State High School Mathematics Tournament

University of South Carolina

Round 2 – February 1, 2025

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- If your answer is wrong, the clock will be restarted. If your opponent doesn't buzz in, they may answer *immediately* after time is called.

Question 2-1

An angle θ between 0 and 90°, satisfies $\cos(\theta) + \sin(\theta) = \sqrt{2}$.

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What is the value of $\log_{\pi^2}(4096 \cdot \theta^6)$?

Solution 2-1

Answer. 3.

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We have $\theta = \pi/4$, so that

$$4096\theta^6 = (4\theta)^6 = \pi^6.$$

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$$4096\theta^6 = (4\theta)^6 = \pi^6.$$

So, $\log_{\pi^2} \pi^6 = 3$.

Question 2-2

What is the units digit of $1 + 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^{2025}$?

Solution 2-2

Answer. 4.

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Answer. 4.

Only showing the **last digits**, we have

$$\begin{aligned} & (1+3+3^2+3^3)+(3^4+3^5+3^6+3^7)+\cdots \\ \equiv\ & (1+3+9+7)+(1+3+9+7)+\cdots \\ \equiv\ & 0+0+\cdots \end{aligned}$$

Every group of four cancels out!

Solution 2-2

Answer. 4.

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Every group of four cancels out! Since 2024 is divisible by 4, we are left with only

$$3^{2024}+3^{2025}\equiv 1+3=4.$$

Question 2-3

Answer as an integer, in simplified form.

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How many triangles have integer side lengths, with the longest side having a length of 2025?

Solution 2-3

Answer. 1026169.

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Let a and b be the lengths of the middle and shortest sides.

$$2025 + 2023 + \cdots + 1 = 1013 \cdot 1013 = (1000 + 13)^2 = 1026169.$$

Solution 2-3

Answer. 1026169.

Let a and b be the lengths of the middle and shortest sides.

- If $a = 2025$, then $1 \le b \le 2025$ (2025 possibilities)

$$2025 + 2023 + \cdots + 1 = 1013 \cdot 1013 = (1000 + 13)^2 = 1026169.$$

Solution 2-3

Answer. 1026169.

Let a and b be the lengths of the middle and shortest sides.

- If $a = 2025$, then $1 \leq b \leq 2025$ (2025 possibilities)
- If $a = 2024$, then $2 \leq b \leq 2024$ (2023 possibilities)

$$2025 + 2023 + \cdots + 1 = 1013 \cdot 1013 = (1000 + 13)^2 = 1026169.$$

Solution 2-3

Answer. 1026169.

Let a and b be the lengths of the middle and shortest sides.

- If $a = 2025$, then $1 \leq b \leq 2025$ (2025 possibilities)
- If $a = 2024$, then $2 \leq b \leq 2024$ (2023 possibilities)
- ...
- If $a = 1013$, then $1013 \leq b \leq 1013$ (1 possibility)

$$2025 + 2023 + \cdots + 1 = 1013 \cdot 1013 = (1000 + 13)^2 = 1026169.$$

Question 2-4

What is the sum of all integers in the set $\{n : n^3 \leq 2025 \leq 3^n\}$?

Solution 2-4

Answer. 57.

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Since

$$12^3 = 1728 < 2025 < 13^3 = 2197,$$

$$3^6 = 729 < 2025 < 3^7 = 2187,$$

the sum is

$$7 + 8 + 9 + 10 + 11 + 12 = 57.$$

Question 2-5

There are unique integers a and b for which

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There are unique integers a and b for which

$$(2-\sqrt{3})^3 = a + b\sqrt{3}.$$

What is $a+b$?

Solution 2-5

Answer. 11.

Solution 2-5

Answer. 11. We have

$$(2-\sqrt{3})^3 = 8 - 12\sqrt{3} + 6(\sqrt{3})^2 - (\sqrt{3})^3 = 26 - 15\sqrt{3}.$$

Question 8

The equation $2^x = x^2$ has three real solutions. What is the nearest integer to their sum?

Solution 8

Answer. 5

$x = 2$, $x = 4$, and $x = -.76\ldots$

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$x = 2$, $x = 4$, and $x = -.76\ldots$
For the negative solution, note that $2^{-\frac{1}{2}} > (-\frac{1}{2})^2$, so $x < -\frac{1}{2}$.

Question 9

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2021 - 2022 + 2023 - 2024?$$

Solution 9

Answer. -1012.

Solution 9

Answer. -1012.

Write it as

$$(1-2)+(3-4)+\cdots+(2023-2024)=(-1)\times 1012.$$

Question 10

How many positive integers $n \leq 10$ satisfy $\cos(n) > 0$?

(Assume radian measure.)

Solution 10

Answer. 4.

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$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

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$$n \in (0, 1.57\ldots) \cup (4.71\ldots, 7.85\ldots)$$

Solution 10

Answer. 4.

$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

$$n \in (0, 1.57\ldots) \cup (4.71\ldots, 7.85\ldots)$$

$$n \in \{1, 5, 6, 7\}$$

Question 11

Simplify:

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$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}}$$

Solution 11

Answer. $\frac{17}{28}$.

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- $1 + \frac{1}{5} = \frac{6}{5}$

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- $1 + \frac{1}{1+\frac{1}{5}} = \frac{11}{6}$

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Answer. $\frac{17}{28}$.

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- $1 + \frac{1}{1+\frac{1}{5}} = \frac{11}{6}$
- $1 + \frac{1}{1+\frac{1}{1+\frac{1}{5}}} = \frac{17}{11}$

Solution 11

Answer. $\frac{17}{28}$.

- $1 + \frac{1}{5} = \frac{6}{5}$
- $1 + \frac{1}{1+\frac{1}{5}} = \frac{11}{6}$
- $1 + \frac{1}{1+\frac{1}{1+\frac{1}{5}}} = \frac{17}{11}$
- $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}} = \frac{28}{17}$

Solution 11

Answer. $\frac{17}{28}$.

- $1+\frac{1}{5}=\frac{6}{5}$
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- $1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}=\frac{17}{11}$
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Notice the pattern: $\frac{6}{5}, \frac{11}{6}, \frac{17}{11}, \frac{28}{17}$