Examination 1 - Math 141, Frank Thorne (thornef@mailbox.sc.edu)

Wednesday, September 23, 2015, 10:50 a.m.

Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper; you should staple this sheet to your work (put this on top) and turn in everything together.

The first question is 16 points and the last six are 14 each.

(1) Give the definition of the *derivative* of a function $f(x)$ at the point $x = a$. (Please give the algebraic definition, using an equation.)

Draw a picture and explain why your equation gives the slope of the tangent line to the graph of $f(x)$ at $x = a$.

(2) Compute

$$\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 - 1}.$$

(3) Compute

$$\lim_{x \to -\infty} \left(\frac{1 - x^3}{x^2 + 7x} \right)^5.$$

(4) Differentiate the function

$$f(x) = x + \frac{9}{x},$$

and find the slope of the tangent line at $x = -3$.

For this problem, use the definition of the derivative or the "alternative formula for the derivative" described in Section 3.2. Do not use the power rule or the quotient rule.

(5) Differentiate:

$$g(x) = \frac{x^2 - 4}{x + 0.5}.$$

For this problem, you may apply any relevant differentiation rules which you know.

(6) Find $\frac{dy}{dx}$ if

$$y = \frac{\cos x}{1 + \sin x}.$$

For this problem, you may apply any relevant differentiation rules which you know.

(7) (See overhead.)

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Wednesday, September 23, 2015, noon

Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper; you should staple this sheet to your work (put this on top) and turn in everything together.

The first question is 16 points and the last six are 14 each.

(1) Compute

$$\lim_{t \to -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}.$$

(2) Compute

$$\lim_{x \to \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}.$$

(3) Give the definition of the *derivative* of a function $f(x)$ at the point $x = a$. (Please give the algebraic definition, using an equation.)

Draw a picture and explain why your equation gives the slope of the tangent line to the graph of $f(x)$ at $x = a$.

(4) Differentiate:

$$f(t) = \frac{t^2 - 1}{t^2 + t - 2}.$$

For this problem, you may apply any relevant differentiation rules which you know.

(5) Differentiate the function

$$f(x) = x + \frac{9}{x},$$

and find the slope of the tangent line at $x = -3$.

For this problem, use the definition of the derivative or the "alternative formula for the derivative" described in Section 3.2. Do not use the power rule or the quotient rule.

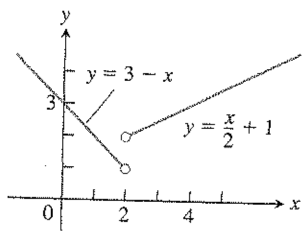
(6) Find $\frac{dy}{dx}$ if

$$y = \frac{4}{\cos x} + \frac{1}{\tan x}.$$

For this problem, you may apply any relevant differentiation rules which you know.

(7) (See overhead.)

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2. \end{cases}$



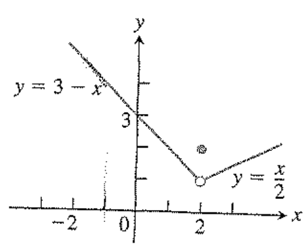
a. Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.

b. Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

d. Does $\lim_{x \to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \frac{x}{2}, & x > 2. \end{cases}$



a. Find $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$, and $f(2)$.

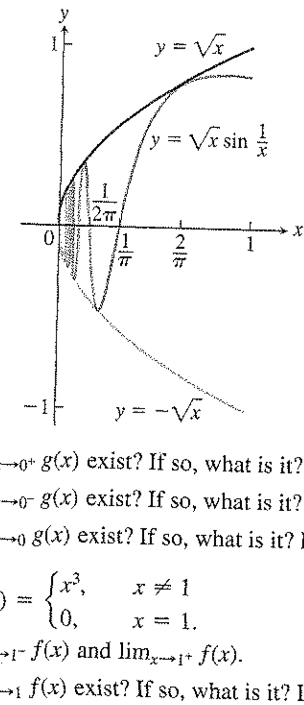
b. Does $\lim_{x \to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$.

d. Does $\lim_{x \to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \frac{1}{x}, & x > 0. \end{cases}$

6. Let $g(x) = \sqrt{x} \sin(1/x)$.



a. Does $\lim_{x \to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x \to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x \to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

c. Does $\lim_{x \to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

c. At what points does only the left-hand limit exist?

d. At what points does only the right-hand limit exist?

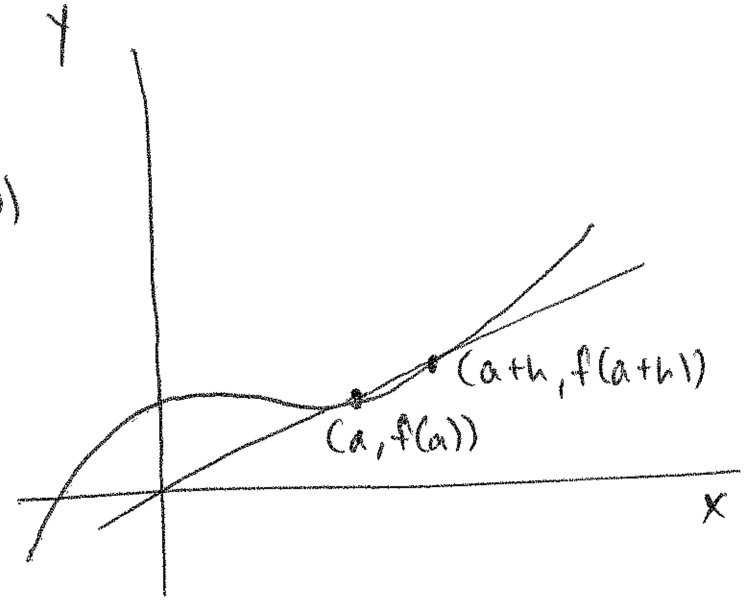
9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

(HW 4, prob. (c)) Problem 1.5.

The <u>derivative</u> of $f(x)$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$



The slope of the graph between $(a, f(a))$ and $(a+h, f(a+h))$ is

$$\frac{\text{rise}}{\text{run}} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$= \frac{f(a+h) - f(a)}{h},$$

the quantity inside the limit. As h approaches zero, the two points both approach $(a, f(a))$, so that the line in the picture approaches the tangent line to the graph at $(a, f(a))$. Thus, the derivative is the limit of this slope, i.e. the slope of the tangent line.

12:00, #1. (2.2, #28)

$$\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2} = \lim_{t\to -1} \frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim_{t\to -1} \frac{t+2}{t-2} = \frac{-1+2}{-1-2}$$
$$= \frac{1}{-3} = \boxed{\frac{-1}{3}}$$

#2. (2.6 # 23)

$$\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}} = \lim_{x\to\infty} \sqrt{\frac{\frac{8x^2}{x^2} - \frac{3}{x^2}}{\frac{2x^2}{x^2} + \frac{x}{x^2}}} = \lim_{x\to\infty} \sqrt{\frac{8 - \frac{3}{x^2}}{2 + \frac{1}{x}}}$$
$$= \sqrt{\frac{8-0}{2+0}} = \sqrt{4} = 2.$$

#3. Appeared also on 10:50 exam. See there

#4. $\frac{d}{dt}\left(\frac{t^2-1}{t^2+t-2}\right) = \frac{(t^2+t-2)\frac{d}{dt}(t^2-1) - (t^2-1)\frac{d}{dt}(t^2+t-2)}{(t^2+t-2)^2}$

(3.3, #20)

$$= \frac{(t^2+t-2)(2t) - (t^2-1)(2t+1)}{(t^2+t-2)^2}$$

$$= \frac{2t^3+2t^2-4t-2t^3+2t-t^2+1}{(t^2+t-2)^2}$$

$$= \frac{t^2-2t+1}{(t^2+t-2)^2} = \frac{(t-1)^2}{((t-1)(t+2))^2}$$

11 pts. if you get this far.

$$= \frac{1}{(t+2)^2}.$$

OR: $\frac{d}{dt}\left(\frac{t^2-1}{t^2+t-2}\right) = \frac{d}{dt}\left(\frac{(t-1)(t+1)}{(t-1)(t+2)}\right) = \frac{d}{dt}\left(\frac{t+1}{t+2}\right)$

$$= \frac{(t+2)\frac{d}{dt}(t+1) - (t+1)\frac{d}{dt}(t+2)}{(t+2)^2}$$

$$= \frac{t+2-(t+1)}{(t+2)^2} = \frac{1}{(t+2)^2}.$$

#5. Appeared also on 10:50 exam, see there

#6. (3.5 #13)

Sol'n 1

$$\frac{d}{dx}\left(\frac{4}{\cos x} + \frac{1}{\tan x}\right)$$

$$= \frac{(\cos x)\frac{d}{dx}(4) - 4\frac{d}{dx}(\cos x)}{\cos^2 x} + \frac{(\tan x)\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(\tan x)}{\tan^2 x}$$

$$= \frac{(\cos x)\cdot 0 - 4\cdot(-\sin x)}{\cos^2 x} + \frac{(\tan x)\cdot 0 - 1\cdot \sec^2 x}{\tan^2 x}$$

$$= \frac{4\sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x}$$

$$= \frac{4\sin x}{\cos^2 x} - \frac{1/\cos^2 x}{\sin^2 x/\cos^2 x}$$

$$= \frac{4\sin x}{\cos^2 x} - \frac{1}{\sin^2 x}.$$

Sol'n 2. Since $\frac{1}{\cos x} = \sec x$ and $\frac{1}{\tan x} = \cot x$

the answer is

$$\frac{d}{dx}(4\sec x + \cot x) = 4\sec x\tan x - \csc^2 x$$

which is the same answer in a different form.

10:50/2.

(2.2 #27) $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1} = \lim_{t\to 1} \frac{(t-1)(t+2)}{(t-1)(t+1)}$

$= \lim_{t\to 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \boxed{\frac{3}{2}}$

3. $\lim_{x\to -\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5 = \lim_{x\to -\infty} \left(\frac{\frac{1}{x^2} - \frac{x^3}{x^2}}{\frac{x^2}{x^2} + \frac{7x}{x^2}}\right)^5$

(2.6 #25)

$= \lim_{x\to -\infty} \left(\frac{\frac{1}{x^2} - x}{1 + \frac{7}{x}}\right)^5$

$= \lim_{x\to -\infty} \left(\frac{-x}{1}\right)^5 = \lim_{x\to -\infty} (-x)^5$

because $\frac{1}{x^2}$ and $\frac{7}{x} \to 0$ as $x \to -\infty$.

As $x \to -\infty$, this is the fifth power of a positive number which gets larger and larger, so

$$\lim_{x\to -\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5 = \infty.$$

4. (3.2 #13)

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h) + \frac{9}{x+h} - x - \frac{9}{x}}{h}$

$= \lim_{h\to 0} \frac{h + \frac{9(x) - 9(x+h)}{(x+h)x}}{h}$

$= \lim_{h\to 0} \frac{h + \frac{9x - 9x - 9h}{(x+h)x}}{h}$

$= \lim_{h\to 0} \frac{h(x+h)x - 9h}{h(x+h)x}$

10:50, #4 cont.

$$= \lim_{h \to 0} \frac{(x+h)x - 9}{(x+h)x} = \frac{x^2 - 9}{x^2}.$$

$$\text{So } f'(-3) = \frac{(-3)^2 - 9}{(-3)^2} = \frac{9-9}{9} = 0$$

and so the slope of the tangent line at $x = -3$ is 0.

#5. $$\frac{d}{dx}\left(\frac{x^2-4}{x+0.5}\right) = \frac{(x+0.5)\frac{d}{dx}(x^2-4) - (x^2-4)\frac{d}{dx}(x+0.5)}{(x+0.5)^2}$$

(3.3, #19)

$$= \frac{(x+0.5)(2x) - (x^2-4)\cdot 1}{(x+0.5)^2}$$

$$= \frac{2x^2 + x - x^2 + 4}{(x+0.5)^2} = \frac{x^2+x+4}{(x+0.5)^2}.$$

#6. If $y = \frac{\cos x}{1+\sin x}$, $$\frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\cdot\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

(3.5 #12)

$$= \frac{(1+\sin x)(-\sin x) - \cos x \cdot \cos x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x}.$$

problem 7 (comes pg 4)

3. $\lim_{x \to 2^+} f(x) = 2$ and $\lim_{x \to 2^-} f(x) = 1$ because these are the values which $f(x)$ is approaching from the right and left respectively.

$\lim_{x \to 2} f(x)$ does not exist because those two values are different.

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = \lim_{x \to 4} f(x) = 3$ because f is continuous at $x = 4$ and the function approaches 3 no matter how $x = 4$ is approached.

4. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) = 1$ because the value of this function approaches 1 as x approaches 2, no matter from which direction x approaches 2.

But $f(2) = 2 \neq 1$.

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = \lim_{x \to -1} f(x) = 4$

because f is continuous at $x = -1$ and the function approaches 4 no matter from which direction x approaches -1.