Error Terms in Arithmetic Statistics

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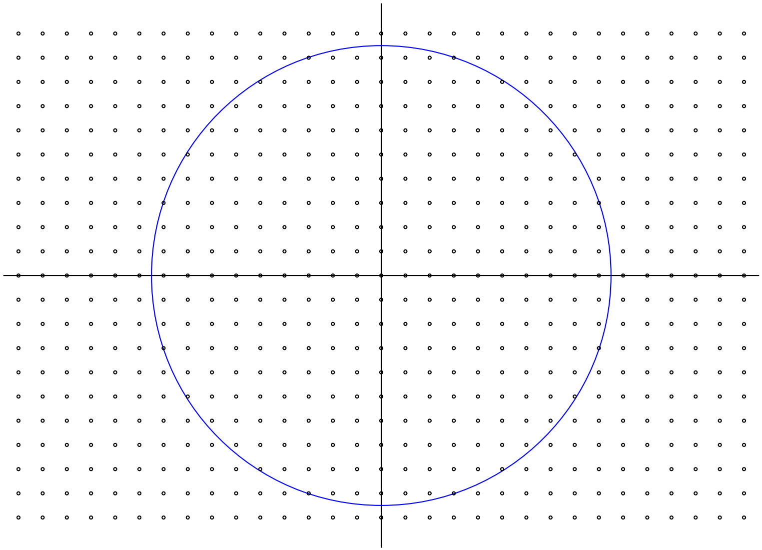
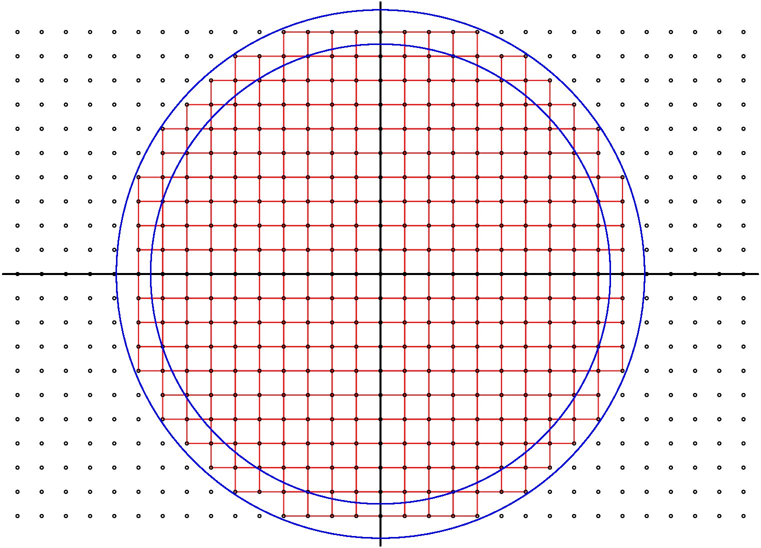
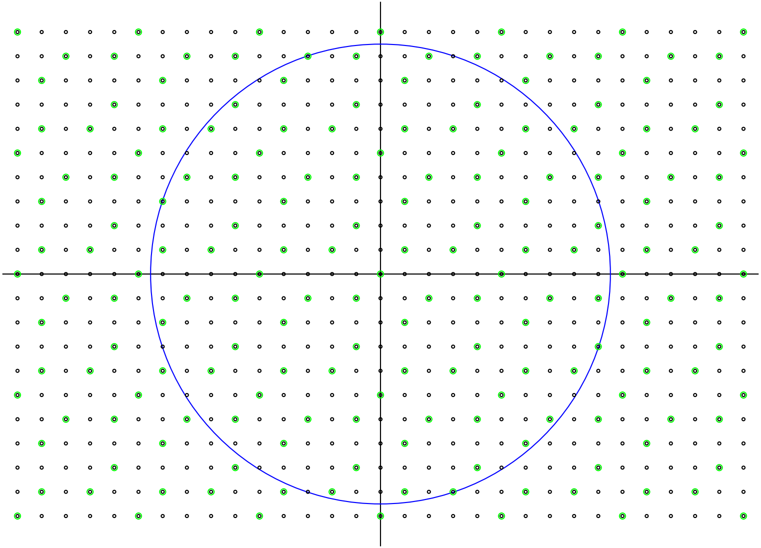
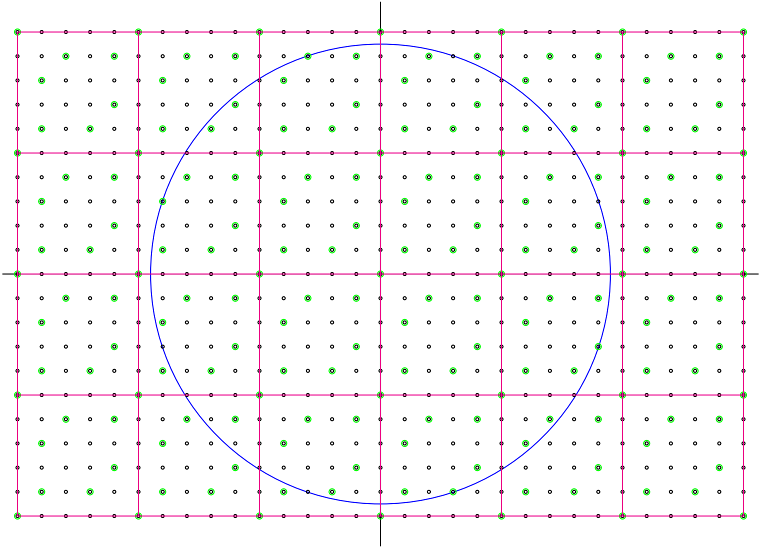
Chennai Mathematical Institute, November 25, 2025

thornef.github.io/cmi-2025.pdf









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Theorem (Davenport-Heilbronn)

We have

$$N_3(X) = \frac{1}{3\zeta(3)} X + o(X).$$

Sample Theorem 2: Counting Quartic and Quintic Fields

Theorem (Bhargava)

We have

$$N_4(X, S_4) \sim \frac{5}{24} \prod_p (1 + p^{-2} - p^{-3} - p^{-4}) X,$$

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Sample Theorem 3: 3-torsion in Quadratic Class Groups

Theorem (Davenport-Heilbronn)

We have

$$\sum_{|D|<X} \#|\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))[3]| = \frac{3+3+1+3}{\pi^2} X + o(X).$$

Sample Theorem 4: 2-Selmer Groups in Elliptic Curves

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Corollary

Their average rank is at most 1.5.

Parametrization: The Basic Metatheorem

Theorem

There exists an explicit, "nice" bijection

$$\{ \textit{ Something nice } \} \longleftrightarrow G(\mathbb{Z}) \backslash V(\mathbb{Z})$$

where V is a f.d. representation of an algebraic group G.

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There exists an explicit, "nice" bijection

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where V is a f.d. representation of an algebraic group G.

Moreover, certain arithmetic properties on the left correspond to congruence conditions on the right.

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Let V be the space of binary cubic forms:

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- $\mathrm{Disc}(gx) = (\det g)^2 \mathrm{Disc}(x)$;
- $\mathrm{Disc}(x) = 0$ if and only if $x(u,v)$ has a repeated root.

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- $\mathrm{Stab}(v)$ *is isomorphic to* $\mathrm{Aut}(R)$*;*
- $\mathrm{Disc}(v) = \mathrm{Disc}(R)$*;*
- *(Davenport-Heilbronn) R is maximal iff, for all primes p, v satisfies a certain congruence condition $\pmod{p^2}$.*

Improved Davenport-Heilbronn

Theorem (DHBBPBSTTTBTT)

We have

$$N_3(X) = \frac{1}{3\zeta(3)}X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{\frac{2}{3}}(\log X)^3).$$

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- ... and more!
  (Bhargava, Ho, Shankar, Varma, X. Wang, Wood, ...)

More Interesting Parametrizations

Table 1: Summary of Higher Composition Laws

#	Lattice ($V_{\mathbb{Z}}$)	Group acting ($G_{\mathbb{Z}}$)	Parametrizes ($\mathcal{C}$)	(k)	(n)	(H)
1.	$\{0\}$	-	Linear rings	0	0	A_0
2.	$\widetilde{\mathbb{Z}}$	$\mathrm{SL}_1(\mathbb{Z})$	Quadratic rings	1	1	A_1
3.	$(\mathrm{Sym}^2\mathbb{Z}^2)^*$ (GAUSS'S LAW)	$\mathrm{SL}_2(\mathbb{Z})$	Ideal classes in quadratic rings	2	3	B_2
4.	$\mathrm{Sym}^3\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})$	Order 3 ideal classes in quadratic rings	4	4	G_2
5.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^2$	Ideal classes in quadratic rings	4	6	B_3
6.	$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^3$	Pairs of ideal classes in quadratic rings	4	8	D_4
7.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^4$	$\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_4(\mathbb{Z})$	Ideal classes in quadratic rings	4	12	D_5
8.	$\wedge^3\mathbb{Z}^6$	$\mathrm{SL}_6(\mathbb{Z})$	Quadratic rings	4	20	E_6
9.	$(\mathrm{Sym}^3\mathbb{Z}^2)^*$	$\mathrm{GL}_2(\mathbb{Z})$	Cubic rings	4	4	G_2
10.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Order 2 ideal classes in cubic rings	12	12	F_4
11.	$\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})^2$	Ideal classes in cubic rings	12	18	E_6
12.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^6$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_6(\mathbb{Z})$	Cubic rings	12	30	E_7
13.	$(\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3)^*$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Quartic rings	12	12	F_4
14.	$\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$	$\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$	Quintic rings	40	40	E_8

Bhargava, *Higher composition laws IV*, Ann. Math., 2008

Still More Interesting Parametrizations

	Group (s.s.)	Representation	Geometric Data	Invariants	Dynkin	§
1.	SL_2	$\mathrm{Sym}^4(2)$	(C, L_2)	$2, 3$	$A_2^{(2)}$	4.1
2.	SL_2^2	$\mathrm{Sym}^2(2) \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_3^{(2)}$	6.1
3.	SL_2^4	$2 \otimes 2 \otimes 2 \otimes 2$	$(C, L_2, L_2', L_2'') \sim (C, L_2, P, P')$	$2, 4, 4, 6$	$D_4^{(1)}$	6.2
4.	SL_2^3	$2 \otimes 2 \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 4, 6$	$B_3^{(1)}$	6.3.1
5.	SL_2^2	$\mathrm{Sym}^2(2) \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_3^{(2)}$	6.3.3
6.	SL_2^2	$2 \otimes \mathrm{Sym}^3(2)$	(C, L_2, P_3)	$2, 6$	$G_2^{(1)}$	6.3.2
7.	SL_2	$\mathrm{Sym}^4(2)$	(C, L_2, P_3)	$2, 3$	$A_2^{(2)}$	6.3.4
8.	$\mathrm{SL}_2^2 \times \mathrm{GL}_4$	$2 \otimes 2 \otimes \wedge^2(4)$	$(C, L_2, M_{2,6})$	$2, 4, 6, 8$	$D_5^{(1)}$	6.6.1
9.	$\mathrm{SL}_2 \times \mathrm{SL}_6$	$2 \otimes \wedge^3(6)$	$(C, L_2, M_{3,6})$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(1)}$	6.6.2
10.	$\mathrm{SL}_2 \times \mathrm{Sp}_6$	$2 \otimes \wedge_0^3(6)$	$(C, L_2, (M_{3,6}, \varphi))$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(2)}$	6.6.3
11.	$\mathrm{SL}_2 \times \mathrm{Spin}_{12}$	$2 \otimes S^+(32)$	$(C \to \mathbb{P}^1(\mathscr{H}_3(\mathbb{H})), L_2)$	$2, 6, 8, 12$	$E_7^{(1)}$	6.6.3
12.	$\mathrm{SL}_2 \times E_7$	$2 \otimes 56$	$(C \to \mathbb{P}^1(\mathscr{H}_3(\mathbb{O})), L_2)$	$2, 6, 8, 12$	$E_8^{(1)}$	6.6.3
13.	SL_3	$\mathrm{Sym}^3(3)$	(C, L_3)	$4, 6$	$D_4^{(3)}$	4.2
14.	SL_3^3	$3 \otimes 3 \otimes 3$	$(C, L_3, L_3') \sim (C, L_3, P)$	$6, 9, 12$	$E_6^{(1)}$	5.1
15.	SL_3^2	$3 \otimes \mathrm{Sym}^2(3)$	(C, L_3, P_2)	$6, 12$	$F_4^{(1)}$	5.2.1
16.	SL_3	$\mathrm{Sym}^3(3)$	(C, L_3, P_2)	$4, 6$	$D_4^{(3)}$	5.2.2
17.	$\mathrm{SL}_3 \times \mathrm{SL}_6$	$3 \otimes \wedge^2(6)$	$(C, L_3, M_{2,6})$ with $L^{\otimes 2} \cong \det M$	$6, 12, 18$	$E_7^{(1)}$	5.5
18.	$\mathrm{SL}_3 \times E_6$	$3 \otimes 27$	$(C \hookrightarrow \mathbb{P}^2(\mathbb{O}), L_3)$	$6, 12, 18$	$E_8^{(1)}$	5.4
19.	$\mathrm{SL}_2 \times \mathrm{SL}_4$	$2 \otimes \mathrm{Sym}^2(4)$	(C, L_4)	$8, 12$	$E_6^{(2)}$	4.3
20.	$\mathrm{SL}_5 \times \mathrm{SL}_5$	$\wedge^2(5) \otimes 5$	(C, L_5)	$20, 30$	$E_8^{(1)}$	4.4

Table 1: Table of coregular representations and their moduli interpretations

Bhargava and Ho, *Coregular spaces and genus one curves*, Camb. J. Math.

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- We might impose congruence conditions as well.

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  Quantitative improvements in Bhargava's averaging method.
  Expect it to work in the same situations.

An explicit evaluation

Theorem (Taniguchi-T., 2011)

We have

$$\widehat{\Phi_{p^2}}(v) = \begin{cases} p^{-2} + p^{-3} - p^{-5} & v/p: \textit{ of type } (0), \\ p^{-3} - p^{-5} & v/p: \textit{ of type } (1^3), (1^2 1), \\ -p^{-5} & v/p: \textit{ of type } (111), (21), (3). \\ p^{-3} - p^{-5} & v: \textit{ of type } (1^3_{**}), \\ -p^{-5} & v: \textit{ of type } (1^3_{*}),\ (1^3_{\max}), \\ 0 & \textit{otherwise.} \end{cases}$$

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So:

$$\frac{1}{p^8} \sum_{v \in V(\mathbb{Z}/p^2\mathbb{Z})} |\widehat{\Phi_{p^2}}(v)| \ll p^{-7}.$$

Consequence – Improved Davenport-Heilbronn

Theorem (DHBBPBSTTTBTT)

We have

$$N_3(X) = \frac{1}{3\zeta(3)}X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{\frac{3}{5}+\epsilon}) + O(X^{1-\frac{1}{8-7+2}+\epsilon}).$$

Application 1: Direct products

Theorem (Wang, Masri-T.-Tsai-Wang)

Let $d \in \{3, 4, 5\}$ and let A be any abelian group. Then

$$N_{d|A|}(X, S_d \times A) \sim c(S_d \times A) X^{1/|A|}.$$

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(This doesn't happen too often.)

Application 2, Counting by other invariants: S_3-cubic fields

$$
\begin{tikzcd}
 & K(\sqrt{D}) \arrow[dd, dash, "(C_3)"] \\
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$$\mathrm{Disc}(K) = \mathrm{D}(K)\mathrm{F}(K)^2,$$

where

$$\mathrm{D}(K) := \mathrm{Disc}(\mathbb{Q}(\sqrt{D})), \quad \mathrm{F}(K) = \mathfrak{f}(K(\sqrt{D})/\mathbb{Q}(\sqrt{D})).$$

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For $\alpha, \beta > 0$, define an expression of the form

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the associated counting function.

Sample counting results

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- $(\alpha = 1, \beta = 2)$:
$$I(\alpha, \beta) = |\mathrm{D}|\mathrm{F}^2$$
is the usual (absolute) discriminant, and
$$N_{|\mathrm{D}|\mathrm{F}^2}(X) = \frac{1}{3\zeta(3)}X + o(X)$$
is the Davenport-Heilbronn theorem.

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- $(\alpha = 1, \beta = 2)$ [Davenport-Heilbronn]
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$$I(\alpha, \beta) = |\mathrm{D}|^3 \mathrm{F}^4$$

is the absolute discriminant of the associated S_3-sextic, and

$$N_{|\mathrm{D}|^3\mathrm{F}^4}(X) = C \cdot X^{1/3} + o(X^{1/3})$$

by Belabas-Fouvry and Bhargava-Wood.

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- $(\alpha = 1, \beta = 1)$:
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Counting by squarefree part

Let $N_C(X)$ count cubic fields where the **squarefree part of the discriminant** is $< X$.

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Theorem (Shankar-T.)

We have

$$N_C(X) = \frac{7}{5} \prod_p \left(1 + \frac{2}{p}\right)\left(1 - \frac{1}{p}\right)^2 X \log X + o(X \log X).$$

Overview of the proof

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where

$$C(\Sigma) = \frac{1}{2\alpha}\Big(\sum_{K \in \Sigma_\infty} \frac{1}{|\mathrm{Aut}(K)|}\Big) \prod_p \Big(\sum_{K \in \Sigma_p} \frac{|\mathrm{D}(K)|_p |\mathrm{F}(K)|_p}{|\mathrm{Aut}(K)|}\Big)\Big(1 - \frac{1}{p}\Big)^2.$$

Also:

$$\#\{K \in \mathcal{F}(\Sigma) : \ \mathrm{D}(K) = -3, \ \mathrm{F}(K) < Z\} \sim C(-3, \Sigma) \cdot Z \log Z.$$

Method 1: Uniform Davenport-Heilbronn

Define

$$N(\mathcal{F}(\Sigma)^{(f)}; Y) := \#\{K \in \mathcal{F}(\Sigma) : \ \mathrm{F}(K) = f, \ |\mathrm{D}(K)| < Y\},$$

Theorem (Davenport-Heilbronn, ...*, Bhargava-Taniguchi-T.)

We have

$$N(\mathcal{F}(\Sigma)^{(f)}; Y) = C_1(\Sigma, f) \cdot Y + C_2(\Sigma, f) \cdot Y^{5/6} + O\big(E(Y; f, \Sigma)\big),$$

with the 'average error bound'

$$\sum_{f \leq F} E(Y_f; f, \Sigma) \ll_\epsilon Y^{2/3+\epsilon} F^{4/3+\epsilon} P_\Sigma^{2/3},$$

uniformly in F and $Y_f \leq Y$.

*: Belabas, Belabas-Bhargava-Pomerance, Bhargava-Shankar-Tsimerman, Taniguchi-T.

Explicit Cohen-Morra

For each nonzero fundamental discriminant d, define a Dirichlet series

$$\Phi_{\Sigma,d}(s) := \frac{1}{2} + \sum_{K \in \mathcal{F}(\Sigma):\ \mathrm{D}(K)=d} \frac{1}{\mathrm{F}(K)^s},$$

Theorem (Cohen-Morra, Cohen-T.)

For Σ not specifying any restriction, we have

$$\Phi_{\Sigma,d}(s) = -\frac{1}{2}+\frac{1}{2}\left(1+\frac{2}{3^{2s}}\right) \prod_{\left(\frac{-3d}{p}\right)=1} \left(1+\frac{2}{p^s}\right)+\sum_{L\in\mathcal{L}_3(d)} M_{3,L}(s) \prod_{\left(\frac{-3d}{p}\right)=1} \left(1+\frac{\omega_L(p)}{p^s}\right),$$

where

$$M_{3,L}(s) := \begin{cases} 1-3^{-2s} & : \mathrm{Disc}(L) = -27d \\ 1+2\cdot 3^{-2s} & : \mathrm{Disc}(L) = -3d, \end{cases}$$

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- $$\omega_L(p) = \begin{cases} -1 & \textit{if } p \textit{ is inert in } L\,, \\ 2 & \textit{if } p \textit{ is totally split in } L\,, \\ 0 & \textit{otherwise.} \end{cases}$$

Analytic Consequence of Cohen-Morra

Define

$$N(\mathcal{F}(\Sigma)_d; Z) := \#\big\{K \in \mathcal{F}(\Sigma) : \mathrm{D}(K) = d,\ \mathrm{F}(K) < Z\big\}.$$

Then:

Theorem (Cohen-Morra)

For $\Sigma = \Sigma_{all}$ and $d \neq -3$, we have

$$N(\mathcal{F}(\Sigma)_d; Z) = \mathrm{Res}_{s=1}\big(\Phi_{\Sigma,d}(s)\big) \cdot Z + O_\epsilon\Big(|\mathcal{L}_3(d)||d|^{1/6} Z^{2/3+\epsilon}\Big).$$

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Proved variations of the above permitting local conditions.