COMPREHENSIVE EXAM IN ALGEBRAIC NUMBER THEORY (FALL 2013)

Recall that the **Minkowski bound** is

$$N(\mathfrak{a}) \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta_K|^{1/2}.$$

1. Let K be the cubic field generated by a root of $x^3 + 2x - 1$. Determine the following data associated to K:
   (a) its **discriminant**;
   (b) the **ring of integers**;
   (c) the number of **real and complex embeddings**;
   (d) the isomorphism class of its **unit group** (your answer should look like, e.g., $\mathbb{Z}^2 \times \mathbb{Z}/(6)$; you do not need to find the actual units);
   (e) the list of **ramified primes**;
   (f) the splitting types of the ideals $(2), (3), (5), (7)$;
   (g) whether or not K is **Galois** over $\mathbb{Q}$; and if not, the Galois group of its Galois closure (i.e., the splitting field of $x^3 + 2x - 1$);
   (h) (bonus) the **the class group**;
   (i) (bonus) the **proportion of primes** which have the splitting types you found above.

2. (a) Define the p-adic integers $\mathbb{Z}_p$.
   (b) (bonus) Give an alternative definition of the p-adic integers $\mathbb{Z}_p$. (further bonus) Prove that these definitions are equivalent.
   (c) Let $p = 7$. Determine which of $\frac{1}{3}$, 7, $\frac{1}{7}$, $\sqrt{2}$, and $\sqrt{5}$ are 7-adic integers. For those that are, compute the 7-adic expansion to at least three decimal places.
   Note that one of the two square roots is in $\mathbb{Z}_7$; give a detailed proof of this, without quoting Hensel's lemma. For the others, a very brief explanation is enough.

3. (a) Determine, with proof, the class group of $\mathbb{Q}(\sqrt{-5})$.
   (b) Is $\mathbb{Q}(\sqrt{100000001})$ a principal ideal domain?

4. (a) Determine the discriminant of a quadratic field $\mathbb{Q}(\sqrt{D})$, for a general D.
   (b) Let ℓ be an odd prime. Then it is known that the discriminant of the **cyclotomic field** $K = Q(\zeta_\ell)$ is equal to $\pm\ell^{\ell-2}$, and also that $(\ell) = (1 - \zeta_\ell)^{\ell-1}$ as ideals of $\mathcal{O}_K$. What fact about the splitting of prime ideals in $\mathcal{O}_K$ is reflected in both of these facts?
   (c) Prove that $\mathbb{Q}(\zeta_\ell)$ contains a unique quadratic subfield, and determine what it is.
   (d) (bonus) Subject to knowing the field is unique, determine what it is in a completely different way.