Modular Forms of Half Integral Weight

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Primary References:

Shimura, Goro. On Modular Forms of Half Integral Weight. The Annals of Mathematics, 2nd Ser., Vol. 97, No. 3 (May, 1973), pp. 440-481.

Other References:

Koblitz, Neal. Introduction to Elliptic Curves and Modular Forms. GTM 97, Springer-Verlag.

Shimura, Goro. Introduction to the Arithmetic Theory of Automorphic Functions. Princeton University Press, 1971.

1 Standard Definitions

We freely make use of the following standard definitions, taken from Shimura's paper and/or Koblitz's book:

We define the square root of a complex number z to have argument in $(-\pi/2, \pi/2]$. Moreover, we define $z^{k/2} = (\sqrt{z})^k$ for any integer k. The fact that we have to make this choice is important in our definition of a half-integral weight modular form.

By a *discrete subgroup* Γ of $SL_2(R)$ we mean a subgroup of $SL_2(R)$ which has the discrete topology under the standard topology on $SL_2(R)$. Generally, we take Γ to be $SL_2(Z)$ or one of its *congruence subgroups* .

For any positive integer N, we define

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (modN)\}$$

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (modN)\}$$

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (mod N)\}.$$

By a *congruence subgroup* of $SL_2(Z)$ we mean any subgroup of $SL_2(Z)$ containing $\Gamma(N)$ for some N. Naturally $\Gamma_1(N)$ and $\Gamma_0(N)$ are examples.

By a (holomorphic) *modular form* of (integer) weight k for a discrete subgroup Γ we mean a holomorphic function f defined on the upper half plane H which satisfies, for all z and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f(\gamma)z = (cz+d)^k f(z).$$

Moreover, we require that f is 'holomorphic at infinity', which means that its Fourier series $f(z) = \Sigma_{n \in Z} a_n e^{2\pi i n z}$ has $a_n = 0$ for $n < 0$. *Meromorphic* modular forms are defined similarly.

2 Modular Forms and Hecke Operators

2.1 A Naive Definition

We try the following:

Wrong Definition: Let k be odd, and let Γ be a discrete subgroup of $GL_2^+(R)$. Assume f is a holomorphic (or meromorphic) function on H, and it satisfies an appropriate condition at the cusps. Then f is a *modular form of weight* $k/2$ if

$$f(\gamma z) = (cz+d)^{k/2} f(z).$$

Suppose we accept such a definition. Then we can prove some sweeping statements.

Proposition: Let k be odd, and let $\Gamma' \subset SL_2(Z)$ be any congruence subgroup. Let f be a modular form of weight $k/2$ relative to the above definition.
Then: $f = 0$.

Proof: Let Γ' be a congruence subgroup of $SL_2(Z)$; i.e., assume that for some $N > 2$

$$\Gamma' \supseteq \Gamma(N) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (mod N)\}.$$

Denote

$$\alpha = \begin{pmatrix} N+1 & N \\ -N & 1-N \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix}.$$

Observe that α and β are in Γ', and compute

$$\alpha\beta = \begin{pmatrix} -N^2+N+1 & N \\ N^2-2N & 1-N \end{pmatrix}.$$

Let f be any nonzero modular form of weight $k/2$; by our definition we require that

$$f(\alpha z) = (-Nz + (1-N))^{k/2} f(z)$$

and

$$f(\beta z) = (-Nz + 1)^{k/2} f(z).$$

Therefore

$$\begin{aligned} f(\alpha\beta z) &= (-N(\beta z) + (1-N))^{k/2} f(\beta z) \\ &= (-Nz + (1-N))^{k/2} (-Nz+1)^{k/2} f(z) \\ &= (\frac{-Nz}{-Nz+1} + (1-N))^{k/2} f(\beta z) \\ &= (-Nz + (1-N))^{k/2} (-Nz+1)^{k/2} f(z). \end{aligned}$$

But applying the definition directly to the matrix $\alpha\beta$, we derive that

$$f(\alpha\beta z) = ((N^2 - 2N)z + (1-N))^{k/2} f(z).$$

Therefore, irregardless of f, we require the following identity:

$$((N^2 - 2N)z + (1-N))^{k/2} = (\frac{-Nz}{-Nz+1} + 1 - N)^{k/2} (-Nz+1)^{k/2}.$$

If k is even we get equality. We had better, because f is by definition a modular form of integral weight $k/2$. But, if k is odd then the equation is wrong by a factor of -1. This can be seen by setting $k = 1$ and plugging in any $z = x + iy$ with $y > 0$.

2.2 A Covering for $GL_2^+(R)$

In hindsight, it is somewhat natural that such a definition should fail; the square root function is multivalued, so our choice of a branch of the square root necessarily led to problems. We may handle this group by requiring that our modular forms act on a covering space of $GL_2^+(R)$, where we allow all branches of the square root simultaneously.

Definition: Let the group $\mathbf{T}$ be either: the set of all complex numbers z with $|z| = 1$ (as in Shimura), or simply the set $1, -1, i, -i$. Then we define a group G to be the set of all ordered pairs $(\alpha, \phi(z))$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(R)$ and $\phi(z)$ is a holomorphic fuction on H satisfying

$$\phi(z)^2 = t(\det\alpha)^{-1/2}(cz + d) \tag{1}$$

for some $t \in T$. We define multiplication of group elements as follows:

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta(z))\psi(z)).$$

Proposition: G is a group under the operation above, with the formula for an inverse given by

$$(\alpha, \phi(z))^{-1} = (\alpha^{-1}, 1/\phi(\alpha^{-1}z)).$$

Proof: This is not obvious, but the calculations are not difficult. See Koblitz, p. 179.

In other words, G is a covering space for $GL_2^+(R)$, either a four-sheeted covering or an infinite covering depending on the definition of T used.

The Slashing Operator:

Definition: Let $\xi = (\alpha, \phi) \in G$. Then we define the action of ξ on H to be the same as that of α, and we define a *slashing operator* by

$$(f|[\xi]_{k/2})(z) = f(\alpha z)(\phi(z))^{-k}.$$

Remark: The notation here is that of Koblitz; Shimura labels this $f|[\xi]_k$.

Note that $f|[\xi\eta]_{k/2} = (f|[\xi]_{k/2})|[\eta]_{k/2}$, as can be verified by calculation.

We now define the groups on which half-integral weight modular forms can act:

Definition: A *Fuchsian subgroup* of G is a subgroup Δ of G satisfying the following:

(1) $P(\Delta)$, the projection of Δ onto $SL_2(R)$ is a discrete subgroup, and this projection is one-to-one.
(2) The fundamental domain $P(\Delta)\ H$ is of finite measure with respect to the invariant measure $y^{-2}dxdy$.
(3) If $-1 \in P(\Delta)$, then its preimage is Δ is $(-1, 1)$.

In other words, if we have a Fuchsian subgroup Δ and its projection in $GL_2{+}(R)$ is Γ, we have mutually inverse isomorphisms $P : \Delta \to \Gamma$ and $L : \Gamma \to \Delta$.

Definition: Suppose that f is a meromorphic (holomorphic) function on H, and Δ is a Fuchsian subgroup of G. Then we call f a *meromorphic (holomorphic) modular form of weight* $k/2$ with respect to Δ if:

(1) $f|[\xi]_{k/2} = f$ for all $\xi \in \Delta$.
(2) f is meromorphic (holomorphic) at every cusp of $P(\Delta)$.

The second condition is not discussed in detail here, but is exactly similar to the same condition on integral weight modular forms: One makes an appropriate condition on the presence of negative Fourier coefficients.

Remark: Shimura uses different notation. What we have called meromorphic modular forms, he calls *automorphic forms* , and holomorphic modular forms he calls *integral forms* . Moreover, we use the notation $G_{k/2}(\Delta)$ to stand for the space of holomorphic modular forms of weight $k/2$ with respect to Δ.

2.3 Modular Forms for Congruence Subgroups of $SL_2(Z)$

For subgroups of $\Gamma_0(4)$ we can define a standard choice of the automorphy factor for each γ, and thereby define a Fuchsian subgroup Δ of G.

Definition: For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ define

$$j(\gamma, z) = \epsilon_d^{-1}(\frac{c}{d})(cz + d)^{1/2}$$

where $\epsilon_d = 1$ if $d \equiv 1(mod4)$ and $\epsilon_d = i$ if $d \equiv 3(mod4)$. Furthermore, define a group isomorphism * of $\Gamma_0(4)$ into G by

$$\gamma^* = (\gamma, j(\gamma, z)).$$

Remarks: We see immediately that $j(\gamma, z)$ satisfies condition (1). We need to check that the definition respects the group operation. Although an elementary proof of this should be possible, but we instead resort, as do Shimura and Koblitz, to *deus ex machina* : $j(\gamma, z)$ satisfies the identity

$$j(\gamma, z) = \Theta(\gamma z)/\Theta(z),$$

where $\Theta(z) = \sum_{n=-\infty}^{n=\infty} e^{2\pi i n^2 z}$, and then the group law is immediate. See Koblitz (p. 148) for a proof of this identity.

Notation: Denote by $\Delta_0(N), \Delta_1(N), \Delta(N)$ the images of $\Gamma_0(N), \Gamma_1(N), \Gamma(N)$ under this isomorphism, for any N divisible by 4. Moreover, we will write $[\gamma]_{k/2}$ for $[\gamma^*]_{k/2}$ so that

$$f|[\gamma]_{k/2} = f(\gamma z) j(\gamma, z)^{-k}.$$

2.4 Actions of Double Cosets on Modular Forms

To define the Hecke operators, we need to define the action of a double coset on a modular form. To do this we review a fact from the theory of modular forms of integral weight:

Proposition: Let Δ be any subgroup of any group G, and let $\xi \in G$ be any element of G so that Δ and $\xi^{-1}\Delta\xi$ are *commensurable;* i.e., their intersection has finite index in either group. Denote $\Delta' = \Delta \cap \xi^{-1}\Delta\xi$. Then $\Delta\xi\Delta$ is a union of right cosets $\Delta'\xi\delta_j$ (with the δ_j in Δ).

Proof: See, for example, Koblitz, p. 165.

Now let Δ be a Fuchsian subgroup of G_1, and α an element of $GL_2^+(R)$ so that $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ, where $\Gamma = P(\Delta)$. Denote by L the inverse map of the projection $P : \Delta \to \Gamma$. And, let $\xi = (\alpha, \phi(z)) \in G$ be any element of G projecting to α.

We may now sensibly define an action of a double coset $\Delta\xi\Delta$ on $G_{k/2}(\Delta)$. We have the set theoretic equality

$$f|[\Delta\xi\Delta]_{k/2} = f|[\cup_v \Delta\xi\delta_v]_{k/2}$$

by virtue of the above. Since Δ acts trivially on f it makes sense to remove the first Δ and define

$$f|[\Delta\xi\Delta]_{k/2} = \sum_v f|[\xi\delta_v]_{k/2}.$$

Because our definition breaks the double cosets $\Delta\xi\Delta$ into right cosets of $\Delta \cap \xi^{-1}\Delta\xi$, we naturally want to understand the subgroups $\Delta \cap \xi^{-1}\Delta\xi$. How do these subgroups behave with respect to the lift? Do we have $\Delta \cap \xi^{-1}\Delta\xi = L(\Gamma \cap \alpha^{-1}\Gamma\alpha)$?

In general, no. To understand the relationship between $\Delta \cap \xi^{-1}\Delta\xi$ and $L(\Gamma \cap \alpha^{-1}\Gamma\alpha)$, we define a map

$$t : \Gamma \cap \alpha^{-1}\Gamma\alpha \to T$$

by the equation

$$\xi L(\gamma)\xi^{-1} = L(\alpha\gamma\alpha^{-1})(1, t(\gamma)).$$

From the condition (1) on the automorphic factors and the group law, we can see that this equation will be true for some t with $t = 1$ depending on γ, α, and possibly ξ.

We derive some properties of the map t:

Proposition:
(1) The map t depends only on γ and α, and not on the choice of ξ.
(2) t is a group homomorphism from $\Gamma \cap \alpha^{-1}\Gamma\alpha$ into T.
(3) $L(Ker(t)) = \Delta \cap \xi^{-1}\Delta\xi$.
(4) $L(\Gamma \cap \alpha^{-1}\Gamma\alpha) = \Delta \cap \xi^{-1}\Delta\xi$ if and only if t is trivial.

Proof:

(1) $\xi = (\alpha, \phi(z))$, where ϕ satisfies

$$\phi(z)^2 = t\det(\alpha)^{-1/2}(cz + d).$$

We renormalize t_ϕ in terms of a 'standard automorphy factor': Defining square roots to be in the right half plane, write

$$\phi(z) = t_\phi \det(\alpha)^{-1/4}(cz + d)^{1/2}$$

where t_ϕ is some complex number with $|t_\phi| = 1$.

Write

$$h(z) = \det(\alpha)^{-1/4}(cz+d)^{1/2},$$

so that $(\alpha, h(z))$ also projects to α, and $\phi(z) = t_\phi h(z)$. We see that

$$\xi^{-1} = (\alpha^{-1}, \frac{1}{\phi(\alpha^{-1}(z)})) = (\alpha^{-1}, \frac{1}{t_\phi}\frac{1}{h(z)}),$$

so that

$$\xi L(\gamma)\xi^{-1} = (\alpha, \phi(z))L(\gamma)(\alpha^{-1}, \frac{1}{\phi(\alpha^{-1}(z)})$$
$$= (\alpha, t_\phi h(z))L(\gamma)(\alpha^{-1}, \frac{1}{t_\phi}\frac{1}{h(z)})$$

. According to the group law, multiplying any automorphy factor by a scalar multiplies the product factor by the same scalar. Therefore the constants t_ϕ and $\frac{1}{t_\phi}$ will cancel, so that the above simplifies to

$$= (\alpha, h(z))L(\gamma)(\alpha^{-1}, \frac{1}{h(z)})$$

which is independent of t_ϕ and therefore of the choice of ξ.

(2) By definition of t,

$$\xi L(\gamma)\xi^{-1} = L(\alpha\gamma\alpha^{-1})(1, t(\gamma))$$
$$\xi L(\gamma')\xi^{-1} = L(\alpha\gamma'\alpha^{-1})(1, t(\gamma'))$$
$$\xi L(\gamma\gamma')\xi^{-1} = L(\alpha\gamma\gamma'\alpha^{-1})(1, t(\gamma\gamma')).$$

But as L defines an isomorphism of Γ into G,

$$\xi L(\gamma\gamma')\xi^{-1} =$$
$$\xi L(\gamma)L(\gamma')\xi^{-1} =$$
$$\xi L(\gamma)\xi^{-1}\xi L(\gamma')\xi^{-1} =$$
$$L(\alpha\gamma\alpha^{-1})(1, t(\gamma))L(\alpha\gamma'\alpha^{-1})(1, t(\gamma'))$$
$$= L(\alpha\gamma\alpha^{-1}\alpha\gamma'\alpha^{-1})(1, t(\gamma))(1, t(\gamma'))$$
$$= L(\alpha\gamma\gamma'\alpha^{-1})(1, t(\gamma)t(\gamma'))$$

so that

$$t(\gamma\gamma') = t(\gamma)t(\gamma').$$

(3). Suppose that $L(\gamma) \in \Delta \cap \xi^{-1}\Delta\xi$. Then $L(\gamma) = \xi^{-1}\delta\xi$, for some $\delta \in \Delta$.

Then, $\delta = \xi L(\gamma)\xi^{-1} = L(\alpha\gamma\alpha^{-1})(1, t(\gamma))$. But if $\delta \in \Delta$, then δ must be the lift of something in Γ, and we see from the previous equation that this something must be $\alpha\gamma\alpha^{-1}$. But then $t(\gamma) = 0$.

Conversely, suppose that $\gamma \in Ker(t)$ for some $\gamma \in \Gamma$. Then $L(\gamma) \in Delta$ because $\Delta = L(\Gamma)$, but if $\gamma \in Ker(t)$ then $L(\alpha\gamma\alpha^{-1}) = \xi L(\gamma)\xi^{-1}$ so that $L(\gamma) = \xi^{-1}L(\alpha\gamma\alpha^{-1})\xi \in \xi^{-1}\Delta\xi$.

(4) is immediate from (3).

Observe that in general $\Delta \cap \xi^{-1}\Delta\xi$ will be a subgroup, perhaps proper, of $L(\Gamma \cap \alpha^{-1}\Gamma\alpha)$.

We now specialize to the case $\Gamma = \Gamma_0(4) \subset SL_2(Z)$, and calculate the function t explcitly for a class of elements $\alpha \in GL_2(Z)$.

As before, we use a star to denote the standard lift $\gamma \to (\gamma, j(\gamma, z))$.

Proposition. (Shimura, 1.2) Let m and n be positive integers. Denote

$$\alpha = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}, \xi = (\alpha, t(n/m)^{1/4})$$

with any $t \in T$. Then, if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \cap \alpha^{-1}\Gamma_0(4)\alpha,$$

we have the relation

$$\xi\gamma^*\xi = (\alpha\gamma\alpha^{-1})^*(1, (\frac{mn}{d})).$$

Note: The notation $(\frac{mn}{d})$ denotes the Legendre symbol. It is extended, by definition, by multiplicativity for d that are not prime.

As in important special case, if mn is a perfect square, then

$$\xi\gamma^*\xi = (\alpha\gamma\alpha)^{-1}.$$

Proof: We calculate that

$$\xi\gamma^*\xi = (\alpha, t(n/m)^{1/4})(\gamma, j(\gamma, z))(\alpha^{-1}, (1/t)(m/n)^{1/4})$$
$$= (\alpha, t(n/m)^{1/4})(\gamma\alpha^{-1}, j(\gamma, \alpha^{-1}z))(1/t)(m/n)^{1/4})$$
$$= (\alpha\gamma\alpha^{-1}, j(\gamma, \alpha^{-1}z)),$$

and

$$j(\gamma, \alpha^{-1}z) = j(\gamma, \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} z)$$

$$= j(\gamma, n/mz)$$
$$= (\frac{c}{d})\epsilon_d^{-1}\sqrt{cn/mz + d}.$$

On the right side,

$$\alpha\gamma\alpha = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} = \begin{pmatrix} a & mb/n \\ nc/m & d \end{pmatrix},$$

so that

$$(\alpha\gamma\alpha^{-1})^*(1, (\frac{mn}{d})) = (\alpha\gamma\alpha^{-1}, j(\alpha\gamma\alpha^{-1}, z)(\frac{mn}{d}))$$
$$= (\alpha\gamma\alpha^{-1}, (\frac{nc/m}{d})\epsilon_d^{-1}\sqrt{(nc/m)z + d}).$$

By the multiplicative property of the Legendre symbol,

$$(\frac{c}{d}) = (\frac{nc/m}{d})(\frac{mn}{d})$$

so that these automorphy factors are equal.

We observe that the resulting map $t = (1, (\frac{mn}{d}))$ depends on both α and γ. Assuming that the case where t is trivial is especially nice, then we will be able to guarantee this condition precisely when mn is a square. If mn is not a square, the Legendre symbol will take on different values for different d.

2.5 Hecke Operators

Hecke operators are defined in much the same way as for modular forms of integral weight, and in the same way they will allow us to establish relations among the Fourier coefficients of modular forms.

Definition: Let m be a positive integer. Write

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, \xi = (\alpha, m^{1/4}).$$

Then the *Hecke operator* $T(m)$ on $G_{k/2}(\Delta_1(N))$ is given by

$$f|T(m) = m^{k/4-1} f|[\Delta_1\xi\Delta_1]_{k/2} = m^{k/4-1}\sum_v f|[\xi\delta_v]_{k/2}.$$

Here we use $\Delta_1 = \Delta_1(N)$; the summation extends over all right cosets of Δ_1 in $\Delta_1\xi\Delta_1$.

We later will discuss twisting by Dirichlet characters, and the definition of Hecke operators acting on twisted modular forms.

We can now prove that the Hecke operators are defined only for square m:

Proposition: Suppose m is not a square, and that m is relatively prime to N. Then $f|T(m) = 0$ if f is a modular form for $\Delta_1(N)$.

Proof:

For simplicity denote $\Delta = \Delta_1(N)$.

From the previous proposition, with

$$\alpha = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}, \xi = (\alpha, t(n/m)^{1/4}), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\xi\gamma^*\xi = (\alpha\gamma\alpha^{-1})^*(1, (\frac{mn}{d})).$$

In this case we consider the matrix $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$, so that $(\frac{mn}{d})$ simplifies to $(\frac{m}{d})$. If m is a perfect square then it will be a quadratic residue modulo any d, and this function will be trivial.

In case m is not a perfect square, then the function $(\frac{m}{d})$ will define a homomorphism from d onto $(1, -1)$. (Recall that m and N are relatively prime, and that the Legendre symbol is extended to arbitrary denominators to make this work.) Moreover, this homomorphism will be nontrivial; if m is not a square there will be some d for which m is not a quadratic residue.

Because we have a nontrivial homomorphism, we conclude that $\Delta \cap \xi^{-1}\Delta\xi$ is a subgroup of $L(\Gamma \cap \alpha^{-1}\Gamma\alpha)$ of index 2. Denoting $\Delta'' = \Delta \cap \xi^{-1}\Delta\xi$ and $\Delta' = L(\Gamma \cap \alpha^{-1}\Gamma\alpha)$, we may write a right coset decomposition $\Delta' = \Delta'' + \Delta''\tau$, where $\tau = \tau'(1, -1)$ with $\tau' \in \Delta \cap \xi^{-1}\Delta\xi$.

To evaluate the action of the double coset $[\Delta\xi\Delta]_{k/2}$ on f, recall that by definition

$$f|[\Delta\xi\Delta]_{k/2} = \sum_j f|[\xi\delta_j]_{k/2},$$

where we sum over all right cosets of $\Delta \cap \xi^{-1}\Delta\xi$ in $\Delta\xi\Delta$. If $\Delta\xi\Delta = \cup_j L(\Gamma \cap \alpha^{-1}\Gamma\alpha)\delta_j = \cup_j \Delta'\delta_j$, then we have a coset decomposition

$$\Delta\xi\Delta = \cup_j(\Delta'' \cup \Delta''\tau)\delta_j$$

$$= \cup_j(\Delta''\delta_j) \cup \cup_j(\Delta''\tau\delta_j)$$

so that

$$f|[\Delta\xi\Delta]_{k/2} = \sum_j f|[\xi\delta_j]_{k/2} + \sum_j f|[\xi\tau\delta_j]_{k/2}$$

$$= \sum_j f|[\xi\delta_j]_{k/2} + \sum_j f|[\xi\tau'(1, -1)\delta_j]_{k/2}$$

$$= \sum_j f|[\xi\delta_j]_{k/2} - \sum_j f|[\xi\tau'\delta_j]_{k/2}.$$

But $\tau' \in \Delta \cap \xi^{-1}\Delta\xi$, so that we may write $\tau' = \xi^{-1}\delta'\xi$, so that the above becomes

$$= \sum_j f|[\xi\delta_j]_{k/2} - \sum_j f|[\delta'\xi\delta_j]_{k/2}$$

$$= \sum_j f|[\xi\delta_j]_{k/2} - \sum_j f|[\xi\delta_j]_{k/2}$$

$$= 0.$$

2.6 Twisting by Dirichlet Characters

Hecke operators are given for modular forms 'twisted' by Dirichlet characters, so we define what this means.

Definition: Let χ be a Dirichlet character modulo N. Denote by $G_{k/2}(N, \chi)$ the set of all $f \in G_{k/2}(\Delta_1(N))$ such that for all $\gamma \in \Gamma_0(N)$,

$$f|[\gamma]_{k/2} = \chi(d)f.$$

We observe that for $\Gamma_1(N)$ the d in each matrix is 1, and $\chi(1) = 1$ for any Dirichlet character χ, so we see that this condition reduces to the condition $f|[\gamma]_{k/2} = f$ for $\gamma \in \Gamma_1(N)$, which is what we expect. Moreover, $[-1]_{k/2}$ gives the identity map, so we assume that $\chi(-1) = 1$; otherwise $G_{k/2}(N, \chi)$ consists only of the zero function.

Hecke Operators on Twisted Modular Forms:

We now define Hecke operators for modular forms in $G_{k/2}(\Delta_0(N))$.

As before write $\Delta_0 = \Delta_0(N)$, etc., and write

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, \xi = (\alpha, p^{1/2}).$$

(We write p^2 for m by virtue of the previous proposition.) Suppose that we have a disjoint union $\Delta_0\xi\Delta_0 = \cup_v \Delta_0\xi_v$, $\Gamma_0\alpha\Gamma_0 = \cup_v \Gamma_0\alpha_v$; then we make the following definition:

Definition: The *Hecke operator* $T(p^2)$ on $G_{k/2}(\Delta_0(N), \chi))$ is given by

$$f|T(p^2) = m^{k/2-2} \sum_v \chi(a_v) f|[\xi_v]_{k/2},$$

where a_v denotes the upper left entry of the matrix $P(\xi_v)$.

One may ask two questions: For starters, what is the a_v doing there? And moreover, is this well defined? The ξ_v are right coset representatives modulo Δ_0, and matrices in Δ_0 are not required to contain 1 in the upper left.

Proposition: The above is well defined. Specifically, if we write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \gamma' = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \gamma, \gamma' \in \Gamma_0(N)$$

and let δ and δ' be their lifts, then we calculate

$$\alpha\gamma = \begin{pmatrix} a & b \\ p^2c & p^2d \end{pmatrix},$$

$$\gamma'\alpha\gamma = \begin{pmatrix} ea + pf^2c & eb + fp^2d \\ ga + p^2ch & gb + p^2h \end{pmatrix}$$

and we have that

$$f|[\xi\delta]_{k/2}\chi(a) = f|[\delta'\xi\delta]_{k/2}\chi(ea + fp^2c).$$

In other words, multiplication of any $\xi\delta$ by δ' on the left does not change the summand in the definition of the Hecke operator.

Proof: Once one realizes that this requires proof, the proof is not so difficult. Because $f \in G_{k/2}(\Delta_0(N), \chi))$ we calculate the right side:

$$f|[\delta'\xi\delta]_{k/2}\chi(ea + fp^2c) =$$

$$f|[\xi\delta]_{k/2}\chi(ea + fp^2c)\chi(h) =$$

$$f|[\xi\delta]_{k/2}\chi(eah + fp^2ch).$$

Looking at the term $eah + fp^2ch$ modulo N, eah is congruent to a because eh is congruent to 1, and c is congruent to 0 so fp^2ch is as well. Therefore, $eah + fp^2ch \equiv a(modN)$, and χ is only defined modulo N, so that the above becomes

$$f|[\xi\delta]_{k/2}\chi(a),$$

as desired.

2.7 Hecke Operators and Fourier Coefficients

We now use the Hecke operators to establish a relationships among the Fourier coefficients of modular forms:

Theorem (Shimura, 1.7). Let p be a prime number, and let $f \in G_{k/2}(N, \chi)$. Denote by $a(n)$ and $b(n)$ the coefficients of the Fourier expansions:

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n$$

$$(f|T(p^2))(z) = \sum_{n=0}^{\infty} b(n)q^n$$

Then

$$b(n) = a(p^2n) + \chi_1(p)(\frac{n}{p})p^{\lambda-1}a(n) + \chi(p^2)p^{k-2}a(n/p^2)$$

where $\lambda = (k-1)/2$, $\chi_1(m) = \chi(m)(\frac{-1}{m})^{\lambda}$, and $a(n/p^2) = 0$ if p^2 does not divide n.

Sketch of Proof[1]: We exclude the case where p divides N. We do note that if $p|N$ the Dirichlet characters will all be zero, so that we are claiming $b(n) = a(p^2n)$. This case follows from other calculations in Shimura not considered here.

So suppose that p does not divide N, and write

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, \xi = (\alpha, p^{1/2}).$$

Then, by definition, we have

$$f|T(p^2) = m^{p/2-2} \sum_{v} \chi(a_v) f|[\xi_v]_{k/2},$$

where the ξ_v are chosen so that the cosets $\Delta_0\xi_v$ run through the complete set of right cosets of Δ_0 in $\Delta_0\xi\Delta_0$, and a_v denotes the upper left entry of the matrix $P(\xi_v)$.

Lemma: With the above notation, we may take as our ξ_v the following elements:

$$\alpha_b = \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} (0 \leq b < p^2)$$

$$\beta_h = \begin{pmatrix} p & h \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ pNs & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \begin{pmatrix} p & h \\ -Ns & r \end{pmatrix} (0 < h < p)$$

$$\sigma = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^2 & -t \\ N & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \begin{pmatrix} p^2d & t \\ -N & 1 \end{pmatrix}.$$

For β_h we choose, for each h, two integers r and s so that $pr + Nsh = 1$; for σ, we choose t and d with $p^2d + Nt = 1$.

We want to define a lifting for such elements; so, for $\gamma\alpha\gamma' \in \Gamma_0\alpha\Gamma_0$ we define $(\gamma\alpha\gamma')^* = \gamma^*\xi\gamma'^*$. Then, by our previous work, this lift defines a one-to-one homomorphism of $\Gamma_0\alpha\Gamma_0$ into $\Delta_0\xi\Delta_0$.

We then calculate the lifts of each of the elements in the lemma above:

$$\alpha_b^* = (\alpha_b, p^{1/2})$$

[1]We do not go into the proof of this in detail, but instead summarize. I confess that I did not verify all the details for myself. We resume the high level of detail for the next theorem.

$$\beta_h^* = (\beta_h, \epsilon_p^{-1}(\frac{-h}{p}))$$

$$\sigma^* = (\sigma, p^{-1/2})$$

So, finally we calculate

$$f|T(p^2) = p^{k/2-2}\sum_v \chi(a_v) f|[\xi_v]_{k/2}$$

$$= p^{k/2-2}(\sum_b f|[\alpha_b^*]_{k/2} + \chi(p)\sum_h f|[\beta_h^*]_{k/2} + \chi(p^2) f|[\sigma]_{k/2}),$$

which yields the theorem. Please see Koblitz, pp. 208-210, for a detailed proof.

Now, assuming that we can find *eigenfunctions* of the Hecke operators, we obtain another useful result of Shimura:

Theorem (Shimura, 1.8). Let t be a positive integer, p be a prime, and $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ be an eigenfunction of the Hecke operator $T(p^2)$ with eigenvalue ω_p. (i.e., assume that $f|T(p^2) = \omega_p f$.)

Suppose that p divides N, or p^2 does not divide t. Then taking λ and χ_1 as before, the following identities hold:

$$\omega_p a(t) = a(p^2 t) + \chi_1(p)(\frac{t}{p})p^{\lambda-1}a(t)$$

$$\omega_p a(p^{2m}t) = a(p^{2m+2}t) + \chi_1(p^2)p^{k-2}a(p^{2m-2}t).$$

Moreover, the Dirichlet series $\sum_{n=1}^{\infty} a(tn^2)n^{-s}$ may be factored as

$$\sum_{n=1}^{\infty} a(tn^2)n^{-s} = (\sum_{(p,n)=1} a(tn^2)n^{-s})\frac{1-\chi_1(p)(\frac{t}{p})p^{\lambda-1-s}}{1-\omega_p p^{-s} + \chi(p)^2 p^{k-2-2s}}.$$

Proof: We follow Shimura's proof, but add considerably more detail.

We first prove a more general form of the first two identities: If n is relatively prime to p, then I claim that the following hold:

$$\omega_p a(tn^2) = a(p^2 tn^2) + \chi_1(p)(\frac{t}{p})p^{\lambda-1}a(tn^2)$$

$$\omega_p a(p^{2m}tn^2) = a(p^{2m+2}tn^2) + \chi_1(p^2)p^{k-2}a(p^{2m-2}tn^2).$$

The identities of the theorem are then achieved by setting $n = 1$. To motivate the proof of the Dirichlet series identity, we note that we may factor any positive integer n_0 as np^m for some $m \geq 0$ and n relatively prime to p.

Recall that since f is an eigenvalue, we may write

$$a(n)\omega_p = a(p^2n) + \chi_1(p)(\frac{n}{p})p^{\lambda-1}a(n) + \chi(p^2)p^{k-2}a(n/p^2).$$

To show the first identity we substitute tn^2 for n and get

$$a(n)\omega_p = a(p^2tn^2) + \chi_1(p)(\frac{tn^2}{p})p^{\lambda-1}a(tn^2) + \chi(p^2)p^{k-2}a(tn^2/p^2).$$

In case p^2 does not divide t, then p^2 does not divide tn^2 as $(n,p)=1$, therefore $a(tn^2/p^2)=0$ and the last term disappears. In case $p|N$, $\chi(p)$ and $\chi_1(p)$ will vanish as they are Dirichlet characters modulo N. In either case,

$$a(n)\omega_p = a(p^2tn^2) + \chi_1(p)(\frac{tn^2}{p})p^{\lambda-1}a(tn^2).$$

We observe that $(\frac{tn^2}{p}) = (\frac{t}{p})(\frac{n^2}{p}) = (\frac{t}{p})$ so that the above becomes

$$a(n)\omega_p = a(p^2tn^2) + \chi_1(p)(\frac{t}{p})p^{\lambda-1}a(tn^2).$$

To obtain the second identity, substitute tn^2p^{2m} for n; we obtain

$$a(n)\omega_p = a(p^{2+2m}tn^2) + \chi_1(p)(\frac{tn^2p^{2m}}{p})p^{\lambda-1}a(tn^2p^{2m}) + \chi(p^2)p^{k-2}a(tn^2p^{2m-2}).$$

We observe that $(\frac{tn^2p^{2m}}{p}) = 0$ as the bottom divides the top so that the second term drops out:

$$a(n)\omega_p = a(p^{2+2m}tn^2) + \chi(p^2)p^{k-2}a(tn^2p^{2m-2}),$$

as desired.

Now we will use these identities and perform power series manipulations to factor the Dirichlet series: As a formal power series, write $H_n(x) = \sum_{n=0}^{\infty} a(tp^2mn^2)x^m$. We now add x times the first identity and, for each $m>0$, x^{m+1} times the second. The left hand side is,

$$x\omega_p a(tn^2) + \sum_{m=1}^{\infty}(x^{m+1}\omega_p a(tp^{2m}n^2))$$

$$= x\omega_p a(tn^2) + \sum_{m=1}^{\infty}(x^m a(tp^{2m}n^2))$$

$$= x\omega_p \sum_{m=0}^{\infty}(x^m a(tp^{2m}n^2))$$

$$= x\omega_p H_n(x).$$

The right hand side is,

$$a(tp^2n^2)x+\chi_1(p)(\frac{t}{p})p^{\lambda-1}a(tn^2)x+\sum_{m=1}^{\infty}[a(tp^{2m+2}n^2)x^{m+1}+\chi(p)^2p^{k-2}a(tp^{2m-2}n^2)x^{m+1}]$$

$$= x\sum_{m=0}^{\infty}a(tp^{2m+2}n^2)x^m+\chi_1(p)(t|p)p^{\lambda-1}a(tn^2)x+\sum_{m=1}^{\infty}\chi(p)^2p^{k-2}a(tp^{2m-2}n^2)x^{m+1}.$$

We simplify these terms as follows:

$$x\sum_{m=0}^{\infty}a(tp^{2m+2}n^2)x^m$$

$$= x\sum_{m=1}^{\infty}a(tp^{2m}n^2)x^{m-1}$$

$$= \sum_{m=1}^{\infty}a(tp^{2m}n^2)x^m$$

$$= \sum_{m=0}^{\infty}a(tp^{2m}n^2)x^m - a(tn^2)$$

$$= H_n(x) - a(tn^2).$$

And,

$$\sum_{m=1}^{\infty}\chi(p)^2p^{k-2}a(tp^{2m-2}n^2)x^{m+1}$$

$$= \chi(p)^2p^{k-2}\sum_{m=1}^{\infty}a(tp^{2m-2}n^2)x^{m+1}$$

$$= \chi(p)^2p^{k-2}\sum_{m=0}^{\infty}a(tp^{2m}n^2)x^{m+2}$$

$$= \chi(p)^2p^{k-2}x^2H_n(x).$$

Accordingly, the right side simplifies to

$$H_n(x) - a(tn^2) + \chi_1(p)(\frac{t}{p})p^{\lambda-1}a(tn^2)x + \chi(p)^2p^{k-2}x^2H_n(x).$$

We have then established the identity

$$x\omega_pH_n(x) = H_n(x) - a(tn^2) + \chi_1(p)(\frac{t}{p})p^{\lambda-1}a(tn^2)x + \chi(p)^2p^{k-2}x^2H_n(x).$$

Collecting terms with an $H_n(x)$ term,

$$H_n(x)[x\omega_p - 1 - \chi(p)^2 p^{k-2}x^2] = -a(tn^2) + \chi_1(p)(\frac{t}{p})p^{\lambda-1}a(tn^2)x$$

and therefore

$$H_n(x) = \frac{a(tn^2)[1 - \chi_1(p)(\frac{t}{p})p^{\lambda-1}x]}{1 - \omega_p x + \chi(p)^2 p^{k-2}x^2}.$$

Substituting p^{-s} for x we obtain

$$H_n(p^{-s}) = \frac{a(tn^2)[1 - \chi_1(p)(\frac{t}{p})p^{\lambda-1-s}]}{1 - \omega_p p^{-s} + \chi(p)^2 p^{k-2-2s}}.$$

But we may write any integer n uniquely as a power of p and a number relatively prime to p, so that

$$\sum_{n=1}^{\infty} a(tn^2)n^{-s}$$

$$= \sum_{(n,p)=1} \sum_{m=0}^{\infty} a(t(p^m n)^2)(p^m n)^{-s}$$

$$= \sum_{(n,p)=1} H_n(p^{-s})n^{-s}$$

so that

$$\sum_{n=1}^{\infty} a(tn^2)n^{-s} = \sum_{(n,p)=1} H_n(p^{-s})n^{-s}$$

$$= \sum_{(n,p)=1} \frac{a(tn^2)[1 - \chi_1(p)(\frac{t}{p})p^{\lambda-1-s}]}{1 - \omega_p p^{-s} + \chi(p)^2 p^{k-2-2s}} n^{-s}$$

$$= [\sum_{(n,p)=1} a(tn^2)n^{-s}] \frac{[1 - \chi_1(p)(\frac{t}{p})p^{\lambda-1-s}]}{1 - \omega_p p^{-s} + \chi(p)^2 p^{k-2-2s}}.$$

We now would like to assert that there are modular forms which are *simultaneous* eigenforms for all the Hecke operators $T(p^2)$, so we do so:

Proposition: There are modular forms which are simultaneous eigenforms for all the Hecke operators $T(p^2)$.

The proof is by the same argument as for integral weight modular forms (see, for example, p. 82 of Shimura's book.)

The following theorem is now immediate[2]:

[2] By 'immediate', I mean that I don't understand how to prove it.

Theorem (Shimura, 1.9). Let $f(x) = \sum_{n=0}^{\infty} a(n)q^n \in G_{k/2}(N, \chi)$. Suppose that f is a common eigenfunction of all the Hecke operators $T(p^2)$, with $f|T(p^2) = \omega_p f$. Let t be a positive integer which has no square factor other than 1 prime to N. Then the Dirichlet series $\sum_{n=1}^{\infty}$ has the following Euler product:

$$\sum_{n=1}^{\infty} = a(t) \prod_{p} \frac{1 - \chi_1(p)(\frac{t}{p})p^{\lambda-1-s}}{1 - \omega_p p^{-s} + \chi(p)^2 p^{k-2-2s}}.$$

3 The Shimura Lifting

We now skip to the main theorem of Shimura's *Annals* paper. It says that if we are given a modular form of half integral weight that is an eigenfunction of all the Hecke operators, we may define a corresponding new function which turns out to be a modular form of integral weight.

Theorem (Shimura). Let $k \geq 3$ be an odd integer, and write $\lambda = (k-1)/2$. Suppose that N is a positive integer divisible by 4, χ is a Dirichlet character modulo N. Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{k/2}(N, \chi)$ be a modular form of weight $k/2$ for $\Gamma_0(N)$ (or more properly, $\Delta_0(N)$), and suppose that f is an eigenfunction of the Hecke operators $T(p^2)$ for all primes p, with eigenvalues ω_p; i.e., $f|T(p^2) = \omega_p f$ for all p. Define a function $F(z) = \sum_{n=1}^{\infty} A(n)q^n$, where the $A(n)$ are defined by the identity

$$\sum_{n=1}^{\infty} A(n)n^{-s} = \prod_{p} [1 - \omega^p p^s + \chi(p)^2 p^{k-2-2s}]^{-1}.$$

Finally, let N_0 be the greatest common divisor of the integers N_t for all square-free t such that $a(t) \neq 0$.

Then $F \in M_{k-1}(N_0, \chi^2)$. Moreover, if $k \geq 5$, then F is a cusp form.

Shimura's theorem was subsequently improved by Niwa:

Theorem (Niwa): We may instead take $N_0 = N/2$.

Shimura proves his theorem using the Euler product above, and the so-called 'converse theorem' of Weil. Shimura quotes Weil's theorem as follows:

Theorem (Weil). Suppose F is given by its Fourier series $F(z) = \sum_{n=1}^{\infty} c_n q^n$, and we construct a corresponding Dirichlet series $D(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. Suppose that M is a positive integer, ϕ is a Dirichlet character modulo M, and P is a set of prime numbers having nonempty intersection $\{a, a+b, a+2b, \ldots\}$ for which $(a, b) = 1, b > 0$. Suppose furthermore that the following conditions are

satisfied:
(i) $D(s)$ converges absolutely in some half plane $Res > \sigma$.
(ii) For every primitive character ψ modulo r with $r \in \{1\} \cup P$ and $(r, M) = 1$, if we write

$$R(s,\phi) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\psi(n)c_n n^{-s},$$

then $R(s,\phi)$ can be continued to a holomorphic function in the whole complex plane, and is bounded in every vertical strip.
(iii) There exists a Dirichlet series $D'(s) = \sum_{n=1}^{\infty} c'_n n^{-s}$ that absolutely converges in some half plane $Res > \sigma'$, and satisfies the following functional equation for all characters ψ considered in (ii):

$$R(k-s,\psi) = C_\psi (r^2 M)^{s-(k/2)}(2\pi)^{-s}\sum_{n=1}^{\infty}\overline{\psi}(n)c'_n n^{-s}$$

where C_ψ is defined by

$$C_\psi = \psi(M)\phi(r)(\psi)^2/r.$$

Then $F \in M_k(M,\phi)$.

It should not surprise the reader that it is even work to prove Shimura's theorem than it is to state all of those hypotheses. (i), it turns out, is not too difficult to prove: One can show that $a(n) = O(n^{1/4})$, which implies that $\sum_m a(tm^2)m^{-s}$ is absolutely convergent for $Res > 1 + k/2$. This implies that the L-series $\sum_{m=1}^{\infty}\chi_t(m)m^{\lambda-1-s}$ is absolutely convergent for $Res > (k-1)/2$, which in turn implies that $\sum_n A_t(n)n^{-s}$ converges absolutely for $Res > 1 + k$. Thus condition (i) is satisfied. Moreover, Weil's theorem has an additional condition under which F is a cusp form, and this is used in Shimura's theorem to prove that if $k \geq 5$ then F is a cusp form there too.

The proof of conditions (ii) and (iii) comprise the next sixteen pages of Shimura's *Annals* paper. To put it mildly, the proof is difficult, so we conclude here and refer the reader to Shimura's paper for the proof.