TATE'S THESIS

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Abstract. These are notes for a talk to be given in the Wisconsin student number theory seminar. The talk is on Tate's thesis [3]. These notes are based closely on Tate's thesis itself. We will be somewhat more explicit about talking about Dirichlet and Hecke L-functions and illustrating how they fit into the general theory.

1. Introduction: the Riemann zeta function

The Riemann zeta function is probably the most important function in all of analytic number theory. We will presume that the reader is familiar with this function and its importance. But we will recall that it has analytic continuation to the entire complex plane, with a functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s). \tag{1.1}$$

In other words the completed zeta function

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{1.2}$$

satisfies the beautiful equation

$$\xi(s) = \xi(1-s). \tag{1.3}$$

This equation is of the utmost simplicity and should perhaps be considered as a miracle. The form of it suggests that $\xi(s)$ is the function we should really be looking at, and the author was motivated to prepare these notes by the following question:

What does the $\pi^{-s/2}\Gamma(s/2)$ mean?

One suggestion is given by the Euler product expansion

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.4}$$

where the product is over all primes.

But is it really? The "highbrow" view of primes is that a **prime** in a number field is an **equivalence class of valuations**, and that therefore there are **infinite primes** as well, given by embeddings of this number field into $\mathbb{R}$ or $\mathbb{C}$. The number of such embeddings is given by the degree of the number field over $\mathbb{Q}$. In particular, $\mathbb{Q}$ has one infinite prime which corresponds to the absolute value $\mathbb{Q}$ receives as a subset of $\mathbb{R}$.

And there is nothing in (1.4) corresponding to this valuation.

Say something profound here! But do the nitty gritty first.

As suggested by (1.4) and the discussion above, we will want to write a zeta function as a product of local zeta functions, and so we will discuss the local theory first and then the global.

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1.1. **The traditional proof.** Before presenting Tate's thesis we will review the traditional proof of the functional equation. We follow Neukirch's presentation [1], but the same proof can be found in many places. We start by writing

$$\pi^{-s}\Gamma(s)\frac{1}{n^{2s}} = \int_0^\infty e^{-\pi n^2 y} y^{s-1} dy, \tag{1.5}$$

which follows by the definition of the gamma function and a change of variables. Summing over all $n \geq 1$, we obtain

$$\pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 y} \right) y^{s-1} dy. \tag{1.6}$$

Here the interchange of summation and integration is justified by absolute convergence.

The **theta function**

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} \tag{1.7}$$

is an example of a **weight 1/2 modular form**[1], and in particular it satisifes the functional equation

$$\theta(-1/z) = \sqrt{z/i}\,\theta(z) \tag{1.8}$$

as a consequence of the Poisson summation formula.

We can thus write

$$\xi(s) = \frac{1}{2}\int_0^\infty (\theta(iy) - 1) y^{s/2} \frac{dy}{y}, \tag{1.9}$$

where the integral is beautifully convergent for large y, but not for small y. Therefore, we write

$$\xi(s) = \frac{1}{2}\int_0^1 (\theta(iy) - 1) y^{s/2} \frac{dy}{y} + \int_1^\infty (\theta(iy) - 1) y^{s/2} \frac{dy}{y} \tag{1.10}$$

and apply the above functional equation to the first integral. We obtain

$$\xi(s) = \frac{1}{2}\int_0^1 (\theta(i/y) - 1) y^{(s-1)/2} \frac{dy}{y} + \int_1^\infty (\theta(iy) - 1) y^{s/2} \frac{dy}{y}. \tag{1.11}$$

We separate the -1 from the $\theta(i/y)$, and do a change of variables $y \to 1/y$ on the $\theta(i/y)$ part, and we obtain

$$\xi(s) = \frac{-1}{s} + \frac{1}{2}\int_1^\infty \theta(iy) y^{(1-s)/2} \frac{dy}{y} + \int_1^\infty (\theta(iy) - 1) y^{s/2} \frac{dy}{y}, \tag{1.12}$$

and this is equal to

$$\xi(s) = \frac{-1}{s} + \frac{-1}{1-s} + \frac{1}{2}\int_1^\infty (\theta(iy) - 1) y^{(1-s)/2} \frac{dy}{y} + \int_1^\infty (\theta(iy) - 1) y^{s/2} \frac{dy}{y}. \tag{1.13}$$

This function converges absolutely and uniformly for all s (except at the two poles), and it is now quite clear that is symmetric in s and $1 - s$.

2. The local theory

Throughout this section let k denote the completion of an algebraic number field at some prime $\mathfrak{p}$ (which may be infinite). We are interested in the multiplicative group $k^\times$. Let $\mathcal{O}$ be the associated valuation ring, and let U denote the subgroup of elements of absolute value 1.

[1]Caution! A lot of people (e.g. Ken) will write $e^{2\pi i n^2 z}$ instead.

2.1. **The additive theory.** Suppose that p is the rational prime divisor which $\mathfrak{p}$ divides. We define a function $\lambda : \mathbb{Q}_p \to \mathbb{R}/\mathbb{Z}$ as follows: If $p = \infty$, we define $\lambda(x) = -x \pmod 1$. If $p \neq \infty$, we let $\lambda(x)$ be the unique rational number mod 1, such that $\lambda(x)$ has only a p-power in the denominator, and $\lambda(x) - x$ is a p-adic integer.

We then define

$$\Lambda(\xi) := \lambda(\mathrm{Tr}_{k/\mathbb{Q}_p}\xi). \tag{2.1}$$

Then we readily check that $\xi \to e^{2\pi i\Lambda(\xi)}$ is a nontrivial character of k (i.e. of the additive group of k). This character will be trivial on $\mathcal{O}$.

We obtain the following

Theorem 2.1. *k is naturally its own character group if we identify the character $\xi \to e^{2\pi i\Lambda(\eta\xi)}$ with the element $\eta \in k$.*

We also have the following

Lemma 2.2. *If $\mathfrak{p}$ is nonarchimedean, the character $e^{2\pi i\Lambda(\eta\xi)}$ is trivial on $\mathcal{O}$ if any only if $\eta \in \mathfrak{d}^{-1}$, the inverse different of k.*

We recall that if L/M is an extension of fields, the **inverse different** $\mathfrak{d}_{L/M}^{-1}$ is the set

$$\mathfrak{d}_{L/M}^{-1} := \{x \in L : Tr(\mathcal{O}_L x) \subseteq \mathcal{O}_M\}. \tag{2.2}$$

The lemma is then essentially a triviality (a bunch of definition chasing).

We then have the standard **Fourier inversion formula:**

Theorem 2.3. *If we define the Fourier transform $\widehat{f}$ of a function $f \in L_1(k)$ by*

$$\widehat{f}(\eta) = \int_k f(\xi)e^{-2\pi i\Lambda(\eta\xi)}d\xi, \tag{2.3}$$

then for a suitable choice of measure

$$f(\xi) = \int_k \widehat{f}(\eta)e^{2\pi i\Lambda(\xi\eta)} = \widehat{\widehat{f}}(-\xi) \tag{2.4}$$

holds.

Moreover, the choice of measure is given by: ordinary Lebesgue measure if k is real, twice Lebesgue measure if k is complex, and if k is $\mathfrak{p}$-adic, the measure for which $\mathcal{O}$ has measure $(\mathbb{N}\mathfrak{d})^{-1/2}$.

2.2. **The multiplicative theory.**

Definition 2.4. *A* **quasicharacter** *of $k^\times$ is a continuous homomorphism $k^\times \to \mathbb{C}^\times$. It is* **unramified** *if it is trivial on U.*

We then have the following classification result for unramified quasicharacters:

Lemma 2.5 (2.3.1)**.** *The unramified quasicharacters of $k^\times$ are the maps of the form $c(\alpha) = |\alpha|^s = e^{s\log|\alpha|}$, where s is any complex number. If $\mathfrak{p}$ is archimedean, then s is determined by c, and otherwise s is determined mod $2\pi i/\log \mathbb{N}\mathfrak{p}$.*

This leads quite directly to a classification theorem for all quasicharacters. We decompose a general element $\alpha \in k^\times$ as $\alpha = \tilde{\alpha}\rho$, where $\alpha \in U$. If $\mathfrak{p}$ is archimedean then ρ is a positive real number, and if $\mathfrak{p}$ is nonarchimedean then ρ is a power of a uniformizer π.

Theorem 2.6 (2.3.1)**.** *The quasicharacters of $k^\times$ are maps of the form $c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s$, where $\tilde{c}$ is any character of U and s is as in the previous lemma.*

We will call two quasicharacters equivalent if their quotient is an unramified quasicharacter. An equivalence class of quasicharacters thus consists of all quasicharacters $c(\alpha) = c_0(\alpha)|\alpha|^s$, where $c_0(\alpha)$ is a fixed representative of the class and s is a complex variable. Therefore, we may view an equivalence class of quasicharacters as a Riemann surface. If $\mathfrak{p}$ is archimedean, we really just get $\mathbb{C}$. If $\mathfrak{p}$ is nonarchimedean, s is determined only modulo $2\pi i/\log(\mathbb{N}\mathfrak{p})$. Accordingly, it makes sense to use the language of complex analysis and talk about meromorphic functions, analytic continuation, and the like.

Definition 2.7. *We let $\mathfrak{z}$ denote the class of all "good" functions defined on k^+, which satisfy the following two properties:*

1. $f(\xi)$ and $\widehat{f}(\xi)$ are continuous, integrable functions of k^+.

2. Denoting by $f(\alpha)$ the restriction to the multiplicative group $k^\times$, $f(\alpha)|\alpha|^\sigma$ and $\widehat{f}(\alpha)|\alpha|^\sigma$ are integrable functions on $k^\times$ for $\sigma > 0$.

We won't think too much about what these conditions mean here; suffice it to say that they are enough to make all the analysis work out.

As Tate says, a ζ-function is "what one might call a multiplicative quasi-Fourier transform of a function $f \in \mathfrak{z}$."

Definition 2.8. *For each $f \in \mathfrak{z}$ we define a function $\zeta(f, c)$ of quasicharacters c, defined for all quasicharacters of exponent greater than 0 by*

$$\zeta(f, c) := \int_{h^\times} f(\alpha)c(\alpha)d^\times\alpha. \tag{2.5}$$

Remark. Tate does not use the notation $d^\times\alpha$ but instead writes $d\alpha$ where the d is in a slightly different typeface. If you read the original then watch out for this!

Definition 2.9. *For a quasicharacter c we define another quasicharacter $\hat{c}$ by*

$$\hat{c}(\alpha) := |\alpha|c^{-1}(\alpha).$$

The main theorem of the local theory is then the following:

Theorem 2.10. *A ζ-function has an analytic continuation to the domain of all quasicharacters, given by a functional equation*

$$\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c}), \tag{2.6}$$

where the factor $\rho(c)$ does not depend on the function f. It is defined for quasicharacters of exponent $\in (0, 1)$ by the functional equation itself, and for all quasicharacters by analytic continuation.

The proof is not too difficult. The main step is a derivation of the identity

$$\zeta(f, c)\zeta(\widehat{g}, \widehat{c}) = \zeta(\widehat{f}, \widehat{c})\zeta(g, c) \tag{2.7}$$

which is valid for any c with exponent $\in (0, 1)$ and any $f, g \in \mathfrak{z}$. This identity is proved by writing each side as a double integral and using Fubini's theorem, and using a change of variables. The whole proof feels (to me) like sleight of hand and there are no difficult steps.

The other part of the proof is the computation of the factor $\rho(c)$ for individual choices of f, which will be undertaken later.

3. Adeles, ideles, characters and measure

The global theory is properly stated in the language of adeles and ideles. We will review this theory; well-written proofs can be found in [3] and elsewhere.

For a number field K and a place v, let K_v be the completion at v and $\mathcal{O}_v$ its ring of integers. Also let U_v denote the unit group (i.e. the elements of valuation 0). The **adele ring** of K is the **restricted product**

$$\mathbb{A}_K := \prod_v{}' K_v, \tag{3.1}$$

where the v range over all places of K, and the product is restricted with respect to the $\mathcal{O}_v$. This means that any adele $x = (x_{v_1}, x_{v_2}, \cdots)$ is required to have $x_{v_i} \in O_{v_i}$ for almost all (= all but finitely many) i.

The **group of ideles** $\mathbb{A}_K^\times$ is simply the multiplicative group of invertible adeles. The ideles are also a restricted product: the restricted product of the multiplicative groups $K_v^\times$ with respect to the U_{v_i}.

We will discuss some generalities about characters and Haar measure and restricted products. In this section $\mathfrak{a} = (a_{v_i})$ will refer to either an adele or an idele. We will adopt Tate's notation: G_v will refer to K_v when we are talking about adeles, and to $K_v^\times$ when we are talking about ideles. Similarly, H_v will refer to $\mathcal{O}_v$ when we are talking about adeles, or U_v when we are talking about ideles. Thus, both groups are now the restricted product of the G_v with respect to the H_v.

We will write G for the group of adeles or ideles, and if S is a finite set of places, we will let G_S refer to the set of adeles or ideles whose finitely many "bad" places are contained in S. Notice that $G_S \subseteq G$ for each S.

The topology is defined as follows: G_S has the product topology for each S. To give a topology for G, it suffices to choose a basis of neighborhoods of 1, and we let this consist of a basis of neighborhoods of 1 in G_S for each S.

This gives a topology on the adeles and on the ideles. Notice that the topology on the ideles is not the subspace topology of that on the adeles.

Continuous quasicharacters on ideles correspond exactly to products of continuous quasicharacters on the G_v. In other words a character $c(\mathfrak{a})$ on G satisfies the product formula

$$c(\mathfrak{a}) = \prod_v c_v(\mathfrak{a}_v), \tag{3.2}$$

where all but finitely many factors of the product are 1. Here c_v denotes the restriction of $c(\mathfrak{a})$ to G_v. Conversely, if we have a collection of characters c_v of the G_v, all but finitely many of which are trivial on H_v, we may multiply them together and obtain a character of G.

Haar measure on the ideles is defined by the product formula

$$d\mathfrak{a} = \prod_p d\mathfrak{a}_p. \tag{3.3}$$

Here the measure $d\mathfrak{a}_p$ refers to the multiplicative measure dx or $d^\times x$ as appropriate. To make this formula more rigorous we define Haar measure on the subgroup G_S by the formula

$$d\mathfrak{a}_S = \Big(\prod_{v\in S} d\mathfrak{a}_v\Big) d\mathfrak{a}_S, \tag{3.4}$$

where $d\mathfrak{a}_S$ is the measure on the (compact!) group $prod_{v\notin S} H_v$ such that $\int_{G_S} d\mathfrak{a}_S = \prod_{v\notin S}\Big(\int_{H_v} d\mathfrak{a}_S\Big)$. We then argue that this uniquely defines a well-defined Haar measure.

We also consider integrable functions. Suppose we are given functions $f_v \in L_1(G_v)$, such that $f_v(\mathfrak{a}_p) = 1$ on H_v for all but finitely many v. Then, we define

$$f(\mathfrak{a}) := \prod_v f_v(\mathfrak{a}_v), \tag{3.5}$$

and thereby obtain a continuous function on G. Moreover, if it satisfies

$$\prod_v \left(\int |f_v(\mathfrak{a}_v)| d\mathfrak{a}_v \right) < \infty, \tag{3.6}$$

then we have the **product formula**

$$\int f(\mathfrak{a}) d\mathfrak{a} = \prod_v \left(\int f_v(\mathfrak{a}_v) d\mathfrak{a} \right). \tag{3.7}$$

4. The global theory

To do... say something about the class number formula?

In this section K will denote a number field, although this theory can be translated to other global fields as well (i.e., to the function field $\mathbb{F}_q(t)$). We will write $\mathbb{A}_K$ for the adele ring of K, and $\mathbb{A}_K^\times$ for the multiplicative group of ideles.

To put the "local-to-global" theory into place we will look at quasicharacters on the ideles. We will insist that these quasicharacters be trivial on $K^\times$, so that we are in fact looking at quasicharacters of the idele class group $\mathbb{A}_K^\times / K^\times$.

We first make a couple of definitions that are the global analogues of the ones considered previously. Tate defines a class of nice functions $\mathfrak{z}$ on the adeles, similarly to before. In particular, f must be integrable and continuous, and $f(\mathfrak{a})|\mathfrak{a}|^\sigma$ must be an integrable function on the ideles for $\sigma > 1$. Moreover, $\sum_{\xi \in k} f(\mathfrak{a}(\mathfrak{x} + \xi))$ must be absolutely convergent, and the convergence must be uniform in certain domains. Finally, all of the same has to be true of $\widehat{f}$.

Here, then, is Tate's definition of a global zeta-function.

Definition 4.1. *For each $f \in \mathfrak{z}$ we define a function $\zeta(f, c)$ of quasicharacters c, defined for all quasicharacters of exponent greater than 1 by*

$$\zeta(f, c) := \int_{\mathbb{A}_K^\times} f(\mathfrak{a}) c(\mathfrak{a}) d^\times \mathfrak{a}. \tag{4.1}$$

Definition 4.2. *For a quasicharacter $c(\mathfrak{a})$ of the ideles we define*

$$\widehat{c}(\mathfrak{a}) := |\mathfrak{a}| c^{-1}(\mathfrak{a}). \tag{4.2}$$

The following is the main theorem of Tate's thesis:

Theorem 4.3. *Any zeta-function $\zeta(f, c)$ has an analytic continuation to the domain of all quasicharacters. The extended function is single-valued and holomorphic, except at $c(\mathfrak{a}) = 1$ and $c(\mathfrak{a}) = |\mathfrak{a}|$ where it has simple poles with residues $-\kappa f(0)$ and $\kappa \hat{f}(0)$ respectively. (Here $\kappa = 2^{r_1}(2\pi)^{r_2} hR/(\sqrt{d}w)$ is the volume of the multiplicative fundamental domain.)*

Moreover, $\zeta(f, c)$ satisfies the functional equation

$$\zeta(f, c) = \zeta(\widehat{f}, \widehat{c}). \tag{4.3}$$

4.1. **Summary of the proof.** To establish the Main Theorem we write

$$\zeta(f,c) = \int_0^\infty \zeta_t(f,c)\frac{dt}{t}, \tag{4.4}$$

where

$$\zeta_t(f,c) := \int_J f(t\mathfrak{b})c(t\mathfrak{b})d^\times \mathfrak{b}. \tag{4.5}$$

Here J is the subgroup of ideles of "absolute value" 1.

We then write

$$\zeta(f,c) = \int_0^1 \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(f,c)\frac{dt}{t}. \tag{4.6}$$

It can be seen easily that the second integral converges. To tackle the first, we use the identity

$$\zeta_t(f,c) + f(0)\int_E c(t\mathfrak{b})d^\times \mathfrak{b} = \zeta_{1/t}(\widehat{f},\widehat{c}) + \widehat{f}(0)\int_E \widehat{c}(\mathfrak{b}/t)d^\times \mathfrak{b}. \tag{4.7}$$

To do: say what E is.

Using this identity and evaluating the integrals over E, we have

$$\zeta(f,c) = \int_1^\infty \zeta_t(f,c)\frac{dt}{t}\int_1^\infty \zeta_t(\widehat{f},\widehat{c})\frac{dt}{t} + \left(\frac{\kappa\widehat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right). \tag{4.8}$$

The two integrals are analytic for all c, and so we obtain analytic continuation. Moreover we can read off the poles and residues directly. Moreover, for $c(\mathfrak{a}) = |\mathfrak{a}|^s$, $\widehat{c}(\mathfrak{a}) = |\mathfrak{a}|^{1-s}$, the form of the expression is unchanged by replacing f and c with $\widehat{f}$ and $\widehat{c}$. This concludes the proof.

We remark that this proof looks a great deal like the traditional proof! We will now say a bit about the proof of the identity (4.7). It starts with the **Poisson summation formula**, which in this setting reads

$$\sum_{\xi\in K}\widehat{f}(\xi) = \sum_{\xi\in K} f(\xi). \tag{4.9}$$

This is of course only conditionally true; in particular, both sides have to converge absolutely (and there are a couple of related technical conditions).

From this we obtain the following theorem:

Theorem 4.4 (Riemann-Roch)**.** *Under appropriate convergence conditions, we have for every idele* $\mathfrak{a}$

$$\frac{1}{|\mathfrak{a}|}\sum_{\xi\in K}\widehat{f}(\xi/\mathfrak{a}) = \sum_{\xi\in K} f(\mathfrak{a}\xi). \tag{4.10}$$

The proof is quite easy (even if one does not shove the technical conditions under the rug, as we are doing here). Write $g(x) = f(\mathfrak{a}x)$, so that Poisson summation implies that

$$\sum_{\xi\in K}\widehat{g}(\xi) = \sum_{\xi\in K} g(\xi). \tag{4.11}$$

We have

$$\widehat{g}(\mathfrak{x}) = \int f(\mathfrak{a}\mathfrak{n})e^{-2\pi i\Lambda(\mathfrak{x}\mathfrak{n})}d\mathfrak{n} = \frac{1}{|\mathfrak{a}|}\int f(\mathfrak{n})e^{-2\pi i\Lambda(\mathfrak{n}\mathfrak{x}/\mathfrak{a})}d\mathfrak{n} = \frac{1}{|\mathfrak{a}|}\widehat{f}(\mathfrak{x}/\mathfrak{a}). \tag{4.12}$$

The theorem does not look too interesting at first; it just looks like a slight variation on the Poisson Summation Formula (which of course is exactly what it is). However, what is *very* interesting is the result you obtain when you let $K = \mathbb{F}_q(t)$. As the name suggests, you indeed recover (for a particular, and fairly simple, choice of f) the classical Riemann-Roch theorem on the dimensions of linear systems associated to algebraic curves.

This is explained in Ramakrishnan and Valenza's book [2]. If time permits I will add a section on this and talk about it.

5. Trivial characters and the Riemann zeta function

Tate computes $\rho(c)$ and the zeta functions for real, complex, and $\mathfrak{p}$-adic places. These computations naturally break up into cases depending on c.

In this section we will consider the simplest possible example. We will take $K = \mathbb{Q}$, and we will consider the trivial character of $\mathbb{A}_K^\times$ (i.e. the character that is defined to be 1 everywhere). Associated to this is the quasicharacter $c(\mathfrak{a}) = |\mathfrak{a}|^s$, which is trivial on $\mathbb{Q}^\times$. (This is a consequence of the product formula.) The nice thing about this family of quasicharacters is that they are unramified everywhere. The main theorem of Tate's thesis will then give an equality which is very familiar.

We begin with the relation

$$\zeta(f, c) = \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}}), \tag{5.1}$$

which is valid for any quasicharacter $c = \prod_{\mathfrak{p}} c_{\mathfrak{p}}$ of exponent greater than 1. This follows from general principles on integration over infinite products.

For the special case of interest, $c(\alpha) = |\alpha|^s$ for a complex parameter s, the above equation is valid when $\Re(s) > 1$.

Accordingly, we will analyze the global zeta-function by looking at each local zeta-function and then multiplying them together. We will need to analyze the $\mathfrak{p}$-adic and real cases separately.

The $\mathfrak{p}$-adic case (unramified, over $\mathbb{Q}$): Let $f(\xi)$ be the characteristic function of $\mathbb{Z}_p$, where $\mathfrak{p} = (p)$. Then,

$$\zeta(f, |\cdot|^s) = \int_{\mathbb{Z}_p} |\alpha|^{-s} d^\times \alpha = \sum_{i \geq 0} p^{-is} \int_U d^\times \alpha. \tag{5.2}$$

Because U has Haar measure 1 (**to do:** figure out why), we sum the geometric series to obtain

$$\zeta(f, |\cdot|^s) = \frac{1}{1 - p^{-s}}. \tag{5.3}$$

$\widehat{f} = f$, so the same calculation gives

$$\zeta(\widehat{f}, \widehat{|\cdot|^s}) = \zeta(\widehat{f}, |\cdot|^{1-s}) = \frac{1}{1 - p^{-(1-s)}}. \tag{5.4}$$

The real case: There are in general two equivalence classes of quasicharacters on the reals: Quasicharacters of the form $|\alpha|^s$, which we denote $|\cdot|^s$, and quasicharacters of the form (sign α)$|\alpha|^s$, which we denote by $\pm|\cdot|^s$. For now we will only need to consider the first kind.

For the character $|\cdot|^s$, we take $f(\xi) = e^{-\pi\xi^2}$, which is a nice Schwarz function and its own Fourier transform. We compute:

$$\zeta(f, |\cdot|^s) = \int_{-\infty}^{\infty} e^{-\pi\alpha^2} |\alpha|^s \frac{d\alpha}{|\alpha|} = 2\int_0^\infty e^{-\pi\alpha^2} \alpha^{s-1} d\alpha = \pi^{-s/2}\Gamma(s/2). \tag{5.5}$$

We also have

$$\zeta(\widehat{f}, \widehat{|\cdot|^s}) = \zeta(f, |\cdot|^{1-s}) = \pi^{-\frac{1-s}{2}}\Gamma((1-s)/2), \tag{5.6}$$

and the ρ factor (of Tate's local theorem) is simply the quotient.

Multiplying the local factors together, we obtain the equation

$$\pi^{-s/2}\Gamma(s/2)\prod_p \frac{1}{1-p^{-s}} = \pi^{-(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\prod_p \frac{1}{1-p^{-(1-s)}}. \tag{5.7}$$

The product on the left is defined only for $\Re(s) > 1$, and the product on the right only for $\Re(s) < 0$. However, Tate's main theorem tells us that these products enjoy a meromorphic continuation to the domain of all complex s, and that this meromorphic continuation is given by the functional equation above outside the "critical strip" $\Re(s) \in (0,1)$.

Q.E.D.

6. Dirichlet, Hecke, and Grössen Characters

Tate's thesis concerns the zeta functions constructed from *Hecke characters*, i.e., characters of the idele class group $\mathbb{A}_K^\times/K^\times$. Accordingly, we will make a detour into Neukirch's book [1] and say more about these characters. In particular, we will want to realize the usual Dirichlet characters on $\mathbb{Q}$ as special cases of Hecke characters, and it is not obvious how to do this.

Once we give the construction, we will be able to use Tate's framework to re-define Dirichlet L-functions and prove analytic continuation and the functional equation for them.

We begin with the following definition:

Definition 6.1. *A* **Hecke character** *of K is a character of the idele class group $\mathbb{A}_K^\times/K^\times$, i.e. a continuous homomorphism*

$$\chi : \mathbb{A}_K^\times \longrightarrow S^1 \tag{6.1}$$

so that $\chi(K^\times) = 1$.

We also have the following related definition of a Grössencharakter. For $\mathfrak{m}$ an integral ideal of the number field K, we let $J^{\mathfrak{m}}$ denote the group of all ideals of K which are coprime to $\mathfrak{m}$.

Definition 6.2. *A* **Grössencharakter** *mod $\mathfrak{m}$ is a character $\chi : J^{\mathfrak{m}} \to S^1$ for which there exists a pair of characters*

$$\chi_f : (\mathcal{O}_K/\mathfrak{m})^\times \longrightarrow S^1, \qquad \chi_\infty : R^\times \longrightarrow S^1, \tag{6.2}$$

such that

$$\chi((a)) = \chi_f(a)\chi_\infty(a) \tag{6.3}$$

for every algebraic integer $a \in \mathcal{O}_K$ coprime to $\mathfrak{m}$.

Here R is the **Minkowski space** $R := K \otimes_{\mathbb{Q}} \mathbb{R}$. If $K = \mathbb{Q}$ then $R = \mathbb{R}$.

And finally we have the notion of a Dirichlet character in a number field. We write $P^{\mathfrak{m}}$ for the group of principal fractional ideals (a) such that $a \equiv 1 \pmod{\mathfrak{m}}$ and a is totally positive. The first condition means that $a = b/c$, where b and c are algebraic integers $\equiv 1 \pmod{\mathfrak{m}}$, and the second means that $\tau(a) > 0$ for every real embedding τ of K.

Definition 6.3. *A* **Dirichlet character** *mod $\mathfrak{m}$ is a character*

$$\chi : J^{\mathfrak{m}}/P^{\mathfrak{m}} \longrightarrow S^1 \tag{6.4}$$

of the **ray class group** *$J^{\mathfrak{m}}/P^{\mathfrak{m}}$ modulo $\mathfrak{m}$.*

Often Dirichlet characters and Grössencharaktere are simply called Hecke characters, so that we have given three (!) definitions above for Hecke characters, which are not obviously equivalent.

In this section we will partially explain the relation between these characters, and we refer to [1] for more details. In particular, we will show how one may start with a usual Dirichlet character (i.e. a character of $(\mathbb{Z}/n\mathbb{Z})^\times$) and obtain an equivalent Hecke character (i.e. a character of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times$). This will allow us to prove the functional equation for Dirichlet L-functions, following Tate. (For clarity and simplicity, we will not prove this in full generality.)

We begin by showing that the above notion of Dirichlet character agrees essentially with the usual one when $K = \mathbb{Q}$.

Proposition 6.4. *There is (for $m \neq 2$) an exact sequence*

$$1 \longrightarrow \pm 1 \longrightarrow (\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow J^\mathfrak{m}/P^\mathfrak{m} \longrightarrow 0, \tag{6.5}$$

where the last map is given by $a \longrightarrow (a)$.

Proof. We first see that it is well-defined, as for any integer k, $\frac{a+km}{a} = \frac{a\overline{a}+km\overline{a}}{a\overline{a}}$, where $\overline{a}$ denotes a multiplicative inverse for a modulo m. The numerator and denominator of this second fraction are then both congruent to 1 mod m.

Injectivity follows as an integer a maps to $P^\mathfrak{m}$ if and only if $(a) \equiv 1 \mod (m)$, and the latter condition is true if either a or $-a$ is $\equiv 1 \mod m$. Surjectivity follows as we can represent any class in $J^\mathfrak{m}/P^\mathfrak{m}$ by an integral ideal (b), where b is coprime to m. □

In more generality, we also have the following proposition:

Proposition 6.5. *Every Dirichlet character χ mod $\mathfrak{m}$ is in fact a Grössencharakter mod $\mathfrak{m}$, which satisfies*

$$\chi((a)) = \chi_f(a)\mathbb{N}((a/|a|)^k) \tag{6.6}$$

for some integer k.

Proof. Omitted for now. □

And finally we want to establish (or at least claim) the correspondence between Hecke characters and Grösencharaktere. To do this we need to introduce notation and a definition.

Suppose that our ideal $\mathfrak{m}$ is factored as $\mathfrak{p} = \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}}$. (Naturally, the product is only over finite primes.) The **unit group** of $\mathcal{O}_\mathfrak{p}$ ($\mathfrak{p} \nmid \infty$) is simply $U_\mathfrak{p}^{(0)} = \mathcal{O}_\mathfrak{p}^\times$, and the **higher unit groups** $U_\mathfrak{p}^{(n)}$ are defined by $U_\mathfrak{p}^{(n)} = 1 + \mathfrak{p}^n$.

We associate to the ideal $\mathfrak{m}$ the subgroup

$$I_f^\mathfrak{m} := \prod_{\mathfrak{p} \nmid \infty} U_\mathfrak{p}^{(n_\mathfrak{p})}. \tag{6.7}$$

Definition 6.6. *We call $\mathfrak{m}$ a* **module of definition** *for the Hecke character χ if*

$$\chi(I_f^\mathfrak{m}) = 1. \tag{6.8}$$

By topological (i.e. continuity) considerations, every Hecke character admits a module of definition. Therefore, any Hecke character is in fact a character

$$\chi : \mathbb{A}_K^\times / I_f^\mathfrak{m} K^\times \longrightarrow S^1. \tag{6.9}$$

The ray class group is a quotient of this group; see Proposition 6.12 of [1].

Given a Hecke character χ with module of definition $\mathfrak{m}$, we may construct a Grössencharakter as follows. For every $\mathfrak{p} \nmid \mathfrak{m}$ we choose a fixed prime element $\pi_\mathfrak{p}$ of $K_\mathfrak{p}$, and define a homomorphism

$$c : J^\mathfrak{m} \longrightarrow \mathbb{A}_K^\times / I_f^\mathfrak{m} K^\times \tag{6.10}$$

which maps a prime ideal $\mathfrak{p} \nmid \mathfrak{m}$ to the class of the idele $< \pi_\mathfrak{p} >= (\dots, 1, \pi_\mathfrak{p}, 1, \dots)$.

We then have the following result, which we will not prove. (See [1], pp. 482-483 for a proof.)

Theorem 6.7. *There is a 1-1 correspondence between Hecke characters with modulus $\mathfrak{m}$ and Grössencharaktere mod $\mathfrak{m}$, given by $\chi \to \chi \cdot c$.*

We will work this out in a particular example. We will apply this correspondence to obtain a Hecke character from a Grössencharakter, which is not the direction that the theorem gives an explicit recipe for.

6.1. **Example: from Dirichlet characters to Hecke characters.** Let χ be a Dirichlet character which is primitive to an odd prime modulus ℓ. We can think in even less generality and imagine that χ is the quadratic character $\left(\frac{\cdot}{\ell}\right)$. In this case $\chi(-1) = 1$ if $p \equiv 1 \mod 4$ and $\chi(-1) = -1$ if $p \equiv 3 \mod 4$.

Our first task is to turn χ into a Grössencharakter, and this is easily done. We define our Grössencharakter χ_G by

$$\chi_G((a)) = \chi(a)\chi_\infty(a), \tag{6.11}$$

where the character $\chi_\infty : \mathbb{R} \to \pm 1$ is the sign character if χ is odd, and the trivial character if χ is even. It is clear that this is well-defined, and it is quite immediate from the definition that this is indeed a Grössencharakter.

We will now obtain a Hecke character in the somewhat naive way. We will take the correspondence for granted, and figure out what Hecke character χ_H should correspond to χ_G.

We will write

$$\chi_H = \prod_{p \leq \infty} \chi_{H,p}, \tag{6.12}$$

where $\chi_{H,p}$ is a character of $\mathbb{Q}_p$.

It turns out that we do not have too much choice. χ_H needs to be trivial on $\mathbb{Q}$, and also on every unit group U_p for $p \neq \ell$. (In other words, χ_H needs to be unramified for $p \neq \ell$.) In addition, χ_H needs to be trivial on the "higher unit group" $1 + \ell\mathbb{Z}_\ell$.

We will determine χ_H one place at a time. **Finite primes $p \neq \ell$.** Consider the idele $\mathfrak{a}_p = (\dots, p, \dots)$, where $p \neq \ell$, and the p is in the $\mathbb{Z}_p$ spot. This is $c(p)$. Our correspondence tells us to define $\chi_H(\mathfrak{a}_p) = \chi_G((p)) = \chi(p)$. Moreover, $\chi_{H,p}$ is trivial on p-adic units, and these two conditions define $\chi_{H,p}$.

The ramified prime $p = \ell$. We know that for a positive prime p, $(p, p, p, \dots)$ maps to 1 (being a diagonally embedded element of $\mathbb{Q}$). Certainly $\chi_{H,\infty}(p) = 1$ for any positive prime p, so this tells us that $\chi_{H,p}(p)\chi_{H,\ell}(p) = 1$, or in other words, $\chi_{H,\ell}(p) = \chi^{-1}(p)$. By multiplicativity we obtain $\chi_{H,\ell}(a) = \chi^{-1}(a)$ for all integers coprime to ℓ.

Moreover, $(\ell, \ell, \ell, \dots)$ maps to 1 so $\chi_{H,\ell}(\ell) = 1$.

To determine $\chi_{H,\ell}$ on $\mathbb{Z}_\ell^\times$ we recall that $\chi_{H,\ell}$ needs to be multiplicatively invariant by $1 + \ell\mathbb{Z}_\ell$. Given any $\alpha \in \mathbb{Z}_\ell^\times$, we may uniquely write $\alpha = b + \gamma\ell$, where b is an integer between 1 and $\ell - 1$, and γ is an element of $\mathbb{Z}_\ell$. We then have

$$\chi_{H,\ell}(\alpha) = \chi_{H,\ell}(b + \gamma\ell) = \chi_{H,\ell}(b\Big(1 + \frac{\gamma}{\beta}\ell\Big)) = \chi_{H,\ell}(b). \tag{6.13}$$

We further notice that for any integer k,

$$\chi_{H,\ell}(b) = \chi_{H,\ell}(b\Big(1 + \frac{k}{b}\ell\Big)) = \chi_{H,\ell}(b + k\ell), \tag{6.14}$$

as we expect. Our conclusion is that $\chi_{H,\ell}(\alpha) = \chi^{-1}(b)$, where $\alpha \in b + \mathbb{Z}_\ell$.

The infinite place. This is now easy, as $\chi_{H,\infty}$ is determined by its value on -1. We know that $\chi_{H,\ell}(-1)\chi_{H,\infty}(-1) = -1$, so that

$$\chi_{H,\infty}(-1) = \chi_{H,\ell}^{-1}(-1) = \chi(-1). \tag{6.15}$$

7. The functional equation for Dirichlet L-functions

We will now use the mechanism of Tate's thesis to explicitly prove analytic contination and the functional equation for Dirichlet L-functions $L(s,\chi)$. As before we will assume that χ is a primitive character whose conductor is an odd prime. We will carry out the analysis for $\zeta(s,c)$, where c ranges over the family of quasicharacters containing the Hecke character χ constructed above. Namely, c will be the quasicharacter $\chi|\cdot|^s$, where s is a complex variable. It is perhaps not obvious that we will get a functional equation for $L(s,\chi)$ in this way. However, this is not something that needs to be proved.

As before we have

$$\zeta(f,c) = \prod_p \zeta_p(f_p, c_p), \tag{7.1}$$

and we will compute each local zeta factor separately.

Finite places $p \neq \ell$. Here χ is unramified. Writing an arbitrary element of $\mathbb{Q}_p$ as $\alpha = p^i u$, where u is a p-adic unit, we have $c_p(p^i u) = \chi(p^i)|p^i|_p^s = \chi(p^i)p^{-is}$. As before, we let f_p be the characteristic function of $\mathbb{Z}_p$. Recall that f_p is its own Fourier transform. The local zeta function is

$$\zeta_p(f_p, c_p) = \int_{\mathbb{Z}_p} \chi(p^i)p^{-is} d^\times\alpha = \sum_{i\geq 0} \int_{ord(\alpha)=i} \chi(p^i)p^{-is} d^\times\alpha. \tag{7.2}$$

Using the multiplicative invariance of Haar measure and the fact that $\mu(\mathbb{Z}_p^\times) = 1$ we obtain

$$\zeta_p(f_p, c_p) = \sum_{i\geq 0} \chi(p)^i p^{-is} = \frac{1}{1-\chi(p)p^{-s}}. \tag{7.3}$$

We have $\widehat{c_p} = |\cdot|c_p^{-1}$, so that $\widehat{c_p}(p^i u) = \chi^{-i}(p)p^{(1-i)s}$, and so we obtain

$$\zeta_p(\widehat{f_p}, \widehat{c_p}) = \frac{1}{1-\chi^{-1}(p)p^{-(1-s)}}. \tag{7.4}$$

The infinite place. We have two possibilities: Either $c_\infty(\alpha) = |\alpha|^s$, or $c_\infty(\alpha) = \operatorname{sgn}(\alpha)|\alpha|^s$. In the former case the local zeta function is exactly as we computed for the Riemann zeta function. We recall the calculation here. We took $f(\xi) = e^{-\pi\xi^2}$, which is its own Fourier transform, and obtained

$$\zeta(f, |\cdot|^s) = \int_{-\infty}^{\infty} e^{-\pi\alpha^2}|\alpha|^s \frac{d\alpha}{|\alpha|} = 2\int_0^\infty e^{-\pi\alpha^2}\alpha^{s-1}d\alpha = \pi^{-s/2}\Gamma(s/2). \tag{7.5}$$

We also had

$$\zeta(\widehat{f}, \widehat{|\cdot|^s}) = \zeta(f, |\cdot|^{1-s}) = \pi^{-\frac{1-s}{2}}\Gamma((1-s)/2). \tag{7.6}$$

In case $c_\infty = \operatorname{sgn}\cdot|\alpha|^s$ the calculation is a little bit different. In this case we choose $f(\xi) = \xi e^{-\pi\xi^2}$, for reasons that will shortly become apparent. We then compute that $\widehat{f}(\xi) = if(\xi)$.

We compute

$$\zeta(f, \operatorname{sgn}|\cdot|^s) = \int_{-\infty}^{\infty} \alpha e^{-\pi\alpha^2}\operatorname{sgn}(\alpha)|\alpha|^s\frac{d\alpha}{|\alpha|} = 2\int_0^\infty e^{-\pi\alpha^2}\alpha^s d\alpha = \pi^{-(s+1)/2}\Gamma((s+1)/2). \tag{7.7}$$

Similarly,

$$\zeta(\widehat{f}, \widehat{\pm|\cdot|^s}) = \zeta(if, \operatorname{sgn}|\cdot|^{1-s}) = i\pi^{-\frac{(1-s)+1}{2}}\Gamma\Big(\frac{(1-s)+1}{2}\Big). \tag{7.8}$$

We therefore see that the form of the functional equation will break up into two cases according to whether χ is even or odd.

The ramified place $p = \ell$. This time our character χ_p is not trivial on $\mathbb{Z}_p^\times$, but it is trivial on $1 + \mathbb{Z}_p$.

We choose our function $f(\xi)$ to be $e^{2\pi i\xi}$ for $\xi \in (p)^{-1}$, and $f(\xi) = 0$ otherwise. We then have

$$\zeta_p(f_p, c_p) = \int_{\frac{1}{p}\mathbb{Z}_p} e^{2\pi i\Lambda(\alpha)}\chi(\alpha)|\alpha|_p^s d^\times\alpha = \sum_{k\geq -1}\int_{\operatorname{ord}(\alpha)=k} p^{-ks}e^{2\pi i\Lambda(\alpha)}\chi(\alpha)d^\times\alpha. \tag{7.9}$$

For $k \neq -1$, the exponential is 1 and so the integral is

$$\int_{\operatorname{ord}(\alpha)=i}\chi(\alpha)d^\times\alpha. \tag{7.10}$$

As χ is nontrivial on units, the integral is zero!

We are left with

$$\zeta_p(f_p, c_p) = p^s\int_{\frac{1}{p}\mathbb{Z}_p^\times} e^{2\pi i\Lambda(\alpha)}\chi(\alpha)d^\times\alpha. \tag{7.11}$$

We may write

$$\frac{1}{p}\mathbb{Z}_p{}^\times = \bigcup_{j=1}^{p-1}\Big(\frac{j}{p} + \mathbb{Z}_p\Big) = \frac{1}{p}j\Big(1 + p\mathbb{Z}_p\Big), \tag{7.12}$$

so that it is apparent that the integrand only depends on j. In particular, $\Lambda(\alpha) = \Lambda(j)$ and $\chi(\alpha) = \chi(j)$. We this obtain

$$\zeta_p(f_p, c_p) = \frac{p^s}{p-1}\sum_{j=1}^{p-1} e^{2\pi ij/p}\chi(j). \tag{7.13}$$

This is far as we can simplify. The sum over j is a **Gauss sum** and is familiar in analytic number theory. For example, it is known that its absolute value is $\sqrt{p}$. For our purposes, we are happy to say that it is what it is.

We now need to handle the Fourier transform. The Fourier transform of f is

$$\widehat{f}(\xi) = \int_{(p)^{-1}} e^{2\pi i\Lambda(\eta)}e^{-2\pi i\Lambda(\xi\eta)}d\eta = \int_{(p)^{-1}} e^{2\pi i\Lambda(\eta(\xi-1))}. \tag{7.14}$$

This is p if $\xi \equiv 1 \mod p$ and 0 otherwise. In other words the value of the integral depends on whether the exponential is trivial or not.

On this set, our quasicharacter $\widehat{c}(\alpha)|\alpha|^{1-s}$ is trivial! We therefore have

$$\zeta(\widehat{f}, \widehat{c}) = p\int_{1+(p)} d^\times\alpha = \frac{p}{p-1}. \tag{7.15}$$

Multiplying the zeta factors, we obtain equations as follows. If χ is even, the zeta function is given by

$$\zeta(f, c) = \frac{p^s}{p-1}\pi^{-s/2}\Gamma(s/2)\Big(\sum_{j=1}^{p-1} e^{2\pi ij/p}\chi(j)\Big)\prod_{p\neq\ell}\frac{1}{1-\chi(p)p^{-s}}. \tag{7.16}$$

We furthermore have

$$\zeta(\widehat{f}, \widehat{c}) = \frac{p}{p-1} \pi^{-(1-s)/2} \Gamma((1-s)/2) \prod_{p \neq \ell} \frac{1}{1 - \chi^{-1}(p) p^{-s}}. \tag{7.17}$$

Denoting $L(s, \chi)$ as usual, we have proved the functional equation

$$p^{s-1} \pi^{-s/2} \Gamma(s/2) \left(\sum_{j=1}^{p-1} e^{2\pi i j/p} \chi(j) \right) L(s, \chi) = \pi^{-(1-s)/2} \Gamma((1-s)/2) L(1-s, \chi^{-1}). \tag{7.18}$$

If χ is odd, then the factor for the real place is different, as we have seen. Multiplying everything together, we obtain for this case

$$p^{s-1} \pi^{-(s+1)/2} \Gamma(\frac{s+1}{2}) \left(\sum_{j=1}^{p-1} e^{2\pi i j/p} \chi(j) \right) L(s, \chi) = i \pi^{\frac{-(1-s)+1}{2}} \Gamma(\frac{(1-s)+1}{2}) L(1-s, \chi^{-1}). \tag{7.19}$$

References

[1] J. Neukirch, *Algebraic number theory.*

[2] D. Ramakrishnan and R. J. Valenza, *Fourier analysis on number fields.*

[3] J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions,* doctoral thesis, May 1950. Google "Cassels Frohlich", download the ensuing really big .pdf, it's on page 305.

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