# State High School Mathematics Tournament

University of South Carolina

January 25, 2020

# Tournament Round 2 – Rules

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- If your answer is wrong, the clock will be restarted. If your opponent doesn't buzz in, they may answer *immediately* after time is called.

# Question 2-1

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$$\begin{aligned} 33 &= 8866128975287528^3 \\ &+ (-877840544286223?)^3 \\ &+ (-2736111468807040)^3. \end{aligned}$$

# Solution 2-1

**Answer.** 9.

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Modulo 10, we have

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so

$$1 \equiv (-?)^3,$$

for which $-? \equiv 1$ is the unique solution. So $? = 9$.

# Question 2-1

Solve for $x$:

$$\log_3(9x) + \log_9(3x) = 7$$

# Solution 2-2

**Answer.** 27.

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Rewrite the equation as

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$$\frac{5}{2} + \frac{3}{2}\log_3(x) = 7.$$

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or

$$\frac{5}{2} + \frac{3}{2}\log_3(x) = 7.$$

So

$$\frac{3}{2}\log_3(x) = 7 - \frac{5}{2} = \frac{9}{2},$$

and $\log_3(x) = 3$, so $x = 27$.

# Question 2-3

What is the smallest value of $r$ for which the following is true?

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The parabola $y = x^2$ intersects (in at least one point) the circle with center $(0, 1)$ and radius $r$.

# Solution 2-3

**Answer.** $\frac{\sqrt{3}}{2}$.

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Solving $y = x^2$ and $x^2 + (y-1)^2 = r^2$ yields

$$y^2 - y + (1 - r^2) = 0,$$

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$$y = \frac{1 \pm \sqrt{-3 + 4r^2}}{2}.$$

This has a nonnegative solution when $-3 + 4r^2 \geq 0$, so when $r^2 \geq \frac{3}{4}$.

# Question 2-4

Your friend rolls two ordinary dice and you roll one.

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What is the probability that your die roll exceeds the total of hers?

# Solution 2-4

**Answer.** $\frac{5}{54}$.

Depending on whether you roll 1, 2, 3, 4, 5, 6, the probability that her total is lower is respectively

$$0, 0, \frac{1}{36}, \frac{3}{36}, \frac{6}{36}, \frac{10}{36}.$$

So the overall probability is

$$\frac{1}{6}\left(\frac{1}{36}+\frac{3}{36}+\frac{6}{36}+\frac{10}{36}\right)=\frac{1}{6}\cdot\frac{20}{36}=\frac{20}{216}=\frac{5}{54}.$$

# Question 2-5

If

$$2\cos^2(x) - \sin^2(x) = \frac{1}{2}$$

and $0 < x < \frac{\pi}{2}$, what is $x$?

# Solution 2-5

**Answer.** $\frac{\pi}{4}$.

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**Answer.** $\frac{\pi}{4}$.

We have

$$3\cos^2(x) = (2\cos^2(x) - \sin^2(x)) + (\cos^2(x) + \sin^2(x)) = \frac{1}{2} + 1 = \frac{3}{2},$$

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We have

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so $\cos^2(x) = \frac{1}{2}$ and $\cos(x) = \frac{\sqrt{2}}{2}$.

So $x = \frac{\pi}{4}$.

# Question 6

You eat a bunch of cupcakes. On the first day, you eat one cupcake; on the second day, you eat two cupcakes; on the third day, you eat three; and so on.

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On what day will you eat your five thousandth cupcake?

# Solution 6

**Answer.** 100.

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After $n$ days, you will have eaten

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

cupcakes. So what is the minimal $n$ for which

$$\frac{n(n+1)}{2} \geq 5000, \quad \text{or} \quad n(n+1) \geq 10000?$$

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$$\frac{n(n+1)}{2} \geq 5000, \quad \text{or} \quad n(n+1) \geq 10000?$$

Since $10000 = 100^2$, we have $n = 100$.

# Question 7

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It depicts an icosahedron: a regular solid with twenty faces, each of which is an identical equilateral triangle.

How many edges are not visible in the logo?

# Solution 7

**Answer.** 12.

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An icosahedron has 30 edges: 20 triangles times 3 edges per triangle, divided by 2 since each edge is shared between two triangles.

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An icosahedron has 30 edges: 20 triangles times 3 edges per triangle, divided by 2 since each edge is shared between two triangles.

You can count that 18 edges are visible in the picture, and $30 - 18 = 12$.

# Question 8

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$$\log_2(33) + \log_{33}(2)?$$

# Solution 8

**Answer**: 6.

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We have

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We have

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We have

$$2^5 = 32, \quad 2^6 = 64,$$

so that $\log_2(33)$ is slightly bigger than 5.
We have

$$\log_{33}(2) = \frac{1}{\log_2(33)} < \frac{1}{5}.$$

The sum of these numbers is less than 6.

How many real solutions $x$ are there to the equation

# Question 9

How many real solutions $x$ are there to the equation

$$x^5 + \sin(x) = 0?$$

# Solution 9

**Answer**: 1.

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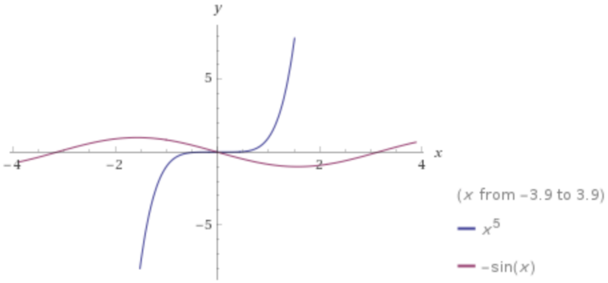
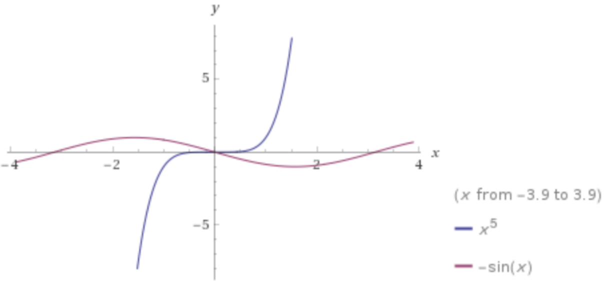



The graphs of $y = x^5$ and $y = -\sin(x)$ don't intersect in $(0, \pi)$ because of opposite signs, or in $[\pi, \infty)$ because $x^5 > 1$. Similarly, there are no intersection points with $x < 0$.

# Solution 9

**Answer**: 1.




The graphs of $y = x^5$ and $y = -\sin(x)$ don't intersect in $(0, \pi)$ because of opposite signs, or in $[\pi, \infty)$ because $x^5 > 1$. Similarly, there are no intersection points with $x < 0$.
So $x = 0$ is the only intersection point.

# Question 10

You toss four coins. What is the probability that at least three of them come up heads?

# Solution 10

**Answer.** $\frac{5}{16}$.

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There are $2^4 = 16$ total ways to flip four coins. The total number with at least three heads is

$$\binom{4}{3} + \binom{4}{4} = 4 + 1 = 5,$$

# Solution 10

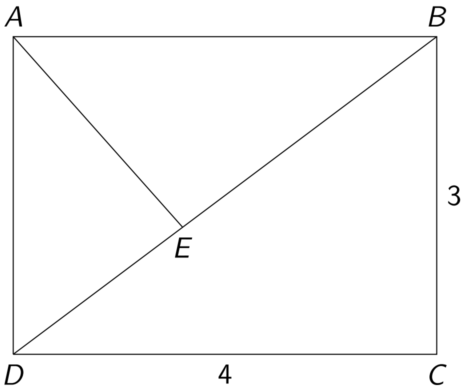
**Answer**. $\frac{5}{16}$.

There are $2^4 = 16$ total ways to flip four coins.
The total number with at least three heads is

$$\binom{4}{3} + \binom{4}{4} = 4 + 1 = 5,$$

$$HHHH, HHHT, HHTH, HTHH, THHH.$$

# Question 11




Given rectangle $ABCD$ as above. If $\angle AEB = 90^\circ$, what is $AE$?

# Solution 11

**Answer.** $\frac{12}{5}$.

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$BD = 5$, and $\triangle ABE \sim \triangle BDC$. So

$$\frac{AE}{AB} = \frac{BC}{BD} = \frac{3}{5}$$

and

$$AE = \frac{3}{5} \cdot AB = \frac{3}{5} \cdot 4 = \frac{12}{5}.$$

# Question 12

How many pairs of positive prime numbers $p, q$ are there with

$$p - q = 21?$$

# Solution 12

**Answer.** 1.

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**Answer.** 1.

All prime numbers other than 2 are odd. The difference of two odd numbers is even. Therefore $q$ must be 2. Since $2 + 21 = 23$ is prime, there is one solution.

# Question 13

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Five games into the baseball season, Cocky Gamecock has a batting average of .435. In his sixth game, he has five at bats and gets hits in all of them.

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Five games into the baseball season, Cocky Gamecock has a batting average of .435. In his sixth game, he has five at bats and gets hits in all of them.
If this raises his batting average to .536, how many at bats does he have through his first six games?

# Solution 13

**Answer.** 28.

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Let $x$ be the number of hits through 6 games, and $y$ the number of at bats. Within a small roundoff error,

$$\frac{x-5}{y-5} = .435, \quad \frac{x}{y} = .536.$$

# Solution 13

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Let $x$ be the number of hits through 6 games, and $y$ the number of at bats. Within a small roundoff error,

$$\frac{x-5}{y-5} = .435, \quad \frac{x}{y} = .536.$$

We have

$$x - 5 = .435(y-5), \quad x = .435y + 5 - 2.175 = .435y + 2.825.$$

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We have

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We thus have

$$x = .536y, \quad .101y = 2.825.$$

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We have

$$x - 5 = .435(y-5), \quad x = .435y + 5 - 2.175 = .435y + 2.825.$$

We thus have

$$x = .536y, \quad .101y = 2.825.$$

Thus, we have

$$y = \frac{2.825}{.101},$$

or $y = 28$ up to the roundoff error.

# Question 14

If you write $\frac{1}{2020}$ as an infinite repeating decimal,

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If you write $\frac{1}{2020}$ as an infinite repeating decimal, what is the sum of the first six digits after the decimal place?

# Solution 14

**Answer.** 18.

$$\frac{1}{2020} = 0.00049504950\ldots$$

# Solution 14

**Answer.** 18.

$$\frac{1}{2020} = 0.00049504950\ldots$$

Note that

$$\frac{1}{101} = .009900990099\ldots,$$

so

$$\frac{1}{1010} = .0009900990099\ldots,$$

$$\frac{1}{2020} = .0004950495049\ldots,$$