The Distribution of G-Weyl CM Fields and the Colmez Conjecture

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The Main Theorem

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Assume a weak form of Malle's Conjecture.

Then, the Colmez conjecture is true for 100% of CM fields of any fixed degree, when ordered by discriminant.

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What do these words mean???

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Conjecture (Colmez '93)

Let E be a CM field, and let X_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$. Then,

$$h_{\mathrm{Fal}}(X_\Phi) = -Z(A^0_{E,\Phi}, 0) - \frac{1}{2}\mu_{\mathrm{Art}}(A^0_{E,\Phi}),$$

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- X_Φ is an abelian variety over $\overline{\mathbb{Q}}$ with complex multiplication by $\mathcal{O}_E$, and CM type for E (....)
- $h_{\mathbf{Fal}}(X_\Phi)$ is the Faltings height of X_Φ, which in fact only depends on Φ;
- The quantity on the right is defined in terms of logarithmic derivatives of Artin L-functions associated to characters defined in terms of the representation theory of $\mathrm{Gal}(\mathbb{Q}^{\mathbf{CM}}/\mathbb{Q})$.
- I could explain all this in more detail, but the margins of these slides are too small.

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- Previous work of my coauthors.

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An adaptation of work of Klüners.

Step 1: Counting quadratic extensions E/F

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To do this, define a Dirichlet series

$$D^{-}_{F,C_2}(s) := \sum_{[E:F]=2} \frac{1}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})^s}$$

where the sum is over all totally imaginary quadratic extensions E/F, and $\mathfrak{D}_{E/F}$ is the relative discriminant.

Theorem (Cohen-Diaz-Olivier '02)

For $\mathrm{Re}(s) > 1$ *we have*

$$D^{-}_{F,C_2}(s) = \frac{1}{\zeta_F(2s)} \sum_{\mathfrak{c}_\infty \subset \mathfrak{m}_\infty} \sum_{\mathfrak{c}|2} \frac{(-1)^{|\mathfrak{c}_\infty|}}{2^{|\mathfrak{c}_\infty|}} \mathcal{N}_{F/\mathbb{Q}}(2/\mathfrak{c})^{1-2s} \sum_{\chi \in Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))} L_F(\chi, s),$$

where $\mathfrak{c}$ *runs over all integral ideals of* F *dividing* 2, $\mathfrak{c}_\infty$ *runs over all subsets of the set of real places* $\mathfrak{m}_\infty$ *of* F, χ *runs over all quadratic characters* $Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))$ *of the ray class group* $\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F)$ *modulo* $\mathfrak{c}^2\mathfrak{c}_\infty$, *and* $L_F(\chi, s)$ *is the L–function of* χ.

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where χ *ranges (more or less) over quadratic characters of* $\mathrm{Cl}(F)$.

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- The quadratic extensions ramified only at 2∞ control the rest.

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Controlling the quadratic extensions

Definition (The constant δ_d)

For each d, choose $\delta_d \geq 0$ so that

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- The conjectured truth: $\delta_d = 0$.
- A little better (BSTTTZ): $\delta_3 = .2784\cdots$ and $\delta_n = \frac{1}{2} - \frac{1}{2n}$ for $n \geq 4$.

Step 2: Counting the base fields

Notation: For $d \geq 1$ and $G \subseteq S_d$, write

$$N_d(X, G) := \#\{K \ : \ [K : \mathbb{Q}] = d, \ \mathrm{Gal}(K^c/\mathbb{Q}) = G\}.$$

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We can restrict the above to totally real K if we like, but this doesn't seem to help us.

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Incorporating everything above, we have

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- $\xi(s)$ is polynomially bounded in vertical strips.

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Application: Look at the Dirichlet series again: no contribution to the residue.

Table: Values of $r_d(G)$ for $d \leq 5$

d	G	Number of fields	Minimal discriminant	Residue	Proportion
2	C_2	100,000	5	0.009856	-
3		25,000	49	3.30×10^{-5}	-
	C_3	107	49	2.29×10^{-5}	0.69
	S_3	24,893	148	1.01×10^{-5}	0.31
4		25,000	725	1.24×10^{-7}	-
	C_4	75	1125	2.41×10^{-8}	0.19
	V_4	289	1600	1.56×10^{-8}	0.13
	D_4	8147	725	5.9×10^{-8}	0.48
	A_4	45	26569	9.3×10^{-11}	0.0008
	S_4	16,444	1957	2.5×10^{-8}	0.20
5		25,000	14641	1.05×10^{-10}	-
	C_5	5	14641	3.08×10^{-11}	0.29
	D_5	28	160801	4.24×10^{-13}	0.003
	F_5	15	2382032	9×10^{-15}	0.00009
	A_5	21	3104644	5×10^{-15}	0.00005
	S_5	24,931	24217	7.4×10^{-11}	0.70