Artificial Intelligence
Project 2 – Kalman Filter

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1 State Space Model

For the state space model it is assumed that the airplane moves at constant velocity and that the airplanes state is position and velocity in 2D space. The radar that is tracking the airplane measures its position but not velocity and makes a measurement at every $\Delta t = 10s$ The measurement has a uniform standard deviation $\sigma_0 = 100m$.

It is assumed that the process is stationary and therefore the transition model, process noise, observation model and observation noise are not time dependent. There is no knowledge of the control input and therefore it is assumed that $u_k = 0$ and $B_k = 0$ for the Kalman filter equation. The model is therefore as follows:

$$x_k = Fx_{k-1} + w$$

$$w \sim \mathcal{N}(0,\, Q)$$

$$z_k = Hx_k + v$$

$$v \sim \mathcal{N}(0,\, R)$$

Since we are tracking position and velocity for x and y the state vector is:

$$x_k = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$$

To track state changes for the airplanes position the state transition matrix is:

$$F = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the radar only measures position the observation transformation matrix is:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Without the noise the predicted state is:

$$x_{k+1} = Fx_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ \dot{x_k} \\ \dot{y_k} \end{bmatrix} = \begin{bmatrix} x_k + \Delta t \\ y_k + \Delta t \\ \dot{x_k} \\ \dot{y_k} \end{bmatrix}$$

$$z_k = Hx_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ \dot{x_k} \\ \dot{y_k} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

For the process noise the co-variance matrix is:

$$Q = \begin{array}{c} \\ x \\ y \\ \dot{x} \\ \dot{y} \end{array} \begin{array}{c} \begin{array}{cccc} \bar{x} & \bar{y} & \bar{\dot{x}} & \bar{\dot{y}} \end{array} \\ \begin{bmatrix} \sigma_x^2 & 0 & \sigma_x\sigma_{\dot{x}} & 0 \\ 0 & \sigma_y^2 & 0 & \sigma_y\sigma_{\dot{y}} \\ \sigma_x\sigma_{\dot{x}} & 0 & \sigma_{\dot{x}}^2 & 0 \\ 0 & \sigma_y\sigma_{\dot{y}} & 0 & \sigma_{\dot{y}}^2 \end{bmatrix} \end{array}$$

When tracking the airplane there is no knowledge of its acceleration but it is assumed that max acceleration is around $3m/s^2$. Therefore acceleration in x and y direction can be thought of as independent mean centered Gaussian random variables with $\sigma_p = 1.5m/s^2$ which leads to following co-variances:

$$\sigma_x^2 = \sigma_y^2 = \frac{1}{4}\Delta t^4 \sigma_p^2$$

$$\sigma_{\dot{x}}^2 = \sigma_{\dot{y}}^2 = \Delta t^2 \sigma_p^2$$

$$\sigma_x\sigma_{\dot{x}} = \sigma_y\sigma_{\dot{y}} = \frac{1}{2}\Delta t^3 \sigma_p^2$$

all other co-variances are 0

The process noise the co-variance matrix is then:

$$Q = \begin{bmatrix} \frac{1}{4}\Delta t^4 & 0 & \frac{1}{2}\Delta t^3 & 0 \\ 0 & \frac{1}{4}\Delta t^4 & 0 & \frac{1}{2}\Delta t^3 \\ \frac{1}{2}\Delta t^3 & 0 & \Delta t^2 & 0 \\ 0 & \frac{1}{2}\Delta t^3 & 0 & \Delta t^2 \end{bmatrix} \sigma_p^2$$

and for the observation noise:

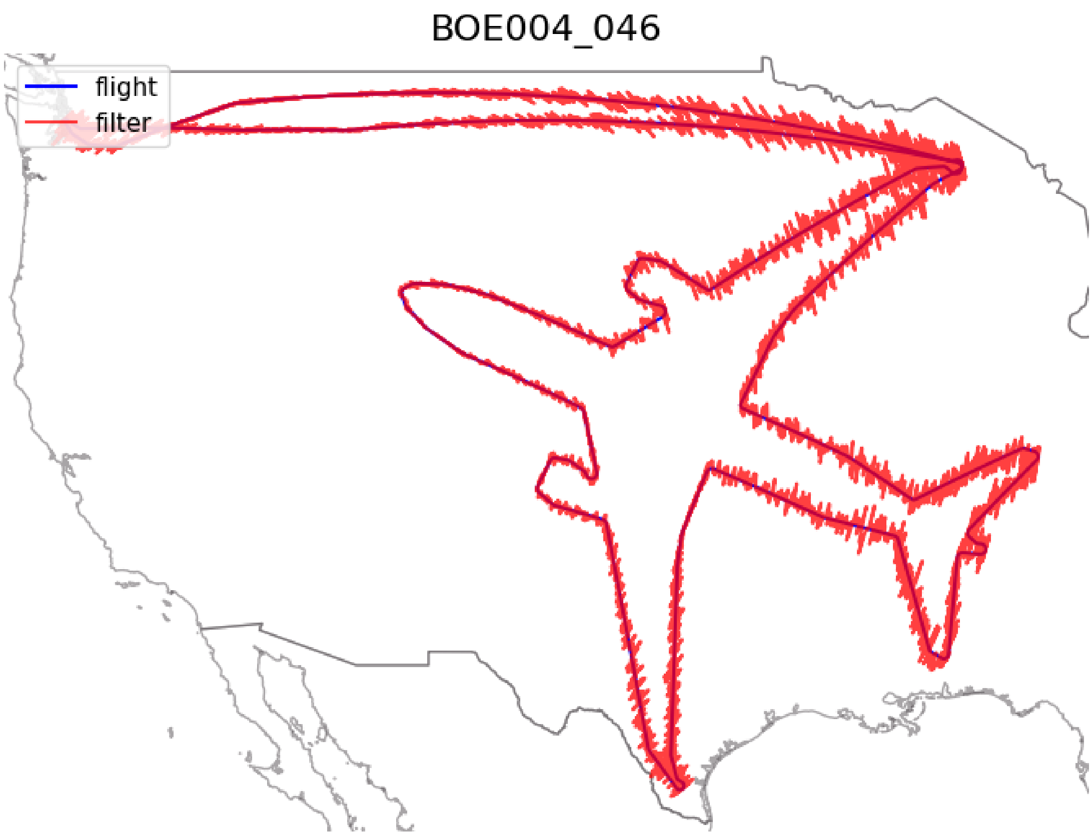
$$R = \begin{array}{c c} & \begin{array}{cc} \bar{x} & \bar{y} \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \end{array}$$

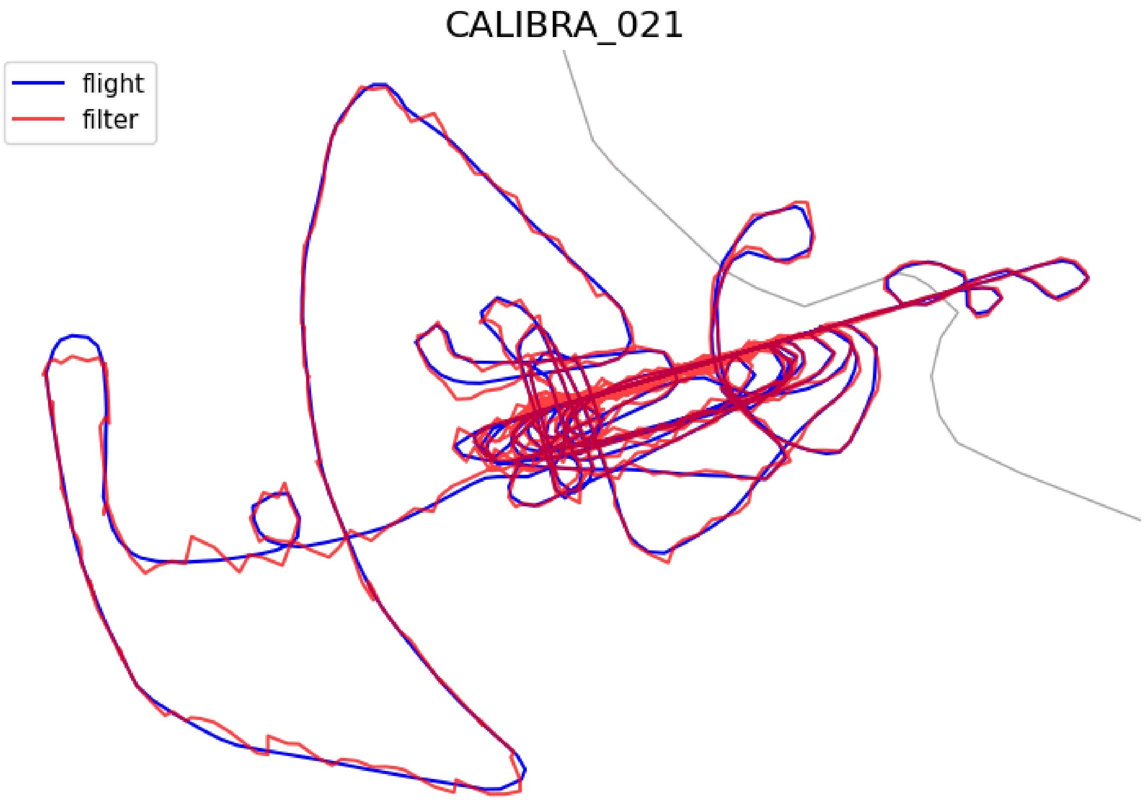
and therefore:

$$R = \begin{bmatrix} \sigma_o^2 & 0 \\ 0 & \sigma_o^2 \end{bmatrix}$$

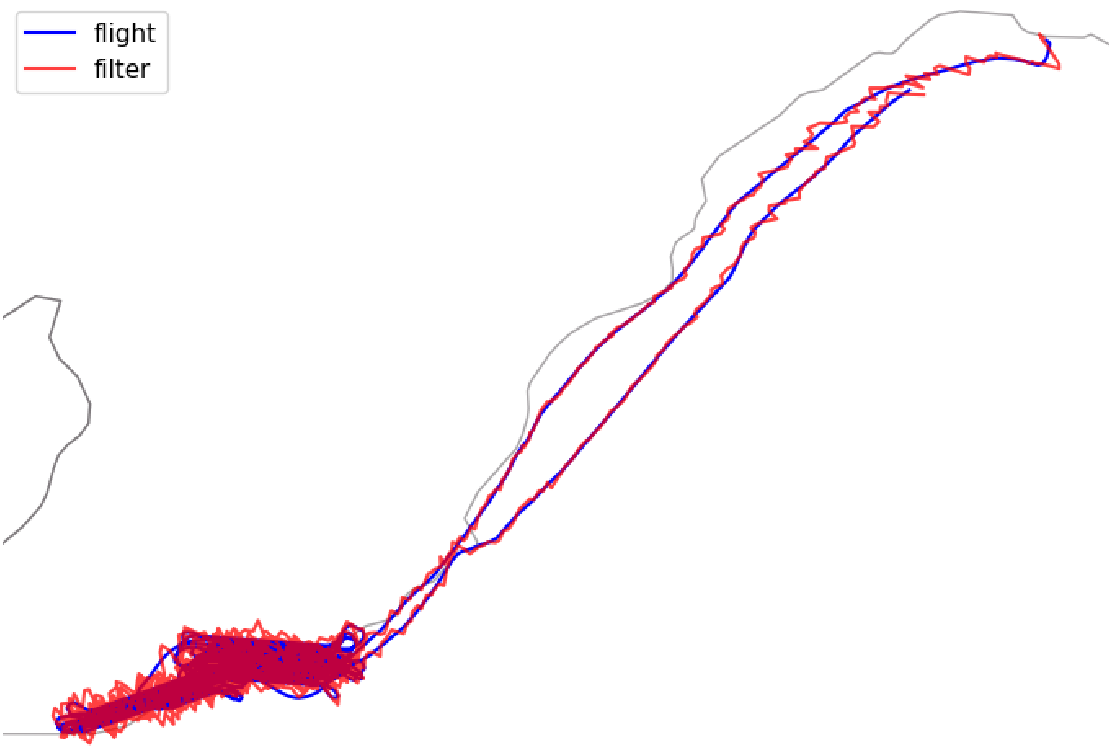
2 Kalman Filter

Having set up the filter it was tested on a few flights, chosen at random. The results for the filter compared to the original flight data can be seen in the images below for 5 random flights:

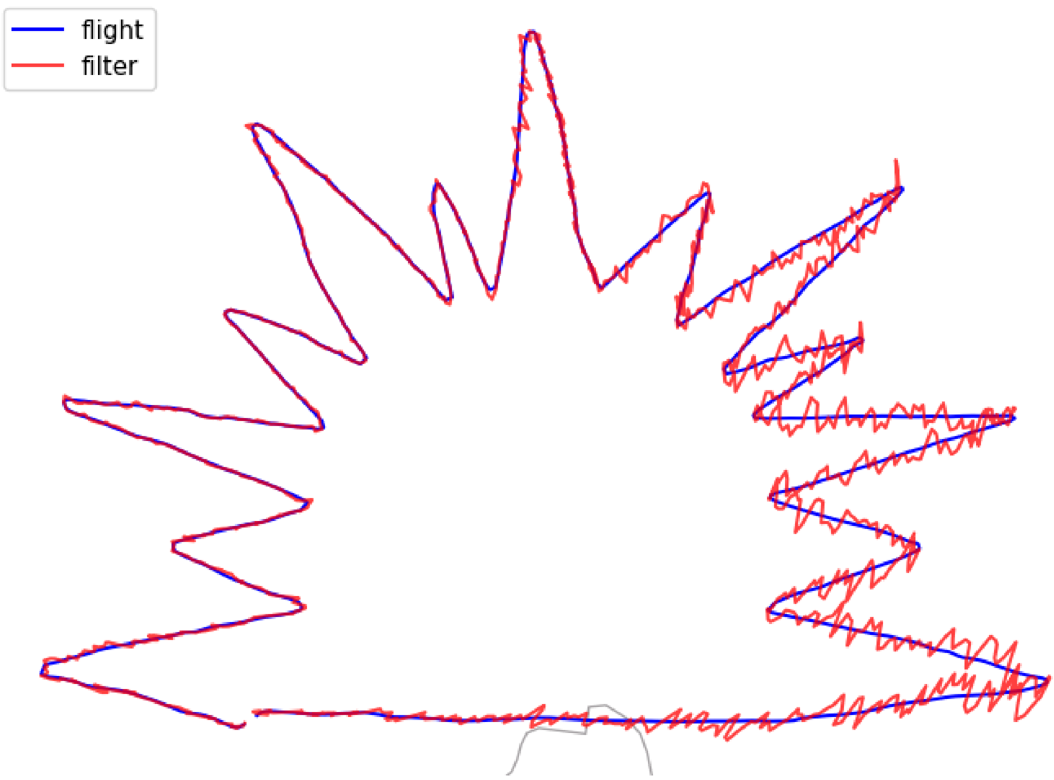

BOE004_046
flight
filter


CALIBRA_021
flight
filter

IGRAD_000



VHOMS_054



Analysing the flight paths the filter starts by having some initial error but that diminishes with the aid of the feedback from the radar observations. Generally it performs best when the airplane is flying at a steady pace but starts to perform worse when there sudden changes in velocity. Since the airplane's position is the only observed information then any changes in velocity that are likely to quickly alter its position are not observed and are therefore likely to show an increase in error.
The error also increases when the airplane has flown far away from the radar as can be seen on the flight path for BOE004-46. As the airplane takes off from the west coast of USA the error in the path begins to decrease but as it starts to get closer to the east coast the error starts to increase again even though the flight path has not changed much. Part of the increase in error could also come from the airplane suddenly changing speed to prepare to form the pattern it later forms with its flight path.
The mean and maximum distance between the filtered and original data was also calculated for the 5 flights shown above along with 5 other additional flights, chosen at random. The results can be seen in the table below:

Flight ID	Mean distance	Maximum distance
ADA4-025	229	1256
BOE004-046	21733	173809
CALIBRA-021	200	1149
D-KWFW-017	1862	13574
FGALN-005	939	5428
IGRAD-000	623	2741
PSWRD35-010	918	5950
VHOMS-054	753	4931
VOR05-034	453	3318
ZEROG-048	2440	16852

3 Process and observation noise experiments

To test the effect of the process noise σ_p and the observation noise σ_o these values were varied to observed how either lowering or increasing these values would have on the mean and maximum error measurements for the previously chosen flights. For σ_p the values chosen were: [0.5, 1.0, 1.5, 2.0, 2.5] and for σ_o: [10, 50, 100, 150, 200]. The effect of the process noise σ_p was first tested by going through the list of chosen values while the observation noise σ_o was set at the initial value of 100. This was then reversed and the process noise σ_p set at its initial value while going through the list of values for the observation noise σ_o. The noise values were then tested by having both starting at low values and increasing together. Lastly the noise values were tested by having the process noise σ_p starting at a low value while the observation noise σ_o started at a high value. The process noise was then increased while the observation noise was decreased. These experiments were run for all of the 10 randomly chosen flights. The results for four of these flights can be seen in the tables below:

Flight ID: ADA4-025

σ_p	σ_o	Mean distance	Maximum distance
0.5	100	265	1225
1.0	100	229	1200
1.5	100	229	1256
2.0	100	233	1287
2.5	100	237	1307
1.5	10	257	1384
1.5	50	240	1320
1.5	100	229	1256
1.5	150	229	1200
1.5	200	237	1151
0.5	10	248	1352
1.0	50	233	1287
1.5	100	229	1256
2.0	150	229	1242
2.5	200	229	1233
0.5	200	388	1482
1.0	150	243	1167
1.5	100	229	1256
2.0	50	245	1339
2.5	10	258	1382

Flight ID: D-KWFW-017

σ_p	σ_o	Mean distance	Maximum distance
0.5	100	1589	11311
1.0	100	1766	12613
1.5	100	1862	13574
2.0	100	1926	14259
2.5	100	1970	14742
1.5	10	2154	17273
1.5	50	2003	15084
1.5	100	1862	13574
1.5	150	1766	12613
1.5	200	1694	12007
0.5	10	2076	15776
1.0	50	1926	14259
1.5	100	1862	13574
2.0	150	1835	13288
2.5	200	1820	13133
0.5	200	1417	10475
1.0	150	1663	11783
1.5	100	1862	13574
2.0	50	2047	15512
2.5	10	2166	17573

Flight ID: IGRAD-000

σ_p	σ_o	Mean distance	Maximum distance
0.5	100	549	2494
1.0	100	596	2567
1.5	100	623	2741
2.0	100	640	2843
2.5	100	653	2901
1.5	10	704	3096
1.5	50	662	2936
1.5	100	623	2741
1.5	150	596	2567
1.5	200	576	2507
0.5	10	682	3005
1.0	50	640	2843
1.5	100	623	2741
2.0	150	615	2689
2.5	200	611	2659
0.5	200	514	2341
1.0	150	568	2506
1.5	100	623	2741
2.0	50	674	2975
2.5	10	708	3105

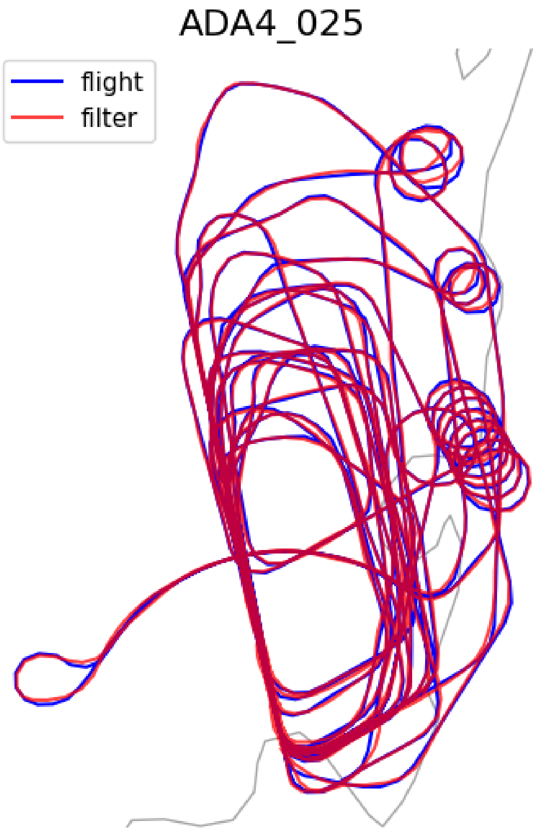
Flight ID: PSWRD35-010

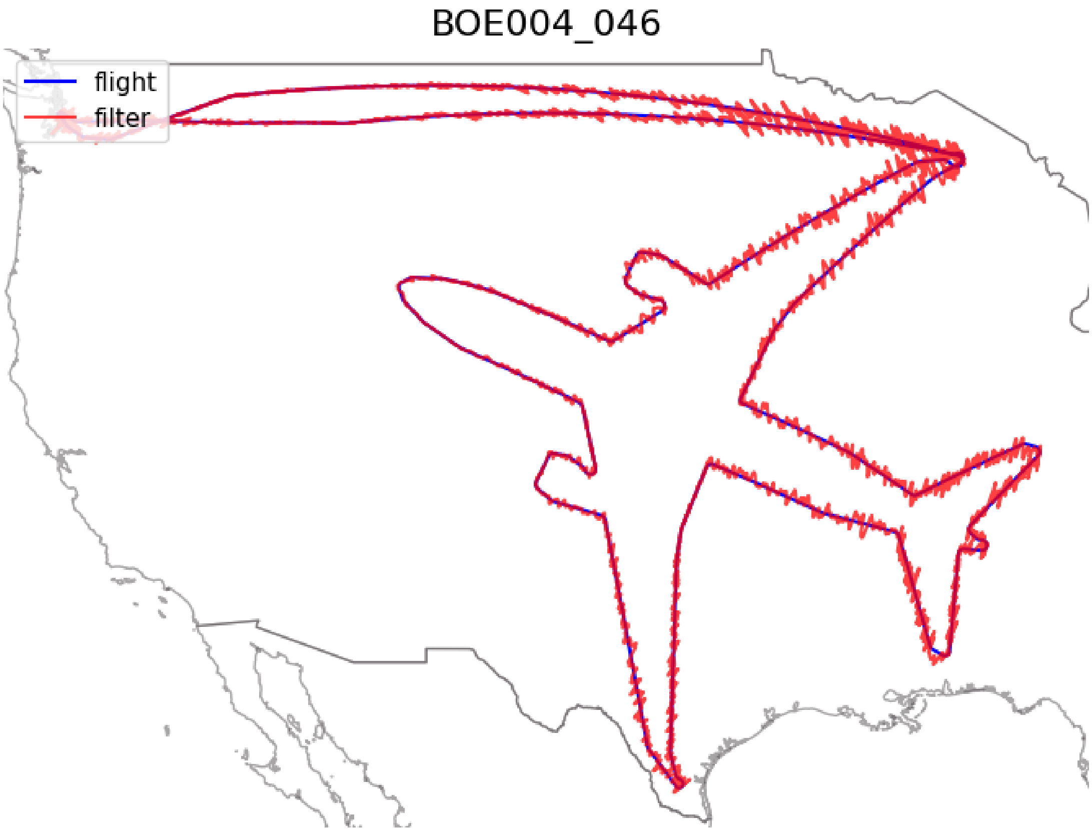
σ_p	σ_o	Mean distance	Maximum distance
0.5	100	803	4924
1.0	100	876	5617
1.5	100	918	5950
2.0	100	946	6181
2.5	100	966	6355
1.5	10	1043	6935
1.5	50	981	6488
1.5	100	918	5950
1.5	150	876	5617
1.5	200	846	5359
0.5	10	1013	6777
1.0	50	946	6181
1.5	100	918	5950
2.0	150	906	5855
2.5	200	899	5802
0.5	200	737	3874
1.0	150	833	5244
1.5	100	918	5950
2.0	50	1001	6669
2.5	10	1047	6921

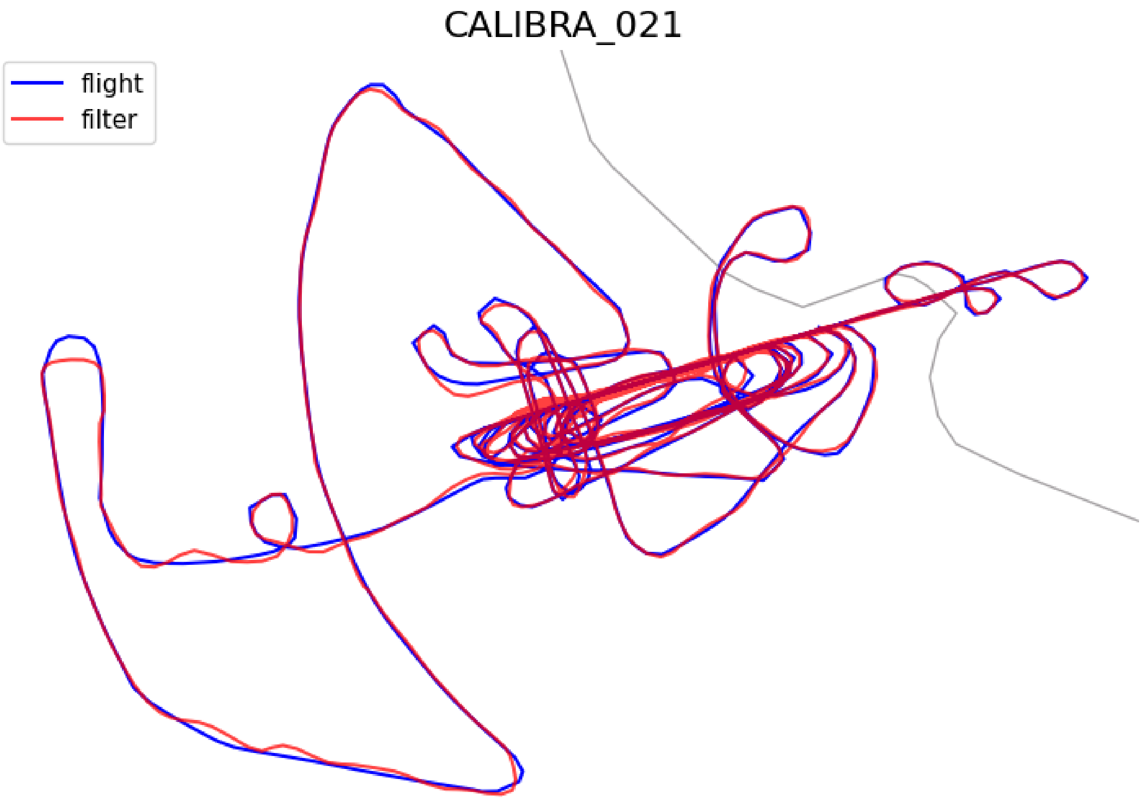
From these result tables it can be seen that generally the error decreases when the value for the process noise σ_p is decreased when the observation noise σ_o was kept constant at 100. The error also decreases when the value for the observation noise σ_o is increased when keeping the process noise σ_p constant at its initial value of 1.5. In line with these results it was observed that for most of the flights the best results were achieved when the process noise σ_p was set at the lowest tested value with the observation noise σ_o increased to the highest tested value. Since the velocity is not observable it follows that the filter will give less error when it is more responsive to observed changes in position but cares less about changes in velocity since that information is unknown.

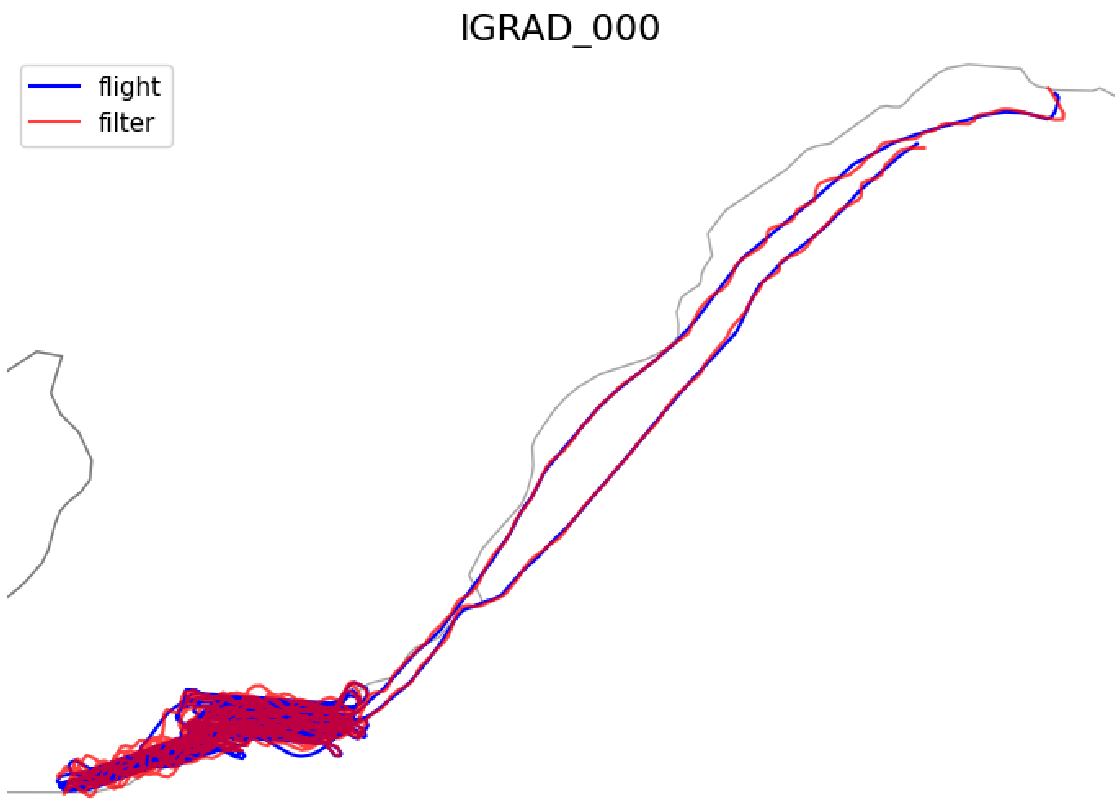
4 Smoothing

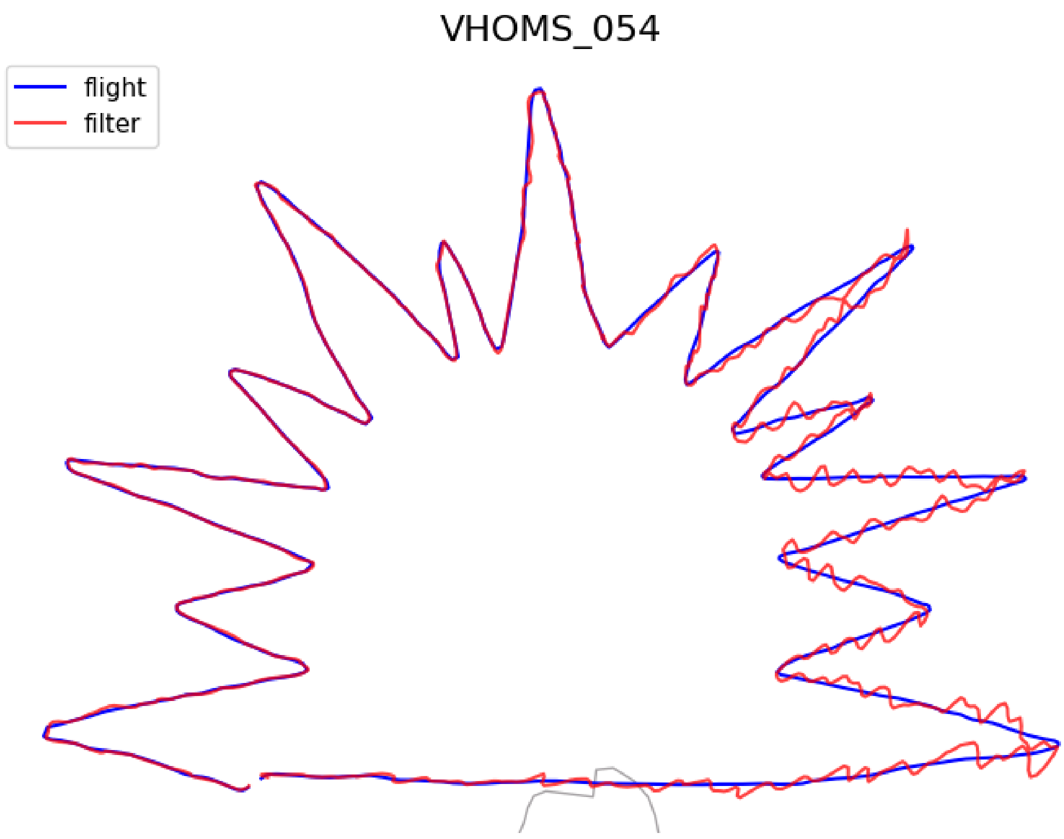
A smoothing was then implemented and tested on the same flights that had been used to test the base Kalman filter. The results for the smoothed data have considerably less noise in the flight path when compared to the original flight data. These results can be seen in the images below for the same 5 random flights as before:




BOE004_046
flight
filter


CALIBRA_021
flight
filter





As before the mean and maximum distance between the smoothed and original data was also calculated for the 5 flights shown above along with the same other 5 additional flights as before for the base filter. As with the plotted flight paths the smoothing gives a significant reduction in the observed error for both the mean and max distances. The results can be seen in the table below:

Flight ID	Mean distance	Maximum distance
ADA4-025	152	910
BOE004-046	12923	96119
CALIBRA-021	151	683
D-KWFW-017	1207	8508
FGALN-005	615	3332
IGRAD-000	427	2054
PSWRD35-010	629	3223
VHOMS-054	491	3955
VOR05-034	311	1611
ZEROG-048	1508	10063

A comparison was also made to try and visualize the difference between the original , radar, filtered and smoothed flight data. The result of this comparison can be seen the image below: