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Equivalence between spin Hamiltonians and boson sampling

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Aaronson and Arkhipov showed that predicting or reproducing the measurement statistics of a general linear optics circuit with a single Fock-state input is a classically hard problem. Here we show that this problem, known as boson sampling, is as hard as simulating the short time evolution of a large but simple spin model with long-range XY interactions. The conditions for this equivalence are the same for efficient boson sampling, namely, having a small number of photons (excitations) as compared to the number of modes (spins). This mapping allows efficient implementations of boson sampling in small quantum computers and simulators and sheds light on the complexity of time evolution with critical spin models.

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I. INTRODUCTION

Boson sampling requires (i) an optical circuit with M modes, randomly sampled from the Haar measure; (ii) an input state with $N \ll M$ photons, with at most one photon per mode; and (iii) photon counters at the output ports that postselect events with at most one photon per port. Under these conditions, the probability distribution for any configuration $\mathbf{n} \in \mathbb{Z}_2^M$, $p(n_1,n_2,\ldots,n_M) = |\gamma_{\mathbf{n}}|^2$, is proportional to the permanent of a complex matrix whose computation is **#P** hard. This result, combined with some reasonable conjectures [1], implies that the output statistics of linear optics circuits with nonclassical inputs cannot be simulated efficiently using classical computers and likely involves an exponential overhead of resources. More recently, boson sampling has been generalized to consider other input states [2,3], extensions to Fourier sampling [4], or trapped-ion implementations [5]. Boson sampling has also been related to practical problems, such as homomorphic encryption of quantum information [6], the prediction of molecular spectra [7], and quantum metrology [8]. Finally, there are other quantum models, such as circuits of commuting quantum gates [9], which also establish potential limits of what can be classically simulated.

The aim of this work is to establish a link between quantum complexity theory and quantum simulation, connecting the boson-sampling problem to a broad family of spin Hamiltonians that appear naturally in different contexts, from trapped ions [10] to superconducting circuits [11]. More precisely, we prove that boson sampling is equivalent to a many-body problem with spins that interact through a long-range, XY coupling and evolve for a very short time, of the order of a single hopping or spin-swap event. The model involves two sets of M input and output spins,

$$H = \sum_{i,j=1}^{M} \sigma^{+}_{\mathrm{out},j} R_{ji} \sigma^{-}_{\mathrm{in},i} + \mathrm{H.c.}, \tag{1}$$

joined by the (unitary) matrix R. We show that the time evolution of an initial state that has only N excited input spins approximates the wave function of the boson-sampling problem after a finite time $t = \pi/2$. All errors in this mapping can be assimilated to bunching of excitations in the optical circuit, and the mapping succeeds whenever boson sampling actually does.

Where does the intuition for Hamiltonian (1) come from? The main idea is the fact that the XY spin model with N excitations in a dilute regime $N \ll M$ has a very small probability of collision between quasiparticles. As we show below, this spin model (1) thus behaves as a linear circuit for noninteracting bosons. The second idea, and the particular choice of Hamiltonian parameters, arises from a Hamiltonian interpretation of a boson sampling circuit where two-level systems $\sigma_{\mathrm{in},j}$ and $\sigma_{\mathrm{out},j}$ act as the photon emitters and photon detectors [cf. Fig. 1(b)]. Building on earlier works with qubits in photonic waveguides [12–14] one can show [11] that a boson-sampling circuit implements a coherent, photon-mediated interaction of the form (1). The combination of these two ideas provides us with a new physical problem that has the same complexity as boson sampling, but which can be mapped to the dynamics of state-of-the-art quantum-simulator experiments and small quantum computers.

The paper is structured as follows. In Sec. II we introduce a Hamiltonian formulation for the boson-sampling problem and compute the time evolution of an arbitrary multiphoton Fock state under this Hamiltonian. In particular, we show that the evolution of the boson-sampling state for a certain time t yields the final state that one would expect at the output of a photonic network. In Sec. III, we formally establish a mapping between boson sampling and spin sampling, i.e., the problem of sampling from the output distribution of a spin register undergoing long-range XY interactions. We give tight bounds for the error norm and variation distance between the evolution of the spin model and a boson-sampling emulator, concluding that simulating the evolution of the spin model for short times is as hard as boson sampling itself. Finally, in Sec. IV we discuss the implications of these formal results in different fields, from quantum simulation, where we can now argue about the quantum advantage of implementing certain simulations in the laboratory, to quantum complexity theory. In particular, the main conclusion of this work is the possibility of establishing formal links between quantum simulation of relevant condensed-matter physics models and rigorous results from complexity theory. We expect that this work will be

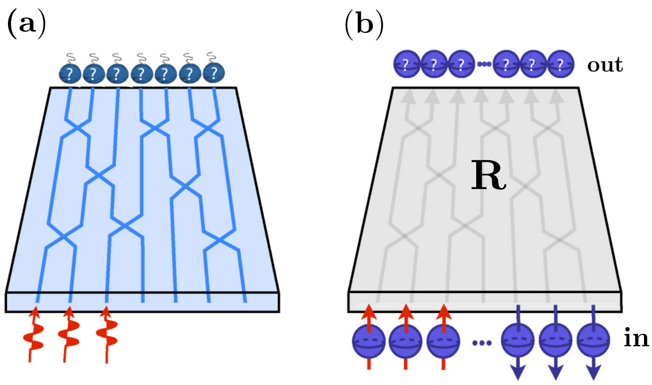

FIG. 1. (a) A setup that consists of beam splitters and free propagation implements boson sampling if the input state has a fixed number of single photons on each port (red wiggles). (b) We can regard those photons as arising from the spontaneous emission of two-level systems onto the circuit (up spins), which after propagation map onto other two-level systems at the end, via the unitary matrix R.

followed by others establishing similar links that consolidate the interest and hardness of quantum-simulation problems.

II. THE BOSON-SAMPLING HAMILTONIAN

The core idea in this work is to relate mathematically the boson-sampling setup to other problems which are framed as evolution under a quantum Hamiltonian. We begin by establishing the analogy between boson sampling and linear Hamiltonians with bosons, leaving for Sec. III the relation between this problem and strongly interacting hard-core bosonic systems and spin models (1).

Let R be a unitary transformation implemented by the linear circuit in Fig. 1(a). R is randomly sampled from $U(M)$ according to the Haar measure. We can associate to this unitary a linear Hamiltonian [16]

$$H_{\rm BS} = \sum_{i,j=1}^{M} (b_j^\dagger R_{ji} a_i + \text{H.c.}) + \sum_{j=1}^{M} \omega (b_j^\dagger b_j + a_j^\dagger a_j) \tag{2}$$

that couples input and output bosonic modes, a and b, of the boson-sampling problem. Evolution with this Hamiltonian, $|\phi(t)\rangle = e^{-iHt}|\phi(0)\rangle$, transforms the initial state of boson sampling,

$$|\phi(0)\rangle = a_1^\dagger \cdots a_N^\dagger |\text{vac}\rangle , \tag{3}$$

into the output of a boson-sampling problem with N excitations:

$$|\phi(\pi/2)\rangle = (-i)^N \prod_{i=1}^{N} \sum_j R_{ji}^* b_j^\dagger |\text{vac}\rangle . \tag{4}$$

The bosonic distribution in the output modes is given by permanents $|\gamma_{\mathbf{n}}|^2 = |\langle \text{vac}| b_1^{\dagger n_1} \cdots b_M^{\dagger n_M} |\phi(\pi/2)\rangle|^2, n_i \in \{0,1\}$, and it is conjectured to be classically hard to simulate.

A. Proof

In order to establish the link between boson sampling and evolution with Hamiltonian (2), we introduce new orthogonal modes

$$c_j^\dagger = \sum_{i=1}^{M} R_{ji} a_i^\dagger . \tag{5}$$

These modes satisfy the appropriate commutation relations, $[c_m, c_n^\dagger] = \sum_i R_{mi}^* R_{ni} = (R^\dagger R)_{m,n} = \delta_{n,m}$. They also diagonalize the previous Hamiltonian, which becomes a sum of beam-splitter models:

$$H_{\rm BS} = \sum_j [b_j^\dagger c_j + c_j^\dagger b_j + \omega(b_j^\dagger b_j + c_j^\dagger c_j)]. \tag{6}$$

Dynamics under $H_{\rm BS}$ involves a swap of excitations from the normal modes c_i into the output modes b_i. After a time $t = \pi/2$ the initial state (3) is transformed into Eq. (4).

Let us prove this statement for an initial state with $N \ll M$ excitations, Eq. (3). We write the Heisenberg equations for operators evolving as $O(t) = e^{-iHt} O e^{iHt}$,

$$\dot{b}_j^\dagger = -i[H, b_j^\dagger] = -i\omega b_j^\dagger - i c_j^\dagger, \tag{7}$$

$$\dot{c}_j^\dagger = -i\omega c_j^\dagger - i b_j^\dagger, \tag{8}$$

which have as solutions

$$b_j^\dagger(t) = e^{-i\omega t}[\cos(t) b_j^\dagger(0) - i \sin(t) c_j^\dagger(0)], \tag{9}$$

$$c_j^\dagger(t) = e^{-i\omega t}[\cos(t) c_j^\dagger(0) - i \sin(t) b_j^\dagger(0)]. \tag{10}$$

Inverting relation (5), we recover

$$\begin{aligned} e^{i\omega t} a_k^\dagger(t) &= \sum R_{jk}^* c_j^\dagger(t) \\ &= \cos(t) a_k^\dagger(0) - i \sin(t) R_{jk}^* b_j^\dagger(0), \end{aligned} \tag{11}$$

where we assume summation over repeated indices. Dynamics under Hamiltonian (6) is coherently transferring population from the a to the b modes, as in

$$|\phi(t)\rangle = e^{-i\omega N t} \prod_{k=1}^{N} [\cos(t) a_k^\dagger - i \sin(t) R_{jk}^* b_j^\dagger] |0\rangle . \tag{12}$$

At time $t = \pi/2$ all population is transferred,

$$|\phi(\pi/2)\rangle = \prod_{k=1}^{N} a_k^\dagger(\pi/2) |0\rangle , \tag{13}$$

with

$$a_k^\dagger(\pi/2) = (-i) e^{-i\omega N\pi/2} \sum_{j=1}^{M} R_{jk}^* b_j^\dagger . \tag{14}$$

This is the outcome anticipated in Eq. (4).

B. Intermediate states

While the output state at $t = \pi/2$ is trivially related to the boson-sampling problem, it is also true that $|\phi(t)\rangle$ may be regarded at other times as a coherent superposition of different boson-sampling instances, $|\xi_{{\rm BS},n}\rangle$, where only $n \in$

$\{0,1,\ldots,N\}$ bosons participate in the M output modes, while $N-n \in \{N,N-1,\ldots,0\}$ remain in the input modes. In other words, we have

$$|\phi(t)\rangle = e^{-i\omega N t}\sum_{n=0}^{N}\binom{N}{n}^{1/2}\cos(t)^{N-n}\sin(t)^{n}\,|\xi_{\mathrm{BS},n}\rangle\,, \quad (15)$$

with normalized states

$$|\xi_{BS,n}\rangle \propto \sum_{\{k,j\}} a^{\dagger}_{k_1}(\pi/2)\cdots a^{\dagger}_{k_n}(\pi/2) \times a^{\dagger}_{j_1}(0)\cdots a^{\dagger}_{j_{N-n}}(0)\,|0\rangle\,. \quad (16)$$

These states consists of $N-n$ excitations that stay in the input modes $[a_j(0)]$ and n excitations that have been fully transferred to $a_k(\pi/2)$, which are linear combinations of the $b_k(0)$ modes. In other words,

$$|\phi(t)\rangle = \sum_{n=0}^{N}\cos(t)^{n}\sin(t)^{N-n}\,|\xi_{n,M}\rangle\,,$$

$$|\xi_{n,M}\rangle \propto \sum_{\pi} c^{\dagger}_{\pi_1}\cdots c^{\dagger}_{\pi_n} a^{\dagger}_{\pi_{n+1}}\cdots a^{\dagger}_{\pi_N}\,|0\rangle\,. \quad (17)$$

Each of these instances $|\xi_{\mathrm{BS},n}\rangle$ is similar to the instances (4), even if they involve a smaller number of particles. Let us now remember that for boson sampling to be "efficient" in the sense of implementation, the number N has to be small enough so that the fraction of states with two or more bosons remains small; otherwise, most experiments would have to be postselected out. Interestingly, from Ref. [17] we know that when that condition is satisfied for N particles in M modes, it is also satisfied for all $n \leqslant N$. This implies that the full wave function $|\phi(t)\rangle$ has the bunching statistics of efficiently sampled boson-sampling problems. This is in sharp contrast with actual intermediate states in a random interferometer, which may contain a lot of bunching before the bosons exit the circuit, and it is due to the fact that the model that we use to recreate the BS output states, H, does not describe those intermediate stages, where the dynamics of individual beam splitters matters.

Finally, let us remark that the intermediate states $|\phi(t)\rangle$ generated by the toy Hamiltonian H_{BS} are not related to the intermediate wave functions in a linear optics circuit. Optical circuits composed of beam splitters and phase shifters will in general lead to intermediate states which are highly bunched and highly entangled, bearing little resemblance to the mathematical construct (12) above.

III. FROM BS TO HARD-CORE BOSONS

Once we have established the equivalent between boson sampling and time evolution of harmonic models, we now relate this problem to a family of strongly correlated models called "hard-core bosons" (HCBs), where particles are not allowed to coexist on the same mode. We show that, under the conditions for boson sampling to be hard, both the output state of the boson-sampling problem (4), as well as the intermediate states (12), are very close to the HCB limit where there is at most exactly one boson per mode. This "closeness" is established on mathematical grounds by comparing the error that we make in replacing $|\phi(t)\rangle$ with a different state $|\psi(t)\rangle$ that results from evolving a HCB state using a HCB version of Eq. (2).

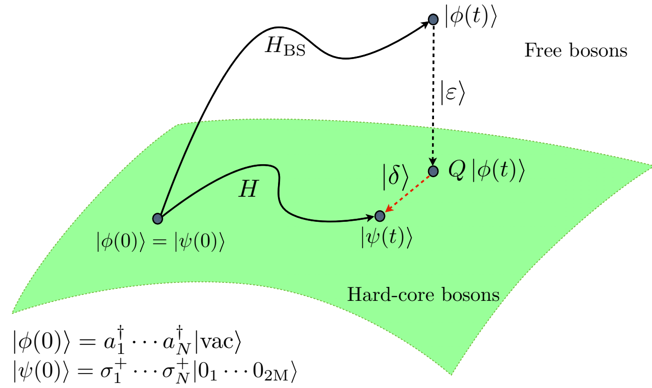

FIG. 2. The distance between the full bosonic state $|\phi(t)\rangle$ and the state approximated with spins, $|\psi(t)\rangle$, is covered by two error vectors: one $|\delta\rangle$ that lives in the hard-core-boson space (red dashed line) and another one that covers the distance between the projected state $Q\,|\phi(t)\rangle$ and the full boson state, $|\phi\rangle$ (black dashed line).

A. Dilute limit

As mentioned above, hard-core bosons are strongly interacting particles with an exclusion principle that prevents them from coexisting in the same mode. This means that, unlike linear models, HCBs can be described using a finite Hilbert space where each degree of freedom is either empty or occupied: $\mathbb{C}^2 = \mathrm{lin}\{|0\rangle\,,|1\rangle\}$. Formally, a HCB problem is assimilated to a spin model, denoting by $|\downarrow\rangle \sim |0\rangle$ and $\sigma^{+}|\downarrow\rangle = |\uparrow\rangle \sim |1\rangle$ the empty and unoccupied sites. The motion of particles between sites, such as Eq. (2), is described by a Hamiltonian of the form (1), where the hopping term $\sigma^{+}_{\mathrm{out},i}\sigma^{-}_{\mathrm{in},j}R_{ij} \sim b^{\dagger}a_j R_{ij}$ is only active when the destination mode i is unoccupied.

The rules of boson sampling [1] demand that we work with a small number of particles in a large number of modes, $N \ll M$. This means that the final state of the linear problem (4) only contains a small probability of two or more bosons accumulating in the same mode. The probability of these coincidences is exactly equal to the probability that during postselection we have to discard the output of the BS circuit. The set of those events where two or more bosons accumulate can be grouped into the error state $|\varepsilon\rangle$ as explained in Fig. 2:

$$|\phi\rangle = Q\,|\phi\rangle + |\varepsilon\rangle\,. \quad (18)$$

Here $Q\,|\phi\rangle$ is the projection onto states with zero or one boson per site. It is therefore a wave function that fits in the HCB Hilbert space. The error $|\varepsilon\rangle$, on the other hand, only contains bunched states that live outside that space. For the errors $|\varepsilon\rangle$ to be eliminated in postselection while maintaining the efficiency of the sampling, the number of modes must be larger than the number of excitations. The suspected ratio [1] at which sampling becomes efficient is $M \simeq N^2$ [15], with bounds being tested theoretically and experimentally [17,18].

B. Spin sampling

The assumption of "diluteness" of excitations, which is needed for the experimental sampling of bosons, ensures that we have a small probability of boson bunching not only at the end of the beam-splitter dynamics, but also at all times. More precisely, the probability of bunching of particles in the state $|\phi(t)\rangle$, estimated by the vector norm-2 $\|Q\phi(t)\|_2^2$, roughly grows in time and is bounded by the final postselection success probability (cf. Appendix A). In other words, boson-sampling dynamics is efficient only when it samples states with at most one boson per mode, the so-called HCB subspace or the spin space. In this situation one would expect that models (1) and (2) become equivalent, with the soft-boson corrections becoming negligible.

Continuing with this line of thought, we now study how well the dynamics of the full bosonic system can be approximated by the hard-core-boson model (1). We regard the spin Hamiltonian as the projection of the full boson-sampling model onto the hard-core-boson subspace, $H = QH_{\rm BS}Q$. We show that the boson sampling dynamics is reproduced by the spin model at short times, with an error that grows with excitation density and which is bounded by $\|\varepsilon\|^2$.

Let us assume that $|\psi\rangle$ is a hard-core-boson state that initially coincides with the starting distribution of the boson-sampling problem, $|\psi(0)\rangle = \sigma^+_{{\rm in},1}\cdots\sigma^+_{{\rm in},N}|0\rangle \sim |\phi(0)\rangle = a_1^\dagger\cdots a_N^\dagger|0\rangle$. This state evolves with the spin model as $i\partial_t|\psi\rangle = QH_{\rm BS}Q|\psi\rangle$. We define the sampling error which is made by working with spins as

$$|\delta\rangle = Q|\phi\rangle - |\psi\rangle. \tag{19}$$

A formal solution for this error is

$$|\delta(t)\rangle = -i\int_0^t e^{-iQH_{\rm BS}Q(t-\tau)}QH_{\rm BS}|\varepsilon(\tau)\rangle\, d\tau. \tag{20}$$

As detailed in Appendix B, bounds for the different terms in this integral provide us with the core result:

$$\|\delta(t)\|_2 \leqslant t\times O\left(\frac{N^2}{\sqrt{M}}\right), \tag{21}$$

which states that the error probability at $t=\pi/2$ is negligible when $N\sim O(M^{1/4})$. In the same appendix we also show numerical evidence that this bound can be at least improved to $N\sim O(M^{1/3})$. Moreover, in Appendix C we show that the same bounds apply for the variation distance between the probability distributions associated to the quantum states $Q|\phi\rangle$ and $|\psi\rangle$, the measure used by Aaronson and Arkhipov in Ref. [1].

Equation (20) shows that the errors due to using a spin model for boson sampling feed from bunching events in the original problem, $|\delta\rangle\propto|\varepsilon\rangle$, which are prevented by the HCB condition. For times short enough, these errors amount to excitations being "backscattered" to the "in" spins. This means that sampling errors can be efficiently postselected in any given realization of these experiments, rejecting measurement outcomes where there contain less than N excitations in the $\sigma^+_{{\rm out},j}$ spins. In this case, what we characterized as an error becomes a postselection success probability, $P_{ok} = 1-\|\delta\|^2$.

C. Spin model bounds

We now sketch the physical intuition used in deriving Eq. (20), leaving the more technical details for Appendix B. We start with the Schrödinger equation for the separation between the HCB and BS models:

$$i\partial_t|\delta\rangle = QH_{\rm BS}Q|\delta\rangle + QH_{\rm BS}|\varepsilon\rangle. \tag{22}$$

As explained above, it shows that the errors in approximating the boson sampling with spins result from the accumulation of processes that, through a single application of $H_{\rm BS}$, undo a pair of bosons from ε, taking this vector into the hard-core-boson sector.

We now bound the maximum error probability as an integral of two norms. For that we realize that out of ε, $QH_{\rm BS}$ cancels all terms that have more than one mode with double occupation. Thus, we can derive an inequality using the operator spectral norms

$$\epsilon^{1/2} = \|\delta\|_2 \leqslant \int_0^t \|QH_{\rm BS}P_{\rm 1bpair}\|_2\|P_{\rm 1bpair}|\varepsilon(\tau)\rangle\|_2 d\tau, \tag{23}$$

where Q is a projector onto HCB states with N particles and $P_{\rm 1bpair}$ is a projector onto the states with $N-2$ isolated bosons and one pair of b bosons on the same site. As explained in the Appendix, the value $\|P_{\rm 1bpair}|\varepsilon(\tau)\rangle\|_2^2 = \|P_{\rm 1bpair}|\phi(\tau)\rangle\|_2^2$ is the probability of finding a single bunched pair in the full bosonic state. Combining a similar bound by Arkhipov [17] with the actual structure of the evolved state, we find

$$\|P_{\rm 1bpair}|\varepsilon(\tau)\rangle\|_2 \leqslant O\left(\frac{N}{\sqrt{M}}\right), \tag{24}$$

which works provided that $N=O(M^{3/4})$. We have also shown (cf. Appendix B) that the operator norm $\|QH_{\rm BS}P_{\rm 1bpair}\|_2$ is strictly smaller than the maximum kinetic energy of N bosons in the original model, $H_{\rm BS}$, so that

$$\|QH_{\rm BS}P_{\rm 1bpair}\|_2 \leqslant N. \tag{25}$$

Combining both bounds, we finally end up with Eq. (21).

The proof also gives us a qualitative understanding of the nature of the errors introduced by the hard-core-boson approximation, $|\delta\rangle$. From the integral we gather that these errors take the rare events in which two bosons are bunched in $|\varepsilon\rangle$ and return those bosons back to the input modes, $a_j^\dagger$. This suggests that errors in the spin or hard-core-boson model $|\delta\rangle$ can be recognized as spin excitations that have not been able to transition from the input modes, $\sigma_{\rm in}$, to the output modes, $\sigma_{\rm out}$.

IV. OUTLOOK AND DISCUSSION

The relation between boson sampling and spin models allows us to draw some interesting conclusions in a variety of fields, from quantum simulation to quantum complexity theory, including existing and possible future experiments. Let us now discuss some of these ideas and close with a recapitulation of the main ideas in this work.

A. Quantum simulation and computation

We can use a quantum simulator with spins to implement spin sampling. As a concrete application, let us assume that we have a quantum simulator that implements the Ising model with arbitrary connectivity and coupling to a transverse magnetic field. We partition the set of $2M$ spins into two subsets,

$$H_{\text{Ising}} = \sum_{i,j=1}^{M} J_{i,j}\sigma^x_{\text{out},i}\sigma^x_{\text{in},j} + B\sum_{j=1}^{M}\left(\sigma^z_{\text{out},j} + \sigma^z_{\text{in},j}\right). \quad (26)$$

In the limit of very large transverse magnetic field, $|B| \gg \|J\|_2$, the evolution of the spin model under this Hamiltonian is effectively described by the XY Hamiltonian (1), up to a constant term proportional to the number of excitations. The coupling matrix is $J_{i,j} = J_{j,i} = R_{ij}$ $(i,j = 1,\dots,M)$ and can represent any random orthogonal transformation.

The interaction H_{Ising} is a possible component of universal quantum computation, because it allows implementing universal two-qubit gates between separate pairs of qubits or spins. However, in our protocol we only require the implementation of a single instance of such random static couplings for a time $t \sim O(\|J\|^{-1})$ which only allows the implementation of one such gate. This is incomparably simpler than combining multiple applications of those couplings in a real algorithm with error correction.

The Ising interaction (26) is already present in trapped-ion quantum simulators with phonon-mediated interactions [19], a setup which has been repeatedly demonstrated in experiments [20–22], even for frustrated models [23,24], extremely large two-dimensional crystals [25], and in particular in the XY limit that we just sketched [10]. While current experiments have only explored a subset of interactions with uniform spatial dependencies, random ion-ion interactions such as the J_{ij} above can be recovered from simple time-dependent controls of the forces that implement the phonon-mediated ion-ion coupling [26].

Another suitable platform for this kind of experiments is the D-wave machine or equivalent superconducting processors with long-range tunable interactions [27–29]. These devices can now randomly sample J from a set of unitaries over a graph that is a subset of the available connectivity graph. Since the number of spins is very large, with over 900 good-quality qubits available, we expect that those simulations would surpass the complexity of the sampling problems that can be modeled in state-of-the-art linear optics circuits.

Our formal results also suggest an efficient implementation of boson sampling in a general-purpose quantum computer using the following algorithm:

(i) Prepare a quantum register with $M + M$ qubits, encoding the input and output spins, in the first and last N positions, respectively.

(ii) Initialize the register to the state $|\psi(0)\rangle = |1_1,\dots,1_N,0_{N+1},\dots,0_{2M}\rangle$.

(iii) Implement the unitary $U = \exp(-iH\pi/2)$, where H is given by Eq. (1).

(iv) Measure the quantum register. In postselected experiments where the N first qubits are at zero, record the resulting state of the output qubits to estimate the sampling probability. For this remember that the relation between $|\psi\rangle$ and $|\phi\rangle$ implies that the sampling probability

$$p(n_1,\dots,n_M) = |\langle 0_1,\dots,0_N,n_1,\dots,n_M|\psi(\pi/2)\rangle|^2 \quad (27)$$

is the same one as the one from boson sampling, modulo the errors introduced by δ. It is worth mentioning that while a general-purpose quantum computer might implement boson sampling via the Schwinger representation of bosons, this would require a larger number of qubit resources, and a greater complexity in implementing beam-splitting operations, while our spin-sampling problem requires a smaller Hilbert space and has a natural implementation on a small quantum computer.

In general, the spin-sampling simulator will have similar sources of error as a boson-sampling device: dephasing, losses, and calibration errors in the interactions. The latter source of errors only has an algebraic contribution, but the former could exponentially damage the sampler. Here is where the use of quantum simulators in the spin case can provide significant advantages. Take the trapped-ion implementation as an example. The encoding of excitations using atomic ground states means losses can be irrelevant. Moreover, working in a subspace with a fixed number of excitations also implies that the device is less sensitive to global dephasing. Both properties have been demonstrated in Ref. [10], an experiment that probes evolution times that are longer than our $Jt = \pi/2$ requirement. Note, however, that both the spin-sampling and boson-sampling problems still have open questions regarding the verification or validation of the final distributions, but this question lays beyond the scope of this work.

B. Complexity theory

Our mapping of boson sampling to spin evolution shows that classically simulating the dynamics of long-range interacting spin models at short times is as hard as the classical simulation of boson sampling. More precisely, if spin sampling could be simulated in a classical computer, then we could approximate the boson-sampling solution with precision poly$(N^2/\sqrt{M})$, which would imply a collapse of the polynomial hierarchy [1].

This idea connects to earlier results that relate the difficulty of classically simulating time evolution due to very fast entanglement growth [30,31]. It also does not contradict the fact that free fermionic problems can be efficiently sampled because model (1) only maps to free fermions for a subclass of tridiagonal matrices R.

There are other remarks to be done about our work and its place in the existing literature. First of all, it can be argued that spins or qubits are the underlying components of a quantum computer whose computation will in general amount to evolution with an effective Hamiltonian. This argument is bogus in that the resulting Hamiltonian will, in general, not be physically implementable, involving interactions with an arbitrary number of spins and distance. Moreover, even if certain models such as (1) are universal and may encode quantum computations [32,33], the time scales of our result amount to a single hopping event, which is scarcely the time to implement a single quantum gate and not an arguably complex computation.

Finally, while at least one work has established connections between the collapse of the polynomial hierarchy and spin models [9], that work builds on the conjecture that the complex partition function of a spin model is already in **#P**, and thus the time evolution of those spin models is hard to be approximated, which is instead the conclusion of this work.

V. CONCLUSIONS

Summing up, we have demonstrated that boson sampling can also be efficiently implemented using spins or qubits interacting through a rather straightforward XY Hamiltonian. We have thus established a family of problems that are efficiently simulatable in a quantum computer but not on a classical one. Our map opens the door to demonstrating quantum supremacy using small quantum simulators of spin models, of which we have offered two examples: trapped ions and superconducting circuits. Interestingly, the connection between boson sampling and spin Hamiltonian simulation hints that the difficulties found in developing certification methods for boson sampling may be also present in the validation of ordinary spin model simulators.

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APPENDIX A: BUNCHING BOUNDS

If we want boson sampling to be efficient, we need to impose that the number of bunching events in $|\phi(\pi/2)\rangle$ remains small with increasing problem size. Such a property is guaranteed on average by the random unitaries U sampled with the Haar measure, as explained by Arkhipov and Kuperberg in the boson-birthday paradox paper [17]. Below we use the fact that the number of bunching events in $|\phi(\pi/2)\rangle = |\xi_{\mathrm{BS},N}\rangle$ indeed upper-bounds the number of bunches in each of its constituents, $|\xi_{\mathrm{BS},n\leqslant N}\rangle$, and we use this idea to draw conclusions on the distance between the true boson-sampling problem and the HCB spin model.

We cannot sufficiently stress the fact that the number of bunching events in $|\phi(t)\rangle$ is not related to the number of bunching events in the intermediate stages of a linear optics circuit. In order to implement a boson sampler one has to combine beam splitters that at different stages of the circuit cause the accumulation of the bosons. However, while those intermediate states in the construct are essential to reach the final boson-sampling states, $|\xi_{\mathrm{BS},n}\rangle$, none of those intermediate states belongs to the family of states at the output of the circuit, $|\xi_{\mathrm{BS},n}\rangle$, which must have a low bunching probability (and which do, as shown by Ref. [17]).

For the purposes of bounding the error from the spin-sampling model, we need to bound the part of the error $|\varepsilon\rangle$ that contains a single pair of bosons on the same site, on top of a background of singly occupied and empty states. We have labeled that component $\|P_{\mathrm{1pair}}\varepsilon\|_2^2$. However, as discussed in Ref. [17], bounding that probability is harder than bounding the probability $p_{\mathrm{HCB}}(N,M)$ of having *no bunching* event in a state with N bosons in M modes, distributed according to the random matrices R_{ji}. This probability is

$$p_{\mathrm{HCB}}(N,M) = \prod_{a=0}^{N} \frac{M-a}{M+a} \simeq e^{-N^2/M} \tag{A1}$$

for dilute systems $N = O(M^{3/4})$. We now use (i) that the state $|\phi(t)\rangle$ in Eq. (12) is made of a superposition of states with $n = 0,1,\ldots,N$ bosons distributed through the M modes, (ii) that due to the randomness of R, each of these components shares the same statistical properties of the boson-sampling states [17], and (iii) that the probability distribution $p_{\mathrm{HCB}}(N,M)$ is monotonously decreasing with N. Using Eq. (15) and this idea we arrive at

$$\begin{aligned}\|Q\phi(t)\|_2^2 &\simeq \sum_{n=0}^{N} \binom{N}{n} \cos(t)^{2(N-n)} \sin(t)^{2n} p_{\mathrm{HCB}}(n,M)\\ &\geqslant \sum_{n=0}^{N} \binom{N}{n} \cos(t)^{2(N-n)} \sin(t)^{2n} p_{\mathrm{HCB}}(N,M)\\ &= [\cos(t)^2 + \sin(t)^2]^N p_{\mathrm{HCB}}(N,M)\\ &= p_{\mathrm{HCB}}(N,M).\end{aligned} \tag{A2}$$

Using the fact that $Q\,|\phi\rangle$ and $|\varepsilon\rangle$ are orthogonal and thus $\|\phi\|_2^2 = \|Q\phi\|_2^2 + \|\varepsilon\|_2^2$, we can find a very loose bound for the error probability of single bunching events,

$$\|P_{\mathrm{1bpair}}\varepsilon\|_2^2 \leqslant \|\varepsilon(t)\|_2^2 \leqslant 1 - p_{\mathrm{HCB}}(N,M). \tag{A3}$$

Note that this bound can be translated into an upper bound of $O(N^2/M)$ using the fact that the exponential falls faster than $1 - N^2/M$.

APPENDIX B: HCB OPERATOR BOUND

In addition to bounding the error vector, we also need to bound the norm of an operator that brings back population from the error subspace into the hard-core-boson subspace. Because $\|P_{\mathrm{1bpair}}\varepsilon\|_2$ is already rather small, we can afford a loose bound for the operator $\|Q H_{\mathrm{BS}} P_{\mathrm{1bpair}}\|$, which is the other part of the integral. The argument is basically as follows. First, we notice that all operators in the product, Q, H_{BS}, and P_{1bpair}, commute with the total number of particles, which in our problem is exactly N. We can thus study the restrictions of these operators to this sector, which we denote as $P_N O P_N$ for each operator, where P_N is the projector onto the space with N particles. We then realize that $\|AB\|_2 \leqslant \|A\|_2\|B\|_2$ and, since

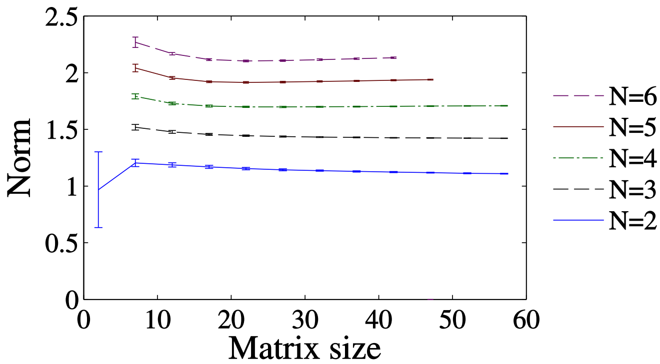

FIG. 3. Numerical estimates of the norm $\|QH_{\rm BS}P_{\rm 1bpair}\|_2$ as a function of the number of bosonic modes, M, for different number of excitations, N.

the projectors have norm 1,

$$\begin{aligned}\|QH_{\rm BS}P_{\rm 1bpair}\|_2 &= \|QP_NH_{\rm BS}P_NP_{\rm 1bpair}\|_2\\ &\leqslant \|Q\|_2\|P_NH_{\rm BS}P_N\|_2\|P_{\rm 1bpair}\|_2\\ &= \|P_NH_{\rm BS}P_N\|_2. \end{aligned} \tag{B1}$$

Notice now that $P_NH_{\rm BS}P_N$ is just the Hamiltonian of N free bosons, without hard-core restrictions of any kind. In other words, it is the restriction of

$$H_{\rm BS} = \sum_k (b_k^\dagger c_k + {\rm H.c.}) \tag{B2}$$

to a situation where $\sum_k c_k^\dagger c_k + b_k^\dagger b_k = N$. We introduce superposition modes, $\alpha_{k\pm} = (c_k \pm b_k)/\sqrt{2}$, and diagonalize

$$H_{\rm BS} = \sum_k (\alpha_{k+}^\dagger\alpha_{k+} - \alpha_{k-}^\dagger\alpha_{k-}), \tag{B3}$$

where the constraint is the same: $\sum_k \alpha_{k+}^\dagger\alpha_{k+} + \alpha_{k-}^\dagger\alpha_{k-} = N$. Since the largest eigenvalues (in modulus) are obtained by filling N of these normal modes with the same frequency sign, we have

$$\|QH_{\rm BS}P_{\rm 1bpair}\|_2 \leqslant \|P_NH_{\rm BS}P_N\|_2 = N. \tag{B4}$$

Note that this proof does not make use of any properties of H such as the fact that it is built from random matrices. As explained in the body of the paper, if we sample $QH_{\rm BS}P_{\rm 1bpair}$ randomly with the Haar measure and average the resulting norms, the bound seems closer to $O(\sqrt{N})$.

We have strong evidence that this bound can be significantly improved using the properties of random matrices R_{ij} and the structure of $QH_{\rm BS}\varepsilon$. In particular, we have numerical evidence that the average norm over the Haar measure is $\|QH_{\rm BS}P_{\rm 1bpair}\|_2 \propto O(N^{1/2})$, which improves the requirement for efficient spin sampling, $N \sim O(M^{1/3})$. Figure 3 shows the average and standard deviation of the operator norm obtained by sampling random bosonic circuits with $N = 2$–6 particles in $M = 7$–60 modes, creating random unitaries according to the Haar measure and estimating the norm of the operator $QH_{\rm BS}P_{\rm 1bpair}$ with a sparse singular value solver. Note how, despite the moderate sample size (200 random matrices for each size), the standard deviation is extremely small, indicating the low probability of large errors and the efficiency of the sampling.

APPENDIX C: VARIATION DISTANCE

Throughout this paper, we have found bounds according to the 2-norm, in contrast to Aaronson and Arkhipov's work, whose results are expressed in terms of the variation distance between probability distributions (that is, 1-norm). However, our proof above can be written in a similar way as the one given by Aaronson and Arkhipov; that is,

$$|p_1 - p_2|_1 := \sum_n |p_1(\mathbf{n}) - p_2(\mathbf{n})|, \tag{C1}$$

which represents the total difference between probabilities for all configurations $\mathbf{n} = (n_1, \ldots, n_M)$ of the occupations at the output ports. In our model, the probability distribution associated to boson sampling would be

$$p_1(\mathbf{n}) = |\langle \mathbf{n}|\phi(t)\rangle|^2 =: |\phi(\mathbf{n})|^2, \tag{C2}$$

where, if we focus on events with $n_i \in \{0,1\}$, we can replace ϕ with $Q\phi$. The corresponding probability for the spin model would be

$$p_2(\mathbf{n}) = |\langle \mathbf{n}|\psi(t)\rangle|^2 =: |\psi(\mathbf{n})|^2. \tag{C3}$$

Using the above expressions for the probability distributions, we can write the following identities for the total variation distance (C1):

$$\begin{aligned}\sum_n |p_1(\mathbf{n}) - p_2(\mathbf{n})| &= \sum_n ||\psi(\mathbf{n})|^2 - |\phi(\mathbf{n})|^2|\\ &= \sum_n |\psi^*(\mathbf{n})\psi(\mathbf{n}) - \phi^*(\mathbf{n})\phi(\mathbf{n})|\\ &= \frac{1}{2}\sum_n |[\langle\psi(\mathbf{n}) + \phi(\mathbf{n})|\delta(\mathbf{n})\rangle\\ &\quad + \langle\delta(\mathbf{n})|\psi(\mathbf{n}) + \phi(\mathbf{n})\rangle]|, \end{aligned} \tag{C4}$$

where $|\delta(\mathbf{n})\rangle = |\psi(\mathbf{n}) - \phi(\mathbf{n})\rangle$. Hence, it follows that

$$\begin{aligned}\sum_n ||\psi(\mathbf{n})|^2 - |\phi(\mathbf{n})|^2| &= \sum_n |{\rm Re}(\langle\psi(\mathbf{n}) + \phi(\mathbf{n})|\delta(\mathbf{n})\rangle)|\\ &= \sum_n |{\rm Re}(2\phi^*(\mathbf{n})\delta(\mathbf{n}) + \delta^*(\mathbf{n})\delta(\mathbf{n})|\\ &\leqslant 2\sum_n |\phi(\mathbf{n})||\delta(\mathbf{n})| + \sum_n |\delta(\mathbf{n})|^2\\ &\leqslant 2\left(\sum_n |\phi(\mathbf{n})|^2\right)^{1/2}\left(\sum_n |\delta(\mathbf{n})|^2\right)^{1/2}\\ &\quad + \|\delta\|_2^2\\ &= 2\|\phi\|_2\|\delta\|_2 + \|\delta\|_2^2. \end{aligned} \tag{C5}$$

Since the boson-sampling wave function is normalized, $\|\phi\|_2 = 1$, and $\|\delta\|_2 \leqslant 1$, we finally get the next tight bound for the variation distance:

$$|p_1 - p_2| \leqslant 3\|\delta\|_2 = O\left(\frac{N^2}{\sqrt{M}}\right). \tag{C6}$$

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