State High School Mathematics Tournament

University of South Carolina

Round 1 – March 23, 2024

Coming Soon!

The **Columbia Math Circle:**

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- Sign up at `https://thornef.github.io/mathcircle/`

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- Contact me at `thorne@math.sc.edu`

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- There will be a tiebreaker if needed.

Question 1-1

How many numbers occur as the last digit of a prime number?

Solution 1-1

Answer. 6.

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$$2, 3, 5, 7, 11, 13, 17, 19, \ldots$$

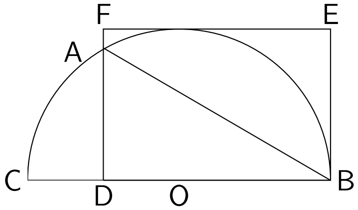
Solution 1-1

Answer. 6.

$$2, 3, 5, 7, 11, 13, 17, 19, \ldots$$

No prime other than 2 is even, so 4, 6, 8, and 0 can't occur.

In a semicircle with center O and diameter CB, a point D is chosen between C and O. A line segment DF of length OB is drawn perpendicular to CB, intersecting the semicircle at A.



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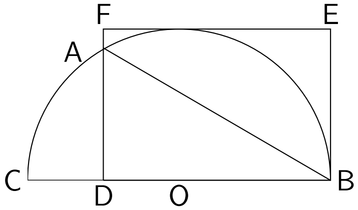

If $AB = 6$, what is the area of the rectangle $DFEB$?

Solution 1-2

Answer. 18.

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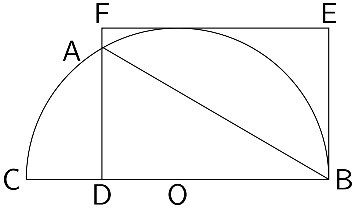
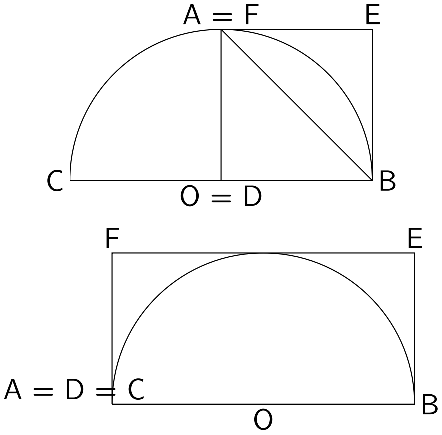


Suppose that the circle has radius r. Both ΔABC and ΔDBA are both right triangles and are similar, so we have $AB^2 = BC \cdot BD$, that is, $BC \cdot BD = 36$. Note that $BE = r$ and $BC = 2r = 2BE$. So the area of the rectangle is $BE \cdot BD = \frac{1}{2} BC \cdot BD = 36/2 = 18$.

"Cheating" Solution 1-2



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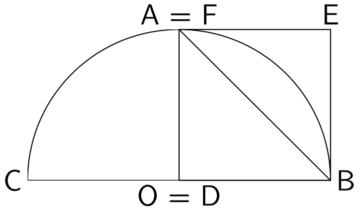
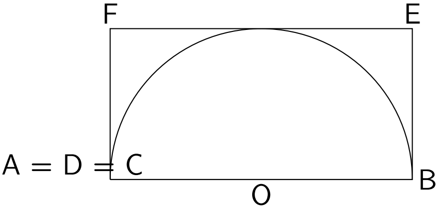
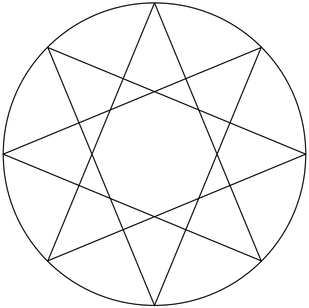


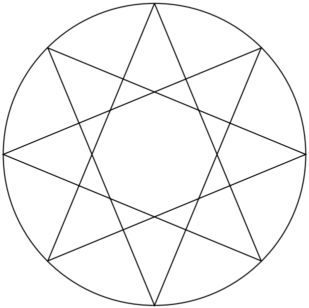


It's 18 no matter where D is.



The above depicts a unit circle, where the endpoints of the depicted line segments are equally spaced.

Question 1-3



The above depicts a unit circle, where the endpoints of the depicted line segments are equally spaced.
What is the area of either of the two visible squares?

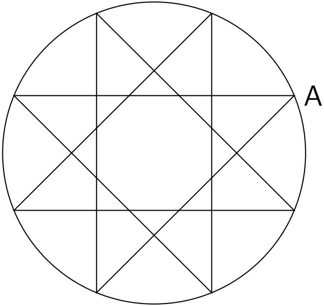
Solution 1-3

Answer. $2 - \sqrt{2}$.

Solution 1-3

Answer. $2-\sqrt{2}$.

Rotate the picture so that the edges of a square are parallel to the coordinate axes:



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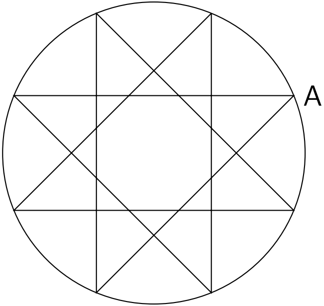

A has coordinates $(\cos\frac{\pi}{8}, \sin\frac{\pi}{8})$, and the square has area

$$4\sin^2\frac{\pi}{8} = 4\cdot\frac{1-\cos\frac{\pi}{4}}{2} = 2-\sqrt{2}.$$

Question 1-4

What is the minimum value assumed by $\sin^4(x) + \cos^4(x)$?

Solution 1-4

Answer. $\frac{1}{2}$.

Solution 1-4

Answer. $\frac{1}{2}$.

We have

$$\begin{aligned} 1 &= (\sin^2 x + \cos^2 x)^2 \\ &= \sin^4(x) + \cos^4(x) + 2\sin^2(x)\cos^2(x) \\ &= \sin^4(x) + \cos^4(x) + \frac{1}{2}\sin^2(2x). \end{aligned}$$

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$\sin^4(x) + \cos^4(x) = \frac{1}{2}$ whenever $\sin(2x) = \pm 1$, for example at $x = \frac{\pi}{4}$.

Question 1-5

You flip three coins and a friend flips three coins.

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What is the probability that you each flip exactly the same number of heads?

Solution 1-5

Answer. $\frac{5}{16}$.

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$$\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{1}{8}\right)^2 = \frac{1+9+9+1}{64} = \frac{5}{16}.$$

Question 1-6

Unit circles C_1 and C_2 intersect at P_1 and P_2. A unit circle C_3 passes through P_2 and has center P_1.

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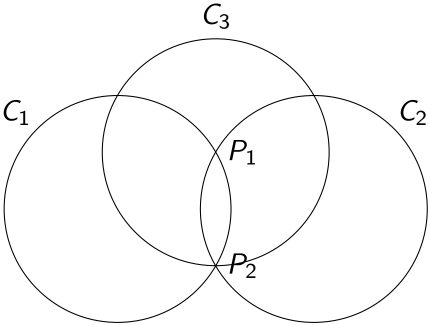
What is the total area covered by the circles?



Solution 1-6

Answer. $\frac{5}{3}\pi + \sqrt{3}$.

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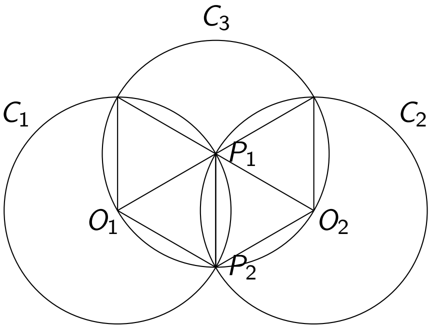
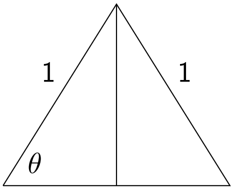


The four triangles have total area $\sqrt{3}$, and the remaining circles have $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ of their areas counted.

If $\triangle ABC$ is an isosceles triangle with $AB = BC = 1$, what should the length of AC be to maximize the triangle's area?

Solution 7

Answer. $\sqrt{2}$



$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

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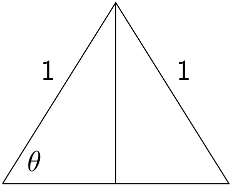

$$\text{Area} = \sin(\theta) \cdot \cos(\theta) = \frac{1}{2}\sin(2\theta).$$

Maximize with $\theta = \frac{\pi}{4}$, so $AC = \sqrt{2}$.

Question 8

The equation $2^x = x^2$ has three real solutions. What is the nearest integer to their sum?

Solution 8

Answer. 5

$x = 2$, $x = 4$, and $x = -.76\ldots$

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For the negative solution, note that $2^{-\frac{1}{2}} > (-\frac{1}{2})^2$, so $x < -\frac{1}{2}$.

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2021 - 2022 + 2023 - 2024?$$

Solution 9

Answer. -1012.

Solution 9

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Write it as

$$(1-2)+(3-4)+\cdots+(2023-2024)=(-1)\times 1012.$$

Question 10

How many positive integers $n \leq 10$ satisfy $\cos(n) > 0$?

(Assume radian measure.)

Solution 10

Answer. 4.

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$$n \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

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$$n \in \{1, 5, 6, 7\}$$

Question 11

Simplify:

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$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{5}}}}}$$

Solution 11

Answer. $\frac{17}{28}$.

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Notice the pattern: $\frac{6}{5}, \frac{11}{6}, \frac{17}{11}, \frac{28}{17}$