Sequence

Given: Positive weights $w = (w_1, \ldots, w_n)$

Define: *weight* of index sequence $x = (x_1, \ldots, x_m)$ as

$$W(x) = w_{x_1} + \ldots + w_{x_m}$$

Define: index sequence $x = (x_1, \ldots, x_m)$ is *good* if $x = (1)$ or

$$x_j = \begin{cases} x_{j-1} + 1 & \textbf{or} \\ x_k \cdot x_l & \textbf{for some } k \le l < j \end{cases}$$

increment (indigo)

multiply (maroon)

Task: given w and index v, compute $\min_{x \in G:\ v \in x} W(x)$

<u>Good</u>:

1			
1	1		
1	2		
1	2	2	
1	2	3	
1	2	4	
1	2	4	5
1	2	4	8
1	2	4	16

<u>Not good</u>:

1 3

Small n: Systematically generate all *short good* sequences by following the two rules

Too slow: Systematically generate all 10^{10} index sequences of length at most 10 and check each for goodness.

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Task: given w and index v, compute $\min\limits_{x \in G:\, v \in x} W(x)$

Observation: $w_i > 0 \Rightarrow$

- Ignore x if it is "supersequence" of other $y \in G$
- Can assume x is *strictly* increasing (speeds up generation)
- Can assume v is *last* index x_m

Good:

1
1 1
1 2
1 2 2
1 2 3
1 2 4
1 2 4 5
1 2 4 8
1 2 4 16

1 2 3
1 2 4
1 2 2 4
1 2 4 5
1 2 3 4 5

Uniform weights

Given: Positive weights $w = (w_1, \ldots, w_n)$ $\boxed{(w_1 = \cdots = w_n)}$

Define: *weight* of index sequence $x = (x_1, \ldots, x_m)$ as

$$W(x) = w_{x_1} + \ldots + w_{x_m} \boxed{= w_1 \cdot |x|}$$

Define: index sequence $x = (x_1, \ldots, x_m)$ is *good* if $x = (1)$ or

$$x_j = \begin{cases} x_{j-1} + 1 & \textbf{or} \\ x_k \cdot x_l & \textbf{for some } k \leq l < j \end{cases}$$

increment (indigo)

multiply (maroon)

Task: given w and index v, compute $\min_{x \in G:\ v \in x} W(x)$ $\boxed{= w_i \cdot \min_{(x_1, \ldots, v) \in G} |x|}$

Good:

1
1 1
1 2
1 2 2
1 2 3
1 2 4
1 2 4 5
1 2 4 8
1 2 4 16

Simpler task: Find shortest good sequence ending in *v.*

v < 300**:** Just do it.

v < 1439**:** Sequence lengths are $|x| \leq 14$. Precompute *all* of them on your machine in an hour.

1 2 3
1 2 4
1 2 2 4
1 2 4 5
1 2 3 4 5

Intended solution

Good:

1
1 1
1 2
1 2 2
1 2 3
1 2 4
1 2 4 5
1 2 4 8
1 2 4 16

Given: Positive weights $w = (w_1, \dots, w_n)$

Define: *weight* of index sequence $x = (x_1, \dots, x_m)$ as

$$W(x) = w_{x_1} + \dots + w_{x_m}$$

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increment (indigo)

multiply (maroon)

Task: given w and index v, compute $\displaystyle\min_{x \in G:\ v \in x} W(x)$

generator *extend*$(x_1, \dots, x_j)$:
    yield$(x_1, \dots, x_j, x_j + 1)$
    for $k \in \{2, \dots, j\}$
        for $j \in \{k, \dots, j\}$
            yield$(x_1, \dots, x_j, x_k \cdot x_l)$

Generalise problem: for index subset $A \subseteq \{1, \dots, n\}$ define

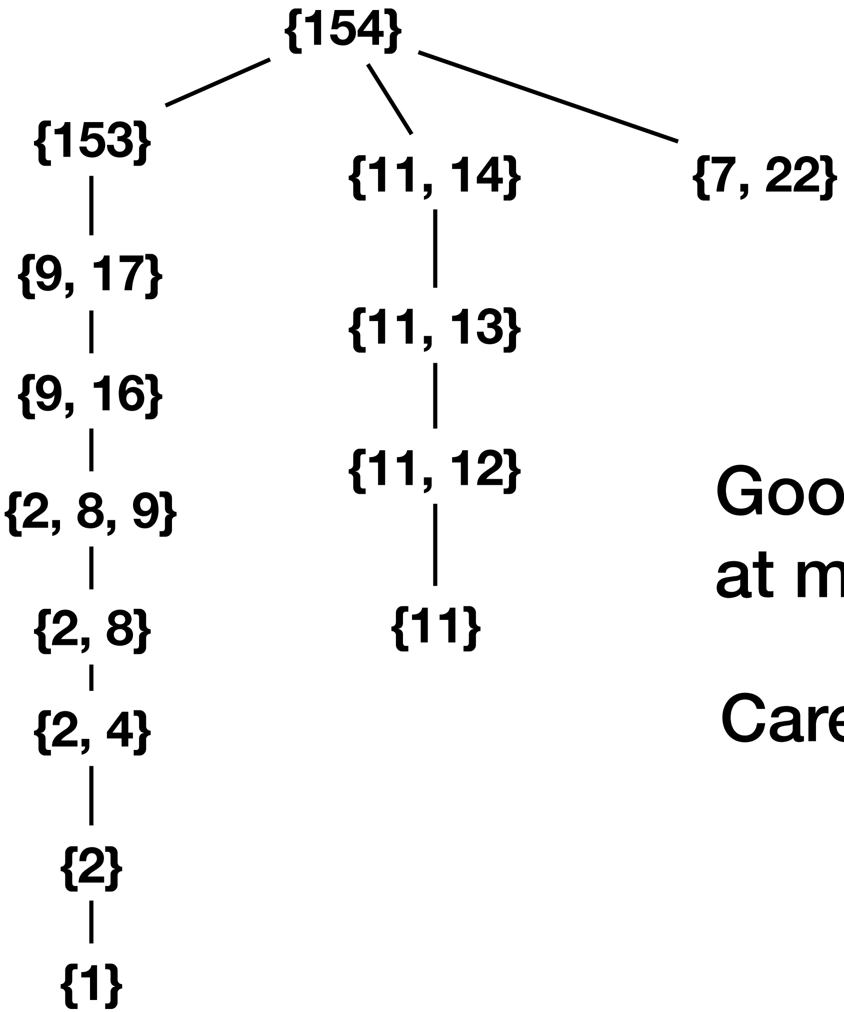
$$F(A) = \min_{x \in G:\ A \subseteq x} W(x)$$

Original problem: compute $F(\{v\})$.

$$F(A \cup \{a_k\}) = w_{a_k} + \min \begin{cases} F(A \cup \{a_k - 1\}) \\ \min_{1 < i \le j \le a_k:\ ij = a_k} F(A \cup \{i, j\}) \end{cases}$$

Why is this fast?

$$F(A \cup \{a_k\}) = w_{a_k} + \min \begin{cases} F(A \cup \{a_k - 1\}) \\ \min_{1 < i \leq j \leq a_k:\ ij = a_k} F(A \cup \{i, j\}) \end{cases}$$



Invariant

$$a_1 \cdot \cdots \cdot a_k \leq v \leq n$$

Good exercise (induction, calculus): at most poly(n) such sets

Careful analysis (Team Poland):

$$ne^{2 \cdot \sqrt{\log n}}$$