Linear Transformations, and matrices.

Def. (p. 400) Let V and U be two vector spaces. A function $T : V \to U$ is a <u>linear</u> <u>transformation</u> if, for all $\vec{v_1}, \vec{v_2} \in V$ and $c \in V$, we have

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$$

$$T(c\vec{v_1}) = c\,T(\vec{v_1}).$$

<u>Examples with</u> $\mathbb{R}^2$.

Define $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

T_2 $\quad T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}.$$

$$T_4\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}.$$

Draw pictures on board.

$$T = T_5\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \cancel{\frac{\sqrt{3}}{2}x + \frac{1}{2}x} \begin{bmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{bmatrix}.$$

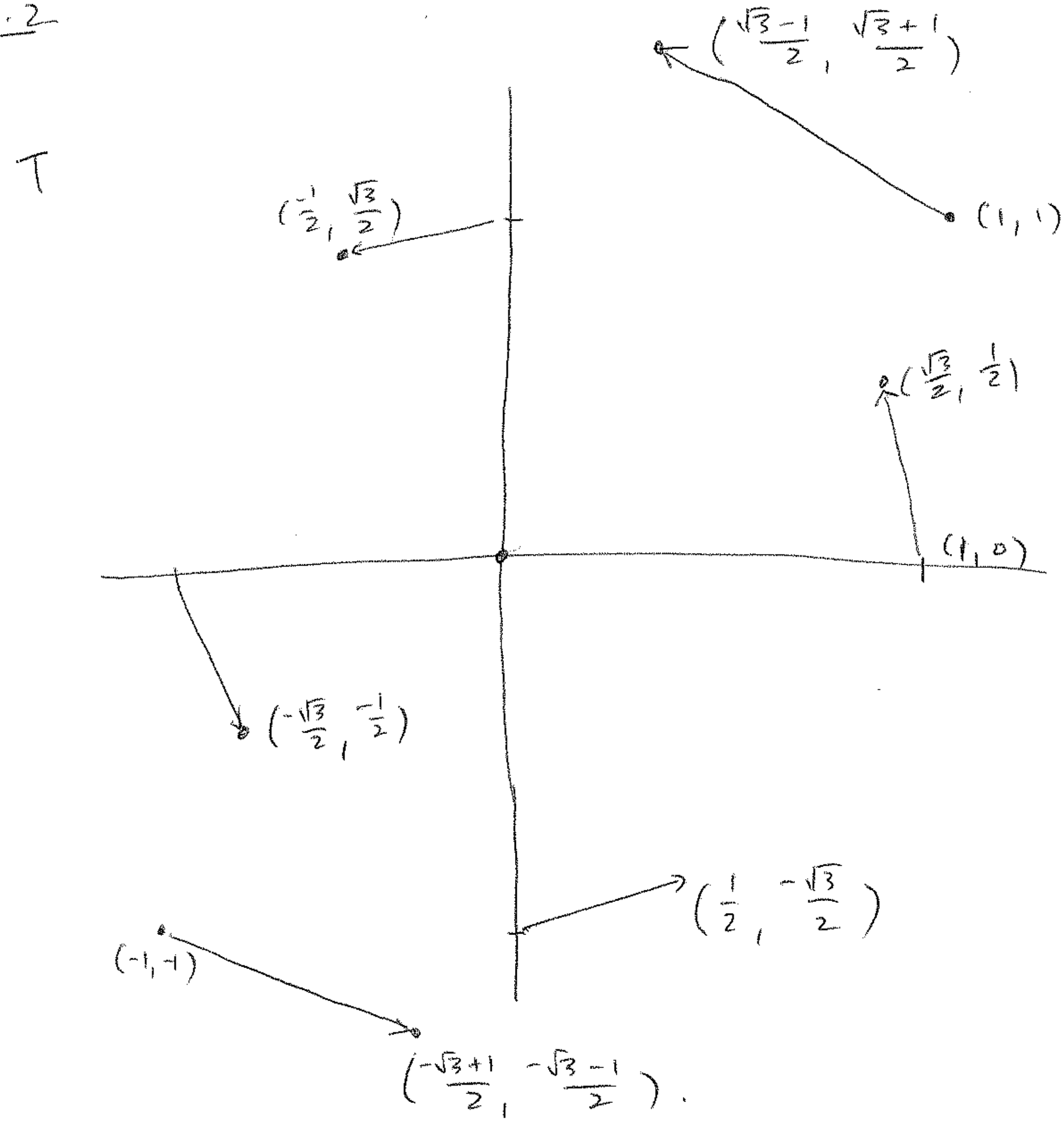
Can we draw a picture?

$$T_5\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$T_5\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$T_5\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \quad T_5\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$T_5\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T_5\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T_5\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}+1}{2} \end{bmatrix}$$



~~Notation Suppose~~

Proposition. T is rotation by 30 degrees.

Proof. Need to check two things:

(1) $|T(\vec{v})| = |\vec{v}|$ for all $\vec{v} \in \mathbb{R}^2$.

(2) $T(\vec{v}) \cdot \vec{v} = |T(\vec{v})| \cdot |\vec{v}| \cos(30^\circ)$.

If $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $|\vec{v}|^2 = x^2 + y^2$,

$$|T(\vec{v})|^2 = \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^2 + \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2$$

$$= \frac{3}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{1}{4}y^2 + \frac{1}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2$$

$$= x^2 + y^2.$$

(2) $T(\vec{v}) \cdot \vec{v} = x \cdot \left(\frac{\sqrt{3}}{2} x - \frac{1}{2} y \right) + y \cdot \left(\frac{1}{2} x + \frac{\sqrt{3}}{2} y \right)$

$= \frac{\sqrt{3}}{2} x^2 - \frac{1}{2} xy + \frac{1}{2} xy + \frac{\sqrt{3}}{2} y^2 = \frac{\sqrt{3}}{2} (x^2 + y^2).$

Exercise. Do the same for

T: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} x + \frac{1}{2} y \\ -\frac{1}{2} x + \frac{\sqrt{3}}{2} y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y \\ \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$.

Figure out what angle rotation they are and prove it in the same manner.

Extra Credit. Figure out the pattern and write down a matrix representing rotation by θ.

Example. Do the same for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}$.

Proposition. Linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ are those which we can write of the form

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \quad \text{for some } a, b, c, d \in \mathbb{R}.$$

Notation. The associated matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ for above.

Observations.

(1) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is any linear transformation,
$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(2) what if T sends every vector to 0?

Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0x + 0y \\ 0x + 0y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Conversely: Suppose $\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$.

Then plug in $x = 1, y = 0$: $a = c = 0$.
Plug in $x = 0, y = 1$: $b = d = 0$.

(3) Suppose $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ are nonzero and parallel,

i.e. $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \vec{w}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \lambda \vec{w}$ for some $\lambda \in \mathbb{R}$.

Then
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= x \cdot \vec{w} + y \cdot \lambda \vec{w} = (x + y\lambda)\vec{w}.$$

This means <u>the image of T lies on a line</u>

(4) Suppose $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ are linearly independent.

Now $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = x\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + y\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

Three observations.

* What if this is 0? <u>BY HYPOTHESIS</u> $x = y = 0$.

  So then $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = 0$ and $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) \neq 0$ if $\begin{bmatrix}x\\y\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$

* Saw before, $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ span $\mathbb{R}^2$.

  This means, <u>the range of T is all of $\mathbb{R}^2$</u>.

* Now, suppose $T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right) = T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)$.

  Then $$T\left(\begin{bmatrix}x_1-x_2\\y_1-y_2\end{bmatrix}\right) = T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right) + T\left(\begin{bmatrix}-x_2\\-y_2\end{bmatrix}\right)$$
  $$= T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right) - T\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) = 0.$$

  By linear independence, $\left.\begin{matrix}x_1 - x_2 = 0\\ y_1 - y_2 = 0\end{matrix}\right\}$ i.e. $x_1 = x_2$, $y_1 = y_2$.

This means T is one-to-one and onto.

If T has matrix $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, its <u>inverse</u> has matrix
$$\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}.$$

Suppose we have an $m \times n$ matrix

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}$$

This represents a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. If we write T for it,

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{m,1} \end{bmatrix}$$

n entries

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{1,2} \\ \vdots \\ a_{m,2} \end{bmatrix}$$

$$\vdots$$

$$T\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{bmatrix}.$$

Therefore,

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + \cdots + x_n T\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 a_{1,1} + \cdots + x_n a_{1,n} \\ \vdots \\ x_1 a_{m,1} + \cdots + x_n a_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Example. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the LT with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}5\\3\\2\\6\end{bmatrix} \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\9\\8\end{bmatrix} \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\5\\3\\1\end{bmatrix}.$$

The associated matrix is $\begin{bmatrix}5 & 1 & 4\\3 & 1 & 5\\2 & 9 & 3\\6 & 8 & 1\end{bmatrix}$.

And, $$T\left(\begin{bmatrix}2\\0\\3\end{bmatrix}\right) = \begin{bmatrix}5 & 1 & 4\\3 & 1 & 5\\2 & 9 & 3\\6 & 8 & 1\end{bmatrix}\begin{bmatrix}2\\0\\3\end{bmatrix} = \begin{bmatrix}22\\21\\13\\15\end{bmatrix}.$$

Example. Compute all $\vec{v}$ for which $T(\vec{v}) = 0$.
This is the *nullspace* or *kernel* of T.

Solution. It's all $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3$ with

$$\begin{bmatrix}5 & 1 & 4\\3 & 1 & 5\\2 & 9 & 3\\6 & 8 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

i.e.

$$\begin{aligned} 5x_1 + x_2 + 4x_3 &= 0\\ 3x_1 + x_2 + 5x_3 &= 0\\ 2x_1 + 9x_2 + 3x_3 &= 0\\ 6x_1 + 8x_2 + x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c}5 & 1 & 4 & 0\\3 & 1 & 5 & 0\\2 & 9 & 3 & 0\\6 & 8 & 1 & 0\end{array}\right].$$

You know how to do this!

10/21, p. 3. ~~10/24 p. 1:~~ More HW: 6.1 A Q1 (a–c), 12–14.

Multiplying matrices.

Given:

$m \times n$ matrix $\quad\quad$ $n \times r$ matrix

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \quad \begin{bmatrix} b_{1,1} & \cdots & b_{1,r} \\ \vdots & & \vdots \\ b_{n,1} & \cdots & b_{n,r} \end{bmatrix}$$

The result is the $m \times r$ matrix.

Entry in row i and column j is $(\text{Row } i)^T \cdot (\text{Column } j)$.

Note that if you regard a vector in $\mathbb{R}^n$ as an $n \times 1$ matrix, matches above.

These stand for functions

$$T_1 : \mathbb{R}^n \to \mathbb{R}^m \qquad T_2 : \underset{\text{\# of columns}}{\mathbb{R}^r} \to \underset{\text{\# of rows}}{\mathbb{R}^n}$$

and the product of the matrices corresponds to the <u>composite function</u>

$$T_1 \circ T_2 : \mathbb{R}^r \to \mathbb{R}^m$$

$$T_1 \circ T_2(\vec{v}) = \underbrace{T_1(\underbrace{T_2(\underset{\text{in } \mathbb{R}^r}{\vec{v}})}_{\text{in } \mathbb{R}^n})}_{\text{in } \mathbb{R}^m}.$$

Matrices and probability.

Suppose:

If ~~today~~ one day is sunny, ~~tomorrow~~ next day will be $\begin{cases} \text{cloudy} & (20\% \text{ chance}) \\ \text{sunny} & (80\% \text{ chance}) \end{cases}$

If one day is cloudy, next day is $\begin{cases} \text{cloudy} & (50\% \text{ chance}) \\ \text{sunny} & (50\% \text{ chance}) \end{cases}$

Ex. Suppose there is a 70% chance Saturday will be sunny. What is the probability Sunday is?

Ans. $\underbrace{0.7 \cdot 0.8}_{\substack{\text{Prob Sat and} \\ \text{Sun both sunny}}} + \underbrace{0.3 \cdot 0.5}_{\substack{\text{Prob Sat cloudy} \\ \text{Sun sunny}}} = 0.71.$

Ex. Today is cloudy. Prob it will be cloudy in two days?

Figure tomorrow first (0.5)

Then two days: $\underbrace{0.5 \cdot 0.5}_{\substack{\text{tomorrow,} \\ \text{two days from now} \\ \text{both sunny}}} + \underbrace{0.5 \cdot 0.2}_{\substack{\text{tomorrow is} \\ \text{sunny} \\ \text{two days from now} \\ \text{cloudy}}} = 0.35.$

To keep track of this, write down a stochastic transition matrix

$$\begin{array}{c c c} & \text{Today} & \\ & \begin{array}{cc} \text{Sunny} & \text{Cloudy} \end{array} & \\ & \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} & \begin{array}{l} \text{Sunny} \\ \text{Cloudy} \end{array} \quad \text{Tomorrow} \end{array}$$

Columns sum to 1, because no matter what today is, prob 100% that something happens tomorrow.

Do above computations w/ matrices.

e.g. if Saturday is sunny w/ prob. 0.7, cloudy w/ prob 0.3 corresponds to a vector $\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

So Sunday probabilities are $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$.

Monday probabilities are $\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 0.74 & 0.65 \\ 0.26 & 0.35 \end{bmatrix}}_{\text{This is a transition matrix that predicts two days in advance.}} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}.$$

Predict three days in advance: $\begin{bmatrix} 0.74 & 0.65 \\ 0.26 & 0.35 \end{bmatrix} \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$

$$= \begin{bmatrix} 0.722 & 0.695 \\ 0.278 & 0.305 \end{bmatrix}$$

and so on.

This is a Markov chain. Consists of:

* A finite set of states. The "system" can be in exactly one at a time.
* A transition matrix, describing the chances that the system goes from state i to state j, as above.
* Transitions are "memoryless".
  e.g. Assume (wrongly) ~~today~~ tomorrow's weather depends only on today.

10/25 p.4.

Further examples.

Politics. Say, ~~three~~ two states : ~~[illegible]~~
A certain district votes democratic or republican.

From

$$\begin{array}{cc}
\begin{array}{cc} D & R \end{array} & \\
\begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} & \begin{array}{c} D \\ R \end{array}
\end{array} \quad \text{To.}$$

Question. What is $\begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}^{10}$? Can you guess?

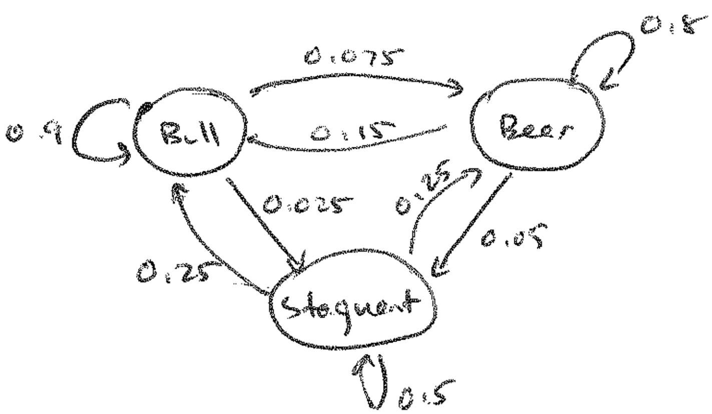
$$\begin{bmatrix} 0.50005\ldots & 0.49999\ldots \\ 0.49999\ldots & 0.50005\ldots \end{bmatrix}$$

Indeed, $\displaystyle\lim_{n\to\infty} \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}^{n} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$.

Stock market : A given week can be a bull market (prices up)
beer market (down)
flat.

Transition matrix

$$\cancel{\begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & \end{bmatrix}} \quad \begin{bmatrix} 0.9 & 0.15 & 0.25 \\ 0.075 & 0.8 & 0.25 \\ 0.025 & 0.05 & 0.5 \end{bmatrix}$$



A transition diagram.

10/25 p.5. (Really 10/27)

Def. If T is a linear transformation from a vector space V to itself, a *steady state* vector $\vec{v}$ is a vector with $T(\vec{v}) = \vec{v}$.

[Later we will be interested in eigenvectors, satisfying the more general property $T(\vec{v}) = \lambda \vec{v}$ for some real λ.]

How to find them?

Write down a matrix M corresponding to T.

Solve $M\vec{v} = \vec{v}$.

Let I be the identity matrix
$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots 0 & 1 \end{bmatrix}$$

which satisfies $I\vec{v} = \vec{v}$ for every $\vec{v}$.

So this is the same as solving
$$M\vec{v} = I\vec{v}$$
or
$$(M - I)\vec{v} = \vec{0}.$$

Ex. With our weather Markov chain we had $M = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$

$$M - I = \begin{bmatrix} -0.2 & 0.5 \\ 0.2 & -0.5 \end{bmatrix}$$ (subtract 1 from diagonal)

Solve $\begin{bmatrix} -0.2 & 0.5 \\ 0.2 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -0.2 & 0.5 & 0 \\ 0.2 & -0.5 & 0 \end{array}\right] \xrightarrow[\text{R2}]{\text{Add R1 to}} \left[\begin{array}{cc|c} -0.2 & 0.5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-5]{\text{Mul R1 by}} \left[\begin{array}{cc|c} 1 & -2.5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is: x_2 anything, $x_1 - 2.5x_2 = 0$

$x_1 = 2.5x_2$

except we want $x_1 + x_2 = 1$.

(we want a probability vector)

So, $2.5x_2 + x_2 = 1$, $x_2 = 2/7$

$x_1 = 5/7$.

Conclusion. $$\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix} = \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix} = \begin{bmatrix} .71428\cdots \\ .28571\cdots \end{bmatrix}$$

Now, $$\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}^5 = \begin{bmatrix} .71498 & .71255 \\ .28502 & .28745 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}^{100} \text{ matches } \begin{bmatrix} 5/7 & 5/7 \\ 2/7 & 2/7 \end{bmatrix}$$

to 20 decimal places.

Conclusion: According to the Markov model, today's weather doesn't have a significant long term effect.

Theorem. If a Markov chain is "irreducible" and "aperiodic" (sufficient condition: no zeroes or ones) then it converges to a matrix whose columns are all the unique steady state.