State High School Mathematics Tournament

University of South Carolina

Round 2 – March 23, 2024

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- If your answer is wrong, the clock will be restarted. If your opponent doesn't buzz in, they may answer *immediately* after time is called.

Question 2-1

You may have seen the formula

$$1 + 2 + \cdots + 100 = 5050.$$

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What is

$$1_8 + 2_8 + \cdots + 100_8,$$

the sum of the integers between 1_8 and 100_8? Answer in base 8.

Solution 2-1

Answer. 4040.

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Answer. 4040.

In base 8, we have

$$\begin{aligned} 1+2+\cdots+100 &= \frac{100\cdot 101}{2} \\ &= \frac{100}{2}\cdot 101 \\ &= 40\cdot 101 \\ &= 4040. \end{aligned}$$

Solution 2-1

Answer. 4040.

In base 8, we have

$$\begin{aligned} 1+2+\cdots+100 &= \frac{100\cdot 101}{2} \\ &= \frac{100}{2}\cdot 101 \\ &= 40\cdot 101 \\ &= 4040. \end{aligned}$$

Or, in base 10,

$$1+2+\cdots+64=\frac{64\cdot 65}{2}=2080=4040_8.$$

Question 2-2

The positive numbers

$$a < b < a + b < ab$$

form an arithmetic progression. What is the sum of these four numbers?

Solution 2-2

Answer. 20.

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We have

$$b - a = (a + b) - b \ \Rightarrow \ b = 2a.$$

Solution 2-2

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So, $b = 2a$, $a + b = 3a$, and $ab = 4a$.

Solution 2-2

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We have

$$b - a = (a + b) - b \;\Rightarrow\; b = 2a.$$

So, $b = 2a$, $a + b = 3a$, and $ab = 4a$.
We have

$$b = 4 \;\Rightarrow\; a = 2 \;\Rightarrow\; 2 + 4 + 6 + 8 = 20.$$

Question 2-3

	O	X
	X	
X		O

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The above shows a Tic-Tac-Toe board, where X has won after five moves.

Question 2-3

	O	X
	X	
X		O

The above shows a Tic-Tac-Toe board, where X has won after five moves.

How many such Tic-Tac-Toe boards (i.e. where X wins after five moves) are there?

Solution 2-3

Answer. 120.

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- 8 possible configurations of Xs: three rows, three columns, two diagonals.

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- For each, $\binom{6}{2} = 15$ ways to place the Os.

Solution 2-3

Answer. 120.

- 8 possible configurations of Xs: three rows, three columns, two diagonals.
- For each, $\binom{6}{2} = 15$ ways to place the Os.
- $8 \times 15 = 120$.

Question 2-4

What is the sum of all integers in the set $\{n : n^3 \leq 2024 \leq 3^n\}$?

Solution 2-4

Answer. 57.

Solution 2-4

Answer. 57.

Since

$$12^3 = 1728 < 2024 < 13^3 = 2197,$$

$$3^6 = 729 < 2024 < 3^7 = 2187,$$

the sum is

$$7 + 8 + 9 + 10 + 11 + 12 = 57.$$

Question 2-5

What is the last digit of 2024^{2024}?

Solution 2-5

Answer. 6.

Solution 2-5

Answer. 6.

Solution. We have

$$2024^{2024} = (2024 \cdot 2024)^{1012}.$$

Solution 2-5

Answer. 6.

Solution. We have

$$2024^{2024} = (2024 \cdot 2024)^{1012}.$$

Since $4 \cdot 4 = 16$, $2024 \cdot 2024$ ends in 6, and the product of numbers ending in 6 will itself end in 6.

Question 2-6

There are unique integers a and b for which

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There are unique integers a and b for which

$$(2-\sqrt{3})^3 = a + b\sqrt{3}.$$

What is $a+b$?

Solution 2-6

Answer. 11.

Solution 2-6

Answer. 11. We have

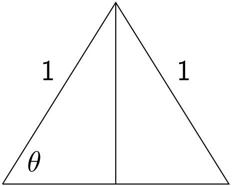
$$(2-\sqrt{3})^3 = 8 - 12\sqrt{3} + 6(\sqrt{3})^2 - (\sqrt{3})^3 = 26 - 15\sqrt{3}.$$

Question 7

If $\triangle ABC$ is an isosceles triangle with $AB = BC = 1$, what should the length of AC be to maximize the triangle's area?

Solution 7

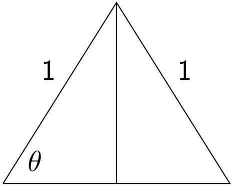
Answer. $\sqrt{2}$



$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

Solution 7

Answer. $\sqrt{2}$



$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

Maximize with $\theta = \frac{\pi}{4}$, so $AC = \sqrt{2}$.

Question 8

The equation $2^x = x^2$ has three real solutions. What is the nearest integer to their sum?

Solution 8

Answer. 5

$x = 2$, $x = 4$, and $x = -.76\ldots$

Solution 8

Answer. 5

$x = 2$, $x = 4$, and $x = -.76\ldots$
For the negative solution, note that $2^{-\frac{1}{2}} > (-\frac{1}{2})^2$, so $x < -\frac{1}{2}$.

Question 9

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2021 - 2022 + 2023 - 2024?$$

Solution 9

Answer. -1012.

Solution 9

Answer. -1012.

Write it as

$$(1-2)+(3-4)+\cdots+(2023-2024)=(-1)\times 1012.$$

Question 10

How many positive integers $n \leq 10$ satisfy $\cos(n) > 0$?

(Assume radian measure.)

Solution 10

Answer. 4.

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Answer. 4.

$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

Solution 10

Answer. 4.

$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

$$n \in (0, 1.57\ldots) \cup (4.71\ldots, 7.85\ldots)$$

Solution 10

Answer. 4.

$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

$$n \in (0, 1.57\ldots) \cup (4.71\ldots, 7.85\ldots)$$

$$n \in \{1, 5, 6, 7\}$$

Question 11

SImplify:

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$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}}$$

Solution 11

Answer. $\frac{17}{28}$.

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- $1 + \frac{1}{5} = \frac{6}{5}$

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- $1 + \frac{1}{5} = \frac{6}{5}$
- $1 + \frac{1}{1+\frac{1}{5}} = \frac{11}{6}$

Solution 11

Answer. $\frac{17}{28}$.

- $1+\frac{1}{5}=\frac{6}{5}$
- $1+\frac{1}{1+\frac{1}{5}}=\frac{11}{6}$
- $1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}=\frac{17}{11}$

Solution 11

Answer. $\frac{17}{28}$.

- $1+\frac{1}{5}=\frac{6}{5}$
- $1+\frac{1}{1+\frac{1}{5}}=\frac{11}{6}$
- $1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}=\frac{17}{11}$
- $1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}=\frac{28}{17}$

Solution 11

Answer. $\frac{17}{28}$.

- $1+\frac{1}{5}=\frac{6}{5}$
- $1+\frac{1}{1+\frac{1}{5}}=\frac{11}{6}$
- $1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}=\frac{17}{11}$
- $1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}=\frac{28}{17}$

Notice the pattern: $\frac{6}{5}, \frac{11}{6}, \frac{17}{11}, \frac{28}{17}$