An Overview of Number Field Counting

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December 11, 2024

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Basic Results

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If $[K : \mathbb{Q}] = d$, then

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In other words,

$$N_d(X) = 0 \ \text{ for } \ X < (5.803 \cdots + o(1))^d.$$

The Inverse Galois Problem

Conjecture

For every d and transitive subgroup $G \subseteq S_d$,

$$X \text{ big enough } \implies N_d(X, G) \neq 0.$$

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Proof.

(Your Name Here)

$\square$

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Conjecture

In fact we have

$$N_d(X, G) \sim c(G) X^{1/a(G)} (\log X)^{b(G)},$$

where $a(G) \geq 1$ and $b(G) \geq 0$ are explicitly described integers.

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So,

$$a(S_d) = 1, \; a(A_d) = 2, \; a(D_4) = 1, \; a(C_p) = p - 1, \ldots$$

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If $p \mid \mathrm{Disc}(K)$, then $p^{a(G)} \mid \mathrm{Disc}(K)$.

Very very very roughly,

$$\sum_{\mathrm{Gal}(\widehat{K}/\mathbb{Q}) \simeq G} |\mathrm{Disc}(K)|^{-s} \sim \prod_p \left(1 + \quad \frac{b(G)}{p^{a(G)}} + \frac{??}{p^{a(G)+1}} + \dots\right)$$

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- Inductive methods (Klüners, Cohen-Diaz-Olivier, ...)
  Obtain old results from new.
  Expand the scope of existing methods.

Generator methods

If $\alpha \in \mathcal{O}_K$ is a generator of $K/\mathbb{Q}$, then $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_K$ and

$$\begin{aligned} |\mathrm{Disc}(\mathcal{O}_K)| &= \mathrm{Disc}(\mathbb{Z}[\alpha]) \cdot [\mathcal{O}_K : \mathbb{Z}[\alpha]]^{-2} \\ &= \mathrm{Disc}(\mathsf{minpoly}_\alpha) \cdot [\mathcal{O}_K : \mathbb{Z}[\alpha]]^{-2}. \end{aligned}$$

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Theorem (Schmidt)

For each d we have

$$N_d(X) \ll X^{\frac{d+2}{4}}.$$

Schmidt's proof

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- By Minkowski's theory, there exists $\alpha \in \mathcal{O}_K$ with trace 0 and $||\alpha||_\sigma \ll |\mathrm{Disc}(K)|^{\frac{1}{2n-2}}$ for all embeddings $\sigma : K \mapsto \mathbb{C}$.

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- Assume that $\mathbb{Q}(\alpha) = K$. (If not, induct.)
- The minimal polynomial of α is

$$\text{minpoly}_\alpha(x) = \prod_\sigma (x - \sigma(\alpha)) = x^n + a_2(\alpha)x^{n-2} + \cdots + a_n(\alpha),$$

$$\text{with} \quad a_i(\alpha) \in \mathbb{Z}, \quad |a_i(\alpha)| \ll |\mathrm{Disc}(K)|^{\frac{i}{2n-2}}.$$

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Theorem (Kummer Theory)

If in addition $\mu_d \subseteq K$, then abelian extensions L/K of exponent d are in bijection with subgroups of $K^\times/(K^\times)^d$.

Cyclic cubic fields

Theorem (Cohn, 1954)

We have

$$\sum_{K \text{ cyclic cubic}} \frac{1}{\mathrm{Disc}(K)^s} = -\frac{1}{2} + \frac{1}{2}\left(1 + \frac{1}{3^{4s}}\right) \prod_{p \equiv 1 \pmod 6} \left(1 + \frac{2}{p^{2s}}\right).$$

Corollary

We have

$$N_3(X, C_3) \sim \frac{11\sqrt{3}}{36\pi} \prod_{p \equiv 1 \pmod 6} \frac{(p+2)(p-1)}{p(p+1)}.$$

General abelian number fields

Theorem (Wright, Mäki, but read Wood's treatment)

Let G be any abelian group of order n. Then we have

$$\sum_{\mathrm{Gal}(K/\mathbb{Q}) \simeq G} \frac{1}{\mathrm{Disc}(K)^s} = \text{ finite sum of Euler products } .$$

Corollary

We have

$$N_{|G|}(X, G) \sim c(G) X^{1/a(G)} (\log X)^{b(G)},$$

where $a(G)$ and $b(G)$ are explicit and $c(G)$ is 'explicit'.

Prime degree (Cohen, Diaz y Diaz, Olivier 2002)

"It is claimed that this constant can be explicitly computed as a a finite product of local adelic integrals, but in practice this has not been done, even for the simplest Abelian groups G, except for $G = C_2$..."

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CDO-prime-cyclic.pdf (page 3 of 41)

Cohen, Diaz y Diaz and Olivier, Cyclic extensions 171

Theorem 1.1. *Let K be a number field of signature (r_1, r_2). Let $\mathcal{R}$ (resp. $\mathcal{D}$) be the set of prime ideals of K which are ramified (resp. totally split) in K_z/K. Then*

$$N_{K,\ell}(C_\ell, X^{\ell-1}) \sim c_1 c_2 c_3 c_4 X \log^{q_z - 1} X$$

with

$$c_1 = \frac{\left(\prod_{d \mid d_z} \zeta_{K_z[d]}(d)^{\mu(d)}\right)^{q_z}}{d_z \ell^{r_2 + r_z} q_z!},$$

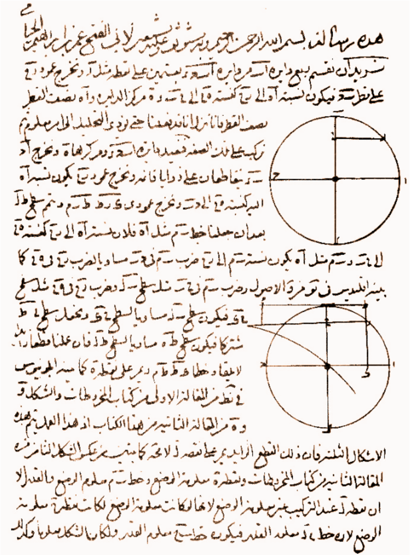
$$c_2 = \prod_{\mathfrak{p} \in \mathcal{D}} \left(\left(1 + \frac{\ell - 1}{\mathcal{N}\mathfrak{p}}\right) \prod_{d \mid d_z} \left(1 - \frac{1}{\mathcal{N}\mathfrak{p}^d}\right)^{(\ell-1)\mu(d)/d}\right),$$

$$c_3 = \left(\prod_{\mathfrak{p} \in \mathcal{R}} \prod_{d \mid d_z} \left(1 - \frac{1}{\mathcal{N}\mathfrak{p}^{df}(\mathfrak{p}_d/\mathfrak{p})}\right)^{g(\mathfrak{p}_d/\mathfrak{p})\mu(d)}\right)^{q_z},$$

$$c_4 = \prod_{\mathfrak{p} \mid \ell,\, \mathfrak{p} \notin \mathcal{D}} \left(1 + \frac{\ell - 1}{\mathcal{N}\mathfrak{p}} - \frac{\ell - 1 - r(e(\mathfrak{p}))(1 - 1/\mathcal{N}\mathfrak{p})}{\mathcal{N}\mathfrak{p}^{\lceil e(\mathfrak{p})/(\ell-1)\rceil}}\right),$$

where $r_z = 0$ if $\zeta_\ell \in K$, while $r_z = r_1 - 1$ otherwise, and by abuse of notation, for any number field L we write $\zeta_L(1)$ for the residue of the Dedekind zeta function $\zeta_L(s)$ at $s = 1$.

The parametrization method



Intersections of conics

Example. Solve $x^4 - x^3 + 3x^2 - 5x + 1 = 0$.

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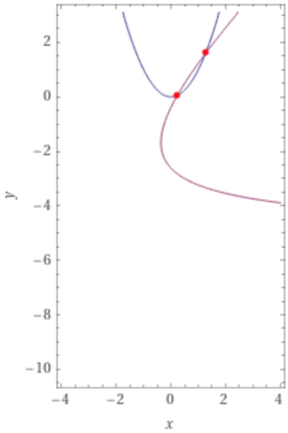

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- $\mathrm{GL}_3(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})$*-orbits on the lattice* $(\mathrm{Sym}^2\mathbb{Z}^3 \otimes \mathbb{Z}^2)$ *of* **pairs of integral ternary quartic forms**.

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- *Pairs* (Q, R), *where* Q *is a* **quartic ring** *and* R *is a* **cubic resolvent** *of* Q.

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- *$G(\mathbb{Z})$-orbits on a lattice $V(\mathbb{Z})$; where G is an* **algebraic group** *acting (often* **prehomogeneously***) on a vector space V;*
- *Some nice class of arithmetic objects we want to count.*

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- Read old papers in representation theory, invariant theory, and commutative algebra for inspiration. (Or Omar Khayyam!)
- Pick your favorite complex representation (G, V) (which should be defined over $\mathbb{Z}$, and for which the invariant theory should be nice).
- Try to prove that the $G(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ parametrize something. Hope to get lucky.

Counting Low Degree Fields

Theorem (Davenport-Heilbronn, Bhargava, et al.)

We have

$$N_3(X) = \frac{1}{3\zeta(3)} X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + O(X^{2/3}(\log X)^{2.09}),$$

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$$N_4(X, S_4) \sim \frac{5}{24}\prod_p (1 + p^{-2} - p^{-3} - p^{-4})X,$$

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These are now lattice point counting problems.

Inductive Methods (New Results From Old)

Theorem (Cohen, Diaz y Diaz, Olivier)

We have

$$N_4(X, D_4) \sim X \cdot \frac{3}{\pi^2} \sum_{D} \frac{2^{-r_2(D)}}{D^2} \frac{L(1, D)}{L(2, D)},$$

where the sum ranges over all fundamental discriminants $\neq 1$.

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Counting by other invariants

Theorem (Belabas-Fouvry, Bhargava-Wood)

We have

$$N_6(X, S_3) \sim \frac{2}{9}\left(\frac{4}{3} + \frac{1}{3^{5/3}} + \frac{2}{3^{7/3}}\right) \prod_{p \neq 3} \left(1 + p^{-1} + p^{-4/3}\right) \cdot X^{1/3}.$$

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Idea: If K is an S_3-cubic with $\mathrm{Disc}(K) = Dn^2$, then $\mathrm{Disc}(\widetilde{K}) = D^3 n^4$ apart from the 2- and 3-adic factors.

Direct products

Theorem (Wang, Masri-T.-Tsai-Wang)

Let $d \in \{3, 4, 5\}$ and let A be any abelian group. Then

$$N_{d|A|}(X, S_d \times A) \sim c(S_d \times A) X^{1/|A|}.$$

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(This doesn't happen too often.)

Wreath products

Theorem (Klüners $+\epsilon$)

Assume a 'weak Malle conjecture' of the form

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Note. $D_4 \simeq C_2 \wr C_2$; subsumes Cohen-Diaz-Olivier as a special case.

Solvable groups

Theorem (Alberts, 2018)

Assume that "the m-torsion in class groups is small on average". Then, for every solvable transitive subgroup $G \subseteq S_d$ we have

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Also: See further related works by Altuğ, Lemke Oliver, Mehta, Shankar, Taniguchi, Varma, Wilson, and previously named authors (in various permutations).

Happy Holidays!