FOURIER COEFFICIENTS OF MODULAR FORMS

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Abstract. These notes describe some conjectures and results related to the distribution of Fourier coefficients of modular forms.

This is a rough draft and these notes should forever be considered incomplete.

1. Introduction and Conjectures

Start with a modular form (probably a cusp form) $f(z) = \sum_{n \geq 1} a_n q^n$. There are a variety of natural questions one could ask about the Fourier coefficients a_n. How are they distributed, and how fast do they grow? In these notes we will describe some of the questions and some of the partial answers. No attempt will be made at being complete.

In general it is nice to assume that $f(z)$ is a Hecke eigenform (all spaces of modular forms have a basis consisting of such), normalized so that $a_1 = 1$. In this case the coefficients are multiplicative, and if we write

$$L(f, s) := \sum_{n \geq 1} a_n n^{-s},$$

then $L(f, s)$ "really is an L-function", meaning that it has analytic continuation, functional equation, is expected to satisfy the Riemann Hypothesis, and obeys lots of other nice properties which you can read about in Iwaniec and Kowalski or elsewhere.

We can play nice games with the L-function. We have the following

Theorem 1.1. *$L(f, s)$ has an Euler product*

$$L(f, s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{(k-1)-2s}} \cdot \prod_p [\cdots],$$

where the latter product is over ramified primes. Throughout, we will be sloppy and essentially ignore the ramified primes, since are only trying to give an overview of the relevant results. (Also, my source material in Iwaniec and Kowalski is similarly sloppy. See page 374!)

I forget who proved this or how, but suffice it to say these facts are highly nontrivial. The nice way to write this (IMHO) is

$$L(f, s) = \prod_p \frac{1}{(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})},$$

where

$$\alpha_p \beta_p = p^{k-1}, \quad \alpha_p + \beta_p = a_p.$$

Furthermore, one has $|\alpha_p| = |\beta_p| = p^{(k-1)/2}$, so that they are in fact complex conjugates. One observes that a_p then determines α_p and hence the a_{p^n} for all prime powers n.

Remark. It is common to normalize the coefficients so that they have absolute value 1. If you do that, then the functional equation will relate s and $1 - s$.

2000 *Mathematics Subject Classification.* 11N25, 11N36.

So then it is really enough to ask questions about the distribution of the a_p. For example, we have the following "prime number theorem":

Theorem 1.2.
$$\sum_{p\leq x}(a_p/p^{(k-1)/2})\log p = -\frac{x^\beta}{\beta} + O(\sqrt{q}x\exp(-c\sqrt{\log x}).$$

Here q is the level, and β is some annoying Siegel zero, if it exists. If you assume GRH then you get a much better result. The point is that once you've written down the L-function, and know that.... well... it's an L-function, then you can just copy the proof of the prime number theorem. If you want, you can even throw in test functions for fun. See Iwaniec and Kowalski for more.

Now let us assume that the a_p are **integers**, which is often the case. For example this will be true if $f(z)$ corresponds to some elliptic curve defined over $\mathbb{Q}$. In other cases, the a_p will at least often be algebraic integers in some fixed number field.

So they are nicely distributed in some sense. In fact, we have the following strong equidistribution conjecture:

Conjecture 1 (Sato-Tate)**.** *Let f be a primitive holomorphic cusp form of weight ≥ 2 which is not of dihedral type (no, I don't know what that means). Define (for each p) a quantity $\theta_p \in [0,\pi]$ by*
$$a_p = 2p^{(k-1)/2}\cos\theta_p.$$
(This is equivalent to writing $\alpha_p = p^{(k-1)/2}e^{i\theta_p}$.) Then, the angles θ_p are equidistributed according to the Sato-Tate measure
$$d\mu_{ST} := \frac{2}{\pi}\sin^2\theta d\theta.$$

What this means is that we have
$$\frac{1}{\mathrm{li}x}\sum_{p\leq x} f(\theta_p) \to \int_0^\pi \frac{2}{\pi}\sin^2\theta d\theta,$$
for any continuous function $f : [0,\pi] \to \mathbb{C}$.

Remark. Although I got this out of Iwaniec and Kowalski (Theorem 21.7), it is **wrong** there. Oops.

Poof. (Sort of like a proof, except that like a magician we are going to pull a rabbit out of a hat.)

For simplicity, following IK, we'll assume that f is the Ramanujan delta-function, whose conductor is 1 (i.e. it is defined on $\mathrm{SL}_2(\mathbb{Z})$), so that there are no ramification issues to deal with (or even to apologize for ignoring).

Recall the definition of $L(f,s)$ before. For convenience, and so I can be sloppy, let's normalize away the powers of p so that $|\alpha_p| = 1$. Then the n**th symmetric power** of $L(f,s)$ is
$$L(Sym^n f, s) := \prod_p \prod_{0\leq j\leq n}(1-\alpha_p^j\beta_p^{n-j}p^{-s})^{-1}.$$
Now the coefficients of this L-function are
$$e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{i(-n)\theta} = \frac{e^{in\theta} - e^{-i(n+2)\theta}}{1-e^{-2i\theta}} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta}-e^{-i\theta}} = \frac{\sin((n+1)\theta_p)}{\sin\theta_p}.$$
This is a polynomial of degree n in $\cos\theta_p$, called the Chebyshev polynomial $P_n(\cos\theta_p)$. We could easily derive recurrence relations for them, etc.

Hmm. Did I just call $L(Sym^n f, s)$ an L-function? It is certainly believed to be, and this is **by no means** obvious. In particular, we have written down some simple combinatorial gobbledygook, and there is no obvious reason it should have an analytic continuation, functional equation, etc. This is true (conjecturally) because the symmetric powers should themselves come from **automorphic representations**. This is unproved in most cases, and quickly gets you into very deep theoretical waters.

So assume this, so we will obtain a conditional proof. (Note that this has been proved for some special cases by Richard Taylor and his collaborators.) The "prime number theorem" for the symmetric powers tells us that

$$\sum_{p \leq x} P_n(\cos\theta_p) = o(\pi(x)),$$

and so

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \leq x} P_n(\cos\theta_p) = 0 = \int_0^\pi P_n(\cos t) d\mu_{ST}(t).$$

Now the latter equality, combined with the crucial (and easy) fact that the Chebyshev polynomials $P_n(\cos t)$ span $C([0,\pi])$, imply Sato-Tate. (You need the Weyl Criterion to make this precise.) □

1.1. **The Atkin-Serre Conjecture.**

Another conjecture is due to Atkin and Serre; it predicts that the coefficients of a newform (without CM) should not be too small. In particular, it predicts that for sufficiently large primes p, we have

$$|a_p| \gg p^{(k-3+\epsilon)/2}.$$

This relates to the Sato-Tate conjecture, but Sato-Tate does not necessarily predict that **no** primes should have small a_p.

This conjecture appears to be quite difficult to prove. For example, we would like to prove Lehmer's conjecture, which states that the coefficients of the Ramanujan Δ-function are never 0, but we don't know how to prove this. (See, however, interesting work of Ono and others on this.)

1.2. **The Lang-Trotter Conjecture.**

Now fix some value of a. What should the asymptotics of the counting function

$$\{p \leq x : a_p = a\}$$

be?

Lang and Trotter made a variety of precise conjectures related to this in a rather lengthy monograph. They assumed that the Fourier coefficients are more or less "randomly" distributed, both in the Sato-Tate sense and l-adically. They predicted, then, that for weight 2 that

$$\{p \leq x : a_p = a\} \sim C_{p,f} \frac{x^{1/2}}{\log x},$$

that for weight 3

$$\{p \leq x : a_p = a\} \sim C_{p,f} \log\log x,$$

and that for weight 4 this counting function should be finite for each a.

The determination of the constants $C_{p,f}$ is rather complicated and involves considerations from class field theory. We do note that (in principle) the Lang-Trotter conjecture should imply Sato-Tate, provided that we understand how to sum the $C_{p,f}$.

2. Known results: Lang-Trotter, etc.

2.1. **The l-adic Galois representation.** Suppose that $E/\mathbb{Q}$ is an elliptic curve, without complex multiplication. Then it is known that $E[l]$, the group of l-division points of E (over the algebraic closure), is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^2$. There is a natural action of $G_{\mathbb{Q}}$, the absolute Galois group of Q, on these points and we obtain the l-adic Galois representation associated to E:

$$\rho_l : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z}).$$

For all but finitely many l, the representation is surjective. (Note: by considering the l-power division points we can obtain a representation on $\mathrm{GL}_2(Z_l)$.) Anyway, by work of Deligne and Serre, we know that for $p \neq l$, the trace of Frobenius at p is equal to a_p mod l, and the determinant is p. (Notice the parallelism with the definition of the L-function above.)

This allows us to prove partial results towards the Lang-Trotter conjecture. If we are only trying to prove bounds on the number of primes p for which $a_p \neq a$, then the only way that occurs to me is rather stupid: pick some integer m, and the proportion of primes which are congruent to a mod ℓ is

$$\frac{\pi(x)}{m} + E(x,m),$$

where $E(x,m)$ is the error term in the Chebotarev density theorem. The first fraction should not really be $1/m$ (as this is not the exact proportion of matrices in $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ with trace equal to a), but it's close enough. Anyway, pick m of a size so that the above two terms are essentially equal, and we're done.

2.2. **Frobenius fields, Cojocaru-Fouvry-Murty, and the square sieve.** Cojocaru, Fouvry, and Murty (and before them, Serre) considered an interesting extension of this problem. Let π_p be a root of the polynomial $x^2 - a_p x + p$, and let K_p be the imaginary quadratic field generated by π_p. We will call K_p the Frobenius field at p. Then we may ask for lower bounds on the number of p for which K_p is some fixed field K.

Cojocaru, Fouvry, and Murty (CFM) proved a variety of different bounds depending on what unproven hypotheses they assumed. (Their work includes an unconditional bound.) For simplicity we will state only one version, which requires GRH but not AC. They proved the upper bound $x^{17/18}\log x$.

To prove this, they observe that if $K_p = \mathbb{Q}(\sqrt{-D})$ then

$$4p - a_p^2 = Dm^2,$$

for some m. Thus, this question is equivalent to asking for an upper bound on the number of squares in the (multi)set

$$\mathcal{A} := \{D(4p - a_p^2) \ : \ p \leq x\}.$$

To do this, they use Heath-Brown's *square sieve*. It says that if $\mathcal{A}$ is some set, and $\mathcal{P}$ is a set of odd primes, and $S(\mathcal{A})$ is the number of squares in $\mathcal{A}$, then

$$S(\mathcal{A}) \leq \frac{\#\mathcal{A}}{\#\mathcal{P}} + \max_{l,q\in\mathcal{P}} \left| \sum_{a\in\mathcal{A}} \left(\frac{a}{lq}\right) \right| + \frac{2}{\#\mathcal{P}} \sum_{a\in\mathcal{A}} \sum_{l\in\mathcal{P},\ (a,l)\neq 1} 1 + \frac{1}{(\#\mathcal{P})^2} \sum_{a\in\mathcal{A}} \Bigg(\sum_{l\in\mathcal{P},\ (a,l)\neq 1} 1 \Bigg)^2.$$

To prove this, write

$$S(\mathcal{A}) \leq \sum_{a\in\mathcal{A}} \frac{1}{(\#\mathcal{P})^2} \Bigg(\sum_{l\in\mathcal{P}} \left(\frac{a}{l}\right) + \sum_{l\in\mathcal{P},(a,l)\neq 1} 1 \Bigg)^2$$

and FOIL.

The question, then, is to bound everything occuring above. We need to look at sums $\left(\frac{4p-a_p^2}{lq}\right)$ for primes $p \leq x$, and this depends only on the residue classes of p and a_p modulo lq. This, in turn, is determined by the Frobenius at p in the extension $\mathbb{Q}(E[lq])/\mathbb{Q} \simeq \mathrm{GL}_2(\mathbb{Z}/lq\mathbb{Z})$. (This isomorphism is deep, and was proved by Serre as long as l and q are sufficiently large.) We then use the Chebotarev density theorem, count matrices in GL_2, and clean up the mess – and we get a bound.

2.3. **Improvements on CFM.** Various improvements were made on the work above, which we will describe here. Although there are unconditional results, for simplicity we will only state the GRH ones.

One improvement was obtained by yours truly. I improved the above GRH result to $x^{9/10}\log x$. The proof improved the “clean up the mess” part. But it was more interesting than it sounds. Essentially the idea is to hold off on the use of the Chebotarev density theorem, and combine the character sums occuring in CFM with the explicit expressions you get for the number of Frobenius primes in the proof of CDT. We get to think about the representation theory of GL_2, which is kind of fun.

An improvement of $x^{4/5}/(\log x)^{1/5}$ was obtained by Cojocaru and David, by using a “mixed representation” approach, using a representation associated both to E and to K. I confess I didn’t understand the motivation all that well, so I won’t try to explain it. But see the end of CFM.

Further improvements were obtained by Zywina. Numerically he obtained the above result, but he analyzed the dependence on the class number h_K and in fact got a bound $x^{4/5}/h_K^{3/5}(\log x)^{1/5}$. In contrast, Cojocaru and David did not state the dependence on h_K, but (Zywina claims) their results give $x^{4/5}h_K^{3/5}/(\log x)^{1/5}$. In particular, this allows him to prove that at least $x^{2/7}(logx)^{-2}$ fields occur.

All of these results were proved to mine, so my paper is going to the dead letter office. But I did discover on the way that I could modestly improve the unconditional CDT, which was kind of interesting.

2.4. **Modular forms of higher weight.** One interesting question that occurs to me is approaching the Lang-Trotter conjecture for higher weight. In particular, fix some a and a modular form $f(z)$; then one ought to be able to derive good bounds on the number of p for which $a(p) = a$. But nothing occurs to me that improves on the results described here. (And I don’t feel optimistic that I could combine them.)

There is an interesting related result of Murty and Murty. (As I write this, I do not have access to their paper, and am writing from memory and from the MathSciNet review.) Suppose that $f(z)$ has $a(p) = 0$ for all sufficiently large primes p. Then, for any odd a, there are only finitely many n for which $a(n) = a$.

This looks like an impressive result at first – but then one realizes that the conclusion is almost true by assumption! However, the hypothesis is known to be true for many large classes of modular forms. For example, it is true for any Hecke eigenform on $\mathrm{SL}_2(\mathbb{Z})$.

The nontrivial portion of the argument is to go from $a(p)$ to $a(p^n)$; after that one can use multiplicativity. And this is indeed nontrivial. One writes $a(p) = \alpha_p + \beta_p$ as usual, and uses interesting results from the theory of Diophantine approximation, etc.

3. Rouse on Atkin-Serre

Jeremy Rouse has improved some interesting results regarding the Atkin-Serre conjecture. Let $H(z)$ be a newform of weight $k \geq 4$ on $\Gamma_0(N)$ without CM.

Theorem 3.1. *Assume that the symmetric power L-functions of H are automorphic and satisfy GRH. Then, for* $0 \leq \alpha \leq 1/8$*,*

$$\#\{p \leq x : |a(p)| \leq p^{(k-1)/2-\alpha} \sim \frac{x^{1-\alpha}}{\log x}.$$

Moreover, the range of α could be extended to $0 \leq \alpha < 1/4$, and to $1/4$ with a square root of log in the denominator, if stronger bounds on the conductors of symmetric power L-functions were known.

We will now sketch the proof. Write

$$a(p) = p^{(k-1)/2}(e^{i\theta} + e^{-i\theta}).$$

Thus, we are asking how often θ is very close to $\pi/2$. Consider the sum

$$\sum_{x \leq p \leq 2x, \quad |\theta_v - \pi/2| < z} \log p.$$

We would like to estimate it accurately, or bound it from above, for z as small as possible.

First, we smooth the sum and instead study

$$\sum_p f_z(\theta_p) g_x(p) \log p,$$

where f and g are nice test functions approximating the above. We write f_z as an absolutely convergent Fourier series, and rewrite the above as

$$\sum_{m=0}^{\infty} a_m(z) \sum_p \cos(m\theta_p) g_x(p) \log p.$$

The next step is to bound the Fourier coefficients of f; Rouse obtains that for nonnegative integers α and β, and $0 < z \leq 1/10$,

$$\sum_{n \geq 0} |a_n(z)| n^{\alpha} \log^{\beta} n = O\left(\frac{1}{z^{\alpha}} \log^{\beta}(1/z) \right).$$

Now, define numbers $\Lambda_m(n)$ by

$$-\frac{L'}{L}(Sym^m(H), s) = \sum_{n \geq 1} \Lambda_m(n) n^{-s},$$

so that for n not a power of any of the ramified primes,

$$\Lambda_m(n) = \sum_{p^k = n} \sum_{i=0}^{m} \alpha_p^{ki} \beta_p^{k(m-i)} \log(p).$$

If p is unramified, and $\alpha_p = e^{i\theta_p}$, then we have

$$\Lambda_m(p) - \Lambda_{m-2}(p) = 2\cos(m\theta_p),$$

where the $m-2$ term is omitted for $m = 0, 1$. In other words, the cosines appearing in the Fourier analysis above can be nicely reinterpreted in terms of the symmetric power L-functions.

In particular, we have

$$\sum_{n \geq 1} \Lambda_m(n) g_x(n) = \sum_p \Big(\cos(m\theta_p) - \cos((m-2)\theta_p) \Big) g_x(p) \log p + O(m\sqrt{x}),$$

where the error term comes from the ramified primes.

Next, we have the "explicit formula"

$$\sum_{n\geq 1}\Lambda_m(n)g_x(n)=\delta_{m,0}G_x(1)-\sum_{\rho}G_x(\rho),$$

where $G_x(s)=\int_0^\infty g_x(y)y^{s-1}dy$ is the Mellin transform of g, and the sum is over the zeroes of $L(Sym^m(H),s)$. (Notice that we are definitely assuming the automorphy of the symmetric power L-functions here.) The proof is by contour integration, counting residues, estimating error terms, etc. as is familiar if you've hacked your way through Chapter 5 of Iwaniec and Kowalski.

Putting all this together, the sum to be understood is

$$\sum_{m\geq 0}a_m(z)\sum_{n}\Big(\Lambda_m(n)-\Lambda_{m-2}(n)\Big)g_x(n)+O(m\sqrt{x}),$$

and plugging in the explicit formula yields

$$\sum_{m\geq 0}a_m(z)\left(\delta_{m,0}G_x(1)-\delta_{m,2}G_x(1)-\sum_{rho}G_x(\rho)+\sum_{\rho'}G_x(\rho')\right),$$

where the first sum is over zeroes of $\mathrm{Sym}^m(H)$, and the second sum is over zeroes of $\mathrm{Sym}^{m-2}(H)$. These two sums contribute (assuming GRH!) $O(\sqrt{x}z^{-3})$; we remark that the odd shape of the bound for Fourier coefficients given previously was motivated by the details of estimating this sum.

Anyway, when the dust clears, the sum under consideration is

$$O(xz+\sqrt{x}z^{-3}),$$

and choosing $z=\frac{1}{2}x^{-\alpha}$ then one obtains the result.

References

[1] J. Lagarias and A. Odlyzko, *Effective versions of the Chebotarev density theorem*, some obscure conference proceedings, unfortunately not available online.

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