State High School Mathematics Tournament

University of South Carolina

February 3, 2018

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Question 1

If $x + y = 6$ and $x^2 + y^2 = 20$, what is $x^3 + y^3$?

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Solution to Question 1

We have

$$36 = (x + y)^2 = x^2 + y^2 + 2xy = 20 + 2xy,$$

so $16 = 2xy$ and $xy = 8$.

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Solution to Question 1

We have

$$36 = (x+y)^2 = x^2 + y^2 + 2xy = 20 + 2xy,$$

so $16 = 2xy$ and $xy = 8$. So,

$$216 = (x+y)^3 = x^3 + y^3 + 3xy(x+y) = x^3 + y^3 + 3 \cdot 8 \cdot 6,$$

and thus

$$x^3 + y^3 = 216 - 3 \cdot 8 \cdot 6 = 216 - 144 = 72.$$

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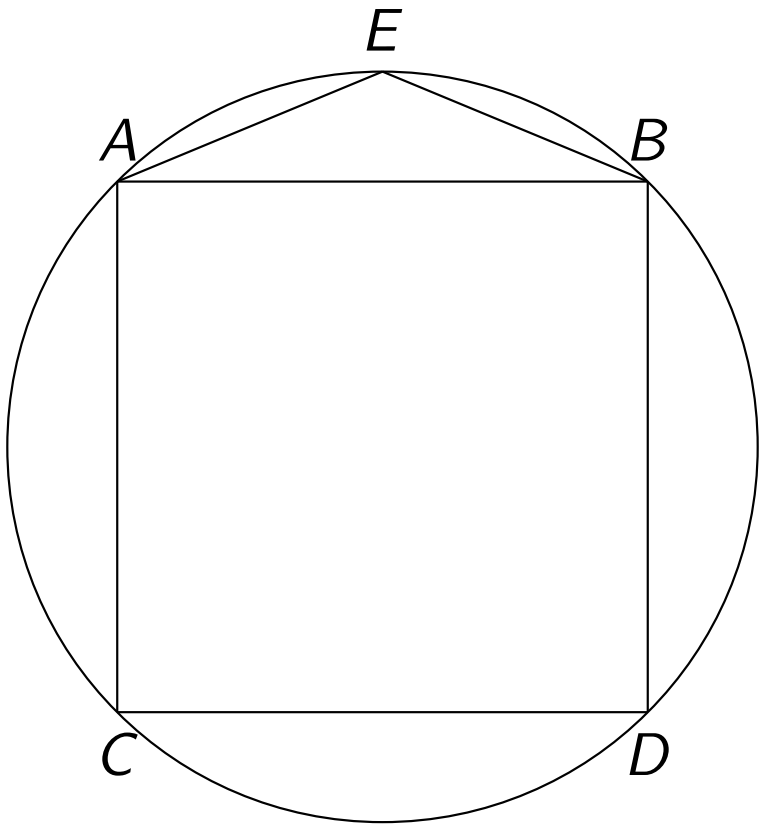
Question 2

A square is inscribed in a circle of radius 1 as follows:



If $\overline{AE} = \overline{BE}$, find the area of $\triangle AEB$.

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Question 3

You flip two coins. One is fair; the other is weighted and is more likely to come up heads than tails.

If the probability of flipping at least one heads is 80%, what is the probability of flipping both heads?

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Solution 3

Let p be the probability that the weighted coin comes up heads. The probability of flipping no heads is

$$\frac{1}{2}(1-p)=\frac{1}{5},$$

so $1-p=\frac{2}{5}$ and $p=\frac{3}{5}$. The probability of flipping two heads is thus

$$\frac{1}{2}\times\frac{3}{5}=\frac{3}{10}.$$

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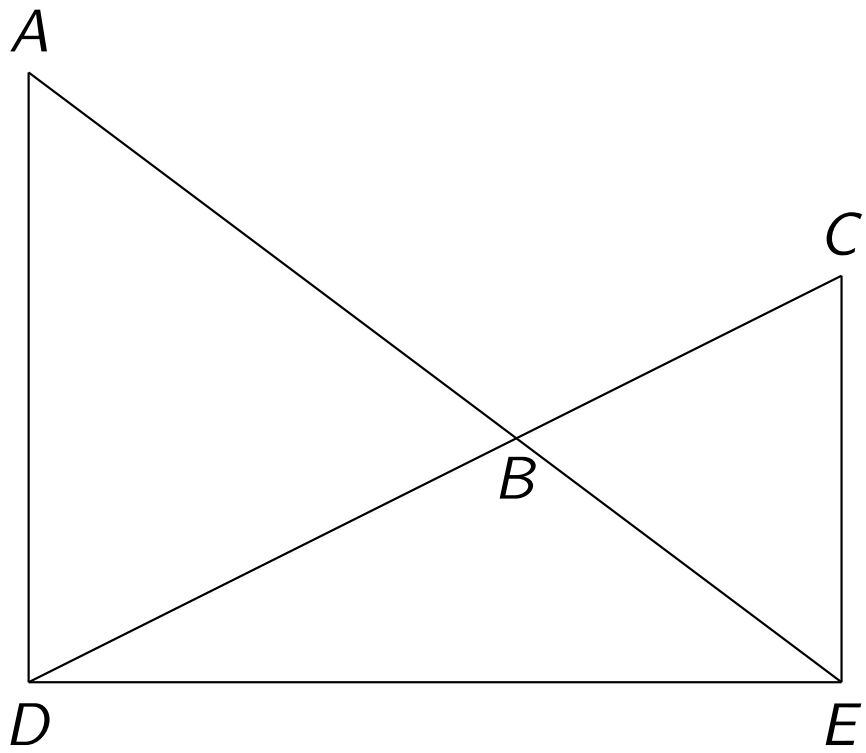
Question 4

In the figure, $\overline{AD}$ and $\overline{CE}$ are perpendicular to $\overline{DE}$; $\overline{AD} = 5$, $\overline{DE} = 3$, and $\overline{CE} = 4$.
Find the area of $\triangle BDE$.



(not drawn to scale)

Solution 4

Answer: 10/3....

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Question 5

How many edges does a 7-dimensional cube have?

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Solution 5

There are $2^7 = 128$ vertices; each is connected by an edge to 7 other vertices, so

$$128 \cdot 7 \cdot \frac{1}{2} = 384.$$

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Question 6

The integer $2^{12} - 1$ has three divisors between 20 and 40. What is their sum?

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Solution 6

We have

$$2^{12} - 1 = (2^6 - 1)(2^6 + 1) = 63 \cdot 65 = 3 \cdot 3 \cdot 7 \cdot 5 \cdot 13.$$

Any product of three or more of these divisors will be too large. So we look for products of two of them, and

$$3 \cdot 7 + \cdot 3 \cdot 13 + \cdot 5 \cdot 7 = 95.$$

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Question 7

A unique circle goes through the following three points:

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Question 7

A unique circle goes through the following three points:

$$(2, 5), (4, 4), (5, 2).$$

What is its diameter?

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Solution 7

Answer: $5\sqrt{2}$

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The three points are symmetric about the line $y = x$, so this line is a diameter.

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Solution 7

Answer: $5\sqrt{2}$

The three points are symmetric about the line $y = x$, so this line is a diameter. The line from $(2, 5)$ to $(5, 2)$ meets this diameter at $(3.5, 3.5)$. (.....)

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Question 9

How many digits are in the base 10 number 20^{18}?

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Solution 9

Answer: 24. (explain....)

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Question 10

In how many points do the graphs of $y = \cos(2\pi x)$ and $y = \frac{x}{2018}$ intersect?

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Solution 10

Answer: 4037: two in every interval $x \in (n, n+1)$ with $n = -2018, \cdots 2017$, and one at $x = 2018$.

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Question 11

What is the minimum value of $f(x) = x^3 - 3x$, over all real numbers x satisfying $x^4 + 36 \leq 13x^2$?

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Solution 11

Answer. -18. We have

$$x^4+36 \leq 13x^2 \iff (x^2-9)(x^2-4) \leq 0 \iff x \in [-3,-2]\cap[2,3].$$

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Solution 11

Answer. -18. We have

$$x^4+36 \le 13x^2 \iff (x^2-9)(x^2-4) \le 0 \iff x \in [-3,-2]\cap[2,3].$$

The function $f(x) = x^3 - 3x = x(x^2 - 3)$ is odd, and is increasing in $[2, 3]$. So it is decreasing in $[-3, -2]$.
We have $f(2) = 2, f(3) = 18, f(-2) = -2, f(-3) = -18$. So the minimum is -18.

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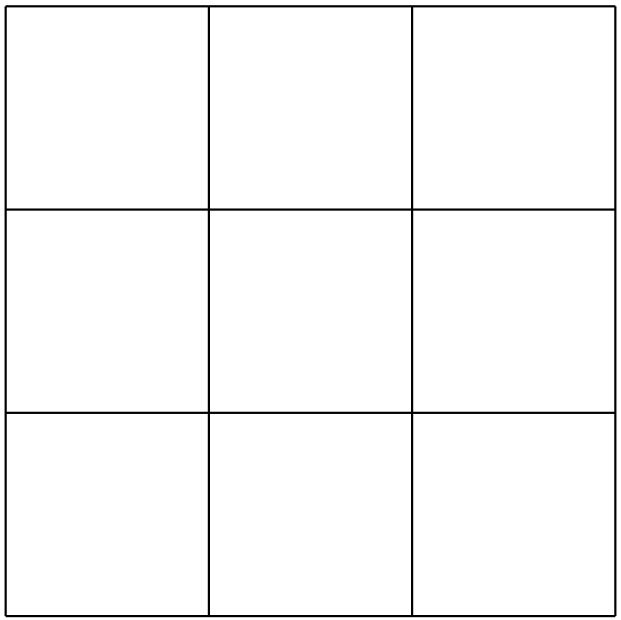
Question 12

Consider (again) a Rubik's cube, where each of the six faces has sixteen *corner points*, illustrated by the intersections of the line segments as follows:



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Question 12

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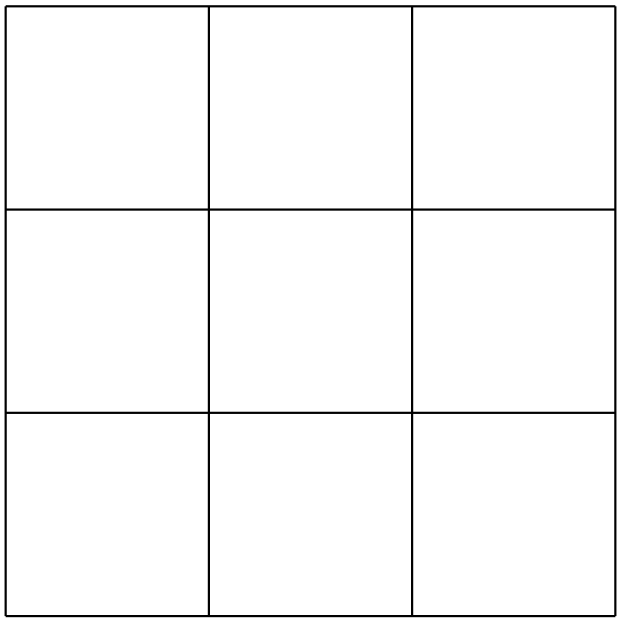

How many corner points are there on the cube total?

Solution 12

Answer. 56.
On each face, there are 16 corner points. Of these:

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Solution 12

Answer. 56.

On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;

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Solution 12

Answer. 56.

On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;

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Solution 12

Answer. 56.

On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;
- 4 are shared with two other faces, and $4 \cdot 2 = 8$.

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Solution 12

Answer. 56.

On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;
- 4 are shared with two other faces, and $4 \cdot 2 = 8$.

$$24 + 24 + 8 = 56.$$

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Question 14

The fraction $\frac{1}{23}$ can be written as a repeating decimal

$$\frac{1}{23} = 0.\overline{0434782608695652173913},$$

where the 22 digits under the bar repeat infinitely many times.

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Question 14

The fraction $\frac{1}{23}$ can be written as a repeating decimal

$$\frac{1}{23} = 0.\overline{0434782608695652173913},$$

where the 22 digits under the bar repeat infinitely many times. What is the sum of these 22 digits?

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Solution 14

Answer. 99.

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$$\frac{1}{23} = 0.\overline{043478260869565217391 3},$$

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Solution 14

Answer. 99.

$$\frac{1}{23} = 0.\overline{0434782608695652173913},$$

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$$\begin{array}{r} 04347826086 \\ +\ 95652173913 \\ \hline 99999999999 \end{array}$$

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Solution 14 (cont.)

$$\frac{1}{23} = 0.\overline{043478260869565217391 3},$$

The 22 digits above are the first digits (after the decimal place) of $\frac{1}{23}, \frac{2}{23}, \cdots, \frac{22}{23}$. Since these occur in pairs $\frac{a}{23}$, $\frac{23-a}{23}$, the digits b, $9-b$ also occur in pairs.

There are 11 pairs of digits which sum to 9, and $11 \cdot 9 = 99$.

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Question 15

An empty pasture has a certain amount of grass, and the grass grows at a constant rate.

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An empty pasture has a certain amount of grass, and the grass grows at a constant rate.

- If you allow 16 cows onto the pasture, they will eat all the grass in 10 days.

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An empty pasture has a certain amount of grass, and the grass grows at a constant rate.

- If you allow 16 cows onto the pasture, they will eat all the grass in 10 days.
- If you instead allow only 10 cows onto the pasture, they will eat all the grass in 22 days.

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Question 15

An empty pasture has a certain amount of grass, and the grass grows at a constant rate.

- If you allow 16 cows onto the pasture, they will eat all the grass in 10 days.
- If you instead allow only 10 cows onto the pasture, they will eat all the grass in 22 days.

If you allow 25 cows onto the pasture, how long will it take them to eat all the grass?

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Question 16

If you expand $(x + 2y)^4$, what is the sum of all the coefficients?

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Solution 16

Answer. 81.

Adding all the coefficients is equivalent to substituting 1 for x and y, and $(1 + 2 \cdot 1)^4 = 3^4 = 81$.

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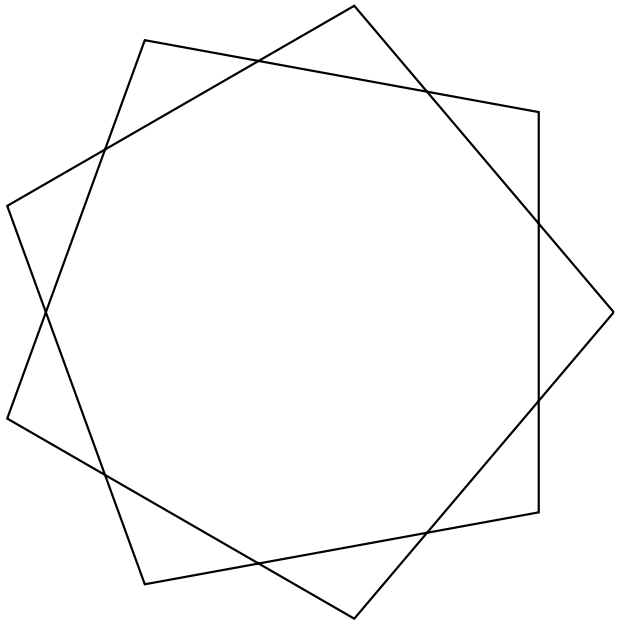
Question 17

The following figure consists of nine line segments:



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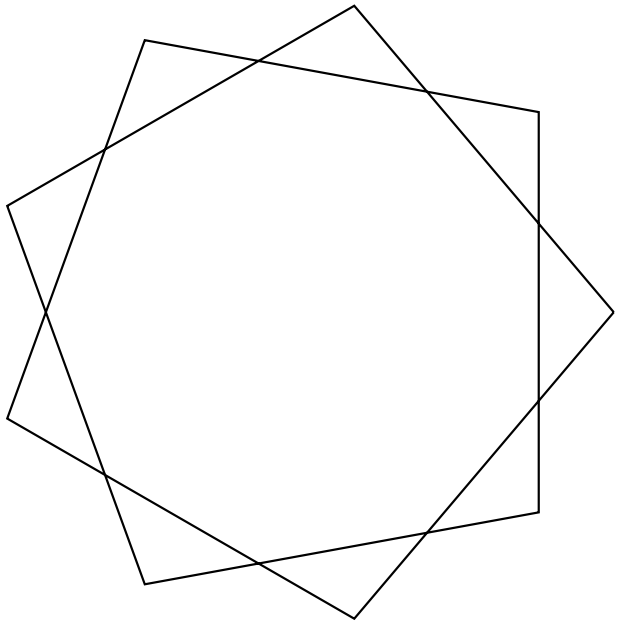
Question 17

The following figure consists of nine line segments:



All of the triangles in the picture are congruent. What is the largest angle in any of these triangles?

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Solution 17

Answer. $\frac{5}{9}\pi$ or 108°.

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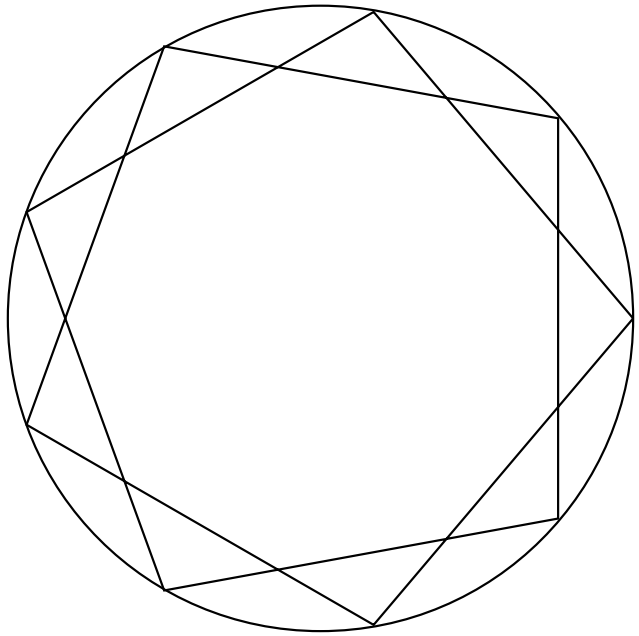
Solution 17

Answer. $\frac{5}{9}\pi$ or 108°.
The figure is symmetric, and can be inscribed in a circle:



Each of these angles is subtended by an arc consisting of $\frac{5}{9}$ of the circle, hence of measure $\frac{5}{9} \cdot 2\pi$.

Question 18

When you expand out and simplify the following product, how many nonzero monomial terms are there?

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Question 18

When you expand out and simplify the following product, how many nonzero monomial terms are there?

$$(x-a)(x-b)(x-c)\cdots(x-z).$$

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Solution 18

Answer. 0.

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Solution 18

Answer. 0.
The product is equal to

$$(x-a)(x-b)(x-c)\cdots(x-x)(x-y)(x-z),$$

and $x - x = 0$, so the product is zero.

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Question 19

If

$$\log_2(x) + \log_8(4x) = 10,$$

what is x?

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Solution 19

Answer: 128.

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