**Homework 1 - Analytic number theory**

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**Due Friday, September 2**

There is a lot of work to do here! Most of the solutions I have not worked out myself in detail. There is at least one problem I don't know how to solve. Doing half the problems every week is pretty good.

1. (5 points) (This problem is boring, but it is good hygiene to do it once in your life.)

   (a) Let $a_n$, $n \geq 1$ be an infinite sequence. Come up with a good notion of what it should mean for the product $\prod_n a_n$ to converge absolutely.

   (b) Using your above definition, rigorously prove that if $s$ is a complex number with $\Re(s) > 1$, the product $\prod_p \frac{1}{1-p^{-s}}$ converges absolutely and is equal to $\sum_n \frac{1}{n^s}$.

2. (5 points) Prove that for $\Re(s) > 1$ we have

$$\zeta(s) = s\int_1^\infty \frac{\lfloor y \rfloor}{y^{s+1}} dy = \frac{s}{s-1} - s\int_1^\infty \frac{\{y\}}{y^{s+1}} dy. \tag{1}$$

   Prove further that the integral on the right converges absolutely for $Re(s) > 0$. (Prove additionally that it is analytic as a function of $s$ if you have the complex analysis background.) This equation allows us to define $\zeta(s)$ whenever $\Re(s) > 0$ and $s \neq 1$.

   The unsolved *Riemann hypothesis* says that if $\zeta(s) = 0$ then $\Re(s) = \frac{1}{2}$. (Prove that, and I will give you a lot of bonus points...)

3. (5 points) This is basically the same proof we saw in class, but arranged slightly differently. (See p. 56 of Davenport.) If we define $T(x) = \sum_{m \leq x} \lfloor x/m \rfloor$, prove that $T(x) = \sum_{n \leq x} \log n$. Prove directly that

$$T(x) - 2T(x/2) \leq \sum_{m \leq x} \Lambda(m) \tag{2}$$

   and conclude that

$$\pi(x) \log x > x(\log 2 - o(1)). \tag{3}$$

   You can do the upper bound too if you like.

4. (10 points) By considering the combination

$$T(x) - T(x/2) - T(x/3) - T(x/5) + T(x/30), \tag{4}$$

   prove better upper and/or lower bounds for $\pi(x)$.

5. (5 points) Suppose that $\pi(x)$ is asymptotic to $C\frac{x}{\log x}$ for some $C$. Prove, using what we already know, that $C$ must be 1.

   (If you prefer, answer this question with $\psi(x)$ and $Cx$ instead.)

6. (5 points) Prove that (as asserted in lecture)

$$\sum_{p^e, \quad e \geq 2} \frac{1}{p^e} = O(1), \tag{5}$$

where the sum is over powers of primes (but excluding the primes themselves).

7. (5 points, **This one is important, please do it even if you skip a lot of questions**) Assuming that $\pi(x) \asymp \frac{x}{\log x}$, conclude (using partial summation) that $\psi(x) \asymp x$.