

6-torsion in class groups of imaginary quadratic fields

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thornef.github.io/6-torsion-ap.pdf



Workshop on Arithmetic Statistics

Lodha Institute, Mumbai, December 15-19, 2025

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The Basic Question

Let $D \neq 0, 1$ be a fundamental discriminant.

Write $\text{Cl}(D) := \text{Cl}(\mathbb{Q}(\sqrt{D}))$ for the associated *ideal class group*.

What can we say about $\text{Cl}(D)$ as D varies?

Dirichlet's Class Number Formula

Theorem

If $D < 0$, we have

$$\text{Cl}(D) = \frac{w\sqrt{|D|}}{2\pi} L(1, \chi_D),$$

and if $D > 0$ we have

$$\text{Cl}(D) = \frac{\sqrt{|D|}}{2\log(\epsilon_D)} L(1, \chi_D),$$

Theorem (Heegner, Baker, Stark)

If $D < 0$ and $\text{Cl}(D) = 1$, then

$$D \in \{-3, -4, -7, -8, -11, -19, -43, -67, -163\}.$$

Class Number One Problem

Theorem (Heegner, Baker, Stark)

If $D < 0$ and $\text{Cl}(D) = 1$, then

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Conjecture

There are **infinitely many positive** D for which $\text{Cl}(D) = 1$.

Question: given $m > 1$, what can we say about

$$\mathrm{Cl}(D)[m] := \{[\mathfrak{a}] \in \mathrm{Cl}(D) : [\mathfrak{a}]^m = [(1)]\} ?$$

2-torsion: Gauss's genus theory

Theorem (Gauss's genus theory ($D < 0$ case))

If $D < 0$, we have

$$\mathrm{Cl}(D)[2] := (\mathbb{Z}/2)^{\omega(D)} - 1,$$

where $\omega(D)$ counts the **unique prime divisors** of D .

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Theorem (Gauss's genus theory ($D > 0$ case))

If $D > 0$, we have

$$\text{Cl}^+(D)[2] := (\mathbb{Z}/2)^{\omega(D)} - 1,$$

and

$$\text{Cl}^+(D)[2] \simeq \text{Cl}(D)[2] \times (\mathbb{Z}/2)$$

if D has any prime divisor $p \equiv 3 \pmod{4}$.

Conjecture (Cohen-Lenstra)

Let p be an odd prime. Then, on average, $\text{Cl}(D)[p]$ is “a random abelian p -group”.

(to do: explain more....)

Asymptotic results

Problem: Given $m \geq 2$, prove asymptotic formulas for

$$\sum_{\pm D < X} \#\mathrm{Cl}(D)[m].$$

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- ▶ $m = 2^k$: Smith (2022)
- ▶ $m = 6$: **New!** KLOST (2025?)

Main Results

Theorem (Koymans, Lemke Oliver, Sofos, T.)

We have

$$\sum_{-X < D < 0} \# \text{Cl}(D)[6] = \frac{3}{\pi^2} \prod_p \left(1 + \frac{1}{p+1}\right) \left(1 - \frac{1}{p}\right) X \log(X) + O(X(\log \log X)^7)$$

$$\sum_{0 < D < X} \# \text{Cl}(D)[6] = \frac{1}{\pi^2} \prod_p \left(1 + \frac{1}{p+1}\right) \left(1 - \frac{1}{p}\right) X \log(X) + o(X \log(X)).$$

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Improves upon (and uses!):

Theorem (Chan, Koymans, Pagano, Sofos)

We have

$$\sum_{0 < \pm D < X} \# \text{Cl}(D)[6] \ll X \log(X).$$

Theorem

We have

$$N_{12}(X, D_6) = CX^{1/6}(\log X)^2 + O(X^{1/6}(\log X)^{1+\epsilon}).$$

Asymptotics for 2-class numbers:

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We have

$$\sum_{0 < -D < X} \# \text{Cl}(D)[2] \sim \frac{3}{2\pi^2} X \log X \cdot \prod_p \left(1 + \frac{1}{p+1}\right) \left(1 - \frac{1}{p}\right).$$

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This is “easy” and “well known”.

Asymptotics for 2-class numbers (2):

Since

$$\mathrm{Cl}(D)[2] := (\mathbb{Z}/2)^{\omega(D)} - 1,$$

we have “morally”

$$\sum_{0 < -D < X} \# \mathrm{Cl}(D)[2] \text{ “} = \text{” } \frac{1}{2} \sum_{\substack{0 < n < X \\ \text{squarefree}}} \tau(n),$$

where $\tau(n)$ counts the positive divisors of n .

Asymptotics for 2-class numbers (3):

We have

$$\begin{aligned}\sum_{n \text{ squarefree}} \tau(n) n^{-s} &= \prod_p \left(1 + \frac{2}{p^s}\right) \\ &= \zeta(s)^2 \cdot \prod_p \left(1 + \frac{2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2},\end{aligned}$$

and

$$\sum_{\substack{0 < n < X \\ \text{squarefree}}} \tau(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)^2 \cdot \prod_p \left(1 + \frac{2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2} X^s \frac{ds}{s}.$$

Asymptotics for 3-class numbers (1):

Theorem (Davenport-Heilbronn, 1971)

We have

$$\sum_{0 < -D < X} \# \text{Cl}(D)[3] \sim \frac{3+3}{\pi^2} X,$$

and

$$\sum_{0 < D < X} \# \text{Cl}(D)[3] \sim \frac{3+1}{\pi^2} X.$$

DH Step 1: Cubic Fields

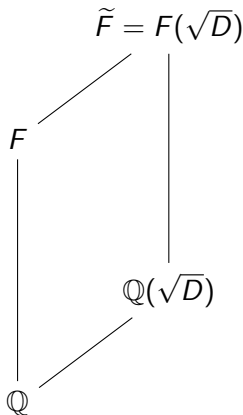
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Let D be a fundamental discriminant. Then, *cubic fields of discriminant D* are in bijection with *subgroups of $\text{Cl}(D)$ of index 3*.

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Let V be the space of binary cubic forms:

$$V := \{x(u, v) = au^3 + bu^2v + cuv^2 + dv^3\}.$$

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for which we have

- ▶ $\text{Disc}(gx) = (\det g)^6 \text{Disc}(x)$;
- ▶ $\text{Disc}(x) = 0$ if and only if $x(u, v)$ has a repeated root.

The Delone-Faddeev correspondence

Definition

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Theorem (Delone-Faddeev, 1964)

There is a 'nice' bijection between:

- ▶ *Cubic rings up to isomorphism; and,*
- ▶ *$\mathrm{GL}_2(\mathbb{Z})$ -equivalence classes of integral binary cubic forms.*

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Theorem (Davenport-Heilbronn)

A cubic ring R is maximal iff its cubic form f satisfies certain congruence conditions $(\bmod p^2)$, for every prime p .

How to count cubic fields

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Techniques:

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Techniques:

- ▶ Geometry of numbers, with **Bhargava's averaging method**;
- ▶ **Shintani zeta functions**.

A Stronger Davenport-Heilbronn Theorem

Theorem (BBDHPSTTT*)

We have

$$\sum_{0 < \pm D < X} \# \text{Cl}(D)[3] = \frac{3 + 3 + 3 + 1}{\pi^2} X + cX^{5/6} + O(X^{2/3+\epsilon}).$$

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- ▶ Counting **all cubic fields** (including non-fundamental discriminants);
- ▶ Allowing for **local specifications**;
- ▶ A 'level of distribution' statement (**important later!!**)

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Answer: Naively.

6-torsion: first steps

We have

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Theorem (Davenport-Heilbronn with divisibility)

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So, we're done, right.....?

Sage Wisdom from Jean-Pierre Serre

“Another theorem says that $|Af|$ is smaller than $C|f|$, where C is a constant. A beautiful word! What is meant is real number, strictly positive, and you call it a constant because it doesn't depend – so it's a real number not depending on some of the data. Now, very often the theorem is taken with ‘Let f be fixed.’ So that, a priori, in normal good mathematics the constant would depend on f . But they do not say. And the constant may depend, certainly on A , but on many other things.

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How?

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How? If you're not careful, badly.

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The Bombieri-Vinogradov Theorem:

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Theorem

For every $A > 0$ there exists $B > 0$ so that

$$\sum_{q < X^{1/2}(\log X)^{-B}} \max_{a \pmod{*} q} \left| \pi(x; q, a) - \frac{1}{\phi(q)} \text{Li}(x) \right| \ll X(\log X)^{-A}.$$

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“GRH on average”.

Davenport-Heilbronn theorem, LOD version

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$$\sum_{d < X^{1/2}(\log X)^{-8}} \left| \sum_{\substack{0 < -D < X \\ d|D}} \#(\text{Cl}(D)[3]-1) - \frac{3}{\pi^2} \prod_{p|D} \frac{p}{p^2-1} \cdot X \right| = o(X).$$

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Got $d \ll X^{1/2-\epsilon}$ in [Bhargava-Taniguchi-T.](#) To improve:

- Squeeze more out of existing proofs.

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Theorem

We have

$$\sum_{d < X^{1/2}(\log X)^{-8}} \left| \sum_{\substack{0 < -D < X \\ d|D}} \#(\text{Cl}(D)[3]-1) - \frac{3}{\pi^2} \prod_{p|D} \frac{p}{p^2-1} \cdot X \right| = o(X).$$

Got $d \ll X^{1/2-\epsilon}$ in [Bhargava-Taniguchi-T.](#) To improve:

- ▶ Squeeze more out of existing proofs.
- ▶ Use $\sum_{\pm D < X} \# \text{Cl}(D)[3]_{\tau}(D) \ll X \log X$ in the guts of the proof.

The Hooley Delta Symbol (1)

Lemma

For any $L > 0$, we have

$$\sum_{0 < \pm D \leq X} \# \text{Cl}(D)[3] \cdot \\ \#\{d : d \mid D, \sqrt{X}(\log X)^{-L} \leq d \leq \sqrt{X}(\log X)^L\} \ll LX(\log \log X)^{7/2}.$$

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Arises from the **Hooley delta function**

$$\Delta(n) = \sup_{u \geq 1} \#\{d \in \mathbb{N} \cap [u, eu] : d \mid n\}.$$

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Theorem (... , de la Bretèche–Tenenbaum)

We have

$$\sum_{n \leq x} \Delta(n) \ll x(\log \log x)^{5/2}. \quad (1)$$

The Hooley Delta Symbol (2)

Our sum in the lemma is reduced to

$$\ll L(\log \log T) \sum_{0 < -D \leq T} (h_3(D) - 1) \Delta(|D|).$$

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Previous work of Chan, Koymans, Pagano, and Sofos treats sums of this type.

Thank you!
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