Answer Sheet for Team Round

1. 11
2. 3791
3. 81
4. 16
5. 8
6. 92
7. 1
8. 46
9. 12
10. 16

1. Define a sequence of integers by $S(1) = 1$, and

$$S(n) = S(1) + \cdots + S(n-1) + n$$

for each integer $n \geq 2$. What is the smallest integer n for which $S(n) > 2022$?

Solution. We see that $S(1) = 1$, $S(2) = 3$, $S(3) = 1+3+3 = 7$, $S(4) = 1+3+7+4 = 15$, etc., and guess that

$$S(n) = 2^n - 1.$$

It is possible to prove this one step at a time (i.e. by mathematical induction): we have that

$$\begin{aligned} S(n+1) &= (2^1 - 1) + (2^2 - 1) + \cdots + (2^n - 1) + (n+1) \\ &= (2^1 + 2^2 + \cdots + 2^n) + 1 \\ &= (2^{n+1} - 2) + 1. \end{aligned}$$

We have $S(11) = 2048 - 1 = 2047$, so the answer is 11.

2. In the game of *Fizz Buzz*, you begin counting $1, 2, 3, \ldots$. If a number is divisible by 3 you say 'Fizz' instead of the number, and if a number is divisible by 5 you say 'Buzz' instead of the number. If a number is divisible by both then you say 'Fizz Buzz'.

The tenth number counted is 17. What will be the 2022th number counted?

Solution. Of the first 15 numbers, you will say $1, 2, 4, 7, 8, 11, 13, 14$ – i.e. 8 of them. (This can be checked directly, or you can compute that $\phi(15) = 8$, where ϕ is the Euler phi function.) The pattern then repeats with every multiple of 15.

Since $2022 = 252 \times 8 + 6$, we get through $15 \times 252 = 3780$, and then count 6 more numbers, and so we arrive at 3791.

3. Consider an ordinary sheet of graph paper, with a grid defined by lines $x = m$ and $y = n$ for every integer m and n.

How many times does the line segment between $(-20, -22)$ and $(20, 22)$ intersect the grid?

Solution: We have:

- The two endpoints;
- We cross 39 vertical lines;
- We cross 43 horizontal lines;
- Three double crossings, at $(\pm 10, \pm 11)$ and $(0, 0)$,

so the answer is

$$2 + 39 + 43 - 3 = 81.$$

4. What is the nearest integer to

$$\log_2(2022) + \log_{20}(2022) + \log_{202}(2022) + \log_{2022}(2022) \ ?$$

(*Hint: Don't try to prove that your answer is correct.*)

Solution: We have

- $\log_2(2022) \approx 11$, because $2^{11} = 2048$ which is very close to 2022.
- $\log_{20}(2022)$ is a little bit bigger than 2.5, because $2022 \approx 20^2 \cdot 5$, and 5 is a little bigger than $\sqrt{20}$.
- $\log_{202}(2022)$ is a little bit smaller than 1.5, because $2022 \approx 202 \cdot 10$, and 10 is a little smaller than $\sqrt{202}$.
- $\log_{2022}(2022) = 1$.

So the answer is

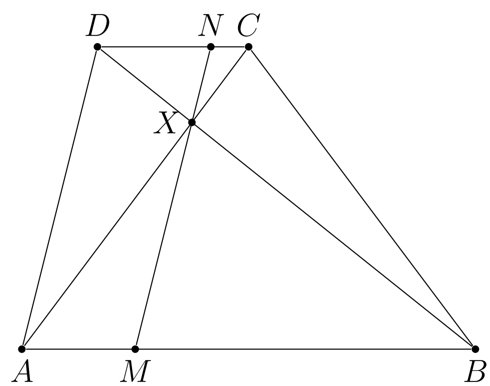
$$11 + 2.5 + 1.5 + 1 = 16.$$

In fact, to three decimals, the answer is

$$10.982 + 2.541 + 1.434 + 1 = 15.957.$$

5. In the trapezoid $ABCD$ with bases AB and DC, let X be the intersection of its diagonals, AC and BD. Let M be a point on the base AB and let the line MX intersect the base CD at N with $\dfrac{AM}{MB} = \dfrac{NX}{XM} = \dfrac{1}{3}$.

   Writing the ratio of the areas of the trapezoids $\dfrac{S_{AMND}}{S_{MBCN}}$ as a fraction $\frac{a}{b}$ in lowest terms, find $a+b$.



Solution (Savu): In a trapezoid, $S_{[AXD]} = S_{[BXC]} = S$. By similarity, $\dfrac{CX}{XA} = \dfrac{1}{3}$, so $\dfrac{S_{[CXD]}}{S} = \dfrac{1}{3}$, so $S_{[CXD]} = \dfrac{S}{3}$. Same argument, $S_{[DXN]} = \dfrac{S}{4}$ and $S_{[CXN]} = \dfrac{S}{12}$. Repeat to get first $S_{[AXB]} = 3S$ and then $S_{[AXM]} = \dfrac{3S}{4}$ and $S_{[BXM]} = \dfrac{9S}{4}$.Then,

$$\frac{S_{[AMND]}}{S_{[BMNC]}} = \frac{S/4 + S + 3S/4}{S/12 + S + 9S/4} = \frac{3}{5},$$

so the answer is 8.

Solution (Thorne): The ratio will be invariant under rescaling in the vertical direction, and shifting and/or rescaling AB and BC. We may therefore assume that we have the coordinates $A(0,0)$, $D(0,4)$, $M(1,0)$, $N(1,4)$, $B(4,0)$, $X(1,3)$. We get that C is $(\frac{4}{3}, 4)$ and finish the problem readily.

6. In a 3×3 grid of numbers, each row, column, and diagonal has the same sum. If four of the numbers are as shown below, what is the sum of the missing five numbers?

20	10	
22	18	

Solution. If x is the common sum, we complete the magic square to

20	10	$x-30$
22	18	$x-40$
$x-42$	$x-28$	$x-38$

Comparing the last row, the last column, and the 'off' diagonal, we get

$$2x - 54 = 3x - 108 = x$$

and hence $x = 54$. The answer is

$$5 \cdot 54 - (30 + 40 + 38 + 28 + 42) = 92.$$

7. You might know the identity

$$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

Similarly, there is a unique degree 6 polynomial f for which we can write $\cos(6\theta) = f(\cos(\theta))$. What is the sum of the coefficients of f?

Solution. One solution is to repeatedly use the angle addition formulas, and derive that

$$\cos(6\theta) = 32\cos(\theta)^6 - 48\cos(\theta)^4 + 18\cos(\theta)^2 - 1.$$

You probably didn't learn this identity in school! Add all the coefficients and you get 1.

If you don't want to do all that work, the question asserts that

$$\cos(6\theta) = a_6\cos(\theta)^6 + a_5\cos(\theta)^5 + a_4\cos(\theta)^4 + a_3\cos(\theta)^3 + a_2\cos(\theta)^2 + a_1\cos(\theta)^1 + a_0.$$

If we plug in $\theta = 0$, we see that

$$1 = \cos(0) = a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0,$$

as desired.

8. A circular table has eight places. In how many ways can you place one or more plates at a subset of the places, if you can't place two plates at adjacent places?

   Solution. You have to brute force this at least partially:

   - There are two arrangements of four plates.
   - The arrangements of three plates: 135, 136, 137, 146, 147, 157; all of these shifted by 1 to start with 2; then 357, 358, 368, 468. So 16 total.
   - The arrangements of 2 plates: there are $\binom{8}{2} = 28$ total ways to place 2 plates, of which 8 have them adjacent, so 20 admissible.
   - You can place one plate in 8 ways.

   $$2 + 16 + 20 + 8 = 46.$$

9. There exists a unique degree 4 polynomial f with integer coefficients and leading coefficient 1, which has $\sqrt{2} + \sqrt{3}$ as a root. What is the sum of the absolute values of the coefficients of f?

   Solution. It is possible to guess that the polynomial is

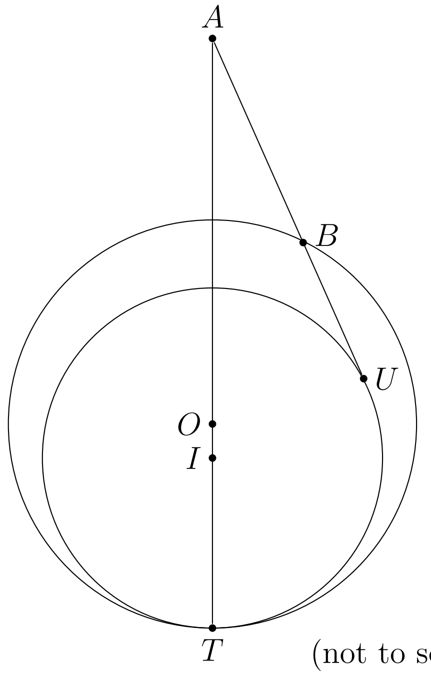
   $$(x - \sqrt{2} - \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} + \sqrt{3}),$$

   and then multiply out to check that the coefficients are indeed all integers. Without guessing, one can compute that

   $$\begin{aligned}(\sqrt{2} + \sqrt{3})^1 &= 0 + 1 \cdot \sqrt{2} + 1 \cdot \sqrt{3} + 0 \cdot \sqrt{6},\\ (\sqrt{2} + \sqrt{3})^2 &= 5 + 0 \cdot \sqrt{2} + 0 \cdot \sqrt{3} + 2 \cdot \sqrt{6},\\ (\sqrt{2} + \sqrt{3})^3 &= 0 + 11 \cdot \sqrt{2} + 9 \cdot \sqrt{3} + 0 \cdot \sqrt{6},\\ (\sqrt{2} + \sqrt{3})^4 &= 49 + 0 \cdot \sqrt{2} + 0 \cdot \sqrt{3} + 20 \cdot \sqrt{6}.\end{aligned}$$

   Our polynomial will be of the form $x^4 + ax^2 + b$, with $49 + 5a + b = 20 + 2a = 0$, and so $a = -10$ and $b = 1$. The polynomial is therefore $x^4 - 10x^2 + 1$, and the answer is 12.

10. The circle of center O and radius R and the circle of center I and radius r, $r < R$, are tangent internally at the point T. Let A be a point on the line TI, with I between A and T, such that $\dfrac{AI}{IT} = \dfrac{5}{3}$, and let the tangent from A to the circle of center I at U, intersect the circle of center O at B with B between A and U. If $AB = \dfrac{7}{5}$ and $BU = \dfrac{3}{5}$, find the inverse of the distance between the centers, $\frac{1}{OI}$.



Solution.

$\triangle AIU$ is a right triangle, $AI = \dfrac{5r}{3}$, $IU = r$, $AU = \dfrac{7}{5} + \dfrac{3}{5} = 2$, Pythagorean Theorem giives $\dfrac{25r^2}{9} - r^2 = 4$, wherefrom $r = \dfrac{3}{2}$. Extend the tangent line AU which intersects the common tangent of the two circles at X.

Construct the other tangent from A to the circle of center I extend to intersect the common tangent at Y. Then I is the incenter of the isosceles triangle $\triangle AXY$. From $\triangle AIU \sim \triangle AXT$, and $AT = \dfrac{8r}{3} = 4$, we find $AX = 5$ and $XT = 3$, so $XY = 6$.

Law of cosine, $\cos A = \dfrac{5^2 + 5^2 - 6^2}{2 \cdot 5 \cdot 5} = \dfrac{7}{25}$ and the projection of AY onto AX is the $AY \cdot \cos A = \dfrac{7}{5} = AB$ (!). So, B is the foot of the altitude from Y in $\triangle AXY$.

Then the circle of center O is centered on the Euler and is circumscribed to the pedal triangle of the $\triangle AXY$. Then this is the nine-point circle of the triangle with radius R equal half of the radius of the circumscribed circle of the triangle, $\dfrac{XY}{2\sin A} = \dfrac{6}{2 \cdot 24/25} = \dfrac{25}{8}$.

Then, $R = \dfrac{25}{8}$ and $OI = R - r = \dfrac{25}{16} - \dfrac{3}{2} = \dfrac{1}{16}$.

Note: There is a more direct approach that doesn't involve constructing the triangle $\triangle AXY$ and knowledge about the nine-point circle. We can extend the tangent until it meets again the larger circle at C, denote $UC = x$, consider the midpoint M of the chord BC and then $BM = \dfrac{2+x}{2}$ and use Pythagoren Theorem in AOM, similarity $\triangle AOM \sim \triangle AIU$ and $\triangle ABO \sim \triangle AOC$ to solve for x, OM, and R.