State High School Mathematics Tournament

University of South Carolina

Round 1 – January 25, 2020

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- There will be a tiebreaker if needed.

Question 1-1

Solve for x:

$$|2x - 1| = |2x - 2|.$$

Solution 1-1

Answer. $\frac{3}{4}$.

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We have

$$2x - 1 = \pm(2x - 2),$$

and since $+$ is impossible, we have

$$2x - 1 = -(2x - 2) = -2x + 2.$$

Solution 1-1

Answer. $\frac{3}{4}$.

We have

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and since $+$ is impossible, we have

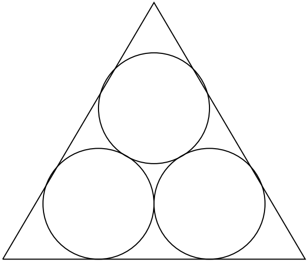
$$2x - 1 = -(2x - 2) = -2x + 2.$$

So,

$$4x = 2 + 1 = 3 \Longrightarrow x = \frac{3}{4}.$$

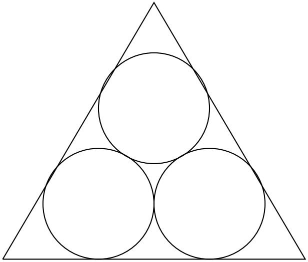
Question 1-2

In the diagram below, three circles of equal size are inscribed in an equilateral triangle, so that they are tangent to each other and to the indicated sides of the triangle.



Question 1-2

In the diagram below, three circles of equal size are inscribed in an equilateral triangle, so that they are tangent to each other and to the indicated sides of the triangle.



If each circle has radius 1, what is the side length of the triangle?

Solution 1-2

Answer. $2 + 2\sqrt{3}$.

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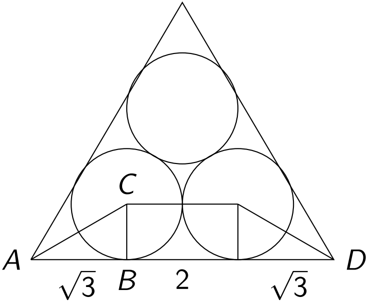

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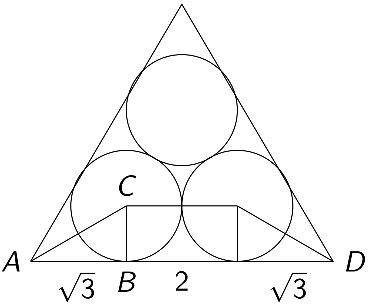

As $\overline{AC}$ lies on an altitude, we have $\angle CAB = 30^\circ$.

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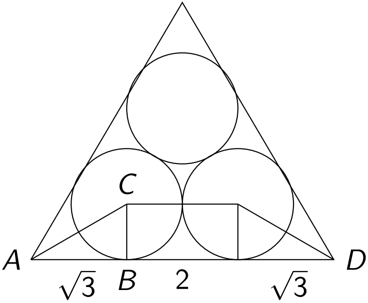

As $\overline{AC}$ lies on an altitude, we have $\angle CAB = 30^\circ$.
Hence $AB = \sqrt{3}$ and so $AD = 2 + 2\sqrt{3}$.

Question 1-3

What is the only real solution x to

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

Solution 1-3

Answer. $-\frac{11}{2}$.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

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$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

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You can solve this for x, or notice that $-\frac{11}{2}$ lies on the midpoint of symmetry of the roots.

Question 1-4

$$123 \times 321 = 41703$$

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In what number base is this equation true?

Solution 1-4

Answer: 8.

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$$(b^2 + 2b + 3) \times (3b^2 + 2b + 1) = \cdots + 8b + 3.$$

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$$(b^2 + 2b + 3) \times (3b^2 + 2b + 1) = \cdots + 8b + 3.$$

So $b \mid 8$. Since the digit 6 appears in the product, $b = 8$.

Question 1-5

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What is the area of the rectangle with these four points as vertices?

Solution 1-5

Answer. $\frac{32}{5}$.

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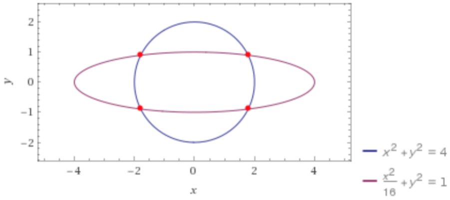

Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$.

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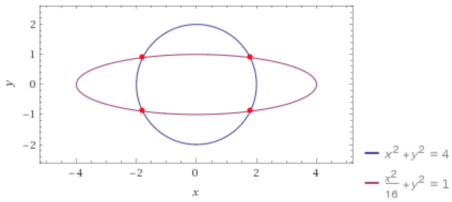

Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$. We have $y^2 = 4 - \frac{16}{5} = \frac{4}{5}$, and so

$$x^2 \cdot y^2 = \frac{16}{5} \cdot \frac{4}{5} = \frac{64}{25},$$

and the unique solution with $x > 0$, $y > 0$ satisfies $xy = \frac{8}{5}$.

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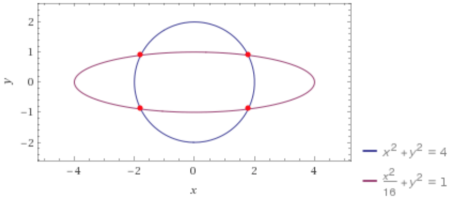

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Since the rectangle is centered at the origin, its area is $4xy = \frac{32}{5}$.

Question 6

You eat a bunch of cupcakes. On the first day, you eat one cupcake; on the second day, you eat two cupcakes; on the third day, you eat three; and so on.

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On what day will you eat your five thousandth cupcake?

Solution 6

Answer. 100.

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After n days, you will have eaten

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

cupcakes. So what is the minimal n for which

$$\frac{n(n+1)}{2} \geq 5000, \quad \text{or} \quad n(n+1) \geq 10000?$$

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Since $10000 = 100^2$, we have $n = 100$.

Question 7

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It depicts an icosahedron: a regular solid with twenty faces, each of which is an identical equilateral triangle.

How many edges are not visible in the logo?

Solution 7

Answer. 12.

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An icosahedron has 30 edges: 20 triangles times 3 edges per triangle, divided by 2 since each edge is shared between two triangles.

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You can count that 18 edges are visible in the picture, and $30 - 18 = 12$.

Question 8

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$$\log_2(33) + \log_{33}(2)?$$

Solution 8

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so that $\log_2(33)$ is slightly bigger than 5.

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$$\log_{33}(2) = \frac{1}{\log_2(33)} < \frac{1}{5}.$$

The sum of these numbers is less than 6.

Question 9

How many real solutions x are there to the equation

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$$x^5 + \sin(x) = 0?$$

Solution 9

Answer: 1.

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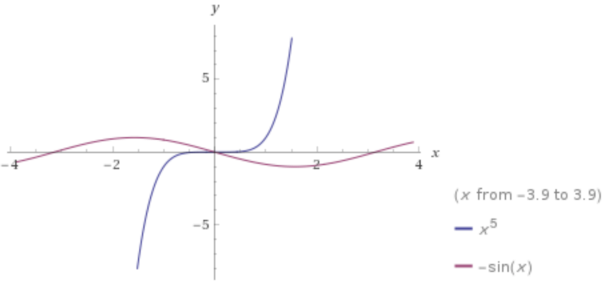

The graphs of $y = x^5$ and $y = -\sin(x)$ don't intersect in $(0, \pi)$ because of opposite signs, or in $[\pi, \infty)$ because $x^5 > 1$. Similarly, there are no intersection points with $x < 0$.

Solution 9

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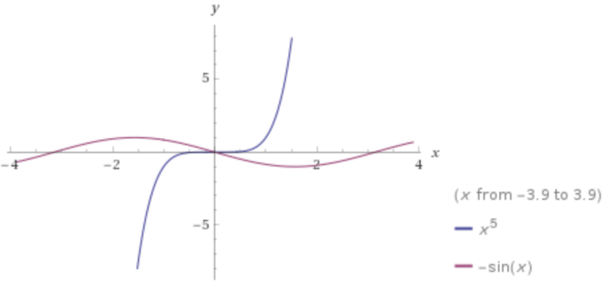

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So $x = 0$ is the only intersection point.

Question 10

You toss four coins. What is the probability that at least three of them come up heads?

Solution 10

Answer. $\frac{5}{16}$.

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There are $2^4 = 16$ total ways to flip four coins.
The total number with at least three heads is

$$\binom{4}{3} + \binom{4}{4} = 4 + 1 = 5,$$

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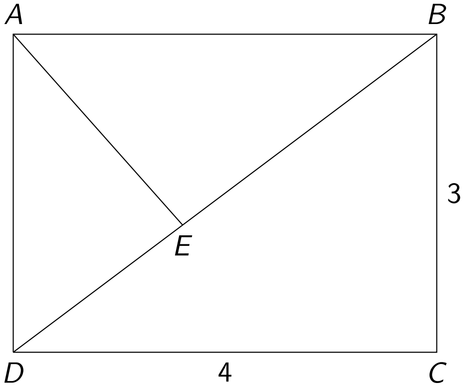
$$HHHH, HHHT, HHTH, HTHH, THHH.$$

Question 11



Given rectangle $ABCD$ as above. If $\angle AEB = 90^\circ$, what is AE?

Solution 11

Answer. $\frac{12}{5}$.

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$BD = 5$, and $\triangle ABE \sim \triangle BDC$. So

$$\frac{AE}{AB} = \frac{BC}{BD} = \frac{3}{5}$$

and

$$AE = \frac{3}{5} \cdot AB = \frac{3}{5} \cdot 4 = \frac{12}{5}.$$

Question 12

How many pairs of positive prime numbers p, q are there with

$$p - q = 21?$$

Solution 12

Answer. 1.

Solution 12

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All prime numbers other than 2 are odd. The difference of two odd numbers is even. Therefore q must be 2. Since $2 + 21 = 23$ is prime, there is one solution.

Question 13

In baseball, an **at bat** *results in either a* **hit** *or an* **out**. *A player's* **batting average** *is their total number of hits divided by at bats, rounded off to the nearest thousandth.*

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If this raises his batting average to .536, how many at bats does he have through his first six games?

Solution 13

Answer. 28.

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Let x be the number of hits through 6 games, and y the number of at bats. Within a small roundoff error,

$$\frac{x-5}{y-5} = .435, \quad \frac{x}{y} = .536.$$

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We have

$$x - 5 = .435(y-5), \quad x = .435y + 5 - 2.175 = .435y + 2.825.$$

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We thus have

$$x = .536y, \quad .101y = 2.825.$$

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We have

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We thus have

$$x = .536y, \quad .101y = 2.825.$$

Thus, we have

$$y = \frac{2.825}{.101},$$

or $y = 28$ up to the roundoff error.

Question 14

If you write $\frac{1}{2020}$ as an infinite repeating decimal,

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If you write $\frac{1}{2020}$ as an infinite repeating decimal, what is the sum of the first six digits after the decimal place?

Solution 14

Answer. 18.

$$\frac{1}{2020} = 0.00049504950\ldots$$

Solution 14

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Note that

$$\frac{1}{101} = .009900990099\ldots,$$

so

$$\frac{1}{1010} = .0009900990099\ldots,$$

$$\frac{1}{2020} = .0004950495049\ldots,$$