State High School Mathematics Tournament

University of South Carolina

February 3, 2018

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- There will be a tiebreaker if needed.

Question

What is the smallest integer larger than

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$$\log_2(36) + \log_{36}(2)?$$

Solution

Answer: 6.

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We have

$$2^5 = 32, \quad 2^6 = 64,$$

so that $\log_2(36)$ is slightly bigger than 5.

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so that $\log_2(36)$ is slightly bigger than 5.

We have

$$\log_{36}(2) = \frac{1}{\log_2(36)} < \frac{1}{5}.$$

Solution

Answer: 6.
We have

$$2^5 = 32, \quad 2^6 = 64,$$

so that $\log_2(36)$ is slightly bigger than 5.
We have

$$\log_{36}(2) = \frac{1}{\log_2(36)} < \frac{1}{5}.$$

The sum of these numbers is less than 6.

Question

How many real solutions x are there to the equation

Question

How many real solutions x are there to the equation

$$x^5 + \sin(x) = 0?$$

Solution

Answer: 1. (to do: explain)

Question

$$123 \times 321 = 41603$$

Question

$$123 \times 321 = 41603$$

In what number base is this equation true?

Solution

Answer: 8.

Solution

Answer: 8.

$$(b^2 + 2b + 3) \times (3b^2 + 2b + 1) = \cdots + 8b + 3.$$

Solution

Answer: 8.

$$(b^2 + 2b + 3) \times (3b^2 + 2b + 1) = \cdots + 8b + 3.$$

So $b \mid 8$. Since the digit 6 appears in the product, $b = 8$.

Question

It takes four people four minutes to saw four square boards into four smaller squares each.

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It takes four people four minutes to saw four square boards into four smaller squares each.

How long does it take nine people to saw nine square boards into nine squares each?

Answer

Answer. Eight minutes.

Answer

Answer. Eight minutes.

Sawing four square boards into four squares each requires 8 cuts. So it takes a person two minutes to make one cut in a board.

Answer

Answer. Eight minutes.

Sawing four square boards into four squares each requires 8 cuts. So it takes a person two minutes to make one cut in a board.

Cutting nine boards into nine squares each requires 36 cuts. Each person can make one every two minutes, so it takes nine people eight minutes each.

Question

If the parabola $y = x^2$ intersects the circle centered at $(0, 1)$ of radius r in exactly two points,

what is r?

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what is r?

Solution

Answer. $\frac{\sqrt{3}}{2}$.

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Solving $y = x^2$ and $x^2 + (y-1)^2 = r^2$ yields

$$y^2 - y + (1 - r^2) = 0.$$

Solution

Answer. $\frac{\sqrt{3}}{2}$.

Solving $y = x^2$ and $x^2 + (y-1)^2 = r^2$ yields

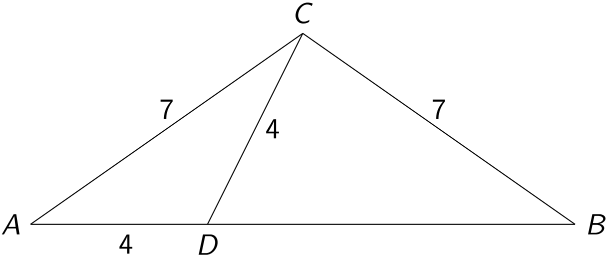
$$y^2 - y + (1 - r^2) = 0.$$

This must have a unique solution. By the quadratic formula,

$$1 - 4(1 - r^2) = 0,$$

so that $r^2 = 3/4$.

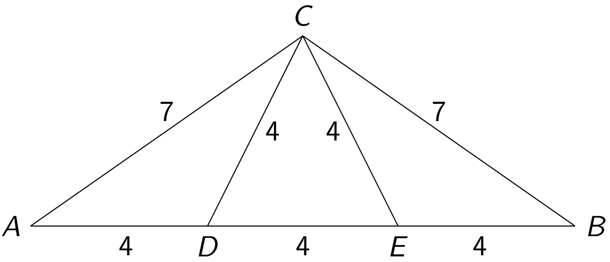
Question



If $\angle CDB = 60^\circ$, what is $\overline{DB}$?

Solution

Answer: 8.



Question

You toss four coins. What is the probability that at least three of them come up heads?

Solution

Answer. $\frac{5}{16}$. (to explain)

Question

The circle $x^2 + y^2 = 4$ intersects the ellipse $\frac{x^2}{16} + y^2 = 1$ in exactly four points.

Question

The circle $x^2 + y^2 = 4$ intersects the ellipse $\frac{x^2}{16} + y^2 = 1$ in exactly four points.

What is the area of the rectangle with these four points as vertices?

Solution

Answer. $\frac{32}{5}$.

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Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$.

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Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$.

We have $y^2 = 4 - \frac{16}{5} = \frac{4}{5}$, and so

$$x^2 \cdot y^2 = \frac{16}{5} \cdot \frac{4}{5} = \frac{64}{25},$$

and the unique solution with $x > 0$, $y > 0$ satisfies $xy = \frac{8}{5}$.

Solution

Answer. $\frac{32}{5}$.

Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$.

We have $y^2 = 4 - \frac{16}{5} = \frac{4}{5}$, and so

$$x^2 \cdot y^2 = \frac{16}{5} \cdot \frac{4}{5} = \frac{64}{25},$$

and the unique solution with $x > 0$, $y > 0$ satisfies $xy = \frac{8}{5}$.

Since the rectangle is centered at the origin, its area is $4xy = \frac{32}{5}$.

Question

If

$$2\cos^2(x) - \sin^2(x) = \frac{1}{2}$$

and $0 < x < \frac{\pi}{2}$, what is x?

Solution

Answer. $\frac{\pi}{4}$.

Solution. We have

$$3\cos^2(x) = (2\cos^2(x) - \sin^2(x)) + (\cos^2(x) + \sin^2(x)) = \frac{1}{2} + 1 = \frac{3}{2},$$

so $\cos^2(x) = \frac{1}{2}$ and $\cos(x) = \frac{\sqrt{2}}{2}$.

So $x = \frac{\pi}{4}$.

Question

What is the only real solution x to

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

Solution

Answer. $-\frac{11}{2}$.

Solution.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

Solution

Answer. $-\frac{11}{2}$.

Solution.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

Solution

Answer. $-\frac{11}{2}$.

Solution.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

$$\frac{1}{(x+4)(x+5)} = \frac{1}{(x+6)(x+7)}$$

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Solution.

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$$(x+4)(x+5) = (x+6)(x+7)$$

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You can solve this for x,

Answer. $-\frac{11}{2}$.

Solution.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

$$\frac{1}{(x+4)(x+5)} = \frac{1}{(x+6)(x+7)}$$

$$(x+4)(x+5) = (x+6)(x+7)$$

You can solve this for x, or notice that $-\frac{11}{2}$ lies on the midpoint of symmetry of the roots.

Question

Solve for x:

$$|2x - 1| = |2x - 2|.$$

Solution

Answer. $\frac{3}{4}$.

Solution. We have

$$2x - 1 = \pm(2x - 2),$$

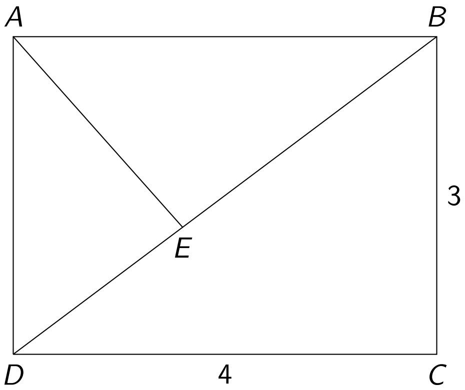
and since $+$ is impossible, we have

$$2x - 1 = -(2x - 2) = -2x + 2.$$

So,

$$4x = 2 + 1 = 3 \Longrightarrow x = \frac{3}{4}.$$

Question



Given rectangle $ABCD$ as above. If $\angle AEB = 90^\circ$, what is $\overline{AE}$?

Solution

Answer. $\frac{12}{5}$.

Solution

Answer. $\frac{12}{5}$.

$\overline{BD} = 5$, and $\triangle ABE \sim \triangle BDC$. So

$$\frac{\overline{AE}}{\overline{AB}} = \frac{\overline{BC}}{\overline{BD}} = \frac{3}{5}$$

and

$$\overline{AE} = \frac{3}{5} \cdot \overline{AB} = \frac{3}{5} \cdot 4 = \frac{12}{5}.$$

Question

How many pairs of prime numbers p, q are there with

$$p - q = 23?$$

Solution

Answer. 0.

Solution

Answer. 0.

All prime numbers other than 2 are odd. The difference of two odd numbers is even. So p or q (hence q) is 2.

$23 + 2 = 25$ is not prime.

Question

(−> team round) Chicken McNuggets are sold in boxes of 4, 6, 10, 20, or 50.

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If you buy exactly three boxes of Chicken McNuggets, how many different quantities of McNuggets can you obtain?

Foo

123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456

Question

If you write $\frac{1}{2020}$ as an infinite repeating decimal,

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If you write $\frac{1}{2020}$ as an infinite repeating decimal, what is the sum of the first six digits after the decimal place?

Solution

Answer. 18.

$$\frac{1}{2020} = 0.00049504950\ldots$$

Solution

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$$\frac{1}{2020} = 0.00049504950\ldots$$

Note that

$$\frac{1}{101} = .009900990099\ldots,$$

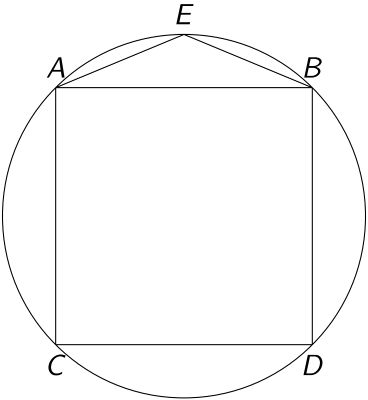
so

$$\frac{1}{1010} = .0009900990099\ldots,$$

$$\frac{1}{2020} = .0004950495049\ldots,$$

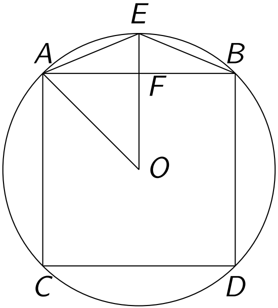
Question 1-2

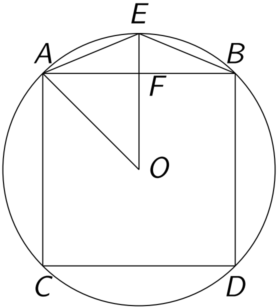
A square is inscribed in a circle of radius 1 as follows:



If $\overline{AE} = \overline{BE}$, find the area of $\triangle AEB$.

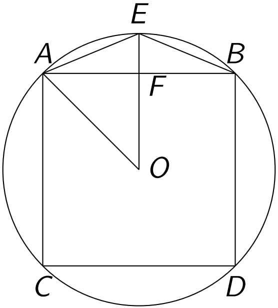
Solution 1-2





We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$,

Solution 1-2



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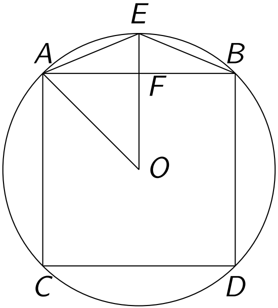
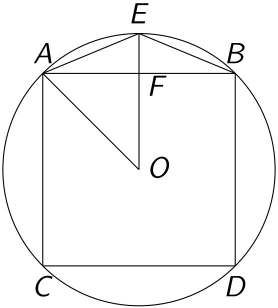


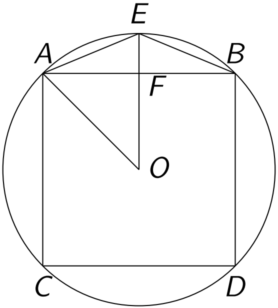
We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$, $\overline{EO} \perp \overline{AB}$, $\overline{OF} = \frac{\sqrt{2}}{2}$,

Solution 1-2



We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$, $\overline{EO} \perp \overline{AB}$, $\overline{OF} = \frac{\sqrt{2}}{2}$, $\overline{EF} = 1 - \frac{\sqrt{2}}{2}$.

Solution 1-2



We have $\overline{AO} = \overline{EO} = 1$, $\overline{AB} = \sqrt{2}$, $\overline{EO} \perp \overline{AB}$, $\overline{OF} = \frac{\sqrt{2}}{2}$, $\overline{EF} = 1 - \frac{\sqrt{2}}{2}$. So $\triangle AEB$ has area

$$\frac{1}{2} \cdot \sqrt{2} \cdot \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2} - 1}{2}.$$

Question 1-3

An empty pasture starts with a certain amount of grass. The grass grows at a constant rate, and cows eat grass at a constant rate.

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- If you instead allow only 10 cows onto the pasture, they will eat all the grass in 22 days.

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- If you allow 16 cows onto the pasture, they will eat all the grass in 10 days.
- If you instead allow only 10 cows onto the pasture, they will eat all the grass in 22 days.

If you allow 25 cows onto the pasture, how long will it take them to eat all the grass?

Solution 1-3

Answer. 5.5 days.

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- Let $f(t)$ be the amount of grass in the pasture after t days, if there are no cows to eat it.

Solution 1-3

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- Let $f(t)$ be the amount of grass in the pasture after t days, if there are no cows to eat it.
- We have $f(t) = a + bt$ for constants a and b. If we measure $f(t)$ in 'cow-days', then $f(10) = 160$ and $f(22) = 220$.

Solution 1-3

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- Since $a + 10b = 160$ and $a + 22b = 220$, we get $b = 5$ and $a = 110$.

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- Since $a + 10b = 160$ and $a + 22b = 220$, we get $b = 5$ and $a = 110$.
- If 25 cows will eat the grass in d days, then $110 + d \cdot 5 = 25d$.

Solution 1-3

Answer. 5.5 days.

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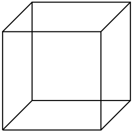
- We have $f(t) = a + bt$ for constants a and b. If we measure $f(t)$ in 'cow-days', then $f(10) = 160$ and $f(22) = 220$.
- Since $a + 10b = 160$ and $a + 22b = 220$, we get $b = 5$ and $a = 110$.
- If 25 cows will eat the grass in d days, then $110 + d \cdot 5 = 25d$.
- So $d = 5.5$.

Question 1-4

A 3-dimensional cube has 12 edges:



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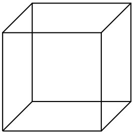

How many edges does a 7-dimensional cube have?

Solution 1-4

Answer. 448.

Solution. There are $2^7 = 128$ vertices; each is connected by an edge to 7 other vertices, so

$$128 \cdot 7 \cdot \frac{1}{2} = 448.$$

Question 1-5

If you expand $(x + 2y)^6$, what is the sum of all the coefficients?

Answer. 729.

Solution 1. We have

$$(x+2y)^6 = x^6+6\cdot 2x^5y+15\cdot 4x^4y^2+20\cdot 8x^3y^3+15\cdot 16x^2y^4+6\cdot 32xy^5+64y^6,$$

Answer. 729.

Solution 1. We have

$$(x+2y)^6 = x^6+6{\cdot}2x^5y+15{\cdot}4x^4y^2+20{\cdot}8x^3y^3+15{\cdot}16x^2y^4+6{\cdot}32xy^5+64y^6,$$

$$1+6{\cdot}2+15{\cdot}4+20{\cdot}8+15{\cdot}16+6{\cdot}32+64 = 1+12+60+160+240+192+64 = 729.$$

Answer. 729.

Solution 1. We have

$$(x+2y)^6 = x^6+6\cdot 2x^5y+15\cdot 4x^4y^2+20\cdot 8x^3y^3+15\cdot 16x^2y^4+6\cdot 32xy^5+64y^6,$$

$$1+6\cdot 2+15\cdot 4+20\cdot 8+15\cdot 16+6\cdot 32+64 = 1+12+60+160+240+192+64 = 729.$$

Solution 2. Adding the coefficients is the same as substituting $x = y = 1$, and

Solution 1-5

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Solution 2. Adding the coefficients is the same as substituting $x = y = 1$, and

$$(1+2)^6 = 3^6 = 729.$$

Question 1-6

Consider the set of integers that can be written in the form $y^2 - x^2$, where x and y are positive integers with $1 \leq x \leq y \leq 10$.

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Consider the set of integers that can be written in the form $y^2 - x^2$, where x and y are positive integers with $1 \leq x \leq y \leq 10$.

How many of them are prime?

Solution 1-6

Answer. 7.

Solution. We have

$$y^2 - x^2 = (y - x)(y + x),$$

so if this is prime then $y = x + 1$ and $y^2 - x^2 = 2x + 1$.

Solution 1-6

Answer. 7.

Solution. We have

$$y^2 - x^2 = (y - x)(y + x),$$

so if this is prime then $y = x + 1$ and $y^2 - x^2 = 2x + 1$. We therefore count the set of prime integers of the form $2x + 1$ (i.e., odd) between 3 and 19, which is

$$\{3, 5, 7, 11, 13, 17, 19\}.$$

Question 7

What is the last digit of 3^{2018}?

Solution 7

Answer. 9.

Solution. Notice that $3^4 = 81$, with last digit 1.

Solution 7

Answer. 9.

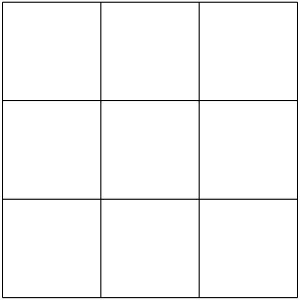
Solution. Notice that $3^4 = 81$, with last digit 1. Since

$$3^{2018} = 3^{4 \cdot 504 + 2} = (81)^{504} \cdot 9,$$

the last digit of 3^{2018} is $1^{504} \cdot 9 = 9$.

Question 8

Consider (again) a Rubik's cube, where each of the six faces has sixteen *corner points*, illustrated by the intersections of the line segments as follows:



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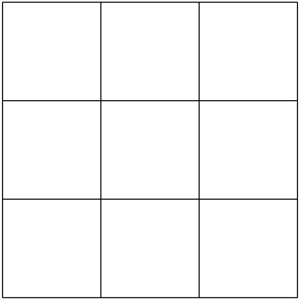

How many corner points are there on the cube total?

Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

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Solution. On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;

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Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;

Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;
- 4 are shared with two other faces, and $4 \cdot 2 = 8$.

Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;
- 4 are shared with two other faces, and $4 \cdot 2 = 8$.

$$24 + 24 + 8 = 56.$$

Question 9

The squares of three consecutive positive integers are added, to obtain 770.
What is the smallest of these integers?

Solution 9

Answer. 15,

$$15^2 + 16^2 + 17^2 = 225 + 256 + 289 = 770.$$

Solution 9

Answer. 15,

$$15^2 + 16^2 + 17^2 = 225 + 256 + 289 = 770.$$

Note that if n denotes the *middle* number, we have

$(n-1)^2+n^2+(n+1)^2 = (n^2-2n+1)+n^2+(n^2+2n+1) = 3n^2+2,$

so $3n^2 = 768$, $n^2 = 256$, and $n = 16$.

Question 10

You flip two coins. One is fair; the other is weighted and is more likely to come up heads than tails.

If the probability of flipping at least one heads is 80%, what is the probability of flipping both heads?

Solution 10

Answer. $\frac{3}{10}$.

Solution. Let p be the probability that the weighted coin comes up heads.
The probability of flipping no heads is

$$\frac{1}{2}(1-p) = \frac{1}{5},$$

so $1 - p = \frac{2}{5}$ and $p = \frac{3}{5}$. The probability of flipping two heads is thus

$$\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}.$$

Question 11

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2017 - 2018?$$

Solution 11

Answer. -1009. Write it as

$$(1-2)+(3-4)+(5-6)+\cdots+(2017-2018),$$

which is -1 added 1009 times.

Question 12

There are unique integers a and b for which

Question 12

There are unique integers a and b for which

$$(1+\sqrt{5})^3 = a + b\sqrt{5}.$$

What is $a+b$?

Solution 12

Answer. 24.

Solution 12

Answer. 24. We have

$$(1+\sqrt{5})^3 = 1 + 3\sqrt{5} + 3(\sqrt{5})^2 + (\sqrt{5})^3 = 16 + 8\sqrt{5}.$$

Question 13

How many digits are in the base 10 number 20^{18}?

Solution 13

Answer: 24.

Solution. We have

$$20^{18} = 262144000000000000000000,$$

which is 2^{18} with 18 zeroes after it.

Solution 13

Answer: 24.

Solution. We have

$$20^{18} = 262144000000000000000000,$$

which is 2^{18} with 18 zeroes after it.

$$2^{18} = 2^{10}2^{8} = 1024 \cdot 256 \sim 1000 \cdot 250 = 250000,$$

with six digits, and $18 + 6 = 24$.