Exercise Set 2.1*

In each of 1–4 represent the common form of each argument using letters to stand for component sentences, and fill in the blanks so that the argument in part (b) has the same logical form as the argument in part (a).

1. a. If all integers are rational, then the number 1 is rational.
All integers are rational.
Therefore, the number 1 is rational.

b. If all algebraic expressions can be written in prefix notation, then ______________________.
______________________.
Therefore, $(a+2b)(a^2-b)$ can be written in prefix notation.

2. a. If all computer programs contain errors, then this program contains an error.
This program does not contain an error.
Therefore, it is not the case that all computer programs contain errors.

b. If ______, then ______.
2 is not odd.
Therefore, it is not the case that all prime numbers are odd.

3. a. This number is even or this number is odd.
This number is not even.
Therefore, this number is odd.

b. ______ or logic is confusing.
My mind is not shot.
Therefore, ______.

4. a. If n is divisible by 6, then n is divisible by 3.
If n is divisible by 3, then the sum of the digits of n is divisible by 3.
Therefore, if n is divisible by 6, then the sum of the digits of n is divisible by 3.
(Assume that n is a particular, fixed integer.)

b. If this function is ____ then this function is differentiable.
If this function is ____ then this function is continuous.
Therefore, if this function is a polynomial, then this function ____.

5. Indicate which of the following sentences are statements.
a. 1,024 is the smallest four-digit number that is a perfect square.
b. She is a mathematics major.
c. $128 = 2^6$ d. $x = 2^6$

Write the statements in 6–9 in symbolic form using the symbols

a. Stocks are increasing but interest rates are steady.
b. Neither are stocks increasing nor are interest rates steady.

7. Juan is a math major but not a computer science major. (m = "Juan is a math major," c = "Juan is a computer science major")

8. Let h = "John is healthy," w = "John is wealthy," and s = "John is wise."
a. John is healthy and wealthy but not wise.
b. John is not wealthy but he is healthy and wise.
c. John is neither healthy, wealthy, nor wise.
d. John is neither wealthy nor wise, but he is healthy.
e. John is wealthy, but he is not both healthy and wise.

9. Either this polynomial has degree 2 or it has degree 3 but not both. (n = "This polynomial has degree 2," k = "This polynomial has degree 3")

10. Let p be the statement "DATAENDFLAG is off," q the statement "ERROR equals 0," and r the statement "SUM is less than 1,000." Express the following sentences in symbolic notation.
a. DATAENDFLAG is off, ERROR equals 0, and SUM is less than 1,000.
b. DATAENDFLAG is off but ERROR is not equal to 0.
c. DATAENDFLAG is off; however, ERROR is not 0 or SUM is greater than or equal to 1,000.
d. DATAENDFLAG is on and ERROR equals 0 but SUM is greater than or equal to 1,000.
e. Either DATAENDFLAG is on or it is the case that both ERROR equals 0 and SUM is less than 1,000.

11. In the following sentence, is the word *or* used in its inclusive or exclusive sense? A team wins the playoffs if it wins two games in a row or a total of *three* games.

Write truth tables for the statement forms in 12–15.

12. $\sim p \wedge q$

13. $\sim(p \wedge q) \vee (p \vee q)$

14. $p \wedge (q \wedge r)$

15. $p \wedge (\sim q \vee r)$

Determine whether the statement forms in 16–24 are logically equivalent. In each case, construct a truth table and include a sentence justifying your answer. Your sentence should show that you understand the meaning of logical equivalence.

16. $p \vee (p \wedge q)$ and p

17. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

18. $p \vee \mathbf{t}$ and $\mathbf{t}$

19. $p \wedge \mathbf{t}$ and p

20. $p \wedge \mathbf{c}$ and $p \vee \mathbf{c}$

25. Hal is a math major and Hal's sister is a computer science major.

26. Sam is an orange belt and Kate is a red belt.

27. The connector is loose or the machine is unplugged.

28. The units digit of 4^{67} is 4 or it is 6.

29. This computer program has a logical error in the first ten lines or it is being run with an incomplete data set.

30. The dollar is at an all-time high and the stock market is at a record low.

31. The train is late or my watch is fast.

Assume x is a particular real number and use De Morgan's laws to write negations for the statements in 32–37.

32. $-2 < x < 7$

33. $-10 < x < 2$

34. $x < 2$ or $x > 5$

35. $x \leq -1$ or $x > 1$

36. $1 > x \geq -3$

37. $0 > x \geq -7$

In 38 and 39, imagine that *num_orders* and *num_instock* are particular values, such as might occur during execution of a computer program. Write negations for the following statements.

38. (*num_orders* > 100 and *num_instock* ≤ 500) or *num_instock* < 200

39. (*num_orders* < 50 and *num_instock* > 300) or (50 ≤ *num_orders* < 75 and *num_instock* > 500)

Use truth tables to establish which of the statement forms in 40–43 are tautologies and which are contradictions.

40. $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$

41. $(p \wedge \sim q) \wedge (\sim p \vee q)$

42. $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$

43. $(\sim p \vee q) \vee (p \wedge \sim q)$

In 44 and 45, determine whether the statements in (a) and (b) are logically equivalent.

44. Assume x is a particular real number.

a. $x < 2$ or it is not the case that $1 < x < 3$.

b. $x \leq 1$ or either $x < 2$ or $x \geq 3$.

math and computer science major.

✱46. In Example 2.1.4, the symbol $\oplus$ was introduced to denote *exclusive or*, so $p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$. Hence the truth table for *exclusive or* is as follows:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

a. Find simpler statement forms that are logically equivalent to $p \oplus p$ and $(p \oplus p) \oplus p$.
b. Is $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$? Justify your answer.
c. Is $(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r)$? Justify your answer.

✱**47.** In logic and in standard English, a double negative is equivalent to a positive. There is one fairly common English usage in which a "double positive" is equivalent to a negative. What is it? Can you think of others?

In 48 and 49 below, a logical equivalence is derived from Theorem 2.1.1. Supply a reason for each step.

48. $(p \wedge \sim q) \vee (p \wedge q) \equiv p \wedge (\sim q \vee q)$ by $\underline{\text{(a)}}$
$\equiv p \wedge (q \vee \sim q)$ by $\underline{\text{(b)}}$
$\equiv p \wedge \mathbf{t}$ by $\underline{\text{(c)}}$
$\equiv p$ by $\underline{\text{(d)}}$

Therefore, $(p \wedge \sim q) \vee (p \wedge q) \equiv p$.

49. $(p \vee \sim q) \wedge (\sim p \vee \sim q)$
$\equiv (\sim q \vee p) \wedge (\sim q \vee \sim p)$ by $\underline{\text{(a)}}$
$\equiv \sim q \vee (p \wedge \sim p)$ by $\underline{\text{(b)}}$
$\equiv \sim q \vee \mathbf{c}$ by $\underline{\text{(c)}}$
$\equiv \sim q$ by $\underline{\text{(d)}}$

Therefore, $(p \vee \sim q) \wedge (\sim p \vee \sim q) \equiv \sim q$.

Use Theorem 2.1.1 to verify the logical equivalences in 50–54. Supply a reason for each step.

50. $(p \wedge \sim q) \vee p \equiv p$

51. $p \wedge (\sim q \vee p) \equiv p$

52. $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$

53. $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

54. $(p \wedge (\sim(\sim p \vee q))) \vee (p \wedge q) \equiv p$

Exercise Set 2.2

Rewrite the statements in 1–4 in if-then form.

1. This loop will repeat exactly N times if it does not contain a **stop** or a **go to.**
2. I am on time for work if I catch the 8:05 bus.
3. Freeze or I'll shoot.
4. Fix my ceiling or I won't pay my rent.

Construct truth tables for the statement forms in 5–11.

5. $\sim p \vee q \rightarrow \sim q$
6. $(p \vee q) \vee (\sim p \wedge q) \rightarrow q$
7. $p \wedge \sim q \rightarrow r$
8. $\sim p \vee q \rightarrow r$
9. $p \wedge \sim r \leftrightarrow q \vee r$
10. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$
11. $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$
12. Use the logical equivalence established in Example 2.2.3, $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$, to rewrite the following statement. (Assume that x represents a fixed real number.)

    If $x > 2$ or $x < -2$, then $x^2 > 4$.

13. Use truth tables to verify the following logical equivalences. Include a few words of explanation with your answers.

    a. $p \rightarrow q \equiv \sim p \vee q$ b. $\sim(p \rightarrow q) \equiv p \wedge \sim q$.

H 14. a. Show that the following statement forms are all logically equivalent.

$$p \rightarrow q \vee r, \quad p \wedge \sim q \rightarrow r, \quad \text{and} \quad p \wedge \sim r \rightarrow q$$

b. Use the logical equivalences established in part (a) to rewrite the following sentence in two different ways. (Assume that n represents a fixed integer.)

If n is prime, then n is odd or n is 2.

15. Determine whether the following statement forms are logically equivalent:

$$p \rightarrow (q \rightarrow r) \quad \text{and} \quad (p \rightarrow q) \rightarrow r$$

In 16 and 17, write each of the two statements in symbolic form and determine whether they are logically equivalent. Include a truth table and a few words of explanation.

16. If you paid full price, you didn't buy it at Crown Books. You didn't buy it at Crown Books or you paid full price.
17. If 2 is a factor of n and 3 is a factor of n, then 6 is a factor of n. 2 is not a factor of n or 3 is not a factor of n or 6 is a factor of n.
18. Write each of the following three statements in symbolic form and determine which pairs are logically equivalent. Include truth tables and a few words of explanation.

    If it walks like a duck and it talks like a duck, then it is a duck.

    Either it does not walk like a duck or it does not talk like a duck, or it is a duck.

    If it does not walk like a duck and it does not talk like a duck, then it is not a duck.

19. True or false? The negation of "If Sue is Luiz's mother, then Ali is his cousin" is "If Sue is Luiz's mother, then Ali is not his cousin."
20. Write negations for each of the following statements. (Assume that all variables represent fixed quantities or entities, as appropriate.)
    a. If P is a square, then P is a rectangle.
    b. If today is New Year's Eve, then tomorrow is January.
    c. If the decimal expansion of r is terminating, then r is rational.
    d. If n is prime, then n is odd or n is 2.
    e. If x is nonnegative, then x is positive or x is 0.
    f. If Tom is Ann's father, then Jim is her uncle and Sue is her aunt.
    g. If n is divisible by 6, then n is divisible by 2 and n is divisible by 3.
21. Suppose that p and q are statements so that $p \rightarrow q$ is false. Find the truth values of each of the following:

    a. $\sim p \rightarrow q$ b. $p \vee q$ c. $q \rightarrow p$

H 22. Write contrapositives for the statements of exercise 20.

H 23. Write the converse and inverse for each statement of exercise 20.

Use truth tables to establish the truth of each statement in 24–27.

24. A conditional statement is not logically equivalent to its converse.
25. A conditional statement is not logically equivalent to its inverse.
26. A conditional statement and its contrapositive are logically equivalent to each other.
27. The converse and inverse of a conditional statement are logically equivalent to each other.

H 28. "Do you mean that you think you can find out the answer to it?" said the March Hare.

"Exactly so," said Alice.

"Then you should say what you mean," the March Hare went on.

"I do," Alice hastily replied; "at least—at least I mean what I say—that's the same thing, you know."

The Hatter is right. "I say what I mean" is not the same thing as "I mean what I say." Rewrite each of these two sentences in if-then form and explain the logical relation between them. (This exercise is referred to in the introduction to Chapter 4.)

If statement forms P and Q are logically equivalent, then $P \leftrightarrow Q$ is a tautology. Conversely, if $P \leftrightarrow Q$ is a tautology, then P and Q are logically equivalent. Use $\leftrightarrow$ to convert each of the logical equivalences in 29–31 to a tautology. Then use a truth table to verify each tautology.

29. $p \to (q \vee r) \equiv (p \wedge \sim q) \to r$

30. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

31. $p \to (q \to r) \equiv (p \wedge q) \to r$

Rewrite each of the statements in 32 and 33 as a conjunction of two if-then statements.

32. This quadratic equation has two distinct real roots if, and only if, its discriminant is greater than zero.

33. This integer is even if, and only if, it equals twice some integer.

Rewrite the statements in 34 and 35 in if-then form in two ways, one of which is the contrapositive of the other.

34. The Cubs will win the pennant only if they win tomorrow's game.

35. Sam will be allowed on Signe's racing boat only if he is an expert sailor.

36. Taking the long view on your education, you go to the Prestige Corporation and ask what you should do in college to be hired when you graduate. The personnel director replies that you will be hired *only if* you major in mathematics or computer science, get a B average or better, and take accounting. You do, in fact, become a math major, get a B$^+$ average, and take accounting. You return to Prestige Corporation, make a formal application, and are turned down. Did the personnel director lie to you?

Some programming languages use statements of the form "r unless s" to mean that as long as s does not happen, then r will happen. More formally:

> **Definition:** If r and s are statements,
>
> **r unless s** means if $\sim s$ then r.

In 37–39, rewrite the statements in if-then form.

37. Payment will be made on the fifth unless a new hearing is granted.

40. Catching the 8:05 bus is a sufficient condition for my being on time for work.

41. Having two 45° angles is a sufficient condition for this triangle to be a right triangle.

Use the contrapositive to rewrite the statements in 42 and 43 in if-then form in two ways.

42. Being divisible by 3 is a necessary condition for this number to be divisible by 9.

43. Doing homework regularly is a necessary condition for Jim to pass the course.

Note that "a sufficient condition for s is r" means r is a sufficient condition for s and that "a necessary condition for s is r" means r is a necessary condition for s. Rewrite the statements in 44 and 45 in if-then form.

44. A sufficient condition for Jon's team to win the championship is that it win the rest of its games.

45. A necessary condition for this computer program to be correct is that it not produce error messages during translation.

46. "If compound X is boiling, then its temperature must be at least 150°C." Assuming that this statement is true, which of the following must also be true?
   a. If the temperature of compound X is at least 150°C, then compound X is boiling.
   b. If the temperature of compound X is less than 150°C, then compound X is not boiling.
   c. Compound X will boil only if its temperature is at least 150°C.
   d. If compound X is not boiling, then its temperature is less than 150°C.
   e. A necessary condition for compound X to boil is that its temperature be at least 150°C.
   f. A sufficient condition for compound X to boil is that its temperature be at least 150°C.

In 47–50 (a) use the logical equivalences $p \to q \equiv \sim p \vee q$ and $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$ to rewrite the given statement forms without using the symbol $\to$ or $\leftrightarrow$, and (b) use the logical equivalence $p \vee q \equiv \sim(\sim p \wedge \sim q)$ to rewrite each statement form using only $\wedge$ and $\sim$.

47. $p \wedge \sim q \to r$

48. $p \vee \sim q \to r \vee q$

49. $(p \to r) \leftrightarrow (q \to r)$

50. $(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$

51. Given any statement form, is it possible to find a logically equivalent form that uses only $\sim$ and $\wedge$? Justify your answer.

equiv

Table 2.3.1 Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Test Yourself

1. For an argument to be valid means that every argument of the same form whose premises ____ has a ____ conclusion.
2. For an argument to be invalid means that there is an argument of the same form whose premises ____ and whose conclusion ____.
3. For an argument to be sound means that it is ____ and its premises ____. In this case we can be sure that its conclusion ____.

Exercise Set 2.3

Use modus ponens or modus tollens to fill in the blanks in the arguments of 1–5 so as to produce valid inferences.

1. If $\sqrt{2}$ is rational, then $\sqrt{2} = a/b$ for some integers a and b.
   It is not true that $\sqrt{2} = a/b$ for some integers a and b.
   $\therefore$ ____________________

2. If $1 - 0.99999\ldots$ is less than every positive real number, then it equals zero.
   ____
   $\therefore$ The number $1 - 0.99999\ldots$ equals zero.

3. If logic is easy, then I am a monkey's uncle.
   I am not a monkey's uncle.
   $\therefore$ ____________________

4. If this figure is a quadrilateral, then the sum of its interior angles is 360°.
   The sum of the interior angles of this figure is not 360°.
   $\therefore$ ____________________

5. If they were unsure of the address, then they would have telephoned.
   ____
   $\therefore$ They were sure of the address.

Use truth tables to determine whether the argument forms in 6–11 are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer. Your explanation should show that you understand what it means for a form of argument to be valid or invalid.

6. $p \to q$
   $q \to p$
   $\therefore p \vee q$

7. p
   $p \to q$
   $\sim q \vee r$
   $\therefore r$

8. $p \vee q$
   $p \to \sim q$
   $p \to r$
   $\therefore r$

9. $p \wedge q \to \sim r$
   $p \vee \sim q$
   $\sim q \to p$
   $\therefore \sim r$

ment are invalid.

a. $p \to q$ b. $p \to q$

q $\sim p$

$\therefore p$ $\therefore \sim q$

(converse error)   (inverse error)

Use truth tables to show that the argument forms referred to in 13–21 are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer. Your explanation should show that you understand what it means for a form of argument to be valid.

13. Modus tollens:

$$p \to q$$
$$\sim q$$
$$\therefore \sim p$$

14. Example 2.3.3(a)   15. Example 2.3.3(b)

16. Example 2.3.4(a)   17. Example 2.3.4(b)

18. Example 2.3.5(a)   19. Example 2.3.5(b)

20. Example 2.3.6   21. Example 2.3.7

Use symbols to write the logical form of each argument in 22 and 23, and then use a truth table to test the argument for validity. Indicate which columns represent the premises and which represent the conclusion, and include a few words of explanation showing that you understand the meaning of validity.

22. If Tom is not on team A, then Hua is on team B.
If Hua is not on team B, then Tom is on team A.
$\therefore$ Tom is not on team A or Hua is not on team B.

23. Oleg is a math major or Oleg is an economics major.
If Oleg is a math major, then Oleg is required to take Math 362.
$\therefore$ Oleg is an economics major or Oleg is not required to take Math 362.

Some of the arguments in 24–32 are valid, whereas others exhibit the converse or the inverse error. Use symbols to write the logical form of each argument. If the argument is valid, identify the rule of inference that guarantees its validity. Otherwise, state whether the converse or the inverse error is made.

24. If Jules solved this problem correctly, then Jules obtained the answer 2.
Jules obtained the answer 2.
$\therefore$ Jules solved this problem correctly.

25. This real number is rational or it is irrational.
This real number is not rational.
$\therefore$ This real number is irrational.

27. If this number is larger than 2, then its square is larger than 4.
This number is not larger than 2.
$\therefore$ The square of this number is not larger than 4.

28. If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite.
The set of all irrational numbers is infinite.
$\therefore$ There are as many rational numbers as there are irrational numbers.

29. If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
Neither of these two numbers is divisible by 6.
$\therefore$ The product of these two numbers is not divisible by 6.

30. If this computer program is correct, then it produces the correct output when run with the test data my teacher gave me.
This computer program produces the correct output when run with the test data my teacher gave me.
$\therefore$ This computer program is correct.

31. Sandra knows Java and Sandra knows C++.
$\therefore$ Sandra knows C++.

32. If I get a Christmas bonus, I'll buy a stereo.
If I sell my motorcycle, I'll buy a stereo.
$\therefore$ If I get a Christmas bonus or I sell my motorcycle, then I'll buy a stereo.

33. Give an example (other than Example 2.3.11) of a valid argument with a false conclusion.

34. Give an example (other than Example 2.3.12) of an invalid argument with a true conclusion.

35. Explain in your own words what distinguishes a valid form of argument from an invalid one.

36. Given the following information about a computer program, find the mistake in the program.
a. There is an undeclared variable or there is a syntax error in the first five lines.
b. If there is a syntax error in the first five lines, then there is a missing semicolon or a variable name is misspelled.
c. There is not a missing semicolon.
d. There is not a misspelled variable name.

37. In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humor and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a–e below) and challenged the reader to use them to figure out the location of the treasure.

a. If this house is next to a lake, then the treasure is not in the kitchen.
b. If the tree in the front yard is an elm, then the treasure is in the kitchen.
c. This house is next to a lake.
d. The tree in the front yard is an elm or the treasure is buried under the flagpole.
e. If the tree in the back yard is an oak, then the treasure is in the garage.

Where is the treasure hidden?

38. You are visiting the island described in Example 2.3.14 and have the following encounters with natives.

a. Two natives A and B address you as follows:
A says: Both of us are knights.
B says: A is a knave.
What are A and B?

b. Another two natives C and D approach you but only C speaks.
C says: Both of us are knaves.
What are C and D?

c. You then encounter natives E and F.
E says: F is a knave.
F says: E is a knave.
How many knaves are there?

H **d.** Finally, you meet a group of six natives, U, V, W, X, Y, and Z, who speak to you as follows:
U says: None of us is a knight.
V says: At least three of us are knights.
W says: At most three of us are knights.
X says: Exactly five of us are knights.
Y says: Exactly two of us are knights.
Z says: Exactly one of us is a knight.
Which are knights and which are knaves?

39. The famous detective Percule Hoirot was called in to solve a baffling murder mystery. He determined the following facts:

a. Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.
b. Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.
c. If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.
d. If Lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.
e. If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.
f. If Sara was in the dining room at the time the murder was committed, then the wine steward killed Lord Hazelton.

Is it possible for the detective to deduce the identity of the murderer from these facts? If so, who did murder Lord Hazelton? (Assume there was only one cause of death.)

40. Sharky, a leader of the underworld, was killed by one of his own band of four henchmen. Detective Sharp interviewed the men and determined that all were lying except for one. He deduced who killed Sharky on the basis of the following statements:

a. Socko: Lefty killed Sharky.
b. Fats: Muscles didn't kill Sharky.
c. Lefty: Muscles was shooting craps with Socko when Sharky was knocked off.
d. Muscles: Lefty didn't kill Sharky.

Who did kill Sharky?

In 41–44 a set of premises and a conclusion are given. Use the valid argument forms listed in Table 2.3.1 to deduce the conclusion from the premises, giving a reason for each step as in Example 2.3.8. Assume all variables are statement variables.

41. a. $\sim p \vee q \to r$
b. $s \vee \sim q$
c. $\sim t$
d. $p \to t$
e. $\sim p \wedge r \to \sim s$
f. $\therefore \sim q$

42. a. $p \vee q$
b. $q \to r$
c. $p \wedge s \to t$
d. $\sim r$
e. $\sim q \to u \wedge s$
f. $\therefore t$

43. a. $\sim p \to r \wedge \sim s$
b. $t \to s$
c. $u \to \sim p$
d. $\sim w$
e. $u \vee w$
f. $\therefore \sim t$

44. a. $p \to q$
b. $r \vee s$
c. $\sim s \to \sim t$
d. $\sim q \vee s$
e. $\sim s$
f. $\sim p \wedge r \to u$
g. $w \vee t$
h. $\therefore u \wedge w$

Answers for Test Yourself

2. Some ways to express the symbol $\forall$ in words are ____.

3. Some ways to express the symbol $\exists$ in words are ____.

5. A statement of the form $\exists x \in D$ such that $Q(x)$ is true if, and only if, $Q(x)$ is ____ for ____.

Exercise Set 3.1*

1. A menagerie consists of seven brown dogs, two black dogs, six gray cats, ten black cats, five blue birds, six yellow birds, and one black bird. Determine which of the following statements are true and which are false.
   a. There is an animal in the menagerie that is red.
   b. Every animal in the menagerie is a bird or a mammal.
   c. Every animal in the menagerie is brown or gray or black.
   d. There is an animal in the menagerie that is neither a cat nor a dog.
   e. No animal in the menagerie is blue.
   f. There are in the menagerie a dog, a cat, and a bird that all have the same color.

2. Indicate which of the following statements are true and which are false. Justify your answers as best as you can.
   a. Every integer is a real number.
   b. 0 is a positive real number.
   c. For all real numbers r, $-r$ is a negative real number.
   d. Every real number is an integer.

3. Let $P(x)$ be the predicate "$x > 1/x$."
   a. Write $P(2)$, $P(\frac{1}{2})$, $P(-1)$, $P(-\frac{1}{2})$, and $P(-8)$, and indicate which of these statements are true and which are false.
   b. Find the truth set of $P(x)$ if the domain of x is $\mathbf{R}$, the set of all real numbers.
   c. If the domain is the set $\mathbf{R}^+$ of all positive real numbers, what is the truth set of $P(x)$?

4. Let $Q(n)$ be the predicate "$n^2 \leq 30$."
   a. Write $Q(2)$, $Q(-2)$, $Q(7)$, and $Q(-7)$, and indicate which of these statements are true and which are false.
   b. Find the truth set of $Q(n)$ if the domain of n is $\mathbf{Z}$, the set of all integers.
   c. If the domain is the set $\mathbf{Z}^+$ of all positive integers, what is the truth set of $Q(n)$?

5. Let $Q(x, y)$ be the predicate "If $x < y$ then $x^2 < y^2$" with domain for both x and y being the set $\mathbf{R}$ of real numbers.
   a. Explain why $Q(x, y)$ is false if $x = -2$ and $y = 1$.
   b. Give values different from those in part (a) for which $Q(x, y)$ is false.
   c. Explain why $Q(x, y)$ is true if $x = 3$ and $y = 8$.
   d. Give values different from those in part (c) for which $Q(x, y)$ is true.

6. Let $R(m, n)$ be the predicate "If m is a factor of n^2 then m is a factor of n," with domain for both m and n being the set $\mathbf{Z}$ of integers.
   a. Explain why $R(m, n)$ is false if $m = 25$ and $n = 10$.
   b. Give values different from those in part (a) for which $R(m, n)$ is false.
   c. Explain why $R(m, n)$ is true if $m = 5$ and $n = 10$.
   d. Give values different from those in part (c) for which $R(m, n)$ is true.

7. Find the truth set of each predicate.
   a. predicate: $6/d$ is an integer, domain: $\mathbf{Z}$
   b. predicate: $6/d$ is an integer, domain: $\mathbf{Z}^+$
   c. predicate: $1 \leq x^2 \leq 4$, domain: $\mathbf{R}$
   d. predicate: $1 \leq x^2 \leq 4$, domain: $\mathbf{Z}$

8. Let $B(x)$ be "$-10 < x < 10$." Find the truth set of $B(x)$ for each of the following domains.
   a. $\mathbf{Z}$ b. $\mathbf{Z}^+$ c. The set of all even integers

Find counterexamples to show that the statements in 9–12 are false.

9. $\forall x \in \mathbf{R}, x > 1/x$.

10. $\forall a \in \mathbf{Z}$, $(a - 1)/a$ is not an integer.

11. $\forall$ positive integers m and n, $m \cdot n \geq m + n$.

12. $\forall$ real numbers x and y, $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$.

13. Consider the following statement:

$\forall$ basketball players x, x is tall.

Which of the following are equivalent ways of expressing this statement?
   a. Every basketball player is tall.
   b. Among all the basketball players, some are tall.
   c. Some of all the tall people are basketball players.
   d. Anyone who is tall is a basketball player.
   e. All people who are basketball players are tall.
   f. Anyone who is a basketball player is a tall person.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol H indicates that only a hint or a partial solution is given. The symbol $\ast$ signals that an exercise is more challenging than usual.

14. Consider the following statement:

$$\exists x \in \mathbf{R} \text{ such that } x^2 = 2.$$

Which of the following are equivalent ways of expressing this statement?
a. The square of each real number is 2.
b. Some real numbers have square 2.
c. The number x has square 2, for some real number x.
d. If x is a real number, then $x^2 = 2$.
e. Some real number has square 2.
f. There is at least one real number whose square is 2.

H 15. Rewrite the following statements informally in at least two different ways without using variables or quantifiers.
a. $\forall$ rectangles x, x is a quadrilateral.
b. $\exists$ a set A such that A has 16 subsets.

16. Rewrite each of the following statements in the form "$\forall$ ______ x, ______."
a. All dinosaurs are extinct.
b. Every real number is positive, negative, or zero.
c. No irrational numbers are integers.
d. No logicians are lazy.
e. The number 2,147,581,953 is not equal to the square of any integer.
f. The number -1 is not equal to the square of any real number.

17. Rewrite each of the following in the form "$\exists$ ______ x such that ______."
a. Some exercises have answers.
b. Some real numbers are rational.

18. Let D be the set of all students at your school, and let $M(s)$ be "s is a math major," let $C(s)$ be "s is a computer science student," and let $E(s)$ be "s is an engineering student." Express each of the following statements using quantifiers, variables, and the predicates $M(s)$, $C(s)$, and $E(s)$.
a. There is an engineering student who is a math major.
b. Every computer science student is an engineering student.
c. No computer science students are engineering students.
d. Some computer science students are also math majors.
e. Some computer science students are engineering students and some are not.

19. Consider the following statement:

$$\forall \text{ integers } n, \text{ if } n^2 \text{ is even then } n \text{ is even.}$$

Which of the following are equivalent ways of expressing this statement?
a. All integers have even squares and are even.
b. Given any integer whose square is even, that integer is itself even.
c. For all integers, there are some whose square is even.
d. Any integer with an even square is even.
e. If the square of an integer is even, then that integer is even.
f. All even integers have even squares.

H 20. Rewrite the following statement informally in at least two different ways without using variables or the symbol $\forall$ or the words "for all."

$\forall$ real numbers x, if x is positive, then the square root of x is positive.

21. Rewrite the following statements so that the quantifier trails the rest of the sentence.
a. For any graph G, the total degree of G is even.
b. For any isosceles triangle T, the base angles of T are equal.
c. There exists a prime number p such that p is even.
d. There exists a continuous function f such that f is not differentiable.

22. Rewrite each of the following statements in the form "$\forall$ ______ x, if ______ then ______."
a. All Java programs have at least 5 lines.
b. Any valid argument with true premises has a true conclusion.

23. Rewrite each of the following statements in the two forms "$\forall x$, if ______ then ______" and "$\forall$ ______ x, ______" (without an if-then).
a. All equilateral triangles are isosceles.
b. Every computer science student needs to take data structures.

24. Rewrite the following statements in the two forms "$\exists$ ______ x such that ______" and "$\exists x$ such that ______ and ______."
a. Some hatters are mad. b. Some questions are easy.

25. The statement "The square of any rational number is rational" can be rewritten formally as "For all rational numbers x, x^2 is rational" or as "For all x, if x is rational then x^2 is rational." Rewrite each of the following statements in the two forms "$\forall$ ______ x, ______" and "$\forall x$, if ______, then ______" or in the two forms "$\forall$ ______ x and y, ______" and "$\forall x$ and y, if ______, then ______."
a. The reciprocal of any nonzero fraction is a fraction.
b. The derivative of any polynomial function is a polynomial function.
c. The sum of the angles of any triangle is 180°.
d. The negative of any irrational number is irrational.
e. The sum of any two even integers is even.
f. The product of any two fractions is a fraction.

26. Consider the statement "All integers are rational numbers but some rational numbers are not integers."
a. Write this statement in the form "$\forall x$, if ______ then ______, but $\exists$ ______ x such that ______."
b. Let Ratl(x) be "x is a rational number" and Int(x) be "x is an integer." Write the given statement formally using only the symbols Ratl(x), Int(x), $\forall$, $\exists$, $\wedge$, $\vee$, $\sim$, and $\rightarrow$.

27. Refer to the picture of Tarski's world given in Example 3.1.13. Let Above(x, y) mean that x is above y (but possibly in a different column). Determine the truth or falsity