Finding Mathematics in Games

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Notes at: `thornef.github.io/last-lecture.pdf`

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Randy Pausch (1960-2008)

Bonkers

https://www.youtube.com/watch?v=mjGEV8s0Dc4

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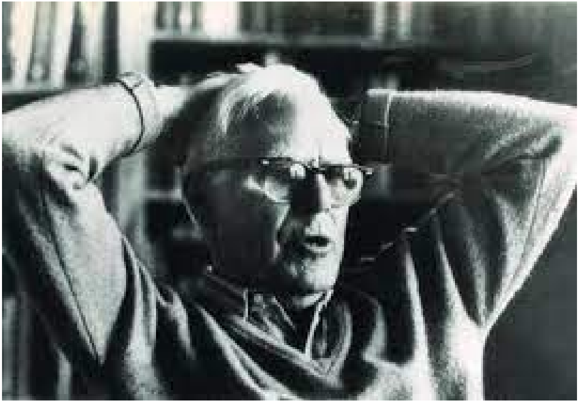
What we'll do today

- Meet my mathematical hero Martin Gardner.
- Explain the concepts of isomorphism and induction.
- Hunt a terrible beast on a dodecahedron.
- Rearrange a few disks in only 1,023 moves.
- Have fun!

Martin Gardner (1914-2010)



Scientific American, May 1957

https://sciam-cms.s3.amazonaws.com/sciam/cache/file/B9DD7F5C-1970-48A6-B9820FF22DE1BB43.pdf

Isomorphism

Definition

Two mathematical objects are **isomorphic** if they *have the same underlying structure*.

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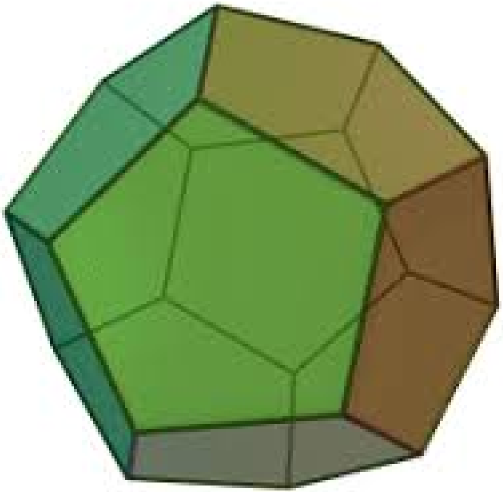
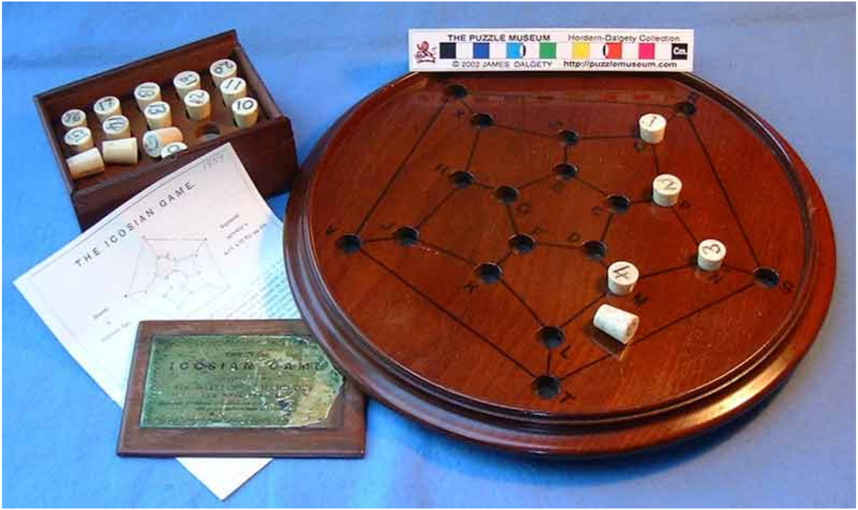
"iso" = same + "morph" = shape

A Dodecahedron



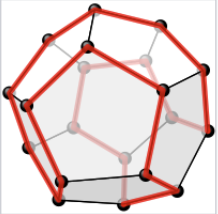
The Icosian Game



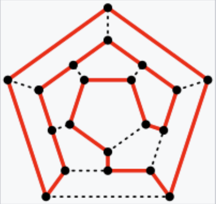
Hunt The Wumpus

https://archive.org/details/Hunt_the_Wumpus_1977_Creative_Computing

The Icosian Game – Solution



One possible Hamiltonian cycle through every vertex of a dodecahedron is shown in red – like all platonic solids, the dodecahedron is Hamiltonian



The above as a two-dimensional planar graph

The Tower of Hanoi

The Tower of Hanoi

- Move the entire stack to another peg, one disk at a time.

The Tower of Hanoi

- Move the entire stack to another peg, one disk at a time.
- You may never place a larger disk over a smaller one.

The Tower of Hanoi – Theorem

Theorem

With ten disks, you can solve the puzzle in exactly 1,023 moves.

Mathematical Induction : Part 1

"A journey of a thousand miles begins with a single step."

– Lao Tzu

The Tower of Hanoi – Theorem

Theorem

With one disk, you can solve the puzzle in exactly 1 move.

Mathematical Induction : Part 2

"One step at a time is all it takes to get you there."

– Emily Dickinson

The Tower of Hanoi – Two Disks

Theorem

With two disks, you can solve the puzzle in exactly 3 moves.

The Tower of Hanoi – Ten Disks

Theorem

With ten disks, you can solve the puzzle in exactly 1023 moves.

The Tower of Hanoi – Ten Disks

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Assuming: With nine disks, your friend knows how to solve the puzzle in 511 moves.

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Solution:

The Tower of Hanoi – Ten Disks

Theorem

With ten disks, you can solve the puzzle in exactly 1023 moves.

Assuming: With nine disks, your friend knows how to solve the puzzle in 511 moves.

Solution:

- Your friend moves the top nine disks in 511 moves.

The Tower of Hanoi – Ten Disks

Theorem

With ten disks, you can solve the puzzle in exactly 1023 moves.

Assuming: With nine disks, your friend knows how to solve the puzzle in 511 moves.

Solution:

- Your friend moves the top nine disks in 511 moves.
- You move the bottom disk (1 move).

The Tower of Hanoi – Ten Disks

Theorem

With ten disks, you can solve the puzzle in exactly 1023 moves.

Assuming: With nine disks, your friend knows how to solve the puzzle in 511 moves.

Solution:

- Your friend moves the top nine disks in 511 moves.
- You move the bottom disk (1 move).
- Your friend moves the top nine disks in 511 moves.

The Tower of Hanoi – Ten Disks

Theorem

With ten disks, you can solve the puzzle in exactly 1023 moves.

Assuming: With nine disks, your friend knows how to solve the puzzle in 511 moves.

Solution:

- Your friend moves the top nine disks in 511 moves.
- You move the bottom disk (1 move).
- Your friend moves the top nine disks in 511 moves.

$$511 + 1 + 511 = 1023.$$

The Tower of Hanoi – Nine Disks

Theorem

With nine disks, you can solve the puzzle in exactly 511 moves.

Assuming: With eight disks, your friend knows how to solve the puzzle in 255 moves.

Solution:

- Your friend moves the top eight disks in 255 moves.
- You move the bottom disk (1 move).
- Your friend moves the top eight disks in 255 moves.

$$255 + 1 + 255 = 511.$$

The Tower of Hanoi – Eight Disks

Theorem

With eight disks, you can solve the puzzle in exactly 255 moves.

Assuming: With seven disks, your friend knows how to solve the puzzle in 127 moves.

Solution:

- Your friend moves the top seven disks in 127 moves.
- You move the bottom disk (1 move).
- Your friend moves the top seven disks in 127 moves.

$$127 + 1 + 127 = 255.$$

The Tower of Hanoi – Seven Disks

Theorem

With seven disks, you can solve the puzzle in exactly 127 moves.

Assuming: With six disks, your friend knows how to solve the puzzle in 63 moves.

Solution:

- Your friend moves the top six disks in 63 moves.
- You move the bottom disk (1 move).
- Your friend moves the top six disks in 63 moves.

$$63 + 1 + 63 = 127.$$

The Tower of Hanoi – n Disks

Theorem

With n disks, you can solve the puzzle in exactly $2^n - 1$ moves.

Assuming: With $n-1$ disks, your friend knows how to solve the puzzle in $2^{n-1} - 1$ moves.

Solution:

- Your friend moves the top $n-1$ disks in $2^{n-1} - 1$ moves.
- You move the bottom disk (1 move).
- Your friend moves the top $n-1$ disks in $2^{n-1} - 1$ moves.

$$(2^{n-1} - 1) + 1 + (2^{n-1} - 1) = 2^n - 1.$$

The Solutions:

The Solutions:

- With one disk: A

The Solutions:

- With one disk: A
- With two disks: ABA

The Solutions:

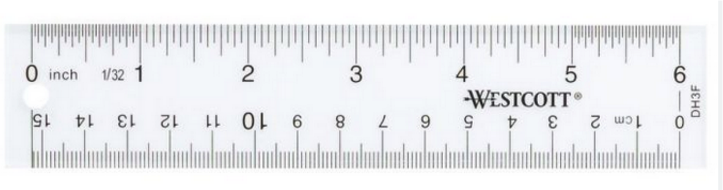
- With one disk: A
- With two disks: ABA
- With three disks: ABACABA

The Solutions:

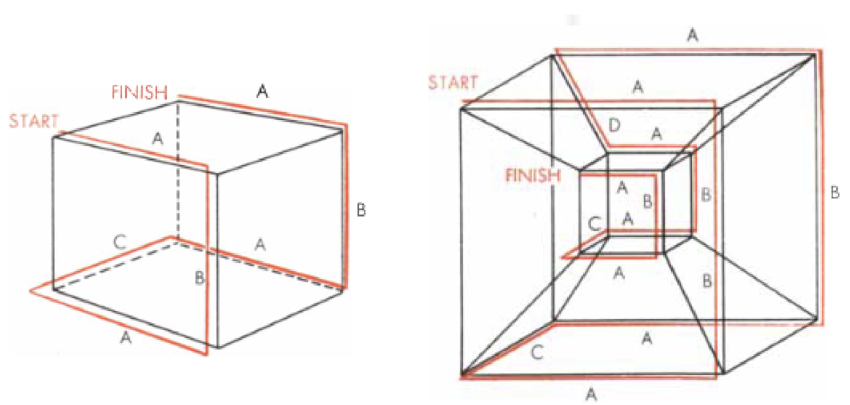
- With one disk: A
- With two disks: ABA
- With three disks: ABACABA
- With four disks: ABACABADABACABA

The Solutions:

- With one disk: A
- With two disks: ABA
- With three disks: ABACABA
- With four disks: ABACABADABACABA



Hamiltonian Path



HAMILTONIAN PATH is traced along the edges of a cube at left. The cube has the coordinates A, B and C; the path follows them in the order ABACABA. At right a Hamiltonian path is traced along the edges of a four-dimensional cube projected in three dimensions. This cube has the coordinates A, B, C and D; the path follows them ABACABADABACABA. This corresponds to the order of transferring four disks in the Tower of Hanoi.

The Morals of the Story

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- Mathematical structure is everywhere!

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- To get good at math, play lots of games!

The Morals of the Story

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- The same structures are everywhere.
- To get good at math, play lots of games!
- When in doubt, take things one step at a time.

Winning at Bonkers

Principle 1: Try Everything and Be Efficient

Can you try all 16 possibilities as quickly as possible?

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Can you try all 16 possibilities as quickly as possible?

Start somewhere, and change 15 times.

Principle 1: Try Everything and Be Efficient

Can you try all 16 possibilities as quickly as possible?

Start somewhere, and change 15 times.

Is there a solution moving only one disk at a time?

Principle 2: Mind The Buzzer

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The 6 is farther away than the 5.

Principle 2: Mind The Buzzer

The 6 is farther away than the 5.

Can we try everything with 6 on top first?

Principle 2: Mind The Buzzer

The 6 is farther away than the 5.

Can we try everything with 6 on top first?

XXXXXXX6XXXXXXX

Principle 2: Mind The Buzzer

Principle 2: Mind The Buzzer

The 4 is almost as far as the 6.

Principle 2: Mind The Buzzer

The 4 is almost as far as the 6.

XXX4XXX6XXX4XXX

Principle 2: Mind The Buzzer

Principle 2: Mind The Buzzer

One closer than the 4 is the 7.

Principle 2: Mind The Buzzer

One closer than the 4 is the 7.

X7X4X7X6X7X4X7X

Principle 2: Mind The Buzzer

Principle 2: Mind The Buzzer

The 5 is closest.

Principle 2: Mind The Buzzer

The 5 is closest.

575457565754575

Deja Vu All Over Again

A B C D

$$575457565754575 = ABACABADABACABA$$