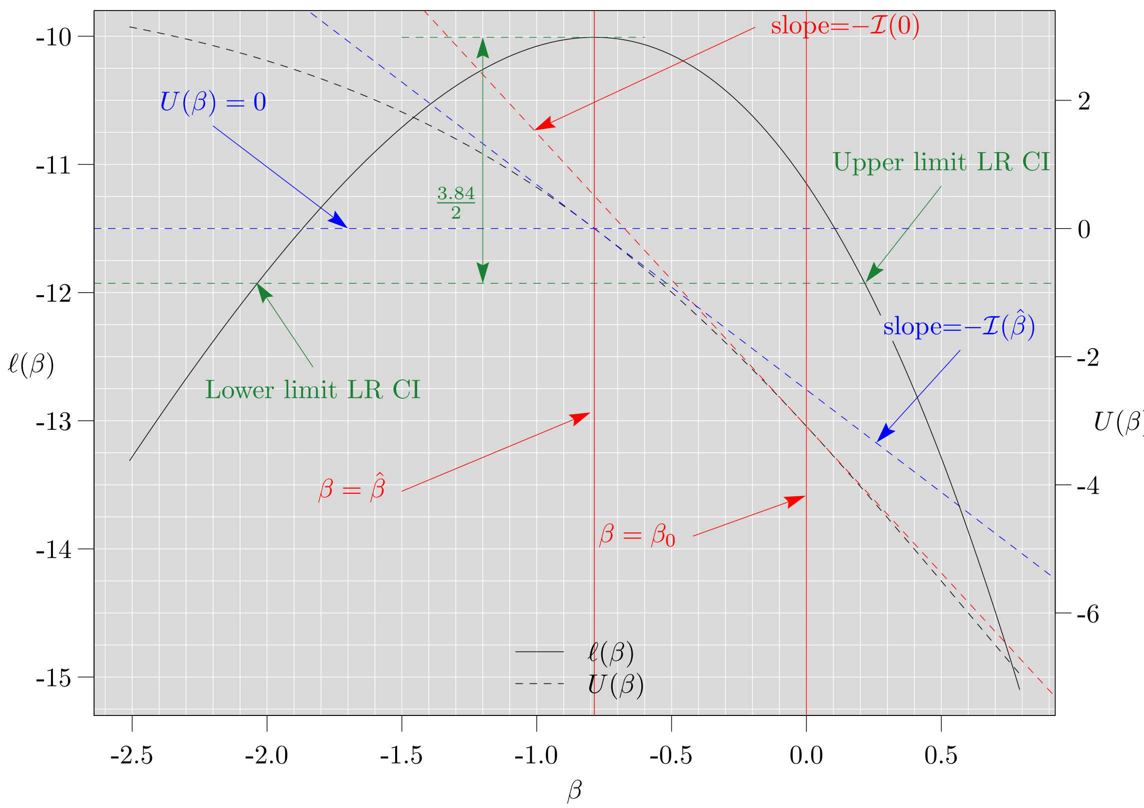
Statistics 641, Spring 2018
Homework #2
Solutions

1. Suppose we have a one parameter family with parameter β and observed data (not shown). The figure below shows the log-likelihood function, $\ell(\beta)$, and score function, $U(\beta)$, given the data. The scale for $\ell(\cdot)$ is shown on the left axis and the scale for $U(\cdot)$ is shown on the right axis.



Using just this figure, estimate the following:

(a) The MLE of β.

The MLE, $\hat{\beta}$ is found either by finding the location of the maximum value of $\ell(\beta)$, or finding where $U(\beta) = 0$. In either case we find $\hat{\beta} \approx -.79$, shown by the vertical red line labeled "$\beta = \hat{\beta}$.

(b) A 95% confidence interval for β based on the likelihood ratio test.

The 95% CI based on the likelihood ratio test consists of values of β_0 such that we won't reject $H_0\colon \beta = \beta_0$ at the 5% level. I.e., values of β_0 for which $-2(\ell(\beta_0) - \ell(\hat{\beta})) < 3.84$, or, equivalently, $\ell(\beta_0) > \ell(\hat{\beta}) - 3.84/2$. The green horizontal dashed line is 3.84/2 units below $\ell(\hat{\beta})$. The limits of the CI are the points where this line crosses $\ell(\beta)$, approximately (-2.04, 0.22).

(c) A 95% confidence interval for β based on the Wald test.

To find the 95% Wald CI, we need $\mathcal{I}(\hat{\beta})$, which is -slope of $U(\hat{\beta})$. By drawing a line tangent to $U(\beta)$ at $\hat{\beta}$, we estimate the slope as -3.2. Hence the standard error $\hat{\beta}$ is $\sqrt{1/3.2} = 0.56$, and the Wald CI is $-0.79 \pm 1.96 \times 0.56 = (-1.88, 0.31)$. Note that this interval is shifted to the slightly to the right of the LR interval.

(d) The likelihood ratio test chi-square statistic for $H_0\colon \beta = 0$.

The likelihood ratio chi-square statistic of $H_0\colon \beta = 0$ is
$-2(\ell(0) - \ell(\hat{\beta})) \approx -2(-10.0 - (-11.15)) = 2.3$.

(e) The Wald chi-square statistic for $H_0\colon \beta = 0$.

The Wald chi-square statistic is $\hat{\beta}^2\mathcal{I}(\hat{\beta}) = -0.79^2 \times 3.2 = 2.00$.

(f) The score test chi-square statistic for $H_0\colon \beta = 0$.

To estimate the score chi-square, we need $U(0)$ and $\mathcal{I}(0)$. From the figure, $U(0) \approx -3.1$. $\mathcal{I}(0)$ is -slope of $U(\beta)$ at $\beta = 0$, or approximately 4.6. Hence the score chi-square statistic is approximately $\dfrac{3.1^2}{4.6} = 2.09$.

The figure is available as a separate file on the homework web page. For each part, please annotate the figure to show how each quantity is estimated. (*Hint: you'll need to numerically estimate derivatives. You can do this by drawing a tangent line on the figure and estimating it's slope. If you have the tools to do this electronically, that's fine.*)

2. Suppose in a mortality trial of treatment A versus treatment B we observe the following table by baseline strata:

	Stratum 1		Stratum 2		Stratum 3		Total	
	A	B	A	B	A	B	A	B
Alive	18	21	8	11	1	4	27	36
Dead	5	1	5	1	12	6	22	8
Total	23	22	13	12	13	10	49	44

(a) Calculate the score test (Pearson chi-square) statistic (uncorrected) for the difference in overall mortality, ignoring strata.

The Pearson chi-square statistic is

$$\frac{(22 - 30 \times 49/93)^2}{63 \times 30 \times 49 \times 44/93^3} = \frac{6.194^2}{5.066} = 7.572$$

Note that if `All` is a table containing the overall cell counts, we can use the `chisq.test` function in `R` to do the test:

```
> All
     [,1] [,2]
[1,]   27   36
[2,]   22    8
```

```
> chisq.test(All, correct=F)

Pearson's Chi-squared test

data:  All
X-squared = 7.5721, df = 1, p-value = 0.005928
```

(b) Calculate the stratified score test for the adjusted difference in overall mortality.

Letting x_k represent the number dead in group A and stratum k, we have

Stratum	$U(0)$	$I(0)$
A	$5 - 6 \times 23/45 = 1.933$	$23 \times 22 \times 39 \times 6/45^3 = 1.299$
B	$5 - 6 \times 13/25 = 1.880$	$13 \times 12 \times 19 \times 6/25^3 = 1.138$
C	$12 - 18 \times 13/23 = 1.827$	$13 \times 10 \times 5 \times 18/23^3 = 0.962$
Total	5.639	3.399

The stratified score chi-square statistic is

$$\frac{5.639^2}{3.399} = 9.356$$

Note that the numerator is slightly smaller than the unadjusted (5.639 versus 6.194), however, the denominator is quite a bit smaller (3.399 versus 5.066), resulting in a larger test statistic. Stratification increases efficiency by reducing variance.

Note that if the variable `Grouped` is a $2 \times 2 \times 3$ array containing the stratified cell counts, we can use the function `mantelhaen.test` to perform a similar, but not identical, test. This function also gives an estimate of the common odds-ratio.

```
> Grouped
, , 1
     [,1] [,2]
[1,]   18   21
[2,]    5    1
, , 2
     [,1] [,2]
[1,]    8   11
[2,]    5    1
, , 3
     [,1] [,2]
[1,]    1    4
[2,]   12    6
```

```
> mantelhaen.test(Grouped, correct=F)
Mantel-Haenszel chi-squared test without continuity correction
data:  Grouped
Mantel-Haenszel X-squared = 9.0354, df = 1, p-value = 0.002648
alternative hypothesis: true common odds ratio is not equal to 1

95 percent confidence interval:
 0.03864222 0.56807600
sample estimates:
common odds ratio
        0.1481611
```

The difference between the test statistics (9.356 versus 9.035) is that `mantelhaen.test` uses the conditional *hypergeometric variance* rather then unconditional variance.

(c) Calculate the "one-step" estimate of the odds ratio based on the score function and compare to the marginal odds ratio from the collapsed table (two right-hand columns in the table above).

From above, the one-step estimate of β, the log-odds ratio is

```
> 5.629/3.399
[1] 1.656075
```

The one-step estimate of the odds ratio is

```
> exp(5.629/3.399)
[1] 5.23871
```

Note that the function `mantelhaen.test` calculates the reciprocal of this odds ratio,

```
>   1/0.1481611
[1] 6.74941
```

which is quite a bit larger than the one-step estimate. Compared to the smoking women example used in class, the one-step estimate in this example is further from the MLE primarily because the MLE is further from the null value, and the quadratic approximation begins to break down.

3. Suppose that we observe two random variables X_1, X_2, each with a Poisson distribution and mean parameters λ_1 and λ_2 respectively. For $i = 1, 2$, let

$$\theta_i = \log \lambda_i = \alpha + \beta z_i$$

where $z_1 = 0$ and $z_2 = 1$. Suppose that we observe $X_1 = x_1$ and $X_2 = x_2$ ($x_i > 0$).

(a) Compute the maximum likelihood estimate of α under the null hypothesis $H_0\colon \beta = 0$.

The density function for a Poisson random variable is $f(x; \lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, so with two RVs, the joint log-likelihood is

$$\ell(\alpha, \beta) = x_1\alpha - e^{\alpha} + x_2(\alpha + \beta) - e^{\alpha+\beta} + \text{stuff}$$

So

$$\begin{aligned} U_\alpha(\alpha, \beta) &= x_1 - e^\alpha + x_2 - e^{\alpha+\beta} \\ U_\beta(\alpha, \beta) &= x_2 - e^{\alpha+\beta} \end{aligned}$$

Under $H_0\colon \beta = 0$, we solve

$$\begin{aligned} 0 &= U_\alpha(\alpha, 0) \\ &= x_1 - e^\alpha + x_2 - e^\alpha \end{aligned}$$

so $\hat{\alpha}_0 = \log \dfrac{x_1 + x_2}{2}$.

(b) Derive the score test for $H_0\colon \beta = 0$.

$$U_\beta(\hat{\alpha}_0, 0) = x_2 - e^{\hat{\alpha}_0} = x_2 - \frac{x_1 + x_2}{2} = \frac{x_2 - x_1}{2}$$

The Fisher information is

$$I_\beta(\hat{\alpha}_0, 0) = \mathrm{Var}\left[U_\beta(\hat{\alpha}_0, 0)\right] = \frac{\mathrm{Var}\left[x_1\right] + \mathrm{Var}\left[x_2\right]}{4}$$

Under H_0, $\mathrm{Var}\left[x_1\right] = \mathrm{Var}\left[x_2\right] = \lambda$, where λ is the common Poisson rate, e^α. Plugging in $\hat{\alpha}_0$,

$$I_\beta(\hat{\alpha}_0, 0) = \frac{x_1 + x_2}{4}$$

Hence the score test is

$$\frac{(x_2 - x_1)^2}{x_2 + x_1}.$$

(c) If $p = \lambda_2/(\lambda_1 + \lambda_2)$, the conditional distribution for X_2 given the sum $m = X_1 + X_2$ is binomial with size m and probability p.

$$f(x_2; p) = \binom{m}{x_2} p^{x_2} (1-p)^{m-x_2}.$$

Derive the score test for $H_0\colon \beta = 0$ using the conditional distribution.

Under $H_0\colon \beta = 0$, $p_0 = 1/2$. Using the results derived in class, the score test is

$$\frac{(x_2 - mp_0)^2}{mp_0(1-p_0)} = \frac{(x_2 - (x_1 + x_2)/2)^2}{(x_1 + x_2)/4} = \frac{(x_2 - x_1)^2}{x_2 + x_1}$$

exactly the same as the unconditional test.

(d) Suppose that in a two-armed randomized trial with equal numbers of subjects per arm we observe 65 and 49 deaths in groups A and B respectively. Assuming that the numbers of deaths follow a Poisson distribution, perform the score test for the null hypothesis that the death rates are the same in the two arms.

Using the above, the score chi-square statistic is

$$\frac{(65-49)^2}{65+49} = \frac{16^2}{114} = 2.25$$

At $\alpha = 0.05$, the critical value is 3.84, so we would not reject H_0

(e) Suppose further that the population in the previous part has two strata, one high risk and one low risk. If the numbers of deaths are 51 and 14 in the high and low risk strata respectively in arm A and 37 and 12 in the high and low risk strata respectively in arm B, perform the stratified score test for the null hypothesis that the death rates are the same in the two arms.

In the high risk stratum, we have

$$U_\beta(\hat{\alpha}, 0) = (51-37)/2 = 14/2, \qquad I_\beta(\hat{\alpha}, 0) = (51+37)/4 = 88/4.$$

and in the high risk stratum, we have

$$U_\beta(\hat{\alpha}, 0) = (14-12)/2 = 2/2, \qquad I_\beta(\hat{\alpha}, 0) = (14+12)/4 = 26/4.$$

The stratified test statistic is

$$\frac{(14+2)^2/4}{(88+26)/4} = \frac{16^2}{114} = 2.25$$

exactly the same as the unstratified test. For Poisson data, stratification does not change the test.

(Note that this procedure gives a "quick and dirty" way of assessing the statistical significance of event rates between two equal sized groups.)