42.1 (Math 702: Spring semester) [MWF 12:00, LC 310.]

Announcements: - Seminar schedule in flux

Math Contest
GAGS, 2/23-25 2-3, Ga Tech

SERMON, 3/10-11

Algebraic Curves, 4/6-8, Madison

Additive combinatorics, 5/21-25

Class time adjustment –

TA Sessions —

Any algebra class requests for next year (talk to Andy or Jesse)

The Tensor Product Construction.

Idea: Given a ring R (with 1, not nec. commutative)

Two (left) R-modules M and N

Will cook up a "product", called $M \otimes N$ or $M \otimes_R N$.

This will be quite different than the direct sum (or cartesian product).

Example. (1) Suppose $R = M = N = \mathbb{C}$.

Then $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ should be $\mathbb{C}$.

If you multiply complex numbers, get complex numbers.

(2) Suppose $M = \mathbb{C}$ and $N = \mathbb{R}[x_1, \ldots, x_n]$.

Then $M \otimes N$ (really, $M \otimes_{\mathbb{R}} N$) should be

$\mathbb{C}[x_1, \ldots, x_n]$.

Let's generalize this and see what we demand.

42.2

Suppose that $R \subseteq S$ are rings and N is a left S-module.

Then N is also a left R-module, and

$$\left.\begin{aligned} (s_1+s_2)n &= s_1 n + s_2 n \\ s(n_1+n_2) &= sn_1 + sn_2 \\ (s_1 s_2)n &= s_1(s_2 n) \end{aligned}\right\} \text{ for all } s, s_1, s_2 \in S,\ n, n_1, n_2 \in N$$

(in particular) $\quad (sr)n = s(rn) \quad (r \in R)$

Can we reverse this? $R \subseteq S$ and N is an R-module.

Can we make it an S-module?

In general, no. e.g. $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Z}$ is a $\mathbb{Z}$-module.

Claim. There is no $\mathbb{Q}$-module structure on $\mathbb{Z}$.

Proof. Suppose there was, and write $\frac{1}{2} \circ 1 = z$ for $z \in \mathbb{Z}$.

Then $\left(\frac{1}{2} + \frac{1}{2}\right) \circ 1 = 1 \circ 1 = 1$ (demand $1m = m$ for all $m \in M$)

$$\| $$

$$\frac{1}{2} \circ 1 + \frac{1}{2} \circ 1 = z + z.$$

You cannot solve $z + z = 1$ in $\mathbb{Z}$!

Revised question. If $R \subseteq S$ and N is an R-module, can we embed N as an R-submodule of an S-module?

Let's try to do this. $\mathbb{Z} \subseteq \mathbb{Q}$ again, and $N = \mathbb{Z}/2$ as a $\mathbb{Z}$-module.

Is there a $\mathbb{Q}$-module containing $\mathbb{Z}/2$ as a $\mathbb{Z}$-submodule?

No: $\mathbb{Q}$-modules are $\mathbb{Q}$-vector spaces and they never have torsion: $k \cdot x = 0 \Rightarrow k$ or $x = 0$.

$$\underset{\mathbb{Z}}{\underset{\uparrow}{k}} \cdot \underset{\mathbb{Q}\text{-vector space}}{\underset{\uparrow}{x}}$$

Another way to think about this. $\mathbb{Z}$ has lots of ideals $\mathbb{Q}$ doesn't.

But: What the hell, let's try anyway.

Want to try to define sn for $s \in S$, $n \in N$. ($R \subseteq S$ with N an R-module.)

Start with the free abelian group (free $\mathbb{Z}$-module) on $S \times N$.

Formal products (s, n) with no relations.

Now we make it into an S-module.

we want:

$$(s_1 + s_2, n) = (s_1, n) + (s_2, n)$$

$$(s, n_1 + n_2) = (s, n_1) + (s, n_2)$$

$$(sr, n) = (s, rn)$$

for all $\begin{cases} s, s_1, s_2 \in S \\ n, n_1, n_2 \in N \\ r \in R. \end{cases}$

Definition #1. If $R \subseteq S$ are rings and N is an R-module, then the tensor product $S \otimes_R N$ is the following quotient group:

$$\frac{\left\{\begin{array}{c}\text{Free abelian group generated by symbols} \\ (s,n) \text{ with } s \in S \text{ and } n \in N\end{array}\right\}}{\left\{\begin{array}{ll}\text{Subgroup generated by} & \\ (s_1+s_2,n)-(s_1,n)-(s_2,n) & \text{for all} \\ (s,n_1+n_2)-(s,n_1)-(s,n_2) & s,s_1,s_2 \in S \\ (sr,n)-(s,rn) & n,n_1,n_2 \in N \\ & r \in R\end{array}\right\}}$$

We write $s \otimes n$ for the coset containing (s,n).

Note: (1) It is possible that $s \otimes n = s' \otimes n'$ even if $s \neq s'$, $n \neq n'$

(2) Not necessarily true that every elt. looks like $s \otimes n$ for some $s \in S$ and $n \in N$.

Proposition. $S \otimes_R N$ is indeed a left S-module, with action

note: these sums will be finite.

$$s\left(\sum s_i \otimes n_i\right) = \sum (ss_i) \otimes n_i.$$

Sketch proof.

(1) Must show this is well-defined, i.e. if $\sum s_i \otimes n_i$ is in the subgroup we're quotienting out by, so is $\sum ss_i \otimes n_i$.

(2) Must show it satisfies the S-module axioms.

These are tedious, but useful for newcomers.

Proposition. There is a natural map

$$\iota : N \longrightarrow S \otimes_R N$$
$$n \longrightarrow 1 \otimes n.$$

It is an R-module homomorphism because

$$r(1 \otimes n) = r \otimes n \quad \text{(by definition)}$$
$$= 1 \cdot r \otimes n$$
$$= 1 \otimes rn \quad \text{(by construction)}$$

and $(1 \otimes n) + (1 \otimes n') = 1 \otimes (n + n')$.

It might not be injective, e.g.

$$\iota : \mathbb{Z}/2 \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2$$

because $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2 = 0$.

Exercise. Prove this. Namely, as abelian groups let

A = free abelian group generated by (s, n) $s \in \mathbb{Q}, n \in \mathbb{Z}/2$

B = subgroup gen by

$$(s_1 + s_2, n) - (s_1, n) - (s_2, n)$$
$$(s, n_1 + n_2) - (s, n_1) - (s, n_2)$$
$$(sr, n) - (s, rn)$$

Then $A = B$.

But $S \otimes_R N$ is the "best possible" S-module to serve as the target of an R-module homomorphism from N.

Proposition (the universal property).

Let $R \subseteq S$ be rings, N a left R-module, and

$$\begin{aligned} \iota : N &\longrightarrow S \otimes_R N \\ n &\longmapsto 1 \otimes n . \end{aligned}$$

Suppose that L is any other left S-module with an R-module homomorphism $N \xrightarrow{\varphi} L$.

Then there exists a unique S-module hom $S \otimes_R N \xrightarrow{\Phi} L$ with $\varphi = \Phi \circ \iota$.

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & S \otimes_R N \arrow[d, dashed, "\Phi"] \\ & L \end{tikzcd}$$

Conversely, given such a Φ, $\varphi = \Phi \circ \iota$ is an R-mod hom $N \longrightarrow L$.

Will be proved later in more generality.

Examples.

1. If N is any left R-module, $R \otimes_R N \cong N$.

Proof. Use the UP with $\varphi = \mathrm{id}$.

$$\begin{tikzcd} & R \otimes_R N \arrow[dr, dashed, "\Phi"] & \\ N \arrow[ur, "\iota"] \arrow[rr, equal, "\mathrm{id}"] & & N \end{tikzcd}$$

ι is injective because id is.

Surjective because $r \otimes n = 1 \otimes rn \in \mathrm{Im}(\iota)$.

2. $R = \mathbb{Z}$, $S = \mathbb{Q}$, A any finite abelian group.

Check: $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$.

3. (To be proved shortly)
If $N \cong R^n$ is a free rank n R-module, then $S \otimes_R N \cong S^n$ is a free rank n S-module.

So, e.g. if $N \cong F^n$ is a vector space, and K/F is an extension field, then $K \otimes_F N$ is a K-vector space with the same basis.

4. If A is any f.g. abelian group, then $A \cong \mathbb{Z}^n \oplus T$ for some nonneg. integer n, finite abelian group T.

Then

$\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q}^n$.

(Will see: tensor products commute w/ direct sums)

Tensor products in general:

Let N be a left R-module

M be a right R-module.

Then $M \otimes_R N$ consists of the quotient

$$\frac{\left\{ \text{free abelian group on } M \times N \right\}}{\left\{ \begin{array}{l} \text{subgroup} \\ \text{gen by} \end{array} \begin{array}{lll} (m_1 + m_2, n) & - (m_1, n) & - (m_2, n) \\ (m, n_1 + n_2) & - (m, n_1) & - (m, n_2) \\ (mr, n) & - (m, rn) & \end{array} \right\}}.$$

Write elements as $m \otimes n$, with

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$
$$mr \otimes n = m \otimes rn.$$

43.1. Last time.

Tensor products $S \otimes_R N$ where $R \subseteq S$ are rings.

Pairs (s, n) subject to bilinearity relations.

Write $s \otimes n$.

Satisfied a universal property

$$\begin{tikzcd} N \arrow[r, "n \mapsto 1 \otimes n"] \arrow[dr, "\forall"'] & S \otimes_R N \arrow[d, dashed, "\exists!"] \\ & L \end{tikzcd}$$

Today: Take the tensor product of two modules.

But: Assume the ring R is commutative.

(DF do the general case. But usually R is commutative.)

Then we need only consider left R-modules.

Definition 1. Let M and N be R-modules.

The tensor product $M \otimes_R N$ consists of the R-module generated by symbols $m \otimes n$, with relations

$$(m + m') \otimes n = m \otimes n + m' \otimes n$$
$$m \otimes (n + n') = m \otimes n + m \otimes n'$$
$$rm \otimes n = m \otimes rn$$

and R-action $r(m \otimes n) = rm \otimes n$.

(Formally: Free abelian group on (m,n), quotient out by relations.)

43.2. Definition. Let M, N, L be R-modules.
A function $\varphi : M \times N \longrightarrow L$ is <u>R-bilinear</u> if
it is an R-module hom in each variable separately:

$$\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$$
$$\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$$
$$\varphi(rm, n) = \varphi(m, rn) = r \cdot \varphi(m, n).$$

<u>Theorem</u>, (the universal property)

The tensor product $M \otimes_R N$ satisfies the following:

(1) The map $M \times N \xrightarrow{\iota} M \otimes_R N$ is R-bilinear. (immediate)

(2) If $M \times N \xrightarrow{\varphi} L$ is any other R-bilinear map, then there is a unique $\Phi : M \otimes_R N \longrightarrow L$ making the diagram commute.

$$\begin{tikzcd}
M \times N \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, dashed, "\Phi"] \\
 & L
\end{tikzcd}$$

43.7.

Some computations (exercises):

$\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$ (do by gens + rel'ns)

$A \otimes_{\mathbb{Z}} B = 0$ if $\begin{cases} A \text{ is } \underline{\text{divisible}} \text{ (given } a \in A,\ n \in \mathbb{Z} - \{0\}, \\ \qquad a/n \in A) \\ B \text{ is } \underline{\text{torsion}} \text{ (every elt. has finite} \\ \qquad \text{order)} \end{cases}$

$\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \equiv \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \neq \mathbb{C}$ (think of these as $\mathbb{R}$-vector spaces ...)

Some basic properties:

Associativity: $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$

Commutativity: $M \otimes_R N \cong N \otimes_R M$

Distributive Laws: $(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$

(and in the second variable too).

Proof of associative law.

The map $(m,n) \to m \otimes (n \otimes l)$, for each fixed l, is R-bilinear, so the UP yields a hom

$$M \otimes_R N \longrightarrow M \otimes_R (N \otimes_R L)$$

satisfying $\quad m \otimes n \longrightarrow m \otimes (n \otimes l)$.

So we get a WD map

$$(M \otimes_R N) \times L \longrightarrow M \otimes_R (N \otimes_R L)$$
$$(m \otimes n, l) \longrightarrow m \otimes (n \otimes l)$$

which is R-bilinear.

Use the UP again, induces a map

$$(M \otimes_R N) \otimes_R L \longrightarrow M \otimes_R (N \otimes_R L)$$
$$(m \otimes n) \otimes l \longrightarrow m \otimes (n \otimes l)$$

Similarly get a map in the opposite direction. So these are mutually inverse isomorphisms.

Distributive Law: $(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$

Consider the map $(M \oplus M') \times N \longrightarrow \text{RHS}$
$$((m, m'), n) \longrightarrow (m \otimes n, m' \otimes n)$$

It is R-bilinear, so get by the UP again
$$(M \oplus M') \otimes N \longrightarrow \text{RHS}$$
$$(m, m') \otimes n \longrightarrow (m \otimes n, m' \otimes n)$$

In the other dir. consider
$$M \times N \to (M \oplus M') \otimes N$$
$$M' \times N \to (M \oplus M') \otimes N$$

$$(m, n) \to (m, 0) \otimes n$$
$$(m', n) \to (0, m') \otimes n \quad \text{resp.}$$

Give homs $\begin{matrix} M \otimes N \\ M' \otimes N \end{matrix} \longrightarrow (M \oplus M') \otimes N$

and hence a hom from the direct sum with

$$(m \otimes n_1, m' \otimes n_2) \longrightarrow (m, 0) \otimes n_1 + (0, m') \otimes n_2.$$

Easily checked: Commutes with the previous one.

44.1. * Review construction and UP of tensor products.
ACD laws + prove dist.

Cor. (Extension of scalars for free modules)
Given $R \subseteq S$ rings, $S \otimes_R R^n \cong S^n$.
[Use distributive law and $S \otimes_R R \cong S$.]

Cor. $R^s \otimes R^t \cong R^{st}$.
Moreover, if bases for R^s, R^t are $m_1 \dots m_s$
$n_1 \dots n_t$
then $m_i \otimes n_j$ form a basis.

Note that vector spaces are free modules over the field.

Another way to think about the UP:

Proposition. For each R-module L, there is a bijection

$$\left\{ \begin{matrix} R\text{-bilinear maps} \\ \phi : M \times N \to L \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} R\text{-module homs} \\ \Phi : M \otimes_R N \to L \end{matrix} \right\}.$$

Proof. Recall:

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\phi"'] & M \otimes_R N \arrow[d, dashed, "\Phi"] \\ & L \end{tikzcd}$$

Any ϕ induces a unique Φ. Conversely, $\phi = \Phi \circ \iota$.

[10.5 important but advanced. will skip for now]

44.3. Definition.

If R is a commutative ring, an <u>R-algebra</u> is, equivalently:

(1) a ring A with a ring hom $f : R \to A$, such that $f(R) \subseteq A$ is in the center of A. (Dummit-Foote)

(2) an R-module A which is simultaneously a ring (wikipedia) with

$$r\cdot(xy) = (r\cdot x)y = x(r\cdot y) \quad \text{in } A.$$

(3) an R-module A with an R-bilinear map $A \times A \to A$ (multiplication) satisfying the associative law.

<u>Exercise</u>. Convince yourself that these are all the same.

Anyway, for any commutative ring R, $T(M)$ is a R-algebra, which multiplication given by the tensor product.

It is <u>not</u> in general commutative, because $x \otimes y \neq y \otimes x$ in general.

[Note that $M \otimes_R N \cong N \otimes_R M$, but this is not the same.]

<u>Theorem</u>. The tensor algebra satisfies the following UP:

If $\varphi : M \to A$ is an R-module homomorphism into any R-algebra A, get a map $\Phi : T(M) \to A$ with

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr] & T(M) \arrow[d, dashed, "\Phi"] \\ & A \end{tikzcd}$$

<u>Proof</u>: Exercise! (Or read DF.)

44.2.

Note also we can define

$$M_1 \otimes_R M_2 \otimes_R M_3 \otimes_R \cdots \otimes_R M_k$$

$$= (((M_1 \otimes_R M_2) \otimes M_3) \otimes \cdots$$

Put the parentheses any way ——

We could have defined this directly (gens + rel'ns) or via a UP

$$\begin{tikzcd}
M_1 \times \cdots \times M_n \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & M_1 \otimes \cdots \otimes M_n \arrow[d, dashed] \\
& L
\end{tikzcd}$$

where φ is R-multilinear: an R-module hom in each variable separately when other variables are constant.

Tensor, Symmetric, and Exterior Algebras (DF, 11.5)

Define $T^0(M) = R$

$$T^k(M) = \underbrace{M \otimes_R M \otimes_R \cdots M}_{k \text{ times}} \quad \text{for each } k \geq 1.$$

$$T(M) = \bigoplus_{k=0}^{\infty} T^k(M). \quad (\text{Identify } M \text{ with } T(M).)$$

This is an R-module containing M as a submodule.

44.4. This is an example of a graded ring:

Def. A ring S is graded if it is the direct sum of additive subgroups

$$S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$$

with $S_i S_j \subseteq S_{i+j}$ for all $i, j \geq 0$.

The elements of S_i are homogeneous of degree i.

Examples. * The tensor algebra, with $T^i(M)$ being the homogeneous component of degree i.

* Polynomial rings $R[x]$ or $R[x_1, \ldots, x_n]$.

Degree is the total poly degree.

Note that polynomial rings are also R-algebras (commutative ring extensions always are).

But wait.

$\cancel{R[x] \cong T(R) \cong T}$

All as R-modules.

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[dr] & T(R) \arrow[d, dashed] \\
 & R[x]
\end{tikzcd}$$

$\leftarrow$ we get a homomorphism!

$$\begin{array}{ccccccccc}
T(R) = & T^0(R) & \oplus & T^1(R) & \oplus & T^2(R) & \oplus & T^3(R) & \oplus \cdots \\
 & \wr\| & & \wr\| & & \wr\| & & \wr\| & \\
 & R & & R & & R \otimes_R R \cong R & & R \otimes R \otimes R \cong R & \\
 & & & & & & & \text{etc.} &
\end{array}$$

OOPS, never mind, I thought we'd construct the symmetric algebra "by accident", but the "obvious" R-module hom $R \hookrightarrow R[x]$ just gives the multiplication map on $T(R)$. Yeah, we can map $R \hookrightarrow R[x]$ by $r \mapsto rx$, but that's less intuitive.

44.5 = 45.1

The symmetric algebra.

Remember that $T(M)$ is not commutative in general, because $x \otimes y \neq y \otimes x$.

What if I don't like this?

Definition. The symmetric algebra $S(M)$ is $T(M)/C(M)$, where $C \leq M$ is the ideal generated by all elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1$.

Remember: Quotienting out by ideals like this $\Updownarrow$ Imposing additional relations.

Some properties.

(1) This ring is commutative.

Why? $T(M)$ is generated as a ring by $R = T^0(M)$ and $M = T^1(M)$.

These commute is $S(M)$ by construction.

(2) The ring is graded, it inherits the grading from $T(M)$.

There is a bit of machinery you can develop here.

$C(M)$ is a graded ideal, $C(M) = \bigoplus_i (C(M) \cap T^i(M))$

Quotient a graded ring by a graded ideal again get a graded ring.

44.6.

3. Once again have $M \hookrightarrow S(M)$.

4. With the grading $T(M) = \bigoplus_i T^i(M)$

have also $S(M) = \bigoplus_i S^i(M)$,

and $$S^k(M) = \frac{M \otimes \cdots\cdots \otimes M}{\{\text{submodule gen. by the } m_1 \otimes \cdots \otimes m_k - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)}\}}.$$
(for $\sigma \in S_n$.)

Proof. We have by the construction
$$S^k(M) = \frac{M \otimes \cdots\cdots \otimes M}{C^k(M)}$$

where $C^k(M)$ are finite sums of elts. of the form

$$m_1 \otimes \cdots \otimes m_{i-1} \otimes (m_i \otimes m_{i+1} - m_{i+1} \otimes m_i) \otimes m_{i+2} \otimes \cdots \otimes m_k.$$

$C^k(M)$ is clearly contained in the submodule above.

Conversely, they've equal because transpositions generate S_n)

44.7. Universal properties.

5. If $\varphi : M \times \cdots \times M \longrightarrow N$ is a symmetric k-multilinear map over R then $\exists$ a unique R-mod hom $\Phi : S^k(M) \longrightarrow N$ with $\varphi = \Phi \circ \iota$:

$$\iota : (m_1, \dots, m_k) \longrightarrow m_1 \otimes \cdots \otimes m_k$$

$$\begin{tikzcd} M \times M \times \cdots \times M \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & S^k(M) \arrow[d, dashed, "\Phi"] \\ & N \end{tikzcd}$$

6. If A is any commutative R-algebra and $\varphi : M \to A$ is an R-mod hom, get a unique R-alg. hom $\Phi : S(M) \to A$ with $\Phi|_M = \varphi$.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, "\varphi"'] & S(M) \arrow[d, dashed] \\ & A \end{tikzcd}$$

44.8. = 45.4

Do we know any commutative R-algebras?

Suppose $M \cong R^n$ is the free rank n R-module.

Write a basis for R^n: $\underset{\substack{\shortparallel \\ (1,0,\dots,0)}}{x_1}, \dots, \underset{\substack{\shortparallel \\ (0,0,\dots,0,1)}}{x_n}$.

Then what is $S^k(M)$?

Consists of linear sums of expressions of the form

$\cancel{(r_1 x_{i_1}) \otimes (r_2 x_{i_2}) \otimes}$

$$(r_1 m_1) \otimes (r_2 m_2) \otimes \cdots \otimes (r_k m_k)$$

modulo rearranging, i.e. $r_1 r_2 \cdots r_k \, m_1 \otimes m_2 \otimes \cdots m_k$

or just $r \, m_1 \otimes \cdots \otimes m_k$.

Write each m_i in terms of the basis and rearrange.

$S^k(M)$ consists of R-linear combinations of expressions $x_{i_1} \otimes \cdots \otimes x_{i_k}$, up to reordering.

But this can be regarded as <u>degree k polynomials</u>.

So, $S(R^n) \cong R[x_1, \dots, x_n]$.

45.5.

The exterior algebra.

Switch around this idea. Demand our algebra anticommute.

<u>Def</u>. Let M be an R-module. The <u>exterior algebra</u> of M is $\Lambda(M) := T(M) / A(M)$, where $A(M)$ is the ideal gen by $m \otimes m$ for $m \in M$.

Again a graded algebra.
The image of $m_1 \otimes m_2 \otimes \cdots \otimes m_k$ in $\Lambda(M)$ is written $m_1 \wedge m_2 \wedge \cdots \wedge m_k$ (a <u>wedge product</u>)

<u>Basic properties</u>.

(1) $m \wedge m' = -m' \wedge m$.

This is because

$$0 = (m+m') \wedge (m+m') = \underbrace{m \wedge m}_{0} + m' \wedge m + m \wedge m' + \underbrace{m' \wedge m'}_{0}$$

(2) $m_1 \wedge m_2 \wedge \cdots \wedge m_k = 0$ if any $m_i = m_j$ (Poke around)

(3) If you switch the places of m_i and m_j, you negate the wedge product. (Prove using above)

45.6

(4) $\Lambda^k(M)$ equals $T^k(M)$ modulo the submodule gen. by elts. of the form $m_1 \otimes m_2 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some i, j.

(5) $\Lambda^k(M)$ satisfies the UP

$$\begin{tikzcd} M \times \cdots \times M \arrow[r, "(m_1, \ldots, m_k) \to m_1 \wedge \cdots \wedge m_k"] \arrow[dr, "\varphi"'] & \Lambda^k(M) \arrow[d, dashed, "\exists !"] \\ & N \end{tikzcd}$$

with respect to alternating multilinear maps $\varphi: M \times \cdots \times M \longrightarrow N$: $\varphi(m_1 \ldots m_k) = 0$ whenever any m_i = any m_j.

(6) Let V be an n-dimensional F-vector space. Then $\Lambda(V)$ is finite dimensional, and $\Lambda^k(V) = 0$ for $k > n$.

why? Consider a generating element in $\Lambda^k(V)$, of the form $v_1 \wedge \cdots \wedge v_k$.

Write the v_i in terms of a fixed basis $w_1, \ldots, w_n$ of V. Get a finite linear sum of terms

$w_{i_1} \wedge \cdots \wedge w_{i_k}$ for $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

Must have repeats if $k > n$.

45.7

Again, if V is an n-dim F-VS with basis $w_1 - w_k$, a basis of $\Lambda^k(V)$ consists of the vectors

$$w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_k} \qquad 1 \le i_1 < i_2 < \cdots < i_k \le n,$$

so that $\dim_F \Lambda^k(V) = \binom{n}{k}$.

Wedge Products and Geometry.

Let V be an n-dimensional F-vector space.

$\varphi : V \longrightarrow V$ an endomorphism.

Then φ also induces an endomorphism on each $\Lambda^k(V)$, by

$$\varphi(v_1 \wedge \cdots \wedge v_k) = \varphi(v_1) \wedge \cdots \wedge \varphi(v_k).$$

$\Lambda^k(V)$ is also a vector space and this map is linear.

The space $\Lambda^n(V)$ is one-dimensional, spanned by

$w_1 \wedge \cdots \wedge w_k$ for any basis elements $w_1 - w_k$.

If φ sends the basis $\{w_i\} \longrightarrow \{x_i\}$, (i.e. if φ is invertible)

then $\varphi(w_1 \wedge \cdots \wedge w_k) = x_1 \wedge \cdots \wedge x_k$

$$= \lambda(\varphi) \cdot w_1 \wedge \cdots \wedge w_k$$

for some scalar $\lambda(\varphi)$ depending on φ.

Moreover, since φ induces linear endomorphisms on $\Lambda^n(V)$, get a homomorphism

$$GL(V) \longrightarrow GL(\Lambda^n(V)) = GL_1(\mathbb{R}).$$

45.8

What is this homomorphism?
Multilinear, alternating, must be the <u>determinant</u>.

Can construct the determinant in a slightly different (but equivalent) way.

Regard det as a map $V^n \longrightarrow F$
by $\det(v_1, v_2, \dots, v_n) = \det\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$.

This is the determinant of the LT sending the standard basis $e_1, \dots, e_n$ to $v_1, \dots, v_n$. (Need to choose a basis)

Have

$$\begin{tikzcd} V^n \arrow[r, "v_1,\dots,v_n \to v_1\wedge\dots\wedge v_n"] \arrow[dr, "\det"'] & \Lambda^n(V^n) \arrow[d, dashed] \\ & F \end{tikzcd}$$

alternating, multilinear.

The UP asserts that the determinant <u>factors through</u> $\Lambda^n(V^n)$.

Here $v_1 \wedge \dots \wedge v_n$ may be considered a (signed) <u>volume element</u> on V.

Also used to construct differential forms.
Start with the vector space consisting of symbols dv, $v \in V$
Take wedge products.

45.9.

Some more geometry.

Let $V = \mathbb{R}^3$ w/ std. basis e_1, e_2, e_3.

If $u = u_1 e_1 + u_2 e_2 + u_3 e_3$

$v = v_1 e_1 + v_2 e_2 + v_3 e_3$

then

$$u \wedge v = (u_1 v_2 - u_2 v_1) e_1 \wedge e_2 + (u_1 v_3 - u_3 v_1) e_1 \wedge e_3 + (u_2 v_3 - v_3 u_2) e_2 \wedge e_3 .$$

Prop. The map $\Lambda^2(\mathbb{R}^3) \longrightarrow (\mathbb{R}^3)^* = \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$

$v \wedge w \longrightarrow \{x \to (v \wedge w) \wedge x\}$

is an isomorphism.

Proof. Both sides are $\mathbb{R}$-vector spaces of dim 3, so enough to prove injective.

But if $v \wedge w \neq 0$, then v, w are nonzero and linearly independent. Let $x \in \mathbb{R}^3$ be such that $\{v, w, x\}$ is a basis. Then $v \wedge w \wedge x = \det [v \ w \ x] \, e_1 \wedge e_2 \wedge e_3$

$\neq 0$.

We can check that
$$e_1 \wedge e_2 \longrightarrow e_3^*$$
$$e_1 \wedge e_3 \longrightarrow -e_2^*$$
$$e_2 \wedge e_3 \longrightarrow e_1^* .$$

If we now identify $\mathbb{R}^3$ with its dual, we see that the wedge coincides with the cross product.