COMPREHENSIVE EXAM IN ANALYTIC NUMBER THEORY

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A passing grade will consist approximately of correct answers to all questions in the first part. Questions answered in the second part will make up for incorrect or skipped questions.

Partial answers are very much encouraged, especially in the second part.

1. The First Part

1. For a primitive Dirichlet character $\chi \pmod q$, define a character sum $S_\chi(t) := \sum_{n \leq t} \chi(n)$. The **Polya-Vinogradov** inequality is the statement that $|S_\chi(t)| \ll \sqrt{q} \log q$.

   A paper of Goldmakher [**?**] offers the following nice proof that $\max_{t \leq q} |S_\chi(t)| \gg \sqrt{q}$:

   "A slick proof of this is to apply partial summation to the Gauss sum $\tau(q) := \sum_{n \leq q} \chi(n) e(n/q)$ and use the classical result that for primitive $\chi \pmod q$, $|\tau(\chi)| = \sqrt{q}$."

   (a.) Explain what the word "primitive" means, and show that it is necessary in the discussion above.
   (b.) Prove that for primitive χ, $|\tau(x)| = \sqrt{q}$. (Hint: evaluate $\tau(\chi)\overline{\tau(\chi)}$.)
   (c.) Spell out the details of Goldmakher's argument.

2. (a.) Define the *convolution* $f * g$ of two arithmetic functions f and g.
   (b.) If f and g are multiplicative, prove that $f * g$ is as well.
   (c.) Let $\mu(n)$ be the Möbius function, and let 1 be the constant function, equal to 1 for every n.

   Evaluate $\mu * 1$, and thus obtain a simple identity for the Dirichlet series $\sum_n \mu(n) n^{-s}$.

3. Assume that x is a real number > 1, not an integer. Then the **explicit formula** reads

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}).$$

   (a.) Explain what the above means. Your answer should define the function $\psi(x)$ and say what the sum over ρ is.
   (b.) Give a sketch of how this formula is proved.
   (c.) Suppose that you have proved that the above terms on the right, except for x, are all $o(x)$, so that $\psi(x) \sim x$. Deduce an asymptotic formula (e.g. *the prime number theorem*) for the number of primes $\leq x$.
   (d.) Given the above formula, prove that the zeta function has at least one nontrivial zero. (What does "nontrivial" mean?)
   (e.) (Bonus.) Given the above formula, prove that the zeta function has infinitely many zeroes.

4. Let $f(n)$ be the characteristic function of integers $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ such that, when written as shown as a product of distinct primes, all the e_i are odd.

   Prove an asymptotic formula for $\sum_{n \leq x} f(n)$ with an explicit error term.

2. The Fun Part

Answer as many questions as you can.

1. The following appears in a still unpublished preprint of Bhargava and Shnidman.

> By Theorem 14 and Lemma 16, it now suffices to count pairs $(b, c) \in L$, up to $SO_Q(\mathbb{Z})$-equivalence, subject to the condition $Q'(b,c)^2 = (b^2 - bc + c^2)^2 < X$. The number of integral points inside the elliptic region cut out by the latter inequality is approximately equal to its area $(2\pi/\sqrt{3})X^{1/2}$, with an error of at most $O(X^{1/4})$. Meanwhile, being the (orientation-preserving) symmetry group of the triangular lattice, $SO_Q(\mathbb{Z})$ is isomorphic to C_6, the cyclic group of order 6. Since this is the cubic action, the cyclic subgroup $C_3 \subseteq SO_Q(\mathbb{Z})$ acts trivially. Up to equivalence, we thus obtain
> $$\frac{2\pi}{2\sqrt{3}} X^{1/2} + O(X^{1/4})$$
> points inside the ellipse.

   Evaluate the infinite sum
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots$$
   and explain why the discussion above is relevant.

2. In his 1859 memoir, Riemann conjectured that the nontrivial zeroes $\rho = \beta + i\gamma$ of the zeta function satisfy $\beta = 1/2$. (If you somehow manage to prove this, you will definitely pass the exam.) Riemann guessed this based on some numerical computations. He found that the first few zeroes are $\rho = \frac{1}{2} + 14.134 \cdots i$, $\rho = \frac{1}{2} + 21.012 \cdots i$, $\rho = \frac{1}{2} + 25.010 \cdots i$, $\ldots$.

   How might he have found these zeroes? Describe a method of proving this, subject to numerical computations that could reasonably by done by hand.

3. Prove that $\zeta(1 + it) \neq 0$ for any t.

4. (a.) Prove, or sketch a proof, subject to the Riemann hypothesis if you like, that every integer ≥ 2 is the sum of 1,000 primes.

   (b.) Conjecture (or prove!) an asymptotic formula for the number of such representations of an integer N, as a function of N.

5. Explain something interesting about character sums which you have read in Iwaniec-Kowalski or elsewhere.

References

[1] L. Goldmakher, `http://arxiv.org/pdf/0911.5547v2.pdf`.