QUADRATIC FORMS REPRESENTING ALL ODD POSITIVE INTEGERS

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ABSTRACT. We consider the problem of classifying all positive-definite integer-valued quadratic forms that represent all positive odd integers. Kaplansky considered this problem for ternary forms, giving a list of 23 candidates, and proving that 19 of those represent all positive odds. (Jagy later dealt with a 20th candidate.) Assuming that the remaining three forms represent all positive odds, we prove that an arbitrary, positive-definite quadratic form represents all positive odds if and only if it represents the odd numbers from 1 up to 451. This result is analogous to Bhargava and Hanke's celebrated 290-theorem. In addition, we prove that these three remaining ternaries represent all positive odd integers, assuming the generalized Riemann hypothesis.

This result is made possible by a new analytic method for bounding the cusp constants of integer-valued quaternary quadratic forms Q with fundamental discriminant. This method is based on the analytic properties of Rankin-Selberg L-functions, and we use it to prove that if Q is a quaternary form with fundamental discriminant, the largest locally represented integer n for which $Q(\vec{x}) = n$ has no integer solutions is $O(D^{2+\epsilon})$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The study of which integers are represented by a given quadratic form is an old one. In 1640, Fermat stated his conjecture that every prime number $p \equiv 1 \pmod 4$ can be written in the form $x^2 + y^2$. In the next century, Euler proved Fermat's conjecture and worked seriously on related problems and generalizations. In 1770, Lagrange proved that every positive integer is a sum of four squares. In 1798, Legendre classified the integers that could be represented as a sum of three squares. This result is deeper and more difficult than either of the two-square or four-square theorems.

Motivated by Lagrange's result, it is natural to ask about the collection of quadratic forms that represent all positive integers, or more generally to fix in advance a collection S of integers, and ask about quadratic forms that represent all numbers in S. The first result in this direction is due to Ramanujan [35], who in 1916 gave a list of 55 quadratic forms of the form

$$Q(x, y, z, w) = ax^2 + by^2 + cz^2 + dw^2,$$

and asserted that this list consisted precisely of the forms (of this prescribed shape) that represent all positive integers. Dickson [11] confirmed Ramanujan's statement (modulo the

2010 *Mathematics Subject Classification.* Primary 11E20; Secondary 11E25, 11E45, 11F30.

The author was supported by NSF grant DMS-0901090.

error that the form $x^2+2y^2+5z^2+5w^2$ was included on Ramanujan's list and represents every positive integer except 15), and coined the term *universal* to describe quadratic forms that represent all positive integers.

A positive-definite quadratic form Q is called *integer-matrix* if it can be written in the form

$$Q(\vec{x}) = \vec{x}^T M \vec{x}$$

where the entries of M are integers. This is equivalent to saying that if

$$Q(\vec{x}) = \sum_{i=1}^{n} \sum_{j \geq i}^{n} a_{ij} x_i x_j,$$

then a_{ij} is even if $i \neq j$. A form Q is called *integer-valued* if the cross-terms a_{ij} are allowed to be odd. In 1948, Willerding [44] classified universal integer-matrix quaternary forms, giving a list of 178 such forms.

The following result classifying integer-matrix universal forms (in any number of variables) was proven by Conway and Schneeberger in 1993.

Theorem ("The 15-Theorem")**.** *A positive-definite integer-matrix quadratic form is universal if and only if it represents the numbers*

$$1, 2, 3, 5, 6, 7, 10, 14, \text{ and } 15.$$

This theorem was elegantly reproven by Bhargava in 2000 (see [2]). Bhargava's approach is to work with integral lattices, and to classify escalator lattices - lattices that must be inside any lattice whose corresponding quadratic form represents all positive integers. As a consequence, Bhargava found that there are in fact 204 universal quaternary integer-matrix forms. Willerding had missed 36 universal forms, listed one universal form twice, and listed nine forms which were not universal.

Bhargava's approach is quite general. Indeed, he has proven that for any infinite set S, there is a finite subset S_0 of S so that any positive-definite integral quadratic form represents all numbers in S if it represents the numbers in S_0. Here the notion of integral quadratic form can mean either integer-matrix or integer-valued (and the set S_0 depends on which notion is used). Bhargava proves that if S is the set of odd numbers, then any integer-matrix form represents everything in S if it represents everything in $S_0 = \{1, 3, 5, 7, 11, 15, 33\}$. He also determines S_0 in the case that S is the set of prime numbers (again for integer-matrix forms); the largest element of S_0 is 73. (These results are stated in [28].)

While working on the 15-Theorem, Conway and Schneeberger were led to conjecture that every integer-valued quadratic form that represents the positive integers between 1 and 290 must be universal. Bhargava and Hanke's celebrated 290-Theorem proves this conjecture (see [1]). Their result is the following.

Theorem ("The 290-Theorem"). *If a positive-definite integer-valued quadratic form represents the twenty-nine integers*

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29$$
$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \textit{ and } 290,$$

then it represents all positive integers.

They also show that every one of the twenty-nine integers above is necessary. Indeed, for every integer t on this list, there is a positive-definite integer-valued quadratic form that represents every positive integer except t. As a consequence of the 290-Theorem, they are able to prove that there are exactly 6436 universal integer-valued quaternary quadratic forms.

A *regular* positive-definite quadratic form is a form $Q(\vec{x})$ with the property that if n is a positive integer and $Q(\vec{x}) = n$ is solvable in $\mathbb{Z}_p$ for all primes p, then $Q(\vec{x}) = n$ is solvable in $\mathbb{Z}$. Willerding and Bhargava make use of regular forms in their work on universal integer-matrix forms.

In the work of Bhargava and Hanke, they switch to using the analytic theory of modular forms, as they need to completely understand more than 6000 quaternary quadratic forms to prove the 290-Theorem. This technique is very general and requires extensive computer computations.

In this paper, we will consider the problem of determining a finite set S_0 with the property that a positive-definite integer-valued quadratic form represents every odd positive integer if and only if it represents everything in S_0. One difference between this problem and the case when S is all positive integers is that there are ternary quadratic forms that represent all odd integers, and it is necessary to classify these. In [26], Kaplansky considers this problem. He proves that there are at most 23 such forms, and gives proofs that 19 of the 23 represent all odd positive integers. He describes the remaining four as "plausible candidates" and indicates that they represent every odd positive integer less than 2^{14}. In [22], Jagy proved that one of Kaplansky's candidates, $x^2 + 3y^2 + 11z^2 + xy + 7yz$, represents all positive odds. The remaining three have yet to be treated.

Conjecture 1. *Each of the ternary quadratic forms*

$$x^2 + 2y^2 + 5z^2 + xz$$
$$x^2 + 3y^2 + 6z^2 + xy + 2yz$$
$$x^2 + 3y^2 + 7z^2 + xy + xz$$

represents all positive odd integers.

We can now state our first main result.

Theorem 2 ("The 451-Theorem"). *Assume Conjecture 1. Then, a positive-definite, integer-valued quadratic form represents all positive odd integers if and only if it represents the 46* *integers*

$$1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 47,$$
$$51, 53, 57, 59, 77, 83, 85, 87, 89, 91, 93, 105, 119, 123, 133, 137,$$
$$143, 145, 187, 195, 203, 205, 209, 231, 319, 385, \textit{and } 451.$$

As was the case for the 290-Theorem, all of the integers above are necessary.

Corollary 3. *For every one of the* 46 *integers t on the list above, there is a positive-definite, integer-valued quadratic form that represents every odd number except t.*

We also have an analogue of results proven in [2] and [1] regarding what happens if the largest number is omitted.

Corollary 4. *Assume Conjecture 1. If a positive-definite, integer-valued quadratic form represents every positive odd number less than* 451*, it represents every odd number greater than* 451*.*

As a consequence of the 451-Theorem, we can classify integer-valued quaternary forms that represent all positive odd integers.

Corollary 5. *Assume Conjecture 1. Suppose that Q is a positive-definite, integer-valued, quaternary quadratic form that represents all positive odds. Then either:*

(a) Q represents one of the 23 *ternary quadratic forms which represents all positive odds, or*

(b) Q is one of 21756 *quaternary forms.*

To prove the 451-Theorem, we must determine the positive, odd, squarefree integers represented by 24888 quaternary quadratic forms Q, and we do this using a combination of four methods. The first method checks to see if a given quaternary represents any of the 23 ternaries listed by Kaplansky. If so, it must represent all positive odds (assuming Conjecture 1).

The second method attempts to find, given the integer lattice L corresponding to Q, a ternary sublattice L' so that the quadratic form corresponding to L' is regular, and the lattice $L' \oplus (L')^{\perp}$ locally represents all positive odds. We make use of the classification of regular ternary quadratic forms due to Jagy, Kaplansky, and Schiemann [23]. This is a version of the technique used by Willerding and Bhargava.

The last two methods are analytic in nature. If Q is a positive-definite, integer-valued quaternary quadratic form, then

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n \in M_2(\Gamma_0(N), \chi), \quad q = e^{2\pi i z}$$

is a modular form of weight 2. We can decompose $\theta_Q(z)$ as

$$\begin{aligned}\theta_Q(z) &= E(z) + C(z)\\ &= \sum_{n=0}^{\infty} a_E(n)q^n + \sum_{n=1}^{\infty} a_C(n)q^n.\end{aligned}$$

Theorem 5.7 of [16] gives the lower bound

$$a_E(n) \geq C_E n \prod_{\substack{p\mid n\\ \chi(p)=-1}} \frac{p-1}{p+1}$$

for some some constant C_E, depending on Q, provided n is squarefree and locally represented by Q. We may decompose the form $C(z)$ into a linear combination of newforms (and the images of newforms under $V(d)$). It is known that the nth Fourier coefficient of a newform of weight 2 is bounded by $d(n)n^{1/2}$ (first proven by Eichler, Shimura, and Igusa in the weight 2 case, and Deligne in the general case), and so there is a constant C_Q so that

$$a_C(n) \leq C_Q d(n) n^{1/2}.$$

If we can compute or bound the constants C_E and C_Q, we can determine the squarefree integers represented by Q via a finite computation.

One method we use is to compute the constant C_Q explicitly, by computing the Fourier expansions of all newforms and expressing $C(z)$ in terms of them. This method is the approach taken by Bhargava and Hanke to all of the cases they consider in [1], and works very well when the coefficient fields of the newforms are reasonably small.

However, in Bhargava and Hanke's cases, the newforms in the decomposition have coefficients in number fields of degree as high as 672, and the computations require weeks of CPU time. In our case, we must consider spaces that have Galois conjugacy classes of newforms of size at least 1312, and for degrees as large as this, this explicit, direct approach is impossible from a practical standpoint.

These large degree number fields only arise in cases when $S_2(\Gamma_0(N),\chi)$ is close to being irreducible as a Hecke module. If the conductor of χ is not primitive, we have a decomposition of $S_2(\Gamma_0(N),\chi)$ into old and new subspaces which are Hecke stable. For this reason, we develop a new method to bound the constant C_Q without explicitly computing the newform decomposition of C which applies when the discriminant of the quadratic form Q is a fundamental discriminant.

Our method allows us to improve significantly the bounds given in the literature on the largest integer n that is not represented by a form Q satisfying appropriate local conditions. For a form $Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$, where A has integer entries and even diagonal entries, let $D(Q) = \det A$ be the discriminant of Q, and let $N(Q)$ be the level of Q. In [38], Schulze-Pillot proves the following result.

Theorem. *Suppose that Q is a positive-definite, integer-valued, quaternary quadratic form with level $N(Q)$. If n is a positive integer so that $Q(\vec{x}) = n$ has primitive solutions in $\mathbb{Z}_p$ for all primes p, and*

$$n \gg N(Q)^{14+\epsilon},$$

then n is represented by Q.

Remark. *We have given a simplified version of Schulze-Pillot's result. The bound Schulze-Pillot gives is completely explicit.*

In [5], Browning and Dietmann use the circle method to study integer-matrix quadratic forms $Q(\vec{x}) = \vec{x}^T A\vec{x}$. A pair Q, k (consisting of a quaternary quadratic form and a positive integer k) satisfies the strong local solubility condition if for all primes p there is a vector $\vec{x} \in \mathbb{Z}^4$ so that

$$Q(\vec{x}) \equiv k \pmod{p^{1+2\tau_p}}$$

and $p \nmid A\vec{x}$. Here τ_p is zero if p is odd and is one if $p = 2$. Their result about quaternary forms is the following.

Theorem. *Assume the notation above and let $\|Q\|$ denote the largest entry in the Gram matrix A of Q. Let $\mathfrak{k}_4^*(Q)$ be the largest positive integer k that satisfies the strong local solubility condition. Then*

$$\mathfrak{k}_4^*(Q) \ll D(Q)^2 \|Q\|^{8+\epsilon}.$$

Remark. *Depending on the quaternary form Q, the bound above is between $D(Q)^{4+\epsilon}$ and $D(Q)^{10+\epsilon}$. For a "generic" quaternary form with small coefficients, we have $\|Q\| \ll D^{1/4}$ and the bound $D^{4+\epsilon}$.*

Our next main result is a significant improvement on the result of Browning and Dietmann in the two cases that $D(Q)$ is a fundamental discriminant, or that $N(Q)$ is a fundamental discriminant and $D(Q) = N(Q)^3$.

Theorem 6. *Suppose that Q is a positive-definite integer-valued quaternary quadratic form with fundamental discriminant D and Gram matrix A. If n is locally represented by Q, but is not represented by Q, then*

$$n \ll D(Q)^{2+\epsilon}.$$

If Q is a form whose level $N(Q)$ is a fundamental discriminant and $D(Q) = N(Q)^3$, then Q represents every locally represented integer

$$n \ll D(Q)^{1+\epsilon}.$$

Remark. *The best results obtainable by the theorem of Browning and Dietmann to the two cases considered above are $n \ll D(Q)^{10+\epsilon}$ and $n \ll D(Q)^{14/3+\epsilon}$, respectively.*

Remark. *The forms Q and Q^* in the statement of the theorem above cannot be anisotropic at any prime, and so we only require a weaker local solubility condition. The result above relies on ineffective lower bounds for L-functions at $s = 1$. However, for any given form Q, we can explicitly compute the bounds, as we will explain in Section 5.*

Our method is similar to the approach of Schulze-Pillot [38] and is based on bounding from above the Petersson norm $\langle C, C\rangle$ of C and bounding from below the Petersson norm of newforms g_i in the space of cusp forms containing C. We obtain upper bounds on $\langle C, C\rangle$ and lower bounds on $\langle g_i, g_i\rangle$ using the theory of Rankin-Selberg L-functions.

If $g_i = \sum_{n=1}^{\infty} a(n)q^n$ and $g_j = \sum_{n=1}^{\infty} b(n)q^n$ are two newforms in $S_2(\Gamma_0(N), \chi)$ with

$$L(g_i, s) = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s}), \text{ and } L(g_j, s) = \prod_p (1-\gamma_p p^{-s})^{-1}(1-\delta_p p^{-s})^{-1},$$

the Rankin-Selberg convolution L-function of f and g is

$$L(g_i \otimes g_j, s) = \prod_{p|N} L_p(g_i \otimes g_j, s) \prod_{p\nmid N} (1-\alpha_p\gamma_p p^{-s})^{-1}(1-\alpha_p\delta_p p^{-s})^{-1}(1-\beta_p\gamma_p p^{-s})^{-1}(1-\beta_p\delta_p p^{-s})^{-1}.$$

Here $L_p(g_i \otimes g_j, s)$ is an appropriate local factor predicted by the local Langlands correspondence. If $g_j = \overline{g_i}$, then $L(g_i \otimes g_j, s)$ has a pole at $s = 1$ with residue equal to an explicit factor times the Petersson norm of g_i. If the factor $L_p(g_i \otimes g_j, s)$ is chosen appropriately, then $L(g_i \otimes g_j, s)$ will have an meromorphic continuation to all of $\mathbb{C}$ (with the only possible pole occurring when $s = 1$ and $g_j = \overline{g_i}$) and a functional equation of the usual type.

In the appendix to [18], Goldfeld, Hoffstein, and Lieman show that $L(g_i \otimes g_i, s)$ has no Siegel zeroes. We work out precisely what the local factors are at primes $p|N$, make effective the result of Goldfeld, Hoffstein and Lieman, and translate this into an explicit lower bound for $\langle g_i, g_i\rangle$. We also give a Dirichlet series representation for $L(g_i \otimes g_j, s)$ in terms of the Fourier coefficients of g_i and g_j.

To give a bound on the Petersson norm of C, we need to extend our theory of Rankin-Selberg L-functions to arbitrary elements of $S_2(\Gamma_0(N), \chi)$. If $f, g \in S_2(\Gamma_0(N), \chi)$ we decompose

$$f = \sum_{i=1}^{u} c_i g_i, \text{ and } g = \sum_{j=1}^{u} d_j g_j$$

into linear combinations of newforms and define

$$L(f \otimes g, s) = \sum_{i=1}^{u}\sum_{j=1}^{u} c_i d_j L(g_i \otimes g_j, s).$$

Unfortunately, the prediction for $L_p(g_i \otimes g_j, s)$ that comes from the local Langlands correspondence makes it so the formula that takes a pair (f, g) and expresses $L(f \otimes g, s)$ in terms of the Fourier coefficients of f and g is not, in general, bilinear. However, we prove that this formula is bilinear if it is restricted to

$$S_2^-(\Gamma_0(N), \chi) = \left\{ \sum_{n=1}^{\infty} a(n)q^n \in S_2(\Gamma_0(N), \chi) : a(n) = 0 \text{ if } \chi(n) = 1 \right\}.$$

Hence, for forms $f \in S_2^-(\Gamma_0(N), \chi)$, $L(f \otimes f, s)$ has an analytic continuation, functional equation, relation between $\mathrm{Res}_{s=1} L(f \otimes f, s)$ and the Petersson norm of f, and a Dirichlet series representation that can be expressed in terms of the coefficients of f. However, for an arbitrary quadratic form Q, the cuspidal part of its theta function C need not be in $S_2^-(\Gamma_0(N), \chi)$. The assumption that $Q = \frac{1}{2}\vec{x}^T A \vec{x}$ where $D(Q) = \det(A)$ is a fundamental discriminant implies that if $Q^* = \frac{1}{2}\vec{x}^T N A^{-1} \vec{x}$, then $\theta_{Q^*} = E^* + C^*$ and $C^* \in S_2^-(\Gamma_0(N), \chi)$. Also, $\langle C^*, C^* \rangle = \frac{1}{\sqrt{N}} \langle C, C \rangle$. Combining these facts, we are able to derive a completely explicit formula for the Petersson norm of C in terms of the Fourier coefficients of C^*.

The method described above gives a much faster algorithm for determining the integers represented by a quadratic form Q with fundamental discriminant. In particular, we can determine the odd squarefree integers represented by a quadratic form Q with $\theta_Q \in M_2(\Gamma_0(6780), \chi_{6780})$ using 26 minutes of CPU time (see Example 5 of Section 5).

Finally, we return to Conjecture 1. There is no general method for determining the squarefree integers represented by a given positive-definite ternary quadratic form Q. The analytic theory gives a formula of the type

$$r_Q(n) = ah(-bn) + B(n)$$

provided n is squarefree and locally represented by Q. Here, $h(-bn)$ is the class number of $\mathbb{Q}(\sqrt{-bn})$, and $B(n)$ is the nth coefficient of a weight $3/2$ cusp form, and the constants a, b, and the form of $B(n)$ depend on the image of n in $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ for primes p dividing the level of Q.

The results of Blomer and Harcos [3] show that $|B(n)| \ll n^{7/16+\epsilon}$, and we have the ineffective bound $h(-bn) \gg n^{1/2-\epsilon}$. This proves that every sufficiently large locally represented squarefree integer is represented, but making this effective is not possible unconditionally. We follow the conditional approach pioneered by Ono and Soundararajan [33], Kane [25], and simplified by Chandee [8].

Theorem 7. *The generalized Riemann hypothesis implies Conjecture 1.*

An outline of the paper is as follows. In Section 2 we will review necessary background about quadratic forms and modular forms. In Section 3 we develop the theory of Rankin-Selberg L-functions which we will use in Section 4 to prove Theorem 6. In Section 5 we will prove the 451-Theorem, and in Section 6 we will prove Theorem 7.

Acknowledgements. *The author used the computer software package Magma* [4] *version 2.17-10 extensively for the computations that prove the 451-Theorem. The author would also like to thank Manjul Bhargava, Jonathan Hanke, David Hansen, Ben Kane, and Ken Ono for helpful conversations. This work was completed over the course of five years at the University of Wisconsin-Madison, the University of Illinois at Urbana-Champaign, and Wake Forest University. The author wishes to thank each of these institutions for their support of this work.*

2. Background and notation

A quadratic form in r variables $Q(\vec{x})$ is integer-valued if it can be written in the form

$$Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x},$$

where A is a symmetric $r \times r$ matrix with integer entries, and even diagonal entries. The matrix A is called the Gram matrix of Q. The quadratic form Q is called positive-definite if $Q(\vec{x}) \geq 0$ for all $\vec{x} \in \mathbb{R}^r$ with equality if and only if $\vec{x} = \vec{0}$. The discriminant of Q is the determinant of A, and the level of Q is the smallest positive integer N so that NA^{-1} has integer entries and even diagonal entries.

Let $\mathbb{H} = \{x+iy : x, y \in \mathbb{R}, y > 0\}$ denote the upper half plane. If k and N are positive integers, and χ is a Dirichlet character mod N, let $M_k(\Gamma_0(N), \chi)$ denote the vector space of modular forms (holomorphic on $\mathbb{H}$ and at the cusps) of weight k that transform according to

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices whose bottom left entry is a multiple of N. Let $S_k(\Gamma_0(N), \chi)$ denote the subspace of cusp forms. If λ is an integer, let $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ denote the vector space of holomorphic half-integer weight modular forms that transform according to

$$g\left(\frac{az+b}{cz+d}\right) = \chi(d)\left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda}(cz+d)^{\lambda+\frac{1}{2}} g(z)$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4N)$. Here $\left(\frac{c}{d}\right)$ is the usual Jacobi symbol if d is odd and positive and $c \neq 0$. We define $\left(\frac{0}{\pm 1}\right) = 1$ and

$$\left(\frac{c}{d}\right) = \begin{cases} \left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c > 0, \\ -\left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c < 0. \end{cases}$$

Finally ϵ_d is 1 if $d \equiv 1 \pmod 4$ and i if $d \equiv 3 \pmod 4$. Let $S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ denote the subspace of cusp forms.

For an integer-valued quadratic form Q, let $r_Q(n) = \#\{\vec{x} \in \mathbb{Z}^r : Q(\vec{x}) = n\}$. The theta series of Q is the generating function

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n, \quad q = e^{2\pi i z}.$$

When r is even, Theorem 10.9 of [19] shows that $\theta_Q(z) \in M_{r/2}(\Gamma_0(N), \chi_D)$, where $D = (-1)^{r/2} \det A$. If r is odd, Theorem 10.8 of [19] gives that $\theta_Q(z) \in M_{r/2}(\Gamma_0(2N), \chi_{2 \det A})$. Here and throughout, χ_D denotes the Kronecker character of the field $\mathbb{Q}(\sqrt{D})$. We may decompose $\theta_Q(z)$ as

$$\theta_Q(z) = E(z) + C(z)$$

where $E(z) = \sum_{n=0}^{\infty} a_E(n) q^n$ is an Eisenstein series, and $C(z) = \sum_{n=1}^{\infty} a_C(n) q^n$ is a cusp form.

If Q is an integer-valued positive definite quadratic form Q, one can associate to Q a lattice L (and vice versa) as follows. We let $L = \mathbb{Z}^r$ and define an inner product on L by

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{2} \left(Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y}) \right).$$

If $\vec{x} \in L$, then $\langle \vec{x}, \vec{x} \rangle = Q(\vec{x})$ is an integer, however arbitrary inner products $\langle x, y \rangle$ with $\vec{x}, \vec{y} \in L$ need not be integral. Suppose that R is an integer-valued quadratic form in $m \leq r$ variables $y_1, y_2, \ldots, y_m$. Then Q represents R if there are linear forms $L_1, L_2, \ldots, L_r$ in the y_i with integer coefficients so that

$$Q(L_1, L_2, \ldots, L_r) = R.$$

It is easy to see that this occurs if and only if there is a dimension m sublattice $L' \subseteq L$ so that L' is isometric to the lattice corresponding to R.

Let $\mathbb{Z}_p$ be the ring of p-adic integers. We say that Q locally represents the non-negative integer m if for all primes p there is a vector $\vec{x}_p \in \mathbb{Z}_p^r$ so that $Q(\vec{x}_p) = m$. We say that m is represented by Q if there is a vector $\vec{x} \in \mathbb{Z}^r$ with $Q(\vec{x}) = m$.

For a quadratic form Q, we let $\mathrm{Gen}(Q)$ denote the finite collection of quadratic forms R so that R is equivalent to Q over $\mathbb{Z}_p$ for all primes p. From the work of Siegel [42] it is known that we can express the Eisenstein series $E(z)$ as a weighted sum over the genus. In particular,

$$E(z) = \frac{\sum_{R \in \mathrm{Gen}(Q)} \frac{\theta_R(z)}{\#\mathrm{Aut}(R)}}{\sum_{R \in \mathrm{Gen}(Q)} \frac{1}{\#\mathrm{Aut}(R)}}.$$

Moreover, the coefficients $a_E(m)$ of $E(z)$ can be expressed as a product

$$a_E(m) = \prod_{p \leq \infty} \beta_p(m)$$

of local densities $\beta_p(m)$. We will make use of the algorithms of Hanke [16] and the formulas of Yang [45] for these local densities.

If Q is a quadratic form over $\mathbb{Q}_p$, Q is equivalent to a diagonal form

$$a_1 x_1^2 + a_2 x_2^2 + \cdots + a_r x_r^2.$$

The discriminant of Q is defined to be $\prod_{i=1}^r a_i$, and is well-defined up to a square in $\mathbb{Q}_p^\times$. We define the ϵ-invariant of Q as in Serre [40] by

$$\epsilon_p(Q) = \prod_{1 \leq i < j \leq r} (a_i, a_j)_p,$$

where $(a,b)_p$ denotes the usual Hilbert symbol. Theorem 4.7 (pg. 39) of [40] proves that two quadratic forms are equivalent over $\mathbb{Q}_p$ if and only if they have the same rank r, the same discriminant, and the same ϵ-invariant.

If Q is an integer-valued quadratic form and p is a prime, we say that Q is anisotropic at p if whenever $\vec{x} \in \mathbb{Z}_p^r$ and $Q(\vec{x}) = 0$, then $\vec{x} = 0$. If the rank of Q is 3 or 4, Q has only finitely many anisotropic primes, and if Q is anisotropic at p, then $p|N$. When $r = 4$, there is a unique $\mathbb{Q}_p$ equivalence class of forms that are anisotropic at p. Such forms have a square discriminant in $\mathbb{Q}_p^\times$, and ϵ-invariant $\epsilon_p(Q) = -(-1,-1)_p$. If the rank of Q is greater than or equal to 5, Q does not have any anisotropic primes.

We will briefly review the theory of integer weight newforms due to Atkin, Lehner, and Li. If d is a positive integer, the map $f(z)|V(d) = f(dz)$ sends $S_k(\Gamma_0(M),\chi)$ to $S_k(\Gamma_0(Md),\chi)$. For forms $f, g \in S_k(\Gamma_0(N),\chi)$, define the Petersson inner product

$$\langle f, g\rangle = \frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \iint_{\mathbb{H}/\Gamma_0(N)} f(x+iy)\overline{g(x+iy)}y^k \frac{dx\,dy}{y^2}.$$

For each prime p, there is a Hecke operator $T(p) : S_k(\Gamma_0(N),\chi) \to S_k(\Gamma_0(N),\chi)$ given by

$$\left(\sum_{n=1}^{\infty} a(n)q^n\right)|T(p) = \sum_{n=1}^{\infty} \left(a(pn) + \chi(p)p^{k-1}a(n/p)\right) q^n.$$

If p is a prime with $\gcd(N,p) = 1$, then the adjoint of the Hecke operator $T(p)$ under the Petersson inner product is $\overline{\chi}(p)T(p)$ (see Theorem 5.5.3 of [10]).

For N fixed, let $S_k^{\mathrm{old}}(\Gamma_0(N),\chi)$ be the subspace of $S_k(\Gamma_0(N),\chi)$ generated by $S_k(\Gamma_0(M),\chi)|V(d)$ over all pairs (d,M) with $dM|N$, $\mathrm{cond}(\chi)|M$ and $M < N$. Let $S_k^{\mathrm{new}}(\Gamma_0(N),\chi)$ be the orthogonal complement of $S_k^{\mathrm{old}}(\Gamma_0(N),\chi)$ with respect to the Petersson inner product.

A newform is a form $f \in S_k^{\mathrm{new}}(\Gamma_0(N),\chi)$ that is a simultaneous eigenform of the operators $T(p)$ for all primes p, and normalized so that if

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n,$$

then $a(1) = 1$. The space $S_k^{\mathrm{new}}(\Gamma_0(N),\chi)$ is spanned by newforms. Deligne's theorem gives the bound

$$|a(n)| \le d(n)n^{\frac{k-1}{2}}$$

on the nth Fourier coefficient of any newform, where $d(n)$ is the number of divisors of n. (In the case of $k = 2$, this result was first established by Eichler, Shimura, and Igusa.) The adjoint formula for the Hecke operators shows that if f and g are two distinct newforms, then $\langle f, g\rangle = 0$. If $\mathrm{cond}(\chi)$ denotes the conductor of the Dirichlet character χ and p is a prime with $p|N$, then the pth coefficient of the newform f satisfies

$$|a(p)| = \begin{cases} p^{\frac{k-1}{2}} & \text{if } \mathrm{cond}(\chi) \nmid N/p \\ p^{\frac{k}{2}-1} & \text{if } p^2 \nmid N \text{ and } \mathrm{cond}(\chi)|N/p \\ 0 & \text{if } p^2|N \text{ and } \mathrm{cond}(\chi)|N/p. \end{cases} \tag{1}$$

(See Theorem 3 of [31].) Finally, define the operator $W_N : S_k^{\mathrm{new}}(\Gamma_0(N), \chi) \to S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ by

$$f|W_N = N^{-k/2} z^{k/2} f\left(-\frac{1}{Nz}\right).$$

We have $W_N^2 = (-1)^k$.

If $\epsilon \in \{\pm 1\}$, define the subspace $M_k^{\epsilon}(\Gamma_0(N), \chi)$ to be the set of forms

$$g(z) = \sum_{n=0}^{\infty} b(n) q^n \in M_k(\Gamma_0(N), \chi)$$

with the property that $b(n) = 0$ if $\chi(n) = -\epsilon$, and let $S_k^{\epsilon}(\Gamma_0(N), \chi) = M_k^{\epsilon}(\Gamma_0(N), \chi) \cap S_k(\Gamma_0(N), \chi)$. Equation (6.57) of [19] shows that if $f(z) = \sum_{n=1}^{\infty} a(n) q^n$ is a newform, and $\gcd(n, N) = 1$, then $a(n) = \chi(n) \overline{a(n)}$. In the case when χ is quadratic, and $\mathrm{cond}(\chi) = N$, the old subspace is trivial, and

$$\dim S_k^{+}(\Gamma_0(N), \chi) = \dim S_k^{-}(\Gamma_0(N), \chi) = \frac{1}{2} \dim S_k(\Gamma_0(N), \chi)$$

and the ϵ-subspace is spanned by $\{f + \epsilon \overline{f} : f \text{ a newform } \}$, where if $f(z) = \sum_{n=1}^{\infty} a(n) q^n$, then $\overline{f}(z) = \sum_{n=1}^{\infty} \overline{a(n)} q^n$.

A newform f of weight $k \geq 2$ is said to have complex multiplication (or CM) if there is Hecke Grössencharacter ξ that corresponds to it. This means that there is an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$, a nonzero ideal $\Lambda \subseteq O_K$, and a homomorphism ξ from the group of all fractional ideals of O_K relatively prime to Λ to $\mathbb{C}^{\times}$ so that

$$\xi(\alpha O_K) = \alpha^{k-1} \quad \text{if } \alpha \equiv 1 \pmod{\Lambda},$$

and so that

$$f(z) = \sum_{\mathfrak{a} \subseteq O_K} \xi(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where the sum if over all integral ideals $\mathfrak{a}$ of O_K and $N(\mathfrak{a}) = \#(O_K/\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. For more details about Hecke Grössencharacters, see Chapter 12 of [19].

3. Rankin-Selberg L-functions

If Q is a positive-definite, quaternary, integer-valued quadratic form, then

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) \in M_2(\Gamma_0(N), \chi)$$

for some positive integer N, and Dirichlet character χ. We may decompose

$$\theta_Q(z) = E(z) + C(z)$$

where

$$C(z) = \sum_{n=1}^{\infty} a_C(n) q^n \in S_2(\Gamma_0(N), \chi).$$

Lower bounds on the coefficients $a_E(n)$ of $E(z)$ are given by Hanke in [16] when n is locally represented by Q (provided n has a priori bounded divisibility by any anisotropic primes) and are of the form $a_E(n) \gg_Q n^{1-\epsilon}$. We may decompose

$$C(z) = \sum_{M|N} \sum_{i=1}^{\dim S_2^{\text{new}}(\Gamma_0(M),\chi)} \sum_{d} c_{d,i,M} g_{i,M} | V(d), \tag{2}$$

where the $g_{i,M}$ are newforms of level M. Applying Deligne's bound, we have that the nth Fourier coefficient of $g_{i,M}|V(d)$ is bounded by

$$d(n/d)\sqrt{n/d} \leq \frac{1}{\sqrt{d}} d(n) \sqrt{n}.$$

Thus, we define the cusp constant C_Q to be

$$C_Q := \sum_{M|N} \sum_{i} \sum_{d} \frac{|c_{d,i,M}|}{\sqrt{d}},$$

and we have that $a_C(n) \leq C_Q d(n) n^{1/2}$ for all n. (When we are considering the representation of odd numbers by Q, we omit those terms in the definition of C_Q for which d is even.)

Combining the lower bound on $a_E(n)$ with the upper bound on $a_C(n)$ shows that Q fails to represent only finitely many positive integers that are locally represented by Q, and have bounded divisibility by any anisotropic primes. We are interested in determining the dependence on the form Q of the constant C_Q, and the implied constant in the estimate for $a_E(n) \gg_Q n^{1-\epsilon}$. These bounds we obtain will prove Theorem 6 and will be the basis of one of the methods we use in Section 5 to prove the 451-Theorem.

For the remainder of this section, we assume that Q is a positive-definite, quaternary quadratic form whose discriminant D is a fundamental discriminant. This implies that $N = D$, and also that χ is a primitive Dirichlet character modulo N. Then the old subspace of $S_2(\Gamma_0(N),\chi)$ is trivial, and the decomposition above simply becomes

$$C(z)=\sum_{i=1}^{u} c_i g_i(z),$$

where $u=\dim S_2(\Gamma_0(N),\chi)$, and the $g_i(z)$ are newforms in $S_2(\Gamma_0(N),\chi)$. Taking the Petersson inner product of C with itself, and using that $\langle g_i, g_j\rangle = 0$ if $i\neq j$ implies that

$$\langle C, C\rangle = \sum_{i=1}^{u} |c_i|^2 \langle g_i, g_i\rangle.$$

Suppose that we have bounds A and B so that $\langle C, C\rangle \leq A$ and $\langle g_i, g_i\rangle \geq B$ for all i. Then, we have

$$\sum_{i=1}^{u} B|c_i|^2 \leq A$$

and so

$$C_Q=\sum_{i=1}^{u}|c_i| \leq \sqrt{u}\sqrt{\sum_{i=1}^{u}|c_i|^2} \leq \sqrt{\frac{Au}{B}}, \tag{3}$$

which follows by the Cauchy-Schwarz inequality. Hence, a bound on C_Q follows from an upper bound on $\langle C, C\rangle$ and a lower bound on $\langle g_i, g_i\rangle$. We will derive bounds on both of these quantities using the theory of Rankin-Selberg L-functions.

Suppose that $f(z)=\sum_{n=1}^{\infty} a(n)q^n$ and $g(z)=\sum_{n=1}^{\infty} b(n)q^n$ are cusp forms. Rankin [36] and Selberg [39] independently developed their convolution L-function

$$\sum_{n=1}^{\infty}\frac{a(n)b(n)}{n^s}$$

and studied its analytic properties. The most relevant property is that the residue of this L-function at $s=1$ is essentially the Petersson inner product $\langle f, g\rangle$.

In order to work out the exact local factors at primes dividing the level, and the gamma factors, it is necessary to transfer to the language of automorphic representations. In this setting, the problem was solved by Jacquet, Piatetskii-Shapiro, and Shalika [21], and the local Langlands correspondence gives a convenient formulation.

A newform g corresponds to a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ (see [6], Chapter 7 for details). Such a representation can be factored as

$$\pi = \otimes_{p\leq\infty}\pi_p$$

where each π_p is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. For any $n\geq 1$, the local Langlands correspondence (proven by Harris and Taylor [17], for more details see Section 10.3 of [6], [30], and [7] for a thorough discussion of the $GL(2)$ case) gives a bijection between the set of smooth, irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ and degree n complex representations of the Weil-Deligne group $W'_{\mathbb{Q}_p}$. Known instances of automorphic lifting maps (including the adjoint square map $r : \mathrm{GL}_2 \to \mathrm{GL}_3$ due to Gelbart and Jacquet [13], the Rankin-Selberg convolution $r : \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ due to Ramakrishnan [34], and the symmetric fourth power map $r : \mathrm{GL}_2 \to \mathrm{GL}_5$ due to Kim [27]) are constructions of automorphic representations

$$\Pi = r(\pi) = \otimes_{p \leq \infty} \Pi_p$$

where Π_p is computed by mapping π_p to a degree 2 complex representation ρ_p of $W'_{\mathbb{Q}_p}$ via the local Langlands correspondence, computing $r(\rho_p)$ and mapping back to the automorphic side (again by the local Langlands correspondence). Since the local Langlands correspondence preserves local L-functions, to compute the Euler factors of $L(\Pi, s)$, it suffices to know the representations $r(\rho_p)$.

Proposition 8. *Assume the notation above, and suppose that f and g are newforms in $S_2(\Gamma_0(N), \chi)$. Write*

$$L(f, s) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$
$$L(g, s) = \prod_p (1 - \gamma_p p^{-s})^{-1} (1 - \delta_p p^{-s})^{-1}.$$

Then, we have

$$L(f \otimes g, s) = \prod_{p \mid N} (1 - p^{-s})^{-2} \prod_{p \nmid N} (1 - \alpha_p \gamma_p p^{-s})^{-1} (1 - \alpha_p \delta_p p^{-s})^{-1} (1 - \beta_p \gamma_p p^{-s})^{-1} (1 - \beta_p \delta_p p^{-s})^{-1},$$

$$L(\mathrm{Ad}^2 f, s) = \prod_p (1 - \alpha_p^2 \chi(p) p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \beta_p^2 \chi(p) p^{-s})^{-1}, \text{ and}$$

$$L(\mathrm{Sym}^4 f, s) = \prod_p (1 - \alpha_p^4 p^{-s})^{-1} (1 - \alpha_p^2 \chi(p) p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \alpha_p^{-2} \chi(p) p^{-s})^{-1} (1 - \alpha_p^{-4} p^{-s})^{-1}.$$

These L-functions are holomorphic (with the exception of a possible pole of $L(f \otimes g, s)$ at $s = 1$), and satisfy the following functional equations.

$$\begin{aligned}
\Lambda(f \otimes g, s) &= N^s \pi^{-2s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right) L(f \otimes g, s)\\
\Lambda(f \otimes g, s) &= \Lambda(f \otimes g, 1-s)\\
\Lambda(\mathrm{Ad}^2 f, s) &= N^s \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right) L(\mathrm{Ad}^2 f, s)\\
\Lambda(\mathrm{Ad}^2 f, 1-s) &= \Lambda(\mathrm{Ad}^2 f, s)\\
\Lambda(\mathrm{Sym}^4 f, s) &= N^s \pi^{-5s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+2}{2}\right)^2 \Gamma\left(\frac{s+3}{2}\right) L(\mathrm{Sym}^4 f, s)\\
\Lambda(\mathrm{Sym}^4 f, 1-s) &= \Lambda(\mathrm{Sym}^4 f, s).
\end{aligned}$$

Proof. Let π_1 and π_2 be the local representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that occur in f and g, respectively. Since the pth Fourier coefficients of f and g have absolute value $p^{1/2}$ (by (1)), the local representations π_1 and π_2 must be principal series representations $\pi(\epsilon_i, \psi\epsilon_i^{-1})$, where ϵ is an unramified character of $\mathbb{Q}_p^\times$ and ψ is the local component of the Dirichlet character χ at p. (This follows from a comparison of the different options for the local L-functions described in Chapter 6, Sections 25 and 26 of [7].)

Applying the local Langlands correspondence, it follows that π_1 and π_2 correspond to representations ρ_1 and ρ_2 of the Weil group that are each the direct sum of two characters. The Weil group $W_{\mathbb{Q}_p}$ is the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ consisting of all elements restricting to some power of the Frobenius on $\overline{\mathbb{F}_p}$. It is a quotient of the Weil-Deligne group. If δ is one of the characters used to construct the principal series representation for π_i, the corresponding constituent of ρ_i is given by

$$\sigma \mapsto \delta(c(\sigma))$$

where $c : \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is the reciprocity law isomorphism of local class field theory.

Now, if r is the tensor product map $r : \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ then

$$r(\delta_1 \oplus \delta_2, \delta_3 \oplus \delta_4) = \delta_1\delta_3 \oplus \delta_1\delta_4 \oplus \delta_2\delta_3 \oplus \delta_2\delta_4.$$

Similarly, if r is the adjoint square map $r : \mathrm{GL}_2 \to \mathrm{GL}_3$ or r is the symmetric fourth power map $r : \mathrm{GL}_2 \to \mathrm{GL}_5$, the the two-dimension representation of $W_{\mathbb{Q}_p}$ maps to a three or five dimension representation of $W_{\mathbb{Q}_p}$ that is a direct sum of characters. The description of the local factors above follows from this, and a simple calculation shows that the product of the local signs over all primes p is equal to 1. The gamma factors are known (see [20] pg. 132 for the Rankin-Selberg L-function, [20] pg. 137 for the adjoint square L-function, and [9] for the symmetric fourth power L-function). $\square$

Remark. *One can obtain numerical confirmation of the result above by checking the stated functional equations using the L-functions package (available in PARI/GP, Magma and Sage) due to Tim Dokchitser (see* [12]*).*

The following formula gives the residue at $s = 1$ of the Rankin-Selberg L-function.

Lemma 9. *Suppose that $f \in S_2(\Gamma_0(N), \chi)$, where $\mathrm{cond}(\chi) = N$. Then,*

$$\mathrm{Res}_{s=1} L(f \otimes \overline{f}, s) = \frac{8\pi^4}{3} \left(\prod_{p|N} 1 + \frac{1}{p} \right) \langle f, f \rangle.$$

Proof. Suppose that

$$L(f, s) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

is the standard L-function attached to f. The "classical" Rankin-Selberg L-function is then given by

$$L(f \otimes \overline{f}, s)_{\mathrm{classical}} = \prod_p (1 - |\alpha_p|^2 |\beta_p|^2 p^{-2s}) \cdot$$
$$\prod_p (1 - |\alpha_p|^2 p^{-s})^{-1} (1 - \alpha_p \overline{\beta_p} p^{-s})^{-1} (1 - \overline{\alpha_p} \beta_p p^{-s})^{-1} (1 - |\beta_p|^2 p^{-s})^{-1}$$
$$= \sum_{n=1}^{\infty} \frac{|c(n)|^2}{n^s}.$$

Equation (13.34) of [19] gives the formula

$$\mathrm{Res}_{s=1} L(f \otimes \overline{f}, s)_{\mathrm{classical}} = \frac{(4\pi)^2}{\Gamma(2)} \langle f, f \rangle.$$

Comparing the Euler factors at primes $p|N$, we see that the "Langlands" Rankin-Selberg L-function is related to the classical one by

$$L(f \otimes \overline{f}, s) = \prod_{p \nmid N} (1 - p^{-2s})^{-1} \prod_{p|N} (1 - p^{-s})^{-1} L(f \otimes \overline{f}, s)_{\mathrm{classical}}.$$

Taking residues at $s = 1$, we have

$$\mathrm{Res}_{s=1} L(f \otimes \overline{f}, s) = \prod_{p \nmid N} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p|N} \left(1 - \frac{1}{p}\right)^{-1} \mathrm{Res}_{s=1} L(f \otimes \overline{f}, s)_{\mathrm{classical}}$$
$$= \frac{8\pi^4}{3} \left(\prod_{p|N} 1 + \frac{1}{p} \right) \langle f, f \rangle,$$

as desired. $\square$

Since $L(f \otimes \overline{f}, s) = \zeta(s) L(\mathrm{Ad}^2 f, s)$, we have $\mathrm{Res}_{s=1} L(f \otimes \overline{f}, s) = L(\mathrm{Ad}^2 f, 1)$. Work of Goldfeld, Hoffstein, and Lieman (see the appendix to [18]) shows that if f is not a CM form, then $L(\mathrm{Ad}^2 f, s)$ cannot have any real zeroes close to $s = 1$, and this implies that $L(\mathrm{Ad}^2 f, 1)$ cannot be too small. Here we give an explicit version of this result that is relevant for the case at hand. (See Lemmas 2 and 3 of [37] for a version in the case that f has level one.)

Proposition 10. *Suppose that χ is a quadratic Dirichlet character with conductor N, and $f \in S_2(\Gamma_0(N), \chi)$ is a newform without complex multiplication. If $N \geq 44$, then $L(\mathrm{Ad}^2 f, s)$ has no real zeroes s with*

$$s > 1 - \frac{5 - 2\sqrt{6}}{4\log(N) - 11}.$$

Proof. Goldfeld, Hoffstein, and Lieman use the auxiliary degree 16 L-function

$$L(s) = \zeta(s)^2 L(\mathrm{Ad}^2 f, s)^3 L(\mathrm{Sym}^4 f, s).$$

The gamma factor is

$$G(s) = N^{4s} \pi^{-16s/2} \Gamma\left(\frac{s}{2}\right)^3 \Gamma\left(\frac{s+1}{2}\right)^7 \Gamma\left(\frac{s+2}{2}\right)^5 \Gamma\left(\frac{s+3}{2}\right),$$

and the completed L-function $\Lambda(s) = s^2(1-s)^2 G(s) L(s)$ has an analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation $\Lambda(s) = \Lambda(1-s)$. (If the newform f has complex multiplication, then $L(\mathrm{Sym}^4 f, s)$ has a pole at $s = 1$ and the argument below breaks down.)

The function $\Lambda(s)$ is an entire function of order 1, and so we let

$$\Lambda(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

be its Hadamard product expansion. Taking the logarithmic derivative gives

$$\sum_{\rho} \frac{1}{s-\rho} + \frac{1}{\rho} = \frac{2}{s} + \frac{2}{s-1} + \frac{G'(s)}{G(s)} + \frac{L'(s)}{L(s)} - B. \tag{4}$$

We take the real part of both sides. Part 3 of Proposition 5.7 of [20] gives that $\mathrm{Re}\,(B) = -\sum_{\rho} \mathrm{Re}\left(\frac{1}{\rho}\right)$. The Dirichlet coefficients of $-L'(s)/L(s)$ are non-negative, and this implies that $L'(s)/L(s) < 0$ if $s > 1$ is real. Taking the real part of (4) gives that

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{s-\rho}\right) \leq \frac{2}{s} + \frac{2}{s-1} + \frac{G'(s)}{G(s)}.$$

We have

$$\frac{G'(s)}{G(s)} = 4\log(N) - 8\log(\pi) + \frac{1}{2}\left[3\psi(s/2) + 7\psi((s+1)/2) + 5\psi((s+2)/2) + \psi((s+3)/2)\right],$$

where $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$. Since $\psi(s)$ is an increasing function of s, we have that $\frac{G'(s)}{G(s)} \leq 4\log(N) - 13$ if $s \leq 1.11$.

We set $s = 1 + \alpha$ where $0 \leq \alpha \leq 0.05$ will be chosen later. If β is a real zero of $L(\mathrm{Ad}^2 f, s)$, then it is a triple zero of $L(s)$, and this means that

$$\frac{3}{\alpha + 1 - \beta} \leq \frac{2}{\alpha + 1} + \frac{2}{\alpha} + \frac{G'(1+\alpha)}{G(1+\alpha)} \leq \frac{2}{\alpha} + (4\log(N) - 11).$$

Choosing α optimally gives that $1 - \beta \geq \frac{5-2\sqrt{6}}{4\log(N)-11}$, provided the corresponding value of s is less than 1.11. This occurs for $N \geq 44$, and shows that

$$\beta \leq 1 - \frac{5 - 2\sqrt{6}}{4\log(N) - 11}.$$

□

We now translate the above result into a lower bound on $L(\mathrm{Ad}^2 f, 1)$ by a similar argument to that in Lemma 3 of [37].

Proposition 11. *Suppose that f is a newform in $S_2(\Gamma_0(N), \chi)$ that does not have complex multiplication. Then*

$$L(\mathrm{Ad}^2 f, 1) > \frac{1}{26\log(N)}.$$

Proof. We consider

$$L(f \otimes \overline{f}, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Then $a(n) \geq 0$ for all $n \geq 1$. Moreover, it is straight-forward to see that $a(n^2) \geq 1$ for all n.

Let $\beta = 1 - \frac{5-2\sqrt{6}}{4\log(N)-11}$ and assume that N is large enough that $\beta \geq 3/4$. Set $x = N^A$, where we let A be a parameter that we will choose optimally at the end of the argument. We consider

$$I = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L(f \otimes \overline{f}, s+\beta) x^s \, ds}{s \prod_{k=2}^{10} (s+k)}.$$

We have that

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s \, ds}{s \prod_{k=2}^{10} (s+k)} = \begin{cases} \frac{(x+9)(x-1)^9}{10! x^{10}}, & \text{if } x > 1 \\ 0, & \text{if } x < 1. \end{cases}$$

Therefore,

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L(f\otimes\overline{f}, s+\beta)}{s\prod_{k=2}^{10}(s+k)} = \sum_{n\leq x} \frac{a(n)(x/n+9)(x/n-1)^9}{10!n^{\beta}(x/n)^{10}}$$
$$\geq \frac{1}{10!}\sum_{n^2\leq x} \frac{(x/n^2+9)(x/n^2-1)^9}{n^2(x/n^2)^{10}}.$$

Since the function $g(z) = \frac{(z+9)(z-1)^9}{z^{10}}$ is increasing for $z > 1$, the above expression is increasing as a function of x. If $x \geq 3989$, then $I \geq \frac{1.6}{10!}$, and if $x \geq 330775$, then $I \geq \frac{1.64}{10!}$.

Now, we move the contour to $\mathrm{Re}\,(s) = \alpha$, where $\alpha = -3/2 - \beta$. There are poles at $s = 1-\beta$, $s = 0$, and $s = -2$. We get

$$I = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L(f\otimes\overline{f}, s+\beta)x^s\,ds}{s\prod_{k=2}^{10}(s+k)} + \frac{L(\mathrm{Ad}^2 f,1)x^{1-\beta}}{(1-\beta)\prod_{k=2}^{10}(1-\beta+k)}$$
$$+\frac{L(f\otimes\overline{f},\beta)}{10!} - \frac{L(f\otimes\overline{f},-2+\beta)x^{-2}}{2\cdot 8!}.$$

There are no zeroes of $L(\mathrm{Ad}^2 f, s)$ to the right of β and so $L(\mathrm{Ad}^2 f,\beta) \geq 0$. Since $\zeta(\beta) < 0$, it follows that $L(f\otimes\overline{f},\beta) \leq 0$. Since the sign of the functional equation of $L(f\otimes\overline{f}, s)$ is 1, it follows that there are an even number of real zeroes in the interval $(0,1)$ and hence $L(f\otimes\overline{f}, 0) < 0$. The only zeroes with $s < 0$ are trivial zeroes, and a simple zero occurs at $s = -1$. Thus, $L(f\otimes\overline{f},-2+\beta) > 0$ and so

$$I - \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L(f\otimes\overline{f}, s+\beta)x^s\,ds}{s\prod_{k=2}^{10}(s+k)} \leq \frac{L(\mathrm{Ad}^2 f,1)x^{1-\beta}}{(1-\beta)\prod_{k=2}^{10}(1-\beta+k)}$$

Now, we apply the functional equation for $L(f\otimes\overline{f}, s)$. It gives that

$$\left|L\left(f\otimes\overline{f}, -\frac{3}{2}+it\right)\right| = \frac{N^4}{(4\pi)^8}|1+2it|^4|3+2it|^3|5+2it|\left|L\left(f\otimes\overline{f},\frac{5}{2}-it\right)\right|.$$

We have that $|L(f\otimes\overline{f},\frac{5}{2}-it)| \leq \zeta(5/2)^4$. We use this to derive the bound

$$\frac{1}{2\pi}\int_{\alpha-i\infty}^{\alpha+i\infty}\left|\frac{L(f\otimes\overline{f}, s+\beta)x^s}{s\prod_{k=2}^{10}(s+k)}\right|\,ds$$
$$\leq \frac{N^{4+A(-3/2-\beta)}\zeta(5/2)^4}{2^{17}\pi^9}\int_{-\infty}^{\infty}\frac{|1+2it|^4|3+2it|^3|5+2it|}{|-3/2-\beta-it|\prod_{k=2}^{10}|k-3/2-\beta+it|}\,dt$$
$$\leq \frac{N^{4+A(-3/2-\beta)}\zeta(5/2)^4}{2^{17}\pi^9}\int_{-\infty}^{\infty}\frac{|1+2it|^4|3+2it|^3|5+2it|}{|1/4+it||9/4+it|\prod_{k=3}^{10}|k-5/2+it|}\,dt.$$

Numerical computation gives the bound

$$\int_{-\infty}^{\infty} \frac{|1+2it|^4|3+2it|^3|5+2it|}{|1/4+it||9/4+it|\prod_{k=3}^{10}|k-5/2+it|}\,dt \leq 2.776686,$$

and this gives

$$\frac{1}{2\pi}\int_{\alpha-i\infty}^{\alpha+i\infty}\left|\frac{L(f\otimes\overline{f},s+\beta)x^s}{s\prod_{k=2}^{10}(s+k)}\right|ds \leq N^{4+A(-3/2-\beta)}\cdot\frac{8.35176\cdot 10^{-3}}{10!}$$

Out of this, we get the lower bound

$$L(\mathrm{Ad}^2 f,1) \geq (1-\beta)\left(\frac{c}{N^{A(1-\beta)}} - \frac{d}{N^{(5/2)A-4}}\right),$$

where $c = 1.6$ or 1.64 depending on whether $3989 \leq x < 330775$ or $x \geq 330775$. If we choose $A = 8/5$ we get

$$L(\mathrm{Ad}^2 f,1) \geq \frac{1}{26\log(N)}.$$

For computational purposes, we use the optimal choice of A, namely

$$A = \frac{1}{\beta+3/2}\left[4 - \frac{\log(1-\beta)+\log(c)-\log(d)-\log(5/2)}{\log(N)}\right]. \tag{5}$$

These bounds suffice when $N \geq 167$. We individually check each of the cases with $N \leq 166$. $\square$

The above proposition implies a lower bound on the Petersson norm of a newform f. We now turn to the problem of bounding from above the Petersson norm $\langle C, C\rangle$. We will give a formula for $\langle C, C\rangle$ using the functional equation for Rankin-Selberg L-functions, and this formula will be used in subsequent sections to prove the 451-Theorem and Theorem 6. First, we give a Dirichlet series representation for the Rankin-Selberg L-function $L(f\otimes g, s)$.

Lemma 12. *Assume the notation above. If m is a positive integer, let $\omega(m)$ denote the number of distinct prime factors of m. If $f, g \in S_2(\Gamma_0(N),\chi)$ are newforms with*

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n, \text{ and } g(z) = \sum_{n=1}^{\infty} b(n)q^n$$

then

$$L(f\otimes g, s) = \sum_{n=1}^{\infty}\left(\sum_{\substack{m\mid n\\ n/m \text{ is a square}}} \frac{2^{\omega(\gcd(m,N))}\mathrm{Re}\,(a(m)b(m))}{m}\right)\frac{1}{n^s}.$$

Proof. Equation (13.1) of [19] states that if

$$\sum_{n=1}^{\infty}\frac{c(n)}{n^s} = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}, \text{ and } \quad \sum_{n=1}^{\infty}\frac{e(n)}{n^s} = \prod_p (1-\gamma_p p^{-s})^{-1}(1-\delta_p p^{-s})^{-1},$$

then

$$\sum_{n=1}^{\infty} \frac{c(n)e(n)}{n^s} = \prod_p (1 - \alpha_p \beta_p \gamma_p \delta_p p^{-2s})$$
$$\prod_p (1-\alpha_p\gamma_p p^{-s})^{-1}(1-\alpha_p\delta_p p^{-s})^{-1}(1-\beta_p\gamma_p p^{-s})^{-1}(1-\beta_p\delta_p p^{-s})^{-1}.$$

If we take $c(n) = a(n)$ and $e(n) = b(n)$, and $L(f, s) = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1}$ and $L(g, s) = \prod_p (1 - \gamma_p p^{-s})^{-1}(1 - \delta_p p^{-s})^{-1}$, it follows that

$$\prod_{p \nmid N} (1-\alpha_p\gamma_p p^{-s})^{-1}(1-\alpha_p\delta_p p^{-s})^{-1}(1-\beta_p\gamma_p p^{-s})^{-1}(1-\beta_p\delta_p p^{-s})^{-1}$$
$$= \prod_{p \nmid N} (1 - p^{-2s})^{-1} \sum_{n \text{ coprime to } N} \frac{a(n)b(n)}{n^{s+1}}.$$

Now, if $p|N$, we have $\beta_p = \delta_p = 0$. Thus

$$(1 - \alpha_p \gamma_p p^{-s})^{-1} = \sum_{k=0}^{\infty} \frac{a(p^k)b(p^k)}{p^{k(s+1)}}. \tag{6}$$

The local factor of $L(f \otimes g, s)$ at p is $(1 - \alpha_p\gamma_p p^{-s})^{-1}(1 - \overline{\alpha_p\gamma_p} p^{-s})^{-1}$. Multiplying (6) by its conjugate, we get

$$\begin{aligned}(1 - \alpha_p\gamma_p p^{-s})^{-1}(1 - \overline{\alpha_p\gamma_p} p^{-s})^{-1} &= \sum_{k=0}^{\infty} \frac{1}{p^{k(s+1)}} \sum_{i=0}^{k} a(p^i)b(p^i)\overline{a(p)^{k-i}b(p)^{k-i}} \\ &= (1 - p^{-2s})^{-1} \left(1 + 2\sum_{k=1}^{\infty} \frac{\mathrm{Re}\left(a(p^k)b(p^k)\right)}{p^{s(k+1)}}\right).\end{aligned}$$

Taking the product of the local factors over all primes p gives us the desired formula. $\square$

If C_1 and C_2 are arbitrary cusp forms in $S_2(\Gamma_0(N), \chi)$, we define $L(C_1 \otimes C_2, s)$ as follows. Write

$$C_1(z) = \sum_{i=1}^{u} c_i g_i(z) \text{ and } C_2(z) = \sum_{j=1}^{u} d_j g_j(z),$$

where the $g_i(z)$, $1 \le i \le u$ are the newforms. Then, let

$$L(C_1 \otimes C_2, s) = \sum_{i=1}^{u} \sum_{j=1}^{u} c_i d_j L(g_i \otimes g_j, s).$$

The formula from Lemma 12 is not, in general, bilinear, and so it cannot equal $L(C_1 \otimes C_2, s)$ for all pairs $C_1, C_2 \in S_2(\Gamma_0(N), \chi)$. The next result is that the formula is valid, provided both C_1 and C_2 are in $S_2^+(\Gamma_0(N), \chi)$ or $S_2^-(\Gamma_0(N), \chi)$.

Lemma 13. *Suppose that f and g are arbitrary cusp forms, with $f, g \in S_2^{\epsilon}(\Gamma_0(N), \chi)$ where $\epsilon \in \{\pm 1\}$ and*

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n, \quad g(z) = \sum_{n=1}^{\infty} b(n)q^n,$$

with $a(n), b(n) \in \mathbb{R}$ for all n. Then

$$L(f \otimes g, s) = \sum_{n=1}^{\infty} \left(\sum_{\substack{m \mid n \\ n/m \text{ is a square}}} \frac{2^{\omega(\gcd(m,N))} a(m) b(m)}{m} \right) \frac{1}{n^s}.$$

Moreover, if

$$\Lambda(f \otimes g, s) = N^s \pi^{-2s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right) L(f \otimes g, s),$$

then $\Lambda(f \otimes g, s) = \Lambda(f \otimes g, 1-s)$, and we have

$$\mathrm{Res}_{s=1} L(f \otimes g, s) = \frac{8\pi^4}{3} \left(\prod_{p \mid N} 1 + \frac{1}{p} \right) \langle f, g \rangle.$$

Proof. All of the statements in the theorem are $\mathbb{R}$-bilinear. For this reason, it suffices to prove them on a collection of basis elements for $S_2^{\epsilon}(\Gamma_0(N), \chi) \cap \mathbb{R}[[q]]$: those of the form $h + \overline{h}$ if $\epsilon = 1$ and $i(h - \overline{h})$ if $\epsilon = -1$. Suppose that h_1 and h_2 are newforms with

$$h_1(z) = \sum_{n=1}^{\infty} a(n)q^n, \quad h_2(z) = \sum_{n=1}^{\infty} b(n)q^n,$$

and set $i_1(z) = h_1(z) + \overline{h_1}(z)$ and $i_2(z) = h_2(z) + \overline{h_2}(z)$ in the case that $\epsilon = 1$ and $i_1(z) = i(h_1(z) - \overline{h_1}(z))$ and $i_2(z) = i(h_2(z) - \overline{h_2}(z))$ in the case that $\epsilon = -1$. A straightforward calculation shows that in both cases,

$$L(i_1 \otimes i_2, s) = \epsilon L(h_1 \otimes h_2, s) + L(\overline{h_1} \otimes h_2, s) + L(h_1 \otimes \overline{h_2}, s) + \epsilon L(\overline{h_1} \otimes \overline{h_2}, s).$$

The formula in Lemma 12 shows that for newforms f and g, $L(\overline{f} \otimes \overline{g}, s) = L(f \otimes g, s)$ and so we have

$$L(i_1 \otimes i_2, s) = 2\epsilon L(h_1 \otimes h_2, s) + 2L(\overline{h_1} \otimes h_2, s).$$

This equality proves all of the claimed results, with the exception of the Dirichlet series representation for $L(i_1 \otimes i_2, s)$.

If $\epsilon = 1$, we have that the numerator of a term in the inner sum of $L(i_1 \otimes i_2, s)$ is

$$\begin{aligned}
&2^{\omega(\gcd(n,N))}\left(2\mathrm{Re}\,(a(n)b(n)) + 2\mathrm{Re}\left(\overline{a(n)}b(n)\right)\right)\\
&= 2^{\omega(\gcd(n,N))}(a(n)b(n) + \overline{a(n)b(n)}) + 2^{\omega(\gcd(n,N))}(\overline{a(n)}b(n) + a(n)\overline{b(n)})\\
&= 2^{\omega(\gcd(n,N))}(a(n) + \overline{a(n)})(b(n) + \overline{b(n)}).
\end{aligned}$$

If $\epsilon = -1$, we have

$$\begin{aligned}
&2^{\omega(\gcd(n,N))}\left(-2\mathrm{Re}\,(a(n)b(n)) + 2\mathrm{Re}\left(\overline{a(n)}b(n)\right)\right)\\
&= 2^{\omega(\gcd(n,N))}(-a(n)b(n) - \overline{a(n)b(n)}) + 2^{\omega(\gcd(n,N))}(\overline{a(n)}b(n) + a(n)\overline{b(n)})\\
&= 2^{\omega(\gcd(n,N))}(ia(n) - i\overline{a(n)})(ib(n) - i\overline{b(n)}).
\end{aligned}$$

It follows that if $i_1(z) = \sum_{n=1}^{\infty} c(n)q^n$ and $i_2(z) = \sum_{n=1}^{\infty} e(n)q^n$, then

$$L(i_1 \otimes i_2, s) = \sum_{n=1}^{\infty} \left(\sum_{\substack{m \mid n \\ n/m \text{ is a square}}} \frac{2^{\omega(\gcd(m,N))}c(m)e(m)}{m} \right) \frac{1}{n^s},$$

which completes the proof. $\square$

Remark. *If $f \in S_2^+(\Gamma_0(N), \chi)$ and $g \in S_2^-(\Gamma_0(N), \chi)$ have real Fourier coefficients, one can see from the definition that $L(f \otimes g, s) = 0$, while the formula from Lemma 12 is typically nonzero. This shows that one cannot use the formula in Lemma 12 in all cases.*

Finally, we give a formula for $\langle C, C\rangle$ under the assumption that $C \in S_2^{\epsilon}(\Gamma_0(N), \chi)$. We follow the approach in [12]. To state our result, let $K_\nu(z)$ denote the usual K-Bessel function of order ν.

Proposition 14. *Suppose that $C(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_2^{\epsilon}(\Gamma_0(N), \chi)$ for $\epsilon \in \{\pm 1\}$. Let*

$$\psi(x) = -\frac{6}{\pi} x K_1(4\pi x) + 24x^2 K_0(4\pi x).$$

Then,

$$\langle C, C\rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n,N))}a(n)^2}{n} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right).$$

Proof. Define (as in [12], pg. 139) the function

$$\Theta(t) = \sum_{n=1}^{\infty} b(n)\phi\left(\frac{nt}{N}\right),$$

where $b(n)$ is the nth Dirichlet coefficient of $L(C \otimes C, s)$, namely

$$b(n) = \sum_{\substack{m \mid n \\ \frac{n}{m} \text{ is a square}}} \frac{2^{\omega(\gcd(m,N))} a(m)^2}{m},$$

and ϕ is the inverse Mellin transform of the gamma factor $\pi^{-2s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)^2\Gamma\left(\frac{s+2}{2}\right)$. Then, $\Theta(t)$ is the inverse Mellin transform of $\Lambda(C \otimes C, s)$. Using the functional equation and shifting the contour to the left gives the formula

$$\Theta\left(\frac{1}{t}\right) = t\Theta(t) + r(t-1) \tag{7}$$

where $r = \mathrm{Res}_{s=1}\Lambda(C \otimes C, s) = -\mathrm{Res}_{s=0}\Lambda(C \otimes C, s)$. Differentiating (7) and setting $t = 1$ gives

$$-\Theta(1) - 2\Theta'(1) = r.$$

Equation (10.43.19) of [32] gives the Mellin transform

$$\int_0^\infty t^{\mu-1} K_\nu(t)\, dt = 2^{\mu-2}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right).$$

Applying the Mellin inversion formula and using that

$$\pi^{-2s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)^2\Gamma\left(\frac{s+2}{2}\right) = (2\pi)^{1-2s}\Gamma(s)\Gamma(s+1)$$

we obtain that

$$\phi(t) = 8\pi^2\sqrt{t}K_1(4\pi\sqrt{t}).$$

Thus,

$$\Theta(t) = \sum_{n=1}^\infty 8\pi^2 b(n)\sqrt{\frac{nt}{N}}K_1\left(4\pi\sqrt{\frac{nt}{N}}\right), \text{ and } \Theta'(t) = \sum_{n=1}^\infty 8\pi^2 b(n)\left(-\frac{2\pi n}{N}K_0\left(4\pi\sqrt{\frac{nt}{N}}\right)\right).$$

Taking the two formulas above, rewriting $b(n)$ as a sum over m and d with $n = md^2$, and switching the order of summation gives the desired formula. $\square$

4. Proof of Theorem 6

In this section, we use the results from Section 3 to prove Theorem 6. Assume as in the previous section that Q is a positive-definite integral-valued quaternary quadratic form with fundamental discriminant $D = D(Q)$ and Gram matrix A. In this case, the level $N = N(Q)$ of Q will equal D, and we will use D and N interchangeably in what follows.

Define the quadratic form Q^* by

$$Q^*(\vec{x}) = \frac{1}{2}\vec{x}^T N A^{-1}\vec{x}.$$

Let

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n = E(z) + C(z), \text{ and}$$
$$\theta_{Q^*}(z) = \sum_{n=0}^{\infty} r_{Q^*}(n) q^n = E^*(z) + C^*(z).$$

Here $E(z), E^*(z)$ are the Eisenstein series and $C(z), C^*(z) \in S_2(\Gamma_0(N), \chi)$. We cannot immediately apply the formulas from Section 3 to estimate $\langle C, C\rangle$ because it is not generally true that $C(z) \in S_2^{\epsilon}(\Gamma_0(N), \chi)$ for $\epsilon = 1$ or $\epsilon = -1$. However, the following result allows us to work with C^* instead.

Proposition 15. *We have $\langle C, C\rangle = N\langle C^*, C^*\rangle$. Moreover, $C^* \in S_2^-(\Gamma_0(N), \chi)$.*

Proof. First, Proposition 10.1 of [19] (pg. 167) shows that

$$\theta_Q | W_N = -\sqrt{N} \theta_{Q^*}.$$

Since each of $E(z)$ and $E^*(z)$ is a linear combination of the theta functions (in the genera of Q and Q^*, respectively), the same formula applies to C and C^*, and so $C|W_N = -\sqrt{N} C^*$. Finally, W_N is an isometry with respect to the Petersson inner product (by Proposition 5.5.2 on page 185 of [10]). It follows that

$$\langle C, C\rangle = \langle C|W_N, C|W_N\rangle = N\langle C^*, C^*\rangle.$$

This proves the first statement.

For the second statement, we will show that $\theta_{Q^*} \in M_2^-(\Gamma_0(N), \chi)$. This implies that $E^* \in M_2^-(\Gamma_0(N), \chi)$, since it is a linear combination of the theta series in $\mathrm{Gen}(Q^*)$, and this in turn implies that $C^* \in S_2^-(\Gamma_0(N), \chi)$.

Proving that $\theta_{Q^*} \in M_2^-(\Gamma_0(N), \chi)$ is a fun exercise using ϵ-invariants. Factor the Dirichlet character χ as

$$\chi = \prod_{p | 2N} \chi_p,$$

where for each prime p, χ_p is a primitive Dirichlet character whose conductor is a power of p. Since $\mathrm{cond}(\chi) = N$, we have that if $p > 2$, $\chi_p(m) = \left(\frac{m}{p}\right)$. We will show that if p is an odd prime dividing N, then $\epsilon_p(Q)$ equals $\chi_p(m)$, where m is any integer relatively prime to N that is represented by Q^*, while for $p = 2$, $\epsilon_2(Q) = -\chi_2(m)$.

From the relation

$$\prod_{p | 2N} \epsilon_p(Q) = 1,$$

we have that if m is represented by Q^* and $\gcd(m, N) = 1$, then

$$\chi(m) = \prod_{p|2N} \chi_p(m) = -\epsilon_2(Q) \prod_{\substack{p|N \\ p>2}} \epsilon_p(Q) = -1.$$

This proves that $\theta_{Q^*} \in M_2^-(\Gamma_0(N), \chi)$.

Suppose that p is an odd prime with $p|N$. Since χ is primitive, it follows that $\mathrm{ord}_p(D) = \mathrm{ord}_p(N) = 1$. It follows that the local Jordan splitting of Q is one of the options listed in the table.

Form	Determinant square-class	ϵ
$x^2 + y^2 + z^2 + pw^2$	p	1
$x^2 + y^2 + nz^2 + npw^2$	p	-1
$x^2 + y^2 + z^2 + npw^2$	np	1
$x^2 + y^2 + nz^2 + pw^2$	np	$-1.$

Here n represents an element of $\mathbb{Z}_p^\times$ that is not a square.

If the local Jordan splitting of the form Q is $ax^2 + by^2 + cz^2 + dw^2$, where d is either p or np, the local splitting of the form Q^* is $Na^{-1}x^2 + Nb^{-1}y^2 + Nc^{-1}z^2 + Nd^{-1}w^2$. It follows that if m is represented by Q^* and m is coprime to p, then $\chi_p(m) = \chi_p(Nd^{-1})$. If N/p is a square mod p, then the determinant square class of Q is p. It follows that Nd^{-1} is a square mod p if and only if $\epsilon = 1$. If N/p is not a square mod p, the determinant square class of Q is np and once again Nd^{-1} is a square mod p if and only if $\epsilon = 1$. This proves that $\chi_p(Nd^{-1}) = \epsilon_p(Q)$ if p is odd.

Over $\mathbb{Z}_2$ every integral quadratic form can be decomposed into a diagonal terms, and multiples of blocks of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (see [24]). If the $D = \det A$ is odd, then its Jordan splitting over $\mathbb{Z}_2$ cannot contain any diagonal components. Therefore its splitting must consist of two blocks. In the case that $D \equiv 1 \pmod 8$, the two blocks must be $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and in the case when $D \equiv 5 \pmod 8$, one block is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the other is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Over $\mathbb{Q}_2$, the blocks $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are equivalent to $2x^2 - 2y^2$ and $2x^2 + 6y^2$. This means that the local Jordan splitting of A is either

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

These are equivalent to $x^2 - y^2 + z^2 - w^2$ and $x^2 - y^2 + z^2 + 3w^2$ respectively, and both of these have $\epsilon = -1$.

When the level is a multiple of 4 but not a multiple of 8, one can see that the quadratic form is equivalent over $\mathbb{Z}_2$ to either

$$\begin{bmatrix} 2a & 0 & 0 & 0 \\ 0 & 2b & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 2a & 0 & 0 & 0 \\ 0 & 2b & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

where $ab \equiv 1 \pmod 4$. A straightforward calculation shows that in this case $\epsilon \equiv a \pmod 4$. The local splitting of Q^* shows that the relevant part (mod 4) is $\frac{N}{4}ax^2 + \frac{N}{4}by^2$. Since $N \equiv 0 \pmod 4$ and $N/4 \equiv 3 \pmod 4$, this shows that the 2-adic squareclass represented by Q^* is $-\epsilon_2(Q)$.

When the level is a multiple of 8, the quadratic form is equivalent over $\mathbb{Z}_2$ to either

$$\begin{bmatrix} 2a & 0 & 0 & 0 \\ 0 & 4b & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 2a & 0 & 0 & 0 \\ 0 & 4b & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The form Q^* represents precisely two odd integers mod 8: $(D/8)b^{-1}$ and $(D/8)(b^{-1} + 2a^{-1})$. A calculation of all 32 options and their ϵ-invariants reveals that the desired result is true in this case as well. This concludes the proof. $\square$

In order to bound the largest locally represented integer not represented by Q, we will require upper and lower bounds on the Eisenstein series coefficients $a_E(n)$ and $a_{E^*}(n)$.

Lemma 16. *For any $\epsilon > 0$, we have*

$$\frac{n^{1-\epsilon}}{N^{1/2}} \ll a_E(n) \ll \frac{n^{1+\epsilon}}{N^{1/2}}$$

if n is locally represented by Q, and

$$\frac{n^{1-\epsilon}}{N^{3/2}} \ll a_{E^*}(n) \ll \frac{n^{1+\epsilon}}{N^{3/2-\epsilon}},$$

if n is locally represented by Q^*. The implied constants depend only on ϵ.

Proof. We have the formula

$$a_E(n) = \prod_{p \leq \infty} \beta_p(n).$$

In [45], formulas are given for the local densities $\beta_p(n)$ (in Yang's notation, these are $\alpha_p(n, \frac{1}{2}A)$). See in particular Theorem 3.1 for $p > 2$ and Theorem 4.1 for $p = 2$. We have $\beta_p(n) = 1$ if $p > 2$ and $p \nmid n$.

If p is odd and $p \nmid N$, then Theorem 3.1 of [45] gives

$$1 - \frac{1}{p} \leq \beta_p(n) \leq 1 + \frac{1}{p}.$$

If p is odd and $p|N$, we get the same bound for the form Q. For the form Q^* we get

$$1 - \frac{1}{p} \leq \beta_p(n) \leq 2$$

provided n is locally represented. Theorem 4.1 of [45] shows that there is an absolute upper bound on $\beta_2(n)$ over all positive integers n and all forms Q and Q^* with discriminants N and N^3, where N is a fundamental discriminant.

Notice that neither Q nor Q^* can be anisotropic at any prime. There is a unique $\mathbb{Q}_p$-equivalence class of quaternary quadratic forms that is anisotropic at p, and such forms must have discriminant a square. The discriminant of Q is N and the discriminant of Q^* is N^3, and neither of these are squares in $\mathbb{Q}_p$ if $p|N$. From this and the recursion formulas of Hanke [16] it follows that there is an absolute lower bound on $\beta_2(n)$ over all quaternary forms Q with fundamental discriminant that locally represent n, and similarly for Q^*. Finally, $\beta_\infty(n) = \frac{\pi^2 n}{\sqrt{D}}$.

Putting these bounds together gives

$$\begin{aligned} \frac{n}{\sqrt{N}} \prod_{p|n} \left(1 - \frac{1}{p}\right) \ll & a_E(n) \ll \frac{n}{\sqrt{N}} \prod_{p|n} \left(1 + \frac{1}{p}\right) \\ \frac{n^{1-\epsilon}}{N^{1/2}} \ll & a_E(n) \ll \frac{n^{1+\epsilon}}{N^{1/2}}. \end{aligned}$$

For Q^* we have

$$\begin{aligned} \frac{n}{\sqrt{N^3}} \prod_{p|n} \left(1 - \frac{1}{p}\right) \ll & a_{E^*}(n) \ll \frac{n}{\sqrt{N^3}} \prod_{p|n} \left(1 + \frac{1}{p}\right) \prod_{p|N} 2 \\ \frac{n^{1-\epsilon}}{N^{3/2}} \ll & a_{E^*}(n) \ll \frac{n^{1+\epsilon}}{N^{3/2-\epsilon}}, \end{aligned}$$

since $\prod_{p|N} 2 \leq d(N) \ll N^\epsilon$. □

Prior to stating and proving our bound on $\langle C, C \rangle$ we need one more preliminary result. This is related to bounding the sum

$$\sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right).$$

Since

$$\psi(x) = -\frac{6}{\pi} x K_1(4\pi x) + 24x^2 K_0(4\pi x),$$

and $K_1(x)$ is positive, it follows that $\psi(x) \leq 24x^2 K_0(4\pi x)$. Using formula (10.32.9) of [32], we have the bound

$$K_0(x) = \int_0^\infty e^{-x\cosh(t)}\,dt \leq \int_0^\infty e^{-x(1+t^2/2)}\,dt = \sqrt{\frac{\pi}{2x}}e^{-x}. \tag{8}$$

It follows that $\psi(x)$ is decreasing exponentially, and hence $\sum_{d=1}^{\infty}\psi(dx)$ is bounded if $x \gg 0$. The following lemma implies that $\sum_{d=1}^{\infty}\psi(dx)$ is bounded $x \to 0$ as well.

Lemma 17. *We have*

$$\lim_{x\to 0}\sum_{d=1}^{\infty}\psi(dx) = \frac{3}{4\pi^2}.$$

Proof. Extend the definition of $\psi(x)$ to all $x \in \mathbb{R}$ by

$$\psi(x) = \begin{cases} \psi(|x|) & \text{if } x \neq 0 \\ -\frac{3}{2\pi^2} & \text{if } x = 0. \end{cases}$$

Note that $\lim_{x\to 0^+}\psi(x) = -\frac{3}{2\pi^2}$ and so $\psi(x)$ is continuous on $(-\infty,\infty)$. It is straightforward to compute that the Fourier transform of $\psi(x)$ is

$$\hat{\psi}(y) = -\frac{9y^2}{\pi^2(4+y^2)^{5/2}}.$$

Therefore, since $\psi(x)$ and $\hat{\psi}(y)$ both are in $L^1((-\infty,\infty))$ and have bounded variation, the Poisson summation formula applies, and gives that

$$\psi(0) + 2\sum_{n=1}^{\infty}\psi(nx) = \hat{\psi}(0) + 2\sum_{n=1}^{\infty}\frac{1}{x}\hat{\psi}(n/x).$$

As $x \to 0$, the right hand side tends to $\hat{\psi}(0) = 0$. This gives

$$\lim_{x\to 0}\sum_{n=1}^{\infty}\psi(nx) = -\frac{1}{2}\psi(0) = \frac{3}{4\pi^2},$$

as desired. □

Our next result gives a bound on $\sum_{n\leq x} r_{Q^*}(n)$ which will be useful in bounding $\langle C, C\rangle$.

Lemma 18. *Assume the notation above. We have*

$$\sum_{n\leq x} r_{Q^*}(n) \ll \max\left(\sqrt{x}, \frac{x^2}{N^{3/2}}\right).$$

Proof. Theorem 2.1.1 of Kitaoka's book [29] shows that we may write the Gram matrix of Q as

$$A = M^T D M,$$

where M is an upper triangular matrix with ones on the diagonal, and D is a diagonal matrix with entries a_1, a_2, a_3, and a_4 where $a_i/a_{i+1} \geq 4/3$ for $i \geq 1$ and $a_1 \geq 1$. This implies that $a_2 \geq 3/4$, $a_3 \geq 9/16$ and $a_4 \geq 27/64$. Since $a_1 a_2 a_3 a_4 = N$, it follows that $a_i \ll N$ for all i.

Taking the inverse and multiplying by N gives

$$A^* = NA^{-1} = M^{-1}(ND^{-1})(M^{-1})^T.$$

If we let $a_i^* = N/a_i$, then we have written

$$Q^*(x_1,x_2,x_3,x_4) = a_1^*(x_1+m_{12}x_2+m_{13}x_3+m_{14}x_4)^2+a_2^*(x_2+m_{23}x_3+m_{24}x_4)^2+a_3^*(x_3+m_{34}x_4)^2+a_4^*x_4^2.$$

We have that $a_i^* \ll N$, $a_i^* \gg 1$, and $a_1^* a_2^* a_3^* a_4^* = N^3$. From the centered equation above, it follows that if $Q^*(x_1,x_2,x_3,x_4) \leq x$, then x_i is in an interval of length at most $2\sqrt{\frac{x}{a_i^*}}$. Thus,

$$\sum_{n \leq x} r_{Q^*}(n) \leq \prod_{i=1}^{4} \left(2\sqrt{\frac{x}{a_i^*}} + 1\right).$$

Since $N^3 = a_1^* a_2^* a_3^* a_4^*$, we have that $a_i a_j \gg N$ and $a_i a_j a_k \gg N^2$. Expanding the product on the right hand side gives that

$$\sum_{n \leq x} r_{Q^*}(n) \ll \begin{cases} \sqrt{x} & x \leq n \\ \frac{x^2}{N^{3/2}} & x \geq n, \end{cases}$$

as desired. □

Now, we bound the Petersson norm of $C(z)$, the cuspidal part of $\theta_Q(z)$. This result is a significant improvement over the result in [38], where it is proven that $\langle C, C\rangle \ll N$, assuming that N is square-free. This improvement is possible because of the exact formula for $\langle C, C\rangle$ from Proposition 14, and a more careful analysis.

Theorem 19. *For all $\epsilon > 0$, we have*

$$\langle C, C\rangle \ll N^{\epsilon}.$$

Proof. Fix $\epsilon > 0$. By Proposition 15, we have $\langle C, C\rangle = N\langle C^*, C^*\rangle$. Proposition 14 then implies that

$$\langle C, C\rangle = \frac{N}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n,N))} a_{C^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right).$$

Note that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \geq N$ and $2^{\omega(\gcd(n,N))} \leq 2^{\omega(N)} \leq d(N) \ll N^{\epsilon/2}$. Since $K_0(x) \leq \sqrt{\frac{\pi}{2x}}e^{-x}$, we have for $x \gg 1$ the estimate

$$\sum_{d=1}^{\infty} \psi(d\sqrt{x}) \ll \sum_{d=1}^{\infty} d^{3/2} x^{3/4} e^{-4\pi d\sqrt{x}} \ll x^{3/4} e^{-4\pi\sqrt{x}}.$$

We have $r_{Q^*}(n) = a_{E^*}(n) + a_{C^*}(n)$ and so $|a_{C^*}(n)| \leq a_{E^*}(n) + r_{Q^*}(n)$. Plugging this estimate in and squaring gives three terms. The first term is

$$\sum_{n=1}^{\infty} \frac{a_{E^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right). \tag{9}$$

The estimates on $a_{E^*}(n)$ from Lemma 16, and the fact that $\sum_{d=1}^{\infty} \psi(d\sqrt{x})$ is bounded as $x \to 0$ give the bound

$$\begin{aligned}&\sum_{n=1}^{N} \frac{n^{1+2\epsilon}}{N^{3-2\epsilon}} + \sum_{n=N}^{\infty} \frac{n^{2+2\epsilon}}{nN^3} \left(\frac{n}{N}\right)^{3/4} e^{-4\pi\sqrt{n/N}} \\ &\ll \frac{1}{N^{1-4\epsilon}} + \frac{1}{N^{2-\epsilon}} \sum_{n=1}^{\infty} \left(\frac{n}{N}\right)^{7/4+\epsilon} e^{-4\pi\sqrt{n/N}}.\end{aligned}$$

The quantity $\frac{1}{N}\sum_{n=1}^{\infty}\left(\frac{n}{N}\right)^{7/4+\epsilon} e^{-4\pi\sqrt{n/N}}$ is a Riemann sum for $\int_0^{\infty} x^{7/4+\epsilon} e^{-4\pi\sqrt{x}}\, dx$ and so it is bounded. It follows that (9) is bounded as $N \to \infty$.

The second term is

$$\sum_{n=1}^{\infty} \frac{2a_{E^*}(n) r_{Q^*}(n)}{n} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right).$$

We split the sum from 1 up to N and from N to ∞. For $1 \leq n \leq N$ we use that $a_{E^*}(n) \ll \frac{n^{1+\epsilon/2}}{N^{3/2}}$ and $r_{Q^*}(n) \ll \sqrt{n}$. This gives

$$\sum_{n=1}^{N} \frac{n^{1/2+\epsilon/2}}{N^{3/2}} \ll N^{\epsilon/2}.$$

For $N \leq n \leq \infty$, we use that $r_{Q^*}(n) \ll \frac{n^2}{N^{3/2}}$ and we get

$$\sum_{n=N}^{\infty} \frac{n^{2+\epsilon/2}}{N^3} \cdot \left(\frac{n}{N}\right)^{3/4} e^{-4\pi\sqrt{n/N}} = \frac{1}{N} \sum_{n=N}^{\infty} n^{\epsilon/2} e^{-2\pi\sqrt{n/N}} \left(\frac{n}{N}\right)^{11/4} e^{-2\pi\sqrt{n/N}}.$$

Note that $n^{\epsilon/2} e^{-2\pi\sqrt{n/N}}$ has its maximum value at $n = \frac{\epsilon N}{2\pi^2}$ and its value there is $\ll N^{\epsilon/2}$. The remainder is a Riemann sum for the convergent integral $\int_1^{\infty} x^{11/4} e^{-2\pi\sqrt{x}}\, dx$, and this shows that the sum for $N \leq n \leq \infty$ is bounded by $N^{\epsilon/2}$.

The third term is

$$\sum_{n=1}^{\infty} \frac{r_{Q^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right).$$

We split this sum into the intervals $k^2 N + 1 \leq n \leq (k+1)^2 N$ for $k \geq 0$. For $n \leq (k+1)^2 N$, we have

$$\sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right) \ll (k+1)^{3/4} e^{-4\pi k}.$$

The contribution from $k^2N+1 \leq n \leq (k+1)^2N$ is therefore bounded by

$$(k+1)^{3/4}e^{-4\pi k}\sum_{n=k^2N+1}^{(k+1)^2N}\frac{r_{Q^*}(n)^2}{n}.$$

The summation by parts identity states that

$$\sum_{m=1}^{n} a_m b_m = \sum_{m=1}^{n-1}(a_m - a_{m+1})\left(\sum_{\ell=1}^{m} b_\ell\right) + a_n\left(\sum_{\ell=1}^{n} b_\ell\right).$$

Applying this to the above sum, we have

$$\begin{aligned}
&(k+1)^{3/4}e^{-4\pi k}\sum_{n=k^2N+1}^{(k+1)^2N}\frac{r_{Q^*}(n)^2}{n}\\
&= (k+1)^{3/4}e^{-4\pi k}\sum_{n=k^2N+1}^{(k+1)^2N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\left(r_{Q^*}(k^2N+1)^2 + r_{Q^*}(k^2N+2)^2+\cdots+r_{Q^*}(n)^2\right)\\
&+\frac{(k+1)^{3/4}e^{-4\pi k}}{(k+1)^2N}\left(r_{Q^*}(k^2N+1)^2 + r_{Q^*}(k^2N+2)^2+\cdots+r_{Q^*}((k+1)^2N)^2\right).
\end{aligned}$$

Since $\sum_{k\leq m} r_{Q^*}(k) \ll \max(\sqrt{m}, \frac{m^2}{N^{3/2}})$, $\sum_{k\leq m} r_{Q^*}(k)^2 \ll \max(m, \frac{m^4}{N^3})$. For $k=0$, we get the bound

$$\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\cdot n + \frac{1}{N}\cdot N \ll \log(N).$$

For $k \geq 1$, the contribution is bounded by

$$\begin{aligned}
&(k+1)^{3/4}e^{-4\pi k}\sum_{n=k^2N+1}^{(k+1)^2N}\frac{1}{n^2}\cdot\frac{n^4}{N^3} + \frac{(k+1)^{3/4}e^{-4\pi k}}{(k+1)^2N}\cdot\frac{(k+1)^8N^4}{N^3}\\
&\ll (k+1)^{3/4}e^{-4\pi k}\cdot(2kN)\cdot\frac{(k+1)^4N^2}{N^3} + k^{27/4}e^{-4\pi k}\\
&\ll k^{27/4}e^{-4\pi k}.
\end{aligned}$$

The sum $\sum_{k=1}^{\infty} k^{27/4}e^{-4\pi k}$ converges, the third term $\ll \log(N)$. We find that the sum of the three terms is $\ll N^{\epsilon/2}$, and using that $d(N) \ll N^{\epsilon/2}$ yields the desired result. □

Finally, we are ready to prove Theorem 6.

Proof of Theorem 6. Fix $\epsilon > 0$. Write

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) = E(z) + C(z),$$

where $E(z) = \sum_{n=0}^{\infty} a_E(n) q^n$ and $C(z) = \sum_{n=0}^{\infty} a_C(n) q^n$. We have

$$a_C(n) \leq C_Q d(n) \sqrt{n}$$

where $C_Q \leq \sqrt{\frac{\langle C, C \rangle u}{B}}$ by (3). Here $u = \dim S_2(\Gamma_0(N), \chi)$ and B is a lower bound for the Petersson norm of a newform in $S_2(\Gamma_0(N), \chi)$. It follows from the work of Hoffstein and Lockhart [18] that $B \gg N^{-\epsilon}$, although this bound is ineffective. From Theorem 19, we have $\langle C, C \rangle \ll N^{\epsilon}$. Combining this with $u \ll N$ gives that $C_Q \ll N^{1/2+\epsilon/2}$.

Now, from Lemma 16, we have $a_E(n) \gg \frac{n^{1-\epsilon/2}}{\sqrt{N}}$ provided n is locally represented by Q. Combining these estimates, we have that $r_Q(n)$ is positive if n is locally represented by Q and

$$\frac{n^{1-\epsilon/2}}{\sqrt{N}} \gg N^{1/2+\epsilon/2} d(n) \sqrt{n}.$$

Since $d(n) \ll n^{\epsilon/2}$, any locally represented n satisfying with $n \gg N^{2+\epsilon}$ is represented.

If f is a newform, then $f|W_N$ is also a newform. It follows from this fact and from Proposition 15 that $C_{Q^*} = \frac{1}{\sqrt{N}} C_Q$. Therefore $r_{Q^*}(n)$ is positive if n is locally represented by Q^* and

$$\frac{n^{1-\epsilon/4}}{N^{3/2-\epsilon/2}} \gg n^{1/2+\epsilon/4}.$$

This implies that $n^{1-\epsilon/2} \gg N^{3/2-\epsilon/2}$, which yield $n \gg N^{3+\epsilon}$. $\square$

5. Proof of the 451-Theorem

If L is a lattice, we say that L is *odd universal* if every odd positive integer is the norm of a vector $\vec{x} \in L$. Such lattices (up to isometry) are in bijection with positive-definite integral quadratic forms Q (up to equivalence) that represent all positive odd integers.

We use the approach (and terminology) pioneered by Bhargava [2] and used in [1] to prove the 290-Theorem. An *exception* for a lattice L is an odd positive integer that does not occur as the norm of a vector in L. If L is a lattice that is not odd universal, we define the *truant* of L to be the smallest positive odd integer t that is the not the norm of a vector in L. An *escalation* of L is a lattice L' generated by L and a vector of norm t. We will study the escalations of the dimension zero lattice, and call all such lattices generated by this process *escalator lattices*. Finally, the 46 odd integers give in the statement of the 451-Theorem are called the *critical integers*.

Note that if L is an odd universal lattice, then there is a sequence of escalator lattices

$$\{0\} = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \subseteq L$$

where L_{i+1} is an escalation of L_i for $0 \leq i \leq n-1$, and L_n is odd universal.

We begin by escalating the zero-dimensional lattice by a vector of norm 1, and getting the unique one-dimensional escalation with Gram matrix $\begin{bmatrix}2\end{bmatrix}$ and quadratic form x^2. This lattice has truant 3 and its escalations have Gram matrices of the form

$$\begin{bmatrix} 2 & a \\ a & 6 \end{bmatrix}.$$

We have $a = 2\langle \vec{x}, \vec{y} \rangle$ where $\vec{x}$ and $\vec{y}$ are vectors of norms 1 and 3. By the Cauchy-Schwarz inequality, we have $|a| \leq 2\sqrt{3}$. Up to isometry, we get four Gram matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

These lattices have truants 5, 7, 5, and 5 respectively. Escalating these four two-dimensional lattices gives rise to 73 three dimensional lattices. Twenty-three of these lattices correspond to the 23 ternary quadratic forms given in [26]. Conjecture 1 states that these represent all positive odds, and we assume Conjecture 1 for the rest of this section.

Escalating the 50 ternary lattices that are not odd universal gives rise to the 24312 *basic* four-dimensional escalators. Of the 24312, 23513 represent every positive odd integer less than 10000. Of the remaining 799, 795 locally represent all odd numbers, and hence represent all but finitely many squarefree odds. The remaining four fail to locally represent all odd integers:

$$\begin{aligned}
&x^2 + 3y^2 + 5z^2 + 7w^2 - 3yw,\\
&x^2 + 3y^2 + 5z^2 + 6w^2 - xw - 2yw + 5zw,\\
&x^2 + 3y^2 + 5z^2 + 11w^2 - xw - 2yw, \text{ and}\\
&x^2 + 3y^2 + 7z^2 + 9w^2 + xy - xw.
\end{aligned}$$

The first three fail to represent integers of the form $5n$, where $n \equiv 3$ or $7 \pmod{10}$ and the last fails to represent integers of the form $7n$ for $n \equiv 3, 5, 13 \pmod{14}$. To handle these four forms, we compute *auxiliary* escalator lattices. The first three lattices have truant 15, and the fourth has truant 21. The auxiliary escalator lattices are those new lattices obtained by escalating $x^2 + 3y^2 + 5z^2$ by 15 (there are 196) and $x^2 + xy + 3y^2 + 7z^2$ by 21 (there are 384). All of these auxiliary lattices locally represent all odds, and every odd universal lattice contains a sublattice isometric to one of the 23 odd universal ternaries, or one of the $24888 = 24312 + 196 + 384 - 4$ four-dimensional escalators (basic or auxiliary). It follows from this that there are only finitely many escalators lattices. We now seek to determine precisely which squarefree positive odd integers are represented by each of these 24888 quadratic forms. When we refer to a form by number, it refers to the index of the form on the list of the 24888 in the file `quatver.txt` (available at `http://www.wfu.edu/~rouseja/451/`).

Method 1: Universal ternary sublattices.

If L is a quaternary lattice with a sublattice L' that is one of the 23 odd universal lattices of dimension 3, then the quadratic form corresponding to L represents all odd integers. Given L it is straightforward to check if such a lattice L' exists, as it must be spanned by vectors of norm 1, 3, 5 and/or 7. If such a lattice exists, then the quadratic form corresponding to L represents all positive odds. This method was not available to Bhargava and Hanke, as there are no ternary quadratic forms that represent all positive integers. The method applies to 2342 of the 24888, and proves that each of these are odd universal.

Example. *Form* 16451 *has level* 2072 *and is the form with largest level to which this method applies. It is given by*

$$Q(x, y, z, w) = x^2 + xy + xw + 3y^2 + 7z^2 + 7w^2.$$

We have $Q(x, y, 0, -z) = x^2 + xy - xz + 3y^2 + z^2$*, which is one of the forms given by Kaplansky in* [26]*. The ternary form* $x^2 + xy - xz + 3y^2 + z^2$ *has genus of size 1 and represents all positive odds. Hence* Q *represents all positive odds.*

Method 2: Nicely embedded regular ternary sublattices.

Recall that a positive-definite quadratic form Q is called *regular* if every locally represented integer m is represented by Q. In [23], Jagy, Kaplansky and Schiemann give a list of 913 ternary quadratic forms. They prove that every regular ternary quadratic form appears on this list, and that 891 of the forms on this list are in fact regular. Of these, 792 are in a genus of size 1, and are hence automatically regular (by the formula of Siegel expressing the Eisenstein series as a weighted sum over the genus). This paper unfortunately does not contain proofs of regularity for the 97 forms, but supplementary documentation is available from Jagy upon request that supplies the necessary proofs.

We say that a quaternary lattice L has a nicely embedded regular ternary if there is a ternary sublattice K whose corresponding quadratic forms is regular, with the property that the quadratic form corresponding to $K \oplus K^{\perp}$ locally represents all positive odds. We may write the quadratic form corresponding to $K \oplus K^{\perp}$ as

$$T(x, y, z) + dw^2$$

where $T(x, y, z)$ is one of the 891 regular ternaries. Our approach for determining the square-free odd numbers not represented by $T(x, y, z) + dw^2$ is then as follows. If n is an odd number, find a representation for n in the form

$$T(x, y, z) + dw^2 = n.$$

Since T is regular, there is then a residue class $a + b\mathbb{Z}$ containing n so that $T(x, y, z) + dw^2$ represents all integers in $a + b\mathbb{Z}$ greater than or equal to n.

To determine the odd integers represented by a quaternary with a nicely embedded regular ternary, we first find a modulus M divisible by all the primes dividing the discriminant of T so that for each $a \in \mathbb{Z}$ with $\gcd(a, M) = 1$ either T locally represents everything in the residue class $a \pmod{M}$ or T does not locally represent any integer in the residue class $a \pmod{M}$.

We then create a queue of residue classes to check, initially containing all $a \pmod{M}$ that T does not locally represent. Within each residue class, we check each number to see if it is represented. If a number is represented with $T(x,y,z) \neq 0$, one can find a residue class $M' \geq M$ so that any number in the residue class $a \pmod{M'}$ is represented. If $M' = M$, we are finished with this residue class. If $M' > M$, the residue classes $a + kM \pmod{M'}$ with $k \neq 0$ that contain squarefree integers are added to the queue. When all of the residue classes have been checked, we are left with a list of odd numbers not represented by $K \oplus K^{\perp}$. It is then necessary to check to see if Q represents these numbers.

Example. *If $Q = x^2+y^2+yz+2z^2+7w^2$, then $T = x^2+y^2+yz+2z^2$ is a nicely embedded regular ternary. The form T represents all positive integers except those of the form $n \equiv 21, 35, 42 \pmod{49}$. We have*

$$\begin{aligned} 21 &= 7 \cdot 1^2 + 14 \\ 35 &= 7 \cdot 2^2 + 7 \\ 42 &= 7 \cdot 2^2 + 14, \end{aligned}$$

and since T represents every positive integer $\equiv 7$ or $14 \pmod{49}$, Q represents all positive integers.

This method applies to 7470 of the quaternaries. Many of these quaternaries are escalations of the regular ternary form $x^2 + xy + 3y^2 + 4z^2$ with truant 77, and some of these escalations have very large level. For example, form 16367

$$Q(x,y,z,w) = x^2 + xy + 3y^2 + 4z^2 + zw + 77w^2$$

has level 13541, the largest of any of the 24888, and form 16350

$$Q(x,y,z,w) = x^2 + xy + xw + 3y^2 + 2yw + 4z^2 - 2zw + 74w^2$$

has $\theta_Q \in M_2(\Gamma_0(12900), \chi_{129})$ and $\dim S_2(\Gamma_0(12900), \chi_{129}) = 2604$ (the largest dimension of $S_2(\Gamma_0(N), \chi)$ for any of the 24888). These forms would be very unpleasant to deal with using other methods. Even though it is occasionally necessary to check a large number of residue classes (as many as 142081), this is method is quite efficient. None of the 7470 quaternaries tested using this method require more than 30 minutes of computation time, and much of this computation time is devoted to checking if Q represents numbers that are not represented by $K \oplus K^{\perp}$.

Method 3: Rankin-Selberg L-functions.

We apply this method for the 8733 quaternaries with fundamental discriminant to which methods 1 and 2 do not apply. We use all of the machinery developed in Section 3, although some modifications are desirable.

Suppose that Q is a positive-definite, integer-valued quadratic form with fundamental discriminant and level N. We use the following procedure to determine which squarefree integers Q represents. First, we compute a lower bound on $\langle g, g \rangle$ for all non-CM newforms in $S_2(\Gamma_0(N), \chi)$ using Proposition 11 (using the optimal choice of the parameter given in equation (5)). Since the lower bound given on $\langle g, g \rangle$ in the proof of Theorem 6 is ineffective, it is necessary to explicitly enumerate the CM forms in $S_2(\Gamma_0(N), \chi)$ and estimate from below their Petersson norms. We do this by finding all negative fundamental discriminants Δ that divide N and all ideals of norm $|N|/|\Delta|$ in the ring of integers of the field $\mathbb{Q}(\sqrt{\Delta})$. All Hecke characters with these moduli are constructed, and then Magma's built-in routines for computing with Hecke Grössencharacters are used to construct the CM forms. Once this is done, Proposition 14 is used to estimate their Petersson norms from below.

We then compute the first $15N$ coefficients of θ_{Q^*}. We pre-compute the local densities associated to Q^* and use these to compute the first $15N$ coefficient of E^*, and from this obtain $C^* = \theta_{Q^*} - E^*$. This data is plugged into Proposition 14. The parts of this formula with $nd^2 \leq 15N$ are explicitly computed. We bound the contribution from terms with $n \leq 15N$ and $nd^2 > 15N$ by using (8), giving that

$$\sum_{d > \sqrt{15N/n}} \psi\left(d\sqrt{\frac{n}{N}}\right) \leq \frac{6\sqrt{2}n^{5/4}}{\sqrt{15N}N^{3/4}} \sum_{d=\lfloor\sqrt{\frac{15N}{m}}+1\rfloor}^{\infty} d^2 e^{-4\pi d\sqrt{n/N}}$$
$$= \frac{6\sqrt{2}n^{5/4}}{\sqrt{15N}N^{3/4}} \left(\frac{e^{-c(a-1)} \cdot (1 + e^{-c} + 2a(e^c - 1) + a^2(e^c - 1)^2)}{(e^c - 1)^3}\right)$$

where $a = \left\lfloor\sqrt{\frac{15N}{n}} + 1\right\rfloor$ and $c = 4\pi\sqrt{n/N}$. We increased the exponent on d in the infinite sum from $3/2$ to 2 to allow the series to be summed in closed form.

For the terms with $n > 15N$, we use that

$$\sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{N}}\right) \leq 6\sqrt{2}\left(\frac{n}{N}\right)^{3/4} \sum_{d=1}^{\infty} d^{3/2} e^{-4\pi d\sqrt{n/N}}.$$

It is easy to see that $\sum_{d=1}^{\infty} d^{3/2} e^{-4\pi d\sqrt{n/N}} \leq 1.000012 e^{-4\pi\sqrt{n/N}}$, and this gives a corresponding bound on the infinite sum of values of ψ. To bound the other terms in the sum, we use that $a_{C^*}(n) \leq C_{Q^*} d(n)\sqrt{n}$ and that $d(n)^2 \leq 7.0609 n^{3/4}$. Plugging all of this in, the terms for $n > 15N$ are bounded by

$$\frac{60 \cdot 2^{\omega(N)} C_{Q^*}^2}{N^{3/4}} \sum_{n=15N+1}^{\infty} n^{3/2} e^{-4\pi\sqrt{n/N}}.$$

Observe that the sum above is at most

$$\left(1 + \frac{1}{15N}\right)^{3/2} \int_{15N}^{\infty} x^{3/2} e^{-4\pi\sqrt{x/N}}\, dx \leq 2.85 \cdot 10^{-20} \left(1 + \frac{1}{15N}\right)^{3/2} N^{5/2}.$$

At the end of this process, we obtain an inequality of the form

$$\langle C^*, C^* \rangle \leq C_1 + C_2 C_{Q^*}^2.$$

We then have

$$C_{Q^*} \leq \sqrt{\frac{u\langle C_*, C_*\rangle}{B}}$$

where B is a lower bound on $\langle g_i, g_i\rangle$. We can solve this inequality above for C_{Q^*} and use the fact that $C_Q = \sqrt{N} C_{Q^*}$ to bound C_Q.

We use a similar method to that of Bhargava and Hanke [1] for computing a lower bound on the Eisenstein series contribution $a_E(n)$, based on part (b) of Theorem 5.7 of [16]. This requires computing the local densities $\beta_p(n)$, which we do according to the procedure given in [16].

The end result is an explicit constant F (which we refer to as the F_4-bound) so that if n is squarefree and

$$F_4(n) = \frac{\sqrt{m}}{d(m)} \prod_{\substack{p \nmid N, p \mid n \\ \chi(p)=-1}} \frac{p-1}{p+1} > F$$

then n is represented by the form Q. We then enumerate all squarefree integers n for which $F_4(n) \leq F$ and check that each of them is represented by Q. To do this, we use a split local cover, a quadratic form

$$R(x, y, z) + dw^2$$

that is represented by Q. If B is the largest number satisfying $F_4(n) \leq F$, we compute an approximation of the theta series of R to precision $C\sqrt{B}$, where C is a constant (which is chosen to depend on the form R). Then, for each squarefree n with $F_4(n) \leq F$, we attempt to find an integer w so that $n - dw^2$ is represented by R. We choose the parameter C so that every $n > 5000$ satisfies this, and we manually check that Q represents every odd number less than 5000.

Example. *Form number 10726 is*

$$Q(x, y, z, w) = x^2 + 3y^2 + 3yz + 3yw + 5z^2 + zw + 34w^2,$$

and has discriminant $N = D = 6780$*, a fundamental discriminant. The dimension of* $S_2(\Gamma_0(N), \chi)$ *is* 1360*. This space has four Galois-orbits of newforms, of sizes* 4*,* 4*,* 40*, and* 1312*. The explicit method of computing the cusp constant that will be described in Method 4 below would be impossible for this form.*

Proposition 11 gives a lower bound

$$\langle g_i, g_i\rangle \geq 0.00001019$$

for non-CM newforms g_i*. We explicitly compute that there are 48 newforms with CM in* $g_i \in S_2(\Gamma_0(N), \chi)$ *and the bound above is valid for them too. Combining Proposition 14 and Proposition 15 with the bounds above, we find that*

$$0.01066 \leq \langle C, C\rangle \leq 0.01079$$

and from this, we derive that $C_Q \leq 1199.86$. *We have that*

$$a_E(n) \geq \frac{28}{151} n \prod_{\substack{p \mid n, p \nmid N \\ \chi(p) = -1}} \frac{p-1}{p+1}.$$

From this, we see that n *is represented by* Q *if* $F_4(n) \geq 6535$. *The computations run in Magma to derive these bounds for* Q *took 3 minutes and 50 seconds.*

A separate program (written in C) verifies that any squarefree number n *satisfying* $F_4(n) \leq 6535$ *has at most* 12 *distinct prime factors, and is bounded by* 8314659320208531. *Of these numbers, it was necessary to check* 4701894614. *This process took 22 minutes and 29 seconds and proves that the form* Q *represents every positive odd integer.*

Method 4: Explicit computation of the cusp constants.

This method is similar to Method 3, except that we do explicit linear algebra computations to compute the constant C_Q. This method is the approach Bhargava and Hanke take for all of the cases they consider in [1], and we apply this method to the 6343 forms Q where none of the first three methods apply.

The following method is used to compute C_Q. If d is a divisor of $N/\mathrm{cond}(\chi)$, we enumerate representatives of the Galois orbits of newforms in $S_2(\Gamma_0(N/d), \chi)$, say $g_1, g_2, \ldots, g_r$. If the Galois orbit of g_i has size k_i, we build a basis for $S_2^{\mathrm{new}}(\Gamma_0(N/d), \chi) \cap \mathbb{Q}[[q]]$ of the form

$$\mathrm{Tr}_{K_i/\mathbb{Q}}(\alpha^j g_i) \text{ for } 1 \leq i \leq r, 0 \leq j \leq k_i - 1,$$

where $K_i = \mathbb{Q}(\alpha)$ is the field generated by adjoining all the Fourier coefficients of g_i to $\mathbb{Q}$. These are then used to build a basis for the image of

$$V(d) : S_2(\Gamma_0(N/d), \chi) \to S_2(\Gamma_0(N), \chi).$$

We do not compute all the coefficients of these forms. Instead we compute coefficients of the form dn where $\gcd(n, N) = 1$ by computing the pth coefficient of all the forms and using the Hecke relations to compute the other coefficients. We repeat this process until the matrix of Fourier expansions has full rank.

Once this basis is built, we solve the linear system (over $\mathbb{Q}$) expressing the cuspidal part C of θ_Q in terms of the basis. To solve this system, we work with one value of d at a time, and only use coefficients of the form dn where $\gcd(n, N) = 1$ to determine the contribution to C of the image of $V(d) : S_2(\Gamma_0(N/d), \chi) \to S_2(\Gamma_0(N), \chi)$. Once we have the representation of C in terms of the full basis for $S_2(\Gamma_0(N), \chi)$, we numerically approximate the embeddings of the α^j and use these to compute C_Q.

Example. *Form 22145 is*

$$Q(x, y, z, w) = x^2 - xz + 2y^2 + yz - 2yw + 5z^2 + zw + 29w^2.$$

This form has level $\theta_Q \in M_2(\Gamma_0(4200), \chi_{168})$. *The dimension of* $S_2(\Gamma_0(4200), \chi_{168})$ *is* 936. *There are 19 Galois conjugacy classes of newforms of levels* 168, 840, *and* 4200, *the largest of which has size* 160.

The $d = 1$ *space has dimension* 752, *and we need to compute the pth coefficient of all newforms of level dividing* 4200 *for* $p \leq 197$. *Once these are computed, it is straightforward to find bases for the* $d = 5$ *and* $d = 25$ *spaces (of dimensions* 156 *and* 28, *respectively). Solving the linear system gives that*

$$C_Q \approx 31.0537.$$

For odd squarefree n, *we have*

$$a_E(n) \geq \frac{28}{117} n \prod_{\substack{p \mid n, p \nmid N \\ \chi(p) = -1}} \frac{p-1}{p+1}.$$

This shows that if n *is a squarefree odd integer and* $F_4(n) > 131.0575$, *then* n *is represented by* Q. *The bound on* F_4 *is quite small, and it is only necessary to test* 638080 *integers. However, computing the bound on* F_4 *required almost a day of computation, due to the difficulty of computing the constant* C_Q. *The result is that* Q *represents all positive odd integers.*

Proof of the 451-Theorem. Assume Conjecture 1. The computations show that every one of the 24888 forms considered locally represents all positive odd integers, and in each case we are able to determine precisely the list of squarefree odd exceptions for each form. Moreover, every odd universal lattice contains one of the 23 odd universal ternary escalators, or one of the 24888. Of the 24888, there are 23519 that represent all positive odds, and 1359 that have exceptions. Of these 1359, there are 15 forms that have an exception which is not a critical integer. (These are forms 1044, 8988, 9011, 9016, 11761, 16366, 16372 17798, 24290, 24311, 24328, 24435, 24463, 24504, and 24817.) It is necessary to check that each escalation of these forms represents all non-critical positive odds. The most time-consuming form to deal with is form 16366,

$$Q(x, y, z, w) = x^2 + xy + 3y^2 + 4z^2 + 77w^2$$

which has truant 143, and fails to represent 187, 231, 385, 451, 627, 935, 1111, 1419, 1903, and 2387. We compute all escalations of it (which requires consideration of more than 10 million Gram matrices), and find among its escalations forms that have truants 187, 231, 385, and 451, but not 627, 935, 1111, 1419, 1903 or 2387. This concludes the proof that every positive-definite quadratic form representing the 46 critical integers represents all positive odd integers. □

Remark. *The program and log files used to prove the 451-Theorem are available at* `http://www.wfu.edu/~rouseja/451`.

We will now show that each critical integer is necessary.

Proof of Corollary 3. Each of the critical integers occurs as the truant for some form Q (see Appendix A). Using the same trick as in [1], if $Q(\vec{x})$ is any form with truant t, consider the form

$$Q' = Q(\vec{x}) + (t+1)y^2 + (t+1)z^2 + (t+1)w^2 + (t+1)v^2 + (2t+1)u^2.$$

This form fails to represent t. However, since every positive integer is expressible as a sum of four squares, if Q represents the odd number a, then every number $\equiv a \pmod{t+1}$ is represented by Q'. This accounts for all odd numbers except those $\equiv t \pmod{t+1}$. Taking $Q = 0$ and $u = 1$, we see that Q' represents all numbers $\equiv t \pmod{t+1}$ that are greater than or equal to $2t+1$. Hence, t is the unique positive odd integer which is not represented by Q'. □

As an application of the 451-Theorem, we will prove Corollary 4.

Proof of Corollary 4. If Q is a quadratic form with corresponding lattice L that represents every positive odd integer less than 451, then L contains as a sublattice one of the 24888 we considered above. Of these, only forms 1048, 16327, 16334, 16336 and 16366 have 451 as an exception. Each of the has a nicely embedded regular ternary, and the application of method 2 shows that each of these represents all odd positive integers n that are not multiples of 11^2, with a finite and explicit set of exceptions. For forms 1048, 16334, 16336 and 16366 it is easy to see that all multiples of 11^2 are represented.

It follows from this that all escalations of forms 1048, 16336, 16336 by exceptions less than 451 represent all odd numbers. For form 16366, we computed all escalations in the course of proving the 451-Theorem and found that none of them have squarefree exceptions greater than 451.

However, form 16327

$$Q(x, y, z, w) = x^2 + xy + 3y^2 + 4z^2 + 66w^2$$

is anisotropic at 11. The form Q represents all odd integers that are not multiples of 11^2 except 319 and 451. A computer calculation shows that $r_Q(121n) = r_Q(n)$ for all positive integers n and hence, the odd integers not represented by Q are those of the form $319 \cdot 11^{2k}$ and $451 \cdot 11^{2k}$. It is therefore necessary to compute all escalations of Q by 319, find those that fail to represent 451, and check that each of these represents $451 \cdot 11^2 = 54571$. We find 21 five-dimensional escalations that fail to represent 451 and each of these represents 54571. □

As an application of the 451-Theorem we will classify those quaternary forms that represent all odd positive integers.

Proof of Corollary 5. The successive minima of quaternary escalator lattices are bounded by 1, 3, 7, and 77. We enumerate all Minkowski-reduced lattices with successive minima less than or equal to these, apply the 451-Theorem to determine which represent all positive odds, and determine those that represent one of the odd universal ternary forms. A list of the 21756 forms that were found is available on the website mentioned above. $\square$

6. Conditional proof of Conjecture 1

We begin by recalling the theory of modular forms of half-integer weight. If λ is a positive integer, let $S_{\lambda+\frac{1}{2}}(\Gamma_0(4N),\chi)$ denote the vector space of cusp forms of weight $\lambda+\frac{1}{2}$ on $\Gamma_0(4N)$ with character χ. We denote by by $T(p^2)$ the usual index p^2 Hecke operator on $S_{\lambda+1/2}(\Gamma_0(4N),\chi)$. Next, we recall the Shimura lifting.

Theorem ([41]). *Suppose that $f(z)=\sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(4N),\chi)$. For each squarefree integer t, let*

$$\mathcal{S}_t(f(z)) = \sum_{n=1}^{\infty}\left(\sum_{d|n}\chi(d)\left(\frac{(-1)^{\lambda}t}{d}\right)d^{\lambda-1}a(t(n/d)^2)\right)q^n.$$

Then, $\mathcal{S}_t(f(z)) \in M_{2\lambda}(\Gamma_0(2N),\chi^2)$. It is a cusp form if $\lambda>1$ and if $\lambda=1$ it is a cusp form if $f(z)$ is orthogonal to all cusp forms $\sum_{n=1}^{\infty}\psi(n)nq^{n^2}$ where ψ is an odd Dirichlet character.

One can show using the definition that if p is a prime and $p \nmid 4tN$, then

$$\mathcal{S}_t(f|T(p^2)) = \mathcal{S}_t(f)|T(p).$$

In [43], Waldspurger relates the Fourier coefficients of a half-integer weight Hecke eigenform f with the central critical L-values of the twists of the integer weight newform F with the same Hecke eigenvalues. If we have a newform

$$F(z)=\sum_{n=1}^{\infty} b(n)q^n \in S_2^{\mathrm{new}}(\Gamma_0(N)),$$

and χ is a quadratic Dirichlet character, we define $F\otimes\chi$ to be the unique newform whose nth Fourier coefficient is $b(n)\chi(n)$ if $\gcd(n, N\cdot \mathrm{cond}(\chi))=1$.

Theorem ([43], Corollaire 2, p. 379). *Suppose that $f \in S_{\lambda+1/2}(\Gamma_0(N),\chi)$ is a half-integer weight modular form and $f|T(p^2)=\lambda(p)f$ for all $p\nmid N$ with Fourier expansion $f(z)=\sum_{n=1}^{\infty} a(n)q^n$. If $F(z)\in S_{2\lambda}(\Gamma_0(N),\chi^2)$ is an integer weight newform with $F(z)|T(p)=\lambda(p)g$ for all $p\nmid N$ and n_1 and n_2 are two squarefree positive integers with $n_1/n_2 \in \left(\mathbb{Q}_p^{\times}\right)^2$ for all $p|N$, then*

$$a(n_1)^2L(F\otimes\chi^{-1}\chi_{n_2(-1)^{\lambda}},1/2)\chi(n_2/n_1)n_2^{\lambda-1/2} = a(n_2)^2L(F\otimes\chi^{-1}\chi_{n_1(-1)^{\lambda}},1/2)n_1^{\lambda-1/2}.$$

If Q is a positive-definite, integer-valued ternary quadratic form, then

$$\theta_Q(z)=\sum_{n=0}^{\infty} r_Q(n)q^n \in M_{3/2}(\Gamma_0(4N),\chi).$$

We may then decompose $\theta_Q(z) = E(z) + C(z)$ where $E(z)$ is a half-integer weight Cohen-Eisenstein series, and $C(z)$ is a cusp form. We have

$$E(z) = \sum_{n=0}^{\infty} a_E(n) q^n,$$

where if $n \geq 1$ is squarefree, then

$$a_E(n) = \frac{24h(-nM)}{Mw(-nM)} \prod_{p \mid 2N} \beta_p(n) \cdot \frac{1 - (1/p)\chi(p)\left(\frac{n}{p}\right)}{1 - 1/p^2}.$$

Here M is a rational number which depends on $n \pmod{8N^2}$ with the property that nM is a fundamental discriminant. Here $h(-nM)$ is the class number of the ring of integers in $\mathbb{Q}(\sqrt{-nM})$ and $w(-nM)$ is half the number of roots of unity in $\mathbb{Q}(\sqrt{-nM})$. From Siegel's work, we have the ineffective lower bound $h(-D) \gg D^{1/2-\epsilon}$, but the strongest effective lower bound we have is due to the work of Goldfeld [14], Gross and Zagier [15] and has the form $h(-D) \gg \log(D)^{1-\epsilon}$. For this reason, there is no general method to determine unconditionally the integers represented by a positive-definite ternary quadratic form.

We may decompose the cusp form contribution as a linear combination of half-integer weight Hecke eigenforms

$$C(z) = \sum_i c_i f_i(z).$$

Each $f_i(z)$ either has the form $\sum \psi(n) n q^{dn^2}$, in which case its nonzero Fourier coefficients are supported on a single square-class, or Waldspurger's theorem applies, and gives that if

$$f_i(z) = \sum_{n=1}^{\infty} b(n) q^n,$$

then

$$|b(n)| = dn^{1/4} |L(F_i \otimes \chi_{bn}, 1/2)|$$

for some constants b and d which depend on the $\mathbb{Q}_p$-square classes of n (provided we can find a value of n in the $\mathbb{Q}_p$-square classes so that the coefficient of f_i and the central L-value of the corresponding twist of F_i are nonzero). The best currently known subconvexity estimate for $L(F_i \otimes \chi_{bn}, 1/2)|$ is due to Blomer and Harcos ([3], Corollary 2) and gives that

$$|b(n)| \ll n^{7/16+\epsilon}.$$

However, the generalized Riemann hypothesis implies that $|b(n)| \ll n^{1/4+\epsilon}$. In [33], Ono and Soundararajan pioneered a method to conditionally determine the integers represented by a ternary quadratic form and used it to prove that Ramanujan's form $x^2 + y^2 + 10z^2$ represents every odd number greater than 2719. This method was generalized by Kane [25] and refined by Chandee [8]. We prove Conjecture 1 by using Theorem 2.1 and Proposition 4.1 of [8] (which assume the generalized Riemann hypothesis) to bound $|L(F_i \otimes \chi_{bn}, 1/2)|$ and

$$L(1, \chi_{nM}) = \frac{\pi h(-nM)}{\sqrt{nM} w(-nM)}.$$

Proof of Theorem 7. For $Q = x^2 + 2y^2 + 5z^2 + xz$, we have

$$\begin{aligned} \theta_Q(z) &= 1 + 2q + 2q^2 + 4q^3 + 2q^4 + 4q^5 + \cdots \\ &= \sum_{n=0}^{\infty} r_Q(n) q^n \in M_{3/2}(\Gamma_0(152), \chi_{152}). \end{aligned}$$

The genus of Q has size 2, and the other form is $R = x^2 + y^2 + 13z^2 - xy - xz + yz$. We have

$$\begin{aligned} E &= \frac{3}{5}\theta_Q + \frac{2}{5}\theta_R = 1 + \frac{18}{5}q + \frac{6}{5}q^2 + \frac{24}{5}q^3 + \frac{18}{5}q^4 + \frac{12}{5}q^5 + \cdots \\ C &= \theta_Q - E = -\frac{8}{5}q + \frac{4}{5}q^2 - \frac{4}{5}q^3 - \frac{8}{5}q^4 + \frac{8}{5}q^5 + \cdots . \end{aligned}$$

The Shimura lift $\mathcal{S}_3 : S_{3/2}(\Gamma_0(152), \chi_{152}) \to M_2(\Gamma_0(76))$ is injective, and $\mathcal{S}_3(C)$ is a constant times the newform

$$F_1(z) = q + q^2 - q^3 + q^4 - 4q^5 - q^6 + \cdots \in S_2(\Gamma_0(38)),$$

which corresponds to the elliptic curve

$$E_1 : y^2 + xy + y = x^3 + x^2 + 1.$$

For each pair $(n_1, n_2) \in (\mathbb{Q}_2^\times / \left(\mathbb{Q}_2^\times\right)^2) \times (\mathbb{Q}_{19}^\times / \left(\mathbb{Q}_{19}^\times\right)^2)$ with $\operatorname{ord}_2(n_1) = 0$, we compute constants a, b, and d so that if n is a squarefree integer with $n/n_1 \in \left(\mathbb{Q}_2^\times\right)^2$ and $n/n_2 \in \left(\mathbb{Q}_{19}^\times\right)^2$, we have

$$r_Q(n) = ah(-bn) \pm dn^{1/4}\sqrt{L(F_1 \otimes \chi_{-152n}, 1/2)}.$$

For $n_1 = n_2 = 1$, we have $a = 3/5$, $b = 152$, and $d \approx 0.9150328989$. This shows that if $r_Q(n) = 0$, then

$$\frac{\sqrt{L(F_1 \otimes \chi_{-152n}, 1/2)}}{L(1, \chi_{-152n})} \geq 2.573276 n^{1/4}.$$

On the other hand, computations using Chandee's theorems give that

$$\frac{\sqrt{L(F_1 \otimes \chi_{-152n}, 1/2)}}{L(1, \chi_{-152n})} \leq 13.848476 \cdot n^{0.1239756}.$$

Comparing these two results, we see that if n is a 2-adic and a 19-adic square, then $r_Q(n) > 0$ if $n \geq 630654$, assuming the generalized Riemann hypothesis. We obtain the same bounds on the other squareclasses (n_1, n_2) where $\operatorname{ord}_{19}(n_2) = 0$. On the squareclasses where $n_2 = 19$, we obtain smaller bounds. Finally, it is possible to prove that C vanishes identically on squareclasses where $n_2 = 38$. To check that every odd number less than this bound is represented, we compute the theta series of $S = x^2 + xz + 5z^2$ up to q^{630654}. For each odd number $n \leq 630654$, we check if $n - 2y^2$ is represented by S for some $y \leq \sqrt{n/2}$. This computation takes 2.79 seconds.

For $Q = x^2 + 3y^2 + 6z^2 + xy + 2yz$, the genus again has size 2, and $\theta_Q \in M_{3/2}(\Gamma_0(248), \chi_{248})$. We find that $\mathcal{S}_1 : S_{3/2}(\Gamma_0(248), \chi_{248}) \to M_2(\Gamma_0(124))$ is injective. If C is the cuspidal part of θ_Q, then $\mathcal{S}_1(C)$ is some constant times the newform $F_2 \in S_2(\Gamma_0(62))$ that corresponds to the elliptic curve

$$E_2 : y^2 + xy + y = x^3 - x^2 - x + 1$$

with conductor 62. Again for each pair $(n_1, n_2) \in \mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \times \mathbb{Q}_{31}^\times/(\mathbb{Q}_{31}^\times)^2$, we find a, b, and d so that

$$r_Q(n) = ah(-bn) \pm dn^{1/4}\sqrt{L(F_2 \otimes \chi_{-248n}, 1/2)}.$$

For $(n_1, n_2) = 1$, we have $a = 3/8$, $b = 248$, and $d \approx 0.6630028204$. If $r_Q(n) = 0$, we get that

$$\frac{\sqrt{L(F_2 \otimes \chi_{-248n}, 1/2)}}{L(1, \chi_{-248n})} \geq 2.835253n^{1/4}$$

and using Chandee's theorems we get

$$\frac{\sqrt{L(F_2 \otimes \chi_{-248n}, 1/2)}}{L(1, \chi_{-248n})} \leq 14.492987 \cdot n^{0.1239756}.$$

This proves that if $n \geq 419230$ and n is a 2-adic and 31-adic square, then $r_Q(n) > 0$ (assuming GRH). We obtain equal or smaller bounds on the other square classes. To check up to this bound, we use that

$$4Q = (2x + y)^2 + 11y^2 + 8yz + 24z^2.$$

If $4n$ is represented by $w^2 + 11y^2 + 8yz + 24z^2$, then $w \equiv y \pmod 2$ and hence if we set $x = \frac{w-y}{2}$, we get that $n = x^2 + 3y^2 + 6z^2 + xy + 2yz$. Hence n is represented by Q if and only if $4n$ is represented by $w^2 + 11y^2 + 8yz + 24z^2$. We compute the theta series for $S = 11y^2 + 8yz + 24z^2$ up to $q^{1680000}$. Then, for each number $m \equiv 4 \pmod 8$ between 4 and 1680000, we check that $m - w^2$ is represented by S for some w. We find that this is true, and the computation takes 2.53 seconds.

Finally, for the form $Q = x^2 + 3y^2 + 7z^2 + xy + xz$, we have $\theta_Q \in M_{3/2}(\Gamma_0(296), \chi_{296})$. We use the maps $\mathcal{S}_1 : S_{3/2}(\Gamma_0(296), \chi_{296}) \to M_2(\Gamma_0(148))$ and $\mathcal{S}_5 : S_{3/2}(\Gamma_0(296), \chi_{296}) \to M_2(\Gamma_0(148))$. We find that neither are injective, but that the intersection of their kernels is zero. If C is the cuspidal part of θ_Q, then C is a linear combination of two eigenforms whose Shimura lifts are the two newforms

$$F_3^+ = q + q^2 + \frac{-1 + \sqrt{5}}{2}q^3 + q^4 + \frac{1 - 3\sqrt{5}}{2}q^5 + \cdots$$
$$F_3^- = q + q^2 + \frac{-1 - \sqrt{5}}{2}q^3 + q^4 + \frac{1 + 3\sqrt{5}}{2}q^5 + \cdots$$

of level 74. For each pair $(n_1, n_2) \in \mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \times \mathbb{Q}_{37}^\times/(\mathbb{Q}_{37}^\times)^2$, we find constants a, b, d_1 and d_2 so that

$$r_Q(n) = ah(-bn) \pm d_1 n^{1/4}\sqrt{L(F_3^+ \otimes \chi_{-296n}, 1/2)} \pm d_2 n^{1/4}\sqrt{L(F_3^- \otimes \chi_{-296n}, 1/2)}.$$

For $n_1 = n_2 = 1$, we have $a = 6/19$, $b = -296$, $d_1 \approx 0.2092923830$ and $d_2 \approx 0.5342698872$. Hence, if $r_Q(n) = 0$, we have

$$d_1 \frac{\sqrt{L(F_3^+ \otimes \chi_{-296n}, 1/2)}}{L(1, \chi_{-296n})} + d_2 \frac{\sqrt{L(F_3^- \otimes \chi_{-296n}, 1/2)}}{L(1, \chi_{-296n})} \geq 1.729392 n^{1/4}.$$

Applying Chandee's theorems, we get

$$\frac{\sqrt{L(F_3^+ \otimes \chi_{-296n}, 1/2)}}{L(1, \chi_{-296n})} \leq 13.678621 n^{0.1239756}, \text{ and}$$
$$\frac{\sqrt{L(F_3^- \otimes \chi_{-296n}, 1/2)}}{L(1, \chi_{-296n})} \leq 15.592398 n^{0.1239756}.$$

It follows that if $r_Q(n) = 0$, then $n \leq 2727720$, assuming GRH. We find equal or smaller bounds on the other square classes. To check up to this bound, we use that

$$4Q = (2x + y + z)^2 + 11y^2 - 2yz + 27z^2.$$

If $4n$ is represented by $w^2 + 11y^2 - 2yz + 27z^2$, then $w \equiv 11y^2 - 2yz + 27z^2 \pmod 2$, which implies that $w \equiv y + z \pmod 2$. Setting $x = \frac{w-(y+z)}{2}$, we obtain $n = x^2 + 3y^2 + 7z^2 + xy + xz$. Thus, n is represented by Q if and only if $4n$ is represented by $w^2 + 11y^2 - 2yz + 27z^2$. We compute the theta series of $S = 11y^2 - 2yz + 27z^2$ up to $q^{10912000}$, and check that for every number $m \equiv 4 \pmod 8$ less than 10912000, $m - w^2$ is represented by S for some integer w. We find that this is true, and the computation takes 16.76 seconds.

This completes the proof of Theorem 7, assuming the generalized Riemann hypothesis. $\square$

Appendix A. Table of quadratic forms with given truants

Form	Truant
$\emptyset$	1
x^2	3
x^2+2y^2	5
x^2+3y^2+xy	7
$x^2+3y^2+4z^2+yz$	11
$x^2+3y^2+6z^2+xy+yz$	13
$x^2+y^2+3z^2$	15
$x^2+2y^2+3z^2+xy+xz+2yz$	17
$x^2+3y^2+7z^2+xy+yz$	19
$x^2+3y^2+3z^2+xy+xz+2yz$	21
$x^2+2y^2+3z^2+yz$	23
$x^2+3y^2+3z^2+xy$	29
$x^2+2y^2+4z^2+yz$	31
$x^2+3y^2+4z^2+10w^2+2yw$	33
$x^2+3y^2+5z^2+3yz$	35
$x^2+2y^2+5z^2+12w^2+xz+xw+yz+3zw$	37
$x^2+3y^2+5z^2+13w^2+xy+xz+yz$	39
$x^2+3y^2+4z^2+xz+2yz$	41
$x^2+3y^2+5z^2+15w^2+xw+yz+2yw+2zw$	47
$x^2+3y^2+4z^2+15w^2+xw+3yz+zw$	51
$x^2+3y^2+5z^2+21w^2+xz+2yz+yw+4zw$	53
$x^2+3y^2+7z^2+9w^2+xy+2yw$	57
$x^2+3y^2+5z^2+16w^2+yz+2yw+3zw$	59
$x^2+y^2+3z^2+yz$	77
$x^2+3y^2+5z^2+21w^2+3yz$	83
$x^2+3y^2+5z^2+23w^2+xw+3yz+4zw$	85
$x^2+3y^2+5z^2+9w^2+xz+3yw$	87
$x^2+3y^2+5z^2+27w^2+xz+2yz+yw+2zw$	89
$x^2+3y^2+7z^2+9w^2+21v^2+xy+yw+7zv$	91
$x^2+2y^2+4z^2+28w^2+yz+zw$	93
$x^2+3y^2+4z^2+11w^2+xw+2zw$	105
$x^2+3y^2+5z^2+31w^2+3yz+3yw$	119
$x^2+3y^2+4z^2+9w^2+3yw$	123
$x^2+3y^2+7z^2+19w^2+57v^2+xy+yz+17wv$	133
$x^2+3y^2+5z^2+26w^2+xw+3yz+3zw$	137
$x^2+y^2+3z^2+47w^2+xw+yz$	143
$x^2+3y^2+3z^2+29w^2+xy+2yz$	145
$x^2+3y^2+3z^2+20w^2+xy+3zw$	187
$x^2+3y^2+6z^2+13w^2+xy+yz$	195
$x^2+2y^2+4z^2+29w^2+58v^2+xz+yz$	203
$x^2+3y^2+4z^2+41w^2+xz+2yz$	205
$x^2+y^2+3z^2+36w^2+xw+yz$	209
$x^2+3y^2+4z^2+77w^2+143v^2+xy+15wv$	231
$x^2+3y^2+4z^2+33w^2+xy$	319
$x^2+3y^2+4z^2+77w^2+143v^2+xy+22wv$	385
$x^2+3y^2+4z^2+77w^2+143v^2+xy+33wv$	451

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