Problem 1

Without loss of generality we will only consider simple graphs. A graph with multiple edges between vertices can be colored the same as a simple graph without duplicate edges.

First we will show that any simple planar graph has a vertex of degree 5 or less. We start with Euler's formula $e = v + f - 2$. Then we notice that each face is bounded by at least 3 edges, and no edge is part of the boundary of more than two faces. This gives $3f \leq 2e$. We can use this to eliminate f from Euler's formula, giving $e \leq 3v - 6$. Now we recall that the sum of the degrees of the vertices is twice the number of edges. So we have

$$\Sigma_i \deg(v_i) \leq 6v - 6$$

If each vertex had degree 6 or more, then the degree sum would be at least $6v$. Therefore there is a vertex of degree 5 or less.

Now we proceed by induction of the number of vertices. Any graph with 5 or fewer vertices can trivially be 5-colored. If we have a larger graph G, pick a vertex v_0 of degree 5 or less. By our inductive hypothesis we can 5-color $G - v_0$. We attempt to extend this coloring to a coloring of G. If there are only 4 (or fewer) colors used among the neighbors of v_0, then we can color v_0 one of the remaining colors.

In the other case, each of v_0's neighbors has a different color. Let u_A, u_B, u_C, and u_D be neighbors of v_0 with colors A, B, C, and D respectively. Without loss of generality, assume that u_C is between u_A and u_B. Now consider the subgraph of $G - v_0$ of all A and B vertices. In any connected component of this subgraph, we can reverse colors A and B to obtain a new coloring. If u_A and u_B are in different connected components of the subgraph, then we can reverse the colors in one of them. Then there will be only 4 colors among neighbors of v_0 and we can color v_0 the fifth color. If u_A and u_B are in the same connected component, then there is a path between them made of of A and B vertices. So, we now consider of subgraph of C and D vertices. As before, we can reverse C and D in any connected component of this subgraph, and if u_C and u_D are in different connected components, then we are done. But they must be in different connected components because u_A, v_0, u_B together with the A and B path between u_A and u_B isolates u_C from u_D. (No path in the C and D subgraph intersects it because none of its vertices are C or D.) Therefore we can 5-color G.

Problem 2

As in Problem 1 we see that any simple planar graph has a vertex of degree 5 or less.

Again, we proceed by induction on the number of vertices. Any graph with 6 or fewer vertices can trivially be 6-colored. If we have a larger graph G, pick a vertex v_0 of degree 5 or less. By our inductive hypothesis we can 6-color $G - v_0$.

We attempt to extend this coloring to a coloring of G. v_0 only has 5 neighbors and we have 6 colors, so we can always choose a valid color for v_0.

Therefore, any planar graph can be 6-colored.

Problem 3

Let G be a graph with 10 vertices and 26 edges. By Turán's theorem, there is at least one triangle in G. Let V_1 be set vertex set of this triangle, and let V_2 be the set containing the other 7 vertices of G.

Consider edges with one endpoint in V_1 and one endpoint V_2. For each pair of edges that share the V_2 endpoint, there is a triangle containing those edges (plus an edge of V_1). Only 7 such edges can exist without any such intersecting pair. Let m_1 be the number of edges between V_1 and V_2. Then we claim that there are at least $m_1 - 7$ triangles that have vertices in both V_1 and V_2 regardless of the total number of edges in the graph. This is obvious for $m_1 \leq 7$. If this is true for graphs that only have $m_1 - 1$ such edges, and if $m_1 > 7$, then we have one such triangle. If we remove an edge from it (not the edge in V_1) we find ourselves in the $m_1 - 1$ case with at least $m_1 - 8$ triangles, so we had at least $m_1 - 7$ triangles to begin with.

Now consider edges that are completely contained in V_2. Let m_2 be the number of such edges. By Turán's theorem, at most 12 such edges can exist before we have a triangle that is completely contained in V_2. We claim that there are at least $m_2 - 12$ such triangles regardless of the total number of edges in the graph. This is obvious for $m_2 - 12$. For $m_2 > 12$ we proceed as above. There exists such a triangle, so we can remove an edge from it and consider the case for a smaller m_2.

Therefore, we have that there are at least $1 + (m_1 - 7) + (m_2 - 12) = m_1 + m_2 - 18$ triangles in the graph. But there are 26 total edges, so $m_1 + m_2 = 23$. Therefore, the G has at least 5 triangles.

We can achieve equality by adding any single edge to $K_{5,5}$.

Problem 4

We will proceed by induction on n. For $n = 0$, there is only one possible graph with 0 vertices, and $K_{0,0}$ is the same as that graph.

If any graph with $n-1$ vertices and $\lfloor \frac{(n-1)^2}{4} \rfloor$ edges is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$, then consider a graph G with n vertices and $\lfloor \frac{n^2}{4} \rfloor$ edges. Although the argument is the almost exactly same in both cases, it is notationally convenient to consider separately the cases when n is odd and when n is even.

Consider the case when n is odd, $n = 2k+1$. Then we have $\frac{n^2-1}{4}$ edges. Thus, the sum of the degrees of the vertices is $\frac{n^2-1}{2}$. Therefore there must be at least one vertex v with degree no more than $\frac{n-1}{2}$. Consider $G - v$. It has no triangles, so by Turán's theorem, it has no more than $\frac{(n-1)^2}{4}$ edges. However, no more than $\frac{n-1}{2}$ edges were removed, so $G - v$ also has at least $\frac{n^2-1}{4} - \frac{n-1}{2} = \frac{(n-1)^2}{4}$ edges. Thus $G - v$ has exactly $\frac{(n-1)^2}{4}$ edges, and we can apply the inductive hypothesis to see that $G - v$ is the same as $K_{k,k}$. Furthermore v has degree exactly $\frac{n-1}{2} = k$. If v was adjacent to a vertex in each component of this $K_{k,k}$ then we would have a triangle, therefore each of v's k edges connects to the same component. Thus G is $K_{k+1,k}$.

Now consider the case when n is even, $n = 2k$. Then we have $\frac{n^2}{4}$ edges. Thus, the sum of the degrees of the vertices is $\frac{n^2}{2}$. Therefore there must be at least one vertex v with degree no more than $\frac{n}{2}$. Consider $G - v$. It has no triangles, so by Turán's theorem, it has no more than $\frac{(n-1)^2+1}{4} = \frac{n^2-2n}{4}$ edges. However, no more than $\frac{n}{2}$ edges were removed, so $G - v$ also has at least $\frac{n^2-2n}{4}$ edges. Therefore $G - v$ has exactly $\frac{n^2-2n}{4}$ edges and v has degree exactly $\frac{n}{2}$. By the inductive hypothesis, $G - v$ is the same as $K_{k,k-1}$. Because there are no triangles, v can only be adjacent to vertices in one component of $K_{k,k-1}$. v therefore connects to each vertex in the component of size k, so G is $K_{k,k}$.

Problem 5

We consider the 27 vertex graph G corresponding to this problem. That is, if each vertex is a point (x, y, z) with $x, y, z \in \{0, 1, 2\}$, then there is an edge between two points whenever the distance between them is exactly 1.

We wish to consider all paths P that visit each vertex exactly once. Each such path will be a subgraph of G with 26 edges. Consider two subsets of the vertices of G: $V_0 = \{(x, y, z) : x, y, z \in \{0, 2\}\}$ and $V_2 = \{(x, y, z) :$ exactly two of x, y, z are $1\}$. Observe that there are no edges that include only vertices in V_0 and V_2. Also notice that V_0 has 8 vertices and V_2 has 6 vertices. For every vertex in G, unless that vertex is an endpoint of P, there will be exactly 2 edges of P adjacent to it. If it is an endpoint then there will be only 1 edge. If we sum over all vertices in V_0 and V_2, there will be 2 edges for each vertex in V_0 and each vertex in V_2, all of which are distinct (exluding endpoints). So, let z be the number of endpoints of P among V_0 and V_2. Then we see that $26 \geq 2(6 + 8) - z = 28 - z$. Therefore $z = 2$.

In other words, any path that visits each vertex of G exactly once has both of its enpoints among V_0 and V_2. In particular, $(1, 1, 1)$ is never the endpoint of such a path.

Problem 6

Consider weighting exactly as before. That is, for a graph G, assign a non-negative weight to each vertex such that the sum of the weights is one. Then define the weight of an edge to be the product of the weights of its vertices. We try to maximize the sum of the weights of the edges.

As in the book, we briefly argue that the sum of the edge weights are maximized when only a complete subgraph of vertices has non-zero weight. Taken individually, each vertex's contribution to edge weight is equal to its weight times the sum of its neighbors' weights. So, given two vertices with no edge between them, it is advantageous to move all the weight to the vertex whose neighbor weigth is higher. Thus, there is a distribution that maximizes weight in which every vertex with positive weight is connected to every other.

Now we consider the maximum edge weight of a complete graph K_n. The best way to distribute weight among vertices is evenly. It is easy to show this for $n = 2$ simply by maximizing $x(1-x)$. For $n \geq 3$ we will proceed by induction with a slight generalization: that the best distribution is even for an arbitrary sum of weights t. Thus the maximum is $\frac{(n(n-1)}{2}(\frac{t}{n})^2 = t^2\frac{n-1}{2n}$. (This is again easy to verify in the 2 case with $x(t-x)$.) Pick a vertex v and a weight $w \leq t$. We will maximize the edge weight among all possibilites where v has weight w. As before, we note that the sum of the contribution of those edges that include v only depends on the weight of v and the sum of the weight of v's neighbors. But both of these are fixed at w and $t-w$ respectively. Therefore it is optimal to maximize the weight among all of v's neighbors. Thus we have edge weight $w(t-w)+(t-w)^2\frac{n-2}{2n-2}$. We can verify that this is maximized at $w = \frac{t}{n}$, so we are done. We know that K_n has its edge weight maximzed at $\frac{n-1}{2n}$. Note that a larger n gives a larger maximum.

Therefore, any graph G on n vertices and m edges without a K_p has its maximum edge weight bounded by the maximum for K_{p-1} which is $\frac{p-2}{2p-2}$. In particular, the even weighting over all vertices gives $\frac{m}{n^2}$ so $\frac{m}{n^2} \leq \frac{p-2}{2p-2}$. If we solve for m, we get

$$m \leq \frac{p-2}{2p-2}n^2$$

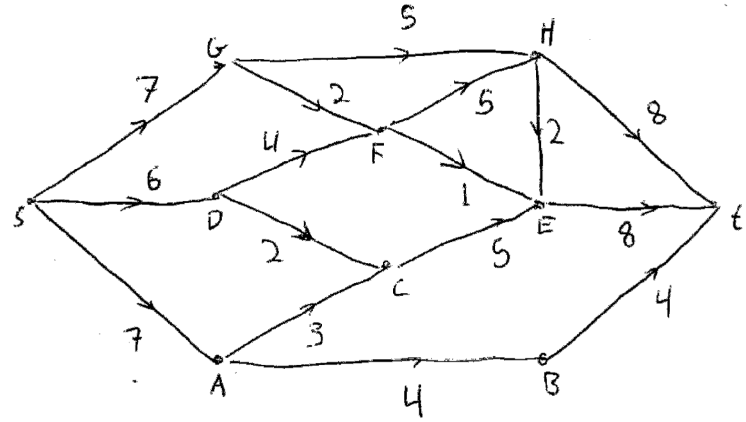
Problem 7

On the next page, we exhibit a flow of 20 along with a cut that has flow 20 across it, proving that 20 is indeed the maximum.

Problem 7

Maximum flow:

(edges not shown have 0 flow)



Minimum cut: