VIETNAM NATIONAL UNIVERSITY HCMC

INTERNATIONAL UNIVERSITY

FINAL REPORT

STUDENT RESEARCH PROJECT

PRICING EUROPEAN BARRIER OPTIONS WITH REBATES

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Abstract

In Vietnam, derivatives market has just started officially very recently, in August, 2017, with futures contracts on VN30 index. It is a very new investment area for Vietnamese investors. This market is however expected to strongly develop soon and then enhance greatly Vietnamese economy. In fact, trading volume on futures contracts on VN30 index is increasing significantly over last few months, and is expected to increase with an even faster rate. One important type of financial derivatives products is options. In Vietnam, covered call (a type of call options) will be traded soon. Options will bring greater leverage to speculators and bring more risk management tools for hedgers. Options thus have great potential chance in Vietnamese financial markets. Understanding clearly the pricing formulas for these products is very urgent, then the object of the thesis formulate the pricing models of European barrier options with rebates using the probabilistic approach. It includes deriving the Black - Scholes - Merton model by the Martingale approach which is model of European options. Then European down and out call options with rebates is formulated by using theory of Wiener process like Reflection Principle, First Passage Time, Markov Property and Stochastic Differential Equations as Itō Calculus and One-Dimensional Diffusion Process. Further, testing real data for normal distribution using R – Studio software. Lastly, using the final formula to calculate the option price of the underlying asset price of stocks in Vietnam. The stock is applied that must be satisfied given conditions by previous testing.

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List of Abbreviations

RV	Random Variable
PDF	Probability Density Function
CDF	Cumulative Density Function
SDE	Stochastic Differential Equation
PDE	Partial Differential Equation
GBM	Geometric Brownian Motion
LN	Log-Normal

Chapter 1

Introduction

In the financial markets around the world, derivative instruments can be derived from Sixth Century B.C. Renowned philosopher of Greece; Thalus is thought to be the first person that had formulated a deal, which is alike to that of option derivative of today, for making profit. A derivative is a security with a price that is dependent upon or derived from one or more underlying assets. The derivative itself is a contract between two or more parties based upon the asset or assets. Value of a derivative changes with price change of its underlying asset (share, commodity, currency and many more). With the passage of time, the need of these instruments increased among various sections of the society for hedging purposes. Later, the speculative motive of the investors also surfaced and led to its popularity.

Considering the given situation, pricing options is very important for the financial markets. In order to solve that issue, the use of mathematical techniques is a force of thing for formulating of the pricing of the option contracts. Mathematical models have been used in economics for a long time. By now, operations research, econometrics and time series analysis constitute major parts of curricula of business schools and economics departments. Therefore, we will go into the complexity of the financial market to see how it needs some our mathematical techniques .

1.1 The structure of the financial market

In the financial markets of economic system, we have found out two different sectors. They are household who saves money and business who needs money for the purpose of prodution or sale of goods and services. Financial market acts as an intermediary between the savers and investors of money where traders buy and sell stocks, bonds, derivatives, foreign exchange and commodies.

There are three conditions to classifying market. Firstly, based on working capital, we have primary market where the new security issues sold to initial buyers. Typically involves an investment bank who underwrites the offering and second market where Securities previously issued are bought and sold. Involves both brokers and dealers. Secondly, based on operating approach, markets be assigned by exchange market that Stock trades conducted via centralized place. Buy/sell is conducted through the exchange; no direct contract between seller and buyer and OTC market (Over-The-Counter) where no centralized place. Trading is done directly between two parties, without the supervision of an exchange. Lastly, based on goods, there are stock market, bond market and derivative instrument. Let's look at stock market, a financial market that enables investors to buy and sell shares of publicly traded companies. There are common stock and preferred stock. Then bond market, a financial market where participants can issue new debt or buy and sell debt securities. The form may be bonds, notes, bills, and so on. For derivative instrument, a financial market that trades securities that derive its value from its underlying asset including stock right, warrant, option. In this research, options will be analized for deriving the formula.

1.2 Options

Options are a type of derivative security. They are a derivative because the price of an option is intrinsically linked to the price of something else. Specifically, options are contracts that grant the right, but not the obligation to buy or sell an underlying asset at a set price on or before a expiration date. The right to buy is called a call option and the right to sell is a put option. A call option, might be thought of as a deposit for a future purpose, is bought if the trader expects the price of the underlying to rise within a certain time frame. For example, let's say you purchase a call option on shares of Intel (INTC) with a strike price which is a fixed at which an asset may be bought or sold of \$40 and an expiration date of May 16th. This option would give you the right to purchase 100 shares of Intel at a price of \$40 on May 16th (the right to do this, of course, will only be valuable if Intel is trading above \$40 per share at that point in time) and you will pay option premium for the writer. Conversely, a put option is bought if the trader expects the price of the underlying to fall within a certain time frame.

At a premium, since the writer of an option is exposed to potential liabilities in the future, he must be compensated with an up-front premium paid by the holder who has bought this option when they together enter into the option contract. When you buy an option, the purchase price is called the premium. If you sell, the premium is the amount you receive. The premium isn't fixed and changes constantly. The premium is likely to be higher or lower today than yesterday or tomorrow. Changing prices reflect the give and take between what buyers are willing to pay and what sellers are willing to accept for the option. The point of agreement becomes the price for that transaction.

In addition, options can be categorized based on the method in which they are traded, their expiration cycle, and the underlying security they relate to. It is listed of some different common types of option: Barrier, American, European, Asian, Lookback... This research will be focus into the combination of European and Barrier option that is more attractive and cheaper than the respective standard European options.

1.2.1 European option

European options are contracts that give the holder the right but not the obligation, to buy or sell the underlying security at a specific price (the strike price) only on the option's expiration date. As the exercise date is limited in the European option, it removes the uncertainty about possible early execution. The lack of this uncertainty is expected to encourage more investors to trade stock options. Theoretically, a European option has lower value than an otherwise, it is because a European option does not enjoy the convenience that arises from flexibility in timing of exercise, which is a premium option. The holder must pay a premium at the initial time. This option is the path independent option.

The path independent option whose payoff depends solely on the events specified to take place upon expiration, rather than the path taken by the underlying variable (price, rate, index, etc). An example for European option, if an investor buys a call option with a strike price of $20, he would benefit if the underlying rises above $20 upon expiration. If not, he would not exercise and the premium paid to buy the option is lost. We consider the payoffs for a European option that is value at the expiration time, as a function of the underlying stock price

$$\text{Call option} = \max(S - K, 0)$$
$$\text{Put option} = \max(K - S, 0)$$

Where

S: the current price

K: the strike price

Let's look at the following diagram which discribes profit of call option. Suppose that we have one call option with exercise price $50 which costs $5, the stock price moves from $0 to $100.

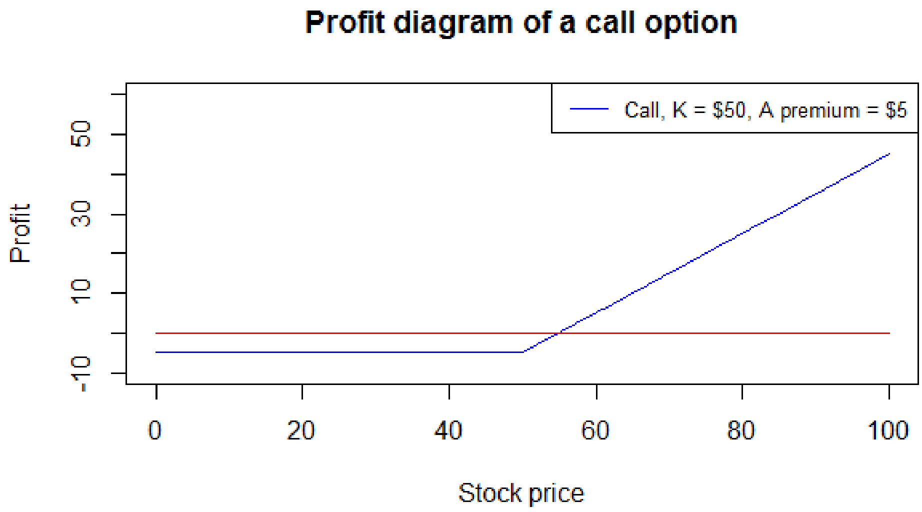

Figure 1.1: Profit of a call option

The above diagram implies that if the stock price belows the strike price, the holder would not exercise the option and had to pay a premium of \$5. Conversely, if the stock price is higher than the strike price and differences between two of them is greater than zero, he would practice the option and receive profit which is equal to the stock price minus the strike price and a premium.

1.2.2 Barrier option

The barrier option is the simpiest popular path dependent option. It is a type of option and its distinctive feature is that the payoff depends not only on the final price of the underlying asset, but also on whether the asset price has breached some barrier level during the life of the option. So, a barrier option's payoff depends on two price levels: the strike price and the so-called barrier price. One of the important things is that the holder of the option may be compensated by a rebate payment for the cancellation of the option, that helps to keep losing all premium he paid before. Let's consider two most common types of barrier options are knock-out and knock-in barrier options.

- Knock-out barrier options is a type of barrier option becomes worthless if the underlying asset price touches the barrier. Moreover, it is also devided into two parts, down and out barrier option which implies if the underlying asset's price falls below the barrier at any point in the option's life, the option will be worthless and up and out barrier option which implies if the underlying asset's price increases above the barrier at any point in the option's life, the option will be worthless.

- Knock-in option has no value until the underlying asset price crosses the in-barrier. This option is classified as down and in barrier option which means if the underlying asset price moves below a barrier at any point in the option's life, the option comes into existence and up and in barrier option which indicates if the price of the underlying asset rises above the barrier at any point in the option's life, the option comes into existence.

Barrier options carry a higher risk to the holder than the more standard types of contracts. With a knock out contract, the holder carries the risk of their investment basically ceasing to exist if the underlying security moves significantly and reaches the knock out price. With a knock in contract, if the underlying security only moves a little in price there may be no profits to be taken. There is, however, one significant advantage that barrier options offer traders, barrier options are generally cheaper than contracts that do not include a barrier price because of the increased risk that the holder has to take. The barrier options are popular and attractive thanks to benefits that they give investors more flexibility to express their view on the asset price movement in the option contract. The buyer can achieve *option premium reduction* through the barrier provision by not paying a premium to cover scenarios he or she views as unlikely. The option writer can limit liabilities when the asset price rises acutely.

1.2.3 European barrier call option

It is a type of option including characters of both European option and barrier option which means the option will be exercised at expiration date unless the underlying asset price reached or exceeded a lower barrier during the option's life. Especially, the option cannot be activated again if the underlying asset price go up after down to a lower barrier. The distinctive features of call options and put options are just conversely like one is the right to buy the asset and hence benefit would be gained as the price of the underlying goes up and one is the right to sell the asset and hence benefit would be gained as the price of the underlying goes down. Therefore, we will focus on only one side and the call option will be analyzed.

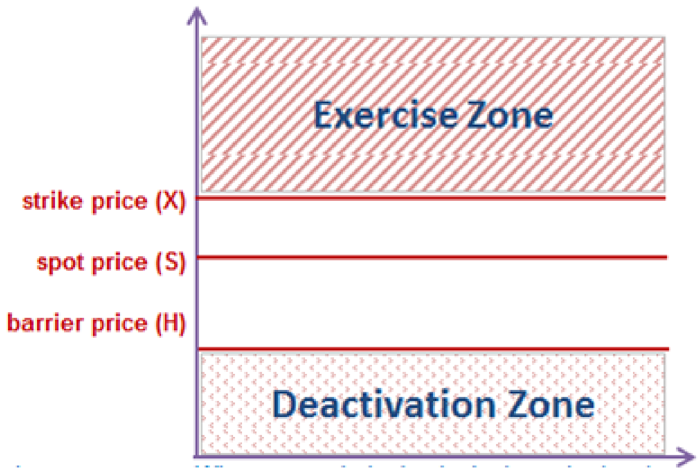

Figure 1.2: Down and Out Call option $X > H$

Let's look at Figure 1.2 which decribes Down and Out Call option when the strike price (X) is greater than the barrier (H). Expectation of the price will go up over the strike price to exercise option and make payoff is equal difference between the stock price at expiration date and the strike price. If the price goes down, the option will be deactivated.

1.2.4 European barrier call option with rebates

This is a specified of European barrier call option which a knock-out occurs, the holder of the option will receive a partial rebate on the premium paid to the option writer. European barrier call option with rebates are cheaper than the respective standard European options because a zero payoff maybe occur before expiry time T. Lower premiums are usually offered for more exotic barrier option, which make them particularly attractive to hedgers in the financial market.

Let's illustrate this type of option by an example. We consider the European down-and-out call option under the underlying asset of XYZ stock, the current price is approximately, \$39 and rebate is \$5. If the XYZ price falls to \$28 which is a barrier level within the next six months. the writer will have to pay \$5 dollars for the holder if XYZ stock is less than or equal to \$28 at any point in the next six months.

Chapter 2

The Mathematical Background

2.1 Probability theory

Probability theory is a part of mathematics that deals with mathematical models of trials whose outcomes depend on chance. For the perspective of mathematical finance, we will go through some basic concepts of probability theory that are needed to begin solving stochastic calculus problems. It is not completed the whole prospect, but be sufficient for this research.

2.1.1 Continuous Random Variables

A continuous random variable is a random variable where the data can take infinitely many values. For example, heights of people in a population. Let's X is a continuous random variable, the cumulative distribution F of X is given by

$$F(x) = P\{X \leq x\}$$

We shall assume that there is some function f such that

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for all real number x, f is known as the PDF for X. The PDF f of a continuous random variable X satifies

1. $f(x) \geq 0$ for all x;
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a \leq X \leq b) = \int_a^b f(x)dx$ for all a, b.

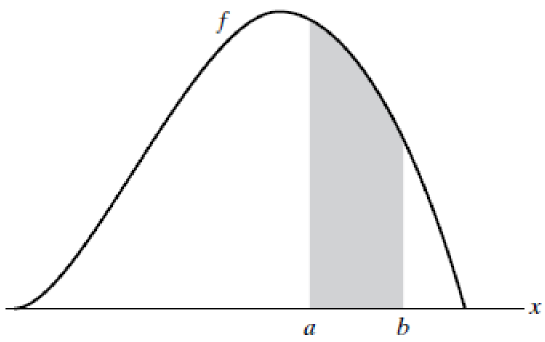

Figure 2.1: $P(a \leq X \leq b)$ = area of shaded region.

Probabilities correspond to areas under the curve $f(x)$. For any single value a, $P(X = a) = 0$. $P(a<X<b) = P(a \leq X<b) = P(a<X \leq b) = P(a \leq X \leq b)$.

The expected value of X,

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

The expected value of any real-valued function g,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

The variance

$$Var(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

$$Var(aX + b) = a^2\sqrt{Var(X)}$$

Where

μ : expected value of X

a, b: real value

2.1.2 Normal Random Variables

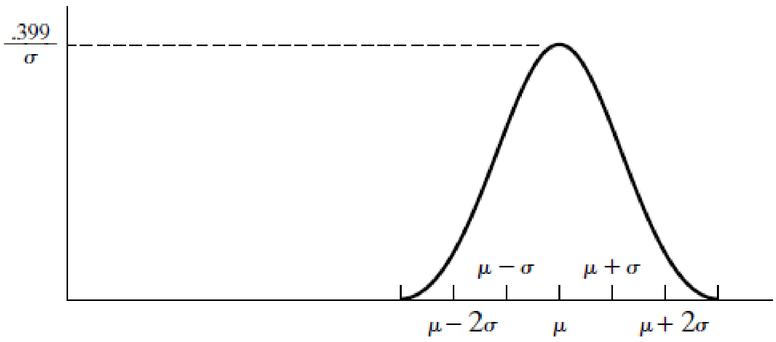

Figure 2.2: Arbitrary μ, σ^2

X is a normal random variable (normally distributed) with parameter μ and σ^2, the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

If $Y = aX + b$, then Y is normally distributed with parameter $a\mu + b$ and $a^2\sigma^2$.

- The cumulative distribution of Y

$$F_Y(x) = P\{Y \leq x\} = F_X\left(\frac{x-b}{a}\right)$$

- The density function of Y

$$f_Y(x) = \frac{1}{\sqrt{2\pi}a\sigma} e^{\frac{-(x-b-a\mu)^2}{2(a\mu)^2}}$$

The standard normal random variable

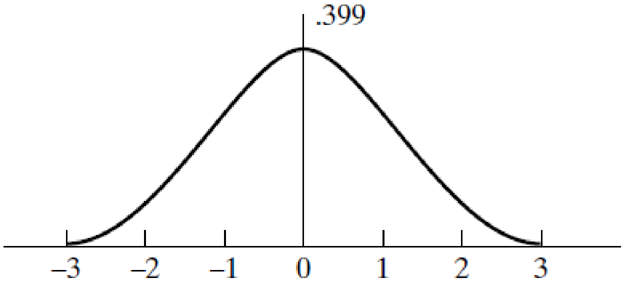

Figure 2.3: $\mu = 0$, $\sigma = 1$

$Z = \dfrac{X-\mu}{\sigma}$ is standard normally distributed with parameters 0 and 1.

Proof

Mean parameter

$$\begin{aligned} E\left(\frac{X-\mu}{\sigma}\right) &= \frac{1}{\sigma}E(X-\mu) = \frac{1}{\sigma}[E(X)-E(\mu)] \\ &= \frac{1}{\sigma}(\mu-\mu) = 0. \end{aligned}$$

Volatility parameter

$$\begin{aligned} Var\left(\frac{X-\mu}{\sigma}\right) &= \frac{1}{\sigma^2}Var(X-\mu) = \frac{1}{\sigma^2}[Var(X)-Var(\mu)] \\ &= \frac{\sigma^2}{\sigma^2} = 1. \end{aligned}$$

The cumulative distribution function of standard normal random variable

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-y^2}{2}}dy \tag{2.1}$$

$$\phi(-x) = 1-\phi(x) \tag{2.2}$$

Proof

Firstly, equation (2.1)

Let $Y = \dfrac{X-\mu}{\sigma}$, then

$$\begin{aligned}
\phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\times 1} e^{\frac{-(y-0)^2}{2\times 1^2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy
\end{aligned}$$

Secondly, equation (2.2)

$$\begin{aligned}
\phi(-x) &= P\{X < -x\} \\
&= P\{X > x\} \\
&= P\{X \in (x, \infty)\} \\
&= \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{\frac{-x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-x^2}{2}} dx \\
&= 1 - \phi(x)
\end{aligned}$$

2.1.3 Lognormal property of stock price

We have a random variable Y

$$Y = e^X \sim \log -N(\mu, \sigma^2) \text{ is distributed log-normally,}$$

$$\text{then } \ln Y = X \sim N(\mu, \sigma^2) \text{ is normally distributed}$$

The lognormal distribution is bounded below by 0 and skewed to the right. It is extremely useful when analyzing stock prices which cannot fall below zero.

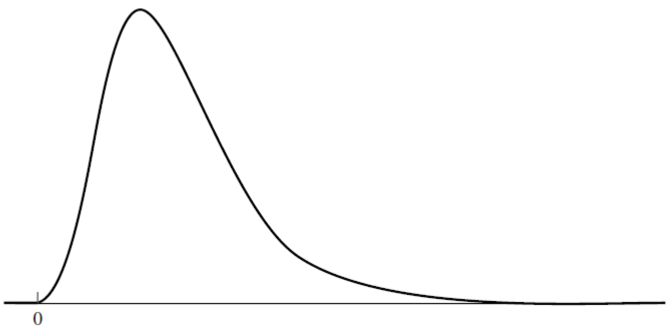

Figure 2.4: Lognormal distribution

The mean value and variance of the log-normal distribution is given by

$$E(Y) = e^{\mu + \frac{1}{2}\sigma^2} \tag{2.3}$$

$$Var(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \tag{2.4}$$

- The probability density function of $LN(\mu, \sigma^2)$ is

$$f(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{\frac{-(lnx-\mu)^2}{2\sigma^2}}$$

- The cumulative distribution function of $LN(\mu, \sigma^2)$ is

$$\begin{aligned}\int_0^\infty f(x)dx &= \frac{1}{x\sqrt{2\pi}\sigma}\int_0^\infty e^{\frac{-(\ln x-\mu)^2}{2\sigma^2}}dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^\infty e^{\frac{-(y-\mu)^2}{2\sigma^2}}dy \\ &= 1.\end{aligned}$$

Where $x = e^y$ or $lnx = y$, this leads to $\frac{dx}{x} = dy$

We consider the stock price S_t as a random process which is only nonnegative. S_T is the stock price at a future time T and S_0 is the stock price at time 0. The variable $\ln S_T$ is normally distributed, so that S_T has a lognormal distribution. The mean of $\ln S_T$ is $\ln S_0 + (\mu - \frac{\sigma^2}{2})T$

and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$. These are expressed by

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T] \tag{2.5}$$

Given an example for the expression 2.5. Consider a stock with an initial price S_0 of \$40, an expected return μ of 16% per annum, and a volatility σ of 20% per annum. The probability distribution of the stock price S_T in 6 months' time is given by

$$\ln S_T \sim \phi[\ln 40 + (0.16 - 0.2^2/2) \times 0.5, 0.2^2 \times 0.5]$$

$$\ln S_T \sim \phi(3.759, 0.02)$$

There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. Therefore, the interval of stock price be able variation with 95% confidence,

$$3.759 - 1.96 \times 0.141 \ < \ \ln S_T \ < \ 3.759 + 1.96 \times 0.141$$

This can be written

$$e^{3.759 - 1.96 \times 0.141} \ < \ S_T \ < \ e^{3.759 + 1.96 \times 0.141}$$

Or

$$32.55 \ < \ S_T \ < \ 56.56$$

Thus, there is a 95% probability that the stock price in 6 months will lie between \$32.55 and \$56.56.

2.2 Wiener Process

In mathematics, a Wiener process is a stochastic process sharing the same behaviour as Brownian motion, which is a physical phenomenon of random movement of particles suspended in a fluid. These financial models are stochastic and continuous in nature, the Wiener process is usually employed to express the random component of the model.

2.2.1 Standard Wiener Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $\{W_t : t \geq 0\}$ is defined to be a standard Wiener process (or $\mathbb{P}$-standard Wiener process) if:

1. $W_0 = 0$;
2. With probability 1, the function $t \to W_t$ is continuous in t.
3. The process $\{W_t\}_{t \geq 0}$ stationary, independent increments.
4. The increment $W_{t+s} - W_s$ has the normal distribution $\mathcal{N}(0, t)$.

The term independent increments means that for every choice of non-negative real numbers

$$0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n < \infty,$$

the increment random variables

$$W_{t_1} - W_{s_1}, W_{t_2} - W_{s_2}, \ldots, W_{t_n} - W_{s_n}$$

are jointly independent; the term stationary increments means that for any $0 < s, t < \infty$ the distribution of the increment $W_{t+s} - W_s$ has the same distribution as $W_t - W_0 = W_t$.

Quadratic Variation

The quadratic varivation property of Bownian motion

- $(dW_t)^2 = dt$
- $dW_t dt = 0$
- $(dt)^2 = 0$
- $(dW_t)^p = 0, \quad p \geq 3$

Where $W_t : t \geq 0$ is a standard Brownian motion and dW_t and dt are the infinitesimal increment of W_t and t, respectively. The significance of the above results constitutes the key ingredients in Itō's formula to find the differential of a stochastic function and also in deriving the Black-Scholes equation to price option.

2.2.2 Martingale Pricing Theory

Martingale pricing is a pricing approach based on the notions of equivalent martingale measure and risk-neutral valuation. The martingale pricing approach is a cornerstone of modern quantitative finance and can be applied to a form of derivatives contracts, e.g. options, future, etc. Without loss of generality, assume we are in the Black–Scholes world with an economy consisting of a risky asset or stock S_t following a geometric Brownian motion and a risk-free asset B_t growing at a continuously compounded interest rate r of the form

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad \frac{dB_t}{B_t} = r dt$$

where μ is the stock drift, σ is the stock volatility and W_t is a standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Radon–Nikody'm Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space satisfying the usual conditions. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{F}, \mathbb{Q})$. Under the assumption that $\mathbb{Q} << \mathbb{P}$, there exists a non-negative random variable Z such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$$

and we call Z the Radon–Nikody'm derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Let's consider the standard Brownian process $W_t^{\mathbb{P}}$ under the measure $\mathbb{P}$. Adding the drift μt to $W_t^{\mathbb{P}}$, μ is a constant, we write

$$W_t^{\mathbb{P}} = W_t^0 + \mu t$$

Here, $W_t^{\mathbb{P}}$ is a Brownian process with drift under $\mathbb{P}$. we can change from measure $\mathbb{P}$ to another measure $\mathbb{Q}$ so that $W_t^{\mathbb{P}}$ becomes a Brownian process with zero drift under $\mathbb{Q}$. Formally, we multiply $d\mathbb{P}$ by a factor $\dfrac{d\mathbb{Q}}{d\mathbb{P}}$ to give $d\mathbb{Q}$. It is postulated that the corresponding Radon–Nikodym derivative for this case is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\mu W_t^{\mathbb{P}} - \frac{\mu^2}{2} t} \tag{2.6}$$

In finance, derivative instruments such as options, swaps or futures can be used for both hedging and speculation purposes. In a hedging scenario, traders can reduce their risk exposure by buying and selling derivatives against fluctuations in the movement of underlying risky asset prices such as stocks and commodities. Conversely, in a speculation scenario, traders can also use derivatives to profit in the future direction of underlying prices. For example, if a trader expects an asset price to rise in the future, then he/she can sell put options (i.e., the purchaser of the put options pays an initial premium to the seller and has the right but not the obligation to sell the shares back to the seller at an agreed price should the share price drop below it at the option expiry date). Given the purchaser of the put option is unlikely to exercise the option, the seller would be most likely to profit from the premium paid by the purchaser. From the point of view of trading such contracts, we would like to price contingent claims (or payoffs of derivative securities such as options) in such a way that there is no arbitrage opportunity (or no risk-free profits). By doing so we will ensure that **even though two traders may differ in their estimate of the stock price direction, yet they will still agree on the price of the derivative security**. In order to accomplish this we can rely on Girsanov's theorem, which tells us how **a stochastic process can have a drift change (but not volatility) under a change of measure**. With the application of this important result to finance we can convert the underlying stock prices under the physical measure (or real-world measure) into **the risk-neutral measure (or equivalent martingale measure) where all the current stock prices are equal to their expected future prices discounted at the risk-free rate**. This is in contrast to using the **physical measure, where the derivative security prices will vary greatly since the underlying assets will differ in degrees of risk from each other**.

Equivalent Martingale Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space satisfying the usual conditions and let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$. The probability measure $\mathbb{Q}$ is said to be an equivalent measure or risk-neutral measure if it satisfies

- $\mathbb{Q} \sim \mathbb{P}$. This means that $\mathbb{Q}$ is equivalent to $\mathbb{P}$, if an event cannot occur under the $\mathbb{P}$ measure then it also cannot occur under the $\mathbb{Q}$ measure and vice versa ($\mathbb{Q}(\beta) = 0 \leftrightarrow \mathbb{P}(\beta) = 0$). Where β is a random variable.
- The discounted risky asset price process $\{B_t^{-1} S_t^{(i)}\}, i = 1, 2, \ldots, m$ are martingales under

$\mathbb{Q}$, that is

$$\mathbb{E}^{\mathbb{Q}}(B_u^{-1}S_u^{(i)}|\mathcal{F}_t) = B_t^{-1}S_t^{(i)}$$

for all $0 \leq t \leq u \leq T$.

Girsanov's Theorem

Theorem 1. *(Girsanov's theorem) Let* $\{(W_t)_{0\leq t\leq T}\}$ *be a* $\mathcal{P}$*-sstandard Brownian motion on the probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *and let* $(\mathcal{F}_t)_{0\leq t\leq T}$ *be the associated filtration. Suppose* $(\theta_t)_{0\leq t\leq T}$ *is an adapted process and consider* $Z_t = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds}$. *If* $E^P(e^{1/2\int_0^T \theta_t^2 dt}) < \infty$ *then* Z_t *is a positive* $\mathcal{P}$*-martingale for* $0 \leq t \leq T$. *By changing the measure* $\mathcal{P}$ *to a measure* $\mathcal{Q}$ *such that* $\frac{d\mathcal{Q}}{d\mathcal{P}}|_{\mathcal{F}_t} = Z_t$ *(*Z_t *is the Radon- Nykodym's derivative of* $\mathcal{Q}$ *with respect to* $\mathcal{P}$ *under the filtration* $\mathcal{F}_t$*) then* $\widetilde{W_t} = W_t + \int_0^t \theta_u du$ $(d\widetilde{W_t} = dW_t + \theta dt)$ *is a* $\mathcal{Q}$*-standard Brownian motion.*

2.3 Stochastic Differential Equations

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms has a random component. Within the context of mathematical finance, SDEs are frequently used to model diverse phenomena such as stock prices, interest rates or volatilities to name but a few. Typically, SDEs have continuous paths with both random and non-random components and to drive the random component of the model they usually incorporate a Wiener process.

To begin with, a one-dimensional stochastic differential equation can be described as

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

where W_t is a standard Wiener process, $\mu(X_t, t)$ is defined as the drift and $\sigma(X_t, t)$ the volatility. Many financial models can be written in this form, such as the lognormal asset random walk, common spot interest rate and stochastic volatility models.

2.3.1 Itō's lemma

Ito's Lemma is a key component in the Ito Calculus, used to determine the derivative of a time-dependent function of a stochastic process.It can be heuristically derived by forming the Taylor series expansion of the function up to its second derivatives and retaining terms up to first order in the time increment and second order in the Wiener process increment. The lemma is widely employed in mathematical finance, and its best known application is in the derivation of the Black–Scholes equation for option values. Therefore, in order to derive Itō's lemma we have to expand a Taylor series and applying the rules of stochastic calculus.

Taylor's Theorem

Taylor's theorem in one real variable.

Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at the point $x_0 \in \mathbb{R}$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x - x_0)^k$$

The partial derivatives through order 2 is given by

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ f(x) - f(x_0) &= f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ \Delta f(x) &= f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 \end{aligned}$$

When $\Delta x \to 0$ and $\Delta f(x) \to 0$. This implies that

$$df(x) = f'(x)dx + \frac{1}{2}f''(x)dx^2$$

Taylor's theorem in two real variable. The partial derivatives of $f(x,t)$ through order 2 is

$$df(x,t) = f'_x(x,t)dx + f'_t(x,t)dt + \frac{1}{2}f''_{x^2}(x,t)dx^2 + \frac{1}{2}f''_{t^2}dt^2 + f''_{xt}dxdt$$

Proof of Itō's lemma

Suppose that $f(X_t,t)$ is a continuous differential function of t and X_t follows

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where W_t is the Brownian motion. The variable X_t has a drift rate of μ_t and has a variance rate of σ_t^2. Applying Taylor's theorem in two variables X_t and t we get

$$df(X_t,t) = f'_x(X_t,t)dX_t + f'_t(X_t,t)dt + \frac{1}{2}f''_{x^2}(X_t,t)dX_t^2 + \frac{1}{2}f''_{t^2}dt^2 + f''_{xt}dX_tdt$$

Based on the above quaratic variation property of Brownian motion we can obtain

$$\begin{aligned}
df(X_t,t) &= f'_x(X_t,t)dX_t + f'_t(X_t,t)dt + \frac{1}{2}f''_{x^2}(X_t,t)dX_t^2 + \frac{1}{2}f''_{t^2}0 + f''_{xt}0 \\
&= f'_x(X_t,t)dX_t + f'_t(X_t,t)dt + \frac{1}{2}f''_{x^2}(X_t,t)dX_t^2 \\
&= f'_x(X_t,t)(\mu_t dt + \sigma_t dW_t) + f'_t(X_t,t)dt + \frac{1}{2}f''_{x^2}(X_t,t)(\mu_t dt + \sigma_t dW_t)^2 \\
&= \left(f'_t(X_t,t) + \mu_t f'_x(X_t,t) + \frac{\sigma_t^2}{2}f''_{x^2}(X_t,t) \right) dt + \sigma_t f'_x(X_t,t)dW_t
\end{aligned}$$

Therefore, Itō's lemma shows that a function f of X and t follows the process

$$df(X_t,t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

Geometric Brownian Motion

The stock price cannot be negative (it is actually always positive, it just turns to zero when its company goes bankrupt). The stock price therefore cannot be governed by a Brownian motion because at some stage the Brownian motion can receive negative values. But if we consider the return of the stock, instead of stock price, it can be negative, and it can be imagined that the movement of return of the stock affected by random factors on the market is similar to the Brownian motion. The following Geometric Brownian motion is used to model the movement of stock price S_t:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

The coefficients μ and σ, representing the drift and volatility of the asset, respectively, are both constant in this model. Applying Ito-Doeblin formula to $f(S_t) = \log S_t$, we obtain:

$$d(\log S_t) = df(S_t) = f'(S_t)dS_t + \frac{1}{2}f''(S_t)dS_t^2 = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{(\mu S_t dt + \sigma S_t dW_t)^2}{S_t^2}$$
$$= \mu dt + \sigma dW_t - \frac{1}{2}\frac{\sigma^2 S_t^2 dW_t^2}{S_t^2} = \mu dt + \sigma dW_t - \frac{\sigma^2 dt}{2} = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$$

It is a standard Brownian motion with a drift term. Taking integral of 2 sides between the limits 0 and t we can write this as:

$$\int_0^t d(\log S_u) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u \Leftrightarrow \log S_u \Big|_0^t = \left(\mu - \frac{1}{2}\sigma^2\right) u \Big|_0^t + \sigma W_u \Big|_0^t$$
$$\Leftrightarrow \log S_t - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \Leftrightarrow \log \frac{S_t}{S_0} = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

Finally, taking the exponential of this equation gives:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{2.7}$$

Geometric Brownian motion on the risk-neutral world

Consider an asset price S_t follows a geometric Brownian motion: $dS_t = \mu S_t dt + \sigma S_t dW_t$, where W_t is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$. According to the Girsanov's theorem, there exists a risk-neutral measure $\mathcal{Q}$ on the filtration $(\mathcal{F}_t)_{t\geq 0}$ such that the process $W_t^{\mathcal{Q}} = W_t + \int_0^t \theta_s ds$ (or $dW_t^{\mathcal{Q}} = dW_t + \theta_t dt$) is a standard Brownian motion on the $\mathcal{Q}$, where θ_t is an adapted process to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

We have

$$dS_t = \mu S_t dt + \sigma S_t dW_t = \mu S_t dt + \sigma S_t (dW_t^{\mathcal{Q}} - \theta_t dt) = (\mu - \sigma\theta) S_t dt + \sigma S_t dW_t^{\mathcal{Q}}.$$

Thus

$$\begin{aligned} d(e^{-rt} S_t) &= -re^{-rt} S_t dt + e^{-rt} dS_t = -re^{-rt} S_t dt + e^{-rt}((\mu - \sigma\theta) S_t dt + \sigma S_t dW_t^{\mathcal{Q}}) \\ &= e^{-rt}((\mu - r - \sigma\theta) S_t dt + \sigma S_t dW_t^{\mathcal{Q}}). \end{aligned}$$

To ensure $\mathcal{Q}$ is an equivalent martingale with $\mathcal{P}$, $e^{-rt} S_t$ has to be a martingale with respect to the standard Brownian motion $W_t^{\mathcal{Q}}$. As a result, we have $\mu - r - \sigma\theta = 0$ or $\theta = \dfrac{\mu - r}{\sigma}$.

Thus, there is a unique θ which makes the discounted asset price process driftless, which is equivalent to saying that there is a unique change of measure which makes the discounted asset price a martingale under the risk-neutral measure.

Hence, under the risk-neutral measure $\mathbb{Q}$, the asset price follows the diffusion process

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t = \mu dt + \sigma(dW_t^{\mathbb{Q}} - \theta dt) = \mu dt + \sigma(dW_t^{\mathbb{Q}} - \frac{\mu - r}{\sigma} dt) = r dt + \sigma dW_t^{\mathbb{Q}}$$

2.3.2 Feynman–Kac Formula for One-Dimensional Diffusion Process

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$. Suppose a price process $(S_t)_{t\geq 0}$ following a generalized geometric Brownian motion: $dS_t = \mu(S_t, t)S_t dt + \sigma(S_t, t)S_t dW_t$. Consider a function $V(S_t, t)$, which is a solution of the PDE

$$\frac{\partial V}{\partial t}(S_t, t) + \frac{\sigma^2(S_t, t)}{2}\frac{\partial^2 V}{\partial S_t^2}(S_t, t) + \mu(S_t, t)\frac{\partial V}{\partial S_t}(S_t, t) - r(t)V(S_t, t) = 0,$$

with the boundary condition $V(S_T, T) = \Phi(S_T)$. Here σ, μ are functions of S_t and t, while r is a function of t and Φ is a function of S_T.

a) Prove that $dZ_u = e^{-\int_t^u r(v)dv}\sigma(S_u, u)\frac{\partial V}{\partial S_u}dW_u$, where $Z_u = e^{-\int_t^u r(v)dv}V(S_u, u), u \geq t$.

b) $V(S_t, t) = E[e^{-\int_t^T r(v)dv}\Phi(S_T)|\mathcal{F}_t]$

Proof. a) Let $g(u) = e^{-\int_t^u r(v)dv} \Rightarrow Z_u = g(u)V(S_u, u) = f(S_u, u)$. Using Ito's formula, we obtain:

$$\begin{aligned}
&dZ_u = d(f(S_u, u)) = f'_{S_u}dS_u + f'_u du + \frac{1}{2}f''_{S_u^2}dS_u^2 \\
&= g(u)\frac{\partial V}{\partial S_u}(S_u, u)dS_u + \big[-g(u)r(u)V(S_u, u) + g(u)\frac{\partial V}{\partial u}(S_u, u)\big]du + \frac{1}{2}g(u)\frac{\partial^2 V}{\partial S_u^2}(S_u, u)\sigma^2(S_u, u)du \\
&= g(u)\frac{\partial V}{\partial S_u}(S_u, u)\sigma(S_u, u)dW_u + g(u)\big[\frac{\partial V}{\partial u}(S_u, u) + \frac{\sigma^2(S_u, u)}{2}\frac{\partial^2 V}{\partial S_u^2}(S_u, u) + \mu(S_u, u)\frac{\partial V}{\partial S_u}(S_u, u) - r(u)V(S_u \\
&= g(u)\frac{\partial V}{\partial S_u}(S_u, u)\sigma(S_u, u)dW_u = e^{-\int_t^u r(v)dv}\sigma(S_u, u)\frac{\partial V}{\partial S_u}dW_u
\end{aligned}$$

b) Integrating both sides of the above equality from t to T, we obtain:

$$\int_t^T dZ_u = \int_t^T g(u)\frac{\partial V}{\partial S_u}(S_u, u)\sigma(S_u, u)dW_u \Rightarrow Z_T - Z_t = \int_t^T g(u)\frac{\partial V}{\partial S_u}(S_u, u)\sigma(S_u, u)dW_u$$

$$E[Z_T - Z_t|\mathcal{F}_t] = E[\int_t^T g(u)\frac{\partial V}{\partial S_u}(S_u,u)\sigma(S_u,u)dW_u|\mathcal{F}_t] = \int_t^t g(u)\frac{\partial V}{\partial S_u}(S_u,u)\sigma(S_u,u)dW_u = 0$$
$$E[e^{-\int_t^T r(v)dv}\Phi(S_T)|\mathcal{F}_t] = E[e^{-\int_t^T r(v)dv}V(S_T,T)|\mathcal{F}_t] = E[Z_T|\mathcal{F}_t] = Z_t = V(S_t,t)$$

□

Remark Consider the stochastic differential equation (SDE)

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t$$

with $\mathbb{E}[(\int_0^t \sigma(X_s,s)^2 ds)^2] < \infty$,then X_t is a martingale if and only if X_t is driftless (i.e., $\mu(X_t,t) = 0$).

Strong Markov property

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion and let τ be a stopping time with respect to $(B_t)_{t\geq 0}$. Define $Z(t) = B_{t+\tau} - B_\tau$ then

- $(Z_t)_{t\geq 0}$ is also a standard Brownian motion.
- For each $t > 0$, $(Z_s)_{0\leq s\leq t}$ is independent with $(B_u)_{0\leq u\leq \tau}$.
- $(B_t)_{t\geq \tau}$ is a Brownian motion start with B_τ.

Example 2.3.1. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let $\tau_m = \min\{t \geq 0|B_t = m\}$. Define $Z(t) = B_{t+\tau_m} - B_{\tau_m}$, then

- $(Z_t)_{t\geq 0}$ is also a standard Brownian motion
- For each $t > 0$, $(Z_s)_{0\leq s\leq t}$ is independent with $(B_u)_{0\leq u\leq \tau_m}$
- $(B_t)_{t\geq \tau_m}$ is a Brownian motion start with m.

Reflection principle of Brownian motion

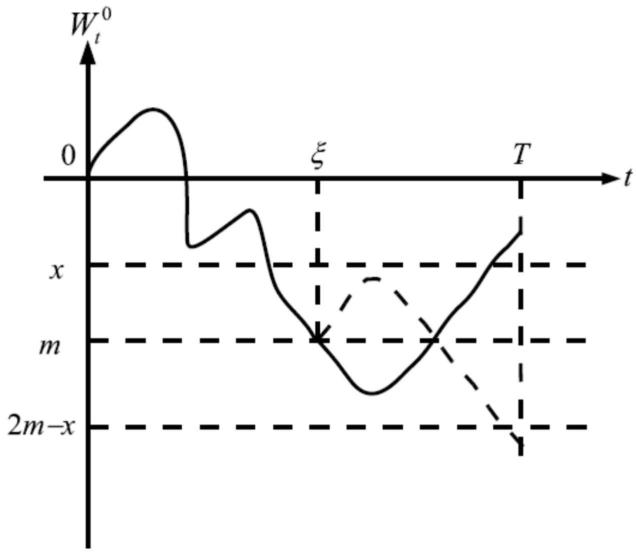

Figure 2.5: A graphical representation of the reflection principle of the Brownian motion W_t.

- $P(\tau_m \leq t, B_t \geq m) = P(\tau_m \leq t, B_t \leq m)$
- $P(\tau_m \leq t, B_t \leq w) = P(B_t \geq 2m - w)$.
- $P(\tau_m \leq t, B_t \geq w) = P(B_t \leq 2m - w)$

First passage time distribution

- For all $m \neq 0$, the random variable τ_m has cumulative distribution function

$$P(\tau_m \leq t) = \frac{2}{\sqrt{2\pi}} \int_{m/\sqrt{t}}^{\infty} e^{-y^2/2} dy$$

and the density function:

$$f_{\tau_m}(t) = \frac{d}{dt} P(\tau_m \leq t) = \frac{|m|}{t\sqrt{2\pi t}} e^{-m^2/(2t)}, t \geq 0.$$

Proof. We first consider the case $m > 0$. We have

$$P(\tau_m \leq t) = P(\tau_m \leq t, B_t \leq m) + P(\tau_m \leq t, B_t \geq m).$$

As clearly $P(\tau_m \leq t, B_t \geq m) = P(B_t \geq m)$ and by the reflection principle

$$P(\tau_m \leq t, B_t \leq m) = P(B_t \geq m),$$

we have $P(\tau_m \leq t) = 2P(B_t \geq m) = 2\frac{1}{\sqrt{2\pi t}}\int_m^\infty e^{-x^2/(2t)}dx = \frac{2}{\sqrt{2\pi}}\int_{m/\sqrt{t}}^\infty e^{-y^2/2}dy$, with $y = x/\sqrt{t}$.

We next consider the case $m < 0$. We have

$$P(\tau_m \leq t) = P(\tau_m \leq t, B_t \leq m) + P(\tau_m \leq t, B_t \geq m).$$

By definition, $P(\tau_m \leq t, B_t \leq m) = P(B_t \leq m)$. In addition, $P(\tau_m \leq t, B_t \geq m) = P(B_t \leq m)$ due to the reflection principle. We can obtain: $P(\tau_m \leq t) = 2P(B_t \leq m) = 2\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^m e^{-x^2/(2t)}dx = 2\frac{1}{\sqrt{2\pi t}}\int_{-m}^\infty e^{-x^2/(2t)}dx = \frac{2}{\sqrt{2\pi}}\int_{-m/\sqrt{t}}^\infty e^{-y^2/2}dy$, with $y = x/\sqrt{t}$. As a result, we can conclude: for $m \neq 0$, $P(\tau_m \leq t) = \frac{2}{\sqrt{2\pi}}\int_{|m|/\sqrt{t}}^\infty e^{-y^2/2}dy$. The density function: $f_{\tau_m}(t) = \frac{d}{dt}P(\tau_m \leq t) = \frac{|m|}{t\sqrt{2\pi t}}e^{-m^2/(2t)}, t \geq 0$. The probability that a standard Brownian motion reaches level m sometime in the interval $[0, T]$ is $P(\tau_m \leq T) = \int_0^T \frac{|m|}{\sqrt{2\pi t^3}}e^{-m^2/(2t)}dt$ □

Minimum of Brownian motion

Let (W_t) be the standard Brownian motion. Then $(m_t)_{t\geq 0}$, where $m_t = \min_{0\leq s\leq t} B_s$, m_t is the minimum value of Brownian motion on $[0, t]$, forms the minimum process of Brownian motion. It can be observed that $m_t \leq a < 0$ if and only if $\tau_a < t$.

$$P(m_t \leq a) = P(\tau_a \leq t) = \int_0^t \frac{-a}{\sqrt{2\pi s^3}}e^{-a^2/(2s)}ds = \int_{-a}^{+\infty} \sqrt{\frac{2}{\pi t}}e^{-x^2/(2t)}dx, \quad x = -\frac{a\sqrt{t}}{\sqrt{s}}.$$

Proposition 1. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $m_t = \min_{0\leq s\leq t} Ws$ be the minimum level reached by a standard Brownian motion $(W_t)_{t\geq 0}$ in the time interval $[0,t]$. Using the reflection principle to show that:

$$\mathcal{P}(m_t \geq a, W_t \geq x) = \begin{cases} N(-\frac{x}{\sqrt{t}}) - N(\frac{2a-x}{\sqrt{t}}), & a \leq 0, a \leq x, \\ N(-\frac{a}{\sqrt{t}}) - N(\frac{a}{\sqrt{t}}), & a \leq 0, a \geq x, \\ 0, & a \geq 0, \end{cases},$$

with $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$ and the joint probability density function of (m_t, W_t) is:

$$f_{m_t,W_t}(a,x) = \begin{cases} \frac{2(x-2a)}{t\sqrt{2\pi t}} \exp(-\frac{1}{2}\frac{(2a-x)^2}{t}), & a \leq 0, a \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For $a < 0$, let $\tau_a = \inf\{t \geq 0 : W_t = a\}$, then $\{m_t \leq a\} \Leftrightarrow \{\tau_a \leq t\}$. The key reflection equality

$$P(\tau_a \leq t, B_t \geq x) = P(B_t \leq 2a - x), \quad \forall x \geq a, a < 0.$$

For $x \geq a$, using the reflection principle, we have:

$$\mathcal{P}(m_t \leq a, W_t \geq x) = \mathcal{P}(\tau_a \leq t, W_t \geq x) = \mathcal{P}(W_t \leq 2a - x) = \mathcal{P}(\sqrt{t}Z \leq 2a - x) = N(\frac{2a-x}{\sqrt{t}}).$$

Thus,

$$\begin{aligned} &\mathcal{P}(m_t \geq a, W_t \geq x) = \mathcal{P}(W_t \geq x) - \mathcal{P}(m_t \leq a, W_t \geq x) \\ &= 1 - N(\frac{x}{\sqrt{t}}) - N(\frac{2a-x}{\sqrt{t}}) = N(-\frac{x}{\sqrt{t}}) - N(\frac{2a-x}{\sqrt{t}}), \text{for } x \geq a, a < 0. \end{aligned}$$

For $a < 0, x \leq a$, as $W_t \geq m_t$, we have:

$$\mathcal{P}(m_t \geq a, W_t \geq x) = \mathcal{P}(m_t \geq a, W_t \geq a) = N(-\frac{a}{\sqrt{t}}) - N(\frac{a}{\sqrt{t}})$$

For $a < 0$, as $m_t \leq W_0 = 0$, thus $\mathcal{P}(m_t \geq a, W_t \geq x) = 0$.

The joint density function of (m_t, W_t) is:

$$f_{m_t,W_t}(a,x) = \begin{cases} \dfrac{2(x-2a)}{t\sqrt{2\pi t}} \exp(-\dfrac{(2a-x)^2}{2t}), & a < 0, x \geq a, \\ 0, & \text{otherwise.} \end{cases}$$

with the notice that $N'(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$. $\square$

Proposition 2. Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$. Let $\widehat{W_t} = \alpha t + W_t$, where α is a constant. Define $m_t = \min_{0\leq s\leq t} \widehat{W}_s$ be the minimum level reached by $(\widehat{W_t})_{t\geq 0}$ in the time interval $[0,t]$. Prove that the joint probability density function of $(m_t, \widehat{W_t})$ under $\mathcal{P}$ is:

$$f_{m_t,\widehat{W}_t}(a,x) = \begin{cases} \dfrac{2(x-2a)}{t\sqrt{2\pi t}} \exp(\alpha x - \dfrac{1}{2}\alpha^2 t - \dfrac{1}{2}\dfrac{(2a-x)^2}{t}), & a \leq 0, a \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We express $\widehat{W_t} = W_t + \displaystyle\int_0^t \alpha s ds$. From the Girsanov's theorem, there exists a risk-neutral measure $\mathcal{Q}$ defined by the Radon-Nykodym's derivative

$$\frac{d\mathcal{Q}}{d\mathcal{P}} = Z = e^{-\frac{1}{2}\int_0^t \alpha^2 ds - \int_0^t \alpha dW_s} = e^{-\frac{1}{2}\alpha^2 t - \alpha W_t} = e^{-\frac{1}{2}\alpha^2 t - \alpha(\widehat{W}_t - \alpha t)} = e^{\frac{1}{2}\alpha^2 t - \alpha \widehat{W}_t}$$

such that $\widehat{W_t}$ is a standard Brownian motion under $\mathcal{Q}$. Let $\widehat{m}_t = \min_{0\leq s\leq t} \widehat{W}_s$, then the joint density of $\widehat{W_t}$ and $\widehat{m}_t$ under measure $\mathcal{Q}$ is given by (Proposition 1):

$$f^{\mathcal{Q}}_{\widehat{m}_t,\widehat{W}_t}(a,x) = \begin{cases} \dfrac{2(x-2a)}{t\sqrt{2\pi t}} \exp(-\dfrac{(2a-x)^2}{2t}), & a < 0, x \geq a, \\ 0, & \text{otherwise.} \end{cases}$$

with the notice that $N'(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$. The joint cumulative distribution of $\widehat{W_t}$ and $\widehat{m}_t$ under measure $\mathcal{P}$ is computed as:

$$\mathcal{P}(\widehat{m}_t \leq a, \widehat{W}_t \leq x) = E^{\mathcal{P}}[\mathcal{I}_{\{\widehat{m}_t \leq a, \widehat{W}_t \leq x\}}] = E^{\mathcal{Q}}[Z^{-1}\mathcal{I}_{\{\widehat{m}_t \leq a, \widehat{W}_t \leq x\}}] = E^{\mathcal{Q}}[e^{-\frac{1}{2}\alpha^2 t + \alpha \widehat{W}_t}\mathcal{I}_{\{\widehat{m}_t \leq a, \widehat{W}_t \leq x\}}]$$
$$= \int_{-\infty}^{a}\int_{-\infty}^{x} e^{-\frac{1}{2}\alpha^2 t + \alpha v} f^{\mathcal{Q}}_{\widehat{m}_t, \widehat{W}_t}(u,v)dvdu = \int_{-\infty}^{a}\int_{a}^{x} e^{-\frac{1}{2}\alpha^2 t + \alpha v}\frac{2(v-2u)}{t\sqrt{2\pi t}}\exp(-\frac{(2u-v)^2}{2t})dvdu, a < 0, x \geq a$$

Therefore,

$$\begin{aligned}
f^{\mathcal{P}}_{\widehat{m}_t, \widehat{W}_t}(a,x) &= \frac{\partial^2 \mathcal{P}(\widehat{m}_t \leq a, \widehat{W}_t \leq x)}{\partial a \partial x} = \frac{\partial^2}{\partial a \partial x}\int_{-\infty}^{a}\int_{a}^{x} e^{-\frac{1}{2}\alpha^2 t + \alpha v}\frac{2(v-2u)}{t\sqrt{2\pi t}}\exp(-\frac{(2u-v)^2}{2t})dvdu \\
&= \frac{\partial}{\partial a}[\frac{\partial}{\partial x}\int_{-\infty}^{a}\int_{a}^{x} e^{-\frac{1}{2}\alpha^2 t + \alpha v}\frac{2(v-2u)}{t\sqrt{2\pi t}}\exp(-\frac{(2u-v)^2}{2t})dvdu] \\
&= \frac{\partial}{\partial a}[\int_{-\infty}^{a} e^{-\frac{1}{2}\alpha^2 t + \alpha x}\frac{2(x-2u)}{t\sqrt{2\pi t}}\exp(-\frac{(2u-x)^2}{2t})du] \\
&= e^{-\frac{1}{2}\alpha^2 t + \alpha x}\frac{2(x-2a)}{t\sqrt{2\pi t}}\exp(-\frac{(2a-x)^2}{2t})du \\
&= \begin{cases} \frac{2(x-2a)}{t\sqrt{2\pi t}}\exp(\alpha x - \frac{1}{2}\alpha^2 t - \frac{(2a-x)^2}{2t}), & a < 0, x \geq a, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.8}$$

□

Chapter 3

The Black - Scholes - Merton model

3.1 European Call Option Valuation

In 1973, Fischer Black and Myron Scholes published their paper "The pricing of options and corporate liabilities" in The Journal of Political Economy with the aim of deriving a theoretical valuation formula to price European options. The principal idea behind their theory is to hedge the option by buying/selling the underlying asset in such a way as to eliminate risk. This type of hedging is called delta hedging. We now make the following assumptions:

- The risk-free interest rate is known and is constant over time.
- The asset price follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

- The asset pays no dividends during the life of the option.
- No transaction costs are associated with buying or selling the asset/option.
- Trading of the asset can take place continuously.
- Short selling is permitted.

Under these assumptions, the option value at any time t, denoted by C_t, depends only on the price of the asset and time, and on parameters which are taken to be known constants, i.e., $C_t = C_t(S_t, t)$.

At time t, we establish a hedging portfolio Π: buy a European call option and short sell Δ shares of the underlying asset. The value of the hedging portfolio Π_t at time t can be expressed by: $\Pi_t = C_t - \Delta S_t$.

In addition, over "short" time intervals, the stochastic part of the change in the option price is perfectly correlated with changes in the stock price, i.e., $d\Pi_t = dC_t - \Delta dS_t$.

By writing the stochastic process of the asset price S_t as a geometric Brownian motion: $dS_t = \mu S_t dt + \sigma S_t dW_t$, where μ is the drift or expected rate of return, σ is a constant volatility and W_t is the standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Applying Ito's lemma, we have:

$$\begin{aligned} dC_t &= \frac{\partial C_t}{\partial t}dt + \frac{\partial C_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(dS_t)^2 + ... & (3.1) \\ &= (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + \mu S_t \frac{\partial C_t}{\partial S_t})dt + \sigma S_t \frac{\partial C_t}{\partial S_t} dW_t. & (3.2) \end{aligned}$$

As a result, we obtain:

$$d\Pi_t = (\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + \mu S_t(\frac{\partial C_t}{\partial S_t} - \Delta))dt + \sigma S_t(\frac{\partial C_t}{\partial S_t} - \Delta)dW_t. \tag{3.3}$$

By choosing the portfolio weight $\Delta = \frac{\partial C_t}{\partial S_t}$, we can eliminate the randomness in the change of the hedging portfolio as follows:

$$d\Pi_t = (\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2})dt. \tag{3.4}$$

The hedging portfolio Π is now a riskless one so that the rate of change in Π_t must be equal to the risk-free interest rate (compounded continuously), i.e., $d\Pi_t = r\Pi_t = r(C_t - \frac{\partial C_t}{\partial S_t}S_t)dt$.

As a result, we obtain the Black-Scholes-Merton equation:

$$\frac{\partial C_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + rS_t \frac{\partial C_t}{\partial S_t} - rC_t = 0.$$

Applying the Feynman-Kac theorem, the time-t value $C(S_t, t)$ can be written as

$$C(S_t, t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[C(S_T, T)|\mathcal{F}_t] \tag{3.5}$$

A European call option $C(S_t; t)$ written on S_t with strike price K has the payoff $C(S_T; T) = \max(S_T - K; 0)$. We substitute this payoff in the Feynman-Kac formula in Equation (3.5). Hence, the time-t value of a European call is

$$C(S_t, t; K, T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\max\{S_T - K, 0\} \mid \mathcal{F}_t]$$

From Girsanov's theorem, under a $\mathbb{Q}$-standard Wiener process we have

$$W_t^{\mathbb{Q}} = W_t + \left(\frac{\mu - r}{\sigma}\right) t$$

And consider equation (2.3.1) which has been appoved in property of GBM

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

Under the risk-neutral measure $\mathbb{Q}$, that equation is equivalent to

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathbb{Q}}} \tag{3.6}$$

where r is the risk-free interest rate. This implies that

$$\begin{aligned} S_t &= e^{logS_0} e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathbb{Q}}} \\ &= e^{logS_0 + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathbb{Q}}} \end{aligned}$$

For $T > t$, $W^{\mathbb{Q}}_{T-t} \sim \mathcal{N}(0, T-t)$. Hence, conditional on S_t we can write

$$S_T \mid S_t \sim \log - \mathcal{N}[\log S_t + (r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)]$$

Under the risk-neutral measure $\mathbb{Q}$ we can write the European call option price as

$$\begin{aligned} C(S_t, t; K, T) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\max\{S_T - K, 0\} \mid \mathcal{F}_t] \\ &= e^{-r(T-t)} \int_0^{\infty} \max\{S_T - K, 0\} f(S_T \mid S_t) dS_t. \end{aligned}$$

Here, for a log normally distributed random variable $\log X \sim \mathcal{N}(\mu, \sigma^2)$ the PDF is

$$f_X(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2} \quad , \quad x > 0$$

Since S_T conditional on S_t have drift is equal to $\log S_t + (r - \dfrac{1}{2}\sigma^2)(T-t)$ and we can thus write

$$f(S_T|S_t) = \frac{1}{S_T\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{\log S_T - \log S_t - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)^2} \quad , \quad S_T > 0$$

or

$$f(S_T|S_t) = \frac{1}{S_T\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{\log S_T - m}{\sigma\sqrt{T-t}}\right)^2} \quad , \quad S_T > 0$$

where $m = \log S_t + (r - \frac{1}{2}\sigma^2)(T-t)$. Therefore,

$$\begin{aligned} C(S_t, t; K, T) &= e^{-r(T-t)} \int_0^{K} \max\{S_T - K, 0\} f(S_T|S_t) dS_t \\ &\quad + e^{-r(T-t)} \int_K^{\infty} \max\{S_T - K, 0\} f(S_T|S_t) dS_t \\ &= e^{-r(T-t)} \int_K^{\infty} (S_T - K) f(S_T|S_t) dS_t \\ &= I_1 - I_2 \end{aligned}$$

where

$$I_1 = e^{-r(T-t)} \int_K^{\infty} S_T f(S_T|S_t) dS_t \quad \text{and} \quad I_2 = e^{-r(T-t)} \int_K^{\infty} K f(S_T|S_t) dS_t$$

Solving I_1 we have

$$\begin{aligned} I_1 &= e^{-r(T-t)} \int_K^\infty S_T f(S_T \mid S_t) dS_t \\ &= e^{-r(T-t)} \int_K^\infty \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{\log S_T - m}{\sigma\sqrt{T-t}}\right)^2} dS_t \end{aligned}$$

and by letting $u = \frac{\log S_T - m}{\sigma\sqrt{T-t}}$ we then have

$$I_1 = \frac{e^{m-r(T-t)}}{\sqrt{2\pi}} \int_{\frac{\log K - m}{\sigma\sqrt{T-t}}}^\infty e^{-\frac{1}{2}u^2 + \sigma u\sqrt{T-t}} du.$$

Using the sum of squares

$$-\frac{1}{2}u^2 + \sigma u\sqrt{T-t} = \frac{1}{2}\left[(u - \sigma\sqrt{T-t})^2 - \sigma^2(T-t)\right]$$

we can simplify I_1 to become

$$\begin{aligned} I_1 &= \frac{e^{m-r(T-t)}}{\sqrt{2\pi}} \int_{\frac{\log K - m}{\sigma\sqrt{T-t}}}^\infty e^{-\frac{1}{2}\left[(u-\sigma\sqrt{T-t})^2 - \sigma^2(T-t)\right]} du \\ &= \frac{e^{m-r(T-t)+\frac{1}{2}\sigma^2(T-t)}}{\sqrt{2\pi}} \int_{\frac{\log K - m}{\sigma\sqrt{T-t}}}^\infty e^{-\frac{1}{2}\left(u-\sigma\sqrt{T-t}\right)^2} du \\ &= e^A \left[\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(u-\sigma\sqrt{T-t}\right)^2} du - \int_{-\infty}^{\frac{\log K - m}{\sigma\sqrt{T-t}}} e^{-\frac{1}{2}\left(u-\sigma\sqrt{T-t}\right)^2} du\right] \end{aligned}$$

Where

$$\begin{aligned} A &= m - r(T-t) + \frac{1}{2}\sigma^2(T-t) \\ &= \log S_t + (r - \frac{1}{2}\sigma^2)(T-t) - r(T-t) + \frac{1}{2}\sigma^2(T-t) = \log S_t \end{aligned}$$

By setting $v = u - \sigma\sqrt{T-t} = \dfrac{\log(K/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and substituting $m = \log S_t + (r + \frac{1}{2}\sigma^2)(T-t)$ we have

$$\int_{-\infty}^{\frac{\log K - m}{\sigma\sqrt{T-t}}} e^{-\frac{1}{2}\left(u-\sigma\sqrt{T-t}\right)^2} du = \int_{-\infty}^{\frac{\log(K/S_t) - (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{1}{2}\nu^2} d\nu$$

Therefore,

$$
\begin{aligned}
I_1 &= e^{\log S_t}\left[1 - \int_{-\infty}^{\frac{\log(K/S_t)-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{1}{2}\nu^2} d\nu\right] \\
&= S_t\left[1 - \Phi\left(\frac{\log(K/S_t)-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \\
&= S_t\Phi\left(\frac{\log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

Similarly for I_2 we have

$$
\begin{aligned}
I_2 &= e^{-r(T-t)}\int_K^{\infty} Kf(S_T)dS_T \\
&= Ke^{-r(T-t)}\int_K^{\infty} \frac{1}{S_T\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{\log S_T - m}{\sigma\sqrt{T-t}}\right)^2} dS_T
\end{aligned}
$$

and by letting $u = \frac{\log S_T - m}{\sigma\sqrt{T-t}}$ and substituting $m = \log S_t + (r - \frac{1}{2}\sigma^2)(T-t)$

$$
\begin{aligned}
I_2 &= e^{-r(T-t)}K\int_{\frac{\log K - m}{\sigma\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \\
&= Ke^{-r(T-t)}\left[1 - \Phi\left(\frac{\log K - m}{\sigma\sqrt{T-t}}\right)\right] \\
&= Ke^{-r(T-t)}\left[1 - \Phi\left(\frac{\log(K/S_t)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \\
&= Ke^{-r(T-t)}\Phi\left(\frac{\log(S_t/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

Therefore,

$$
C(S_t, t; K, T) = S_t e^{-(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{3.7}
$$

where

$$
d_1 = \frac{\log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
$$

$$d_2 = \frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

3.2 Application

3.2.1 Condition of the Black-Sholes-Merton moldel

Six assumptions of the Black-Sholes-Merton model is given by

1. The option is European
2. No dividends are paid out during the option's life
3. Efficient markets
4. There are no transaction costs in buying the option
5. The risk-free rate and volatility of the underlying are known and constant
6. The returns on the underlying are normally distributed

Problem

Consider a European call option on FPT stock without dividends. Market movements cannot be predicted. These information is given 5 first assumptions. We need to check the last one which is the rate of returns are normally distributed. We can get the historical data from source: `http://www.cophieu68.vn/historyprice.php?id=fpt`, the stock price is just calculated in 1 year, 2017. Firstly, we will compute daily return (u_i) of FPT stock.

$$u_i = \ln\frac{S_i}{S_{i-1}} \quad \text{for } \; i = 0, 1, \ldots, n$$

where

$n + 1$: Number of observations

S_i: Stock price at end of ith interval, with $i = 0, 1, \ldots, n$

Testing for normal distribution

The software is using for testing data is R – Studio. The code is shown at the end of report. We will use graphical methods which is normal probability plot (Q-Q plot or quantile quantile plots). Q-Q plot is a graphical technique to help us assess whether or not a data set is approximately normally distributed. If the most part the data points follow along the straight line, this will be normally distributed. We will describe the previous daily return. Here is our result

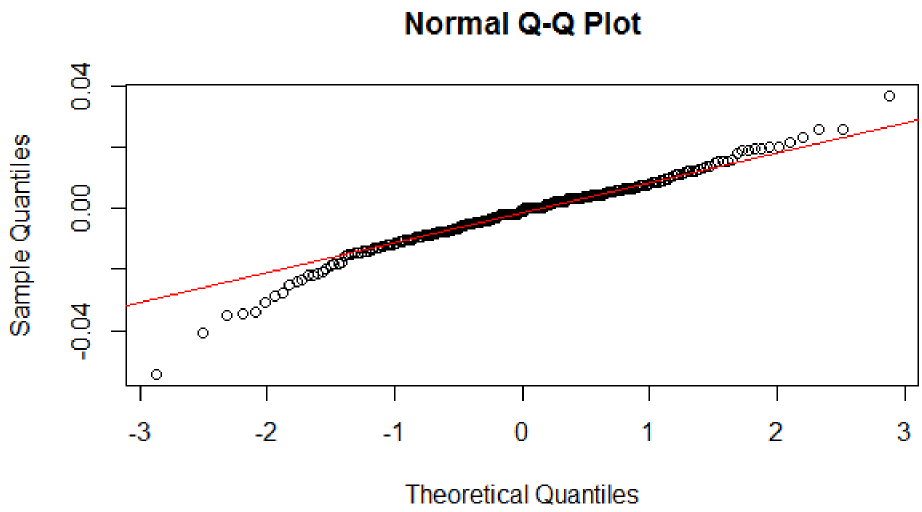

Figure 3.1: The distribution of FPT stock

This graph satisfies conditions of normal distribution. Here the x-axis is the probability and the y-axis is the values of returns. Therefore, the returns is most likely normally distribution.

3.2.2 Apply to the Black-Scholes-Merton model

Let's asumme that FPT stock satisfies conditions of the Black-Scholes-Merton model. We first estimate the volatility of a stock price empirically. The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2}$$

where $\bar{u}$ is the mean of the u_i. An estimate for the volatility per annum as

$$\text{Volatility per annum} = \text{Volatility per trading day} \times \sqrt{\text{Number of trading days per annum}}$$

Using R-studio, we get

- Volatility per trading day $s = 0.015$

- There are 250 trading days of year 2017. Therefore,

  Volatility per annum $= s \times \sqrt{250} = 24\%$

This company currently sells for 59.8 VND per share. The annual stock price volatility is 24%. Assume that the annual continuously compounded risk-free interest rate is 3%. Using the Black-Scholes model for the price of a call option on a company's stock with strike price 62 VND and time for expiration of half a year. Let's cover it.

S_0	Stock price at time zero	59.8 VND
K	Strike price	62 VND
σ	Annual volitility	24%
r	Annual riskless rate	3%
T	Option expiration (in years)	0.5

Table 3.1: FPT stock

We first calculate each components of equation 3.7.

$$d_1 = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -0.04$$
$$d_2 = \frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -0.21$$

Substituting the above variables into equation 3.7, we get

$$C(S_t, t; K, T) = S_t e^{-(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = 3.48$$

This means that the option price or a premium is 3.48 VND for each stock.

Chapter 4

European Barrier Options with Rebates

4.1 European down-and-out call option

We require $S_t > B$ so as to ensure the option would not knock out at the starting time t. We consider the case for the down-and-out call option price when $B \leq K$, as the option holder is more likely to give up their exercise right when the asset below the strike price K.

Under the assumption of Black-Scholes model, the price of a European down-and-out-call option at time t, denoted by $C_{d/o}(S_t, t)$ satisfies the Black-Scholes equation:

$$\frac{\partial C_{d/o}}{\partial t}(S_t, t) + rS_t \frac{\partial C_{d/o}}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_{d/o}}{\partial S^2}(S_t, t) - rC_{d/o} = 0$$

From the Feynman-Kac theorem, $C_{d/o}(S_t, t)$ can be computed from the formula:

$$C_{d/o}(S_t, t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[C_{d/o}(S_T, T)|\mathcal{F}_t],$$

where $C_{d/o}(S_T, T)$ can be expressed as:

$$C_{d/o}(S_T, T) = \max\{S_T - K, 0\}\mathcal{I}_{\min_{t \le u \le T} S_u > B},$$

where $\mathcal{I}_{\min_{t \le u \le T} S_u > B}$ is the indicator of the set $\{S_u > B\}$, i.e.,

$$\mathcal{I}_{\min_{t \le u \le T} S_u > B} = \begin{cases} 1, & \text{if} \quad \min_{t \le u \le T} S_u > B \\ 0, & \text{if} \quad \min_{t \le u \le T} S_u \le B \end{cases}.$$

In other words, the option holder will receive the positive difference of S_T and K if $S_T > K$ and the barrier has not been hit up to time T. Under the risk-neutral measure $\mathbb{Q}$, S_t follows the geometric Brownian motion:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}}$$

such that $W_t^{\mathbb{Q}} = W_t + \left(\frac{\mu - r}{\sigma}\right) t$ is a $\mathbb{Q}$-standard Wiener process. Following equation (3.6) with times from t to T for $T > t$, we obtain

$$\begin{aligned} S_T &= S_t e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma W_{T-t}^{\mathbb{Q}}} \\ &= S_t e^{\sigma \widehat{W}_{T-t}} \end{aligned} \tag{4.1}$$

where $\widehat{W}_{T-t} = \nu(T-t) + W_{T-t}^{\mathbb{Q}}$ and $\nu = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$. By defining

$$m_{T-t} = \min_{t \le u \le T} \widehat{W}_{u-t}$$

we can express

$$\min_{t \le u \le T} S_u = \min_{t \le u \le T} S_t e^{\sigma \widehat{W}_{u-t}} = S_t e^{\sigma \min_{t \le u \le T} \widehat{W}_{u-t}} = S_t e^{\sigma m_{T-t}}.$$

As a result, the payoff can be expressed as:

$$C_{d/o}(S_T, T) = \max\{S_T - K, 0\}\mathcal{I}_{\{\min_{t \le u \le T} S_u > B\}} = \max\{S_t e^{\sigma \widehat{W}_{T-t}} - K, 0\}\mathcal{I}_{\{S_t e^{\sigma m_{T-t}} > B,\}}$$

$$= (S_t e^{\sigma \widehat{W}_{T-t}} - K)\mathcal{I}_{\{S_t e^{\sigma m_{T-t}} > B, S_t e^{\sigma \widehat{W}_{T-t}} > K\}} = (S_t e^{\sigma \widehat{W}_{T-t}} - K)\mathcal{I}_{\{m_{T-t} > \frac{1}{\sigma}\log\left(\frac{B}{S_t}\right), \widehat{W}_{T-t} > \frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)\}}$$

The down-and-out call option price at time t is

$$\begin{aligned}
C_{d/o}(S_t, t; K, B, T) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[C_{d/o}(S_T, T)|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_t e^{\sigma \widehat{W}_{T-t}} - K)\mathcal{I}_{\{m_{T-t} > \frac{1}{\sigma}\log\left(\frac{B}{S_t}\right), \widehat{W}_{T-t} > \frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)\}}\Big|\mathcal{F}_t] \\
&= e^{-r(T-t)} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} \int_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\infty} (S_t e^{\sigma x} - K) f^{\mathbb{Q}}_{m_{T-t}, \widehat{W}_{T-t}}(a, x)\, da\, dx
\end{aligned}$$

where $f^{\mathbb{Q}}_{m_u, \widehat{W}_u}(m, \omega)$ is the joint probability density function of $(m_u, \widehat{W}_u)$. Applying the formula 2.10, we have: $f^{\mathbb{Q}}_{m_u, \widehat{W}_u}(a, x) = \begin{cases} \dfrac{2(x-2a)}{u\sqrt{2\pi u}} \exp(\nu x - \dfrac{1}{2}\nu^2 u - \dfrac{(2a-x)^2}{2u}), & a < 0, x \geq a, \\ 0, \quad \text{otherwise.} \end{cases}$ Thus,

we have: $f^{\mathbb{Q}}_{m_{T-t}, \widehat{W}_{T-t}}(a, x) = \begin{cases} \dfrac{2(x-2a)}{(T-t)\sqrt{2\pi(T-t)}} \exp(\nu x - \dfrac{1}{2}\nu^2 (T-t) - \dfrac{(2a-x)^2}{2(T-t)}), & a < 0, x \geq a, \\ 0, \quad \text{otherwise.} \end{cases}$

For convenience, we recall that $\widehat{W}_{T-t} = \nu(T-t) + W^{\mathbb{Q}}_{T-t}$ and $\nu = \dfrac{1}{\sigma}(r - \dfrac{1}{2}\sigma^2)$. The option price can now be computed as:

$$C_{d/o}(S_t, t; K, B, T = e^{-r(T-t)} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} \int_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\min(0,x)} \frac{2(S_t e^{\sigma x} - K)(x-2a)}{(T-t)\sqrt{2\pi(T-t)}} e^{\nu x - \frac{1}{2}\nu^2(T-t) - \frac{(2a-x)^2}{2(T-t)}}\, da\, dx$$

$$= e^{-r(T-t)} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} \int_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\min(0,x)} \frac{(S_t e^{\sigma x} - K)}{\sqrt{2\pi(T-t)}} d(e^{\nu x - \frac{1}{2}\nu^2(T-t) - \frac{(2a-x)^2}{2(T-t)}})\, dx$$

$$= e^{-r(T-t)} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} \frac{(S_t e^{\sigma x} - K)}{\sqrt{2\pi(T-t)}} e^{\nu x - \frac{1}{2}\nu^2(T-t) - \frac{(2a-x)^2}{2(T-t)}} \Bigg|_{\frac{1}{\sigma}\log\left(\frac{B}{S_t}\right)}^{\min(0,x)} dx$$

$$= e^{-r(T-t)} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} \frac{(S_t e^{\sigma x} - K)}{\sqrt{2\pi(T-t)}} e^{\nu x - \frac{1}{2}\nu^2(T-t)} \left[e^{-\frac{(2\min(0,x)-x)^2}{2(T-t)}} - e^{-\frac{(\frac{2}{\sigma}\log\left(\frac{B}{S_t}\right)-x)^2}{2(T-t)}} \right] dx$$

$$= e^{-r(T-t)} \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} \frac{(S_t e^{\sigma x} - K)}{\sqrt{2\pi(T-t)}} e^{\nu x - \frac{1}{2}\nu^2(T-t)} \left[e^{-\frac{x^2}{2(T-t)}} - e^{-\frac{(\frac{2}{\sigma}\log\left(\frac{B}{S_t}\right)-x)^2}{2(T-t)}} \right] dx$$

$$= e^{-r(T-t)} \int_{\frac{1}{\sigma}log\left(\frac{K}{S_t}\right)}^{\infty} (S_t e^{\sigma x} - K) \frac{1}{\sqrt{2\pi(T-t)}} e^{\nu x - \frac{1}{2}\nu^2(T-t) - \frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2} dx$$

$$
\begin{aligned}
&-e^{-r(T-t)}\int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}(S_te^{\sigma x}-K)\frac{1}{\sqrt{2\pi(T-t)}}e^{\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{\frac{2}{\sigma}\log\frac{B}{S_t}-x}{\sqrt{T-t}}\right)^2}dx\\
&=S_tI_1-KI_2-(S_tI_3-KI_4)
\end{aligned}
$$

Where

$$
\begin{aligned}
I_1&=\frac{1}{\sqrt{2\pi(T-t)}}\int_{x=\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}e^{-r(T-t)+\sigma x+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2}dx\\
I_2&=\frac{1}{\sqrt{2\pi(T-t)}}\int_{x=\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}e^{-r(T-t)+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2}dx\\
I_3&=\frac{1}{\sqrt{2\pi(T-t)}}\int_{x=\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}e^{-r(T-t)+\sigma x+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{\frac{2}{\sigma}\log\frac{B}{S_t}-x}{\sqrt{T-t}}\right)^2}dx\\
I_4&=\frac{1}{\sqrt{2\pi(T-t)}}\int_{x=\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty}e^{-r(T-t)+\nu x-\frac{1}{2}\nu^2(T-t)-\frac{1}{2}\left(\frac{\frac{2}{\sigma}\log\frac{B}{S_t}-x}{\sqrt{T-t}}\right)^2}dx
\end{aligned}
$$

Key Result

$$
\frac{1}{\sqrt{2\pi T}}\int_L^U e^{ax-\frac{1}{2}(\frac{x}{\sqrt{T}})^2}dx=e^{\frac{1}{2}a^2T}\left[\Phi\left(\frac{U-aT}{\sqrt{T}}\right)-\Phi\left(\frac{L-aT}{\sqrt{T}}\right)\right] \tag{4.2}
$$

Proof. Consider the following expression:

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi T}}\int_L^U e^{ax-\frac{1}{2}(\frac{x}{\sqrt{T}})^2}dx&=\frac{1}{\sqrt{2\pi T}}\int_L^U e^{-\frac{1}{2}(\frac{x^2}{T}-2ax)}dx\\
&=\frac{1}{\sqrt{2\pi T}}\int_L^U e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}-a\sqrt{T}\right)^2+\frac{1}{2}a^2T}dx\\
&=\frac{1}{\sqrt{2\pi T}}e^{\frac{1}{2}a^2T}\int_L^U e^{-\frac{1}{2}\left(\frac{x}{\sqrt{T}}-a\sqrt{T}\right)^2}dx
\end{aligned} \tag{4.3}
$$

Let $y=\dfrac{x}{\sqrt{T}-a\sqrt{T}}$. This implies that $dy=\dfrac{dx}{\sqrt{T}}$ or $dx=\sqrt{T}dy$.

When

$$
\begin{aligned}
x=L\rightarrow y&=\frac{L}{\sqrt{T}}-a\sqrt{T}=\frac{L-aT}{\sqrt{T}}\\
x=U\rightarrow y&=\frac{U}{\sqrt{T}}-a\sqrt{T}=\frac{U-aT}{\sqrt{T}}
\end{aligned}
$$

The equation 4.3 becomes

$$\begin{aligned}
\frac{1}{\sqrt{2\pi T}}\int_L^U e^{ax-\frac{1}{2}(\frac{x}{\sqrt{T}})^2}dx &= \frac{1}{\sqrt{2\pi T}}e^{\frac{1}{2}a^2T}\int_{\frac{L-aT}{\sqrt{T}}}^{\frac{U-aT}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\sqrt{T}dy \\
&= e^{\frac{1}{2}a^2T}\frac{1}{\sqrt{2\pi}}\int_{\frac{L-aT}{\sqrt{T}}}^{\frac{U-aT}{\sqrt{T}}} e^{-\frac{1}{2}y^2}dy \\
&= e^{\frac{1}{2}a^2T}\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\frac{U-aT}{\sqrt{T}}} e^{-\frac{1}{2}y^2}dy - \int_{-\infty}^{\frac{L-aT}{\sqrt{T}}} e^{-\frac{1}{2}y^2}dy\right] \\
&= e^{\frac{1}{2}a^2T}\left[\Phi\left(\frac{U-aT}{\sqrt{T}}\right) - \Phi\left(\frac{L-aT}{\sqrt{T}}\right)\right],
\end{aligned}$$

with $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$ is the cumulative distribution of the standard normal random variable. □

Using the equation (4.2) that is proved above. We have

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi(T-t)}}e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)}\int_{x=\frac{1}{\sigma}log\left(\frac{K}{S_t}\right)}^{\infty} e^{(\sigma+\nu)x-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2}dx \\
&= e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)+\frac{1}{2}(\nu+\sigma)^2(T-t)}\left[\Phi(\infty) - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t)-(\nu+\sigma)(T-t)}{\sqrt{T-t}}\right)\right]
\end{aligned}$$

and knowing that $\nu = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$, we have

$$\begin{aligned}
I_1 &= e^{-r(T-t)-\frac{1}{2}\left(\frac{2r-\sigma^2}{2\sigma}\right)^2(T-t)+\frac{1}{2}\left(\frac{2r+\sigma^2}{2\sigma}\right)^2(T-t)}\left[1 - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t)-(\nu+\sigma)(T-t)}{\sqrt{T-t}}\right)\right] \\
&= e^0\left[1 - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t)-(\nu+\sigma)(T-t)}{\sqrt{T-t}}\right)\right] = 1 - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t)-(\nu+\sigma)(T-t)}{\sqrt{T-t}}\right) \\
&= \Phi\left(-\frac{\frac{1}{\sigma}\log(K/S_t)-(\nu+\sigma)(T-t)}{\sqrt{T-t}}\right) = \Phi\left(\frac{\log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

Similarly we can deduce

$$I_2 = \frac{1}{\sqrt{2\pi(T-t)}}e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)}\int_{x=\frac{1}{\sigma}log\left(\frac{K}{S_t}\right)}^{\infty} e^{\nu x-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2}dx$$

$$= e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)+\frac{1}{2}\nu^2(T-t)} \left[\Phi(\infty) - \Phi\left(\frac{\frac{1}{\sigma}log(K/S_t) - \nu(T-t)}{\sqrt{T-t}}\right)\right]$$

$$= e^{-r(T-t)} \left[1 - \Phi\left(\frac{\frac{1}{\sigma}log(K/S_t) - \nu(T-t)}{\sqrt{T-t}}\right)\right]$$

$$= e^{-r(T-t)}\Phi\left(\frac{log(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$I_3 = \frac{1}{\sqrt{2\pi(T-t)}} e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}log^2\left(\frac{B}{S_t}\right)} \times \int_{\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{\left[\nu+\sigma+\frac{2}{\sigma(T-t)}log\left(\frac{B}{S_t}\right)\right]x-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2} dx$$

$$= e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}\log^2\left(\frac{B}{S_t}\right)+\frac{1}{2}\left[\nu+\sigma+\frac{2}{\sigma(T-t)}\log\left(\frac{B}{S_t}\right)\right]^2(T-t)}$$

$$\times \left[\Phi(\infty) - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t) - \left[\nu+\sigma+\frac{2}{\sigma(T-t)}\log(B/S_t)\right](T-t)}{\sqrt{T-t}}\right)\right]$$

$$= \left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}} \left[1 - \Phi\left(\frac{\log(S_tK/B^2) - (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right]$$

$$= \left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}} \Phi\left(\frac{\log(B^2/(S_tK)) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$I_4 = \frac{1}{\sqrt{2\pi(T-t)}} e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}\left(\log\left(\frac{B}{S_t}\right)\right)^2}$$

$$\times \int_{x=\frac{1}{\sigma}\log\left(\frac{K}{S_t}\right)}^{\infty} e^{\left[\nu+\frac{2}{\sigma(T-t)}\log\left(\frac{B}{S_t}\right)\right]x-\frac{1}{2}\left(\frac{x}{\sqrt{T-t}}\right)^2} dx$$

$$= e^{-r(T-t)-\frac{1}{2}\nu^2(T-t)-\frac{2}{\sigma^2(T-t)}\left(\log\left(\frac{B}{S_t}\right)\right)^2+\frac{1}{2}\left[\nu+\frac{2}{\sigma(T-t)}\log\left(\frac{B}{S_t}\right)\right]^2(T-t)}$$

$$\times \left[\Phi(\infty) - \Phi\left(\frac{\frac{1}{\sigma}\log(K/S_t) - \left[\nu+\frac{2}{\sigma(T-t)}\log(B/S_t)\right](T-t)}{\sqrt{T-t}}\right)\right]$$

$$= e^{-r(T-t)} \left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}} \left[1 - \Phi\left(\frac{\log(S_tK/B^2) - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right]$$

$$= e^{-r(T-t)} \left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}} \Phi\left(\frac{\log(B^2/(S_tK)) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

Therefore,

$$C_{d/o}(S_t, t; K, B, T)$$

$$
\begin{aligned}
=&S_tI_1 - KI_2 - (S_tI_3 - KI_4)\\
=&S_t\Phi\left(\frac{log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&-Ke^{-r(T-t)}\Phi\left(\frac{log(S_t/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&-S_t\left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}}\Phi\left(\frac{log(B^2/(S_tK))+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&+Ke^{-r(T-t)}\left(\frac{S_t}{B}\right)^{-1-\frac{2r}{\sigma^2}}\Phi\left(\frac{log(B^2/(S_tK))+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\\
=&C_{bs}(S_t,t;K,T)-\left(\frac{S_t}{B}\right)^{2\lambda}C_{bs}(\frac{B^2}{S_t},t;K,T)
\end{aligned}
$$

where $\lambda = \frac{1}{2}\left(1-\frac{r}{\frac{1}{2}\sigma^2}\right)$ and

$$
\begin{aligned}
C_{bs}(S_t,t;K,T) &= S_tN(d_1)-Ke^{-r(T-t)}N(d2)\\
d_1 &= \frac{log(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\\
d_2 &= d_1-\sigma\sqrt{T-t} = \frac{log(S_t/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\\
C_{bs}(\frac{B^2}{S_t},t;K,T) &= \frac{B^2}{S_t}N(d_3)-Ke^{-r(T-t)}N(d4)\\
d_3 &= \frac{log(B^2/(S_tK))+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\\
d_4 &= d_3-\sigma\sqrt{T-t} = \frac{log(B^2/(S_tK))+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Illustration

We will make an European call option under FPT stock with time for expiration of half year. Let's take the data in Table 3.1 and the knock-out occurs when a barrier is crossed and in our case this barrier is 55 VND. We will summarize this data. Look at the following table

S_0	Stock price at time zero	59.8 VND
K	Strike price	62 VND
B	Down and out barrier price	55 VND
σ	Annual volitility	24%
r	Annual riskless rate	3%
T	Option expiration (in years)	0.5

Table 4.1: Option Pricing Parameters

Subtitute the above data we can get

$$d_1 = \frac{log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -0.03964939$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = -0.21$$

$$d_3 = \frac{log(B^2/(S_tK)) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -0.209355$$

$$d_4 = d_3 - \sigma\sqrt{T-t} = -1.195445$$

$$\lambda = \frac{1}{2}\left(1 - \frac{r}{\frac{1}{2}\sigma^2}\right) = -0.02083$$

$$C_{bs}(S_t, t; K, T) = S_tN(d_1) - Ke^{-r(T-t)}N(d2) = 3.480033$$

$$C_{bs}(\frac{B^2}{S_t}, t; K, T) = \frac{B^2}{S_t}N(d_3) - Ke^{-r(T-t)}N(d4) = 0.63234662$$

$$C_{bs}(S_t, t; K, T) - \left(\frac{S_t}{B}\right)^{2\lambda} C_{bs}(\frac{B^2}{S_t}, t; K, T) = 2.849887$$

Therefore, the value of European down-and-out call option without rebates is 2.85 VND.

4.2 Rebates Value

Rebates is a part on the premium paid to the option holder. This leads to European barrier call option with rebates which is cheaper than the respective standard European options. Thus, let's R is a payable rebate at knock-out time $\tau, t \leq \tau \leq T$. Let $C^R_{d/o}(S_t, t; K, B, T)$ be the European down-and-out option respectively with common barrier B, strike price K, rebate R and expiry time T.

We have

$$C^R_{d/o}(S_t, t; K, B, T) = C_{d/o}(S_t, t; K, B, T) + \tilde{C}^R_{d/o}(S_t, t; K, B, T)$$

where $C_{d/o}(S_t, t; K, B, T)$ is the European down-and-out option prices without rebates while $\tilde{C}^R_{d/o}(S_t, t; K, B, T)$ is the corresponding option prices associated with immediate rebate at knock-out time. Thus, it is important for finding out rebates value.

Let's consider the reflection principle of Brownian motion from figure 2.5 with given assumptions, we obtain the joint distribution function for the zero-drift case as follows:

$$\begin{aligned}\mathbb{P}(W^0_T > x, m^T_0 < m) &= \mathbb{P}(\widetilde{W}^0_T < 2m - x) = \mathbb{P}(W^0_T < 2m - x)\\ &= \Phi\left(\frac{2m - x}{\sigma\sqrt{T}}\right), \quad m \leq \min(x, 0).\end{aligned}$$

Next, we apply the Girsanov Theorem to effect the change of measure for finding the above joint distribution when the Brownian motion has nonzero drift. Suppose under the measure $\mathbb{P}$,W^μ_t is a Brownian motion with drift rate μ. We change the measure from $\mathbb{P}$ to $\mathbb{Q}$ such that W^μ_t becomes a Brownian process with zero drift under $\mathbb{Q}$. From equation 2.6 of the Randon-Nikodym derivative, we have the following joint distribution

$$\begin{aligned}\mathbb{P}(W^\mu_T > x, m^T_0 < m) &= E_{\mathbb{P}}\left[\mathcal{H}_{\{W^\mu_T > x\}}\mathcal{H}_{\{m^T_0 < m\}}\right]\\ &= E_{\mathbb{Q}}\left[\mathcal{H}_{\{W^\mu_T > x\}}\mathcal{H}_{\{m^T_0 < m\}}e^{\frac{\mu W^\mu_T}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}}\right],\end{aligned}$$

Then, by applying the reflection principle and observing that W^μ_T is a zero-drift Brownian motion under $\mathbb{Q}$, we obtain

$$\begin{aligned}\mathbb{P}(W^\mu_T > x, m^T_0 < m) &= E_{\mathbb{Q}}\left[\mathcal{H}_{\{W^\mu_T < 2m - x\}}e^{\frac{\mu W^\mu_T}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}}\right]\\ &= E_{\mathbb{Q}}\left[\mathcal{H}_{\{2m - W^\mu_T > x\}}e^{\frac{\mu}{\sigma^2}(2m - W^\mu_T) - \frac{\mu^2 T}{2\sigma^2}}\right]\\ &= e^{\frac{2\mu m}{\sigma^2}}E_{\mathbb{Q}}\left[\mathcal{H}_{\{W^\mu_T < 2m - x\}}e^{-\frac{\mu}{\sigma^2}W^\mu_T - \frac{\mu^2 T}{2\sigma^2}}\right]\end{aligned}$$

$$
\begin{aligned}
&= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{z^2}{2\sigma^2 T}} e^{-\frac{\mu}{\sigma^2} W_T^{\mu} - \frac{\mu^2 T}{2\sigma^2}} dz \\
&= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{\left(-\frac{(z+\mu T)^2}{2\sigma^2 T}\right)} dz \\
&= e^{\frac{2\mu m}{\sigma^2}} \Phi\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right), \quad m \leq min(x, 0).
\end{aligned}
$$

Suppose the Brownian motion W_t^{μ} has a downstream barrier m over the period $[0, T]$ so that $m_0^T > m$, we would like to derive the joint distribution

$$
\mathbb{P}(W_T^{\mu} > x, m_0^T > m), \text{ where } m \leq \min(x, 0).
$$

By applying the law of total probabilities, we obtain

$$
\begin{aligned}
&\mathbb{P}(W_T^{\mu} > x, m_0^T > m) \\
=&\mathbb{P}(W_t^{\mu} > x) - \mathbb{P}(W_T^{\mu} > x, m_0^T < m) \\
=&\Phi\left(\frac{-x + \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} \Phi\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right), \quad m \leq \min(x, 0).
\end{aligned}
$$

Under the special case $m = x$, since W_T^{μ} is implicitly implied from $m_0^T > m$, we have

$$
\mathbb{P}(m_0^T > m) = \Phi\left(\frac{-m + \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} \Phi\left(\frac{m + \mu T}{\sigma\sqrt{T}}\right) \tag{4.4}
$$

First Passage Time Density Functions

Let $Q(u; m)$ denote the density function of the first passage time at which the downstream barrier m is first hit by the Brownian path W_t^{μ}, that is, $Q(u; m)du = \mathbb{P}(\tau_m \in du)$ which means that probability Brownian path hits m within interval of time $(u, u + du)$. First, we determine the distribution function $\mathbb{P}(\tau_m \in du)$ by observing that $\{\tau_m > du\}$ and $m_0^u > m$ are equivalent events. By equation (4.4), we obtain

$$
\mathbb{P}(\tau_m > du) = \mathbb{P}(m_0^u > m)
$$

$$= \Phi\left(\frac{-m+\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}}\Phi\left(\frac{m+\mu T}{\sigma\sqrt{T}}\right)$$

The density function $Q(u;m)$ is then given by

$$\begin{aligned} Q(u;m)du &= \mathbb{P}(\tau_m \in du) \\ &= -\frac{\partial}{\partial u}\left[\Phi\left(\frac{-m+\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}}\Phi\left(\frac{m+\mu u}{\sigma\sqrt{u}}\right)\right] du\mathcal{H}_{\{m<0\}} \end{aligned} \tag{4.5}$$

Let $V_1 = \dfrac{-m+\mu m}{\sigma\sqrt{u}}$ and $V_2 = \dfrac{m+\mu m}{\sigma\sqrt{u}}$. Then equation 4.5 becomes

$$\begin{aligned} Q(u;m)du &= -\left[\frac{\partial}{\partial V_1}\Phi(V_1)\frac{\partial}{\partial u}V_1 - e^{\frac{2\mu m}{\sigma^2}}\frac{\partial}{\partial V_2}\Phi(V_2)\frac{\partial}{\partial u}V_2\right] \\ &= -\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\frac{-m+\mu m}{\sigma\sqrt{u}}\right)^2}{2}}\left(\frac{-m}{2\sigma\sqrt{u^3}}+\frac{\mu}{2\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\frac{m+\mu m}{\sigma\sqrt{u}}\right)^2}{2}}\left(\frac{m}{2\sigma\sqrt{u^3}}+\frac{\mu}{2\sigma\sqrt{u}}\right)\right] \\ &= \frac{-m}{\sqrt{2\pi\sigma^2 u^3}}e^{-\frac{(m-\mu u)^2}{2\sigma^2 u}}du\mathcal{H}_{\{m<0\}} \end{aligned} \tag{4.6}$$

Suppose the asset price S_t follows the GBM under the risk neutral measure such that $\ln\frac{S_t}{S} = W_t^{\mu}$, where S is the asset price at time zero and the drift rate $\mu = r - \frac{\sigma^2}{2}$. We write B as the barrier level, then equation 4.6 becomes

$$Q(u;B) = -\frac{\ln\frac{B}{S}}{\sqrt{2\pi\sigma^2 u^3}}e^{-\frac{[\ln\frac{B}{S}-(r-\frac{\sigma^2}{2})u]^2}{2\sigma^2 u}}$$

A rebate R is paid to the option holder upon breaching the barrier at level B by the asset price path at time t, $0 < t < T$. Since the expected rebate payment over the time interval $[u, u+du]$ is given by $R\,Q(u;B)du$, the expected present value of the rebate is given by

$$\begin{aligned} \text{rebates value} &= R\int_0^T e^{-ru}Q(u;B)du \\ &= -R\int_0^T e^{-ru}\frac{\ln\frac{B}{S}}{\sqrt{2\pi\sigma^2 u^3}}e^{-\frac{[\ln\frac{B}{S}-(r-\frac{\sigma^2}{2})u]^2}{2\sigma^2 u}}du \\ &= R\left[\left(\frac{B}{S}\right)^{\alpha_+}\Phi\left(-\frac{\ln\frac{B}{S}+\beta T}{\sigma\sqrt{T}}\right) + \left(\frac{B}{S}\right)^{\alpha_-}\Phi\left(-\frac{\ln\frac{B}{S}-\beta T}{\sigma\sqrt{T}}\right)\right] \end{aligned}$$

where

$$\beta = \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}, \quad \alpha_{\pm} = \frac{r - \frac{\sigma^2}{2} \pm B}{\sigma^2}$$

Therefore, the final result of European barrier call option with rebates is

$$C^R_{d/o}(S_t, t; K, B, T) = C_{bs}(S_t, t; K, T) - \left(\frac{S_t}{B}\right)^{2\lambda} C_{bs}(\frac{B^2}{S_t}, t; K, T)$$
$$+ R\left[\left(\frac{B}{S}\right)^{\alpha_+} \Phi\left(-\frac{\ln\frac{B}{S} + \beta T}{\sigma\sqrt{T}}\right) + \left(\frac{B}{S}\right)^{\alpha_-} \Phi\left(-\frac{\ln\frac{B}{S} - \beta T}{\sigma\sqrt{T}}\right)\right]$$

4.3 Application

Let's look at an example to make the previous formula more clearly. Given a constant rebate \$1 is paid any time the option are knocked-out within the lives of the down-and-out barrier option. Other parameters remain the same as Table 4.1. Let's summarize the data:

We already get the value of European down-and-out call option without rebates is 2.85 VND.

S	Stock price at time zero	59.8 VND
K	Strike price	62 VND
B	Down and out barrier price	55 VND
σ	Annual volitility	24%
r	Annual riskless rate	3%
T	Option expiration (in years)	0.5
R	Rebates	10

Table 4.2: Option Pricing Parameters with rebates

Then, rebates value will be calculated. We will first calculate each components:

$$\beta = \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2} = 0.0588$$

$$\alpha_+ = \frac{r - \frac{\sigma^2}{2} + B}{\sigma^2} = 1.041667$$

$$\alpha_- = \frac{r - \frac{\sigma^2}{2} - B}{\sigma^2} = -1$$

$$\Phi\left(-\frac{\ln\frac{B}{S}+\beta T}{\sigma\sqrt{T}}\right)=\Phi(0.009144972)=-0.3198036$$

$$\Phi\left(-\frac{\ln\frac{B}{S}-\beta T}{\sigma\sqrt{T}}\right)=\Phi(0.7162518)=-0.666286$$

$$\text{rebates value}=6.179553$$

Thus, the final result we obtain

$$C^{R}_{d/o}(S_t,t;K,B,T)=9.02944$$

This means that the European barrier down-and-out call option has price of 9.03 VND.

Chapter 5

Conclusion

Currently, the financial market in Vietnam becomes exciting than ever thanks to the launch of Derivatives market which is to complete the stock market and financial market in Vietnam, helping to improve the investor base and attract more foreign investors, especially institutional investors, therefore, to promote liquidity in the underlying market. Up to now, Seven securities companies have been approved to become members of the Derivatives market of Hanoi Stock Exchange and connect with the derivatives trading system of Hanoi Stock Exchange including: VPBank Securities, Ho Chi Minh City Securities Corporation, Saigon Securities Inc., Vietnam Bank for Investment and Development Securities, VNDirect Securities Joint Stock Company, Viet Capital Securities JSC, MB Securities JSC. Investors can open derivatives trading accounts at those 07 securities companies. The first derivatives product on the market is the VN30Index Futures Contract. Vietnam will be the fifth country to have derivatives markets in the ASEAN region besides Singapore, Malaysia, Indonesia and Thailand and the 42nd country in the world with this high-end financial market.

Therefore, it will open more potentiality for sort of derivatives and what this thesis tend to is the option that is is an agreement between two parties to facilitate a potential transaction on the underlying security at a preset price, referred to as the strike price, prior to the expiration date. Pricing option is one of the most important of its surviving in trading in stock market. As a result, the mathematical techniques will be given to solve that problem. In particalar, the probability approaches will be used in this context. Firstly, the overall view of the option, is chosen, is European barrier call option with rebates that gives the holder the right but not the obligation, to buy or sell the underlying security at a specific price (the strike price) only on the option's expiration date. Especially, for barrier to make it more attractive, the trader will predict the limit of trend changing of stock price in order to reduce risk for themself and the holder have ability to reveice a part of a premium they would paid before that is called rebates. To deal with this option, this paper is mentioned general theory related to the neccessary approaches. The normal distribution of continuous random variables is which cannot be missed in the probability theory. Then, using the main property of Wiener process express the random component of the model with the martingale pricing theory, which supports very powerfully for convert the real data into the mathematical techniques for solving problems easily. Then, combining the probability theory, Wiener process and Stochastic differential equation with the financial theory logically to find out the formula of the option. Lastly, applying the option of the underlying stock into the result.

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Appendix A

R Code

A.1 Sketch payoff diagram of a call option

```
> #Figure 1.1: Payoff of call option

> getwd() #print the current working directory
[1] "C:/Users/tapud_000/Documents"

> setwd("E:/Thesis/R") #change to mydirectory

> S<-read.csv("ex1.csv") #Read CSV into R

> attach(S) #make the data available to the R search path

>plot(Stock.price,Payoff,xlim=c(0,100),ylim = c(-10,60),type = "l",col="blue",
    xlab="Stock price", ylab = "Profit",main = "Profit diagram of a call
    option",lwd=1.5) #sketch diagram of call option

> q<-rep(0,101) # Profit is according to stock price

>lines(Stock.price,q,col="red",lwd=1.5) #sketch the line describes payoff is
    equal zero to help comparison easily

>plot(Stock.price,Payoff,xlim=c(0,100),ylim = c(-10,60),type = "l",col="blue",
    ylab = "Profit",main = "Profit diagram of a call option",lwd=1.5,
    legend("topright", legend=c("Call, K = $50, A premium = $5"),col=("blue"),
```

```
lty=1:2, cex=0.8))# add legend to plot in R
```

A.2 Calculate the daily return

```
>  # FPT stock

> FPT=read.csv("fpt.csv") #Read CSV into R

> attach(FPT) #make the data available to the R search path

> names(FPT) #show name of objects is stored in R
[1] "Stock" "Date" "Price"

> Si<-Price[2:250]

> Sk<-Price[1:249]

> ui<-log(Si/Sk) #This is the daily return
```

A.3 Calculate the volatility of FPT stock

```
>  # Volatility per annum

> FPT=read.csv("fpt.csv") #Read CSV into R

> attach(FPT) #make the data available to the R search path

> names(FPT) #show name of objects is stored in R
[1] "Stock" "Date" "Price"

> Si<-Price[2:250]

> Sk<-Price[1:249]

> ui<-log(Si/Sk)

>s<-sqrt((1/148)*sum((ui-mean(ui))^2))

> s
[1] 0.01518166
```

A.4 Check data normality

```
>  #Figure 3.1: The distribution of FPT stock

> FPT=read.csv("stock.csv") #Read CSV into R

> attach(FPT) #make the data available to the R search path

> names(FPT) #show name of objects is stored in R
[1] "Stock"                  "Day(i)"
[3] "Stock.price" "Daily.Return"
[5] "Volatility"

> DailyReturn=na.omit(Daily.Return) #remove NAs value from Daily.Return

>  #Using quantitle-quantitle (QQ) plots to check data normality in R

> qqnorm(DailyReturn) #produces a normal QQ plot of the variable

> qqline(DailyReturn,col=2,lwd=1.5) #adds a reference line
```

A.5 Black-Scholes-Merton model

```
> #Black Scholes Model

> #Assigning variables

> s<-59.8 #Stock Price

> k<-62 #Strike Price

> sigma<-0.24 #Volatility

> r<-0.03 #Risk free rate

> T<-0.5 #Expiration date

> #Calculate d1

>d1<-(log(s/k)+(r+(sigma^2)/2)*T)/(sigma*sqrt(T))

> d1
[1] -0.03964939

> #Calculate d2

>d2<-(log(s/k)+(r-(sigma^2)/2)*T)/(sigma*sqrt(T))

> d2
[1] -0.209355

> #Calculate Call Option price

>c<-s*pnorm(d1)-exp(-r*T)*k*pnorm(d2)
> c
[1] 3.480033
```

A.6 European barrier call option

```
> #European down-and-out call option without rebates

> #Assigning variables

> S<-59.8 #Stock price at time zero

> K<-62 #Strike price

> B<-55 #Down and out barrier price

> sigma<-0.24 #Annual volitility

> r<-0.03 #Annual riskless rate

> T<-0.5 #Option expiration

>d1<-(log(S/K)+(r+(sigma^2)/2)*T)/(sigma*sqrt(T))

> d1
[1] -0.03964939

> d2=d1-sigma*sqrt(T)

> d2
[1]  -0.209355

>d3=(log((B^2)/(S*K))+(r+(sigma^2)/2)*T)/(sigma*sqrt(T))

> d3
[1] -1.025739

> d4=d3-sigma*sqrt(T)

> d4
[1]  -1.195445

>lamda<-(sigma^2-2*r)/(2*sigma^2)
```

```
> lambda
[1]  -0.02083333

>C1<-S*pnorm(d1)-K*exp(-r*T)*pnorm(d2)

> C1
[1] 3.480033

>C2<-((B^2)/S)*pnorm(d3)-K*exp(-r*T)*pnorm(d4)

> C2
[1] 0.6323466

>C<-C1-((S/B)^(2*alpha))*C2

> C
[1] 2.849887 #The European down-and-out call option without rebates
```

A.7 European barrier call option

```
> #Pricing of European down-and-out call option with rebates

> #Assigning variables

> S<-59.8 #Stock price at time zero

> K<-62 #Strike price

> B<-55 #Down and out barrier price

> sigma<-0.24 #Annual volitility

> r<-0.03 #Annual riskless rate

> T<-0.5 #Option expiration

>R<-10

> #We already calculate the European down-and-out call option without rebates
   is $2.85. Next it will be for rebates and final result.

>beta<-sqrt((r-(sigma^2)/2)^2+2*r*(sigma^2))

> beta
[1] 0.0588

>alpha1<-(r-((sigma^2)/2)+beta)/(sigma^2)

> alpha1
[1] 1.041667

>alpha2<-(r-((sigma^2)/2)-beta)/(sigma^2)

> alpha2
[1] -1

>l1<-(log(B/S)+beta*T)/(sigma*sqrt(T))
```

```
> l1
[1] -0.3198036

>l2<-(log(B/S)-beta*T)/(sigma*sqrt(T))

> l2
[1] -0.666286

> R<-1

>RV<-R*(((B/s)^alpha1)*pnorm(l1)+((B/s)^alpha2)*pnorm(l2))
> RV
[1] 6.179553

> Result<-C+RV

> Result
[1] 9.02944
```