6-torsion in class groups of imaginary quadratic fields

Frank Thorne
j/w Peter Koymans, Robert Lemke Oliver, and Efthymios Sofos

University of South Carolina, Columbia, SC, USA
Lodha Mathematical Sciences Institute, Mumbai

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thornef.github.io/6-torsion-ap.pdf

Conference Announcement

Workshop on Arithmetic Statistics

Lodha Institute, Mumbai, December 15-19, 2025

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Contact `thorne@math.sc.edu` for more information.

The Basic Question

Let $D \neq 0, 1$ be a fundamental discriminant.

Write $\mathrm{Cl}(D) := \mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$ for the associated *ideal class group*.

What can we say about $\mathrm{Cl}(D)$ as D varies?

Dirichlet's Class Number Formula

Theorem

If $D < 0$, we have

$$\mathrm{Cl}(D) = \frac{w\sqrt{|D|}}{2\pi} L(1, \chi_D),$$

and if $D > 0$ we have

$$\mathrm{Cl}(D) = \frac{\sqrt{|D|}}{2\log(\epsilon_D)} L(1, \chi_D),$$

Class Number One Problem

Theorem (Heegner, Baker, Stark)

If $D < 0$ and $\mathrm{Cl}(D) = 1$, then

$$D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}.$$

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Conjecture

There are **infinitely many positive** D for which $\mathrm{Cl}(D) = 1$.

Class Group Torsion

Question: given $m > 1$, what can we say about

$$\mathrm{Cl}(D)[m] := \{[\mathfrak{a}] \in \mathrm{Cl}(D) : [\mathfrak{a}]^m = [(1)]\}\,?$$

2-torsion: Gauss's genus theory

Theorem (Gauss's genus theory ($D < 0$ case))

If $D < 0$, we have

$$\mathrm{Cl}(D)[2] \simeq (\mathbb{Z}/2)^{\omega(D)-1},$$

where $\omega(D)$ counts the **unique prime divisors of D.**

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Theorem (Gauss's genus theory ($D > 0$ case))

If $D > 0$, we have

$$\mathrm{Cl}^{+}(D)[2] \simeq (\mathbb{Z}/2)^{\omega(D)-1},$$

and

$$\mathrm{Cl}^{+}(D)[2] \simeq \mathrm{Cl}(D)[2] \times (\mathbb{Z}/2)$$

if D has any prime divisor $p \equiv 3 \pmod 4$.

Conjectures on p-torsion

Conjecture (Cohen-Lenstra)

Let p be an odd prime. Then, on average, $\mathrm{Cl}(D)[p]$ is "a random abelian p-group".

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Conjecture (Brumer-Silverman)

For any number field K and prime p, we have

$$\#\mathrm{Cl}(D)[p] \ll |\mathrm{Disc}(D)|^{\epsilon}.$$

Asymptotic results

Problem: Given $m \geq 2$, prove asymptotic formulas for

$$\sum_{\pm D < X} \#\mathrm{Cl}(D)[m].$$

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- $m = 2^k$: Smith (2022)
- $m = 6$: New! KLOST (2025?)

Main Results

Theorem (Koymans, Lemke Oliver, Sofos, T.)

We have

$$\sum_{-X<D<0} \#\mathrm{Cl}(D)[6] = \frac{3}{\pi^2}\prod_p \left(1+\frac{1}{p+1}\right)\left(1-\frac{1}{p}\right) X\log(X) + O\left(X(\log\log X)^7\right)$$

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Improves upon (and uses!):

Theorem (Chan, Koymans, Pagano, Sofos)

We have

$$\sum_{0<\pm D<X} \#\mathrm{Cl}(D)[6] \ll X\log(X).$$

Another result

Theorem

We have

$$N_{12}(X, D_6) = CX^{1/6}(\log X)^2 + O(X^{1/6}(\log X)^{1+\epsilon}).$$

The Basic Idea

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Count $\mathrm{Cl}(D)[3]$ and $\mathrm{Cl}(D)[2]$ "at the same time".

Asymptotics for 2-class numbers:

Theorem

We have

$$\sum_{0<-D<X} \#\mathrm{Cl}(D)[2] \sim \frac{3}{2\pi^2} X \log X \cdot \prod_p \left(1 + \frac{1}{p+1}\right)\left(1 - \frac{1}{p}\right).$$

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This is “easy” and “well known”.

Asymptotics for 2-class numbers (2):

Since

$$\mathrm{Cl}(D)[2] \simeq (\mathbb{Z}/2)^{\omega(D)-1},$$

we have "morally"

$$\sum_{0<-D<X} \#\mathrm{Cl}(D)[2] \text{ `` = '' } \frac{1}{2} \sum_{\substack{0<n<X \\ \text{squarefree}}} \tau(n),$$

where $\tau(n)$ **counts the positive divisors of** n.

Asymptotics for 2-class numbers (3):

We have

$$\sum_{n \text{squarefree}} \tau(n) n^{-s} = \prod_p \left(1 + \frac{2}{p^s}\right)$$
$$= \zeta(s)^2 \cdot \prod_p \left(1 + \frac{2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2},$$

and

$$\sum_{\substack{0<n<X \\ \text{squarefree}}} \tau(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)^2 \cdot \prod_p \left(1 + \frac{2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2} X^s \frac{ds}{s}.$$

Asymptotics for 3-class numbers (1):

Theorem (Davenport-Heilbronn, 1971)

We have

$$\sum_{0<-D<X} \#\mathrm{Cl}(D)[3] \sim \frac{3+3}{\pi^2} X,$$

and

$$\sum_{0<D<X} \#\mathrm{Cl}(D)[3] \sim \frac{3+1}{\pi^2} X.$$

DH Step 1: Cubic Fields

Proposition

Let D be a fundamental discriminant. Then, *cubic fields of discriminant* D are in bijection with *subgroups of* $\mathrm{Cl}(D)$ *of index* 3.

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$$
\begin{tikzcd}
 & \widetilde{F} = F(\sqrt{D}) \arrow[dl, dash] \arrow[dd, dash] \\
F \arrow[dd, dash] & \\
 & \mathbb{Q}(\sqrt{D}) \arrow[dl, dash] \\
\mathbb{Q} &
\end{tikzcd}
$$

Let V be the space of binary cubic forms:

$$V := \{x(u, v) = au^3 + bu^2v + cuv^2 + dv^3\}.$$

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- $\mathrm{Disc}(gx) = (\det g)^6 \mathrm{Disc}(x)$;
- $\mathrm{Disc}(x) = 0$ if and only if $x(u, v)$ has a repeated root.

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There is a 'nice' bijection between:

- *Cubic rings up to isomorphism; and,*
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Theorem (Davenport-Heilbronn)

A cubic ring R is maximal iff its cubic form f satisfies certain congruence conditions $\pmod{p^2}$*, for every prime p.*

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- Shintani zeta functions.

A Stronger Davenport-Heilbronn Theorem

Theorem (BBDHPSTTT*)

We have

$$\sum_{0<\pm D<X} \#\mathrm{Cl}(D)[3] = \frac{3+3+3+1}{\pi^2} X + cX^{5/6} + O(X^{2/3+\epsilon}).$$

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Variations:

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Variations:

- Counting all cubic fields (including non-fundamental discriminants);
- Allowing for local specifications;
- A 'level of distribution' statement (important later!!)

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Answer: Naively.

6-torsion: first steps

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$$\frac{1}{2} \sum_{0<-D<X} \#\mathrm{Cl}(D)[3] \cdot 2^{\omega(D)-1} = \sum_{0<d<\sqrt{X}} \mu^2(d) \sum_{\substack{d^2<\mathrm{sfk}(-D)<X \\ d|D}} \#\mathrm{Cl}(D)[3].$$

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So, we're done, right......?

Sage Wisdom from Jean-Pierre Serre

"Another theorem says that $|Af|$ is smaller than $C|f|$, where C is a constant. A beautiful word! What is meant is real number, strictly positive, and you call it a constant because it doesn't depend – so it's a real number not depending on some *of the data. Now, very often the theorem is taken with 'Let f be fixed.' So that,* a priori, *in normal good mathematics the constant would depend on f. But they do not say. And the constant may depend, certainly on A, but on many other things.*

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How? If you're not careful, badly.

What is a level of distribution?

The Bombieri-Vinogradov Theorem:

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Theorem

For every $A > 0$ there exists $B > 0$ so that

$$\sum_{q < X^{1/2}(\log X)^{-B}} \max_{a \pmod{^* q}} \left| \pi(x; q, a) - \frac{1}{\phi(q)} \mathrm{Li}(x) \right| \ll X (\log X)^{-A}.$$

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"The primes have level of distribution $X^{1/2}(\log X)^{-B}$."

"GRH on average".

Theorem

We have

$$\sum_{d < X^{1/2}(\log X)^{-8}} \left| \sum_{\substack{0 < -D < X \\ d \mid D}} (\#\mathrm{Cl}(D)[3] - 1) - \frac{3}{\pi^2} \prod_{p \mid D} \frac{p}{p^2 - 1} \cdot X \right| = o(X).$$

Davenport-Heilbronn theorem, LOD version

Theorem

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$$\sum_{d < X^{1/2} (\log X)^{-8}} \left| \sum_{\substack{0 < -D < X \\ d \mid D}} (\#\mathrm{Cl}(D)[3] - 1) - \frac{3}{\pi^2} \prod_{p \mid D} \frac{p}{p^2 - 1} \cdot X \right| = o(X).$$

Got $d \ll X^{1/2 - \epsilon}$ in Bhargava-Taniguchi-T. To improve:

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Theorem

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$$\sum_{d < X^{1/2} (\log X)^{-8}} \left| \sum_{\substack{0 < -D < X \\ d \mid D}} (\#\mathrm{Cl}(D)[3] - 1) - \frac{3}{\pi^2} \prod_{p \mid D} \frac{p}{p^2 - 1} \cdot X \right| = o(X).$$

Got $d \ll X^{1/2 - \epsilon}$ in Bhargava-Taniguchi-T. To improve:

- Squeeze more out of existing proofs.

Davenport-Heilbronn theorem, LOD version

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- Squeeze more out of existing proofs.
- Use $\sum_{\pm D < X} \#\mathrm{Cl}(D)[3] \tau(D) \ll X \log X$ in the guts of the proof.

The Hooley Delta Symbol (1)

Lemma

For any $L > 0$, we have

$$\sum_{0 < \pm D \leq X} \#\mathrm{Cl}(D)[3] \cdot \#\{d : d \mid D, \sqrt{X}(\log X)^{-L} \leq d \leq \sqrt{X}(\log X)^{L}\} \ll LX(\log \log X)^{7/2}.$$

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Arises from the Hooley delta function

$$\Delta(n) = \sup_{u\geq 1} \#\{d \in \mathbb{N} \cap [u, \mathrm{e}u] : d \mid n\}.$$

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Theorem (..., de la Bretèche–Tenenbaum)

We have

$$\sum_{n \leq x} \Delta(n) \ll x(\log\log x)^{5/2}. \tag{1}$$

The Hooley Delta Symbol (2)

Our sum in the lemma is reduced to

$$\ll L(\log\log T) \sum_{0 < -D \leq T} (h_3(D) - 1)\Delta(|D|).$$

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Previous work of Chan, Koymans, Pagano, and Sofos treats sums of this type.

Thank you!

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