Number Field Counting, L-Functions, and Automorphic Forms

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Modular Forms and Related Topics, Beirut, May 2018

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so that

$$N_d(X) = \sum_{\substack{G \subseteq S_d \\ \text{transitive}}} N_d(X, G).$$

Basic Results

Theorem (Finiteness – Hermite)

For each d and X, $N_d(X)$ is finite.

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If $[K : \mathbb{Q}] = d$, then

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$$|\mathrm{Disc}(K)| \geq \left(\frac{d^d}{d!}\right)^2 \left(\frac{\pi}{4}\right)^d .$$

In other words,

$$N_d(X) = 0 \ \text{ for } \ X < (5.803 \cdots + o(1))^d.$$

The Inverse Galois Problem

Conjecture

For every d and transitive subgroup $G \subseteq S_d$,

$$X \text{ big enough } \implies N_d(X, G) \neq 0.$$

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Proof.

????????

$\square$

Three Methods to Count Number Fields

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- Polynomial methods (upper and lower bounds);
- Abelian methods (upper bounds and asymptotics);
- Parametrization methods (asymptotics in limited cases).

Polynomial methods

If $\alpha \in \mathcal{O}_K$ is a generator of $K/\mathbb{Q}$, then $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_K$ and

$$\begin{aligned} |\mathrm{Disc}(\mathcal{O}_K)| &= \mathrm{Disc}(\mathbb{Z}[\alpha]) \cdot [\mathcal{O}_K : \mathbb{Z}[\alpha]]^{-2} \\ &= \mathrm{Disc}(\min{}_\alpha(X)) \cdot [\mathcal{O}_K : \mathbb{Z}[\alpha]]^{-2}. \end{aligned}$$

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Theorem (Schmidt)

For each d we have

$$N_d(X) \ll X^{\frac{d+2}{4}}.$$

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See Ellenberg-Venkatesh (2006), Dummit (2017) for refinements.

Abelian methods

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Theorem (Class Field Theory)

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Abelian extensions L/K of degree d and conductor dividing $\mathfrak{m}$ are in bijection with index d quotients of $\mathrm{Cl}_{\mathfrak{m}}(L)$.

Theorem (Kummer Theory)

If in addition $\mu_d \subseteq K$, then abelian extensions L/K of exponent d are in bijection with subgroups of $K^\times/(K^\times)^d$.

Sample theorem

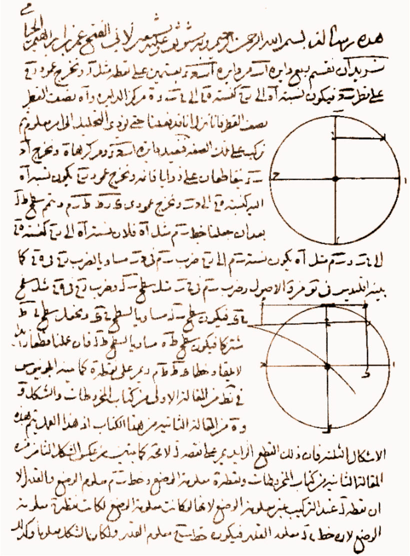
Definition

If K is an S_3- cubic field, its *quadratic resolvent* is $\mathbb{Q}(\sqrt{\mathrm{Disc}(K)})$, the unique quadratic subfield of K^c.

Theorem (Cohen, Morra, T.)

Let $D \neq 0, 1$ be a fundamental discriminant. Then,

$$\sum_{\substack{[K:\mathbb{Q}]=3 \\ \mathbb{Q}(\sqrt{D}) \text{ is the} \\ \text{quadratic resolvent of } K}} |\mathrm{Disc}(K)| = \{\text{explicit finite sum of Euler products}\}.$$



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There exists an explicit, discriminant preserving bijection between the following two sets:

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- $\mathrm{GL}_3(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})$*-orbits on the lattice* $(\mathrm{Sym}^2\mathbb{Z}^3 \otimes \mathbb{Z}^2)$ *of* **pairs of integral ternary quartic forms**.

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- *Pairs* (Q, R), *where* Q *is a* **quartic ring** *and* R *is a* **cubic resolvent** *of* Q.

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- $G(\mathbb{Z})$*-orbits on a lattice* $V(\mathbb{Z})$*; where* G *is an* **algebraic group** *acting (often* **prehomogeneously***) on a vector space* V*;*

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- $G(\mathbb{Z})$*-orbits on a lattice* $V(\mathbb{Z})$*; where* G *is an* **algebraic group** *acting (often* **prehomogeneously***) on a vector space* V*;*
- *Some nice class of arithmetic objects we want to count.*

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- Read old papers in representation theory, invariant theory, and commutative algebra for inspiration. (Or Omar Khayyam!)
- Pick your favorite complex representation (G, V) (which should be defined over $\mathbb{Z}$, and for which the invariant theory should be nice).
- Try to prove that the $G(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ parametrize something. Hope to get lucky.

An Arithmetic Statistics Theorem

Theorem (Davenport-Heilbronn 1971 + BBPSTTT*)

We have

$$N_3(X) = \frac{1}{3\zeta(3)} X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + O(X^{2/3+\epsilon}).$$

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*: Belabas (1999), Belabas, Bhargava, Pomerance (∼2006), Bhargava, Shankar, Tsimerman (2010), Taniguchi, T. (2011), Bhargava, Taniguchi, T. (2018? hopefully??)

To be (briefly) discussed today:

- The Colmez conjecture;
- Automorphy of Sato-Shintani zeta functions;
- Equidistribution of shapes of number fields.

Connection 1: The Colmez Conjecture

(work with Adrian Barquero-Sanchez and Riad Masri)

The Main Theorem

Theorem

Assume a weak form of Malle's Conjecture.

Then, the Colmez conjecture is true for 100% of CM fields of any fixed degree, when ordered by discriminant.

The Colmez Conjecture

Conjecture (Colmez '93)

Let E be a CM field, and let X_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$. Then,

$$h_{\mathrm{Fal}}(X_\Phi) = -Z(A^0_{E,\Phi}, 0) - \frac{1}{2}\mu_{\mathrm{Art}}(A^0_{E,\Phi}),$$

where in the above

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- X_Φ is an abelian variety over $\overline{\mathbb{Q}}$ with complex multiplication by $\mathcal{O}_E$, and CM type for E (....)
- $h_{\mathbf{Fal}}(X_\Phi)$ is the Faltings height of X_Φ, which in fact only depends on Φ;
- The quantity on the right is defined in terms of logarithmic derivatives of Artin L-functions associated to characters defined in terms of the representation theory of $\mathrm{Gal}(\mathbb{Q}^{\mathbf{CM}}/\mathbb{Q})$.
- I could explain all this in more detail, but the margins of these slides are too small.

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Definition

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- Previous work of my coauthors.

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That is: 100% of CM fields have Galois group as big as it can be.

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- For each K counted by $N_d(X, G)$ we have

$$|\mathrm{Cl}(K)[2]| \ll_{\epsilon, G} |\mathrm{Disc}(K)|^{\delta_G + \epsilon}.$$

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Say the Weak Malle Conjecture holds for (d, G) if

$$M(G) + \delta_G < 2.$$

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- Adapting work of Klüners, use the abelian method to count the number of F for each E.

Step 1: Counting quadratic extensions E/F

Define a Dirichlet series

$$D^{-}_{F,C_2}(s) := \sum_{[E:F]=2} \frac{1}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{d}_{E/F})^s}$$

where the sum is over totally imaginary quadratic extensions E/F.

Theorem (Cohen-Diaz-Olivier '02)

For $\mathrm{Re}(s) > 1$ we have

$$D^{-}_{F,C_2}(s) = \frac{1}{\zeta_F(2s)} \sum_{\mathfrak{c}_\infty \subset \mathfrak{m}_\infty} \sum_{\mathfrak{c} | 2} \frac{(-1)^{|\mathfrak{c}_\infty|}}{2^{|\mathfrak{c}_\infty|}} \mathcal{N}_{F/\mathbb{Q}}(2/\mathfrak{c})^{1-2s} \sum_{\chi \in Q(\mathrm{Cl}_{\mathfrak{c}^2 \mathfrak{c}_\infty}(F))} L_F(\chi, s),$$

where $\mathfrak{c}$ runs over all integral ideals of F dividing 2, $\mathfrak{c}_\infty$ runs over all subsets of the set of real places $\mathfrak{m}_\infty$ of F, χ runs over all quadratic characters $Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))$ of the ray class group $\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F)$ modulo $\mathfrak{c}^2\mathfrak{c}_\infty$, and $L_F(\chi, s)$ is the L–function of χ.

Counting quadratic extensions: morally

Theorem (Cohen-Diaz-Olivier '02)

For $\mathrm{Re}(s) > 1$ *we have*

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We obtain

$$\#\{E \ : \ \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{d}_{E/F}) \leq Y\} \asymp Y + \#\mathrm{Cl}(F)[2] \cdot o(Y),$$

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$$\sum_E d_E^{-s} = \sum_F \frac{1}{d_F^{2s}} D_{F,C_2}(s)^{-s}.$$

Putting it all together

Incorporating everything above, we have

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Proposition

If the Weak Malle Conjecture is true for whatever family of fields we're summing F over, then:

- $\xi(s)$ converges absolutely in $\Re(s) > 1$;
- $\xi(s)$ has meromorphic continuation to a half-plane $\Re(s) > \alpha$, with $\alpha < 1$; it has a simple pole at $s = 1$, with residue

$$\sum_{F} \frac{\mathrm{Res}_{s=1}\zeta_F(s)}{2^d d_F^2 \zeta_F(2)} \text{ ``}\approx\text{''} \sum_{F} \frac{1}{d_F^2}.$$

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- $\xi(s)$ is polynomially bounded in vertical strips.

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Application: Look at the Dirichlet series again: no contribution to the residue.

Table: Values of $r_d(G)$ for $d \leq 5$

d	G	Number of fields	Minimal discriminant	Residue	Proportion
2	C_2	100,000	5	0.009856	-
3		25,000	49	3.30×10^{-5}	-
	C_3	107	49	2.29×10^{-5}	0.69
	S_3	24,893	148	1.01×10^{-5}	0.31
4		25,000	725	1.24×10^{-7}	-
	C_4	75	1125	2.41×10^{-8}	0.19
	V_4	289	1600	1.56×10^{-8}	0.13
	D_4	8147	725	5.9×10^{-8}	0.48
	A_4	45	26569	9.3×10^{-11}	0.0008
	S_4	16,444	1957	2.5×10^{-8}	0.20
5		25,000	14641	1.05×10^{-10}	-
	C_5	5	14641	3.08×10^{-11}	0.29
	D_5	28	160801	4.24×10^{-13}	0.003
	F_5	15	2382032	9×10^{-15}	0.00009
	A_5	21	3104644	5×10^{-15}	0.00005
	S_5	24,931	24217	7.4×10^{-11}	0.70

Connection 2: Sato-Shintani Zeta Functions and Automorphy

Binary cubic forms I: Definitions

The lattice of *binary cubic forms* is

$$V(\mathbb{Z}) := \{au^3 + bu^2v + cuv^2 + dv^3 \ : \ a, b, c, d \in \mathbb{Z}\}.$$

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$$(g \circ f)(u, v) = \frac{1}{\det g} f((u, v)g),$$

which satisfies

$$\mathrm{Disc}(g \circ f) = (\det g)^2 \mathrm{Disc}(f),$$

$$\mathrm{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

Theorem (Levi 1914, Delone-Faddeev 1940, Gan-Gross-Savin 2002)

There is an explicit, discriminant-preserving bijection between the set of $\mathrm{GL}_2(\mathbb{Z})$*-orbits on* $V(\mathbb{Z})$ *and the set of* **cubic rings**.

A Sato-Shintani zeta function

Definition

The *(cubic) Sato-Shintani zeta function* associated to this (G, V) is

$$\xi^{\pm}(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}$$

$$\text{“} = \text{”} \sum_{\pm \mathrm{Disc}(\mathcal{O}) > 0} |\mathrm{Disc}(\mathcal{O})|^{-s},$$

where $\mathcal{O}$ ranges over orders in cubic fields.

A Sato-Shintani zeta function

Definition

The *(cubic) Sato-Shintani zeta function* associated to this (G, V) is

$$\xi^{\pm}(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}$$

$$\text{``}=\text{''} \sum_{\pm \mathrm{Disc}(\mathcal{O}) > 0} |\mathrm{Disc}(\mathcal{O})|^{-s},$$

where $\mathcal{O}$ ranges over orders in étale cubic algebras.

You can ignore this if it looks scary.

Let $\xi(s) := \sum_n a(n) n^{-s}$. Then Perron's formula states that

$$\sum_{n<X} a(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \xi(s) X^s \frac{ds}{s},$$

subject (only) to absolute convergence.

How to extract arithmetic density results

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If $\xi(s)$ is a 'nice zeta function', we can hope for asymptotics with explicit error terms.

The functional equation

Theorem (Shintani, 1971)

The above zeta functions converge absolutely for $\Re(s) > 1$. *They continue to functions holomorphic in the plane except for simple poles at* 1 *and* $5/6$, *and satisfy the functional equation*

$$\begin{pmatrix} \xi^+(1-s) \\ \xi^-(1-s) \end{pmatrix} = \Gamma\left(s - \frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s + \frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s} \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3\sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \widehat{\xi}^+(s) \\ \widehat{\xi}^-(s) \end{pmatrix}.$$

Moreover,

$$\mathrm{Res}_{s=1}\xi^{\pm}(s) = \frac{\pi^2(3 + C^{\pm})}{36}, \quad \mathrm{Res}_{s=\frac{5}{6}}\xi^{\pm}(s) = K^{\pm}\frac{\zeta(1/3)\Gamma(1/3)^3}{4\sqrt{3}\pi}.$$

Sato and Shintani's general theorem

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Let G be a reductive group acting prehomogeneously on a finite dimensional vector space V, with irreducible singular set.
Then, the associated Sato-Shintani zeta functions

$$\xi^{(i)}(s) := \sum_{\substack{x \in G(\mathbb{Z}) \backslash V^{(i)}(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}$$

enjoy analytic continuations and functional equations.

Sato-Shintani: the fine print

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- If $|\mathrm{Stab}(x)|$ isn't finite, replace it with a regulator.
- Sato-Shintani don't give a recipe for writing down the explicit functional equation.
- The residues (and possibly even the location and multiplicity of the poles) are nontrivial to compute.

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- Explicit determination of the functional equations, poles, and residues. (Also determined previously by Shintani.)
- Much else besides.

A negative theorem

Theorem (T.)

The Sato-Shintani zeta function associated to the space of binary cubic forms is **not** *a finite sum of Euler products.*

The abelian method again

Theorem (Cohen, Morra, T.)

Let $D \neq 0, 1$ be a fundamental discriminant. Then,

$$\sum_{\substack{[K:\mathbb{Q}]=3 \\ \mathrm{Disc}(K)=Dn^2}} |\mathrm{Disc}(K)|^{-s} = -\frac{1}{2}+\frac{1}{2}\left(1+\frac{2}{3^{2s}}\right) \prod_{\left(\frac{-3D}{p}\right)=1} \left(1+\frac{2}{p^s}\right)+$$

$$\sum_{L \in \mathcal{L}_3(D)} M_{3,2,L}(s) \prod_{\left(\frac{-3D}{p}\right)=1} \left(1+\frac{\omega_L(p)}{p^s}\right),$$

$$M_{3,2,L}(s) := \begin{cases} 1-3^{-2s} & : \mathrm{Disc}(L) = -27D \\ 1+2\cdot 3^{-2s} & : \mathrm{Disc}(L) = -3D, \end{cases}$$

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- *$\mathcal{L}_3(D)$ is the set of cubic fields of discriminant $-3D$ or $-27D$;*
- *$\omega_L(p) = 2$ if p splits completely in L, and $\omega_L(p) = -1$ otherwise.*

Connection 3: Twisting Sato-Shintani Zeta Functions by Automorphic Forms

(work of Bob Hough)

Sato-Shintani zeta functions reimagined

The Sato-Shintani zeta function associated to the space of binary cubic forms:

$$\xi^{\pm}(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}.$$

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For simplicity, consider the positive zeta function only:

$$\xi^{+}(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V^{+}(\mathbb{Z}) \\ (\mathrm{Disc}(x) > 0)}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}.$$

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Choose (for example) $v_+ = uv(u+v) \in V^+(\mathbb{R})$.

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$$g \longrightarrow g \cdot v_+$$

is surjective and exactly 6-to-1.

Pretend for today: This map is a bijection.

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- Pushing forward, identify ϕ as a function on $\mathrm{GL}_2(\mathbb{Z})\backslash V^+(\mathbb{R})$.
- Define the ϕ-twisted Shintani $\mathscr{L}$-function

$$\xi^+(s,\phi) := \sum_{x \in \mathrm{GL}_2(\mathbb{Z})\backslash V^+(\mathbb{Z})} \frac{\phi(x)}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}.$$

Analytic Continuation

Theorem (Hough)

The function $\xi^+(s, \phi)$, when properly defined, has analytic continuation to $\Re(s) > \frac{1}{8}$.

The shape of a number field

Given a number field K with real embeddings $\sigma_1, \dots, \sigma_r$ and complex embeddings $\tau_1, \tau_1', \cdots, \tau_s, \tau_s'$, its **shape** is the quadratic form

$$q(x) := \sigma_1(x)^2 + \cdots + \sigma_r(x)^2 + 2|\tau_1(x)|^2 + \cdots + 2|\tau_s(x)|^2,$$

restricted to the $(n-1)$-dimensional lattice

$$\{x \in \mathbb{Z} + n\mathcal{O}_K \ : \ \mathrm{Tr}(x) = 0\},$$

which we may consider as an element of

$$\mathrm{GL}_{n-1}(\mathbb{Z}) \backslash \mathrm{GL}_{n-1}(\mathbb{R}) / \mathrm{GO}_{n-1}(\mathbb{R}).$$

Equidistribution of shapes

Theorem (Terr, Bhargava-Harron)

Let $n = 3, 4, 5$. The shapes of number fields counted by $N_n(X, S_n)$ become equidistributed as $X \to \infty$.

Theorem (Hough)

Let ϕ be a cuspidal automorphic form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ (satisfying suitable technical hypotheses). Then, for any smooth, compactly supported test function $F : (0, \infty) \to \mathbb{R}$ we have

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- Recover shape equidistribution via spectral decomposition of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$.
- Results for $n = 4$ too.