Exam of Automatic Control (9 credits) - January 18, 2022

Surname: Immatriculation number:

Name: Place and date of birth:

Exercise ZSE (5 points) Consider the two discrete-time LTI systems characterized by the following matrices:

$$A_1 = 1,\ B_1 = -1,\ C_1 = 1,\ D_1 = -1; \qquad A_2 = \begin{bmatrix} -2 & 1.5 \\ -5 & 3.5 \end{bmatrix},\ B_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix},\ C_2 = \begin{bmatrix} -5 & 3 \end{bmatrix},\ D_2 = -1$$

and say whether they are zero state equivalent and whether each of them is reachable, observable, stabilizable and detectable.

Exercise LIN (6 points) Consider the continuous-time plant:

$$\begin{aligned} \dot{x}_1 &= -4x_1 - 3 + x_2^2, \\ \dot{x}_2 &= x_2 - 2x_1 \end{aligned}$$

compute all the equilibrium points and the linearization of the dynamics about each one of them.
Based on the linearization, say whether the specific equilibrium is: 1) exponentially stable; 2) unstable; 3) nothing can be concluded.

Exercise MEC (5 points) Consider the continuous-time linear system characterized by $A = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$, $e^{At} = \begin{bmatrix} 3e^t - 2e^{2t} & 6e^{2t} - 6e^t \\ e^t - e^{2t} & 3e^{2t} - 2e^t \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and compute, whenever possible, the control law taking values in the set $[t_0, t_1] = [0, 2]$ and taking the state from $x(t_0) = x(0) = 0$, to $x(t_1) = x(2) = x_{final}$, where x_{final} takes the following values

$$a)\ x_{final} = x_a = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad b)\ x_{final} = x_b = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad c)\ x_{final} = x_c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

In particular, for cases a), b) and c) indicate whether such a control exists and, if it exists, give its explicit form.

Exercise KAL (6 points) Consider the continuous-time linear dynamics described by:

$$\left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ -3 & 4 & -6 & 5 \\ -3 & 3 & -6 & 2 \\ \hline 1 & -1 & 2 & \end{array}\right]$$

and compute its Kalman decomposition. Specify if the system is controllable, observable, stabilizable and detectable.

Solution to exercise ZSE

$$G_1(z) = -\frac{z}{z-1}$$

$$G_2(z) = -\frac{1}{z-1} - 1 = -\frac{z}{z-1}$$

System 1 is
- ☒ Reachable
- ☒ Stabilizable
- ☒ Observable
- ☒ Detectable

System 2 is
- ☐ Reachable
- ☒ Stabilizable
- ☐ Observable
- ☒ Detectable

The two systems:
- ☒ Are ZSE
- ☐ Are NOT ZSE

Instructions for filling up the exam sheet

This solutions sheet MAY be compiled with a pencil

For the exam, **ONLY** this solutions sheet should be turned in. The exam will be graded using this sheet only.
In case of lack of time, students may turn in their exam *without* (or only partially) compiling the solutions sheet. This will however result in a -3 **points penalty** in the final grade.

Solution to exercise LIN

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -4 & 2x_2 \\ -2 & 1 \end{bmatrix} \qquad \text{Equilibria: } x_a = \begin{bmatrix} 3/2 \\ 3 \end{bmatrix} \; x_b = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix} \ldots$$

Equilibrium	x_a	x_b	x_c	x_d
Linearization matrix A	$\begin{bmatrix} -4 & 6 \\ -2 & 1 \end{bmatrix}$	$\begin{bmatrix} -4 & -2 \\ -2 & 1 \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$
Exponentially Stable	☒	☐	☐	☐
Unstable	☐	☒	☐	☐
Can't say	☐	☐	☐	☐

Solution to exercise MEC

$$e^{At}BB^\top e^{A^\top t} = e^{4t}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \qquad W_R(t_0, t_1) = \frac{e^8 - 1}{4}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Case a): ☒ A solution **DOES NOT EXIST**; ☐ A solution **EXISTS** and is

$\eta_1 =$ $\qquad$ $u_a(t) =$

Case b): ☐ A solution **DOES NOT EXIST**; ☒ A solution **EXISTS** and is

$$\eta_1 = \frac{4}{1-e^8}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad u_b(t) = B^T e^{A^T(t_1 - t)}\eta_1 = \frac{4}{1-e^8} e^{4-2t}$$

Case c): ☐ A solution **DOES NOT EXIST**; ☒ A solution **EXISTS** and is

$$\eta_1 = \frac{4}{e^8-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad u_c(t) = \frac{4}{e^8-1} e^{4-2t}$$

Solution to exercise KAL (Only fill up the relevant parts)

$$R = \begin{bmatrix} 1 & 5 & 5 \\ 5 & 5 & 5 \\ 2 & 0 & 0 \end{bmatrix} \qquad \text{1) Basis of Im } R \cap \text{Ker } O = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \text{2) Completion of Im } R = \times$$

$$O = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 4 & -4 & 8 \end{bmatrix} \qquad \text{3) Completion of Ker } O = \times \qquad \text{4) Completion of } T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad T^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right] = \begin{array}{c} \begin{array}{cc} \underbrace{\qquad\quad}_{c\bar{o}} & \bar{c}o \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 2 & \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} c\bar{o} \\ \bar{c}o \\ \\ \end{array}$$

The system is: Reachable (Y☐ N☒), Stabilizable (Y☒ N☐), Observable (Y☐ N☒), Detectable (Y☐ N☒)