The Mathematics of Game Shows

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1 Introduction

We will begin by watching a few game show clips and seeing a little bit of the math behind them.

1.1 Example: The Price Is Right, Contestants' Row

We begin with the following clip from The Price Is Right:

`https://www.youtube.com/watch?v=TmKP1a03E2g`

Game Description (Contestants' Row - The Price Is Right): *Four contestants are shown an item up for bid. In order, each guesses its price (in whole dollars). You can't use a guess that a previous contestant used. The winner is the contestant who bids the closest to the actual price without going over.*

In this clip, the contestants are shown some scuba equipment, and they bid 750, 875, 500, and 900 in that order. The actual price is \$994, and the fourth contestant wins. What can we say about the contestants' strategy?

- As a first step, it is useful to precisely describe the results of the bidding: the first contestant wins if the price is in $[750, 874]$[1]; the second, if the price is in $[875, 899]$; the third, in $[500, 749]$; the fourth, in $[900, \infty)$. If the price is less than \$500, then all the bids are cleared and the contestants start over.

  We can see who did well before we learn how much the scuba gear costs. Clearly, the fourth contestant did well. If the gear is worth anything more than \$900 (which is plausible), then she wins. The third contestant also did well: he is left with a large range of winning prices – 250 of them to be precise. The second contestant didn't fare well at all: although his bid was close to the actual price, he is left with a very small winning range. This is not his fault: it is a big disadvantage to go early.

[1] Recall that $[a, b]$ is mathematical notation for all the numbers between a and b.

- The next question to ask is: could any of the contestants have done better?

  We begin with the fourth contestant. Here the answer is *yes*, and her strategy is **dominated** by a bid of \$876, which would win in the price range $[876, \infty)$. In other words: *a bid of \$876 would win every time a bid of \$900 would, but not vice versa.* Therefore it is better to instead bid \$876 if she believes the scuba gear is more than \$900.

  Taking this analysis further, we see that there are exactly four bids that make sense: 876, 751, 501, or 1. Note that each of these bids, except for the one-dollar bid, screws over one of her competitors, and this is not an accident: Contestant's Row is a **zero-sum game** – if someone else wins, you lose. If you win, everyone else loses.

- The analysis gets much more subtle if we look at the *third* contestant's options. **Assume that the fourth contestant will play optimally.** (Of course this assumption is very often not true in practice.

  Suppose, for example, that the third contestant believes that the scuba gear costs around \$1000. The previous bids were \$750 and \$875. Should he follow the same reasoning and bid \$876? Maybe, but this exposes him to a devastating bid of \$877.

There is much more to say here, but we go on to a different example.

1.2 Deal or No Deal

Here is a clip of the game show **Deal or No Deal**:

`https://www.youtube.com/watch?v=I3BzYiCSTo8`

The action starts around 4:00.

Game Description (Deal or No Deal): *There are 26 briefcases, each of which contains a variable amount of money from \$0.01 to \$1,000,000, totalling \$3,418,416.01, and averaging \$131477.53. The highest prizes are \$500,000, \$750,000, and \$1,000,000.*

The contestant chooses one briefcase and keeps it. Then, one at a time, the contestant chooses other briefcases to open, and sees how much money is in each (and therefore establishes that these are not the prizes in his/her own briefcase). Periodically, the 'bank' offers to buy the contestant out, and give him/her a fixed amount of money to quit playing. The contestant either accepts or says 'no deal' and continues playing.

The **expected value** of the game is the average amount of money you expect to win. (We'll have much more to say about this.) So, at the beginning, the expected value of the game is \$131477.53, presuming the contestant rejects all the deals. In theory, that means that the contestant should be equally happy to play the game or to receive \$131477.53. (Of course, this might not be true in practice.)

Now let's look at the game after he chooses six briefcases. The twenty remaining contain a total of \$2936366, or an average of \$146818. The expected value has gone up, because the contestant eliminated mostly small prizes and none of the three biggest. If he wants to maximize his expected value (and I repeat that this won't necessarily be the case), then all he has to know is that

$$146818 > 51000$$

and so he keeps playing.

The show keeps going like this. After five more cases are eliminated, he again gets lucky and is left with fifteen cases containing a total of \$2808416, so an average of \$187227. The bank's offer is \$125,000 which he refuses. And it keeps going.

1.3 Jeopardy – Final Jeopardy

Here we see the Final Jeopardy round of the popular show Jeopardy:

`https://www.youtube.com/watch?v=DAsWPOuF4Fk`

Game Description (Jeopardy, Final Round): *Three contestants start with a variable amoung of money (which they earned in the previous two rounds). They are shown a category, and are asked how much they wish to wager on the final round. The contestants make their wagers privately and independently.*

After they make their wagers, the contestants are asked a trivia question. Anyone answering correctly gains the amount of their wager; anyone answering incorrectly loses it.

Perhaps here an English class would be more useful than a math class! This game is difficult to analyze; unlike our two previous examples, the players play *simultaneously* rather than *sequentially*.

In this clip, the contestants start off with \$9,400, \$23,000, and \$11,200 respectively. It transpires that nobody knew who said that *the funeral baked meats did coldly furnish forth the marriage tables*. (Richard II? Really? When in doubt, guess Hamlet.) The contestants big respectively \$1801, \$215, and \$7601.

We will save a thorough analysis for later, but we will make one note now: the second contestant can obviously win. If his bid is less than \$600, he will end up with more than \$22,400.

2 Probability

2.1 Sample Spaces and Events

At the foundation of any discussion of game show strategies is a discussion of *probability*. You have already seen this informally, and we will work with this notion somewhat more formally.

Definition 2.1 *1. A* **sample space** *is the set of all possible outcomes of a some process.*

2. *An* **event** *is any subset of the sample space.*

Example 2.2 *You roll a die. The sample space consists of all numbers between one and six. Using formal mathematical notation, we can write*

$$S = \{1, 2, 3, 4, 5, 6\}.$$

We can use the notation $\{\dots\}$ to describe a set and we simply list the elements in it.

Let E be the event that you roll an even number. Then we can write

$$E = \{2, 4, 6\}.$$

Alternatively, we can write

$$E = \{x \in S \ : \ x \text{ is even}\}.$$

Both of these are correct.

Example 2.3 *You choose at random a card from a poker deck. The sample space is the set of all 52 cards in the deck. We could write it*

$$\begin{aligned} S = \{ & A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, 9\clubsuit, 8\clubsuit, 7\clubsuit, 6\clubsuit, 5\clubsuit, 4\clubsuit, 3\clubsuit, 2\clubsuit, \\ & A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit, 9\diamondsuit, 8\diamondsuit, 7\diamondsuit, 6\diamondsuit, 5\diamondsuit, 4\diamondsuit, 3\diamondsuit, 2\diamondsuit, \\ & A\heartsuit, K\heartsuit, Q\heartsuit, J\heartsuit, 10\heartsuit, 9\heartsuit, 8\heartsuit, 7\heartsuit, 6\heartsuit, 5\heartsuit, 4\heartsuit, 3\heartsuit, 2\heartsuit, \\ & A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit, 9\spadesuit, 8\spadesuit, 7\spadesuit, 6\spadesuit, 5\spadesuit, 4\spadesuit, 3\spadesuit, 2\spadesuit\} \end{aligned}$$

but writing all of that out is annoying. An English description is probably better.

Example 2.4 *You choose two cards at random from a poker deck. Then the sample space is the set of all pairs of cards in the deck. For example, $A\spadesuit A\heartsuit$ and $7\clubsuit 2\diamondsuit$ are elements of this sample space,*

This is definitely too long to write out every element, so here an English description is probably better. (There are exactly 1,326 elements in this sample space.) Some events are easier to describe – for example, the event that you get a pair of aces can be written

$$E = \{A\spadesuit A\heartsuit, A\spadesuit A\diamondsuit, A\spadesuit A\clubsuit, A\heartsuit A\diamondsuit, A\heartsuit A\clubsuit, A\clubsuit A\diamondsuit\}$$

and has six elements. If you are playing Texas Hold'em, your odds of being dealt a pair of aces is exactly $\frac{6}{1326} = \frac{1}{221}$, or a little under half a percent.

Our next example is taken from the following TPIR clip:

`https://www.youtube.com/watch?v=TR7Smevj1AQ`

Game Description (Squeeze Play (The Price Is Right)): *You are shown a prize, and a five- or six-digit number. The price of the prize is this number with one of the digits removed, other than the first or the last.*

The contestant is asked to remove one digit. If the remaining number is the price, the contestant wins the prize.

In this clip the contestant is shown the number 114032. Can we describe the game in terms of a sample space?

It is imporrtant to recognize that **this question is not precisely defined. Your answer will depend on your interpretation of the question!** This is probably very much *not* what you are used to from a math class.

Here's one possible interpretation. Either the contestant wins or loses, so we can describe the sample space as

$$S = \{\text{you win}, \text{you lose}\}.$$

Logically there is nothing wrong with this. But it doesn't tell us very much about the structure of the game, does it?

Here is an answer I like better. We write

$$S = \{14032, 11032, 11432, 11402\},$$

where we've written 14032 as shorthand for 'the price of the prize is 14032'.

Another correct answer is

$$S = \{2, 3, 4, 5\},$$

where here 2 is shorthand for 'the price of the prize has the second digit removed.'

Still another correct answer is

$$S = \{1, 4, 0, 3\},$$

where here 1 is shorthand for 'the price of the prize has the 1 removed.'

All of these answers make sense, and all of them require an accompanying explanation to understand what they mean.

The contestant chooses to have the 0 removed. So the event that the contestant wins can be described as $E = \{11432\}$, $E = \{4\}$, or $E = \{0\}$, depending on which way you wrote the sample space. (Don't mix and match! Once you choose how to write your sample space, you need to describe your events in the same way.) If all the possibilities are equally likely, the contestant has a one in four chance of winning.

The contest guesses correctly and is on his way to Patagonia!

Notation 2.5 *If S is any set (for example a sample space or an event), write $N(S)$ for the number of elements in it. In this course we will always assume this number is finite.*

Probability Rule: All Outcomes are Equally Likely. Suppose S is a sample space in which all outcomes are equally likely, and E is an event in S. Then the **probability of E, denoted** $P(E)$, is

$$P(E) = \frac{N(E)}{N(S)}.$$

Example 2.6 *You roll a die, so $S = \{1, 2, 3, 4, 5, 6\}$.*

1. *Let E be the event that you roll a 4, i.e., $E = \{4\}$. Then $P(E) = \frac{1}{6}$.*
2. *Let E be the event that you roll an odd number, i.e., $E = \{1, 3, 5\}$. Then $P(E) = \frac{3}{6} = \frac{1}{2}$.*

Example 2.7 *You draw one card from a deck, with S as before.*

1. *Let E be the event that you draw a spade. Then $N(E) = 13$ and $P(E) = \frac{13}{52} = \frac{1}{4}$.*
2. *Let E be the event that you draw an ace. Then $N(E) = 4$ and $P(E) = \frac{4}{52} = \frac{1}{13}$.*
3. *Let E be the event that you draw an ace or a spade. What is $N(E)$? There are thirteen spades in the deck, and there are three aces which are not spades. Don't double count the ace of spades!*

   So $N(E) = 16$ and $P(E) = \frac{16}{52} = \frac{4}{13}$.

Example 2.8 *In a game of Texas Hold'em, you are dealt two cards at random in first position. You decide to raise with a pair of sixes or higher, ace-king, or ace-queen, and to fold otherwise.*

The sample space has 1326 elements in it. The event of two-card hands which you are willing to raise has 86 elements in it. (If you like, write them all out. Later we will discuss how this number can be computed more efficiently!)

Since all two card hands are equally likely, the probability that you raise is $\frac{86}{1326}$, or around one in fifteen.

Now, here is an important example: You roll two dice and sum the totals. What is the probability that you roll a 7?

The result can be anywhere from 2 to 12, so we have

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

and $E = \{7\}$. ☠☠☠: Therefore, $P(E) = \frac{N(E)}{N(S)} = \frac{1}{11}$.

Here is another solution. We can roll anything from 1 to 6 on the first die, and the same for the second die, so we have

$$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36,\\ 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}.$$

We list all the possibilities that add to 7:

$$E = \{16, 25, 34, 43, 52, 61\}$$

And so $P(E) = \frac{6}{36} = \frac{1}{6}$.

☠☠☠ We solved this problem two different ways and got two different answers. **The point is that not every outcome in a sample space will be equally likely**. We know that a die (if it is equally weighted) is equally likely to come up 1, 2, 3, 4, 5, or 6. So we can see that, according to our second interpretation, all the possibilities are still equally likely because all combinations are explicitly listed. But there is no reason why all the sums should be equally likely.

Note that it is often true that all outcomes are *approximately* equally likely, and we *model* this scenario by assuming that they are. *If our assumptions are close to the truth, so is our answer.*

For example, consider the trip to Patagonia. If we assume that all outcomes are equally likely, the contestant's guess has a 1 in 4 chance of winning. But the contestant correctly guessed that over \$14,000 was implausibly expensive, and around \$11,000 was more reasonable.

Another example comes from the TPIR game **Rat Race**:

`https://www.youtube.com/watch?v=Kp8rhV5PUMw`

Game Description (Rat Race (The Price Is Right)): *The game is played for three prizes: a small prize, a medium prize, and a car.*

There is a track with five wind-up rats (pink, yellow, blue, orange, and green). The contestant attempts to price three small items, and chooses one rat for each successful attempt. The rats then race. If he picked the third place rat, she wins the small prize; if she picked the second place rat, she wins the medium prize; if he picked the first place rat, she wins the car.

(Note that it is possible to win two or even all three prizes.)

Note that except for knowing the prices of the small items, there is no strategy. The rats are (we presume) equally likely to finish in any order.

In this example, the contestant correctly prices two of the items and picks the pink and orange rats.

Problem 1. *Compute the probability that she wins the car.*

Here's the painful solution: describe all possible orderings in which the rats could finish. We can describe the sample space as

$$S = \{POB, POR, POG, PBR, PBG, PRG, \dots, \dots\}$$

where the letters indicate the ordering of the first three rats to finish. Any such ordering is equally likely. The sample space has sixty elements, and twenty-four of them start with P or G. So the probability is $\frac{24}{60} = \frac{2}{5}$.

Do you see the easier solution? To answer the problem we were asked, we only care about the **first** rat. So let's ignore the second and third finishers, and write the sample space as

$$S = \{P, O, B, R, G\}.$$

The event that she wins is

$$E = \{P, G\},$$

and so $P(E) = \frac{N(E)}{N(S)} = \frac{2}{5}$.

Here's a possible solution that was suggested in class. It doesn't work, and it's very instructive to think about why it doesn't work. As the sample space, take all combinations of one rat and which order it finishes in:

$$\begin{aligned} S = \{&\text{Pink rat finishes first,} \\ &\text{Pink rat finishes second,} \\ &\text{Pink rat finishes third,} \\ &\text{Pink rat finishes fourth,} \\ &\text{Pink rat finishes fifth,} \\ &\text{Yellow rat finishes first,} \\ &\text{etc.}\} \end{aligned}$$

This sample space indeed lists a lot of different things that could happen. But how would you describe the event that the contestant wins? If the pink or orange rat finishes first, certainly she wins. But what if the yellow rat finishes third? Then maybe she wins, maybe she loses. There are several problems with this sample space:

- The events are not mutually exclusive. It can happen that **both** the pink rat finishes second, **and** the yellow rat finishes first. A sample space should be described so that **exactly one of the outcomes will occur**.

  Of course, a meteor could strike the television studio, and Drew, the contestant, the audience, and all five rats could explode in a giant fireball. But we're building *mathematical models* here, and so we can afford to ignore remote possibilities like this.

- In addition, you can't describe the event 'the contestant wins' as a subset of the sample space. What if the pink rat finishes fifth? The contestant also has the orange rat. It is ambigious whether this possibility should be part of the event or not.

Altogether, (from the learner's perspective) a very good wrong answer! Once you are very experienced, you will be able to skip straight to the correct answer. When you are just learning the material, your first idea will often be incorrect. Your willingness to critically examine your ideas, and to revise or reject them when needed, will lead you to the truth.

Problem 2. *Compute the probability that she wins both the car and the meal delivery.*

Here we care about the first *two* rats. We write

$$S = \{PO, PB, PR, PG, OP, OB, OR, OG, BP, BO, BR, BG, RP, RO, RB, RG, GP, GO, GB, GR\}.$$

The sample space has twenty elements in it. ($20 = 5 \times 4$: there are 5 possibilities for the first place finisher, and (once we know who wins) 4 for the second. More on this later.) The event that she wins is

$$\{PO, OP\}$$

and $P(E) = \frac{N(E)}{N(S)} = \frac{2}{20} = \frac{1}{10}$.

Problem 3. *Compute the probability that she wins all three prizes.*

Zero. Duh. She only won two rats! Sorry.

2.2 The Addition and Multiplication Rules

The Addition Rule (1). Suppose E and F are two *disjoint* events – i.e., they don't overlap. Then

$$P(E \text{ or } F) = P(E) + P(F).$$

Example 2.9 *You roll a die. Compute the probability that you roll either a 1, or a four or higher.*

Let $E = \{1\}$ be the event that you roll a 1, and $E = \{4, 5, 6\}$ be the event that you roll a 4 or higher. Then

$$P(E \text{ or } F) = P(E) + P(F) = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}.$$

Example 2.10 *You draw a poker card at random. What is the probability you draw either a heart, or a black card which is a ten or higher?*

Let E be the event that you draw a heart. As before, $P(E) = \frac{13}{52}$.
Let F be the event that you draw a black card ten or higher, i.e.,

$$F = \{A\clubsuit, K\clubsuit, Q\clubsuit, J\clubsuit, 10\clubsuit, A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit\}.$$

Then $P(F) = \frac{10}{52}$.
So we have

$$P(E \text{ or } F) = \frac{13}{52} + \frac{10}{52} = \frac{23}{52}.$$

Example 2.11 *You draw a poker card at random. What is the probability you draw either a heart, or a red card which is a ten or higher?*

This doesn't have the same answer, because hearts are red. If we want to apply the addition rule, we have to do so carefully.
Let E be again the event that you draw a heart, with $P(E) = \frac{13}{52}$.
Now let F be the event that you draw a diamond which is ten or higher:

$$F = \{A\diamondsuit, K\diamondsuit, Q\diamondsuit, J\diamondsuit, 10\diamondsuit\}.$$

Now together E and F cover all the hearts and all the red cards at least ten, and there is no overlap. So we can use the addition rule.

$$P(E \text{ or } F) = P(E) + P(F) = \frac{13}{52} + \frac{5}{52} = \frac{18}{52}.$$

We can also use the addition rule with more than two events, as long as they don't overlap.

Example 2.12 *Consider the Rat Race contestant from earlier. What is the probability that she wins any two of the prizes?*

Solution 1. *We will give a solution using the addition rule. (Later, we will give another solution using the Multiplication Rule.)*

Recall that her chances of winning the car and the meal delivery were $\frac{1}{10}$. Let us call this event CM instead of E.

Now what are her chances of winning the car and the guitar? (Call this event CG.) Again $\frac{1}{10}$. If you like, you can work this question out in the same way. But it is best to observe that there is a natural symmetry in the problem. The rats are all alike and any ordering is equally likely. They don't know which prizes are in which lanes. So the probability has to be the same.

Finally, what is $P(MG)$, the probability that she wins the meal service and the guitar? Again $\frac{1}{10}$ for the same reason.

Finally, observe these events are all disjoint, because she can't possibly win more than two. So the probability is three times $\frac{1}{10}$, or $\frac{3}{10}$.

Here is a contrasting situation. Suppose the contestant had picked all three small prices correctly, and got to choose three of the rats. In this case, the probability she wins both the car and the meal service is $\frac{3}{10}$, rather than $\frac{1}{10}$. (You can either work out the details yourself, or else take my word for it.)

But this time the probability that she wins two prizes is *not* $\frac{3}{10} + \frac{3}{10} + \frac{3}{10}$, because now the events CM, CG, and MG are not disjoint: it is possible for her to win all three prizes, and if she does, then all of CM, CG, and MG occur!

It turns out that in this case the probability that she wins *at least* two is $\frac{7}{10}$, and the probability that she wins *exactly* two is $\frac{3}{5}$.

The Multiplication Rule. The multiplication rule computes the probability that both E **and** F occur. The formula is the following:

$$P(E \text{ and } F) = P(E) \times P(F).$$

It is not always valid, but it is valid in either of the following circumstances:

- The events E and F are *independent*.
- The probability given for F assumes that the event E occurs (or vice versa).

Example 2.13 *You flip a coin twice. What is the probability that you flip heads both times?*

We can use the multiplication rule for this. The probability that you flip heads if you flip a coin once is $\frac{1}{2}$. Since coin flips are independent (flipping heads the first time doesn't make it more or less likely that you will flip heads the second time) we multiply the probabilities to get $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Alternatively, we can give a direct solution. Let

$$S = \{HH, HT, TH, TT\}$$

and

$$E = \{HH\}.$$

Since all outcomes are equally likely,

$$P(E) = \frac{N(E)}{N(S)} = \frac{1}{4}.$$

We can also use the multiplication rule for more than two events.

Example 2.14 *You flip a coin twenty times. What is the probability that you flip heads every time?*

If we use the multiplication rule, we see at once that the probability is

$$\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \frac{1}{2^{20}} = \frac{1}{1048576}.$$

This example will illustrate the second use of the Multiplication Rule.

Example 2.15 *Consider the Rat Race example again (as it happened in the video). What is the probability that the contestant wins both the car and the meal service?*

Solution. The probability that she wins the car is $\frac{2}{5}$, as it was before. So we need to now compute the probability that she wins the meal service, *given that she won the car*.

This time the sample space consists of *four* rats: we leave out whichever one won the car. The event is that her remaining one rat wins the meal service, and so the probability of this event is $\frac{1}{4}$.

By the multiplication rule, the total probability is

$$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}.$$

Example 2.16 *Suppose a Rat Race contestant prices all three prizes correctly and has the opportunity to race three rats. What is the probability she wins all three prizes?*

Solution. The probability she wins the car is $\frac{3}{5}$, as before: the sample space consists of the five rats, and the event that she wins consists of the three rats she chooses. (Her probability is $\frac{3}{5}$ no matter which rats she chooses, under our assumption that they finish in a random order.)

Now assume that she wins the first prize. Assuming this, the probability that she wins the meals is $\frac{2}{4} = \frac{1}{2}$ The sample space consists of the four rats *other than the first place* *finisher,* and the event that she wins the meals consists of the two rats *other than the first place finishers.*

Now assume that she wins the first and second prizes. The probability she wins the guitar is $\frac{1}{3}$: the sample space consists of the three rats *other than the first two finishers,* and the event that she wins the meals consists of the single rat *other than the first two finishers.*

There is some subtlety going on here! To illustrate this, consider the following:

Example 2.17 *Suppose a Rat Race contestant prices all three prizes correctly and has the opportunity to race three rats. What is the probability she wins the meal service?*

Solution. There are five rats in the sample space, she chooses three of them, and each of them is equally likely to finish second. So her probability is $\frac{3}{5}$ (same as her probability of winning the car).

But didn't we just compute that her odds of winning the car are $\frac{1}{2}$? What we're seeing is something we'll investigate much more later. This probability $\frac{1}{2}$ is a **conditional** probability: it assumes that one of the rats finished first, and illustrates what is hopefully intuitive: if she wins first place with one of her three rats, she is less likely to also win second place.

In particular, this reasoning illustrates the following misapplication of the multiplication rule. Suppose we compute again the probability that she wins all three prizes with three rats. She has a $\frac{3}{5}$ probability of winning first, a $\frac{3}{5}$ probability of winning second, and a $\frac{3}{5}$ probability of winning third. By the multiplication rule, the probability that all of these events occur is

$$\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}.$$

What is wrong with this reasoning is that these events are *not independent.*

Michael Larson. Here is a bit of game show history. The following clip comes from the game show Press Your Luck on May 19, 1984.

`https://www.youtube.com/watch?v=UzggoA4lLwk`

Here Michael Larsen smashed the all-time record by winning $110,237. The truly fascinating clip starts at around 17:00, where Larson continues to press his luck, to the host's increasing disbelief. On 28 consecutive spins, Larson avoided all the whammies and each time hit a space that afforded him an extra spin. There are eighteen squares on the board, and on average there are approximately five spaces worth money and an extra spin.

Example 2.18 *Assume for simplicity that each time there are exactly five spaces (out of eighteen) that Larson wants to hit, and that the outcome is random and that each square is equally likely to occur.*

If Larson spins twenty-eight times, compute the probability that he hits a good spot every time.

Solution. *This is a straightforward application of the multiplcation rule. The answer is* $\left(\frac{5}{18}\right)^{28}$*, or approximately one in*

$$3,771,117,128,139,603.$$

Either Larson got very, *very*, **very**, **VERY** lucky....... or else the pattern is not random and he figured it out.

Card Sharks. Here is another game show from the eighties that leads to interesting probability computations.

In brief, the rules of the show are as follows. The contestants each get a line-up of five cards and their objective is to make it to the end. They take turns (based on their answers to some trivia questions).

When a contestant gets to go, their marker is placed on the first card. The first card is revealed and the contestant may replace it with another random card if so desired. Then h/she must guess higher or lower. In any case the next card is revealed and if correct the contestant may either 'freeze' or keep going.

Wherever the contestant freezes, his/her marker is placed on that card and h/she starts from there on their next turn.

Here is a typical clip:

`https://www.youtube.com/watch?v=bUv0CRU6t5o`

Here is our objective: *Assuming that the trivia questions are a 50-50 tossup, determine the optimal strategy in all situations.* This problem is somewhat difficult (and our mental heuristics for it are fairly spot on). But at least in principle, it is possible to give a complete solution to this problem.

We won't try to achieve this all at once. Instead, we'll ask a number of probability questions to get started:

Example 2.19 *Consider Cynthia's first turn, where she guesses 'lower'. Compute the probability that she is correct.*

Answer. *The sample space consists of the* 51 *cards other than the king of clubs. Of these, only seven are not lower: the four aces, and the three remaining kings. So* $51-7=44$ *cards are lower, and her chances are* $\frac{44}{51}$.

We also compute the probabilities at the next two rounds. She guesses the third card will be higher than a 2. There are 50 cards remaining, and 47 of them are higher than a 2, so her odds are $\frac{47}{50}$.

The next card was a 9. Of the 49 remaining cards, 27 are lower than a 9 and 19 are higher. (And the three remaining nines are neither higher nor lower – so she would lose no matter what she picked). If she chose to play, her odds of winning the next card would be $\frac{27}{49}$, or slightly better than 50-50. She quite reasonably chooses to freeze and lock in her position.

Now we skip ahead to Royce's second round (when both Royce and Cynthia have frozen on the third of five cards).

Here are several questions we can ask:

- Given that Royce has replaced his nine with a three, compute the probability that he can win the round (assuming he doesn't freeze).
- Before Royce sees the three, compute the probability that he can win the round.
- Given that Royce's card is a five, compute the probability that he wins if he doesn't choose to freeze.
- If Royce chooses to freeze, answers the next trivia question correctly, and gets to go again, compute the probability that he wins on his next attempt.

  (Note that this is not the total probability he wins: he could lose on his next attempt, but then answer another trivia question correctly and get yet another try.)
- If Royce chooses to freeze and Cynthia answers the next trivia question correctly, what is the probability that she wins the next round (if she doesn't freeze)?

These questions get us closer to the question we're *really* interested in: should Royce freeze on the five or not? As is often the case, the question we are interested in is quite difficult and we build up to being able to answer it.

We tackle the first question.

Example 2.20 *Given that Royce has replaced his nine with a three, compute the probability that he can win the round. Assume that he doesn't choose to freeze, and that his higher/lower guess is always optimal.*

Note that there are 48 cars left in the deck: a three, a four, a six, and a nine are all missing.

It is easy to compute the probability that Royce's *first* guess is correct: out of 48 remaining cards, 41 are higher, so the probability is $\frac{41}{48}$. Now, **assuming that Royce's first guess is correct**, what is the probability that his second guess is correct?

Well ...... we don't know. It depends on what the first card **is**. Later, we will see some clever tricks for carrying out this sort of computation more easily. But for now, we outline a 'brute force' computation:

- Royce's first guess will be correct if the first card is a four, five, six, seven, eight, nine, ten, jack, queen, king, or ace.
- Based on Royce's first guess, we can determine what Royce should guess for the second card and the probability that this guess will be correct.

  Let's do an example of this. Suppose the first card is a four; the probability of this occurring is $\frac{3}{48}$. (This reduces to $\frac{1}{16}$, but the pattern will be clearer if we do not reduce our fractions to lowest terms.)

Then Royce should clearly guess that the second will be higher. There are 47 remaining cards, of which 38 are higher than a four. So *assuming that the first card is a four*, the probability that Royce wins is $\frac{38}{47}$. Therefore, the probability that *the first card is a four* **and** *Royce wins* is $\frac{3}{48} \times \frac{38}{47}$.

- We will therefore use **both** the addition and the multiplication rules by **dividing into cases**: For each possible first card n (that doesn't lose Royce the round immediately), we compute the probability that the first card **is** n **and that** Royce wins the round. This is the multiplication rule.

  Since all of these possibilities are mutually exclusive, but one of them has to occur if Royce is to win, we see that the probability that Royce wins is the total of the probabilities we computed in the first step. This is the addition rule!

Let's roll up our sleeves and do it. The proof won't be pretty, but it is not as scary as it looks.

- With probability $\frac{3}{48}$ the first card will be a four. Then Royce should guess higher, and with probability $\frac{38}{47}$ the next card will be higher.
- With probability $\frac{4}{48}$ the first card will be a five. Then Royce should guess higher, and with probability $\frac{34}{47}$ the next card will be higher.
- With probability $\frac{3}{48}$ the first card will be a six. Then Royce should guess higher, and with probability $\frac{31}{47}$ the next card will be higher.
- With probability $\frac{4}{48}$ the first card will be a seven. Then Royce should guess higher, and with probability $\frac{27}{47}$ the next card will be higher.
- With probability $\frac{4}{48}$ the first card will be an eight. Then Royce should guess higher, and with probability $\frac{23}{47}$ the next card will be higher.
- With probability $\frac{3}{48}$ the first card will be a nine. Then Royce should guess lower, and with probability $\frac{24}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a ten. Then Royce should guess lower, and with probability $\frac{27}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a jack. Then Royce should guess lower, and with probability $\frac{31}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a queen. Then Royce should guess lower, and with probability $\frac{35}{47}$ the next card will be lower.
- With probability $\frac{4}{48}$ the first card will be a king. Then Royce should guess lower, and with probability $\frac{39}{47}$ the next card will be lower.

- With probability $\frac{4}{48}$ the first card will be an ace. Then Royce should guess lower, and with probability $\frac{43}{47}$ the next card will be lower.

(Note that all of the cases look more or less the same. Often, this is an indication that you can look for shortcuts – but we won't do so here.)

The total probability that Royce wins is therefore

$$\frac{3}{48}\cdot\frac{38}{47}+\frac{4}{48}\cdot\frac{34}{47}+\frac{3}{48}\cdot\frac{31}{47}+\frac{4}{48}\cdot\frac{27}{47}+\frac{4}{48}\cdot\frac{23}{47}+\frac{3}{48}\cdot\frac{24}{47}+\frac{4}{48}\cdot\frac{27}{47}+\frac{4}{48}\cdot\frac{31}{47}+\frac{4}{48}\cdot\frac{35}{47}+\frac{4}{48}\cdot\frac{39}{47}+\frac{4}{48}\cdot\frac{43}{47}.$$

This is equal to $\frac{1315}{2256}$, which is already in lowest terms. Yeah, I know. You were hoping it would be nice and simple, and that in retrospect you could have solved the problem in your head. You couldn't have. Neither could I. Sometimes math is like that.

This is roughly 58.2%, which is not bad at all.

2.3 Factorials, permutations, and combinations

This video illustrates a playing of the Price Is Right game **Ten Chances**:

`https://www.youtube.com/watch?v=iY_gmGcDKXE`

Game Description (Ten Chances (The Price Is Right)): *The contestant is shown a small prize, a medium prize, and a large prize. She has ten chances to win as many prizes as she can.*

The price of small prize has two numbers in it, and the contestant is shown three different numbers. She then guesses the price of the first prize. She takes as many chances as she needs to.

Once she wins the small prize, she attempts to win the medium prize. The price of the medium prize has three numbers in it, and the contestant is shown four.

Finally, if she wins the medium prize, she attempts to win the car. Its price has five numbers in it, and the contestant is shown these five.

Example 2.21 *The price of the pasta maker contains two digits from* $\{0, 6, 9\}$. *Suppose that each possibility is equally likely to be the price of the pasta maker.*

If the contestant has one chance, what are her odds of winning?

Solution 1. We can give a straightforward solution by simply enumerating the sample space of all possibilities. It is

$$\{06, 09, 60, 69, 90, 96\}.$$

The contestant's choice describes an event with one of these possibilities in it. Since we hypothesized that each was equally likely to occur, her odds of winning are $\frac{1}{6}$.

Solution 2. We use the multiplication rule. There are three different possibilities for the first digit, and exactly one of them is correct. The probability that she gets the first digit correct is therefore $\frac{1}{3}$.

Now, **assume she got the first digit correct.** (If she didn't, she might have used up the correct second digit already, and be doomed to botch that one also!) Then there are two remaining digits, and the probability that she picks the correct one is $\frac{1}{2}$.

Thus the probability of getting both correct is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

Notice, incidentally, that our assumption that the possibilities are equally likely is not realistic. Surely the pasta maker's price is not 06 dollars? Especially since you'd write it 6 and not 06? (Indeed, if you have watched the show a lot, you know that when there is a zero the price always ends with it. Knowing this fact is a *big* advantage.)

Now, she is going to use up at most six of her chances on the pasta maker, so she gets to move on to the mower. Here the price contains three digits from $\{0, 6, 8, 9\}$. This problem can be solved in the same way. The relevant sample space is

$\{068, 069, 086, 089, 096, 098, 608, 609, 680, 689, 690, 698, 806, 809, 860, 869, 890, 896, 906, 908, 960, 968, 980, 986$

which has 24 elements in it, so her probability of winning is $\frac{1}{24}$. The analogue of solution 2 gives $\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$.

Finally, the price of the car has the digits $\{0, 1, 5, 6, 8\}$ and this time she uses all of them. The sample space is too long to effectively write out. So we work out the analogue of Solution 2: Her odds of guessing the first digit are $\frac{1}{5}$. If she does so, her odds of guessing the second digit is $\frac{1}{4}$ (since she has used one up). If both these digits are correct, her odds of guessing the third digit is $\frac{1}{3}$. If these three are correct, her odds of guessing the fourth digit are $\frac{1}{2}$. Finally, **if** the first four guesses are correct then the last digit is automatically correct by process of elimination. So the probability she wins is

$$\frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{120}.$$

Here the number 120 is equal to 5!, or 5 **factorial**. In math, an exclamation point is read 'factorial' and it means the product of all the numbers up to that point. We have

$$\begin{aligned}
1! &= 1 && = 1 \\
2! &= 1 \times 2 && = 2 \\
3! &= 1 \times 2 \times 3 && = 6 \\
4! &= 1 \times 2 \times 3 \times 4 && = 24 \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 && = 120 \\
6! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 && = 720 \\
7! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 && = 5040 \\
8! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 && = 40320 \\
9! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 && = 362880 \\
10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 && = 3628800,
\end{aligned}$$

and so on. We also write $0! = 1$. Why 1 and not zero? 0! means 'don't multiply anything', and we think of 1 as the starting point for multiplication. (It is the *multiplicative identity*, satisfying $1 \times x = x$ for all x.) So when we compute $0!$ it means we didn't leave the starting point.

These numbers occur **very** commonly in the sorts of questions we have been considering, for reasons we will shortly see.

Example 2.22 *The lucky contestant wins the first two prizes in only three chances, and has seven chances left over. If each possibility for the price of the car is equally likely, then what is the probability that she wins it?*

The answer is seven divided by $N(S)$, the number of elements in the sample space. So if we could just compute $N(S)$, we'd be done.

Here there is a trick! She guesses 16580, and we know that the probability that this is correct is $\frac{1}{N(S)}$: one divided by the number of total possible guesses. But we already computed the probability: it's $\frac{1}{120}$. Therefore, we know that $N(S)$ is 120, without actually writing it all out!

The mathematical discipline of **combinatorics** is the art of *counting without counting.* We just solved our first combinatorics problem: we figured out that there were 120 ways to rearrange the numbers $0, 1, 5, 6, 8$ without actually listing them. We now formalize this principle.

Definition 2.23 *Let T be a* **string**. *For example,* 01568 *and* 22045 *are strings of numbers, ABC and xyz are strings of letters, and $\otimes - \oplus\clubsuit$☠ is a string of symbols. Order matters:* 01568 *is not the same string as* 05186*.*

A **permutation** *of T is any reordering of T.*

So, for example, if T is the string 1224, then 2124, 4122, 1224, and 2142 are all permutations of T. Note we *do* consider T itself to be a permutation of T, for the same reason that we consider 0 a number. It is called the **trivial permutation**.

We have the following:

Proposition 2.24 *Let T be a string with n distinct symbols. Then there are exactly $n!$ distinct permutations of T.*

In math, a *proposition* (or a *theorem*) is a statement of something true. We have stated lots of true facts in these notes; here the title 'Proposition' indicates that this one is particularly important and worth your attention.

Please read the statement carefully. In particular, the conclusion is only guaranteed to hold when the hypotheses also hold. If the hypotheses don't hold, then the conclusion may or may not be true. For example, if T is the string 122, then the set of all permutations of it is

$$\{122, 212, 221\}$$

which has 3 elements, and $3 \neq 3! = 6$.

Note also that this solves our earlier Ten Chances question. The contestant's guesses are all permutations of the string 01568, of which there are $5! = 120$. The sample space S

consists of all 120 permutations. The contestant can make seven guesses, so let E be the set of these 7 permutations. Since we have assumed that each possible guess is equally likely to be correct, her odds (probability) of winning are $\frac{7}{120}$.

We will now offer a **proof** of the proposition. Please don't be too scared by the word 'proof': it just means a convincing explanation of why it is true. This course will not focus on *writing* proofs, but it is good to gain practice reading them.

Proof: Suppose T is a string with n distinct symbols, and we want to construct a permutation of T. symbol We first choose the first symbol. Since T has n distinct symbols, we have n choices for the first symbol.

No matter what we choose for the first symbol, there are $n - 1$ choices for the second symbol (all but the one we picked already), so that there are $n \times (n - 1)$ choices for the first two.

Similarly, there are $n - 2$ choices for the third symbol, and so on. This continues until the last (the nth) symbol, for which there is exactly one choice. $\square$

In math we often end proofs with a little square. If you like, you can end proofs with the phrase **QED**, which is an abbreviation for 'quod erat demonstrandum' – Latin for 'that which was to be shown'. In practice, saying or writing 'QED' serves the same purpose as a football player spiking the ball after he has scored a touchdown.

If you are especially observant, you will notice that the proof is very similar to our explanation of the multiplication rule for probability. There is a good reason for this: the same principle underlies both, and counting and probability are two sides of the same coin.

We now return to our Ten Chances contestant. Recall that she has seven chances to win the car.

Example 2.25 *Suppose that the contestant has watched The Price Is Right a lot and so knows that the last digit is the zero. Compute the probability that she wins the car, given seven chances.*

Solution. Here her possible guesses consist of permutations of the string 1568, followed by a zero. There are $4! = 24$ of them, so her winning probability is $\frac{7}{24}$.

Her winning probability went up by a factor of exactly 5 – corresponding to the fact that $\frac{1}{5}$ of the permutations of 01568 have the zero in the last digit. Equivalently, a random permutation of 01568 has probability $\frac{1}{5}$ of having the zero ias the last digit.

Now, a smart contestant can do better. Suppose, for example, that she guessed 85610. Mathematically it looks like a good guess .... but she is playing for a Chevy Cavalier. I mean, really. We can rule out the 8 as the first digit, as well as the 6 and the 5.

Example 2.26 *Suppose that the contestant knows that the the last digit is the zero and the first digit is the one. Compute the probability that she wins the car, given seven chances.*

Solution. Her guesses now consist of permutations of the string 568, with a 1 in front and followed by a zero. There are $3! = 6$ of them. Assuming that the assumptions are correct and that she doesn't screw up, she is a sure bet to win the car.

Mathematically, her probability of winning is 1 (which is the same as 100%). Please *don't* answer that her probability is $\frac{7}{6}$. This doesn't make much sense!

Note that it is only true of Ten Chances that car prices always end in zero – not of The Price Is Right in general. Here is a contestant who is very excited until she realizes the odds she is against:

`https://www.youtube.com/watch?v=AAIU6knD7BA`

2.4 Exercises

Most of these should be relatively straightforward, but there are a couple of quite difficult exercises mixed in here for good measure.

1. Card questions. In each question, you choose at random a card from an ordinary deck. What is the probability you –
   (a) Draw a spade?
   (b) Draw an ace?
   (c) Draw a face card? (a jack, queen, king, or an ace)
   (d) Draw a spade or a card below five?

2. Dice questions:
   (a) You roll two dice and sum the total. What is the probability you roll exactly a five? At least a ten?
   (b) You roll three dice and sum the total. What is the probability you roll at least a 14? (This question is kind of annoying if you do it by brute force. Can you be systematic?)
   (c) The dice game of *craps* is (in its most basic form) played as follows.

       You roll two dice. If you roll a 7 or 11 on your first roll, you win immediately, and if you roll a 2, 3, or 12 immediately, you lose immediately. Otherwise, your total is called "the point" and you continue to roll again until you roll either the point (again) or a seven. If you roll the point, you win; if you roll a seven, you lose.

       In a game of craps, compute the probability that you win on your first roll and the probability that you lose on your second roll.
   (d) In a game of craps, compute the probability that the game goes to a second round and you win on the second round.

(e) In a game of craps, compute the probability that the game goes to a second round and you lose on the second round.

(f) In a game of craps, compute the probability that you win.

3. Consider the game Press Your Luck described above. Assume (despite rather convincing evidence to the contrary) that the show is random, and that you are equally likely to stop on any square on the board.

   (a) On each spin, estimate the probability that you hit a Whammy. Justify your answer.

   (Note: This is mostly not a math question. You have to watch the video clip for awhile to answer it.)

   (b) On each spin, estimate the probability that you *do not* hit a Whammy.

   (c) If you spin three times in a row, what is the probability you don't hit a whammy? Five? Ten? Twenty-eight? (If your answer is a power of a fraction, please also use a calculator or a computer to give a decimal approximation.)

4. Consider the game Rat Race described above.

   (a) Suppose that the contestant only prices one item correctly, and so gets to pick one rat. What is the probability that she wins the car? That she wins *something*? That she wins nothing?

   (b) What if the customer prices all three items correctly? What is the probability that she wins the car? Something? Nothing? All three items?

   (c) Consider now the first part of the game, where the contestant is pricing each item. *Assume* that she has a 50-50 chance of pricing each item correctly. What is the probability she prices no items correctly? Exactly one? Exactly two? All three?

   Comment on whether you think this assumption is realistic.

   (d) Suppose now that she has a 50-50 chance of pricing each item correctly, and she plays the game to the end. What is the probability she wins the car?