Comprehensive exam syllabus for Math 782, Analytic Number Theory (Fall 2011)

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The exam will consist of **two parts**. The first part will consist of **core material** that is foundational in analytic number theory. All students should thoroughly master this material. The second part will consist of additional questions which test additional material covered during the course (and possibly outside it), and which will give you a chance to show off what else you know. Some part II questions may be individualized – you are invited to e-mail me (at least two weeks before the exam, please!) and tell me what topics beyond Part I you have studied.

The cutoff for passing will be roughly 80 points. Part I will be worth 100 points, and Part II will be worth a substantial number of additional points. You should plan on doing most or all of Part I correctly, and getting at least something in Part II.

Formulas will be provided where appropriate, including Stirling's formula and the Hadamard product formula for the zeta function.

Syllabus for Part I

1. Definition of the Riemann zeta function. The Dirichlet series and Euler product. Elementary manipulation such as taking logarithmic derivatives, etc. Prove there are only finitely many primes.

2. Arithmetic functions, especially $\Lambda(n)$ and $\mu(n)$. Be able to prove identities such as $\log(n) = \sum_{d|n} \Lambda(d)$.

3. Partial summation (or, equivalently, Stieltjes integration). Equivalence of various forms of the prime number theorem. Proof that $\sum_{p \leq x} \frac{1}{p}$ is asymptotic to $\log\log x$.

4. Dirichlet characters. Computations (e.g. list all the Dirichlet characters of any modulus), orthogonality relations (with proofs), real Dirichlet characters, primitive and induced characters.

5. Dirichlet L-functions. Prove of the prime number theorem for arithmetic progressions, given $L(1,\chi) \neq 0$. Prove that $L(1,\chi) \neq \infty$.

6. The divisor function $d(n)$. Estimates for $\sum_{n \leq x} d(n)$. (Know at least one proof)

7. Möbius inversion (with proof). Applications such as bounding the number of primes in $[x, x+y]$; counting squarefree integers $\leq x$ (or related).

8. Convolution of arithmetic sequences, and relation to Euler products.

9. Additional arithmetic functions such as the Euler ϕ function and the sum of divisors function $\sigma(n)$. Proofs of identities tying these functions together.

10. Positive definite integral binary quadratic forms. The equivalence relation. Discriminants of quadratic forms. Automorphs. Definition of the class number. Computation of class numbers.

11. Elementary complex analysis. Know what a meromorphic function is, what a pole is (and what the order of a pole is); know and be able to apply Cauchy's residue theorem. Evaluation of (relatively simple) contour integrals.

12. Understand the statement of Riemann's theorem on analytic continuation and functional equation of the zeta function. (Proofs won't be covered in Part I.)

13. Understand what the *explicit formula* is. Explain, in broad terms, how it is proved and what it says about the primes. Understand what Perron's formula says.

14. Gauss sums. Evaluation of G^2.

15. Good homework problems to review (these or similar would be fair game for Part I on the exam): 1 (1, 2, 3, 6, 7), 2 (1, 4, 6), 3 (3, 5), 4 (1-4), 5 (2-5), 6b (1-5), 7 (1, 2), 8 (1), 9 (1, 6-8).

This leaves **a lot** of material for Part II, including the proofs of analytic statements about zeta and L-functions (including the prime number theorem), the Poisson summation formula, Dirichlet's class number formula, Polya-Vinogradov, the circle method, the Riemann hypothesis and its consequences, and much more! You certainly don't have to know all of this, but you should know at least a little bit of it. There will be enough of a variety of questions that you should feel safe picking and choosing.