Homework 8 - Analytic number theory

Frank Thorne, thornef@mailbox.sc.edu

Due Friday, October 21

1. (5 points) It was proved in lecture (see also Ch. 18 of Davenport) that

$$\psi(x) = x + O\Big(x \exp(-C_1 (\log x)^{1/2}) (\log x) + x \exp(-(\log x)^{1/2}) (\log x)^2 \Big) \tag{1}$$

for an effective constant C_1. Prove that this implies that

$$\psi(x) = x + O\Big(x \exp(-C_2 (\log x)^{1/2}) \Big) \tag{2}$$

and further

$$\pi(x) = \frac{x}{\log x} + O\Big(x \exp(-C_3 (\log x)^{1/2}) \Big), \tag{3}$$

where in your proof you determine values for C_2 and C_3 in terms of C_1. (No need to determine the best constants.)

2. (5 points) It was partially shown in class, and is shown entirely in Davenport, that

$$\sum_{\gamma > 0} \frac{\beta}{\beta^2 + \gamma^2} = \frac{-B}{2},$$

where $-B$ is an explicit constant > 0.022.

From this formula, prove explicit (no $\ll$, big-O) bounds for the number of zeroes in $\zeta(s)$ in the critical strip with $\Im(s) < T$.

(*Caution:* We already implicitly used similar bounds to establish the existence of the infinite product for $\xi(s)$, so this is not another approach to proving such bounds.)

3. (5 points) The formula of the previous problem implicitly assumes that the zeroes of $\zeta(s)$ occur in complex conjugate pairs. Accordingly, you need to prove separately that $\zeta(s)$ doesn't have any real zeroes between 0 and 1.

Prove this.

4. (5 points) Imagine that $\zeta(s)$ satisfied the Riemann hypothesis, except for four zeroes off the critical line located at, say, $(.5 \pm .4) \pm 15i$. Show that this implies a version of the prime number theorem of the form

$$\psi(x) = x + * x^{9/10} + O(x^{1/2} \log^2 x). \tag{4}$$

(You should determine what goes in the place of the asterisk.)

For 5 more points, compute, present, and explain numerical evidence that this version of the prime number theorem is wrong.

5. (5+ points) Download the software "L" from Mike Rubinstein's webpage (easily found via Google), and/or Tim Dokchitser's "ComputeL". Run a variety of numerical experiments on the zeroes of the zeta function, the values of $\zeta(1+it)$, or anything else you're prepared to argue is interesting. Report your findings.

6. (5 points) If a_i, for $1 \le i \le k$, is a sequence of positive numbers between 0 and 1, prove that $\sum_i a_i < 1 + \sum_{i<j} a_j a_j$.