COMPREHENSIVE EXAM IN ANALYTIC NUMBER THEORY (FALL 2013)

1. Refer to p. 8 of Davenport's book. Assume that q is an odd prime $\equiv 3 \pmod 4$,

$$L(s) = \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) n^{-s},$$

   and

$$G(n) = \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \exp(2\pi i m n/q).$$

   (a) Justify that $G \neq 0$ by computing G^2.
   (b) Prove (1) on p. 8.
   (c) Prove that the series in (2) converges for $|z| \leq 1$ and $z \neq 1$.
   (d) Prove (6).
   (e) Verify the statement immediately after (7).
   (f) Dirichlet's class number formula for imaginary quadratic fields says that, for $d < 0$,

$$h(d) = \frac{w\sqrt{|d|}}{2\pi} L(s, \chi_d),$$

   where $h(d)$ is the class number of $\mathbb{Q}(\sqrt{d})$, and

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s}.$$

   Using all of these formulas (all of which were proved by Dirichlet), compute the class number of $\mathbb{Q}(\sqrt{-31})$. (Pay close attention to how these L-functions are defined.)

2. The nonvanishing of $L(1, \chi)$ is such a beautiful theorem that it's worth a second proof. Refer to p. 37 of Iwaniec-Kowalski.
   (a) Justify the equation after (2.29).
   (b) IK refer to 'opening the convolution (2.29)'. Define the convolution of two arithmetic functions, and say what two functions (2.29) is the convolution of.
   (c) Justify each step of the proof after 'We obtain', in substantially more detail than Iwaniec and Kowalski do. For each of the O-terms, specify what variables, if any, the implied constants depend on.
   (Hint: My hand-written note 'Require ...', which I wrote a few years ago, might be a useful hint, but I wrote it in a somewhat sloppy way.)

3. Explicitly describe all of the Dirichlet characters modulo 10.

4. Refer to Chapter 7 of Shakarchi and Stein's book.
   (a) Summarize the proof given there of the prime number theorem.
   (b) With Lemma 2.4 as a model, prove a similar formula for
   $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^s}{s(s+1)(s+2)}.$$
   For convergence issues, you are welcome, and indeed strongly encouraged, to use facts proved on pp. 192-193, e.g. to compare your integrals to theirs rather than to imitate their proof.
   (c) Formulate an analogue of Proposition 2.3 which uses your variant of Lemma 2.4.
   (d) Another analogue of Lemma 2.4 (Perron's formula) is that the integral
   $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^s}{s}ds$$
   is equal to 0 if $0 < a < 1$, and 1 if $a > 1$. Do not prove this, but do prove an analogue of Proposition 2.3 which directly gives a formula for $\psi(x)$.
   (e) (bonus) All of this is related to a general phenomenon of interest to Fourier analysts. How?
   (f) Using this last analogue, if possible, would have made Stein and Shakarchi's proof of the prime number theorem simpler, for example because it would eliminate the need for Proposition 2.2. However, there is one point at which their argument would fail. Find it, and explain.