Homework 9 - Analytic number theory

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1. (5+ points) Consider a variant Gauss sum

$$G(p^2) = \sum_{m=1}^{p^2} \left(\frac{m}{p}\right) e(m/p^2), \tag{1}$$

where p is an odd prime. Is $|G(p^2)| = p$ always? Prove or disprove. Further prove as much as you can about this Gauss sum.

2. (5 points; Iwaniec and Kowalski, (4.24)) If u and v are real numbers with v positive, prove the Poisson summation variant

$$\sum_{m \in \mathbb{Z}} f(vm + u) = \frac{1}{v} \sum_{n \in \mathbb{Z}} \widehat{f}(n/v) e(un/v). \tag{2}$$

(The easiest proof starts with the Poisson summation formula and does a change of variables.)

3. (5 points) For any $N > 0$, rigorously justify the existence of the integral $\int_{-\infty}^{\infty} e(Nx^2)dx$. (Note that it does not converge absolutely.)

(**Bonus,** 5 points) Evaluate it without any appeal to Gauss sums. (Warning: Off the top of my head I don't know how to do this.)

4. (5 points) Prove, for any $\alpha \in \mathbb{R}$, and any $x > 0$, that

$$\sum_{n \in \mathbb{Z}} e^{-(n+\alpha)^2 \pi / x} = x^{1/2} \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x + 2\pi i n \alpha}. \tag{3}$$

5. (5 points) Prove that $\zeta(0) = -1/2$.

6. (5 points) If χ is a primitive, nonprincipal character mod q, prove that

$$\chi(n)\tau(\overline{\chi}) = \sum_{m=1}^{q} \overline{\chi}(m) e(mn/q) \tag{4}$$

if $(n, q) = 1$.

7. (3 points) If χ is primitive mod q, prove that

$$L(1, \chi) = \sum_{n \leq x} \frac{\chi(n)}{n} + O(q^{1/2} \log q / x) \tag{5}$$

for any $x \geq 1$ and $q \geq 1$.

8. (4 or 8 points) Conclude from the previous problem that

$$\sum_{\chi \neq \chi_0} L(1, \chi) = \phi(q) + O(q^{1/2} \log q), \tag{6}$$

where the summation is over all nontrivial characters modulo q. (For 4 points, assume q is an odd prime; for 8 points, don't.)