$

$N(x)$ is the cumulative probability distribution function for a variable with a standard normal distribution
c is the European call price
S_0 is the stock price at time zero
K is the strike price
r is the continuously compounded risk-free rate
σ is the stock price volatility
T is the time to maturity of the option.

Consider S_T is a lognormally distributed and Standard Deviation of lnS_T denotes by σ. Fistly, Proving the another key result that is useful for the main formula.

Key Result

The expected payoffs of the option is followed by equation:

$$E[\max(S_T - K), 0] = E(S_T)N(d_1) - KN(d_2) \tag{4.7.4}$$

Where

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \frac{\sigma^2}{2}}{\sigma} \tag{4.7.5}$$

$$d_2 = \frac{ln\left(\frac{S_0}{K}\right) - \frac{\sigma^2}{2}}{\sigma} \tag{4.7.6}$$

Proof

Define $g(S_T)$ as the probability density function of S_T. Then

$$E[\max(S_T - K, 0)] = \int_K^\infty (S_T - K)g(S_T)d(S_T) \tag{4.7.7}$$

The variable $ln(S_T)$ is normally distributed with standard deviation σ. From the properties of the lognormal distribution (from equation (4.3.1)), the mean of $ln(S_T)$ is μ, where

$$E[S_T] = e^{\mu + \frac{\sigma^2}{2}}$$

$$ln[E(S_T)] = \mu + \frac{\sigma^2}{2}$$

$$\mu = ln[E(S_T)] - \frac{\sigma^2}{2} \tag{4.7.8}$$

Define a variable

$$Z = \frac{ln(S_T) - \mu}{\sigma} \tag{4.7.9}$$

This variable is standard normally distributed with a mean of μ and a standard deviation of 1.0. The desity function for Z by $h(Z)$ so that

$$h(Z) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(Z-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-Z^2}{2}}$$

Using equation (4.7.9) to convert the expression on the right-hand side of the equation (4.7.7) from integral over V to an integral over Z, we get

$$E[\max(S_T - K), 0] = \int_{\frac{lnK-\mu}{\sigma}}^\infty (e^{Z\sigma - \mu} - K)h(Z)dZ$$

$$= \int_{\frac{lnK-\mu}{\sigma}}^\infty e^{Z\sigma - \mu}h(Z)dZ - K\int_{\frac{lnK-\mu}{\sigma}}^\infty h(Z)dZ \tag{4.7.10}$$

Now

$$
\begin{aligned}
e^{Z\sigma-\mu}h(Z) &= \frac{1}{\sqrt{2\pi}}e^{\frac{(-Z^2+2Z\sigma+2\mu)}{2}} \\
&= \frac{1}{\sqrt{2\pi}}e^{\frac{[-(Z-\sigma)^2+2\mu+\sigma^2]}{2}} \\
&= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{2\pi}}e^{\frac{[-(Z-\sigma)^2]}{2}} \\
&= e^{\mu+\frac{\sigma^2}{2}}h(Z-\sigma)
\end{aligned}
$$

This means that equation (4.7.10) becomes

$$
E[\max(S_T-K),0] = e^{\mu+\frac{\sigma^2}{2}}\int_{\frac{lnK-\mu}{\sigma}}^{\infty} h(Z-\sigma)dZ - K\int_{\frac{lnK-\mu}{\sigma}}^{\infty} h(Z)dZ \tag{4.7.11}
$$

The cumulative density function of standard normal distribution N(x) that a variable with mean of zero and a standard deviation of 1.0 is less than x, the first integral in equation (4.7.11)

$$
\begin{aligned}
\int_{\frac{lnK-\mu}{\sigma}}^{\infty} h(Z-\sigma)dZ &= 1-N(Z-\sigma) \\
&= 1-N[\left(\frac{lnK-\mu}{\sigma}-\sigma\right)] \\
&= N[-\left(\frac{lnK-\mu}{\sigma}-\sigma\right)] \\
&= N[\frac{-lnK+\mu}{\sigma}+\sigma]
\end{aligned}
$$

Substituting for μ from equation (4.3.1) leads to

$$
\begin{aligned}
N[\frac{-lnK+\mu}{\sigma}+\sigma] &= N[\frac{-lnK+ln[E(S_T)]-\frac{\sigma^2}{2}}{\sigma}+\sigma] \\
&= N[\frac{ln\frac{E(S_T)}{K}+\frac{\sigma^2}{2}}{\sigma}] \\
&= N(d_1)
\end{aligned}
$$

Similarly the second integral in equation (4.7.11)

$$\begin{aligned}\int_{\frac{lnK-\mu}{\sigma}}^{\infty} h(Z)dZ &= 1 - N(\frac{lnK-\mu}{\sigma})\\ &= N[\frac{-lnK + ln[E(S_T) - \frac{\sigma^2}{2}]}{\sigma}]\\ &= N[\frac{ln\frac{E(S_T)}{K} - \frac{\sigma^2}{2}}{\sigma}]\\ &= N(d_2)\end{aligned}$$

Therefore, equation (4.7.11) becomes

$$\begin{aligned}E[\max(S_T - K), 0] &= e^{\mu+\frac{\sigma^2}{2}}N(d_1) - KN(d_2)\\ &= e^{ln[E(S_T)]-\frac{\sigma^2}{2}+\frac{\sigma^2}{2}}N(d_1) - KN(d_2)\\ &= E(S_T)N(d_1) - KN(d_2)\end{aligned}$$

The Black-Scholes-Merton Result

A non-dividend-paying stock maturing at time T, the call price c is given by

$$c = e^{-rT}E[\max(S_T - K), 0] \tag{4.7.12}$$

E denotes the expectation in a risk-neutral world. From equation (4.3.1) and (4.3.4), $E(S_T) = e^{\mu+\frac{1}{2}\sigma^2}$ and the standard deviation of lnS_T is $\sigma\sqrt{T}$.
From the key result, equation (4.7.12) implies

$$c = e^{-rT}S_0e^{rT}N(d_1) - KN(d_2) = S_0N(d_1) - Ke^{-rT}N(d_2)$$

Where

$$\begin{aligned}d_1 &= \frac{ln\frac{E(S_T)}{K} + \frac{\sigma^2T}{2}}{\sigma\sqrt{T}} = \frac{ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\\ d_2 &= \frac{ln\frac{E(S_T)}{K} - \frac{\sigma^2T}{2}}{\sigma\sqrt{T}} = \frac{ln\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\end{aligned}$$

This is the Black–Scholes–Merton result.