AN EXPLICIT THETA CORRESPONDENCE FOR $(\widetilde{\mathrm{SL}_2}, PB^\times)$

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Abstract. These are rough notes for a talk to be given in the Wisconsin student number theory seminar. The talk is on Section 3 of Kartik Prasanna's paper [5]. The talk and this paper are meant to describe the background as throughly as possible, and then briefly describe what Prasanna proved.

Disclaimer: While I have attempted to be clear and correct, I prepared this in a short amount of time, and I am not an expert in this field. In particular, when I was unsure of something I frequently guessed.

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1. Analysis on v-adic spaces

We will say a little bit about some of the analysis that appears in Prasanna's paper, and is no doubt essential background for reading Tate's thesis, Jacquet-Langlands, etc.

1.1. **Haar measure on $\mathbb{Q}_v$ and $\mathbb{Q}_v$-vector spaces.** Reference: [9].

$\mathbb{Q}_v$ is a *locally compact topological field* . This means that every $x \in \mathbb{Q}_v$ has a neighborhood whose closure is compact. To prove this, we first prove that $\mathbb{Z}_v$ is compact, and then $x + \mathbb{Z}_v$ is too (addition is a homeomorphism).

An *additive Haar measure* μ on $\mathbb{Q}_v$ (i.e. one that satisfies $\mu(U) > 0$ for any open U, and $\mu(x + U) = \mu(U)$ for any x) therefore exists. Furthermore, it is unique up to a scalar multiple.

This measure can be easily described. Suppose normalize the measure so that $\mu(\mathbb{Z}_v) = 1$. Then for any $k > 0$, $\mathbb{Z}_v$ breaks up into cosets of $p^k\mathbb{Z}_v$, which must all have equal measure, so that $\mu(v^k\mathbb{Z}_v) = v^{-k}$ for any positive k. We quickly check that the same must be true for negative k as well. We can then use additivity and translation invariance to figure out the measure of any subset of $\mathbb{Q}_v$.

There is also a *multiplicative* Haar measure that instead satisfies $\mu(xU) = \mu(U)$.

Any $\mathbb{Q}_v$-vector space is locally compact as well, as are the adeles. Therefore these have a Haar measure as well, which is just the product measure, and we can normalize it as we please.

1.2. **Characters on the adeles.** Let ψ be the "usual" character of $\mathbb{Q}\backslash\mathbb{A}$. This is defined as follows. For an adele $x = (x_v)$, we write

$$\psi(x) = \prod_{v \le \infty} \psi_v(x_v),$$

where

$$\psi_\infty(x_\infty) = e^{2\pi i x}$$

and the definition at the finite places is taken so that ψ will be trivial on $\mathbb{Q}$. In particular, if $a \in \mathbb{Q}_v$ is given by

$$a = a_{-m}v^{-m} + \cdots + a_{-1}v^{-1} + a_0 v^0 + \cdots ,$$

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then we write
$$x(a) = a_{-m}v^{-m} + \cdots + a_{-1}v^{-1}$$
and
$$\psi_v(a) = \exp(-2\pi i x(a)).$$
By constuction, we can prove that ψ is indeed trivial on $\mathbb{Q}$.

We have further the

Proposition 1.1. *Every character ϕ on $\mathbb{Q}\backslash\mathbb{A}$ is of the form $\phi(x) = \psi(ax)$ for some $a \in \mathbb{Q}$.*

Sketch. We first see that any character ϕ on the adeles must be of the form $\phi = \prod_{v\leq\infty}\phi_v$ for characters ϕ_v on $\mathbb{Q}_v$. Then, we check that ϕ_∞ must be of the form stated. Finally, we check that if ϕ is trivial on $\mathbb{Q}$, then ϕ_∞ determines ϕ_v at all $v < \infty$.

See p. 13 of [4] for a complete proof. □

1.3. **The Fourier transform.** Let ψ be a character of $\mathbb{Q}_v$. The **Fourier transform** of a function $\phi : \mathbb{Q}_v \to \mathbb{C}$ is given by

$$\hat{\varphi}(y) = \int_{\mathbb{Q}_v} \varphi(x)\psi(xy)dx. \tag{1.1}$$

Similarly, for a finite-dimensional $\mathbb{Q}_v$-vector space V with a bilinear form $\langle -, - \rangle$, the Fourier transform of a function $\phi : V \to \mathbb{C}$ with respect to the pairing $(x, y) \to \psi(\langle x, y\rangle)$ is given by

$$\hat{\varphi}(y) = \int_{\mathbb{Q}_v} \varphi(x)\psi(\langle x, y\rangle)dx. \tag{1.2}$$

We shall also denote this function (following Prasanna) as $\mathcal{F}_\psi(\varphi)$ or simply $\mathcal{F}_\psi(\varphi)$.

The **Fourier inversion formula** says that (in both cases) we will have
$$\hat{\hat{\varphi}}(y) = \varphi(-y),$$
up to some constant. This constant depends on our choice of normalization of the Haar measure, and we may (and will) take this constant to be 1. In Prasanna's language, the Haar measure on V is picked to be **autodual** with respect to the pairing $(x, y) \to \psi(\langle x, y\rangle)$.

The same may be done over the adeles as well.

1.4. **The Schwartz space.** The **Schwartz space** of functions on $\mathbb{R}^n$ is defined to be the set of all C^∞ functions f, such that f and all its derivatives decay faster than any polynomial. One nice feature about the Schwartz class is that the classical Fourier transform maps the Schwartz class to itself.

Over the v-adic numbers we can make a similar definition: (See [4]) We define the Schwartz space of $\mathbb{Q}_v$ to consist of those functions which have compact support and are locally constant.

Finally we define the Schwartz space $\mathcal{S}(\mathbb{A}^n)$ of the adeles: A "pure" tensor is a function $f = \prod_{v\leq\infty} f_v$ on $\mathbb{A}$, so that each $f_v \in \mathcal{S}(\mathbb{Q}_v^n)$, and all but finitely many f_v are equal to the characteristic function of $\mathbb{Z}_v$. Then, $\mathcal{S}(\mathbb{A}^n)$ is equal to the span of these pure tensors.

2. The algebraic setup

2.1. **Quaternion algebras.** Let B be an *indefinite quaternion algebra* over $\mathbb{Q}$. We recall the definition of a quaternion algebra: B is a four-dimensional algebra over $\mathbb{Q}$, whose center is isomorphic to $\mathbb{Q}$, with no nontrivial ideals. B can be realized as
$$B = < 1, i, j, k \mid i^2 = a,\ j^2 = b, ij = -ji = k >,$$
and we write $B = < a, b >$ for short.

The condition that B be *indefinite* means that

$$B \otimes \mathbb{R} \simeq M_2(\mathbb{R}). \tag{2.1}$$

Equivalently, a and b are not both negative. We can realize this isomorphism explicitly as follows: (Assume without loss of generality that $a > 0$.)

$$1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i \longrightarrow \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \qquad j \longrightarrow \begin{pmatrix} 0 & \pm\sqrt{|b|} \\ \sqrt{|b|} & 0 \end{pmatrix}.$$

The $\pm$ in the last matrix is the same as the sign of b.

We let $V \subset B$ denote the subspace of trace 0 elements, i.e., the 3-dimensional subspace spanned by i, j, and k. We have a *quadratic form* on V given by $Q(x) = -N(x)$ and so we think of V as a quadratic space. Here N is the norm, and if we are thinking of B as matrices then the norm is equal to the determinant.

We further have a blinear form $\langle -, - \rangle$ on V given by $Q(x) = \frac{1}{2}\langle x, x \rangle$ and so we check that

$$\langle x, y \rangle = Q(x+y) - Q(x) - Q(y) = -N(x+y) + N(x) + N(y). \tag{2.2}$$

Remark. This conflicts with notation elsewhere. In particular, I have seen the alternative definition $\langle x, x \rangle = Q(x)$, but Prasanna makes the definition above.

The **orthogonal group** is then defined, with respect to this bilinear form, as

$$O(V) := \{\tau \in GL(V) | \langle \tau u, \tau v \rangle = \langle u, v \rangle\}.$$

Caution: It is natural to think of V as matrices in light of the isomorphism (2.1), and it is natural to think of $O(V)$ as matrices as well. But V has dimension 3 and so $O(V)$ consists of 3x3 matrices.

By Proposition 6.15 of [1] we have (since the dimension of V is odd)

$$O(V) = SO(V) \times \{\pm 1\},$$

and the center of $SO(V)$ is trivial.

Proposition 2.1. *Let $\beta \in PB^\times$ act on V by $R(\beta)(v) = \beta v \beta^{-1}$. Then this representation yields an isomorphism*

$$PB^\times \times \{\pm 1\} \simeq O(V).$$

Remark. It is good to look at the definition of this action and remind ourselves that it makes sense. In particular, we recall that V is a subspace of B, and so the multiplication $\beta v \beta^{-1}$ makes sense in B. The trace of $\beta v \beta^{-1}$ will be identical to that of v, i.e., 0, so in fact we get an element of V.

Proof. The action given describes a representation

$$B^\times \longrightarrow GL(V)$$

and see immediately that the scalars act trivially, so that we in fact have a representation $PB^\times \longrightarrow GL(V)$. We now wish to claim that the image is in fact in $SO(V)$. First we claim that it is in $O(V)$, in other words that $\langle \beta u \beta^{-1}, \beta v \beta^{-1} \rangle = \langle u, v \rangle$. To do this we write out the definition of the bilinear form (2.2) in terms -of norms and then use the multiplicativity of norms. To check that the image is in $SO(V)$ we argue topologically: $PB^\times$ is connected and the map is continuous, so it must be into a connected component of $O(V)$ (and $SO(V)$ is the connected component of 1.) To show that this map is injective, we write out the multiplication $\beta v \beta^{-1}$. This is easiest if we tensor with $\mathbb{R}$ and write out everything out as matrices; we observe that only the scalar matrices act trivially.

To do: Justify that the representation is surjective. ☐

2.2. $W \otimes V$ **and its metaplectic cover.** Let W be a symplectic space of dimension 2. This means that W is equipped with an alternating (i.e., $\langle w_1, w_2\rangle = -\langle w_2, w_1\rangle$) bilinear form. We also have $W = X \oplus Y$, where X and Y are each maximal isotropic subspaces (of dimension 1 each.)

We recall that V is an orthogonal space, equipped with a symmetric bilinear form. The tensor product $W \otimes V$ is then naturally a symplectic space, where the bilinear form is given by

$$\langle w_1 \otimes v_1, w_2 \otimes v_2\rangle := \langle w_1, w_2\rangle\langle v_1, v_2\rangle.$$

This is easily seen to be alternating.

Moreover we have embeddings

$$O(V) \longrightarrow Sp(W \otimes V), \qquad Sp(W) \longrightarrow Sp(W \otimes V)$$

which are obtained by letting $\sigma \in O(V)$ (similarly in $Sp(W)$) act on $w \otimes v$ by $\sigma(w \otimes v) = w \otimes \sigma(v)$.

We say that $Sp(W)$ and $O(V)$ form a *dual reductive pair* in $Sp(W \otimes V)$. This means that (1) $Sp(W)$ and $O(V)$ centralize each other in $Sp(W \otimes V)$, and (2) these groups act reductively.

2.3. **The Hilbert symbol.** The Hilbert symbol

$$(-,-)_v : \mathbb{Q}_v^\times \times \mathbb{Q}_v^\times \longrightarrow \{\pm 1\}$$

is defined so that $(a,b)_v = 1$ if $z^2 = ax^2 + by^2$ has a solution with $x, y, z \in \mathbb{Q}_v$, and $(a,b)_v = -1$ otherwise. It is symmetric and multiplicative. Moreover, except at 2 and ∞, if the v-adic valuation of both a and b is 0 then $(a,b)_v = 1$. Therefore it also makes sense to define

$$(a,b) := \prod_v (a,b)_v \tag{2.3}$$

when a and b are rational numbers, or (more generally) elements of $\mathbb{A}^\times$. If a and b are both rational then $(a,b) = 1$, as a consequence of quadratic reciprocity.

3. The metaplectic cover and Weil representation: Generalities

3.1. **Motivating example: modular forms of half-integral weight.** (see [8]) We recall the motivating example from Shimura's work on half-integral weight modular forms. Suppose we tried to define such modular forms as holomorphic functions, with appropriate conditions at the cusps, that satisfied for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in some nice group, the condition

$$f(\gamma z) = (cz + d)^{k/2} f(z).$$

We check that we don't get any modular forms. We almost do, except the group action goes wrong by a minus sign, due to the fact that the choice of square root (of $cz + d$) is ambiguous.

Shimura remedies the problem as follows. Define a group G to be the set of all ordered pairs $(\alpha, \phi(z))$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}_2(R)$ (or some nice subgroup thereof) and $\phi(z)$ is a holomorphic fuction on $\mathbb{H}$ satisfying

$$\phi(z)^2 = t(cz + d) \tag{3.1}$$

for some $t \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Multiplication in G is then defined by the equation

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta(z))\psi(z)).$$

One checks that G is indeed a group, an S_1-sheeted cover of $\mathrm{SL}_2(\mathbb{R})$, so that we obtain an exact sequence

$$1 \longrightarrow S^1 \longrightarrow G \longrightarrow \mathrm{SL}_2(\mathbb{R}) \longrightarrow 1.$$

The modularity transformation law then becomes, for $(\alpha, \phi) \in G$,

$$f(\alpha z) = (\phi(z))^k f(z).$$

Moreover, for subgroups Γ of $\Gamma_0(4)$ Shimura defines an injection of Γ into G by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (\gamma, j(\gamma, z)),$$

where

$$j(\gamma, z) = \epsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{1/2},$$

where $\epsilon_d = 1$ if $d \equiv 1 \mod 4$ and $\epsilon_d = i$ if $d \equiv 3 \mod 4$.

Note that the value of t will always be a fourth root of unity. This is perhaps a bit ugly, but this "multiplier system" takes care of the ugliness involving square roots, and we nicely obtain modular forms for G.

3.2. **The Weil representation and metaplectic cover.** We will now follow the book of Kudla, Rapoport, and Yang [3] for a while. The setup and choice of notation are not exactly consistent with [5], but they are close.

Let $G = \mathrm{Sp}_{2n}(W)$, where $W = X \oplus Y$ is a symplectic $\mathbb{Q}_v$-vector space ($v < \infty$) and X and Y are maximal isotropic subspaces. Also, let ψ be a character of $\mathbb{Q}_v$ which is **unramified**, that is, trivial precisely on $\mathbb{Z}_v$ and not on $v^{-1}\mathbb{Z}_v$.

Then we obtain a nice operation of G on the Schwartz space $\mathcal{S}(X)$, as follows. Write $g \in$ as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are n-by-n matrices such that a acts on X, d acts on Y, b sends X to Y, and c sends Y to X. Then

$$g \cdot \varphi(x) := \int_{Y/\ker(c)} \psi\Big(\frac{1}{2}(xa, xb) + (xb, yc) + \frac{1}{2}(yc, yd)\Big)\varphi(xa + yc) d_g(y). \tag{3.2}$$

Here x and y are row vectors, and $(x, y) = x^T y$. The measure $d_g(y)$ is normalized appropriately so as to make this operator unitary – we want $(g \cdot \varphi, g \cdot \varphi) = (\varphi, \varphi)$.

In doing this we obtain a **projective representation** of G on $\mathcal{S}(X)$. By "projective" we mean

$$\rho(g_1)\rho(g_2) = *\rho(g_1 g_2)$$

where $*$ denotes some complex number of absolute value 1. (In fact, it is given by the Leray cocycle $c_L(g_1, g_2)$ which will be described shortly.) In other words, "projective" means "almost".

However, in light of Shimura's example, we see what we can hope to do. We construct the **metaplectic extension** $\widetilde{G}$ of G which fits into an exact sequence

$$1 \longrightarrow S^1 \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1.$$

$\widetilde{G}$ consists of all pairs $[g, z]$ with $g \in G$ and $|z| = 1$, and we have multiplication $[g_1, z_1]_L [g_2, z_2]_L = [g_1 g_2, z_1 z_2 c_L(g_1, g_2)]_L$. Here c_L is the **Leray cocycle**. A definition of this can be found in the paper of Rao [7]; the definition is involved and involves some considerations from symplectic geometry. (See Section 2 for the geometry, and Theorem 4.1 of [7] for the final result.) For our purposes, the point is that we obtain a map $G \rtimes S_1 \longrightarrow \widetilde{G}$ given by $(g, z) \longrightarrow [g, z]_L$, and then letting elements of the form $[1, z]$ act on $\mathcal{S}(X)$ as scalars, we obtain a representation of $\widetilde{G}$ on $\mathcal{S}(X)$.

Computation in Sp_2**.** The formula (3.2) can be readily used for computations. We will illustrate this for Sp_2, so that $\mathcal{S}(X) = \mathcal{S}(\mathbb{Q}_v)$, and compare our results with those stated by Prasanna.

Looking ahead, we write

$$\mathbf{n}(n) := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{d}(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The action of $\mathbf{n}(n)$ is given by

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot \varphi(x) := \int_{\mathbb{Q}_v/\ker(c)} \psi\Big(\frac{1}{2}(x, nx) + (x, 0) + \frac{1}{2}(0, y)\Big)\varphi(xa + yc) d_g(y).$$

As $c = 0$, its kernel is everything (We view c as a linear transformation from $\mathbb{Q}_v$ to itself, or more precisely, from X to Y, where these are the maximal isotropic subspaces of our two-dimensional symplectic space.) Therefore, the integral is over a single point.

Of course $(x, 0) = (0, y)$ and $(x, nx) = n(x, x)$. (We could just as well write $(x, x) = x^2$ here, but this way the analogy with higher-dimensional spaces is clearer.) And $xa + yc = x$. We obtain

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot \varphi(x) = \psi\Big(\frac{n}{2}(x, x)\Big)\varphi(x).$$

Similarly, we compute

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \varphi(x) = \int_{Y/\ker(c)} \psi\Big(\frac{1}{2}(xa, 0) + (0, 0) + \frac{1}{2}(0, ya^{-1})\Big)\varphi(xa + 0) d_g(y).$$

As $\psi(0) = 1$, we see that

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \varphi(x) = \varphi(xa).$$

Finally, we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \varphi(x) = \int_{Y/\ker(c)} \psi\Big(\frac{1}{2}(0, x) + (x, -y) + \frac{1}{2}(-y, 0)\Big)\varphi(0 - y) d_g(y).$$

This time $\ker(c)$ is trivial, so the integral is not, and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \varphi(x) = \int_{\mathbb{Q}_v} \psi((x, y))\varphi(y) dy,$$

which is exactly the Fourier transform.

Change of coordinates. The above coordinates are known as the Leray coordinates. It is also possible to renormalize and use different coordinates. In particular we may construct a different map $G \rtimes S_1 \longrightarrow \widetilde{G}$, taking (g, z) to $[g, z] := [g, \lambda(g)z]_L$, for some choice of function $\lambda(g)$. With this different map, the choice of cocycle becomes different. For example, KRY define a choice of coordinates (see p. 322) so that the cocycle is trivial on $\mathrm{Sp}_{2n}(\mathbb{Z}_v) \times \mathrm{Sp}_{2n}(\mathbb{Z}_v)$. Similarly, KRY define the **Rao coordinates** so that the cocycle takes values in ± 1. This is also Prasanna's choice of coordinates. Thus with this normalization, we may consider a smaller metaplectic extension $\widetilde{G}$, which is simply a 2-fold cover of G.

4. The metaplectic cover and Weil representation in Prasanna's notation

4.1. **The metaplectic cover.** Here is how Prasanna defines the metaplectic cover of $\mathrm{SL}_2(\mathbb{Q}_v)$ (**note:** $\mathrm{SL}_2 = \mathrm{Sp}_2$) and of $\mathrm{SL}_2(\mathbb{A})$. Somewhat predictably, the definition is a mess.

The metaplectic cover of $\mathrm{SL}_2(\mathbb{Q}_v)$ (and of $\mathrm{SL}_2(\mathbb{A})$) is defined as the set of pairs (σ, ϵ) with $\sigma \in SL_2, \epsilon \in \{\pm 1\}$, with multiplication

$$(\sigma, \epsilon)(\sigma', \epsilon') = (\sigma\sigma', \epsilon\epsilon'\beta(\sigma, \sigma')), \tag{4.1}$$

where the cocycle β is given using the Rao coordinates. We can give an explicit definition as follows. Suppose that $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q}_v)$. Let $x(\sigma) = c$ if $c \neq 0$, and $x(\sigma) = d$ otherwise. If v is the infinite prime, let $s_v(\sigma) = 1$. If v is a finite prime, let $s_v(\sigma) = (c, d)_v$ if $cd \neq 0$ and $\mathrm{ord}_v(c)$ is odd, and let $s_v(\sigma) = 1$ otherwise. Then,

$$\beta_v(\sigma, \sigma') := (x(\sigma)x(\sigma'))_v(-x(\sigma)x(\sigma'), x(\sigma\sigma'))_v s_v(\sigma)s_v(\sigma')s_v(\sigma\sigma').$$

Although this does not look pretty, it is totally explicit.

Presumably in the adelic case you use (2.3) and use the global Hilbert symbols instead.

We will also consider the metaplectic cover of $\widetilde{\mathrm{Sp}}(W \otimes V)$. Presumably, as discussed before, this is the cover chosen to make the Weil representation work out and the cocycle may be determined by a process similar to the above. Prasanna does not actually define the cocycle in this case (or if he does, then I missed it.)

Proposition 4.1. *The metaplectic cover $\widetilde{\mathrm{Sp}}(W \otimes V)$ splits over $O(V)$; in other words there is an embedding*

$$O(V) \longrightarrow \widetilde{\mathrm{Sp}}(W \otimes V).$$

Proof. We can give a hand-waving proof, although we have not really given a proper definition of $\widetilde{\mathrm{Sp}}(W \otimes V)$. In particular, assume that the definition of the metaplectic cover has been chosen specifically so that the Weil representation works out. Then, we have a nice action of $O(V)$ on Schwartz functions $\varphi(x)$, where $x \in X \otimes V = V$: If $\tau \in O(V)$, then $\tau(\varphi)(x) := \varphi(\tau^{-1}(x))$. We also see that this action commutes with the action of $\widetilde{\mathrm{Sp}}(W)$ as defined below, so that this representation of $O(V)$ is a subrepresentation of the Weil representation. □

4.2. **The Weil representation.** We want to define the Weil representation of $\widetilde{\mathrm{Sp}}(W \otimes V)_v$. (Here the subscript v just means tensor with $\mathbb{Q}_v$.) In particular we recall that W is a symplectic space of dimension 2, V is an orthogonal space of dimension 3, and therefore $W \otimes V$ is a symplectic space of dimension 6. We can write $W = X \oplus Y$, where X and Y are maximal isotropic subspaces, each of dimension 1. In a similar way we write

$$W \otimes V = (X \otimes V) \oplus (Y \otimes V),$$

and as X is one-dimensional, we can (and will) identify $X \otimes V$ with V.

With these identifications, the Weil representation of $\widetilde{\mathrm{Sp}}(W \otimes V)_v$ is a representation

$$\widetilde{\mathrm{Sp}}(W \otimes V)_v \longrightarrow \mathcal{S}(V \otimes \mathbb{Q}_v).$$

We will also write V_v for $V \otimes \mathbb{Q}_v$.

We then obtain a representation of $\widetilde{Sp}(W_v) \times PB_v^\times$ as follows: We have an inclusion

$$\iota : \widetilde{\mathrm{Sp}}(W_v) \times O(V_v) \longrightarrow \widetilde{\mathrm{Sp}}(W \otimes V)_v,$$

where the inclusion of $\widetilde{\mathrm{Sp}}(W_v)$ into $\widetilde{\mathrm{Sp}}(W \otimes V)_v$ lifts that of $\mathrm{Sp}(W_v)$ into $\mathrm{Sp}(W \otimes V)_v$, and the inclusion of $O(V_v) \to \widetilde{\mathrm{Sp}}(W \otimes V)_v$ is given by Proposition 4.1. This gives us a representation of $\widetilde{Sp}(W_v) \times O(V_v)$. We further have an inclusion $PB_v^\times \to O(V_v)$ given by Proposition 2.1, and composing this inclusion with our representation, we obtain the desired representation. The representation depends on a fixed character ψ' of $\mathbb{Q}\backslash\mathbb{A}$, and we will denote this representation $\omega_{\psi'}$.

Restricting this representation to $\widetilde{\mathrm{Sp}}(W_v)$, we obtain a genuine representation of $\widetilde{\mathrm{Sp}}(W_v)$, which we call $r_{\psi'}$, and which satisfies the following properties:

$$r_{\psi'}(\mathbf{n})\varphi(x) = \psi'(n\langle x, x\rangle)\varphi(x) \tag{4.2}$$

$$r_{\psi'}(\mathbf{d}(a))\varphi(x) = \mu_{\psi'}(a)(a,-1)_v|a|^{3/2}\varphi(ax) \tag{4.3}$$

$$r_{\psi'}(w,\epsilon)\varphi(x) = \epsilon\gamma_{\psi',Q}\mathcal{F}_{\psi'}(\varphi) = \pm\epsilon\gamma_{\psi'}\mathcal{F}_{\psi'}(\varphi). \tag{4.4}$$

Explanation:

The function $\varphi(x)$ is an (arbitrary) **Schwartz function** on $V \otimes \mathbb{Q}_v$, and so are all the functions on the right. This representation is on the space of such functions, so these formulas give the action of a set of generators for $\widetilde{\mathrm{Sp}}(W_v)$ on $\mathcal{S}_{\psi'}(V \otimes \mathbb{Q}_v)$.

$\mathbf{n}$, $\mathbf{d}$, and w refer to matrices in $\mathrm{Sp}_2(\mathbb{Q}_v)$, as follows:

$$\mathbf{n}(n) := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{d}(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Typically if $\sigma \in \mathrm{Sp}_2(\mathbb{Q}_v)$, then we also use σ to refer to $(\sigma, 1) \in \widetilde{\mathrm{Sp}_2}(\mathbb{Q}_v)$ $(=: \widetilde{S}_v.)$ Therefore the first two equations tell us the representation on $(\mathbf{n}, 1)$ and $(\mathbf{d}(a), 1)$ respectively. In the third equation $\epsilon \in \{\pm 1\}$, and so we can figure out the representation on $(\mathbf{n}, -1)$, etc. Really this just implies that

$$r_{\psi'}(1,-1)\varphi(x) = -\varphi(x).$$

It is important to keep in mind that $(\sigma, 1)$ refers to $[\sigma, 1]_R$ in the Rao coordinates; i.e., we have already changed coordinates so that the cocycle takes values in ± 1. This explains the presence of the constants $\gamma_{\psi'}, \mu_{\psi'}$, etc., which did not appear in our computation in Section 3.2.

These constants are defined as follows: (We shall just give Prasanna's definitions without attempting to motivate them.) The constant $\gamma_{\psi'}(t)$ is "the constant associated by Weil" to the character ψ' and the quadratic form tx^2, which is ("may be computed to be") for $v < \infty$

$$\gamma_{\psi'}(t) = \lim_{n\to\infty} \int_{v^{-n}\mathbb{Z}_v} \psi'(tx^2) d_t x,$$

where $d_t x$ is Haar measure chosen to be autodual with respect to the pairing $(x, y) \to \psi'(txy)$. We also write $\gamma_{\psi'} = \gamma_{\psi'}(1)$. We also have

$$\mu_{\psi'}(t) := (t,t)_v \gamma_{\psi'}(t)\gamma_{\psi'}(1)^{-1} = \gamma_{\psi'}(1)\gamma_{\psi}(t)^{-1}.$$

In the third equation (4.4) the sign $+$ or $-$ depends on whether v is ramified or unramified in B. The definition of $\gamma_{\psi',Q}$ is

$$\gamma_{\psi',Q} := \prod_{i=1}^{3} \gamma(\psi'^{a_i}),$$

where $Q(x) = \sum a_i x_i^2$, where the x_i form an orthogonal basis.

Remark. It should not be difficult to verify that the equality in (4.4) and this definition are equivalent.

Remark. I had a little bit about automorphic forms and Hecke algebras in here, but I didn't go into very much depth so I simply deleted the section.

5. Prasanna's work

5.1. **Some theta integrals.** Suppose $s \in \mathcal{A}_0$. In other words, s is a cusp form on $PB_{\mathbb{Q}}^\times \backslash PB_{\mathbb{A}}^\times$. Also suppose $t \in \tilde{\mathcal{A}}_0$; i.e. t is a cusp form on $S_{\mathbb{Q}} \backslash \widetilde{S}_{\mathbb{A}}$. Also, let $\varphi \in \mathcal{S}_{\psi'}(V \otimes \mathbb{A})$. Then we write

$$\theta(\psi', \varphi, \sigma, \beta) := \sum_{x \in V} r_{\psi'}(\sigma) R(\beta) \varphi(x).$$

Explanation: Here β is a quaternion, σ is an element of $\widetilde{\mathrm{SL}}_2$, and so the sum is over different elements of $\mathcal{S}_{\psi'}(V_{\mathbb{A}})$, as acted on by the representation of $\widetilde{\mathrm{SL}}_2 \times PB^\times$.

We need to define the action of β on Schwartz functions. This is easy; we have $R(\beta)x = \beta x\beta^{-1}$, so we define $R(\beta)\phi(x) = \varphi(\beta^{-1}x)$. In particular,

$$R(\beta)\varphi(x) := R(\beta)\varphi(\beta^{-1}x\beta).$$

We can now see that the actions commute. There are two ways to see this. The easiest is to just check the definitions of the actions (i.e., refer to the definition of the Weil representation). We may also note that we've claimed to have a representation of $\widetilde{S}_v \otimes V$, so the actions of $\widetilde{S}_v$ and V had better commute.

With this in mind, I would prefer to write the definition as

$$\theta(\psi', \varphi, \sigma, \beta) := \sum_{x\in V} r_{\psi'}(\sigma)\varphi(\beta^{-1}x\beta).$$

Notice that we are summing over elements of an infinite vector space so there is a question of convergence. It is perhaps not obvious that this converges, but for φ an appropriate Schwartz function (i.e., something with rapid decay), it does indeed converge. This won't be justified here.

We now define

$$t_{\psi'}(\varphi, \sigma, s) := \int_{PB_{\mathbb{Q}}^\times \backslash PB_{\mathbb{A}}^\times} \theta(\psi', \varphi, \sigma, \beta)s(\beta)d^\times\beta,$$

$$T_{\psi'}(\varphi, \beta, t) := \int_{SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})} \overline{\theta(\psi', \varphi, \sigma, \beta)}t(\sigma)d\sigma.$$

Here $t_{\psi'}$ is a function of $\sigma \in \widetilde{S}_{\mathbb{A}}$ and $T_{\psi'}$ is a function of $\beta \in PB_{\mathbb{A}}^\times$.

Remark. At first, the second equation looks wrong; it looks like we should be integrating over $\mathrm{SL}_2(\mathbb{Q})\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A})$ rather than $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})$. However, for any $\sigma \in \mathrm{SL}_2(\mathbb{A})$, the integrand does not depend on the choice of lift $(\sigma, \pm1)$ in $\widetilde{SL}_2(\mathbb{A})$, so that in fact we obtain a well-defined integral over $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})$.

In other words, suppose that a character ψ' and a Schwartz function φ have been fixed. Then $t_{\psi'}(\varphi, \sigma, s)$ gives, for each automorphic form s on $PB_{\mathbb{A}}^\times$, a function of σ (i.e., of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$), which is an automorphic form. Conversely, $T_{\psi'}(\varphi, \beta, t)$ gives, for each automorphic form t on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, a function of β (i.e. of $PB_A^\times$) which is again an automorphic form.

Here is how Prasanna puts it: We let $\mathcal{V}$ and $\widetilde{V}$ be representations for the Hecke algebras of $PB_{\mathbb{A}}^\times$ and $\widetilde{S}_{\mathbb{A}}$. In other words, $\mathcal{V}$ consists of automorphic forms on $\mathcal{A}_0$ and $\widetilde{V}$ consists of automorphic forms on $\widetilde{\mathcal{A}}_0$. We are considering these spaces of automorphic forms, along with the actions of the respective Hecke algebras.

Define

$$\Theta(\mathcal{V}, \psi') := \{t_{\psi'}(\varphi, \cdot, s) : s \in \mathcal{V}, \varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})\},$$
$$\Theta(\tilde{V}, \psi') := \{T_{\psi'}(\varphi, \cdot, t) : t \in \tilde{V}, \varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})\}.$$

Then these are representation spaces for the Hecke algebras of $\tilde{S}_{\mathbb{A}}$ and $PB_{\mathbb{A}}^\times$ respectively. In other words, integrating against this theta function allows us to go back and forth.

Some notation. We let ν be an odd quadratic discriminant and $\delta \in \{\pm1\}$ be the sign of ν.

We now make a particular choice of ψ': We write

$$\psi' = \psi^{1/|\nu|},$$

where ψ is the usual additive character on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$, as described earlier.

Also, let τ be such that $\delta = (-1)^\tau$. (In other words, τ is 0 if ν is positive and 1 if ν is negative.)

5.2. **The theta correspondence.** We will now give an overview of the remainder of Section 3 of Prasanna's paper. We will cite his results, and bring in the relevant details as needed. The proofs are complicated and cite lemmas in Waldspurger's paper repeatedly, so we will not go into them here. Rather, we shall simply attempt to explain what Prasanna proves.

We assume that a holomorphic newform $f \in S_{2k}(\Gamma_0(N))$ is fixed, where N is an odd squarefree integer, and that a character χ' of conductor N' dividing $4N$ is given. We write $M = lcm(4, N'N)$. We define an additional character χ by $\chi = \chi' \cdot \left(\frac{-1}{\cdot}\right)^{k+\tau}$, and let f_χ be a newform in $\pi_f \otimes \chi$ with conductor M.

We have the following definition from Waldspurger's work:

Definition 5.1 ([10], I.2)**.** *Let f_χ be, as above, a newform of weight $2k$ and character χ. The space $S_{k+\frac{1}{2}}(M, \chi', f_\chi) \subseteq S_{k+\frac{1}{2}}(M, \chi')$ is defined to be*

$$S_{k+\frac{1}{2}}(M, \chi', f_\chi) := \{g \in S_{k+\frac{1}{2}}(M, \chi') : T(p^2)g = \lambda_p(f_\chi)g \text{ for almost all } p \nmid N\}.$$

Remark. I have written χ' where Prasanna writes a boldface χ.

By work of Waldspurger, the space $S_{k+\frac{1}{2}}(M, \chi', f_\chi)$ is two-dimensional (assuming that χ is not ramified at 2). Furthermore it admits a one-dimensional subspace, the *Kohnen plus space*, whose Fourier coefficients are only supported on integers congruent to 0 or 1 mod 4.

Let $h_{f,\chi}$ denote an element of this Kohnen plus space (so that it is uniquely defined up to a constant.) Write

$$h_{f,\chi} = \sum_{\xi > 0} a_\xi(h_{f,\chi}) q^\xi.$$

Then Waldspurger proved, subject to some technical conditions, that

$$a_\xi(h_{f,\chi})^2 = A|\xi|^{k-1/2} L(1/2, \pi \otimes \chi_{\xi_0}),$$

for a number A which is independent of ξ. Here $\xi_0 = (-1)^\tau \xi$.

Prasanna then proves that the form $h_{f,\chi}$ can be constructed as a theta lift from $PB^\times$:

Proposition 5.2 ([5], Proposition 3.3)**.** *Suppose that $L(\frac{1}{2}, \pi \otimes \chi_\nu) \neq 0$, and that χ' is a character of conductor dividing $4N$ with $\chi'(-1) = 1$, satisfying some technical conditions.*

Then writing $\chi = \chi' \cdot \left(\frac{-1}{\cdot}\right)^{k+\tau}$, one has that χ is unramified at 2 and $S_{k+\frac{1}{2}}(M, \chi', f_\chi) \subseteq \Theta(\pi' \otimes \chi_v, \psi')$.

In other words, the Shimura lifts of f_χ as considered by Waldspurger are contained in the image of this theta correspondence. Here π' is a representation of $PB^\times$, so the image of it will be in $\widetilde{SL}_2$.

A choice of Schwartz function. At this point (Section 3.2 of Prasanna's paper), he chooses a particular Schwartz function $\varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})$. We write $\varphi = \prod_v \varphi_v$, and the determination of each φ_v is quite involved. The dependence on ν enters here.

We also pick (for each q dividing N^-, which will be the discriminant of B and will divide N) a particular choice of quaternion algebra.

Finally, we make a particular choice of $s \in \pi' \otimes \chi_\nu$, which satisfies some properties not described here, such that $t(\psi', \phi, \cdot, s)$ equals a multiple of $h_{f,\chi}$.

All of these choices will allow us to be more explicit in what follows.

A function t_h. Suppose $h \in S_{k+\frac{1}{2}}(M)$. Then we can associate to it a unique continuous function t_h on $S_{\mathbb{Q}} \backslash \widetilde{S}_{\mathbb{A}}$ satisfying some technical hypotheses, so that the map $h \to t_h$ induces an isomorphism

$$S_{k+\frac{1}{2}}(M, \chi') \simeq \mathcal{A}_{k+\frac{1}{2}}(M, \chi_0).$$

See Proposition 2.1 of [5].

Remark. What looks like a k in Proposition 2.1 of [5] is in fact a κ. κ is defined to be $2k+1$. I am unsure why Prasanna uses the extra notation.

This is, presumably, a version of the standard dictionary between modular and automorphic forms.

We can now say more about the lifts of $PB^{\times}$ to $\widetilde{SL}_2$ described in Proposition 5.2. Keep in mind that a particular s has been chosen.

Proposition 5.3 ([5], Proposition 3.4)**.** *Let $t' = t_{\psi'}(\varphi, \omega, s)$. Then we have*

(1) $t' \in \widetilde{\mathcal{A}}_{k+\frac{1}{2}}(M, \chi_0)$.
(2) Let $h' \in S_{k+\frac{1}{2}}$ *be such that* $t' = t_{h'}$*. Then* $h' \in S_{k+\frac{1}{2}}(M, \chi', f_\chi)$.

In other words, (1) tells us that the map $t_{\psi'}$ gives us a nice automorphic form, and (2) tells us (if I understand correctly...) what its Shimura lift is.

Furthermore, Prasanna also shows (in Section 4) that h' and t' are nonzero, and that some nonzero multiple of h' has Fourier coefficients in the field generated over $\mathbb{Q}$ by the eigenvalues of f and the values of χ.

Let h_χ be this scalar multiple, and assume further that h_χ is λ-adically normalized. This means that the ideal generated by the Fourier coefficients of h_χ is an integral ideal in $\mathbb{Q}(f, \chi)$ and prime to λ. (Presumably λ is just some fixed prime.)

Let $t = t_{h_\chi}$ and let $s' = T_{\psi'}(\varphi, g, t)$.

Proposition 5.4 ([5], Proposition 3.5, 4.2)**.** *$s' = \beta s$ for a nonzero constant β.*

In other words, the operation $T_{\psi'} \circ t_{\psi'}$ is multiplication by a nonzero constant.

References

[1] Larry Grove, *Classical Groups and Geometric Algebra.*
[2] Stephen Kudla, *From modular forms to automorphic representations,* in *Introduction to the Langlands Program.*
[3] S. Kudla, M. Rapoport, and Tonghai Yang, *Modular forms and special cycles on Shimura curves.*
[4] S. Miller, Lecture notes, http://www.math.rutgers.edu/ sdmiller/math574/math574notes.pdf
[5] K. Prasanna, *Arithmetic properties of the Shimura-Shintani-Waldspurger correspondence.*
[6] K. Prasanna, *Integrality of a ratio of Petersson norms.*
[7] R. Ranga Rao, *On some explicit formulas in the theory of Weil representation.*
[8] G. Shimura, *On modular forms of half integral weight.* See also my notes on my webpage.
[9] W. Stein, *Lecture notes*, http://modular.fas.harvard.edu/papers/ant/html/node73.html
[10] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier.*

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