State High School Mathematics Tournament

University of South Carolina

Round 1 – February 1, 2025

Thanks

George Androulakis, Teegan Bailey, Tapas Bhowmik, Matthew Boylan, George Brooks, Mitchel Colebank, Jen Crooks-Monastra, Steven Derochers, Benji Dial, Maria Girardi, Bryan Gentry, AJ Greene, Ashlee Greene, Hossein Haj-Hariri, Siming He, Alec Helm, Alison Hogue, Isaiah Hollars, Aditya Iyer, Beneisha Johnson, Xinfeng Liu, Linyuan Lu, Ruth Luo, Steven Lynn, Jonah Klein, Andy Kustin, Megan McKay, Josiah McKay, Caleb McWhorter, DeeAnn Moss, Edsel Pena, Chris Portwood, Joel Samuels, Ronda Sanders, Dan Savu, Henry Simmons, Wilma Sims, Pankaj Singh, Will Smith, Jan Smoak, Gabe Staton, Rhonda Stephens, Swati, Wei-Lun Tsai, Paula Vasquez, Xiaofeng Yang, Sean Yee, Haonan Zhang

More Opportunities!

If you enjoyed today, check out:

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If you enjoyed today, check out:

- The **Columbia Math Circle**, contact me at `thorne@math.sc.edu`

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- The **All-State Math Team**, go to `scmathteam.com`

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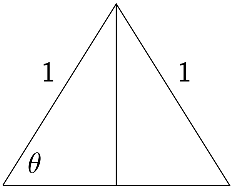
- There will be a tiebreaker if needed.

If $\triangle ABC$ is an isosceles triangle with $AB = BC = 1$, what should the length of AC be to maximize the triangle's area?

Solution 1-1

Answer. $\sqrt{2}$



$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

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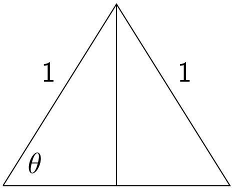

$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

Maximize with $\theta = \frac{\pi}{4}$, so $AC = \sqrt{2}$.

Question 1-2

What is the sum of all integer solutions $A > 1$ to the equation

$$\log_4 A + \log_A 4 = \frac{5}{2}?$$

Solution 1-2

Answer. 18.

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We have $\frac{\ln A}{\ln 4} + \frac{\ln 4}{\ln A} = \frac{5}{2}$, which is equivalent to

$$\ln^2 A - \frac{5}{2}(\ln 4)\ln A + (\ln 4)^2 = 0. \tag{1}$$

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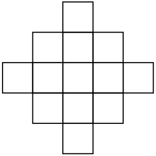
The solutions are $\ln A = 2\ln 4,\ \frac{1}{2}\ln 4$, or $A = 16, 2$.

On this morning's individual test this morning, you saw the following figure, where each small square is a square of the same size.



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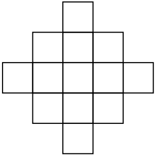
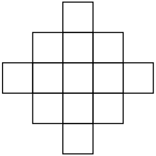


You determined (I hope!) that there are 18 squares of all sizes in the figure.

Question 1-3



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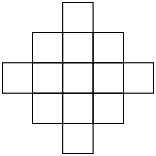

We may interpret this as squares of side length 1, centered at each (i, j) with $i, j \in \mathbb{Z}$ and $|i| + |j| \leq 2$.

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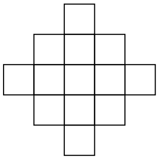

We may interpret this as squares of side length 1, centered at each (i, j) with $i, j \in \mathbb{Z}$ and $|i| + |j| \leq 2$.

Consider a 3-dimensional analogue, with cubes of side length 1, centered at every (i, j, k) with $i, j, k \in \mathbb{Z}$ and $|i| + |j| + |k| \leq 2$.

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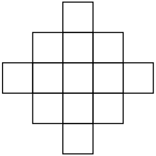

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How many cubes of all sizes are in this three-dimensional analogue?

Solution 1-3

Answer. 25.

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In three dimensions, there are only the unit cubes!

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- 13 cubes in the center layer;

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- 13 cubes in the center layer;
- 5 cubes one layer above, and 5 one layer below;

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- 13 cubes in the center layer;
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- One cube on top, and one on the bottom.

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- 13 cubes in the center layer;
- 5 cubes one layer above, and 5 one layer below;
- One cube on top, and one on the bottom.

$$13 + 5 + 5 + 1 + 1 = 25.$$

Question 1-4

Unit circles C_1 and C_2 intersect at P_1 and P_2. A unit circle C_3 passes through P_2 and has center P_1.

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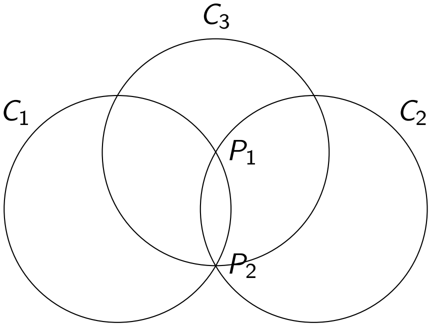
What is the total area covered by the circles?



Solution 1-4

Answer. $\frac{5}{3}\pi + \sqrt{3}$.

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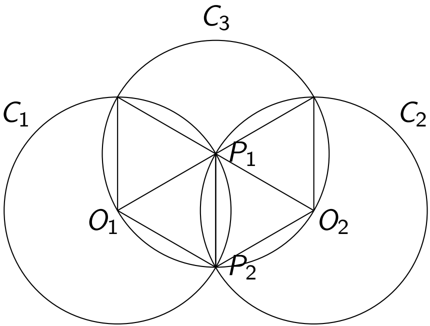

The four triangles have total area $\sqrt{3}$, and the remaining circles have $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ of their areas counted.

Question 1-5

You flip three coins and a friend flips three coins.

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What is the probability that you each flip exactly the same number of heads?

Solution 1-5

Answer. $\frac{5}{16}$.

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$$\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{1}{8}\right)^2 = \frac{1+9+9+1}{64} = \frac{5}{16}.$$

Question 8

The equation $2^x = x^2$ has three real solutions. What is the nearest integer to their sum?

Solution 8

Answer. 5

$x = 2$, $x = 4$, and $x = -.76\ldots$

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$x = 2$, $x = 4$, and $x = -.76\ldots$
For the negative solution, note that $2^{-\frac{1}{2}} > (-\frac{1}{2})^2$, so $x < -\frac{1}{2}$.

Question 9

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2021 - 2022 + 2023 - 2024?$$

Solution 9

Answer. -1012.

Solution 9

Answer. -1012.

Write it as

$$(1-2)+(3-4)+\cdots+(2023-2024)=(-1)\times 1012.$$

Question 10

How many positive integers $n \leq 10$ satisfy $\cos(n) > 0$?

(Assume radian measure.)

Solution 10

Answer. 4.

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$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

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$$n \in \{1, 5, 6, 7\}$$

Question 11

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$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}}$$

Solution 11

Answer. $\frac{17}{28}$.

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Notice the pattern: $\frac{6}{5}, \frac{11}{6}, \frac{17}{11}, \frac{28}{17}$