Team Round – University of South Carolina Math Contest, 2022

1. This is a team round. You have one hour to solve these problems as a team, and you should submit one set of answers for your team as a whole. Working together is, of course, encouraged.
2. Drawings are not necessarily drawn to scale.
3. All answers are **positive integers.**
4. There are ten questions, all independent from each other. Some are difficult. Do not be discouraged if you don't get them right away.
5. Some of the solutions involve some trial and error. None of them is intended to be solved using *only* trial and error, although that may sometimes work too.
6. **No calculators, books, notes, googling, asking people who aren't your teammates, etc.**
7. At the conclusion, send solutions by **private** Zoom chat or to `thorne@math.sc.edu`.

GOOD LUCK!

1. Define a sequence of integers by $S(1) = 1$, and
$$S(n) = S(1) + \cdots + S(n-1) + n$$
for each integer $n \geq 2$. What is the smallest integer n for which $S(n) > 2022$?

2. In the game of *Fizz Buzz*, you begin counting $1, 2, 3, \ldots$. If a number is divisible by 3 you say 'Fizz' instead of the number, and if a number is divisible by 5 you say 'Buzz' instead of the number. If a number is divisible by both then you say 'Fizz Buzz'.

   The tenth number counted is 17. What will be the 2022th number counted?

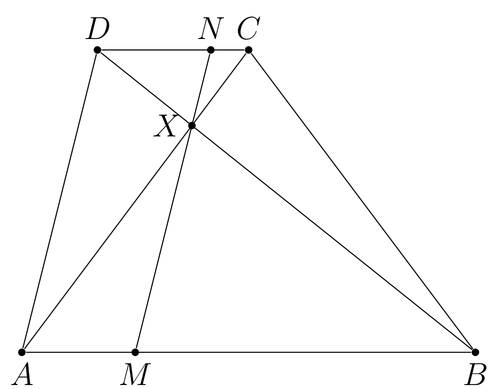
3. Consider an ordinary sheet of graph paper, with a grid defined by lines $x = m$ and $y = n$ for every integer m and n.

   How many times does the line segment between $(-20, -22)$ and $(20, 22)$ intersect the grid?

4. What is the nearest integer to
$$\log_2(2022) + \log_{20}(2022) + \log_{202}(2022) + \log_{2022}(2022)\ ?$$
(*Hint: Don't try to prove that your answer is correct.*)

5. In the trapezoid $ABCD$ with bases AB and DC, let X be the intersection of its diagonals, AC and BD. Let M be a point on the base AB and let the line MX intersect the base CD at N with $\dfrac{AM}{MB} = \dfrac{NX}{XM} = \dfrac{1}{3}$.

   Writing the ratio of the areas of the trapezoids $\dfrac{S_{AMND}}{S_{MBCN}}$ as a fraction $\frac{a}{b}$ in lowest terms, find $a + b$.



6. In a 3×3 grid of numbers, each row, column, and diagonal has the same sum. If four of the numbers are as shown below, what is the sum of the missing five numbers?

20	10	
22	18	

7. You might know the identity

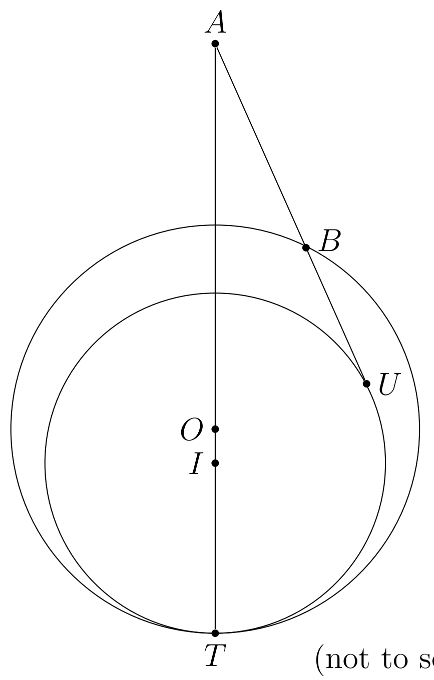
$$\cos(2\theta) = 2\cos^2(\theta) - 1.$$

Similarly, there is a unique degree 6 polynomial f for which we can write $\cos(6\theta) = f(\cos(\theta))$. What is the sum of the coefficients of f?

8. A circular table has eight places. In how many ways can you place one or more plates at a subset of the places, if you can't place two plates at adjacent places?

9. There exists a unique degree 4 polynomial f with integer coefficients and leading coefficient 1, which has $\sqrt{2}+\sqrt{3}$ as a root. What is the sum of the absolute values of the coefficients of f?

10. The circle of center O and radius R and the circle of center I and radius r, $r < R$, are tangent internally at the point T. Let A be a point on the line TI, with I between A and T, such that $\frac{AI}{IT} = \frac{5}{3}$, and let the tangent from A to the circle of center I at U, intersect the circle of center O at B with B between A and U. If $AB = \frac{7}{5}$ and $BU = \frac{3}{5}$, find the inverse of the distance between the centers, $\frac{1}{OI}$.



(not to scale)