The Mathematics of Game Shows

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1 Introduction

We will begin by watching a few game show clips and seeing a little bit of the math behind them.

1.1 Example: The Price Is Right, Contestants' Row

We begin with the following clip from The Price Is Right:

`https://www.youtube.com/watch?v=TmKP1a03E2g`

Game Description (Contestants' Row - The Price Is Right): *Four contestants are shown an item up for bid. In order, each guesses its price (in whole dollars). You can't use a guess that a previous contestant used. The winner is the contestant who bids the closest to the actual price without going over.*

In this clip, the contestants are shown some scuba equipment, and they bid 750, 875, 500, and 900 in that order. The actual price is $994, and the fourth contestant wins. What can we say about the contestants' strategy?

- As a first step, it is useful to precisely describe the results of the bidding: the first contestant wins if the price is in $[750, 874]$[1]; the second, if the price is in $[875, 899]$; the third, in $[500, 749]$; the fourth, in $[900, \infty)$. If the price is less than $500, then all the bids are cleared and the contestants start over.

  We can see who did well before we learn how much the scuba gear costs. Clearly, the fourth contestant did well. If the gear is worth anything more than $900 (which is plausible), then she wins. The third contestant also did well: he is left with a large range of winning prices – 250 of them to be precise. The second contestant didn't fare well at all: although his bid was close to the actual price, he is left with a very small winning range. This is not his fault: it is a big disadvantage to go early.

[1] Recall that $[a, b]$ is mathematical notation for all the numbers between a and b.

- The next question to ask is: could any of the contestants have done better?

  We begin with the fourth contestant. Here the answer is *yes*, and her strategy is **dominated** by a bid of \$876, which would win in the price range $[876, \infty)$. In other words: *a bid of \$876 would win every time a bid of \$900 would, but not vice versa.* Therefore it is better to instead bid \$876 if she believes the scuba gear is more than \$900.

  Taking this analysis further, we see that there are exactly four bids that make sense: 876, 751, 501, or 1. Note that each of these bids, except for the one-dollar bid, screws over one of her competitors, and this is not an accident: Contestant's Row is a **zero-sum game** – if someone else wins, you lose. If you win, everyone else loses.

- The analysis gets much more subtle if we look at the *third* contestant's options. **Assume that the fourth contestant will play optimally.** (Of course this assumption is very often not true in practice.

  Suppose, for example, that the third contestant believes that the scuba gear costs around \$1000. The previous bids were \$750 and \$875. Should he follow the same reasoning and bid \$876? Maybe, but this exposes him to a devastating bid of \$877.

There is much more to say here, but we go on to a different example.

1.2 Deal or No Deal

Here is a clip of the game show **Deal or No Deal**:

`https://www.youtube.com/watch?v=I3BzYiCSTo8`

The action starts around 4:00.

Game Description (Deal or No Deal): *There are 26 briefcases, each of which contains a variable amount of money from \$0.01 to \$1,000,000, totalling \$3,418,416.01, and averaging \$131477.53. The highest prizes are \$500,000, \$750,000, and \$1,000,000.*

The contestant chooses one briefcase and keeps it. Then, one at a time, the contestant chooses other briefcases to open, and sees how much money is in each (and therefore establishes that these are not the prizes in his/her own briefcase). Periodically, the 'bank' offers to buy the contestant out, and give him/her a fixed amount of money to quit playing. The contestant either accepts or says 'no deal' and continues playing.

The **expected value** of the game is the average amount of money you expect to win. (We'll have much more to say about this.) So, at the beginning, the expected value of the game is \$131477.53, presuming the contestant rejects all the deals. In theory, that means that the contestant should be equally happy to play the game or to receive \$131477.53. (Of course, this might not be true in practice.)

Now let's look at the game after he chooses six briefcases. The twenty remaining contain a total of \$2936366, or an average of \$146818. The expected value has gone up, because the contestant eliminated mostly small prizes and none of the three biggest. If he wants to maximize his expected value (and I repeat that this won't necessarily be the case), then all he has to know is that

$$146818 > 51000$$

and so he keeps playing.

The show keeps going like this. After five more cases are eliminated, he again gets lucky and is left with fifteen cases containing a total of \$2808416, so an average of \$187227. The bank's offer is \$125,000 which he refuses. And it keeps going.

1.3 Jeopardy – Final Jeopardy

Here we see the Final Jeopardy round of the popular show Jeopardy:

`https://www.youtube.com/watch?v=DAsWPOuF4Fk`

Game Description (Jeopardy, Final Round): *Three contestants start with a variable amoung of money (which they earned in the previous two rounds). They are shown a category, and are asked how much they wish to wager on the final round. The contestants make their wagers privately and independently.*

After they make their wagers, the contestants are asked a trivia question. Anyone answering correctly gains the amount of their wager; anyone answering incorrectly loses it.

Perhaps here an English class would be more useful than a math class! This game is difficult to analyze; unlike our two previous examples, the players play *simultaneously* rather than *sequentially*.

In this clip, the contestants start off with \$9,400, \$23,000, and \$11,200 respectively. It transpires that nobody knew who said that *the funeral baked meats did coldly furnish forth the marriage tables*. (Richard II? Really? When in doubt, guess Hamlet.) The contestants big respectively \$1801, \$215, and \$7601.

We will save a thorough analysis for later, but we will make one note now: the second contestant can obviously win. If his bid is less than \$600, he will end up with more than \$22,400.

2 Probability

2.1 Sample spaces and events

At the foundation of any discussion of game show strategies is a discussion of *probability*. You have already seen this informally, and we will work with this notion somewhat more formally.

Definition 2.1 *1. A* **sample space** *is the set of all possible outcomes of a some process.*

2. *An* **event** *is any subset of the sample space.*

Example 2.2 *You roll a die. The sample space consists of all numbers between one and six. Using formal mathematical notation, we can write*

$$S = \{1, 2, 3, 4, 5, 6\}.$$

We can use the notation $\{\dots\}$ to describe a set and we simply list the elements in it.
Let E be the event that you roll an even number. Then we can write

$$E = \{2, 4, 6\}.$$

Alternatively, we can write

$$E = \{x \in S \ : \ x \text{ is even}\}.$$

Both of these are correct.

Example 2.3 *You choose at random a card from a poker deck. The sample space is the set of all 52 cards in the deck. We could write it*

$$\begin{aligned} S = \{&A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, 9\clubsuit, 8\clubsuit, 7\clubsuit, 6\clubsuit, 5\clubsuit, 4\clubsuit, 3\clubsuit, 2\clubsuit, \\ &A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit, 9\diamondsuit, 8\diamondsuit, 7\diamondsuit, 6\diamondsuit, 5\diamondsuit, 4\diamondsuit, 3\diamondsuit, 2\diamondsuit, \\ &A\heartsuit, K\heartsuit, Q\heartsuit, J\heartsuit, 10\heartsuit, 9\heartsuit, 8\heartsuit, 7\heartsuit, 6\heartsuit, 5\heartsuit, 4\heartsuit, 3\heartsuit, 2\heartsuit, \\ &A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit, 9\spadesuit, 8\spadesuit, 7\spadesuit, 6\spadesuit, 5\spadesuit, 4\spadesuit, 3\spadesuit, 2\spadesuit\} \end{aligned}$$

but writing all of that out is annoying. An English description is probably better.

Example 2.4 *You choose two cards at random from a poker deck. Then the sample space is the set of all pairs of cards in the deck. For example, $A\spadesuit A\heartsuit$ and $7\clubsuit 2\diamondsuit$ are elements of this sample space,*

This is definitely too long to write out every element, so here an English description is probably better. (There are exactly 1,326 elements in this sample space.) Some events are easier to describe – for example, the event that you get a pair of aces can be written

$$E = \{A\spadesuit A\heartsuit, A\spadesuit A\diamondsuit, A\spadesuit A\clubsuit, A\heartsuit A\diamondsuit, A\heartsuit A\clubsuit, A\clubsuit A\diamondsuit\}$$

and has six elements. If you are playing Texas Hold'em, your odds of being dealt a pair of aces is exactly $\frac{6}{1326} = \frac{1}{221}$, or a little under half a percent.

Our next example is taken from the following TPIR clip:

`https://www.youtube.com/watch?v=TR7Smevj1AQ`

Game Description (Squeeze Play (The Price Is Right)): *You are shown a prize, and a five- or six-digit number. The price of the prize is this number with one of the digits removed, other than the first or the last.*

The contestant is asked to remove one digit. If the remaining number is the price, the contestant wins the prize.

In this clip the contestant is shown the number 114032. Can we describe the game in terms of a sample space?

It is imporrtant to recognize that **this question is not precisely defined. Your answer will depend on your interpretation of the question!** This is probably very much *not* what you are used to from a math class.

Here's one possible interpretation. Either the contestant wins or loses, so we can describe the sample space as

$$S = \{\text{you win}, \text{you lose}\}.$$

Logically there is nothing wrong with this. But it doesn't tell us very much about the structure of the game, does it?

Here is an answer I like better. We write

$$S = \{14032, 11032, 11432, 11402\},$$

where we've written 14032 as shorthand for 'the price of the prize is 14032'.

Another correct answer is

$$S = \{2, 3, 4, 5\},$$

where here 2 is shorthand for 'the price of the prize has the second digit removed.'

Still another correct answer is

$$S = \{1, 4, 0, 3\},$$

where here 1 is shorthand for 'the price of the prize has the 1 removed.'

All of these answers make sense, and all of them require an accompanying explanation to understand what they mean.

The contestant chooses to have the 0 removed. So the event that the contestant wins can be described as $E = \{11432\}$, $E = \{4\}$, or $E = \{0\}$, depending on which way you wrote the sample space. (Don't mix and match! Once you choose how to write your sample space, you need to describe your events in the same way.) If all the possibilities are equally likely, the contestant has a one in four chance of winning.

The contest guesses correctly and is on his way to Patagonia!

Notation 2.5 *If S is any set (for example a sample space or an event), write $N(S)$ for the number of elements in it. In this course we will always assume this number is finite.*

Probability Rule: All Outcomes are Equally Likely. Suppose S is a sample space in which all outcomes are equally likely, and E is an event in S. Then the **probability of E, denoted** $P(E)$, is

$$P(E) = \frac{N(E)}{N(S)}.$$

Example 2.6 *You roll a die, so $S = \{1, 2, 3, 4, 5, 6\}$.*

1. *Let E be the event that you roll a 4, i.e., $E = \{4\}$. Then $P(E) = \frac{1}{6}$.*
2. *Let E be the event that you roll an odd number, i.e., $E = \{1, 3, 5\}$. Then $P(E) = \frac{3}{6} = \frac{1}{2}$.*

Example 2.7 *You draw one card from a deck, with S as before.*

1. *Let E be the event that you draw a spade. Then $N(E) = 13$ and $P(E) = \frac{13}{52} = \frac{1}{4}$.*
2. *Let E be the event that you draw an ace. Then $N(E) = 4$ and $P(E) = \frac{4}{52} = \frac{1}{13}$.*
3. *Let E be the event that you draw an ace or a spade. What is $N(E)$? There are thirteen spades in the deck, and there are three aces which are not spades. Don't double count the ace of spades!*

   So $N(E) = 16$ and $P(E) = \frac{16}{52} = \frac{4}{13}$.

Example 2.8 *In a game of Texas Hold'em, you are dealt two cards at random in first position. You decide to raise with a pair of sixes or higher, ace-king, or ace-queen, and to fold otherwise.*

The sample space has 1326 elements in it. The event of two-card hands which you are willing to raise has 86 elements in it. (If you like, write them all out. Later we will discuss how this number can be computed more efficiently!)

Since all two card hands are equally likely, the probability that you raise is $\frac{86}{1326}$, or around one in fifteen.

Now, here is an important example: You roll two dice and sum the totals. What is the probability that you roll a 7?

The result can be anywhere from 2 to 12, so we have

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

and $E = \{7\}$. ☠☠☠: Therefore, $P(E) = \frac{N(E)}{N(S)} = \frac{1}{11}$.

Here is another solution. We can roll anything from 1 to 6 on the first die, and the same for the second die, so we have

$$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36,$$
$$41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}.$$

We list all the possibilities that add to 7:

$$E = \{16, 25, 34, 43, 52, 61\}$$

And so $P(E) = \frac{6}{36} = \frac{1}{6}$.

☠☠☠ We solved this problem two different ways and got two different answers. **The point is that not every outcome in a sample space will be equally likely**. We know that a die (if it is equally weighted) is equally likely to come up 1, 2, 3, 4, 5, or 6. So we can see that, according to our second interpretation, all the possibilities are still equally likely because all combinations are explicitly listed. But there is no reason why all the sums should be equally likely.

Note that it is often true that all outcomes are *approximately* equally likely, and we *model* this scenario by assuming that they are. *If our assumptions are close to the truth, so is our answer.*

For example, consider the trip to Patagonia. If we assume that all outcomes are equally likely, the contestant's guess has a 1 in 4 chance of winning. But the contestant correctly guessed that over \$14,000 was implausibly expensive, and around \$11,000 was more reasonable.

Another example comes from the TPIR game **Rat Race**:

`https://www.youtube.com/watch?v=Kp8rhV5PUMw`

Game Description (Rat Race (The Price Is Right)): *The game is played for three prizes: a small prize, a medium prize, and a car.*

There is a track with five wind-up rats (pink, yellow, blue, orange, and green). The contestant attempts to price three small items, and chooses one rat for each successful attempt. The rats then race. If he picked the third place rat, she wins the small prize; if she picked the second place rat, she wins the medium prize; if he picked the first place rat, she wins the car.

(Note that it is possible to win two or even all three prizes.)

Note that except for knowing the prices of the small items, there is no strategy. The rats are (we presume) equally likely to finish in any order.

In this example, the contestant correctly prices two of the items and picks the pink and orange rats.

Problem 1. *Compute the probability that she wins the car.*

Here's the painful solution: describe all possible orderings in which the rats could finish. We can describe the sample space as

$$S = \{POB, POR, POG, PBR, PBG, PRG, \dots, \dots\}$$

where the letters indicate the ordering of the first three rats to finish. Any such ordering is equally likely. The sample space has sixty elements, and twenty-four of them start with P or G. So the probability is $\frac{24}{60} = \frac{2}{5}$.

Do you see the easier solution? To answer the problem we were asked, we only care about the **first** rat. So let's ignore the second and third finishers, and write the sample space as

$$S = \{P, O, B, R, G\}.$$

The event that she wins is

$$E = \{P, G\},$$

and so $P(E) = \frac{N(E)}{N(S)} = \frac{2}{5}$.

Problem 2. *Compute the probability that she wins both the car and the meal delivery.*

Here we care about the first *two* rats. We write

$$S = \{PO, PB, PR, PG, OP, OB, OR, OG, BP, BO, BR, BG, RP, RO, RB, RG, GP, GO, GB, GR\}.$$

The sample space has twenty elements in it. ($20 = 5 \times 4$: there are 5 possibilities for the first place finisher, and (once we know who wins) 4 for the second. More on this later.) The event that she wins is

$$\{PO, OP\}$$

and $P(E) = \frac{N(E)}{N(S)} = \frac{2}{20} = \frac{1}{10}$.

Problem 3. *Compute the probability that she wins all three prizes.*

Zero. Duh. She only won two rats! Sorry.

2.2 Some Rules for Probability

The Addition Rule (1). Suppose E and F are two *disjoint* events – i.e., they don't overlap. Then

$$P(E \text{ or } F) = P(E) + P(F).$$

Example 2.9 *You roll a die. Compute the probability that you roll either a 1, or a four or higher.*

Let $E = \{1\}$ be the event that you roll a 1, and $E = \{4, 5, 6\}$ be the event that you roll a 4 or higher. Then

$$P(E \text{ or } F) = P(E) + P(F) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}.$$

Example 2.10 *You draw a poker card at random. What is the probability you draw either a heart, or a black card which is a ten or higher?*

Let E be the event that you draw a heart. As before, $P(E) = \frac{13}{52}$.
Let F be the event that you draw a black card ten or higher, i.e.,

$$F = \{A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit\}.$$

Then $P(F) = \frac{10}{52}$.

So we have

$$P(E \text{ or } F) = \frac{13}{52} + \frac{10}{52} = \frac{23}{52}.$$

Example 2.11 *You draw a poker card at random. What is the probability you draw either a heart, or a red card which is a ten or higher?*

This doesn't have the same answer, because hearts are red. If we want to apply the addition rule, we have to do so carefully.

Let E be again the event that you draw a heart, with $P(E) = \frac{13}{52}$.

Now let F be the event that you draw a diamond which is ten or higher:

$$F = \{A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit\}.$$

Now together E and F cover all the hearts and all the red cards at least ten, and there is no overlap. So we can use the addition rule.

$$P(E \text{ or } F) = P(E) + P(F) = \frac{13}{52} + \frac{5}{52} = \frac{18}{52}.$$

We can also use the addition rule with more than two events, as long as they don't overlap.

Example 2.12 *Consider the Race Game contestant from earlier. What is the probability that she wins any two of the prizes?* **Solution 1.** *We will give a solution using the addition rule. (Later, we will give another solution using the Multiplication Rule.)*

Recall that her chances of winning the car and the meal delivery were $\frac{1}{10}$. Let us call this event CM instead of E.

Now what are her chances of winning the car and the guitar? (Call this event CG.) Again $\frac{1}{10}$. If you like, you can work this question out in the same way. But it is best to observe that there is a natural symmetry in the problem. The rats are all alike and any ordering is equally likely. They don't know which prizes are in which lanes. So the probability has to be the same.

Finally, what is $P(MG)$, the probability that she wins the meal service and the guitar? Again $\frac{1}{10}$ for the same reason.

Finally, observe these events are all disjoint, because she can't possibly win more than two. So the probability is three times $\frac{1}{10}$, or $\frac{3}{10}$.

Here is a contrasting situation. Suppose the contestant had picked all three small prices correctly, and got to choose three of the rats. In this case, the probability she wins both the car and the meal service is $\frac{3}{10}$, rather than $\frac{1}{10}$. (You can either work out the details yourself, or else take my word for it.)

But this time the probability that she wins two prizes is *not* $\frac{3}{10} + \frac{3}{10} + \frac{3}{10}$, because now the events CM, CG, and MG are not disjoint: it is possible for her to win all three prizes, and if she does, then all of CM, CG, and MG occur!

It turns out that in this case the probability that she wins *at least* two is $\frac{7}{10}$, and the probability that she wins *exactly* two is $\frac{3}{5}$.

The Multiplication Rule. The multiplication rule computes the probability that both E **and** F occur. The formula is the following:

$$P(E \text{ and } F) = P(E) \times P(F).$$

It is not always valid, but it is valid in any of the following three circumstances:

- The events E and F are *independent*.
- The probability given for F assumes that the event E occurs.
- The probability given for E assumes that the event F occurs.

Example 2.13 *You flip a coin twice. What is the probability that you flip heads both times?*

We can use the multiplication rule for this. The probability that you flip heads if you flip a coin once is $\frac{1}{2}$. Since coin flips are independent (flipping heads the first time doesn't make it more or less likely that you will flip heads the second time) we multiply the probabilities to get $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Alternatively, we can give a direct solution. Let

$$S = \{HH, HT, TH, TT\}$$

and

$$E = \{HH\}.$$

Since all outcomes are equally likely,

$$P(E) = \frac{N(E)}{N(S)} = \frac{1}{4}.$$

We can also use the multiplication rule for more than two events.

Example 2.14 *You flip a coin twenty times. What is the probability that you flip heads every time?*

If we use the multiplication rule, we see at once that the probability is

$$\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \frac{1}{2^{20}} = \frac{1}{1048576}.$$

This example will illustrate the second use of the Multiplication Rule.

Example 2.15 *Consider the Rat Race example again (as it happened in the video). What is the probability that the contestant wins both the car and the meal service?*

The probability that she wins the car is $\frac{2}{5}$, as it was before. So we need to now compute the probability that she wins the meal service, given that she won the car.

This time the sample space consists of four rats: we leave out whichever one won the car. The event is that her remaining one rat wins the meal serive, and so the probability of this event is $\frac{1}{4}$.

By the multiplication rule, the total probability is

$$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}.$$