3. Given any two real numbers, there is a real number in between.
   a. Given any two real numbers a and b, there is a real number c such that c is ____.
   b. For any two ____, ____ such that $a < c < b$.
4. Given any real number, there is a real number that is greater.
   a. Given any real number r, there is ____ s such that s is ____.
   b. For any ____, ____ such that $s > r$.
5. The reciprocal of any positive real number is positive.
   a. Given any positive real number r, the reciprocal of ____.
   b. For any real number r, if r is ____, then ____.
   c. If a real number r ____, then ____.
6. The cube root of any negative real number is negative.
   a. Given any negative real number s, the cube root of ____.
   b. For any real number s, if s is ____, then ____.
   c. If a real number s ____, then ____.
7. Rewrite the following statements less formally, without using variables. Determine, as best as you can, whether the statements are true or false.
   a. There are real numbers u and v with the property that $u + v < u - v$.
   b. There is a real number x such that $x^2 < x$.
   c. For all positive integers n, $n^2 \geq n$.
   d. For all real numbers a and b, $|a + b| \leq |a| + |b|$.

In each of 8–13, fill in the blanks to rewrite the given statement.

8. For all objects J, if J is a square then J has four sides.
   a. All squares ____.
   b. Every square ____.
   c. If an object is a square, then it ____.
   d. If J ____, then J ____.
   e. For all squares J, ____.
9. For all equations E, if E is quadratic then E has at most two real solutions.
   a. All quadratic equations ____.
   b. Every quadratic equation ____.
   c. If an equation is quadratic, then it ____.
   d. If E ____, then E ____.
   e. For all quadratic equations E, ____.
10. Every nonzero real number has a reciprocal.
    a. All nonzero real numbers ____.
    b. For all nonzero real numbers r, there is ____ for r.
    c. For all nonzero real numbers r, there is a real number s such that ____.
11. Every positive number has a positive square root.
    a. All positive numbers ____.
    b. For any positive number e, there is ____ for e.
    c. For all positive numbers e, there is a positive number r such that ____.
12. There is a real number whose product with every number leaves the number unchanged.
    a. Some ____ has the property that its ____.
    b. There is a real number r such that the product of r ____.
    c. There is a real number r with the property that for every real number s, ____.
13. There is a real number whose product with every real number equals zero.
    a. Some ____ has the property that its ____.
    b. There is a real number a such that the product of a ____.
    c. There is a real number a with the property that for every real number b, ____.

Answers for Test Yourself

1. true; all elements of a set 2. is true; also has to be true 3. there is at least one thing

1.2 The Language of Sets

. . . when we attempt to express in mathematical symbols a condition proposed in words. First, we must understand thoroughly the condition. Second, we must be familiar with the forms of mathematical expression. —George Polyá (1887–1985)

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as

Exercise Set 1.2

1. Which of the following sets are equal?

$$A = \{a, b, c, d\} \qquad B = \{d, e, a, c\}$$
$$C = \{d, b, a, c\} \qquad D = \{a, a, d, e, c, e\}$$

2. Write in words how to read each of the following out loud.
   a. $\{x \in \mathbf{R}^+ \mid 0 < x < 1\}$
   b. $\{x \in \mathbf{R} \mid x \leq 0 \text{ or } x \geq 1\}$
   c. $\{n \in \mathbf{Z} \mid n \text{ is a factor of } 6\}$
   d. $\{n \in \mathbf{Z}^+ \mid n \text{ is a factor of } 6\}$

3. a. Is $4 = \{4\}$?
   b. How many elements are in the set $\{3, 4, 3, 5\}$?
   c. How many elements are in the set $\{1, \{1\}, \{1, \{1\}\}\}$?

4. a. Is $2 \in \{2\}$?
   b. How many elements are in the set $\{2, 2, 2, 2\}$?
   c. How many elements are in the set $\{0, \{0\}\}$?
   d. Is $\{0\} \in \{\{0\}, \{1\}\}$?
   e. Is $0 \in \{\{0\}, \{1\}\}$?

H 5. Which of the following sets are equal?

$$A = \{0, 1, 2\}$$
$$B = \{x \in \mathbf{R} \mid -1 \leq x < 3\}$$
$$C = \{x \in \mathbf{R} \mid -1 < x < 3\}$$
$$D = \{x \in \mathbf{Z} \mid -1 < x < 3\}$$
$$E = \{x \in \mathbf{Z}^+ \mid -1 < x < 3\}$$

H 6. For each integer n, let $T_n = \{n, n^2\}$. How many elements are in each of T_2, T_{-3}, T_1 and T_0? Justify your answers.

7. Use the set-roster notation to indicate the elements in each of the following sets.
   a. $S = \{n \in \mathbf{Z} \mid n = (-1)^k, \text{ for some integer } k\}$.
   b. $T = \{m \in \mathbf{Z} \mid m = 1 + (-1)^i, \text{ for some integer } i\}$.
   c. $U = \{r \in \mathbf{Z} \mid 2 \leq r \leq -2\}$
   d. $V = \{s \in \mathbf{Z} \mid s > 2 \text{ or } s < 3\}$
   e. $W = \{t \in \mathbf{Z} \mid 1 < t < -3\}$
   f. $X = \{u \in \mathbf{Z} \mid u \leq 4 \text{ or } u \geq 1\}$

8. Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.
   a. Is $B \subseteq A$? b. Is $C \subseteq A$?
   b. Is $C \subseteq C$? d. Is C a proper subset of A?

9. a. Is $3 \in \{1, 2, 3\}$? b. Is $1 \subseteq \{1\}$?
   c. Is $\{2\} \in \{1, 2\}$? d. Is $\{3\} \in \{1, \{2\}, \{3\}\}$?
   e. Is $1 \in \{1\}$? f. Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?
   g. Is $\{1\} \subseteq \{1, 2\}$? h. Is $1 \in \{\{1\}, 2\}$?
   i. Is $\{1\} \subseteq \{1, \{2\}\}$? j. Is $\{1\} \subseteq \{1\}$?

10. a. Is $((-2)^2, -2^2) = (-2^2, (-2)^2)$?
    b. Is $(5, -5) = (-5, 5)$?
    c. Is $(8 - 9, \sqrt[3]{-1}) = (-1, -1)$?
    d. Is $\left(\frac{-2}{-4}, (-2)^3\right) = \left(\frac{3}{6}, -8\right)$?

11. Let $A = \{w, x, y, z\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:
    a. $A \times B$ b. $B \times A$
    c. $A \times A$ d. $B \times B$

12. Let $S = \{2, 4, 6\}$ and $T = \{1, 3, 5\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:
    a. $S \times T$ b. $T \times S$
    c. $S \times S$ d. $T \times T$

Answers for Test Yourself

1. does not matter 2. the set of all real numbers 3. the set of all integers 4. the set of all rational numbers 5. the set of all x such that $P(x)$ 6. every element in A is an element in B 7. the set of all ordered pairs (a, b) where a is in A and b is in B

1.3 The Language of Relations and Functions

Mathematics is a language. — Josiah Willard Gibbs (1839–1903)

There are many kinds of relationships in the world. For instance, we say that two people are related by blood if they share a common ancestor and that they are related by marriage if one shares a common ancestor with the spouse of the other. We also speak of the relationship between student and teacher, between people who work for the same employer, and between people who share a common ethnic background.

Similarly, the objects of mathematics may be related in various ways. A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B, or if A and B have at least one element in common. A number x may be said to be related to a number y if $x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$. Two identifiers in a computer

a. Draw arrow diagrams for U, V, and W.
b. Indicate whether any of the relations U, V, and W are functions.

9. a. Find all relations from $\{0,1\}$ to $\{1\}$.
b. Find all functions from $\{0,1\}$ to $\{1\}$.
c. What fraction of the relations from $\{0,1\}$ to $\{1\}$ are functions?

10. Find four relations from $\{a, b\}$ to $\{x, y\}$ that are not functions from $\{a, b\}$ to $\{x, y\}$.

11. Define a relation P from $\mathbf{R}^+$ to $\mathbf{R}$ as follows: For all real numbers x and y with $x > 0$,

$$(x, y) \in P \quad \text{means that} \quad x = y^2.$$

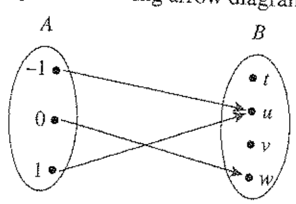
Is P a function? Explain.

12. Define a relation T from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all real numbers x and y,

$$(x, y) \in T \quad \text{means that} \quad y^2 - x^2 = 1.$$

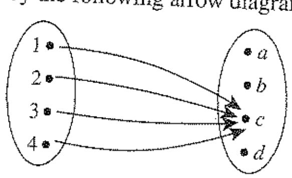
Is T a function? Explain.

13. Let $A = \{-1, 0, 1\}$ and $B = \{t, u, v, w\}$. Define a function $F: A \to B$ by the following arrow diagram:



a. Write the domain and co-domain of F.
b. Find $F(-1)$, $F(0)$, and $F(1)$.

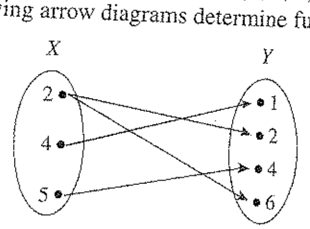
14. Let $C = \{1, 2, 3, 4\}$ and $D = \{a, b, c, d\}$. Define a function $G: C \to D$ by the following arrow diagram:



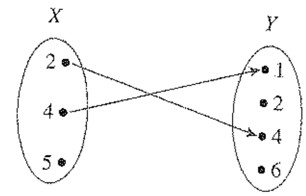
a. Write the domain and co-domain of G.
b. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

15. Let $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Which of the following arrow diagrams determine functions from X to Y?

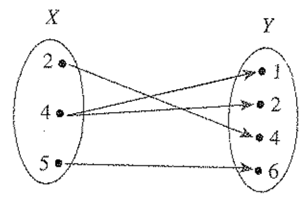
a.



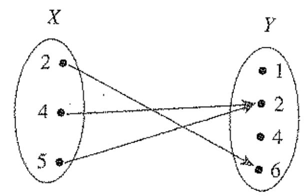
b.



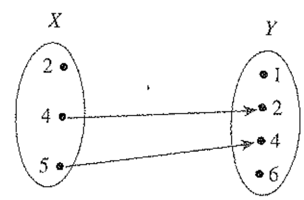
c.



d.



e.



16. Let f be the squaring function defined in Example 1.3.6. Find $f(-1)$, $f(0)$, and $f\left(\frac{1}{2}\right)$.

17. Let g be the successor function defined in Example 1.3.6. Find $g(-1000)$, $g(0)$, and $g(999)$.

18. Let h be the constant function defined in Example 1.3.6. Find $h\left(-\frac{12}{5}\right)$, $h\left(\frac{0}{1}\right)$, and $h\left(\frac{9}{17}\right)$.

19. Define functions f and g from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For all $x \in \mathbf{R}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = \frac{2x^3 + 2x}{x^2 + 1}.$$

Does $f = g$? Explain.

20. Define functions H and K from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For all $x \in \mathbf{R}$,

$$H(x) = (x - 2)^2 \quad \text{and} \quad K(x) = (x - 1)(x - 3) + 1.$$

Does $H = K$? Explain.

Answers for Test Yourself

1. a subset of the Cartesian product $A \times B$ 2. a. an element y of B such that $(x, y) \in F$ (i.e., such that x is related to y by F) b. $(x, y) \in F$ and $(x, z) \in F$; $y = z$ 3. the unique element of B that is related to x by F

Exercise Set 2.1*

In each of 1–4 represent the common form of each argument using letters to stand for component sentences, and fill in the blanks so that the argument in part (b) has the same logical form as the argument in part (a).

1. a. If all integers are rational, then the number 1 is rational.
All integers are rational.
Therefore, the number 1 is rational.

b. If all algebraic expressions can be written in prefix notation, then _______________________________.
____.
Therefore, $(a+2b)(a^2-b)$ can be written in prefix notation.

2. a. If all computer programs contain errors, then this program contains an error.
This program does not contain an error.
Therefore, it is not the case that all computer programs contain errors.

b. If ________, then ________.
2 is not odd.
Therefore, it is not the case that all prime numbers are odd.

3. a. This number is even or this number is odd.
This number is not even.
Therefore, this number is odd.

b. ________ or logic is confusing.
My mind is not shot.
Therefore, ________.

4. a. If n is divisible by 6, then n is divisible by 3.
If n is divisible by 3, then the sum of the digits of n is divisible by 3.
Therefore, if n is divisible by 6, then the sum of the digits of n is divisible by 3.
(Assume that n is a particular, fixed integer.)

b. If this function is ______ then this function is differentiable.
If this function is ______ then this function is continuous.
Therefore, if this function is a polynomial, then this function ______.

5. Indicate which of the following sentences are statements.
a. 1,024 is the smallest four-digit number that is a perfect square.
b. She is a mathematics major.
c. $128 = 2^6$ d. $x = 2^6$

Write the statements in 6–9 in symbolic form using the symbols ~, ∨, and ∧ and the indicated letters to represent component statements.

6. Let s = "stocks are increasing" and i = "interest rates are steady."
a. Stocks are increasing but interest rates are steady.
b. Neither are stocks increasing nor are interest rates steady.

7. Juan is a math major but not a computer science major. (m = "Juan is a math major," c = "Juan is a computer science major")

8. Let h = "John is healthy," w = "John is wealthy," and s = "John is wise."
a. John is healthy and wealthy but not wise.
b. John is not wealthy but he is healthy and wise.
c. John is neither healthy, wealthy, nor wise.
d. John is neither wealthy nor wise, but he is healthy.
e. John is wealthy, but he is not both healthy and wise.

9. Either this polynomial has degree 2 or it has degree 3 but not both. (n = "This polynomial has degree 2," k = "This polynomial has degree 3")

10. Let p be the statement "DATAENDFLAG is off," q the statement "ERROR equals 0," and r the statement "SUM is less than 1,000." Express the following sentences in symbolic notation.
a. DATAENDFLAG is off, ERROR equals 0, and SUM is less than 1,000.
b. DATAENDFLAG is off but ERROR is not equal to 0.
c. DATAENDFLAG is off; however, ERROR is not 0 or SUM is greater than or equal to 1,000.
d. DATAENDFLAG is on and ERROR equals 0 but SUM is greater than or equal to 1,000.
e. Either DATAENDFLAG is on or it is the case that both ERROR equals 0 and SUM is less than 1,000.

11. In the following sentence, is the word *or* used in its inclusive or exclusive sense? A team wins the playoffs if it wins two games in a row or a total of *three* games.

Write truth tables for the statement forms in 12–15.

12. $\sim p \wedge q$ 13. $\sim(p \wedge q) \vee (p \vee q)$

14. $p \wedge (q \wedge r)$ 15. $p \wedge (\sim q \vee r)$

Determine whether the statement forms in 16–24 are logically equivalent. In each case, construct a truth table and include a sentence justifying your answer. Your sentence should show that you understand the meaning of logical equivalence.

16. $p \vee (p \wedge q)$ and p 17. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

18. $p \vee \mathbf{t}$ and $\mathbf{t}$ 19. $p \wedge \mathbf{t}$ and p

20. $p \wedge \mathbf{c}$ and $p \vee \mathbf{c}$

21. $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol $*$ signals that an exercise is more challenging than usual.

22. $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$

23. $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$

24. $(p \vee q) \vee (p \wedge r)$ and $(p \vee q) \wedge r$

Use De Morgan's laws to write negations for the statements in 25–31.

25. Hal is a math major and Hal's sister is a computer science major.

26. Sam is an orange belt and Kate is a red belt.

27. The connector is loose or the machine is unplugged.

28. The units digit of 4^{67} is 4 or it is 6.

29. This computer program has a logical error in the first ten lines or it is being run with an incomplete data set.

30. The dollar is at an all-time high and the stock market is at a record low.

31. The train is late or my watch is fast.

Assume x is a particular real number and use De Morgan's laws to write negations for the statements in 32–37.

32. $-2 < x < 7$

33. $-10 < x < 2$

34. $x < 2$ or $x > 5$

35. $x \leq -1$ or $x > 1$

36. $1 > x \geq -3$

37. $0 > x \geq -7$

In 38 and 39, imagine that *num_orders* and *num_instock* are particular values, such as might occur during execution of a computer program. Write negations for the following statements.

38. (*num_orders* > 100 and *num_instock* $\leq$ 500) or *num_instock* < 200

39. (*num_orders* < 50 and *num_instock* > 300) or (50 $\leq$ *num_orders* < 75 and *num_instock* > 500)

Use truth tables to establish which of the statement forms in 40–43 are tautologies and which are contradictions.

40. $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$

41. $(p \wedge \sim q) \wedge (\sim p \vee q)$

42. $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$

43. $(\sim p \vee q) \vee (p \wedge \sim q)$

In 44 and 45, determine whether the statements in (a) and (b) are logically equivalent.

44. Assume x is a particular real number.

a. $x < 2$ or it is not the case that $1 < x < 3$.

b. $x \leq 1$ or either $x < 2$ or $x \geq 3$.

45. a. Bob is a double math and computer science major and Ann is a math major, but Ann is not a double math and computer science major.

b. It is not the case that both Bob and Ann are double math and computer science majors, but it is the case that Ann is a math major and Bob is a double math and computer science major.

✱46. In Example 2.1.4, the symbol $\oplus$ was introduced to denote *exclusive or,* so $p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$. Hence the truth table for *exclusive or* is as follows:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

a. Find simpler statement forms that are logically equivalent to $p \oplus p$ and $(p \oplus p) \oplus p$.

b. Is $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$? Justify your answer.

c. Is $(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r)$? Justify your answer.

✱**47.** In logic and in standard English, a double negative is equivalent to a positive. There is one fairly common English usage in which a "double positive" is equivalent to a negative. What is it? Can you think of others?

In 48 and 49 below, a logical equivalence is derived from Theorem 2.1.1. Supply a reason for each step.

48.
$$\begin{aligned}(p \wedge \sim q) \vee (p \wedge q) &\equiv p \wedge (\sim q \vee q) && \text{by } \underline{\ (a)\ } \\ &\equiv p \wedge (q \vee \sim q) && \text{by } \underline{\ (b)\ } \\ &\equiv p \wedge \mathbf{t} && \text{by } \underline{\ (c)\ } \\ &\equiv p && \text{by } \underline{\ (d)\ }\end{aligned}$$

Therefore, $(p \wedge \sim q) \vee (p \wedge q) \equiv p$.

49.
$$\begin{aligned}&(p \vee \sim q) \wedge (\sim p \vee \sim q) \\ &\equiv (\sim q \vee p) \wedge (\sim q \vee \sim p) && \text{by } \underline{\ (a)\ } \\ &\equiv \sim q \vee (p \wedge \sim p) && \text{by } \underline{\ (b)\ } \\ &\equiv \sim q \vee \mathbf{c} && \text{by } \underline{\ (c)\ } \\ &\equiv \sim q && \text{by } \underline{\ (d)\ }\end{aligned}$$

Therefore, $(p \vee \sim q) \wedge (\sim p \vee \sim q) \equiv \sim q$.

Use Theorem 2.1.1 to verify the logical equivalences in 50–54. Supply a reason for each step.

50. $(p \wedge \sim q) \vee p \equiv p$

51. $p \wedge (\sim q \vee p) \equiv p$

52. $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$

53. $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

54. $(p \wedge (\sim(\sim p \vee q))) \vee (p \wedge q) \equiv p$