State High School Mathematics Tournament

University of South Carolina

Round 1 – April 22, 2023

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- There will be a tiebreaker if needed.

Question 1-1

What is the nearest integer to

$$\log_2(3) \times \log_3(4) \times \log_4(5) \times \cdots \times \log_{2022}(2023)?$$

Solution 1-1

Answer. 11.

Solution 1-1

Answer. 11.

The quantity is

$$\frac{\ln(3)}{\ln(2)} \times \frac{\ln(4)}{\ln(3)} \times \cdots \times \frac{\ln(2023)}{\ln(2022)} = \frac{\ln(2023)}{\ln(2)} = \log_2(2023).$$

Solution 1-1

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Since $2^{11} = 2048 \approx 2023$, we have $\log_2(2023) \approx 11$.

Question 1-2

MAKE YOUR MOVE
3 2 9 9 6 1 2 4 2
GRILL SET
BISTRO TABLE
LED LCD HDTV

Question 1-2

On an episode of *The Price Is Right*, a contestant must guess the prices of a TV, a bistro table, and a grill set.

These are 4, 3, and 2 digits respectively, and must be chosen consecutively from a sequence of nine digits. All nine digits must be used.

Question 1-2

In the above setup, if every possible outcome is equally likely, what is the probability that the TV has the 1 in its price?

Solution 1-2

Answer. $\frac{2}{3}$.

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- 329961242
- 329961242
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Solution 1-2

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The probability that the 1 is green, if one of the above six possibilities is randomly selected.

Question 1-3

If $\sin(x) + \cos(x) = \frac{5}{4}$, what is $\sin^3(x) + \cos^3(x)$?

Solution 1-3

Answer. $\frac{115}{128}$.

Solution 1-3

Answer. $\frac{115}{128}$.

With $s := \sin(x)$ and $c := \cos(x)$,

$$s^3 + c^3 = (s + c)(s^2 - sc + c^2) = \frac{5}{4}(1 - sc)$$

Solution 1-3

Answer. $\frac{115}{128}$.

With $s := \sin(x)$ and $c := \cos(x)$,

$$s^3 + c^3 = (s + c)(s^2 - sc + c^2) = \frac{5}{4}(1 - sc)$$

$$\frac{25}{16} = (s + c)^2 = (s^2 + 2sc + c^2) = 1 + 2sc,$$

so $sc = \frac{9}{32}$.

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so $sc = \frac{9}{32}$.

$$s^3 + c^3 = \frac{5}{4} \cdot \left(1 - \frac{9}{32}\right) = \frac{5}{4} \cdot \frac{23}{32}.$$

Question 1-4

You roll four ordinary dice. What is the probability that the sum of some subset of the dice rolls equals the sum of some other subset of the dice rolls?

Solution 1-4

Answer. 1 (or 100%).

Solution 1-4

Answer. 1 **(or** 100%**).**

There are $1296 = 6 \times 6 \times 6 \times 6$ total possibilities.

Solution 1-4

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- All with a repeated dice roll are included.

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There are $1296 = 6 \times 6 \times 6 \times 6$ total possibilities.

- All with a repeated dice roll are included.
- Any with a 1 is included (consecutive numbers or 1246).

Solution 1-4

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There are $1296 = 6 \times 6 \times 6 \times 6$ total possibilities.

- All with a repeated dice roll are included.
- Any with a 1 is included (consecutive numbers or 1246).
- Any with a 2 is included (contains $2, 4, 6$ or $2, 3, 5$).

Solution 1-4

Answer. 1 **(or** 100%**).**

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- 3456 is included.

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- All with a repeated dice roll are included.
- Any with a 1 is included (consecutive numbers or 1246).
- Any with a 2 is included (contains $2, 4, 6$ or $2, 3, 5$).
- 3456 is included.
- This will *always* happen!

Question 1-5

Unit circles C_1 and C_2 intersect at P_1 and P_2. A unit circle C_3 passes through P_2 and has center P_1.

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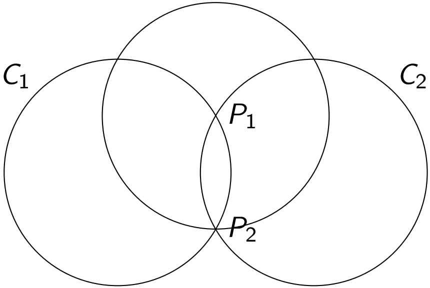
What is the distance between the centers of C_1 and C_2?



Solution 1-5

Answer. $\sqrt{3}$.

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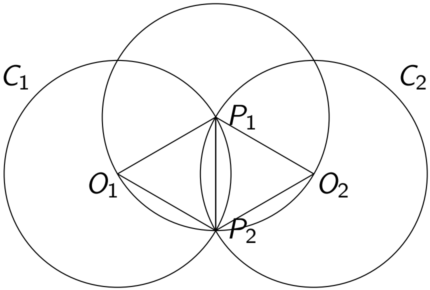

Both triangles shown are equilateral of side length 1 and altitude $\frac{\sqrt{3}}{2}$.

Question 1-6

Same setup: Unit circles C_1 and C_2 intersect at P_1 and P_2. A unit circle C_3 passes through P_2 and has center P_1.

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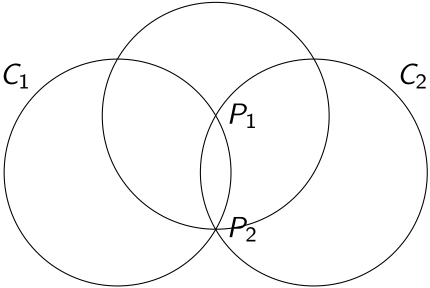
What is the total area covered by the circles?



Solution 1-6

Answer. $\frac{5}{3}\pi + \sqrt{3}$.

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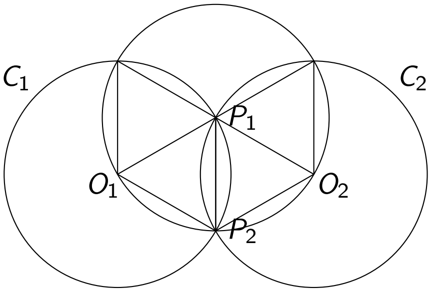


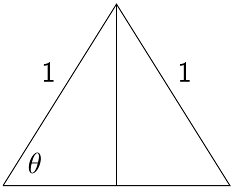
The four triangles have total area $\sqrt{3}$, and the remaining circles have $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{3}$ of their areas counted.

Question 7

If $\triangle ABC$ is an isosceles triangle with $AB = BC = 1$, what should the length of AC be to maximize the triangle's area?

Solution 7

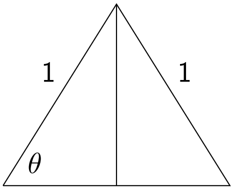
Answer. $\sqrt{2}$



$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

Solution 7

Answer. $\sqrt{2}$



$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

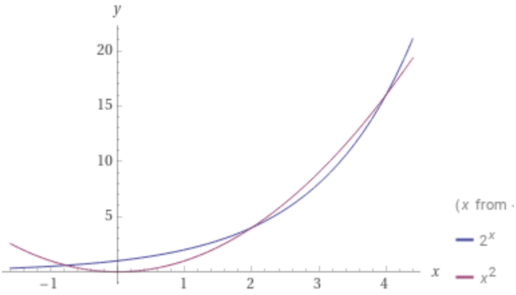
Maximize with $\theta = \frac{\pi}{4}$, so $AC = \sqrt{2}$.

Question 8

The equation $2^x = x^2$ has three real solutions. What is the nearest integer to their sum?

Solution 8

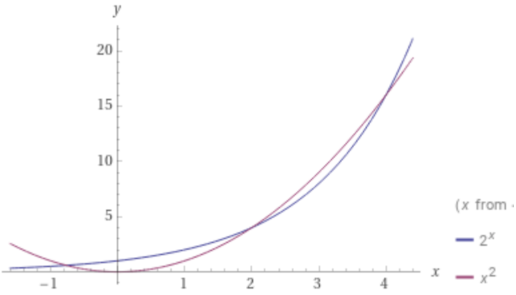
Answer. 5



$x = 2$, $x = 4$, and $x = -.76\ldots$

Solution 8

Answer. 5



$x = 2$, $x = 4$, and $x = -.76\ldots$
For the negative solution, note that $2^{-\frac{1}{2}} > (-\frac{1}{2})^2$, so $x < -\frac{1}{2}$.

Question 9

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2021 - 2022 + 2023?$$

Solution 9

Answer. 1012.

Solution 9

Answer. 1012.

Write it as

$$(1-2)+(3-4)+\cdots+1012+\cdots+(-2020+2021)+(-2022+2023).$$

We have, e.g.,

$$1-2-2022+2023=0.$$

Question 10

How many positive integers $n \leq 10$ satisfy $\cos(n) > 0$?

(Assume radian measure.)

Solution 10

Answer. 4.

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Answer. 4.

$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

Solution 10

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$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

$$n \in (0, 1.57\ldots) \cup (4.71\ldots, 7.85\ldots)$$

Solution 10

Answer. 4.

$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

$$n \in (0, 1.57\ldots) \cup (4.71\ldots, 7.85\ldots)$$

$$n \in \{1, 5, 6, 7\}$$

Question 11

There are unique integers a and b for which

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$$(2 - \sqrt{3})^3 = a + b\sqrt{3}.$$

What is $a + b$?

Solution 11

Answer. 11.

Solution 11

Answer. 11. We have

$$(2 - \sqrt{3})^3 = 8 - 12\sqrt{3} + 6(\sqrt{3})^2 - (\sqrt{3})^3 = 26 - 15\sqrt{3}.$$

Question 12

SImplify:

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$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}}$$

Solution 12

Answer. $\frac{17}{28}$.

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- $1 + \frac{1}{5} = \frac{6}{5}$

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- $1 + \frac{1}{5} = \frac{6}{5}$
- $1 + \frac{1}{1+\frac{1}{5}} = \frac{11}{6}$

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Answer. $\frac{17}{28}$.

- $1 + \frac{1}{5} = \frac{6}{5}$
- $1 + \frac{1}{1+\frac{1}{5}} = \frac{11}{6}$
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Solution 12

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- $1 + \frac{1}{1+\frac{1}{5}} = \frac{11}{6}$
- $1 + \frac{1}{1+\frac{1}{1+\frac{1}{5}}} = \frac{17}{11}$
- $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}} = \frac{28}{17}$

Solution 12

Answer. $\frac{17}{28}$.

- $1+\frac{1}{5}=\frac{6}{5}$
- $1+\frac{1}{1+\frac{1}{5}}=\frac{11}{6}$
- $1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}=\frac{17}{11}$
- $1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}=\frac{28}{17}$

Notice the pattern: $\frac{6}{5}, \frac{11}{6}, \frac{17}{11}, \frac{28}{17}$

Question 2-1

How many times does the graph of $y = x^6 + 6x^4 + 11x^2 + 6$ cross the x-axis?

Solution 2-1

Answer. 0.

Solution 2-1

Answer. 0.

$$x^6 + 6x^4 + 11x^2 + 6 \geq 0 + 0 + 0 + 6$$

Question 2-2

If you expand out

and simplify, how many terms will the resulting polynomial have?

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If you expand out

$$(x + y)^{10} + (x - y)^{10}$$

and simplify, how many terms will the resulting polynomial have?

Solution 2-2

Answer. 6.

Solution 2-2

Answer. 6.

$$(x + y)^{10} = x^{10} + 10x^9y + 55x^8y^2 + 120x^7y^3 + \cdots + y^{10},$$

Solution 2-2

Answer. 6.

$$(x+y)^{10} = x^{10} + 10x^9y + 55x^8y^2 + 120x^7y^3 + \cdots + y^{10},$$

$$(x+y)^{10} = x^{10} - 10x^9y + 55x^8y^2 - 120x^7y^3 + \cdots + y^{10}.$$

Solution 2-2

Answer. 6.

$$(x+y)^{10} = x^{10} + 10x^9y + 55x^8y^2 + 120x^7y^3 + \cdots + y^{10},$$

$$(x+y)^{10} = x^{10} - 10x^9y + 55x^8y^2 - 120x^7y^3 + \cdots + y^{10}.$$

The odd terms cancel and the even terms remain.

Question 2-3

How many zeroes does 2023! end in?

Solution 2-3

Answer. 503.

Solution 2-3

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Solution. The answer is the number of factors of 5 in 2023!.

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- $\lfloor \frac{2023}{5} \rfloor = 404$ integers $n \leq 2023$ are divisible by 5.

Solution 2-3

Answer. 503.

Solution. The answer is the number of factors of 5 in 2023!.

- $\lfloor \frac{2023}{5} \rfloor = 404$ integers $n \leq 2023$ are divisible by 5.
- 80 integers $n \leq 2023$ are divisible by 5^2.

Solution 2-3

Answer. 503.

Solution. The answer is the number of factors of 5 in 2023!.

- $\lfloor \frac{2023}{5} \rfloor = 404$ integers $n \leq 2023$ are divisible by 5.
- 80 integers $n \leq 2023$ are divisible by 5^2.
- 16 integers $n \leq 2023$ are divisible by 5^3.

Solution 2-3

Answer. 503.

Solution. The answer is the number of factors of 5 in 2023!.

- $\lfloor \frac{2023}{5} \rfloor = 404$ integers $n \leq 2023$ are divisible by 5.
- 80 integers $n \leq 2023$ are divisible by 5^2.
- 16 integers $n \leq 2023$ are divisible by 5^3.
- 3 integers $n \leq 2023$ are divisible by 5^4.

Solution 2-3

Answer. 503.

Solution. The answer is the number of factors of 5 in 2023!.

- $\lfloor \frac{2023}{5} \rfloor = 404$ integers $n \leq 2023$ are divisible by 5.
- 80 integers $n \leq 2023$ are divisible by 5^2.
- 16 integers $n \leq 2023$ are divisible by 5^3.
- 3 integers $n \leq 2023$ are divisible by 5^4.

$$404 + 80 + 16 + 3 = 503.$$

Question 2-4

What is the sum of the real number solutions to $x^6 - 7x^3 - 8 = 0$?

Solution 2-4

Answer. 1.

Solution 2-4

Answer. 1.

$$x^6 - 7x^3 - 8 = (x^3 - 8)(x^3 + 1)$$

Solution 2-4

Answer. 1.

$$x^6 - 7x^3 - 8 = (x^3 - 8)(x^3 + 1)$$

The two factors have unique roots $x = 2$ and $x = -1$ respectively.

Question 2-5

	O	X
	X	
X		O

Question 2-5

	O	X
	X	
X		O

The above shows a Tic-Tac-Toe board, where X has won after five moves.

Question 2-5

	O	X
	X	
X		O

The above shows a Tic-Tac-Toe board, where X has won after five moves.

How many such Tic-Tac-Toe boards are there?

Solution 2-5

Answer. 120.

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- 8 possible configurations of Xs: three rows, three columns, two diagonals.

Solution 2-5

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- 8 possible configurations of Xs: three rows, three columns, two diagonals.
- For each, $\binom{6}{2} = 15$ ways to place the Os.

Solution 2-5

Answer. 120.

- 8 possible configurations of Xs: three rows, three columns, two diagonals.
- For each, $\binom{6}{2} = 15$ ways to place the Os.
- $8 \times 15 = 120$.

Question 2-6

What is the last digit of 2023^{2023}?

Solution 2-6

Answer. 7.

Solution 2-6

Answer. 7.

Solution. The last digit of 2023^{2023} equals the last digit of 3^{2023}.

Solution 2-6

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$3^4 = 81$ and $2023 = 4 \cdot 505 + 3$, so

Solution 2-6

Answer. 7.

Solution. The last digit of 2023^{2023} equals the last digit of 3^{2023}.

$3^4 = 81$ and $2023 = 4 \cdot 505 + 3$, so

$$3^{2023} = 3^{4 \cdot 505 + 3} = (81)^{505} \cdot 3^3 = (\ldots ??1) \cdot 27,$$

which ends in 7.