How much work is done on an airplane travelling through a hurricane?

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Introduction – Rationale & Aim

In Physics class we started studying Topic 5: “Electricity” and as a class tried a simulation by PHET Colorado[1] (Figure 1). The simulation was called “Electric Field Hockey”, where you place positive and negative charges to attract and repel a particle past some barriers and score a goal. Although this was very fun to play, there was a feature tucked away with a button at the bottom that enabled “Field”. This in turn added arrows to the board and it was immediately clear what those arrows represented. It was the force vector that would be acting on the particle if it landed on that spot. I became fascinated with it and spent an hour after class playing around with the simulation.

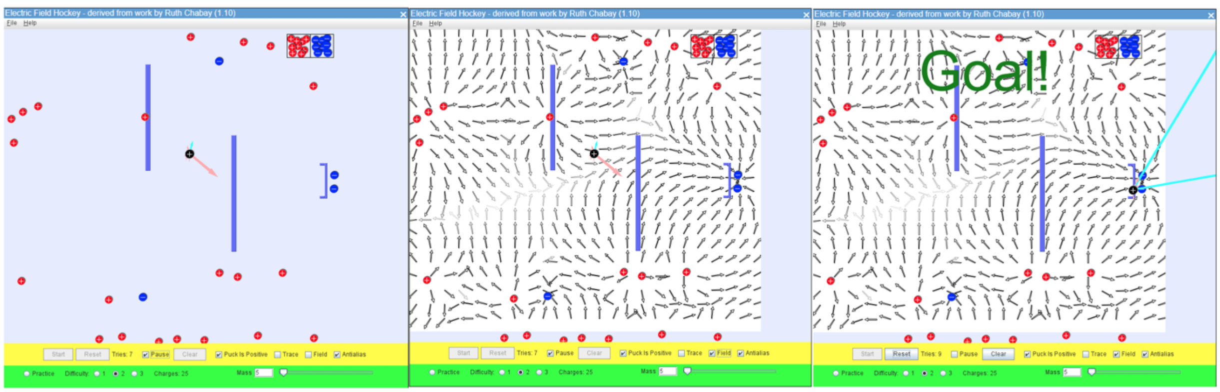

Figure 1: Screenshots from the PHET Colorado simulation

A few weeks later, in Math class, a similar image came up on the board and I was again absorbed by it. I knew I recognized it from the magnetic fields we had just learnt, so after school I researched the topic and learnt it was known as ‘Vector Fields’. Through this research, I realized it could not only be used to describe magnetism, electrostatics and more, but it could also be used to describe fluid flow such as wind, which had fascinated me before with equations such as Navier-Stokes.

Increasingly thrilled, I found out many winds were in fact described using vector fields, even in everyday life such as weather forecasts. The ability to describe all the forces acting on an object in a field fascinated me and I immediately made a connection between this idea and its application with airplanes, especially with how the forces acting on them could be described with

[1] https://phet.colorado.edu/en/simulations/electric-hockey

a vector field. I think everyone has always found hurricanes scary but, in some way, also captivating. So, I thought being able to describe the effects of a hurricane on an airplane would be quite interesting. After all, could an airplane use hurricanes as a natural boost? (If of course it could withstand it). In this exploration I aim to find out how much work a hurricane's wind does on an airplane travelling through it.

Background Theory

Vector field

Given that a vector is an object that has a magnitude and direction, a vector field F in $\mathbb{R}^N$ is a function that assigns to each point $(x, y, z, \dots)$ an Nth-dimensional vector given by $\vec{F}(x, y, z, \dots)$ (Caculus Volume 3 - 38 Vector Fields , n.d.) (Dawkins, 2018). In this case, I will be using a 2-dimensional vector field for a rough approximation. This means the vector fields will assign each point (x, y) a 2-dimensional vector given by $\vec{F}(x, y)$. Vector fields are commonly written with cartesian rectangular unit vector notation.

Because of my interest in computer science, I like to think of them as functions for vectors. You can input the coordinates $(x, y, z, \dots)$ of a point and get the vector at that particular point in the vector field.

Vector fields are usually represented as a diagram with arrows at each coordinate pointing in the direction of the vector direction. Making the arrows the size of its magnitude would result in a very messy diagram and so the arrows are usually scaled down relative to the other arrows' magnitude. An example is shown with Figure 2 below:

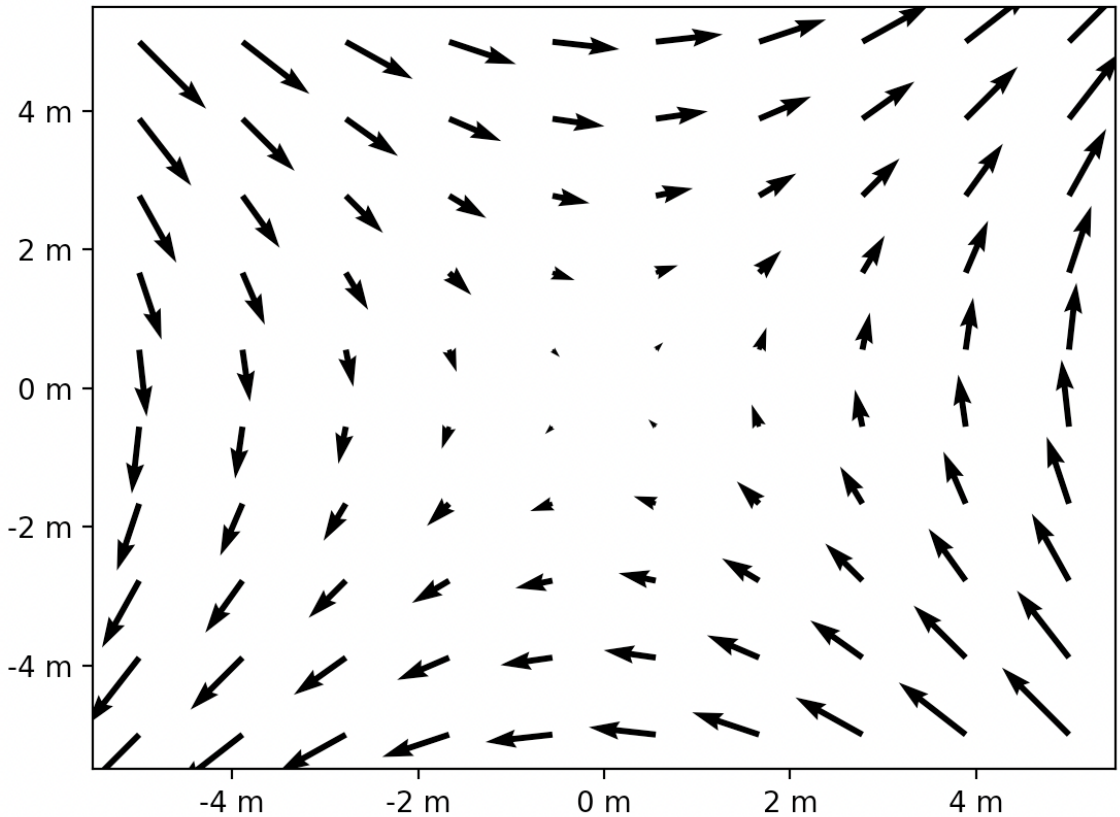

Figure 2: Diagram of field vector $\vec{F}(x, y) = (y)\mathbf{i} + (x)\mathbf{j}$ plotted using python code found in appendix 1.

For this essay, I will plot the vector fields with python code I wrote. These plots could also be done by hand through sampling the x and y of each point in the field. For example: at $(-4, -4)$, there would be vector:

$$-4\mathbf{i} - 4\mathbf{j}$$

This can be seen in Figure 2 above.

Work done

Work done given by Newtonian physics is defined by Equation 1 as follows:

$$Work\ done\ (W) = Force\ (F) \cdot Displacement\ (s) \cdot \cos\theta$$

Equation 1: θ is the angle between the force and the displacement

This is usually simplified to $Work\ done = Force\ \times Displacement$, due to θ being 0 as the work being done is in the same direction as the force. In this case, as the work being done is in the same direction as the force, it will also be simplified to

$$W = F \times s$$

Line integrals

Determining a line integral is when a vector field is integrated along a curve. "Intuitively this is summing up all vector components in line with the tangents to the curve, expressed as their scalar products." (Wikipedia, n.d.)

Line integrals to find work done

Since

$$W = F \times s$$

We can therefore deduce that "the dot product of a force vector and a displacement vector tells us how much work the force did on the object as it moved from the tail of its displacement vector to the tip" (Schlicker, Keller, & Long)

Hence, giving

$$W = \int_C \vec{F} \cdot d\vec{s}$$

Equation 2: line integral for work done (whitman. edu), *(Corral, 2009)*

Answering the Question

To start, online, I found the vector field:

$$\vec{F}(\mathrm{x},\mathrm{y}) = \left(\frac{y}{x^2+y^2}\right)\mathbf{i} - \left(\frac{x}{x^2+y^2}\right)\mathbf{j}$$

Equation 3 (Caculus Volume 3 - 38 Vector Fields , n.d.)

This vector field equation is similar to a hurricane as the vectors in the center have a bigger magnitude than the vectors on the outside, so I'll use it as the model for the hurricane. This vector field can be visualized with Figure 3 below:

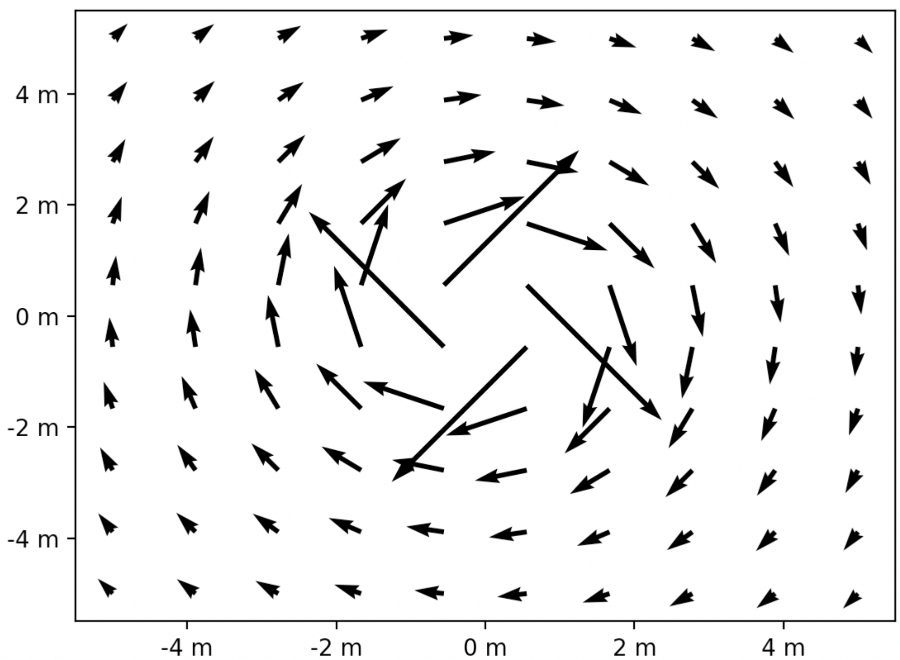

Figure 3: Diagram of force vector field $\vec{F}(x,y) = \left(\frac{y}{x^2+y^2}\right)\mathbf{i} - \left(\frac{x}{x^2+y^2}\right)\mathbf{j}$ plotted using python code found in appendix 2.

This model is obviously an oversimplification and doesn't even take into account the eye of the storm having no wind; however, it might be a good first estimate as to how to describe a hurricane. Using line integration, we can find the work done on the airplane with formula

$$W = \int_C \vec{F} \cdot d\vec{s}$$

Derived from the equation

$$W = F \times S$$

If we want the airplane to go from point (-4, -4) to (-4,4) we can first represent the path we'll take using curve

$$C: x = -4$$

Restricted at

$$-4 \leq y \leq 4$$

This can be converted to parametric form with variable t

$$C: x(t) = -4$$
$$y(t) = t$$

Hence also restricted at

$$-4 \leq t \leq 4$$

We can then represent this with cartesian rectangular unit vector notation as vector

$$\vec{s}(t) = (-4)\mathbf{i} + (\mathbf{t})\,\mathbf{j}$$

Next, to get $d\vec{s}$, we can differentiate to get

$$\frac{d\vec{s}}{dt} = \vec{s}'(t) = 0\mathbf{i} + \mathbf{j}$$
$$d\vec{s} = (0)\mathbf{i} + (dt)\mathbf{j}$$

The force vector field

$$\vec{F}(\mathrm{x},\mathrm{y}) = \left(\frac{y}{x^2 + y^2}\right)\mathbf{i} - \left(\frac{x}{x^2 + y^2}\right)\mathbf{j}$$

Can then also be represented parametrically using variable t, as force vector field by substituting y as t and x as -4

$$\vec{F}(t) = \left(\frac{(\mathrm{t})}{(-4)^2 + (\mathrm{t})^2}\right)\mathbf{i} - \left(\frac{(-4)}{(-4)^2 + (t)^2}\right)\mathbf{j}$$
$$\vec{F}(t) = \left(\frac{\mathrm{t}}{16 + \mathrm{t}^2}\right)\mathbf{i} + \left(\frac{4}{16 + t^2}\right)\mathbf{j}$$

Again, using the line integral for work done formula

$$W = \int_C \vec{F} \cdot d\vec{s}$$

We can substitute the values we have

$$W = \int_{t=-4}^{t=4} \left(\left(\frac{\mathrm{t}}{16 + \mathrm{t}^2}\right)\mathbf{i} + \left(\frac{4}{16 + t^2}\right)\mathbf{j}\right) \cdot \left((0)\mathbf{i} + (dt)\mathbf{j}\right)$$

And the dot product of these two vectors gives

$$W = \int_{t=-4}^{t=4} \left(0 + \frac{4}{16 + t^2}\right) dt$$

Simplifying

$$W = \int_{-4}^{4} \frac{4}{16 + t^2} dt$$

$$W = \left[\tan^{-1}\left(\frac{t}{4}\right)\right] \begin{matrix} 4 \\ -4 \end{matrix}$$

$$W = \left[\tan^{-1}\left(\frac{4}{4}\right)\right] - \left[\tan^{-1}\left(\frac{-4}{4}\right)\right]$$

$$W = \left(\frac{\pi}{4}\right) - \left(-\frac{\pi}{4}\right)$$

$$W = \frac{\pi}{2}\ joules$$

This, however, obviously was not what I was expecting; it can't only have given $\frac{\pi}{2}\ joules$. This made me realize I hadn't taken a lot into consideration: I hadn't calculated the work done on an airplane flying through a hurricane but instead a particle through this specific force field. To produce an answer to the question I had posed, I would have to collect rough real-life estimates for hurricane and plane values. I wasn't able to find any premade force fields for hurricanes, so instead I needed to fit my original vector field to match estimated wind velocities. Then I would be able to turn that into a force field.

First, I needed to change my plot range from (-4, 4) to (-200000m, 200000m), as hurricanes are, on average, a bit bigger than 400 km in diameter (US National Weather Service), hence 200 km in radius. Next, since I was going to calculate how much work would be done on an airplane flying through the edges of a hurricane, I would have to find realistic estimates of hurricane wind speeds. On average, I found wind speeds at around 200 km from the eye of the storm to be at around $15ms^{-1}$ (ivo, 2014). To adjust my velocity field to fit this size, I added the constant $\boldsymbol{a}$ to the original equation so I could scale up the model and set out the following equation:

$$\left|\vec{V}(\mathrm{x}, \mathrm{y})\right| = \sqrt{\left(\frac{ay}{x^2 + y^2}\right)^2 + \left(\frac{ax}{x^2 + y^2}\right)^2}$$

Thus, substituting real-life values and rearranging to get $\boldsymbol{a}$:

$$\left|\vec{V}(-2\cdot 10^5, 0)\right| = 15 = \sqrt{\left(\frac{a\cdot(0)}{(-2\cdot 10^5)^2+(0)^2}\right)^2 + \left(\frac{a\cdot(-2\cdot 10^5)}{(-2\cdot 10^5)^2+(0)^2}\right)^2}$$

$$15 = \sqrt{0+\left(\frac{(-2\cdot 10^5)\cdot a}{(-2\cdot 10^5)^2+0}\right)^2}$$

$$15 = \sqrt{\left(\frac{(-2\cdot 10^5)\cdot a}{(-2\cdot 10^5)^2}\right)^2}$$

$$15 = \pm\frac{(-2\cdot 10^5)\cdot a}{(-2\cdot 10^5)^2}$$

$$a = \pm 15\cdot(-2\cdot 10^5)$$

I found the value of $\boldsymbol{a}$:

$$a = \pm 3\cdot 10^6$$

Seeing as we want a clockwise rotation:

$$a = 3\cdot 10^6$$

Substituting back into the original equation I set out, I get

$$\vec{V}(\mathrm{x},\mathrm{y}) = \left(\frac{(3\cdot 10^6)\cdot y}{x^2+y^2}\right)\mathbf{i} - \left(\frac{(3\cdot 10^6)\cdot x}{x^2+y^2}\right)\mathbf{j}$$

To see the scope of the model's accuracy, I tested it at the center, just outside of the eye, where the strongest winds should be. I substituted the values:

$$\left|\vec{V}(-2\cdot 10^4, 0)\right| = \sqrt{\left(\frac{(3\cdot 10^6)\cdot(0)}{(-2\cdot 10^4)^2+(0)^2}\right)^2 + \left(\frac{(3\cdot 10^6)\cdot(-2\cdot 10^4)}{(-2\cdot 10^4)^2+(0)^2}\right)^2}$$

and simplifying, I got

$$\left|\vec{V}(-2\cdot 10^4, 0)\right| = 150\, ms^{-1}$$

In reality though, it should be closer to ~46 ms^{-1} (ivo, 2014), showing that this model doesn't work very well for the center. This is likely because it's not a real hurricane model and doesn't have the same change in magnitude between the center and edges as a real hurricane would. However, seeing as the path I've set is only near the edges, this shouldn't be too much of a problem; the scope has been reduced and this issue should therefore be insignificant. If I wanted to find a different route, I would have to readjust the model to fit my path. This might be tricky if

I were to want a path that went through a very big change in winds such as diagonally or closer to the center. Figure 4 below, is the adjusted vector field:

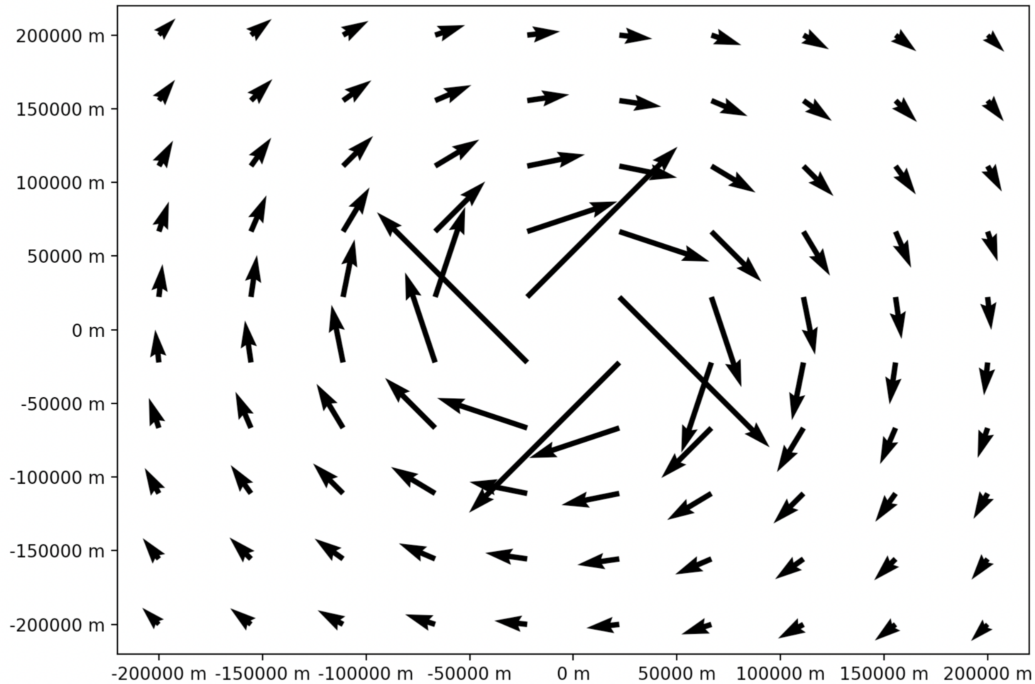

Figure 4: Diagram of velocity vector field of a hurricane $\vec{V}(x,y) = \left(\frac{(3\cdot10^6)\cdot y}{x^2+y^2}\right)\mathbf{i} - \left(\frac{(3\cdot10^6)\cdot x}{x^2+y^2}\right)\mathbf{j}$ *plotted using python code found in appendix 3*

As I said earlier, next, I needed to turn the velocity field into a force field; this is because to find the work done, I need to find the force. To do this I found the general formula for the drag force of wind:

$$F_d = \frac{1}{2}\rho u^2 c_d A$$

Equation 4: The Drag Equation (NASA)

$\rho: Air\ Density, u: Velocity, c_d: Drag\ Coefficient, A: Area$

Then I found estimates of each constant.

Constant $\frac{1}{2}\rho = 0.613$ for Air

(V, 2019)

Constant $c_d = 0.031$ for a Boeing 747

(Scott, 2004)

Constant A = 158.3 m^2 for a Boeing 747

(Withers, 2012)

Hence,

$$F_d = 3.006 \cdot u^2$$

First, I tried substituting

$$\vec{V}(\mathrm{x}, \mathrm{y}) = \left(\frac{(3 \cdot 10^6) \cdot y}{x^2 + y^2}\right)\mathbf{i} - \left(\frac{(3 \cdot 10^6) \cdot x}{x^2 + y^2}\right)\mathbf{j}$$

As $\boldsymbol{u}$ (because $\vec{V}(\mathrm{x}, \mathrm{y})$ represents the wind velocity) resulting in

$$\overrightarrow{F_d}(x, y) = 3.006 \cdot \left(\left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)\mathbf{i} - \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)\mathbf{j}\right)^2$$

$$\overrightarrow{F_d}(x, y) = 3.006 \cdot \left(\left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)^{\mathbf{2}}\mathbf{i} - \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)^{\mathbf{2}}\mathbf{j}\right)$$

This however plotted the force field in Figure 5, below:

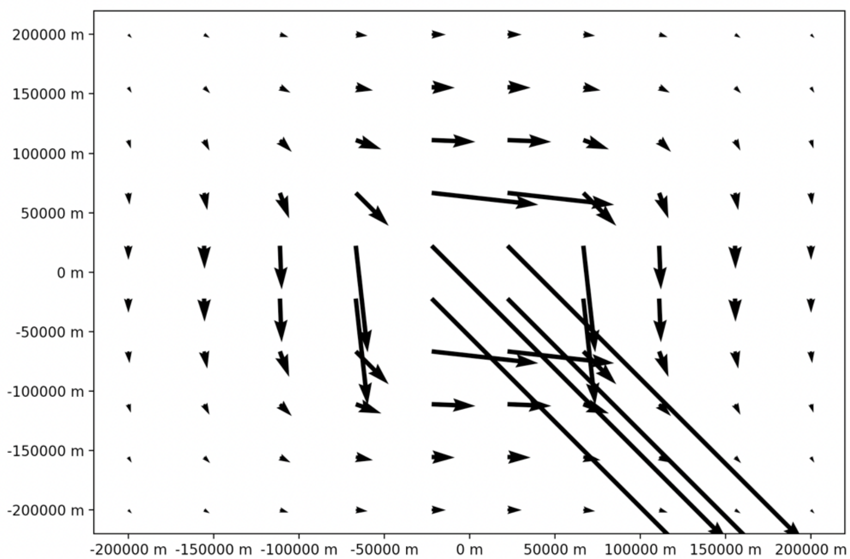

Figure 5: Diagram of force vector field of a hurricane $\overrightarrow{F_d} = 3.006 \cdot \left(\left(\frac{(3 \cdot 10^6)y}{x^2+y^2}\right)^{\mathbf{2}}\mathbf{i} - \left(\frac{(3 \cdot 10^6)x}{x^2+y^2}\right)^{\mathbf{2}}\mathbf{j}\right)$ *plotted using python code found in appendix 4*

This was very different to what I was expecting; in theory and intuitively the force and velocity should be in the same direction, so I hypothesized this was because while squaring it, the direction was lost.

To remedy this, I turned

$$\vec{V}(\mathrm{x,y}) = \left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)\mathbf{i} - \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)\mathbf{j}$$

Into polar form

$$\left|\vec{V}(x,y)\right| = \sqrt{\left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)^2 + \left(\frac{-(3 \cdot 10^6)x}{x^2 + y^2}\right)^2}$$

$$\tan\theta = \frac{\left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)}{\left(\frac{-(3 \cdot 10^6)y}{x^2 + y^2}\right)}$$

$$\tan\theta = -\frac{x}{y}$$

$$\theta = \tan^{-1}\left(-\frac{x}{y}\right)$$

$$\vec{V}(\mathrm{x,y}) = \langle\sqrt{\left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)^2 + \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)^2} \cdot \boldsymbol{cis}\left(\tan^{-1}\left(-\frac{x}{y}\right)\right)\rangle$$

Again, substituting

$$\vec{V}(\mathrm{x,y}) = \langle\sqrt{\left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)^2 + \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)^2} \cdot \boldsymbol{cis}\left(\tan^{-1}\left(-\frac{x}{y}\right)\right)\rangle$$

As $\boldsymbol{u}$, in

$$F_d = 3.006 \cdot u^2$$

Gives

$$\overrightarrow{F_d}(x,y) = 3.006 \cdot \left(\langle\sqrt{\left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)^2 + \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)^2} \cdot \boldsymbol{cis}\left(\tan^{-1}\left(-\frac{x}{y}\right)\right)\rangle\right)^2$$

Here, the magnitude can be squared individually without affecting the direction and therefore

$$\overrightarrow{F_d}(x,y) = 3.006 \cdot \left(\langle\left(\sqrt{\left(\frac{(3 \cdot 10^6)y}{x^2 + y^2}\right)^2 + \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2}\right)^2}\right)^{\mathbf{2}} \cdot \boldsymbol{cis}\left(\tan^{-1}\left(-\frac{x}{y}\right)\right)\rangle\right)$$

$$\overrightarrow{F_d}(x,y) = 3.006 \cdot \left(\langle \left(\left(\frac{(3 \cdot 10^6)y}{x^2+y^2} \right)^2 + \left(\frac{(3 \cdot 10^6)x}{x^2+y^2} \right)^2 \right) \cdot \boldsymbol{cis} \left(\tan^{-1}\left(-\frac{x}{y}\right) \right) \rangle \right)$$

Plotting $\overrightarrow{F_d}(x,y)$ results in Figure 6 below:

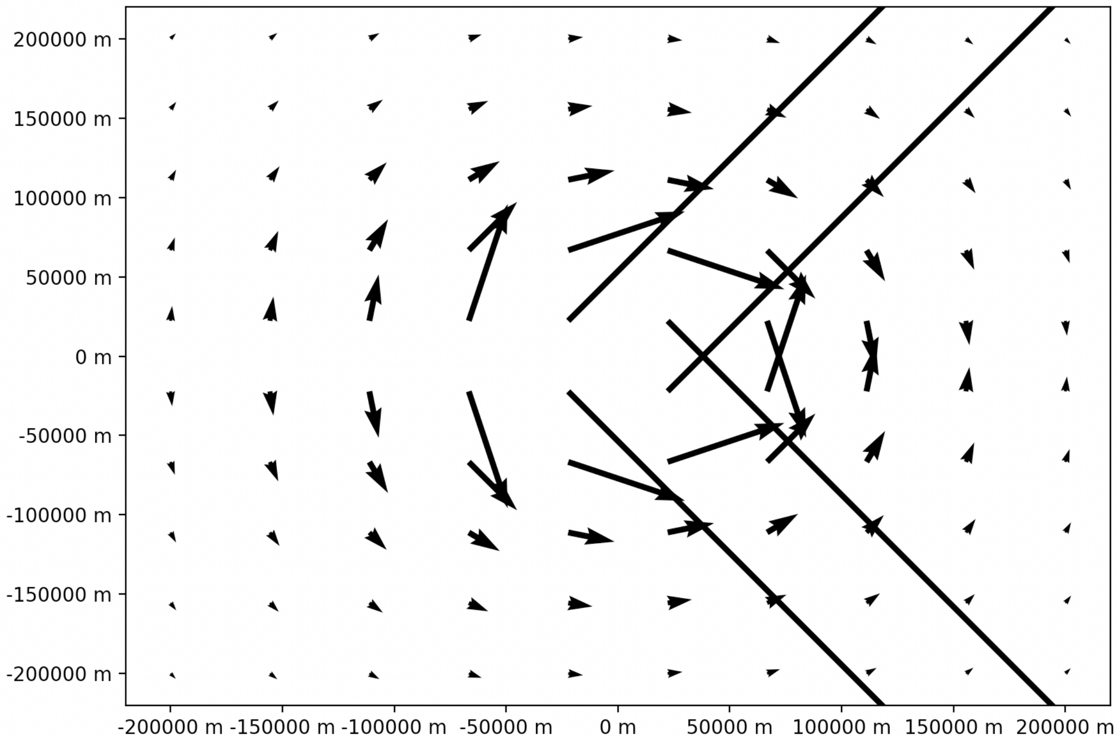

Figure 6:Diagram of force vector field of a hurricane $\overrightarrow{F_d}(x,y) = 3.006 \cdot \left(\langle \left(\left(\frac{(3 \cdot 10^6)y}{x^2+y^2} \right)^2 + \left(\frac{(3 \cdot 10^6)x}{x^2+y^2} \right)^2 \right) \cdot \boldsymbol{cis} \left(tan^{-1}\left(-\frac{x}{y}\right) \right) \rangle \right)$ *plotted using python code found in appendix 5*

While this is better than Figure 5, this still isn't what I was expecting. It seems that the bottom half is in the wrong direction. After a lot of headaches and research, I found that the reason for the problem was the regular arctangent is flawed in that it can't distinguish between vectors that have diametrically opposite directions. To solve this, programmers created a solution with the function $atan2(y,x)$ first appearing in Fortran. Programmers had also had trouble converting from cartesian form to polar form and this function therefore took this into account. For now, I will continue using $\tan^{-1}\left(\frac{-x}{y}\right)$ in the mathematical equations and $atan2(-x,y)$ when writing the code to plot the vector fields.

Using $atan2(-x, y)$ instead of $\tan^{-1}\left(\frac{-x}{y}\right)$, I was able to produce Figure 6, the following plot:

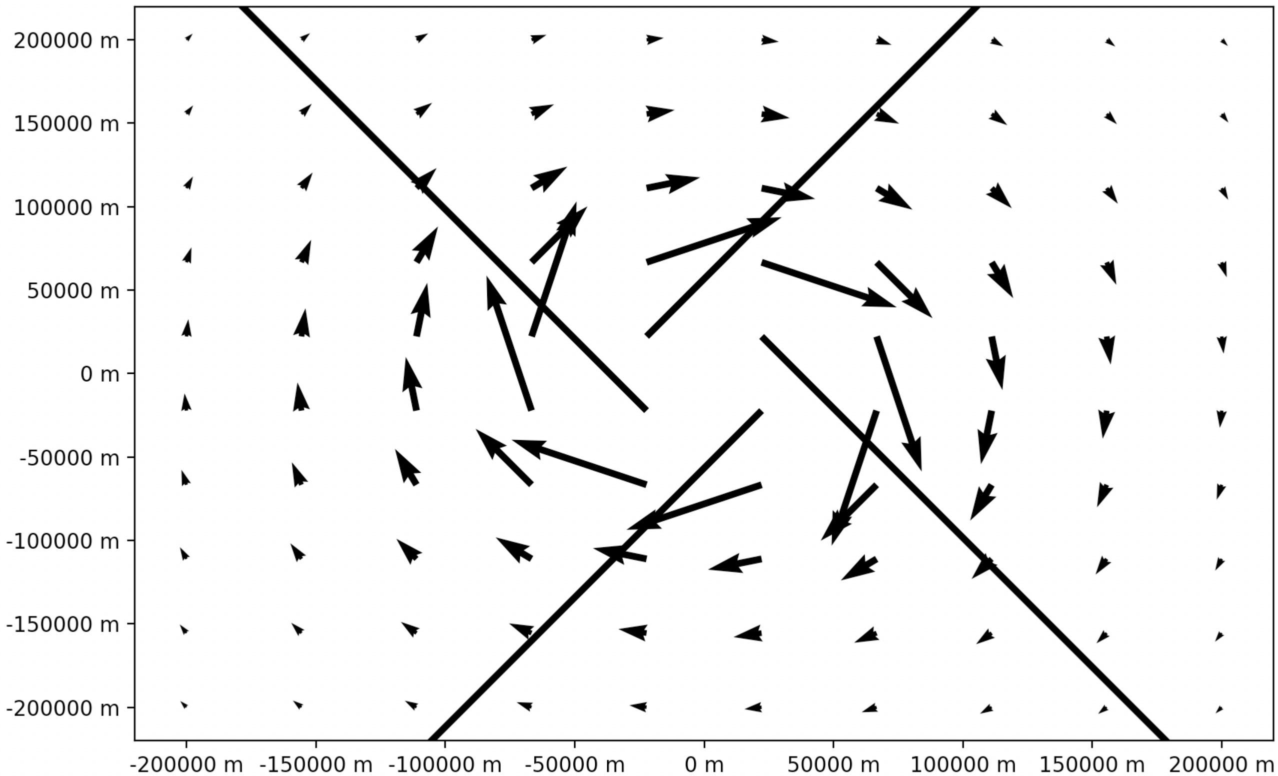

Figure 7: Diagram of force vector field of a hurricane $\overrightarrow{F_d}(x,y) = 3.006 \cdot \left(\langle\left(\left(\frac{(3\cdot 10^6)y}{x^2+y^2}\right)^2 + \left(\frac{(3\cdot 10^6)x}{x^2+y^2}\right)^2\right) \cdot \boldsymbol{cis}\left(tan^{-1}\left(-\frac{x}{y}\right)\right)\rangle\right)$ *plotted using python code found in appendix 6*

This looks much better and fits what I originally expected it to look like.

Next, if we want the airplane to go from point $(-2 \cdot 10^5, -2 \cdot 10^5)$ to $(-2 \cdot 10^5, 2 \cdot 10^5)$ we can use curve

$$C: x = -2 \cdot 10^5$$

Restricted at

$$-2 \cdot 10^5 \le y \le 2 \cdot 10^5$$

This can again be converted to parametric form with variable t

$$C: x(t) = -2 \cdot 10^5$$

$$y(t) = t$$

Restricted at

$$-2 \cdot 10^5 \le t \le 2 \cdot 10^5$$

This can be represented with cartesian rectangular unit vector notation as vector

$$\vec{s}(t) = (-2 \cdot 10^5)\mathbf{i} + (\mathbf{t})\,\mathbf{j}$$

Differentiating, we get

$$\frac{d\vec{s}}{dt} = \vec{s}'(t) = 0\mathbf{i} + \mathbf{j}$$

$$d\vec{s} = (0)\mathbf{i} + (dt)\mathbf{j}$$

The force vector field

$$\overrightarrow{F_d}(x, y) = 3.006 \cdot \left(\langle \left(\frac{(3 \cdot 10^6)y}{x^2 + y^2} \right)^2 + \left(\frac{(3 \cdot 10^6)x}{x^2 + y^2} \right)^2 + \boldsymbol{cis}\left(\tan^{-1}\left(-\frac{x}{y} \right) \right) \rangle \right)$$

Can then also be represented parametrically using variable t, as force vector field

$$\overrightarrow{F_d}(t) = 3.006 \cdot \left(\langle \left(\left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2 \right) + \boldsymbol{cis}\left(\tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right) \right) \rangle \right)$$

This can be converted from polar form back to cartesian rectangular unit vector form with

$$x = r \cdot \cos(\theta)$$

$$y = r \cdot \sin(\theta)$$

In this case

$$r = \left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2$$

$$\theta = \tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right)$$

Hence,

$$\overrightarrow{F_d}(t) = 3.006 \cdot \left(\left(\left(\left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2 \right) \cdot cos\left(\tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right) \right) \right) \mathbf{i} + \left(\left(\left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2 \right) \cdot sin\left(\tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right) \right) \right) \mathbf{j} \right)$$

and

$$d\vec{s} = (0)\mathbf{i} + (dt)\mathbf{j}$$

Again, using the line integral for work done formula

$$W = \int_C \vec{F} \cdot d\vec{s}$$

We can substitute the values we have

$$W = 3.006 \cdot \int_{t=-2\cdot 10^5}^{t=2\cdot 10^5} \left[\left(\left(\left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2 \right) \cdot cos\left(\tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right) \right) \right) \mathbf{i} + \left(\left(\left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2 \right) \cdot sin\left(\tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right) \right) \right) \mathbf{j} \right] \cdot \left((0)\mathbf{i} + (dt)\mathbf{j} \right)$$

And finding the dot product, we get

$$W = 3.006 \cdot \int_{t=-2\cdot10^5}^{t=2\cdot10^5} \left(\left(\left(\left(\frac{(3\cdot10^6)t}{(-2\cdot10^5)^2+t^2} \right)^2 + \left(\frac{(3\cdot10^6)\cdot(-2\cdot10^5)}{(-2\cdot10^5)^2+t^2} \right)^2 \right) \cdot sin\left(\tan^{-1}\left(-\frac{(-2\cdot10^5)}{t} \right) \right) \mathbf{i} \right) \mathrm{dt}$$

$$W = 3.006 \cdot \int_{t=-2\cdot10^5}^{t=2\cdot10^5} \left(\left(\left(\left(\frac{(3\cdot10^6)t}{(-2\cdot10^5)^2+t^2} \right)^2 + \left(\frac{(3\cdot10^6)\cdot(-2\cdot10^5)}{(-2\cdot10^5)^2+t^2} \right)^2 \right) \cdot sin\left(\tan^{-1}\left(-\frac{(-2\cdot10^5)}{t} \right) \right) \right) dt$$

Integrating gives:

$$W = 3.006 \cdot \left[\frac{4.5\cdot10^7 \cdot \sqrt{\frac{4\cdot10^{10}}{t^2}+1} \cdot t^2}{t^2+4\cdot10^{10}} \right]_{-2\cdot10^5}^{2\cdot10^5}$$

And substituting to solve the area bounded by the integral:

$$W = 3.006 \cdot \left[\left[\frac{4.5\cdot10^7 \cdot \sqrt{\frac{4\cdot10^{10}}{2\cdot{10^5}^2}+1} \cdot 2\cdot{10^5}^2}{2\cdot{10^5}^2+4\cdot10^{10}} \right] - \left[\frac{4.5\cdot10^7 \cdot \sqrt{\frac{4\cdot10^{10}}{-2\cdot{10^5}^2}+1} \cdot -2\cdot{10^5}^2}{-2\cdot{10^5}^2+4\cdot10^{10}} \right] \right]$$

Simplifying, we get

$$W = 0$$

This again is not what I expected as there should obviously be some work done on the airplane; as seen in Figure 8 below, there is bounded area.

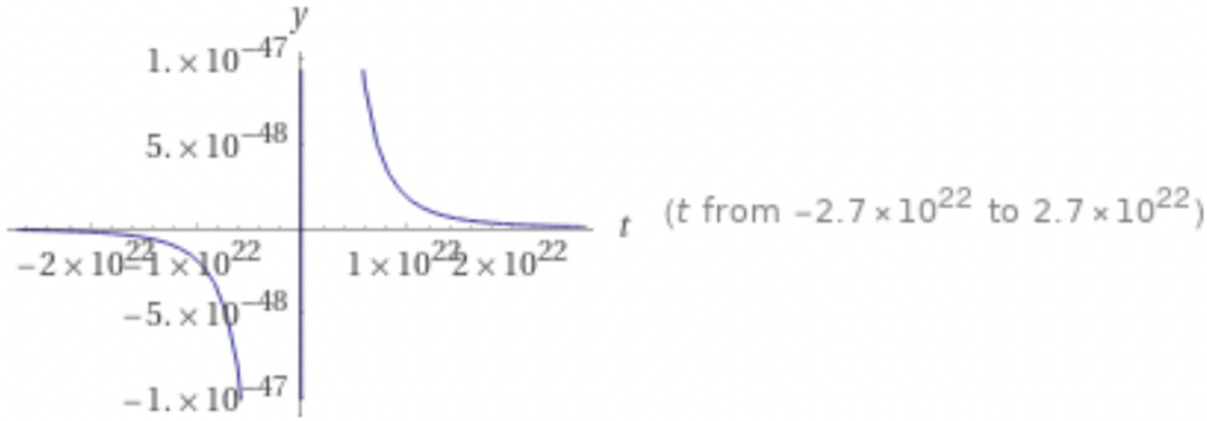

Figure 8: Equation $\left(\left(\frac{(-3\cdot10^6)t}{(-2\cdot10^5)^2+t^2}\right)^2+\left(\frac{(-3\cdot10^6)\cdot(-2\cdot10^5)}{(-2\cdot10^5)^2+t^2}\right)^2\right)\cdot sin\left(tan^{-1}\left(-\frac{(-2\cdot10^5)}{t}\right)\right)$ *plotted using WolframAlpha*

After some thought, I realized it was because the negative-x side of the function was under the x-axis leading to a negative area and the two areas cancelling. To solve this, I separated the integrals and followed the method in appendix 7, resulting in adding the two areas together:

$$W = 3.006 \cdot \left[\left[\frac{4.5\cdot10^7\cdot\sqrt{\frac{4\cdot10^{10}}{(2\cdot10^5)^2}+1}\cdot(2\cdot10^5)^2}{(2\cdot10^5)^2+4\cdot10^{10}}\right]\right.$$

$$\left.+\left[\frac{4.5\cdot10^7\cdot\sqrt{\frac{4\cdot10^{10}}{(-2\cdot10^5)^2}+1}\cdot(-2\cdot10^5)^2}{(-2\cdot10^5)^2+4\cdot10^{10}}\right]\right]$$

Simplifying, we get

$$W = 3.006\cdot(6.36396\cdot10^7)$$

$$W = 1.913006376 \times 10^8\ Joules$$

$$= 0.1913006376\ \ Gigajoules$$

$$\approx 0.191\ \ Gigajoules$$

Conclusion

To conclude, by modelling a hurricane's wind pattern and velocity with a vector field, I was able to calculate a force field which I then used, in conjunction with the path the airplane would take, to calculate the work done, on the airplane by the hurricane winds, with the line integral. I succeeded in my aim and the final number came out to be $\approx \boldsymbol{0.19\ Gigajoules}$ of energy (provided by the tail wind, seeing as it's in the same direction as the airplane). It's important to note that this does not include the energy produced by the airplane engines but only just the estimated theoretical energy a hurricane would provide as a 'boost' to the airplane.

This was obviously just an estimate of the energy provided to the airplane by the hurricane as many variables were not taken into account. The most important would probably be the aerodynamics of the airplane and the very rough estimates of each value, especially the wind velocities. Moreover, as this was only a 2D representation, this model isn't as accurate as a 3D model might be. Additionally, I only considered the y-axis force from the hurricane; in reality, because of the whirlwind-like pattern of the hurricane, there would also be an x-axis force to the left. Thus, while the plane would receive a helping hand from the hurricane as tailwind, it would also have to correct for the horizontal displacement.

For a future study it would be interesting to improve on this estimate by using another more realistic and possibly more complicated model to describe the hurricane's wind velocities; perhaps the hurricane's eye could be considered, or it could even be 3-Dimensional.

This math might not be extremely useful in the real world, considering there are probably also measuring devices that are located around the globe that can provide more accurate answers to this question at a specific location. However, this math is probably extremely useful in film CGI and video games: not only in, for example, flight simulators, but also any other game that might have realistic fluids such as a river that pushes the character around. This might however be too complicated for a computer to calculate in real time, unless the math is done before and it's a premade model, meaning the values only have to be inputted into a formula or equation. I'm very happy I took on this challenging topic because of all the techniques and different concepts I learnt about.

Appendix

Appendix 1: Python code

```
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.ticker import FormatStrFormatter
x,y = np.meshgrid(np.linspace(-5,5,10),np.linspace(-5,5,10))

u = y
v = x

plt.quiver(x,y,u,v)
plt.gca().xaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.gca().yaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.show()
```

Appendix 2: Python code

```
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.ticker import FormatStrFormatter
x,y = np.meshgrid(np.linspace(-5,5,10),np.linspace(-5,5,10))

u = (y/((x**2)+(y**2)))
v = -(x/((x**2)+(y**2)))

plt.quiver(x,y,u,v)
plt.gca().xaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.gca().yaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.show()
```

Appendix 3: Python code

```
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.ticker import FormatStrFormatter
x,y = np.meshgrid(np.linspace(-200000,200000,10),np.linspace(-200000,200000,10))

u = (3000000*y/((x**2)+(y**2)))
v = -(3000000*x/((x**2)+(y**2)))

plt.quiver(x,y,u,v)
plt.gca().xaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.gca().yaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.show()
```

Appendix 4: Python code

```
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.ticker import FormatStrFormatter
x,y = np.meshgrid(np.linspace(-200000,200000,10),np.linspace(-200000,200000,10))

u = 3.006*((3000000*y/((x**2)+(y**2)))**2)
v = -3.006*((3000000*x/((x**2)+(y**2)))**2)

plt.quiver(x,y,u,v)
plt.gca().xaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.gca().yaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.show()
```

Appendix 5: Python code

```
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.ticker import FormatStrFormatter
x,y = np.meshgrid(np.linspace(-200000,200000,10),np.linspace(-200000,200000,10))

u = 3.006*(((((-3000000)*y)/((x**2)+(y**2)))**2)+((((-3000000)*x)/((x**2)+(y**2)))**2)) * (np.cos((np.arctan(-x/y))))
v = 3.006*(((((-3000000)*y)/((x**2)+(y**2)))**2)+((((-3000000)*x)/((x**2)+(y**2)))**2)) * (np.sin((np.arctan(-x/y))))

plt.quiver(x,y,u,v)
plt.gca().xaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.gca().yaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.show()
```

Appendix 6: Python code

```
import matplotlib.pyplot as plt
import numpy as np
from matplotlib.ticker import FormatStrFormatter
x,y = np.meshgrid(np.linspace(-200000,200000,10),np.linspace(-200000,200000,10))

u = 3.006*(((((-3000000)*y)/((x**2)+(y**2)))**2)+((((-3000000)*x)/((x**2)+(y**2)))**2)) * (np.cos((np.arctan2(-x,y))))
v = 3.006*(((((-3000000)*y)/((x**2)+(y**2)))**2)+((((-3000000)*x)/((x**2)+(y**2)))**2)) * (np.sin((np.arctan2(-x,y))))

plt.quiver(x,y,u,v)
plt.gca().xaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.gca().yaxis.set_major_formatter(FormatStrFormatter('%d m'))
plt.show()
```

Appendix 7: Method for simplifying integral

$$W = 3.006 \cdot \int_{t=-2\cdot 10^5}^{t=0} \left(\left(\left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2 \right) \cdot sin\left(\tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right) \right) \right) dt +$$

$$3.006 \cdot \int_{t=-2\cdot 10^5}^{t=0} \left(\left(\left(\frac{(3 \cdot 10^6)t}{(-2 \cdot 10^5)^2 + t^2} \right)^2 + \left(\frac{(3 \cdot 10^6) \cdot (-2 \cdot 10^5)}{(-2 \cdot 10^5)^2 + t^2} \right)^2 \right) \cdot sin\left(\tan^{-1}\left(-\frac{(-2 \cdot 10^5)}{t} \right) \right) \right) dt$$

Integrating and substituting gives:

$$W = 3.006 \cdot \left[\left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{(0)^2} + 1 \cdot (0)^2}}{(0)^2 + 4 \cdot 10^{10}} \right] - \left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{(-2 \cdot 10^5)^2} + 1 \cdot (-2 \cdot 10^5)^2}}{(-2 \cdot 10^5)^2 + 4 \cdot 10^{10}} \right] \right.$$

$$\left. + \left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{(2 \cdot 10^5)^2} + 1 \cdot (2 \cdot 10^5)^2}}{(2 \cdot 10^5)^2 + 4 \cdot 10^{10}} \right] - \left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{0^2} + 1 \cdot 0^2}}{(0)^2 + 4 \cdot 10^{10}} \right] \right]$$

This results in

$$W = 3.006 \cdot \left[0 - \left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{(2 \cdot 10^5)^2} + 1} \cdot (2 \cdot 10^5)^2}{(2 \cdot 10^5)^2 + 4 \cdot 10^{10}} \right] + \left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{(-2 \cdot 10^5)^2} + 1} \cdot (-2 \cdot 10^5)^2}{(-2 \cdot 10^5)^2 + 4 \cdot 10^{10}} \right] - 0 \right]$$

As area has to be positive, we can simplify this to

$$W = 3.006 \cdot \left[\left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{(2 \cdot 10^5)^2} + 1} \cdot (2 \cdot 10^5)^2}{(2 \cdot 10^5)^2 + 4 \cdot 10^{10}} \right] + \left[\frac{4.5 \cdot 10^7 \cdot \sqrt{\frac{4 \cdot 10^{10}}{(-2 \cdot 10^5)^2} + 1} \cdot (-2 \cdot 10^5)^2}{(-2 \cdot 10^5)^2 + 4 \cdot 10^{10}} \right] \right]$$

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