Homework 6b - Analytic number theory

Frank Thorne, thornef@mailbox.sc.edu

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Complex analysis practice. If you work on this, please ignore Homework 6a.

1. (3 points) If f and g are holomorphic functions, then so are $f+g$, fg (and, by induction, f^n for any positive integer n), and cf, where c is a constant.

2. (5 points) Suppose that $f(z) = \sum_{n \geq 0} a(n)(z-z_0)^n$ converges absolutely whenever $z - z_0 < \rho$. (In fact, if it converges throughout this region, then it converges absolutely).

   Prove that f is holomorphic in this region.

3. (5 points) Prove the Cauchy-Riemann equations.

4. (3 points) Prove that $f(z) = \overline{z}$ is not holomorphic.

5. (10 points) **Important. Carefully** evaluate the integral

$$\int_{2-i\infty}^{2+i\infty} \frac{dz}{z^3+1}, \tag{1}$$

   via the following procedure. First of all, reduce the problem to an evaluation of

$$\int_{2-iT}^{2+iT} \frac{dz}{z^3+1}, \tag{2}$$

   and explain why this is justified. Then, "shift the contour" to a rectangular contour from $2-iT$ to $A-iT$ (where A is a big real number), to $A+iT$ to $2+iT$. Explain what "shifting the contour" means. Then, bound the value of this contour from above. Your bound will be in terms of A and T. Prove that this can be made arbitrarily small by choosing A and T appropriately.

   Colloquially this is referred to as "shifting the contour all the way to the right". Why is this a good way to think about the problem?

6. (10+ points) (Some of this will be done in lecture, but not all. I haven't decided exactly what yet. Please fill in all the gaps.)

   The gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} ds. \tag{3}$$

   Note that if $s \in \mathbb{C}$, $t^s := \exp(s \log t)$. There is no ambiguity if t is positive real.

   (a) Determine (with proof) the set of all s for which this integral converges, and the set of all s for which it defines a holomorphic function of s.

(b) Integrating by parts, prove the functional equation

$$\Gamma(s+1) = s\Gamma(s) \tag{4}$$

in the region of absolute convergence. Explain why this continues $\Gamma(s)$ to a function holomorphic on the whole complex plane, except for simple poles at $s = 0, -1, -2, -3, \cdots$

(c) Evalute the residues at the poles.

(d) It can be shown that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{5}$$

(For bonus points, prove this.) Using this identity, prove that $\Gamma(s)$ is never equal to zero.

7. I might add some more problems if I come up with anything good. (But tackling all of this is already good.)