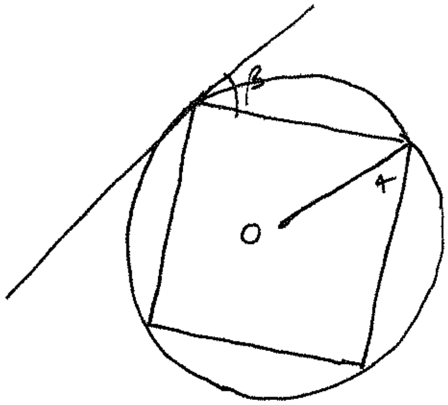
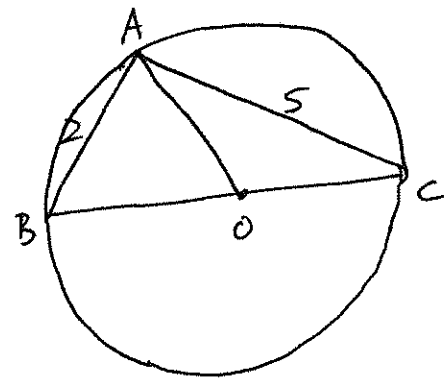
Math 531 Exam 1.
(Version 1)
Hand in six solutions.

Note: O is always used for the centers of circles.

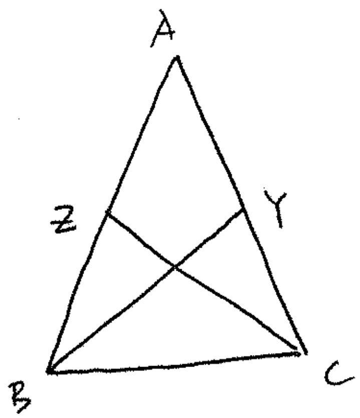
1. A square is inscribed in a circle, and a tangent line is drawn. Find α and β.



2. Find $\overline{AO}$.
(Here $\overline{AB} = 2$, $\overline{AC} = 5$)



3.

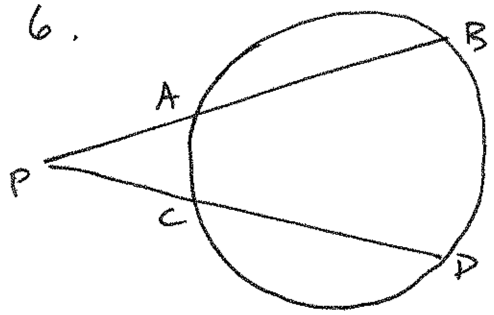


Medians BY and CZ have been drawn in isosceles $\triangle ABC$ with base BC. Prove BY = CZ.

4. Prove that a parallelogram with equal diagonals is a rectangle.

5. In $\triangle ABC$, prove that $\angle A$ is a right angle if and only if the length of the median from A to BC is exactly half the length of BC.

6.



Prove (as proved in class)

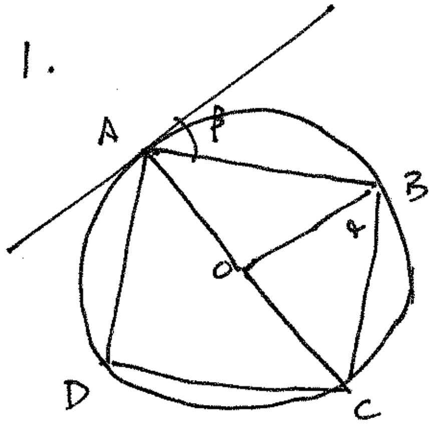
$\angle P \overset{\circ}{=} \widehat{BD} - \widehat{AC}$.

7. Given ΔABC, let X, Y and Z be the midpoints of sides BC, AC, and AB, respectively, and draw ΔXYZ.

Show that this divides the original triangle into four congruent triangles.

Exam solutions.

1.



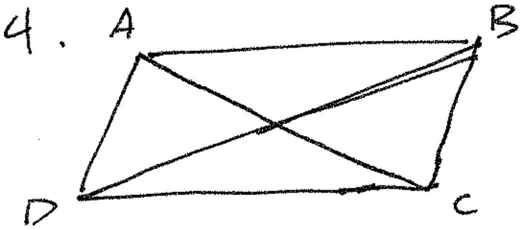
We have $\triangle AOB \cong \triangle COB$ by SSS, so $\angle OBC = \angle OBA$. These add to $90°$, so $\angle OBC = \varphi = 45°$.

Also $\beta = 45°$ because it subtends a $90°$ arc and $45° = \frac{1}{2} \cdot 90°$.

2. A triangle inscribed in a circle where one edge of the triangle is the diameter must be right. So $\angle BAC$ is $90°$. By the Pythagorean Theorem, $BC = \sqrt{2^2 + 5^2} = \sqrt{29}$ and $AO = \frac{1}{2} BC = \frac{\sqrt{29}}{2}$.

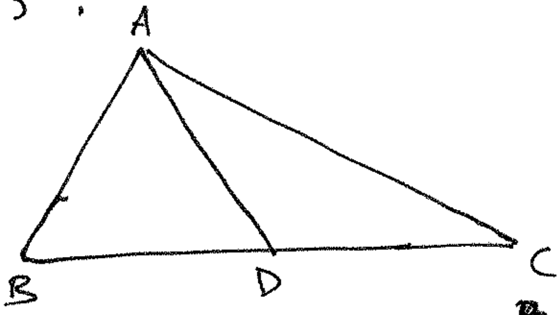
3. By the pons asinorum $\angle ABC = \angle ACB$, and $AB = AC$, and $ZB = \frac{1}{2} AB$ and $CY = \frac{1}{2} AC$, so $ZB = CY$. of course $BC = BC$, so by SAS $\triangle ZBC \cong \triangle YCB$, so $CZ = BY$.

4.



(not to scale)

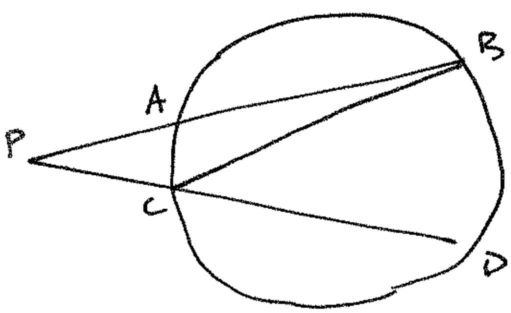
We have $AB = CD$ and $AD = BC$. (Theorem proved in class)
Also $DB = AC$ by hypothesis.
So $\triangle ACD \cong DBA$ by SSS, so $\angle CDA = \angle BAD$. These angles are supplementary since $AB \parallel CD$, so each is half of $180°$, or $90°$.
Similarly $\angle ABC$ is supplementary to $\angle BAD$, so $90°$, and the same is also true of $\angle BCD$.
Thus $ABCD$ is a rectangle.

5. 

Draw the circumcircle around $\triangle ABC$. Then as proved in class A is right if and only if BC is the diameter.

If BC is the diameter, then since AD is the median, $\overline{BD} = \overline{DC}$ and D is the center, so $\overline{AD} = \overline{BD} = \overline{DC}$ and so $\overline{AD} = \frac{1}{2}\overline{BC}$.

Conversely, if $AD = \frac{1}{2} BC$, then we know $BD = DC$ so $AD = BD = DC$. Draw the circle of radius BD with center D and it must be the same circle as before. Since BD and CD are radii on the same line, BC is a diameter.

6. 

We know $\angle BCD \doteq \frac{1}{2}\widehat{BD}$

$\angle ABC = \frac{1}{2}\widehat{AC}$,

$\angle P + \angle PBC + \angle BCP = 180^\circ$.

We also know $\angle BCP + \angle BCD = 180^\circ$, so

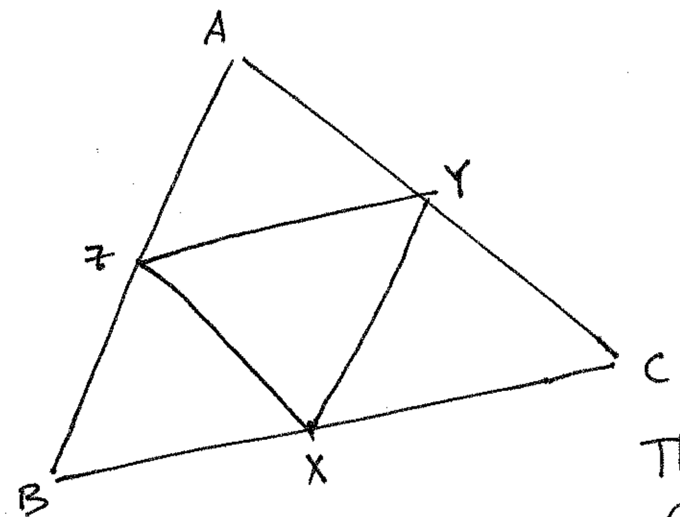
$$\angle P + \angle PBC + (180^\circ - \angle BCD) = 180^\circ$$

$$\angle P + \angle PBC - \angle BCD = 0$$

$$\angle P = \angle BCD - \angle PBC$$

$$= \frac{1}{2}\widehat{BD} - \frac{1}{2}\widehat{AC} \quad \text{as desired.}$$

tron sol p 3.



We know that $ZY \parallel BC$ because ZY divides AB and AC into equal proportions. Therefore $\angle AZY = \angle ZBX$ (they are corresponding angles) and so $\triangle AZY \sim \triangle ABC$ by AA.

Thus ~~AZ=AB~~ $ZY = \frac{1}{2} BC$ (because $\frac{ZY}{BC} = \frac{AZ}{AB} = \frac{1}{2}$). And so $ZY = BX = XC$

By the same reasoning $ZX = \frac{1}{2} AC$, $YX = \frac{1}{2} AB$.

So we have
$AZ = ZB = YX$
$AY = CY = ZX$
$ZY = BX = XC$

so all four small triangles are congruent by SSS.