Analytic Properties of Shintani Zeta Functions

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January 19, 2010

A famous problem

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Some questions:

- How many K are there with fixed degree d and $|\Delta_K| < X$?
- How many with fixed degree and $|\Delta_K| = X$?

Asymptotics for cubic fields

Folk Conjecture: For each d,

$$n_d(X) \sim C_d X,$$

for some constant C_d.

Counting fields of low degree

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- $d = 4$: (Bhargava) $C_4 = \frac{5}{24} \prod_p (1 + p^{-2} - p^{-3} - p^{-4})$.

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- $d = 5$: (Bhargava) $C_5 = \frac{13}{120}\prod_p(1 + p^{-2} - p^{-4} - p^{-5})$.
- $d > 5$: Open. C_d conjectured by Bhargava; $n_d(X) \ll X^{n^\epsilon}$ due to Ellenberg-Venkatesh.

From cubic fields to cubic forms

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To count cubic fields:

- First count cubic *rings* and then apply a sieve method;
- Understand cubic rings by means of binary cubic forms.

Binary cubic forms

Let V be the space of binary cubic forms:

$$V := \{x(u,v) = x_1 u^3 + x_2 u^2 v + x_3 uv^2 + x_4 v^3;\ x_1, x_2, x_3, x_4 \in \mathbb{R}\}.$$

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$$(gx)(u,v) = x((u,v)g).$$

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$\mathrm{GL}_2(\mathbb{R})$ acts on V by

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This makes V into a **prehomogeneous vector space:** There are GL_2-orbits of the same dimension as V.

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Theorem (Delone-Faddeev, 1964)

There is a canonical, explicit bijection between the set of cubic rings up to isomorphism and the set of $\mathrm{GL}_2(\mathbb{Z})$*-equivalence classes of integral binary cubic forms.*

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Theorem (Delone-Faddeev, 1964)

There is a canonical, explicit bijection between the set of cubic rings up to isomorphism and the set of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral binary cubic forms.

This bijection preserves the discriminant.

Quartic and quintic rings and fields

Bhargava: Related but more sophisticated techniques apply.

Dirichlet series for cubic rings

Shintani 1972: Associate a Dirichlet series.

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Let $a(n) :=$ number of cubic rings of discriminant n.

Theorem (Shintani, 1972)

The Dirichlet series $\sum_{n \geq 1} a(n) n^{-s}$ and $\sum_{n \geq 1} a(-n) n^{-s}$ have meromorphic continuation to all of $\mathbb{C}$ and satisfy functional equations.

Binary cubic forms, cont.

L is the lattice of **integral** binary cubic forms:

$$L := \{x(u,v) = x_1 u^3 + x_2 u^2 v + x_3 u v^2 + x_4 v^3;\ x_1, x_2, x_3, x_4 \in \mathbb{Z}\}.$$

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$\widehat{L}$ is the **dual lattice**

$$L := \{x(u,v) = x_1 u^3 + x_2 u^2 v + x_3 uv^2 + x_4 v^3 \in L;\ x_2, x_3 \in 3\mathbb{Z}\}.$$

$\mathrm{SL}_2(\mathbb{Z})$ acts on L and $\widehat{L}$ and preserves the discriminant.

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The class number $h(n)$ (resp. $\widehat{h}(n)$) is the number of $\mathrm{SL}_2(\mathbb{Z})$-orbits on L (resp. $\widehat{L}$) of discriminant n.

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The class number $h(n)$ (resp. $\widehat{h}(n)$) is the number of $\mathrm{SL}_2(\mathbb{Z})$-orbits on L (resp. $\widehat{L}$) of discriminant n.

Technical point: Actually, we need to adjust for orbits with nontrivial stabilizers.

The Shintani zeta functions defined

Define

$$\xi_+(L, s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \qquad \xi_-(L, s) := \sum_{n=1}^{\infty} \frac{h(-n)}{n^s},$$

$$\xi_+(\widehat{L}, s) := \sum_{n=1}^{\infty} \frac{\widehat{h}(n)}{n^s}, \qquad \xi_-(\widehat{L}, s) := \sum_{n=1}^{\infty} \frac{\widehat{h}(-n)}{n^s},$$

the **Shintani zeta functions.**

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We have the formula

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- $c(\mathcal{O}) = 3$ if $\mathcal{O}$ is three multiples of $\mathbb{Z}$.
- If we limit $\mathcal{O} \otimes \mathbb{Q}$ to direct sums of cyclic Galois extensions, this can be understood by other means.

Theorem (Shintani, 1972)

The above series converge absolutely for $\Re(s) > 1$, have meromorphic continuation to all of $\mathbb{C}$ with poles at $s = 1$ and $s = 5/6$, and satisfy the functional equation

$$\begin{pmatrix} \xi_+(L, 1-s) \\ \xi_-(L, 1-s) \end{pmatrix} = \Gamma\left(s - \frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s + \frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s} \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \xi_+(\widehat{L}, s) \\ \xi_-(\widehat{L}, s) \end{pmatrix}. \quad (1)$$

Diagonalization and Simplification

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Also, Ohno conjectured, and Nakagawa proved, the relations

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$$\xi_-(\widehat{L}, s) = 3^{1-3s}\xi_+(L, s).$$

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Putting all this together...

The "Nice" Shintani zeta function

Write

$$\xi_{\mathrm{add}}(s) := 3^{1/2}\xi_+(L, s) + \xi_-(L, s),$$
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$$\Lambda_{\mathrm{add}}(s) := \left(\frac{432}{\pi^4}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{12}\right)\Gamma\left(\frac{s}{2}-\frac{1}{12}\right)\xi_{\mathrm{add}}(s),$$
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Then

$$\Lambda_{\mathrm{add}}(s) = \Lambda_{\mathrm{add}}(1-s),$$
$$\Lambda_{\mathrm{sub}}(s) = \Lambda_{\mathrm{sub}}(1-s).$$

Both have poles at $s=1$, and Λ_{add} also has a pole at $s=5/6$.

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Today: *begin* to answer these questions.

The location of the zeros

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$$N(T) = \frac{T}{\pi} \log \left(\frac{432 T^4}{(2\pi e)^4} \right) + O(\log T).$$

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Folliows by Stirling's formula and contour integration.

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$$0.5 + 4.745125599327 \cdots i$$

$$0.5 + 6.962286575567 \cdots i$$

$$0.5 + 8.4742944491274 \cdots i$$

$$0.5 + 10.152261066735 \cdots i$$

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$$0.81420 \cdots + 7.05984 \cdots i.$$

Euler products and non-Euler products

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$$\sum_{(u,v)\neq(0,0)} (au^2 + buv + cv^2)^{-s}.$$

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For $a, b, c \in Z$ and $b^2 - 4ac < 0$, special case of above.

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Open question: Are almost all the zeros on $Re(s) = 1/2$?

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See Hejhal, *Epstein zeta functions and supercomputers*.

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The good news:

- We can try to work in general.

Zeros outside the critical strip

Theorem (Soundararajan-T.)

The Shintani zeta functions ξ_{add}, ξ_{sub}, *and* ξ_{-} *have infinitely many zeros to the right of* $\Re(s) = 1$.

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Our results follow from a more general method.

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If a zeta function has zeros in $\Re s > 1$, then this *proves* the zeta function doesn't have an Euler product.

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Stark: Solve the class number 1 problem this way.

Stark's challenge

Theorem (Folk Theorem)

Write

$$\zeta(s, Q) = \sum_{(u,v)\neq(0,0)} (u^2 + uv + cv^2)^{-s}.$$

If $c > 41$, $\zeta(s, Q)$ *has a zero* s *with* $\sigma > 1$.

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Problems:

- The only "proof" uses the class number one determination.
- It hasn't been proved at all for algebraic numbers other than integers.

Stark's challenge (cont.)

"So the challenge is clear: prove the folk theorem, but better still, FIND A PURELY ANALYTIC PROOF OF THE FOLK THEOREM."

Stark's challenge (cont.)

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Stark's all caps, not mine.

Sketch proof of theorem

It suffices to find *one* zero outside the critical strip.

Sketch proof of theorem

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- For ξ_{-}, we have to work harder.

A zero of $\xi_-(s)$ in $\Re s > 1$

To "find a zero":

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- If we can, then we have zeros outside the critical strip.

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For appropriate χ,

$$\sum_{n \leq 10^6} a(n)\chi(n)n^{-1.3} = -.162\cdots, \quad \sum_{n > 10^6} |a(n)|n^{-1.3} < 0.1.$$

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Conclusion: Compute 100,000,000,000,000 coefficients and try again.

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We have technical partial results in this direction.

All the Shintani zeta functions

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Cubic rings and sieve methods

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for a **multiplicative** $\omega(d)$ and any $\alpha < 1$.

Or even better:

$$\sum_{n \leq X; d | n} b(n) = C\omega(d)X + C'\nu(d)X^{5/6} + O(X^{\alpha}),$$

where $\nu(d)$ is also multiplicative and $\alpha < 5/6$.

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- An analytic proof should detect the secondary term.

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Proposition (preliminary)

We have

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as before, with $\alpha = 3/5 + \epsilon$.

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Ongoing work: Reduce or eliminate the dependence on d.

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Note: Taniguchi has also considered this.

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- Possible generalization to quartic and quintic fields.

The Fourier transform

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Question: Is this Fourier transform something “nice”?

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Ellenberg and Venkatesh:

$$a(n) \ll n^{1/3+\epsilon}.$$

Improving this would allow all sorts of analytic techniques to work.

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- The hope: use this to prove something.