Fourier Analysis in Bhargava's Averaging Method
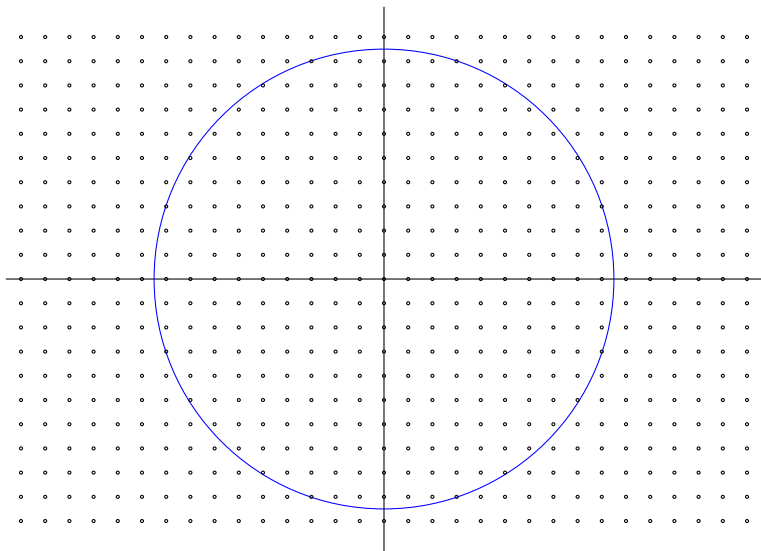
Frank Thorne
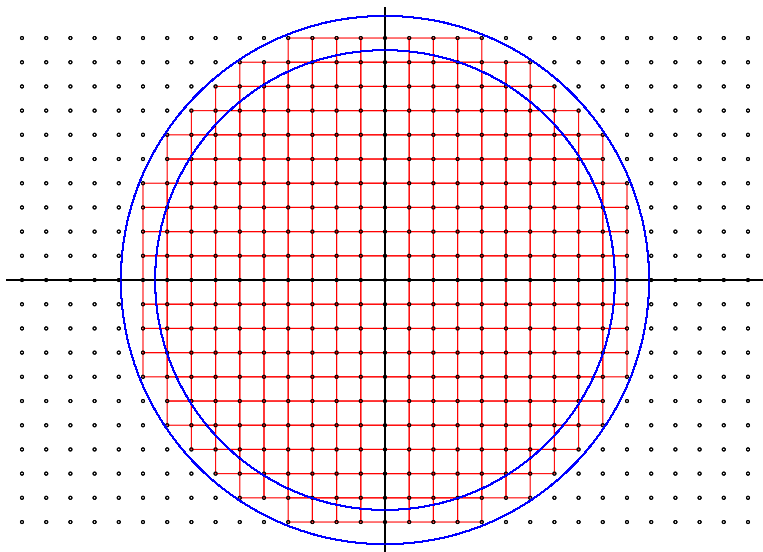
University of South Carolina
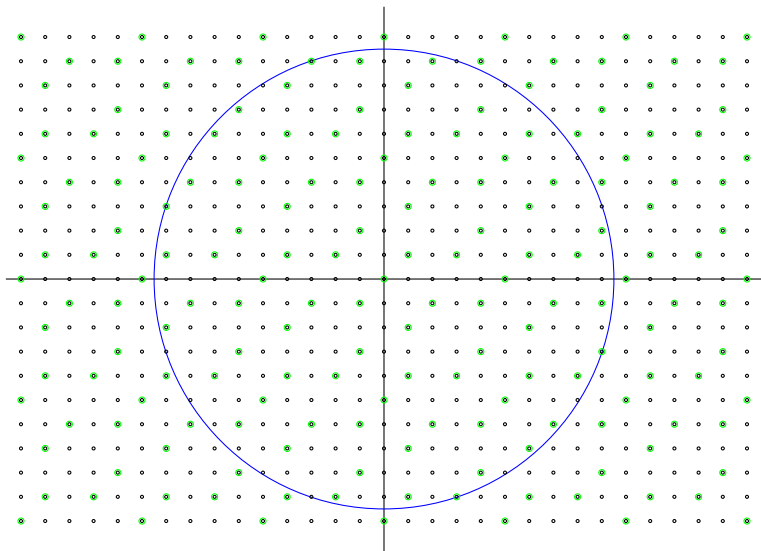
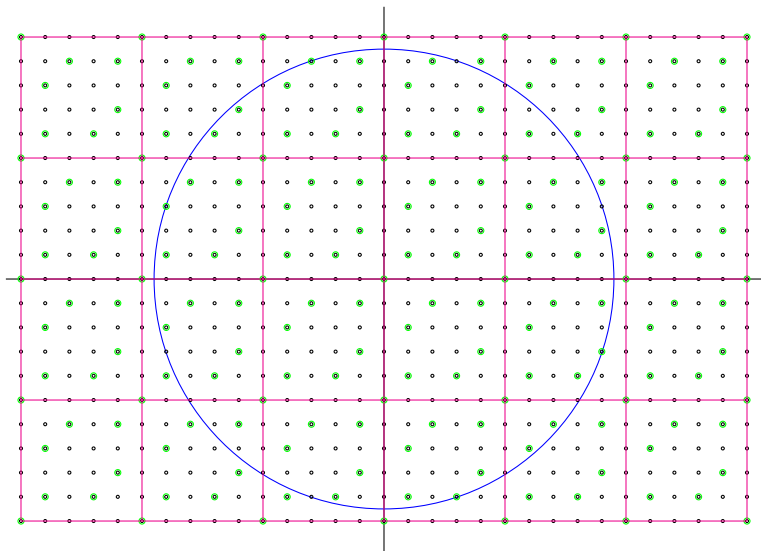Sendai Conference on Automorphic Forms, June 22, 2025
`thornef.github.io/japan-fourier-2025.pdf`

What is "arithmetic statistics"?

Sample Theorem 1: Counting Cubic Fields

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For any integer $d \geq 1$, write

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We have

$$N_3(X) = \frac{1}{3\zeta(3)} X + o(X).$$

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$$N_4(X, S_4) \sim \frac{5}{24} \prod_p (1 + p^{-2} - p^{-3} - p^{-4})X,$$

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Theorem (Davenport-Heilbronn)

We have

$$\sum_{|D|<X} \#|\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))[3]| = \frac{3+3+1+3}{\pi^2}X + o(X).$$

Sample Theorem 4: 2-Selmer Groups in Elliptic Curves

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When elliptic curves E are ordered by height, the average size of their 2-Selmer groups is 3.

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Corollary

Their average rank is at most 1.5.

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Moreover, certain arithmetic properties on the left correspond to congruence conditions on the right.

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Let V be the space of binary cubic forms:

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- $\mathrm{Disc}(gx) = (\det g)^2\mathrm{Disc}(x)$;
- $\mathrm{Disc}(x) = 0$ if and only if $x(u,v)$ has a repeated root.

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- Stab(v) *is isomorphic to* $\mathrm{Aut}(R)$;
- $\mathrm{Disc}(v) = \mathrm{Disc}(R)$;
- *(Davenport-Heilbronn)* R *is* maximal *iff, for all primes* p, v *satisfies a certain congruence condition* (mod p^2).

Improved Davenport-Heilbronn

Theorem (DHBBPBSTTTBTT)

We have

$$N_3(X) = \frac{1}{3\zeta(3)}X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{\frac{2}{3}}(\log X)^3).$$

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- ... and more!
  (Bhargava, Ho, Shankar, Varma, X. Wang, Wood, ...)

More Interesting Parametrizations

Table 1: Summary of Higher Composition Laws

#	Lattice ($V_\mathbb{Z}$)	Group acting ($G_\mathbb{Z}$)	Parametrizes ($\mathcal{C}$)	(k)	(n)	(H)
1.	$\{0\}$	-	Linear rings	0	0	A_0
2.	$\widetilde{\mathbb{Z}}$	$SL_1(\mathbb{Z})$	Quadratic rings	1	1	A_1
3.	$(\mathrm{Sym}^2\mathbb{Z}^2)^*$ (GAUSS'S LAW)	$SL_2(\mathbb{Z})$	Ideal classes in quadratic rings	2	3	B_2
4.	$\mathrm{Sym}^3\mathbb{Z}^2$	$SL_2(\mathbb{Z})$	Order 3 ideal classes in quadratic rings	4	4	G_2
5.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^2$	$SL_2(\mathbb{Z})^2$	Ideal classes in quadratic rings	4	6	B_3
6.	$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$	$SL_2(\mathbb{Z})^3$	Pairs of ideal classes in quadratic rings	4	8	D_4
7.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^4$	$SL_2(\mathbb{Z}) \times SL_4(\mathbb{Z})$	Ideal classes in quadratic rings	4	12	D_5
8.	$\wedge^3\mathbb{Z}^6$	$SL_6(\mathbb{Z})$	Quadratic rings	4	20	E_6
9.	$(\mathrm{Sym}^3\mathbb{Z}^2)^*$	$GL_2(\mathbb{Z})$	Cubic rings	4	4	G_2
10.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3$	$GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$	Order 2 ideal classes in cubic rings	12	12	F_4
11.	$\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$	$GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})^2$	Ideal classes in cubic rings	12	18	E_6
12.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^6$	$GL_2(\mathbb{Z}) \times SL_6(\mathbb{Z})$	Cubic rings	12	30	E_7
13.	$(\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3)^*$	$GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$	Quartic rings	12	12	F_4
14.	$\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$	$GL_4(\mathbb{Z}) \times SL_5(\mathbb{Z})$	Quintic rings	40	40	E_8

Bhargava, *Higher composition laws IV*, Ann. Math., 2008

	Group (s.s.)	Representation	Geometric Data	Invariants	Dynkin	§
1.	SL_2	$Sym^4(2)$	(C, L_2)	$2, 3$	$A_2^{(2)}$	4.1
2.	SL_2^2	$Sym^2(2) \otimes Sym^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_2^{(2)}$	6.1
3.	SL_2^4	$2 \otimes 2 \otimes 2 \otimes 2$	$(C, L_2, L_2', L_2'') \sim (C, L_2, P, P')$	$2, 4, 4, 6$	$D_4^{(1)}$	6.2
4.	SL_2^3	$2 \otimes 2 \otimes Sym^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 4, 6$	$B_3^{(1)}$	6.3.1
5.	SL_2^2	$Sym^2(2) \otimes Sym^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_3^{(2)}$	6.3.3
6.	SL_2^2	$2 \otimes Sym^3(2)$	(C, L_2, P_3)	$2, 6$	$G_2^{(1)}$	6.3.2
7.	SL_2	$Sym^4(2)$	(C, L_2, P_3)	$2, 3$	$A_2^{(2)}$	6.3.4
8.	$SL_2^2 \times GL_4$	$2 \otimes 2 \otimes \wedge^2(4)$	$(C, L_2, M_{2,6})$	$2, 4, 6, 8$	$D_5^{(1)}$	6.6.1
9.	$SL_2 \times SL_6$	$2 \otimes \wedge^3(6)$	$(C, L_2, M_{3,6})$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(1)}$	6.6.2
10.	$SL_2 \times Sp_6$	$2 \otimes \wedge_0^3(6)$	$(C, L_2, (M_{3,6}, \varphi))$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(2)}$	6.6.3
11.	$SL_2 \times Spin_{12}$	$2 \otimes S^+(32)$	$(C \to \mathbb{P}^1(\mathcal{H}_3(\mathbb{H})), L_2)$	$2, 6, 8, 12$	$E_7^{(1)}$	6.6.3
12.	$SL_2 \times E_7$	$2 \otimes 56$	$(C \to \mathbb{P}^1(\mathcal{H}_3(\mathbb{O})), L_2)$	$2, 6, 8, 12$	$E_8^{(1)}$	6.6.3
13.	SL_3	$Sym^3(3)$	(C, L_3)	$4, 6$	$D_4^{(3)}$	4.2
14.	SL_3^3	$3 \otimes 3 \otimes 3$	$(C, L_3, L_3') \sim (C, L_3, P)$	$6, 9, 12$	$E_6^{(1)}$	5.1
15.	SL_3^2	$3 \otimes Sym^2(3)$	(C, L_3, P_2)	$6, 12$	$F_4^{(1)}$	5.2.1
16.	SL_3	$Sym^3(3)$	(C, L_3, P_2)	$4, 6$	$D_4^{(3)}$	5.2.2
17.	$SL_3 \times SL_6$	$3 \otimes \wedge^2(6)$	$(C, L_3, M_{2,6})$ with $L^{\otimes 2} \cong \det M$	$6, 12, 18$	$E_7^{(1)}$	5.5
18.	$SL_3 \times E_6$	$3 \otimes 27$	$(C \hookrightarrow \mathbb{P}^2(\mathbb{O}), L_3)$	$6, 12, 18$	$E_8^{(1)}$	5.4
19.	$SL_2 \times SL_4$	$2 \otimes Sym^2(4)$	(C, L_4)	$8, 12$	$E_6^{(2)}$	4.3
20.	$SL_5 \times SL_5$	$\wedge^2(5) \otimes 5$	(C, L_5)	$20, 30$	$E_8^{(1)}$	4.4

Table 1: Table of coregular representations and their moduli interpretations

Bhargava and Ho, *Coregular spaces and genus one curves*, Camb. J. Math.

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- We might impose congruence conditions as well.

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