Fourier Analysis in Arithmetic Statistics

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IIT Bombay, December 3, 2025

thornef.github.io/iitb-2025.pdf

frankthorne — gp — 80×24

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Last login: Wed Dec  3 09:31:01 on ttys000
Franks-MacBook-Pro-2:~ frankthorne$ gp
          GP/PARI CALCULATOR Version 2.8.0 (development 18205-1c269ec)
               i386 running darwin (x86-64 kernel) 64-bit version
compiled: Nov 11 2015, Apple LLVM version 6.0 (clang-600.0.54) (based on LLVM 3.
5svn)
                            threading engine: single
               (readline not compiled in, extended help enabled)

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Type ? for help, \q to quit.
Type ?15 for how to get moral (and possibly technical) support.

parisize = 8000000, primelimit = 500000
? default(realprecision, 50)
? exp(Pi*sqrt(163))
%2 = 262537412640768743.99999999999925007259719818568888
? 
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Sample Theorem 1: Counting Cubic Fields

For any integer $d \geq 1$, write

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Theorem (Davenport-Heilbronn)

We have

$$N_3(X) = \frac{1}{3\zeta(3)} X + o(X).$$

Sample Theorem 2: Counting Quartic and Quintic Fields

Theorem (Bhargava)

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$$N_4(X, S_4) \sim \frac{5}{24} \prod_p (1 + p^{-2} - p^{-3} - p^{-4}) X,$$

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Sample Theorem 3: 3-torsion in Quadratic Class Groups

Theorem (Davenport-Heilbronn)

We have

$$\sum_{|D|<X} \#|\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))[3]| = \frac{10}{\pi^2} X + o(X).$$

Sample Theorem 4: 2-Selmer Groups in Elliptic Curves

Theorem (Bhargava-Shankar)

When elliptic curves E are ordered by height, the average size of their 2-Selmer groups is 3.

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Corollary

Their average rank is at most 1.5.

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Theorem

There exists an explicit, "nice" bijection

$$\{ \textit{ Something nice } \} \longleftrightarrow G(\mathbb{Z}) \backslash V(\mathbb{Z})$$

where V is a f.d. representation of an algebraic group G.

Parametrization: The Basic Metatheorem

Theorem

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Moreover, certain arithmetic properties on the left correspond to congruence conditions on the right.

Example: Binary Cubic Forms

Let V be the space of binary cubic forms:

$$V := \{x(u, v) = au^3 + bu^2 v + cuv^2 + dv^3\}.$$

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- $\mathrm{Disc}(gx) = (\det g)^2 \mathrm{Disc}(x)$;
- $\mathrm{Disc}(x) = 0$ if and only if $x(u,v)$ has a repeated root.

Example: Binary Cubic Forms (2)

Theorem (Levi, Delone-Faddeev, Gan-Gross-Savin)

$G(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ parametrize cubic rings. Further, if $v \leftrightarrow R$,

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- $\text{Stab}(v)$ *is isomorphic to* $\mathrm{Aut}(R)$;
- $\mathrm{Disc}(v) = \mathrm{Disc}(R)$;
- *(Davenport-Heilbronn) R is maximal iff, for all primes p, v satisfies a certain congruence condition* $\pmod{p^2}$.

Some parametrizations

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- ... and more!
  (Bhargava, Ho, Shankar, Varma, X. Wang, Wood, ...)

More Interesting Parametrizations

Table 1: Summary of Higher Composition Laws

#	Lattice ($V_{\mathbb{Z}}$)	Group acting ($G_{\mathbb{Z}}$)	Parametrizes ($\mathcal{C}$)	(k)	(n)	(H)
1.	$\{0\}$	-	Linear rings	0	0	A_0
2.	$\widetilde{\mathbb{Z}}$	$\mathrm{SL}_1(\mathbb{Z})$	Quadratic rings	1	1	A_1
3.	$(\mathrm{Sym}^2\mathbb{Z}^2)^*$ (GAUSS'S LAW)	$\mathrm{SL}_2(\mathbb{Z})$	Ideal classes in quadratic rings	2	3	B_2
4.	$\mathrm{Sym}^3\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})$	Order 3 ideal classes in quadratic rings	4	4	G_2
5.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^2$	Ideal classes in quadratic rings	4	6	B_3
6.	$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^3$	Pairs of ideal classes in quadratic rings	4	8	D_4
7.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^4$	$\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_4(\mathbb{Z})$	Ideal classes in quadratic rings	4	12	D_5
8.	$\wedge^3\mathbb{Z}^6$	$\mathrm{SL}_6(\mathbb{Z})$	Quadratic rings	4	20	E_6
9.	$(\mathrm{Sym}^3\mathbb{Z}^2)^*$	$\mathrm{GL}_2(\mathbb{Z})$	Cubic rings	4	4	G_2
10.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Order 2 ideal classes in cubic rings	12	12	F_4
11.	$\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})^2$	Ideal classes in cubic rings	12	18	E_6
12.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^6$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_6(\mathbb{Z})$	Cubic rings	12	30	E_7
13.	$(\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3)^*$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Quartic rings	12	12	F_4
14.	$\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$	$\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$	Quintic rings	40	40	E_8

Bhargava, *Higher composition laws IV*, Ann. Math., 2008

Still More Interesting Parametrizations

	Group (s.s.)	Representation	Geometric Data	Invariants	Dynkin	§
1.	SL_2	$\mathrm{Sym}^4(2)$	(C, L_2)	$2, 3$	$A_2^{(2)}$	4.1
2.	SL_2^2	$\mathrm{Sym}^2(2) \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_3^{(2)}$	6.1
3.	SL_2^4	$2 \otimes 2 \otimes 2 \otimes 2$	$(C, L_2, L_2', L_2'') \sim (C, L_2, P, P')$	$2, 4, 4, 6$	$D_4^{(1)}$	6.2
4.	SL_2^3	$2 \otimes 2 \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 4, 6$	$B_3^{(1)}$	6.3.1
5.	SL_2^2	$\mathrm{Sym}^2(2) \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_3^{(2)}$	6.3.3
6.	SL_2^2	$2 \otimes \mathrm{Sym}^3(2)$	(C, L_2, P_3)	$2, 6$	$G_2^{(1)}$	6.3.2
7.	SL_2	$\mathrm{Sym}^4(2)$	(C, L_2, P_3)	$2, 3$	$A_2^{(2)}$	6.3.4
8.	$\mathrm{SL}_2^2 \times \mathrm{GL}_4$	$2 \otimes 2 \otimes \wedge^2(4)$	$(C, L_2, M_{2,6})$	$2, 4, 6, 8$	$D_5^{(1)}$	6.6.1
9.	$\mathrm{SL}_2 \times \mathrm{SL}_6$	$2 \otimes \wedge^3(6)$	$(C, L_2, M_{3,6})$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(1)}$	6.6.2
10.	$\mathrm{SL}_2 \times \mathrm{Sp}_6$	$2 \otimes \wedge_0^3(6)$	$(C, L_2, (M_{3,6}, \varphi))$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(2)}$	6.6.3
11.	$\mathrm{SL}_2 \times \mathrm{Spin}_{12}$	$2 \otimes S^+(32)$	$(C \to \mathbb{P}^1(\mathscr{H}_3(\mathbb{H})), L_2)$	$2, 6, 8, 12$	$E_7^{(1)}$	6.6.3
12.	$\mathrm{SL}_2 \times E_7$	$2 \otimes 56$	$(C \to \mathbb{P}^1(\mathscr{H}_3(\mathbb{O})), L_2)$	$2, 6, 8, 12$	$E_8^{(1)}$	6.6.3
13.	SL_3	$\mathrm{Sym}^3(3)$	(C, L_3)	$4, 6$	$D_4^{(3)}$	4.2
14.	SL_3^3	$3 \otimes 3 \otimes 3$	$(C, L_3, L_3') \sim (C, L_3, P)$	$6, 9, 12$	$E_6^{(1)}$	5.1
15.	SL_3^2	$3 \otimes \mathrm{Sym}^2(3)$	(C, L_3, P_2)	$6, 12$	$F_4^{(1)}$	5.2.1
16.	SL_3	$\mathrm{Sym}^3(3)$	(C, L_3, P_2)	$4, 6$	$D_4^{(3)}$	5.2.2
17.	$\mathrm{SL}_3 \times \mathrm{SL}_6$	$3 \otimes \wedge^2(6)$	$(C, L_3, M_{2,6})$ with $L^{\otimes 2} \cong \det M$	$6, 12, 18$	$E_7^{(1)}$	5.5
18.	$\mathrm{SL}_3 \times E_6$	$3 \otimes 27$	$(C \hookrightarrow \mathbb{P}^2(\mathbb{O}), L_3)$	$6, 12, 18$	$E_8^{(1)}$	5.4
19.	$\mathrm{SL}_2 \times \mathrm{SL}_4$	$2 \otimes \mathrm{Sym}^2(4)$	(C, L_4)	$8, 12$	$E_6^{(2)}$	4.3
20.	$\mathrm{SL}_5 \times \mathrm{SL}_5$	$\wedge^2(5) \otimes 5$	(C, L_5)	$20, 30$	$E_8^{(1)}$	4.4

Table 1: Table of coregular representations and their moduli interpretations

Bhargava and Ho, *Coregular spaces and genus one curves*, Camb. J. Math.

Theorem (DHBBPBSTTTBTT*)

We have

$$N_3(X) = \frac{1}{3\zeta(3)}X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{\frac{2}{3}}(\log X)^3).$$

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*: Davenport-Heilbronn, Belabas, Belabas-Bhargava-Pomerance, Bhargava-Shankar-Tsimerman, Taniguchi-T., Bhargava-Taniguchi-T.

A fundamental task in this subject is to give some quantitative measures of additive structure in a set, and then investigate to what extent these measures are equivalent to each other. For example, one could try to quantify each of the following informal statements as being some version of the assertion "A has additive structure":

- $A + A$ is small;
- $A - A$ is small;
- $A - A$ can be covered by a small number of translates of A;
- kA is small for any fixed k;
- there are many quadruples $(a_1, a_2, a_3, a_4) \in A \times A \times A \times A$ such that $a_1 + a_2 = a_3 + a_4$;
- there are many quadruples $(a_1, a_2, a_3, a_4) \in A \times A \times A \times A$ such that $a_1 - a_2 = a_3 - a_4$;
- the convolution $1_A * 1_A$ is highly concentrated;
- the subset sums $FS(A) := \{\sum_{a \in B} a : B \subseteq A\}$ have high multiplicity;
- the Fourier transform $\widehat{1_A}$ is highly concentrated;
- the Fourier transform $\widehat{1_A}$ is highly concentrated in a cube;
- A has a large intersection with a generalized arithmetic progression, of size comparable to A;
- A is contained in a generalized arithmetic progression, of size comparable to A;
- A (or perhaps $A - A$, or $2A - 2A$) contains a large generalized arithmetic progression.

The reader is invited to investigate to what extent these informal statements are true for sets such as progressions and cubes, and false for sets such as random sets. As it turns out, once one makes the above assertions more quantitative, there are a number of deep and important equivalences between them; indeed, to oversimplify tremendously, all of the above criteria for additive structure are "essentially"

Tao and Vu, *Additive Combinatorics*, Cambridge Univ. Press, 2006

The overall program

Let $\Phi_{p^2} : V(\mathbb{Z}/p^2\mathbb{Z}) \to \mathbb{C}$ be, for example, the characteristic function of nonmaximal cubic rings.

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- Show that Φ_{p^2} is nice in some way we can quantify;
- Develop lattice point counting methods which use the niceness.

An explicit evaluation

Theorem (Taniguchi-T., 2011)

We have

$$\widehat{\Phi_{p^2}}(v) = \begin{cases} p^{-2} + p^{-3} - p^{-5} & v/p : \textit{ of type } (0), \\ p^{-3} - p^{-5} & v/p : \textit{ of type } (1^3), (1^2 1), \\ -p^{-5} & v/p : \textit{ of type } (111), (21), (3). \\ p^{-3} - p^{-5} & v : \textit{ of type } (1^3_{**}), \\ -p^{-5} & v : \textit{ of type } (1^3_{*}),\ (1^3_{\max}), \\ 0 & \textit{otherwise.} \end{cases}$$

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So:

$$\frac{1}{p^8} \sum_{v \in V(\mathbb{Z}/p^2\mathbb{Z})} |\widehat{\Phi_{p^2}}(v)| \ll p^{-7}.$$

Consequence – Improved Davenport-Heilbronn

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$$N_3(X) = \frac{1}{3\zeta(3)}X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{\frac{3}{5}+\epsilon}) + O(X^{1-\frac{1}{8-7+2}+\epsilon}).$$

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- The morphism method (Ishitsuka, Taniguchi, T., Xiao).

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Thank you!

धन्यवाद!