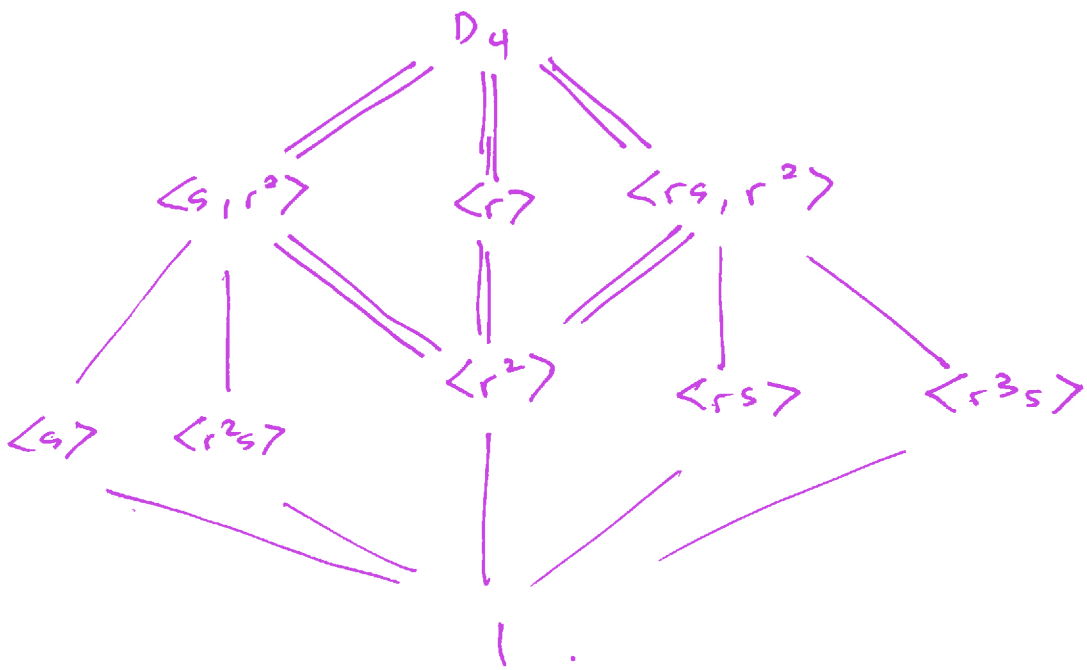
13.5.

Example. Subgroup lattice of D_4 and $D_4/\langle r^2\rangle$.



Double lines: subgroup lattice of $D_4/\langle r^2\rangle$.

Note. Knowing G/N and N does not determine N.

"Factoring through".

Suppose we have homomorphisms $\varphi : G \to H$ and ~~a normal subgroup N~~ $\psi : G \to K$.

We say φ factors through ψ if we can make the diagram commute.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & H \\ & K \arrow[u, dashed, "\theta"'] \end{tikzcd}$$

i.e. we can make $\varphi = \theta \circ \psi$ in the picture.

13.6. In particular, if $N \triangleleft G$, φ factors through G/N if it factors through the quotient homomorphism $\pi : G \to G/N$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, dashed, "\exists\, \theta"'] & \end{tikzcd}$$

<u>Lemma</u>. φ factors through G/N if and only if it is trivial on N.

<u>Proof</u>. Same as 1st iso thm!

Notice also that θ is uniquely determined.

<u>This construction is ubiquitous</u>.

Proof 2. Let Sym(n) formally manipulate polynomials in $x_1, \dots, x_n$ in the same way, i.e. $\sigma(x_i) = x_{\sigma(i)}$.

That is, Sym(n) acts on $\mathbb{Z}[x_1, \dots, x_n]$.

Now define $$\Delta := \prod_{1 \le i < j \le n} (x_i - x_j)$$

Then $\sigma\Delta = \Delta$ or $-\Delta$ for each σ, and if σ is a transposition, then $\sigma\Delta = -\Delta$.

Why? Suppose σ switches r and s with $r < s$.
Then the terms which appear with opposite sign are:

* $x_r - x_s$
* $x_i - x_s$ with $r < i < s$
* $x_r - x_i$ with $r < i < s$

The latter two occur in pairs, so cancel.

We define $\varepsilon(\sigma) = \frac{\sigma\Delta}{\Delta}$, which is a homomorphism since we have a group action.

$$\frac{\sigma\tau\Delta}{\Delta} = \underbrace{\frac{\sigma\tau\Delta}{\tau\Delta}} \cdot \frac{\tau\Delta}{\Delta}$$

This is $\frac{\sigma\Delta}{\Delta}$ since $\tau\Delta$ is Δ or $-\Delta$. (It doesn't matter which.)

Prop. A permutation is odd (i.e. not in A_n) iff the number of cycles of even length in its cycle decomposition is odd.

[Check it!]

Note that odd cycles are even permutations and even cycles are odd permutations:

$(x_1 \cdots x_k)$ is a product of $k-1$ transpositions.