Mean 3-torsion in the class group of imaginary quadratic fields

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October 4, 2012

Abstract

We prove an asymptotic with two main terms for the mean number of 3-torsion elements in the class group of imaginary quadratic fields when the fields are counted with a smooth weight on the discriminant. This gives an alternate proof of a recent result of Taniguchi-Thorne and Bhargava-Shankar-Tsimerman.

1 Introduction

Let $-D$, $D > 0$ be a fundamental discriminant and write $h_3(-D)$ for the number of order 3 elements in the class group $H(-D)$. Davenport and Heilbronn [7] first estimated the mean

$$\sum_{0<D\leq X} h_3(-D) = \frac{3}{\pi^2}X + o(X) \tag{1}$$

as a consequence of their asymptotic count for cubic fields. The error term has been improved several times [1], [2], and recently has been established with a negative secondary main term of order $X^{\frac{5}{6}}$, separately by Taniguchi and Thorne [14] and Bhargava, Shankar and Tsimerman [4], via rather different methods: both proofs follow [7] in counting cubic fields through binary cubic forms, then passing to 3-torsion by class field theory; Tanaguchi-Thorne get to binary cubic forms through the associated zeta function [12], whereas Bhargava-Shankar-Tsimerman build on Bhargava's techniques in the geometry of numbers [3]. In this article we give a third proof of the secondary main term, albeit while counting discriminants with a smooth weight [we also restrict to discriminants of form $-D = -4d, d \equiv 2 \bmod 4$ to simplify exposition].

Theorem 1.1. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function having compact support and let $\epsilon > 0$ arbitrary. As $X \to \infty$,*

$$\sum_{\substack{d\equiv 2 \bmod 4 \\ \textit{squarefree}}} \phi\left(\frac{d}{X}\right) h_3(-4d) = \frac{2}{\pi^2}\tilde{\phi}(1)X + c_3\tilde{\phi}\left(\frac{5}{6}\right)X^{\frac{5}{6}} + O(X^{\frac{13}{16}+\epsilon})$$

with $\tilde{\phi}(s)$ *the Mellin transform* $\tilde{\phi}(s) = \int_0^\infty \phi(x) x^s \frac{dx}{x}$ *and*

$$c_3 = \frac{\Gamma\left(\frac{1}{6}\right)}{3\pi^{\frac{3}{2}}\Gamma\left(\frac{2}{3}\right)} \zeta\left(\frac{1}{3}\right)\left[2^{\frac{2}{3}} - 2^{\frac{1}{3}} + 1\right] \prod_{p\ odd}\left[1 - \frac{p^{\frac{1}{3}}+1}{p^2+1}\right].$$

Remark. The current record error term for this Theorem, which does not require smoothing, is $O(X^{\frac{18}{23}+\epsilon})$, from [14].

Our proof passes along somewhat more classical lines in analytic number theory. A 3-torsion ideal in $\mathbb{Q}(\sqrt{-d})$ cubes to a principal ideal $(x+y\sqrt{-d}),\ y \neq 0$ which results in the norm equation

$$m^3 = x^2 + dy^2. \tag{2}$$

Using this equation we describe the Heegner points associated to 3-torsion ideals, then use Poisson summation to count points, and bound the resulting exponential sums. The ideas build upon our earlier preprint [8] and Soundararajan's work on divisibility of class numbers [13]. In particular, the negative secondary main term appears visibly in our method from the trivial lower bound $N\mathfrak{a} \geq d^{\frac{1}{3}}$ valid for any 3-torsion ideal, which results in an absense of Heegner points in the part of the cusp of $PSL_2(\mathbb{Z})\backslash\mathbb{H}$ with $\Im(z) \gg X^{\frac{1}{6}}$, a set of hyperbolic volume $\asymp X^{\frac{-1}{6}}$.

The asymptotic (1) and it's secondary main term have a bit of history, which we now describe. The notion of a negative second term has origins in work of Shintani [12] and later Datskovsky and Wright [6], who showed the the zeta function associated to binary cubic forms has, in addition to a pole at 1, a second pole at $\frac{5}{6}$. One reason for the Davenport-Heilbronn Theorem's continued prominence is that it remains the only established evidence for Cohen and Lenstra's heuristics [5] regarding the class groups of imaginary quadratic fields. Researchers interested in testing out the heuristics numerically were surprised to find that actual counts for fixed odd torsion were always rather less than expected based upon the heuristics, and in the case of 3-torsion, less than guaranteed by the Davenport-Heilbronn Theorem. Seeking to explain this discrepancy led Roberts to go back to the work of Datskovsky-Wright, and to conjecture the secondary main term for 3-torsion [11].

In this connection we draw the reader's attention to the outline of our proof in Section 5.3, where we decompose the count for 3-torsion into a diagonal term and two terms containing exponential sums. The diagonal corresponds to a volume calculation – it comes from the 0th Fourier coefficient under Poisson summation – and contributes the two main terms, while the exponential sums are shown to give error terms. A similar process can be carried out in counting k torsion for odd primes $k > 3$, and also for odd composite k, although the situation becomes more complicated. Already for $k = 5$ the resulting exponential sums appear too difficult to estimate by present methods, but the resulting diagonal term can be evaluated much as in Section 6, yielding the same linear main term as in (1), consistent with the Cohen-Lenstra Heuristic, and a negative secondary term, now of size $X^{\frac{1}{2}+\frac{1}{k}}$.[1] In the case of $k = 5, 7$ this diagonal asymptotic is in good agreement with counts of k torsion even for moderately small discriminants ($-D$ with $D < 5 \times 10^8$), see Tables 1 and 2. For larger k and the same range of discriminants, actual torsion counts tend to be rather less than the diagonal asymptotic. In the case of composite k, genuine main terms should arise from the exponential sums, at least if the result is to be in agreement with Cohen-Lenstra.

The paper is organized as follows. In the next section we set down our conventions for Fourier and Mellin transforms and prove some basic estimates. In the Section 3 we give the precise parameterization of 3-torsion ideals along the lines of equation (2). In Section 4 we describe the results that we will use concerning the geometry of $PSL_2(\mathbb{Z})\backslash\mathbb{H}$. In Section 5 we outline our argument, including the introduction of a global and local parametrization. Section 6 evaluates the diagonal main terms, while Section 7 bounds the error from exponential sums. We conclude in Section 8 with our sieve for fundamental discriminants.

2 Function notation and properties

We adopt the following notations regarding Fourier and Mellin transforms. For one variable integrable function f on $\mathbb{R}^+$, we denote its Mellin transform by $\tilde{f}(s)$, defined where convergent by

$$\tilde{f}(s) = \int_0^\infty f(x)x^{s-1}dx, \qquad s \in \mathbb{C}$$

and possibly extended by analytic continuation elsewhere. For a function $f(x, y)$ in two variables we denote by

$$f^1(u, y) = \int_{-\infty}^{\infty} f(x, y)e(-ux)dx, \qquad f^2(x, v) = \int_{-\infty}^{\infty} f(x, y)e(-vy)dy,$$
$$f^{1,2}(u, v) = \int_{-\infty}^{\infty} f(x, y)e(-ux - vy)dxdy$$

respectively the function with the Fourier transform taken in the first, second, or both first and second slots.

Lemma 2.1. *Let $F \in \mathcal{S}(\mathbb{R}^2)$ be a Schwarz-class function and set*

$$f(x, y) = F(A + Bx, C + Dx + Ey), \qquad B, E \neq 0.$$

[1]When counting with a smooth weight ϕ, the constant on the $X^{\frac{1}{2}+\frac{1}{k}}$ term has the Mellin transform $\tilde{\phi}$ evaluated at $\frac{1}{2} + \frac{1}{k}$ and replaces c_3 with c_k given by

$$c_k = \frac{\Gamma(\frac{k-2}{2k})\zeta(\frac{k-2}{k})}{k\pi^{\frac{3}{2}}\Gamma(\frac{k-1}{k})}\left[1 - 2^{\frac{1}{k}} + 2^{\frac{k-1}{k}}\right]\prod_{p \text{ odd}}\left[1 + \frac{1}{p+1}\left(\frac{1}{p^{\frac{1}{k}}} - \frac{1}{p^{\frac{k-2}{k}}} - \frac{1}{p^{\frac{k-1}{k}}} - \frac{1}{p}\right)\right].$$

D < ...	Total 5-torsion	Cohen-Lenstra	Discrep. w. C-L	Diagonal asymp.	Discrep. diagonal
4000000	194464	202642	-8178	194509	-45
8000000	392996	405284	-12288	392073	922
16000000	791328	-810569	19241	789107	2220
32000000	1588520	-1621138	32618	1586274	2245
64000000	3186224	3242277	-56053	3185640	583
128000000	6393960	6484555	-90595	6392548	1411
256000000	12818136	12969111	-150975	12819645	-1509
512000000	25673816	25938223	-264407	25695414	-21598

Table 1: Aggregate 5-torsion elements in fields of discriminant $-D = -4d$, where $d \equiv 2 \bmod 4$ is squarefree, and comparison with both Cohen-Lenstra main term, and the diagonal asymptotic including the negative secondary term.

D < ...	Total 7-torsion	Cohen-Lenstra	Discrep. w. C-L	Diagonal asymp.	Discrep. diagonal
4000000	197094	202642	-5548	196899	194
8000000	397902	405284	-7382	396317	1584
16000000	796266	-810569	14303	796568	-302
32000000	1595088	1621138	-26050	1599277	-4189
64000000	3201048	3242277	-41229	3208142	-7094
128000000	6427098	6484555	-57457	6431256	-4158
256000000	12870768	12969111	-98343	12885889	-15121
512000000	25832964	25938223	-105259	25808278	24685

Table 2: Aggregate 7-torsion elements in fields of discriminant $-D = -4d$, where $d \equiv 2 \bmod 4$ is squarefree, and comparison with both Cohen-Lenstra main term, and the diagonal asymptotic including the negative secondary term.

Then

$$f^{1,2}(u,v)=\frac{1}{BE}e\left(\frac{A}{B}u+\left(\frac{C}{E}-\frac{AD}{BE}\right)V\right)F^{1,2}\left(\frac{1}{B}u-\frac{D}{BE}v,\frac{1}{E}v\right).$$

Throughout, $\phi\in C_c^\infty(\mathbb{R}^+)$ is the smooth function of the Theorem. In order to make partitions of unity, we also fix once for all time a function $\sigma\in C_c^\infty(\mathbb{R})$ satisfying $\sigma\geq 0$, $\operatorname{supp}(\sigma)\subset[-2,2]$ and $\sum_{n\in\mathbb{Z}}\sigma(n+x)=1$. Letting $\sigma^\times(x)=\sigma(\log x)$ we obtain a related positive function of compact support on $\mathbb{R}^+$ satisfying $\sum_{n\in\mathbb{Z}}\sigma^\times(e^n x)=1$.

In addition to the fixed ϕ, let $\{\psi_k\}_{k\in\mathbb{Z}\cup\{\infty\}}$ be smooth functions on $\mathbb{R}^+$ satisfying either

$$\begin{aligned}\psi=\psi_\infty: &\qquad \operatorname{supp}(\psi_\infty)\subset[e^{-6},\infty],\quad z>e^6\Rightarrow\psi_\infty(z)\equiv 1,\\ &\text{or}\\ \psi=\psi_k, k\text{ finite}: &\qquad \operatorname{supp}(\psi_k)\subset[e^{-6},e^6].\end{aligned}\tag{3}$$

Note in particular that ϕ and ψ_k have Mellin transforms $\tilde{\phi},\tilde{\psi}_k$ that are entire, with the exception of a simple pole at 0 of residue -1 in the case of ψ_∞.

For positive parameters X,Y,a and a general function ψ we consider the function

$$\Phi_{X,Y,a}(x,y|\psi)=\phi\left(\frac{x^3-y^2}{Xa^2}\right)\psi\left(\frac{x^3-y^2}{Y^2a^2x^2}\right),\tag{4}$$

which will package the Archimedean data of our eventual parameterizations. We also set

$$\Phi_M(x,y|\psi)=\phi\left(x^3-y^2\right)\psi\left(\left(x^3-y^2\right)\frac{M}{x^2}\right).\tag{5}$$

The appropriate function ψ and relevant parameters will generally be clear from the context, in which case they are omitted. Note that for fixed x, and for ψ_k (k finite or infinite) satisfying support condition (3), $\Phi_M(\psi_k)$ is supported on $x^3-y^2\asymp 1$ so that

$$m(y:\Phi_M(x,y|\psi_k)\neq 0)\ll x^{\frac{-3}{4}}.$$

Also, $\Phi_M(\psi_k)$ is supported in $x\ll\sqrt{M}$ and for $k<\infty$, $\Phi_M(\psi_k)$ is supported on $x\asymp\sqrt{M}$. In particular, for finite k we have the bound

$$\|\Phi_M(\psi_k)\|_1\ll M^{\frac{-1}{4}}.\tag{6}$$

Lemma 2.2. *The Fourier transform of $\Phi_{X,Y,a}(x,y)$ satisfies*

$$\Phi^{1,2}_{X,Y,a}(u,v)=\left(a^2X\right)^{\frac{5}{6}}\Phi_M^{1,2}\left(a^{\frac{2}{3}}X^{\frac{1}{3}}u,aX^{\frac{1}{2}}v\right),\qquad M=\frac{X^{\frac{1}{3}}}{a^{\frac{4}{3}}Y^2}.$$

Moreover, for $\psi=\psi_k$ satisfying (3) the function Φ_M satisfies

$$D_1^{i_1}D_2^{i_2}\Phi_M(x,y|\psi)\ll M^{i_1+\frac{3i_2}{4}}\|\phi\|_{C^{i_1+i_2}}\|\psi\|_{C^{i_1+i_2}},$$

and therefore

$$\Phi_M^{1,2}(u,v|\psi) \ll \left(\frac{M}{u}\right)^{i_1}\left(\frac{M^{\frac{3}{4}}}{v}\right)^{i_2}\|\phi\|_{C^{i_1+i_2}}\|\psi\|_{C^{i_1+i_2}}.$$

If, in addition, $k<\infty$, then we have the stronger bound

$$\Phi_M^{1,2}(u,v|\psi) \ll M^{\frac{-1}{4}}\left(\frac{M}{u}\right)^{i_1}\left(\frac{M^{\frac{3}{4}}}{v}\right)^{i_2}\|\phi\|_{C^{i_1+i_2}}\|\psi\|_{C^{i_1+i_2}}.$$

Proof. We have

$$\begin{aligned}\Phi_{X,Y,a}^{1,2}(u,v) &= \int_{\mathbb{R}^2}\phi\left(\frac{x^3-y^2}{a^2X}\right)\psi\left(\frac{x^3-y^2}{x^2Y^2a^2}\right)e\left(-ux-vy\right)dxdy\\ &= \left(a^2X\right)^{\frac{5}{6}}\int_{\mathbb{R}^2}\phi\left(x^3-y^2\right)\psi\left(\frac{x^3-y^2}{x^2}\frac{X^{\frac{1}{3}}}{Y^2a^{\frac{4}{3}}}\right)\\ &\qquad\qquad\times e\left(-\left(a^2X\right)^{\frac{1}{3}}ux-\left(a^2X\right)^{\frac{1}{2}}vy\right)dxdy\\ &= \left(a^2X\right)^{\frac{5}{6}}\Phi_M^{1,2}\left(a^{\frac{2}{3}}X^{\frac{1}{3}}u, aX^{\frac{1}{2}}v\right).\end{aligned}$$

The bounds on the derivatives are straightforward from the observation $x \ll \sqrt{M}$ and $y \ll x^{\frac{3}{2}}$, and the bound on the Fourier transform is deduced by integration by parts. The stronger bound when k is finite takes into account the restriction

$$\operatorname{supp}\left(\Phi_M(\psi_k)\right) \subset \{x : x \asymp \sqrt{M}\}.$$

□

Lemma 2.3. *Let $k<\infty$ and let ψ_k satisfy (3). Assume $B>0$, $M>1$ and $K_2>\frac{1}{B\sqrt{M}}$. We have*

$$\sum_{K_1<a\le K_1+K_2}\left|\Phi_M^{1,2}\left(A+Ba,C\right)\right|^2 \ll M^{\frac{-1}{2}}\left(1+\frac{1}{B\sqrt{M}}\right)\log K_2.$$

Proof. The sum is

$$\begin{aligned}&\int_{\mathbb{R}^2}\Phi_{M,k}^2\left(x_1,C\right)\overline{\Phi_{M,k}^2\left(x_2,C\right)}\sum_{K_1<a\le K_1+K_2}e\left(-\left(A+Ba\right)\left(x_1-x_2\right)\right)dx_1dx_2\\ &\quad\ll \int_{\mathbb{R}^2}|\Phi_{M,k}^2|\left(x_1,C\right)|\Phi_{M,k}^2|\left(x_2,C\right)\cdot\min\left(K_2,\|B\left(x_1-x_2\right)\|^{-1}\right)dx_1dx_2\\ &\quad\ll M^{\frac{-3}{2}}\int_{x_1,x_2\asymp M^{\frac{1}{2}}}\min\left(K_2,\|B\left(x_1-x_2\right)\|^{-1}\right)dx_1dx_2\\ &\quad\ll M^{\frac{-1}{2}}\left(1+\frac{1}{B\sqrt{M}}\right)\log K_2.\end{aligned}$$

□

Lemma 2.4. *Let $\psi \in C^\infty(\mathbb{R}^+)$ supported in $[\delta, \infty]$, $\delta > 0$ satisfying $D^j(\psi(x) - 1)x^a \to 0$ as $x \to \infty$ for all j, a. Set*

$$H(z) = \Phi^{1,2}_{z^{-1}}(0, 0|\psi).$$

Then for $s \neq 0$ and $\frac{2}{3}s + \frac{1}{6} \neq -n$, $n \in \mathbb{Z}_{\geq 0}$

$$\tilde{H}(s) = \frac{1}{3}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{6}+\frac{2}{3}s\right)}{\Gamma\left(\frac{2}{3}+\frac{2}{3}s\right)}\tilde{\phi}\left(\frac{5}{6}+\frac{s}{3}\right)\tilde{\psi}(-s).$$

Proof. For $\Re(s) > 0$ we have

$$\begin{aligned}
\tilde{H}(s) &= 2\int_0^\infty \int_0^\infty \int_0^\infty \phi\left(x^3 - y^2\right)\psi\left(\frac{x^3-y^2}{x^2 z}\right) xyz^s \frac{dz}{z}\frac{dy}{y}\frac{dx}{x} \\
&= \int_0^\infty \int_0^1 \int_0^\infty \phi\left(x^3(1-y)\right)\psi\left(x(1-y)z\right)x^{\frac{5}{2}}y^{\frac{-1}{2}}z^{-s}\frac{dz}{z}dy\frac{dx}{x} \\
&= \frac{1}{3}\int_0^\infty \phi(x)\, x^{\frac{5}{6}+\frac{s}{3}}\frac{dx}{x} \cdot \int_0^1 (1-y)^{\frac{2s}{3}-\frac{5}{6}} y^{\frac{-1}{2}} dy \cdot \int_0^\infty \psi(z)\, z^{-s}\frac{dz}{z} \\
&= \frac{1}{3}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{6}+\frac{2}{3}s\right)}{\Gamma\left(\frac{2}{3}+\frac{2}{3}s\right)}\tilde{\phi}\left(\frac{5}{6}+\frac{s}{3}\right)\tilde{\psi}(-s).
\end{aligned}$$

The conditions on ψ guarantee that $\tilde{\psi}$ extends to a meromorphic function, with a single simple pole of residue -1 at $s = 0$. The formula thus holds for s not equal to a pole of the right hand side, by analytic continuation. □

3 Parametrizations

Let $d \equiv 2 \bmod 4$ be positive and squarefree. An integral ideal $\mathfrak{a}$ of $\mathbb{Q}(\sqrt{-d})$ is said to be primitive if $\mathfrak{a} = n \cdot \mathfrak{b} \Rightarrow |n| = 1$ for any integral ideal $\mathfrak{b}$. Each primitive ideal may be expressed as a $\mathbb{Z}$-module of the form $[N\mathfrak{a}, b + \sqrt{-d}]$, where $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. To primitive $\mathfrak{a}$ we associate the 'Heegner point' $z_\mathfrak{a} \in \Gamma_\infty\backslash\mathbb{H}$, $z_\mathfrak{a} = \frac{b+\sqrt{-d}}{N\mathfrak{a}}$. The following proposition parametrizes primitive ideals of a fixed class in $\mathbb{Q}(\sqrt{-d})$ as associated to the image of a fixed Heegner point under the standard action of $\Gamma_\infty\backslash\Gamma$ on the strip $\Gamma_\infty\backslash\mathbb{H}$.

Proposition 3.1. *Let $d \equiv 2 \bmod 4$ be positive and squarefree. Fix ideal class $[\mathfrak{a}] \in H(-4d)$. The map $\mathfrak{b} \mapsto z_\mathfrak{b}$ is a bijection between primitive ideals of class $[\mathfrak{b}] = [\mathfrak{a}]$ and the set $\Gamma_\infty\backslash\Gamma \cdot z_\mathfrak{a}$.*

Proof. See e.g. the beginning of Chapter 22 of [10]. □

The next proposition parametrizes non-principal primitive ideals of $\mathbb{Q}(\sqrt{-d})$ whose class has order 1 or 3 in the class group.

Proposition 3.2. *Let $d \equiv 2 \bmod 4$ be positive and squarefree. The set*

$$\mathcal{I}_d = \{(\ell, m, n, t) \in (\mathbb{Z}^+)^4 : \ell m^3 = \ell^2 n^2 + t^2 d, \ell | d, (\ell m n, t) = 1\}$$

is in bijection with primitive ideal pairs $(\mathfrak{a}, \overline{\mathfrak{a}})$ with $\mathfrak{a} \neq (1)$ and $\mathfrak{a}^3$ principal in $\mathbb{Q}(\sqrt{-d})$. Explicitly, the ideal $\mathfrak{a}$ is given as a $\mathbb{Z}$-module by

$$\mathfrak{a} = [\ell m, \ell n t^{-1} + \sqrt{-d}],$$

where $N\mathfrak{a} = \ell m$ and t^{-1} is the inverse of t modulo m.

Proof. In [8] this is proven with the condition $(\ell m n, t) = 1$ on $\mathcal{I}_d$ replaced by $(m, ntd) = 1$. The two conditions are equivalent: given $(\ell m n, t) = 1$ then clearly $(m, nd) = 1$ since $p|(m, nd) \Rightarrow p^2|t^2 d \Rightarrow p|t$. On the other hand, if $(m, ntd) = 1$ then $(\ell n, t) = 1$ since $p|(\ell n, t) \Rightarrow p^2|\ell m^3 \Rightarrow p|m$. □

The previous parametrization asserts that 3-torsion principal ideals (essentially) lead to a local relation

$$m^3 = n^2 + t^2 d \qquad \Rightarrow \qquad m^3 \equiv n^2 \bmod t^2.$$

Our final proposition in this section parametrizes local solutions to this type of equation.

Proposition 3.3. *Let $N > 0$ be an integer. Define*

$$S_N = \{(m, n) \in ((\mathbb{Z}/N\mathbb{Z})^\times)^2 : m^3 \equiv n^2 \bmod N\}$$

and

$$S'_N = \{(m, n) \in ((\mathbb{Z}/4N\mathbb{Z})^\times)^2 : m^3 - n^2 \equiv 2N \bmod 4N\}.$$

Then we have the local parametrization

$$S_N = \{(w^2, w^3) : w \in (\mathbb{Z}/N\mathbb{Z})^\times\},$$

and furthermore

$$S'_N = \{(m + 2N, n) : (m, n) \in S_{4N}\}.$$

Also, given $(m, n) \in \mathbb{Z}^2$, $(mn, N) = 1$ solving $m^3 \equiv n^2 \bmod N$, we have

$$(m + aN)^3 \equiv (n + a'N)^2 \bmod N^2 \qquad \Leftrightarrow \qquad 3am^2 \equiv 2a'n^2 \bmod N.$$

Proof. To prove the parametrization, note that $w \mapsto (w^2, w^3)$ and $(m, n) \mapsto m^{-1}n$ are inverse maps between $(\mathbb{Z}/N\mathbb{Z})^\times$ and S_N. The remaining claims are simple modular arithmetic. □

4 Geometry

We will prove the Theorem by counting Heegner points in $\Gamma_\infty\backslash\mathbb{H}$. Since for a fixed d, each ideal class of $H(-4d)$ contains one Heegner point in any fundamental domain $\mathcal{F}$ for $\Gamma\backslash\mathbb{H}$, one could proceed naively by counting 3-torsion points in a fixed fundamental domain. One then seeks an 'equidistribution' result for Heegner points in the strip $\Gamma_\infty\backslash\mathbb{H}$ near the standard fundamental domain. Combined with Proposition 3.1 the following lemma asserts that it is in fact sufficient to understand the distribution of only the imaginary component of Heegner points in the strip $\Gamma_\infty\backslash\mathbb{H}$ having at imaginary part $\geq \frac{1}{(\log D)^{1+\epsilon}}$, see (7) of the next section.

Lemma 4.1. *Let* $\Psi_0(x) = (4\pi x - 2)e^{-\pi x}$ *and* $\Psi(x) = \sum_{n=1}^\infty \Psi_0(m^2 x)$. *Then* $\Psi(x) \in C^\infty$ *satisfies*

1. $$\Psi^j(z) \ll_j (1+z)e^{-z}$$

2. $$\Psi(z) - 1 \ll \frac{1}{z^2}e^{\frac{-1}{z}}$$

3. $$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Psi\left(\Im^{-1}(\gamma\cdot z)\right) = 1.$$

Proof. The Mellin transform of Ψ is

$$\tilde{\Psi}(s) = (4s-2)\pi^{-s}\Gamma(s)\zeta(2s),$$

and so by Mellin inversion,

$$\Psi(z) = \frac{1}{2\pi i}\int_{(1)} \tilde{\Psi}(s) z^{-s} ds = 1 + \frac{1}{2\pi i}\int_{(\frac{-1}{2})} \tilde{\Psi}(s) z^{-s} ds.$$

Therefore $\Psi(z) = 1 + o(1)$ as $z \to 0$. The first bound now follows (for small z, differentiate under the integral).

The second bound is a consequence of the third together with Lemma 4.2, below.

To prove the third, recall that the non-holomorphic Eisenstein series is defined by

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma\cdot z)^s, \qquad \Re(s) > 1.$$

This has meromorphic continuation to all of $\mathbb{C}$ with polynomial growth in vertical strips. Moreover, the completed Eisenstein series

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$$

has only a pair of simple poles at 1 and 0, each of residue $\frac{1}{2}$ and satisfies the functional equation

$$E^*(z,s) = E^*(z,1-s).$$

Thus, by Mellin inversion

$$\begin{aligned}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Psi\left(\Im^{-1}(\gamma\cdot z)\right) &= \frac{1}{2\pi i}\int_{(2)} E(z,s)\tilde{\Psi}(s)ds \\ &= \frac{1}{2\pi i}\int_{(2)} (4s-2)E^*(z,s)ds \\ &= 1 + \frac{1}{2\pi i}\int_{(\frac{1}{2})} (4s-2)E^*(z,s)ds.\end{aligned}$$

But the integral on the half line vanishes, since by the functional equation

$$\frac{1}{2\pi i}\int_{(\frac{1}{2})} (4s-2)E^*(z,s)ds = \frac{1}{2\pi i}\int_{(\frac{1}{2})} (2-4s)E^*(z,s)ds = 0.$$

□

The following simple geometric bound may be found in [9].

Lemma 4.2. *We have the bound*

$$\#\left\{\gamma\in\Gamma_\infty\backslash\Gamma : \Im(\gamma\cdot z) > \frac{1}{Y}\right\} \ll 1+Y.$$

Also, for z real and $\gamma\in\Gamma_\infty\backslash\Gamma$, $\gamma\neq I$

$$\Im(\gamma\cdot iz) \ll \frac{1}{z}.$$

Corollary 4.1. *Given quadratic extension $\mathbb{Q}(\sqrt{-d})$ and class $[\mathfrak{a}]\in H(-4d)$ we have the bound*

$$\#\left\{\mathfrak{b}\ \textit{primitive} : [\mathfrak{b}] = [\mathfrak{a}], N\mathfrak{b} \ll Y\sqrt{d}\right\} \ll 1+Y.$$

Proof. Apply Proposition 3.1 and Lemma 4.2. □

5 Initial manipulations

In this section we give the initial argument for the Theorem, up to the point at which we need cancellation in exponential sums.

Say $\mathcal{A}$ is the desired sum of 3-torsion elements in the theorem. Also, given squarefree $d \equiv 2 \bmod 4$, $d>0$, write $\mathcal{P}(-d)$ for the set of primitive ideals in $\mathbb{Q}(\sqrt{-d})$. By Proposition 3.1 and Lemma 4.1 we have

$$\mathcal{A} = \sum_{\substack{d\equiv 2 \quad (4)\\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) h_3(-4d) = \sum_{\substack{d\equiv 2 \quad (4)\\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) \sum_{\substack{\mathfrak{a}\in\mathcal{P}(-d)\\ [\mathfrak{a}]\neq[(1)],[\mathfrak{a}]^3=[(1)]}} \Psi\left(\frac{N\mathfrak{a}}{\sqrt{d}}\right). \quad (7)$$

By adding and subtracting out the contribution of the principal class,

$$\mathcal{A} = \sum_{\substack{d \equiv 2 \pmod 4 \\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) \left[\Psi\left(d^{\frac{-1}{2}}\right) + 2 \left(\sum_{\substack{(\mathfrak{a}, \overline{\mathfrak{a}}) \in \mathcal{P}(-d)^2 \\ \mathfrak{a} \neq (1), [\mathfrak{a}]^3 = [(1)]}} \Psi\left(\frac{N\mathfrak{a}}{\sqrt{d}}\right) \right) - 1 \right].$$

By rapid convergence of Ψ to 1 at small argument ((2) of Lemma 4.1), this leaves

$$\mathcal{A} + o(1) = 2 \sum_{\substack{d \equiv 2 \pmod 4 \\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) \sum_{\substack{(\mathfrak{a}, \overline{\mathfrak{a}}) \in \mathcal{P}(-d)^2 \\ \mathfrak{a} \neq (1), [\mathfrak{a}]^3 = [(1)]}} \Psi\left(\frac{N\mathfrak{a}}{\sqrt{d}}\right).$$

In order to gain a bit of control we introduce a smooth partition of unity localizing the count of Heegner points to points having imaginary part lying in dyadic intervals. We write

$$\mathcal{A} + o(1) = 2 \sum_{\substack{d \equiv 2 \pmod 4 \\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) \sum_{\substack{(\mathfrak{a}, \overline{\mathfrak{a}}) \in \mathcal{P}(-d)^2 \\ \mathfrak{a} \neq (1), [\mathfrak{a}]^3 = [(1)]}} \Psi\left(\frac{N\mathfrak{a}}{\sqrt{d}}\right) \sum_{k=-\infty}^{\infty} \sigma^{\times}\left(\frac{N\mathfrak{a}}{\sqrt{d}} e^{-k}\right).$$

We may rearrange this as

$$\begin{aligned}
&\mathcal{A} + o(1) = \\
&2 \sum_{\frac{\log^2 X}{X^{\frac{1}{6}}} \leq e^k < \log^3 X} \sum_{\substack{d \equiv 2 \pmod 4 \\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) \sum_{\substack{(\mathfrak{a}, \overline{\mathfrak{a}}) \in \mathcal{P}(-d)^2 \\ \mathfrak{a} \neq (1), [\mathfrak{a}]^3 = [(1)]}} \Psi\left(\frac{N\mathfrak{a}}{\sqrt{d}}\right) \sigma^{\times}\left(\frac{N\mathfrak{a}}{\sqrt{d}} e^{-k}\right) \\
&\quad + 2 \sum_{\substack{d \equiv 2 \pmod 4 \\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) \sum_{\substack{(\mathfrak{a}, \overline{\mathfrak{a}}) \in \mathcal{P}(-d)^2 \\ \mathfrak{a} \neq (1), [\mathfrak{a}]^3 = [(1)]}} \left(\Psi\left(\frac{N\mathfrak{a}}{\sqrt{d}}\right) \sum_{e^k < \frac{\log^2 X}{X^{\frac{1}{6}}}} \sigma^{\times}\left(\frac{N\mathfrak{a}}{\sqrt{d}} e^{-k}\right) \right) \\
&= o(1) + \left(\sum_{\frac{\log^2 X}{X^{\frac{1}{6}}} \leq e^k < \log^3 X} \mathcal{A}_k \right) + \mathcal{A}_\infty,
\end{aligned} \tag{8}$$

so that $\mathcal{A}_\infty$ collects together all the points near the cusp.[2]

In the case of $\mathcal{A}_\infty$, set $Y_\infty = \frac{X^{\frac{1}{6}}}{\log^2 X}$ and define function ψ_∞ by

$$\psi_\infty(z) = \Psi\left(\frac{1}{z^{\frac{1}{2}} Y}\right) \sum_{e^k < \frac{1}{Y}} \sigma^{\times}\left(\frac{1}{z^{\frac{1}{2}} Y e^k}\right). \tag{9}$$

---

[2]Notice that primitive 3-torsion ideal $\mathfrak{a}$ satisfies

$$\mathfrak{a}^3 = (x + y\sqrt{-d}),\ y \neq 0 \qquad \Rightarrow \qquad N\mathfrak{a}^3 = x^3 + dy^2 \geq d,$$

so that there are no Heegner points with imaginary part $\gg d^{\frac{1}{6}}$.

In the case of $\mathcal{A}_k$, $\frac{\log^2 X}{X^{\frac{1}{6}}} \le e^k < \log^3 X$, set $Y_k = e^{-k}$ and define function ψ_k by

$$\psi_k(z) = \Psi\left(\frac{1}{z^{\frac{1}{2}} Y}\right) \sigma^{\times}\left(\frac{1}{z^{\frac{1}{2}}}\right). \tag{10}$$

Thus Y_k is the size of the imaginary part of Heegner points captured by $\mathcal{A}_k$. Note that ψ_k and ψ_∞ satisfy support condition (3) and also the derivative bound

$$\phi_k^{(j)} \ll_{j,\epsilon} X^{\epsilon}.$$

For k either finite or infinite we obtain,

$$\mathcal{A}_k = 2 \sum_{\substack{d \equiv 2 \quad (4) \\ \square-\text{free}}} \phi\left(\frac{d}{X}\right) \sum_{\substack{(\mathfrak{a}, \overline{\mathfrak{a}}) \in \mathcal{P}(-d)^2 \\ \mathfrak{a} \neq (1), [\mathfrak{a}]^3 = [(1)]}} \psi_k\left(\frac{d}{Y^2 N \mathfrak{a}^2}\right).$$

5.1 Global parametrization

Solving

$$d = \frac{\ell m^3 - \ell^2 n^2}{t^2}$$

in the parametrization of Proposition 3.2 gives

$$\mathcal{A}_k = 2 \sum_{\substack{(\ell,m,n,t) \in (\mathbb{Z}^+)^4 \\ (\ell m n, t) = 1 \\ d = \frac{\ell m^3 - \ell^2 n^2}{t^2} \text{ satisfies} \\ d \equiv 2 \quad (4), \square\text{-free}}} \phi\left(\frac{\ell m^3 - \ell^2 n^2}{t^2 X}\right) \psi_k\left(\frac{\ell m^3 - \ell^2 n^2}{Y^2 \ell^2 m^2 t^2}\right)$$

$$= 2 \sum_{\substack{(\ell,m,n,t) \in (\mathbb{Z}^+)^4 \\ \ell\ \square\text{-free}, m \text{ odd} \\ (\ell m n, t) = 1 \\ \ell m^3 - \ell^2 n^2 \equiv 2t^2 \quad (4t^2)}} \Phi_{X,Y,\ell t}(\ell m, \ell^2 n | \psi_k) \times \left(\sum_{\substack{s^2 | \frac{\ell m^3 - \ell^2 n^2}{t^2} \\ s < Z}} + \sum_{\substack{s^2 | \frac{\ell m^3 - \ell^2 n^2}{t^2} \\ s \ge Z}} \right) \mu(s)$$

Note that the support of ϕ and ψ_k allow us to truncate the sums to the ranges $\ell m \ll \frac{\sqrt{X}}{Y}$ and $\ell t \ll \frac{X^{\frac{1}{4}}}{Y^{\frac{3}{2}}}$.

The sieving parameter Z is at our disposal; we choose $Z = X^{\frac{3}{16}}$. By the argument in Proposition 8.1, the sum over large s contributes $\ll \frac{X^{1+\epsilon}}{YZ} + X^{\frac{3}{4}+\epsilon} Y^{\frac{-5}{2}}$, and therefore these terms contribute to $\mathcal{A}$ an error of size $\ll X^{\frac{13}{16}+\epsilon}$.

The main term may be arranged as

$$\sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ (\ell,t)=1\\ \ell\ \square\text{-free}}} \sum_{\substack{s=s_1s_2s_3s_4<Z,\text{ odd}\\ s_1|\ell, s_2|t\\ (s_3s_4,\ell t)=1}} \mu(s) \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (mn,s_4t)=1\\ s_3|(m,n)\\ m\text{ odd}\\ \ell m^3-\ell^2n^2\equiv 2s^2t^2 \quad (4s^2t^2)}} \Phi_{\ell t}(\ell m,\ell^2 n)$$

$$= \sum_{\substack{s=s_1s_2s_3s_4<Z\\ \text{odd}}} \mu(s) \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ \ell\ \square\text{-free}\\ (\ell,st)=(t,s_1s_3s_4)=1}} \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (mn,s_2s_4t)=1\\ m\text{ odd}\\ \ell s_1^2s_3m^3-\ell^2n^2\\ \equiv 2s_2^4s_4^2t^2 \quad (4s_2^4s_4^2t^2)}} \Phi_{s_1s_2\ell t}(s_1^2s_3\ell m, \ell^2s_1^2s_3n).$$

When $\ell = 2\ell'$ is even, the condition at 2 is guaranteed, and we obtain

$$\begin{aligned} \mathcal{A}_{k,e} &= \sum_{\substack{s<Z\\ \text{odd}}} \mu(s) \sum_{\substack{(\ell',t)\in(\mathbb{Z}^+)^2\\ 2\ell'\ \square\text{-free}\\ (2\ell',st)=(t,s_1s_3s_4)=1}} \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (mn,s_2s_4t)=1\\ m\text{ odd}\\ (2s_1^2s_3\ell'm)^3\equiv(4s_1^2s_3\ell'^2n)^2\\ \text{mod } s_2^4s_4^2t^2}} \Phi(\cdot,\cdot) \\ &= \sum_{\substack{s<Z\\ \text{odd}}} \mu(s) \sum_{\substack{(\ell',t)\in(\mathbb{Z}^+)^2\\ 2\ell'\ \square\text{-free}\\ (2\ell',st)=(t,s_1s_3s_4)=1}} \mathcal{A}_{k,e,s,\ell',t}. \end{aligned} \tag{11}$$

When ℓ is odd, we have simply

$$\begin{aligned} \mathcal{A}_{k,o} &= \sum_{\substack{s<Z\\ \text{odd}}} \mu(s) \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ \ell\ \square\text{-free, odd}\\ (\ell,st)=(t,s_1s_3s_4)=1}} \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (mn,s_2s_4t)=1\\ m\text{ odd}\\ (s_1^2s_3\ell m)^3-(s_1^2s_3\ell^2n)^2\\ \equiv 2s_2^4s_4^2t^2 \quad (4s_2^4s_4^2t^2)}} \Phi(\cdot,\cdot) \\ &= \sum_{\substack{s<Z\\ \text{odd}}} \mu(s) \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ \ell\ \square\text{-free, odd}\\ (\ell,st)=(t,s_1s_3s_4)=1}} \mathcal{A}_{k,o,s,\ell,t}. \end{aligned}$$

We treat $\mathcal{A}_{k,e}$ in detail. The term $\mathcal{A}_{k,o}$ may be handled similarly.

5.2 Local parametrization

By Proposition 3.3, we may parametrize the sum over (m,n) in $\mathcal{A}_{k,e}$ by setting

$$\begin{aligned} 2s_1^2s_3\ell m &= (4\ell s_1s_3w)^2 + (2a+1)\cdot 2\ell s_1^2s_3\cdot s_2^2s_4t \\ 4s_1^2s_3\ell^2 n &= (4\ell s_1s_3w)^3 + (2a+1)\cdot 12\ell^2s_1^3s_3^2w\cdot s_2^2s_4t + b\cdot 4\ell^2s_1^2s_3\cdot s_2^4s_4^2t^2, \\ &\qquad a,b\in\mathbb{Z}, w\in\left(\mathbb{Z}/s_2^2s_4t\mathbb{Z}\right)^\times. \end{aligned}$$

Thus we express

$$\mathcal{A}_* = \sum_{\substack{w\in\mathbb{Z}\\(w,s_2s_4t)=1}} \sigma\left(\frac{w}{s_2^2s_4t}\right) \sum_{(a,b)\in\mathbb{Z}^2} \Phi\Big((4\ell s_1s_3w)^2 + (2a+1)\cdot 2\ell s_1^2s_3\cdot s_2^2s_4t,$$
$$(4\ell s_1s_3w)^3 + (2a+1)\cdot 12\ell^2s_1^3s_3^2w\cdot s_2^2s_4t + b\cdot 4\ell^2s_1^2s_3\cdot s_2^4s_4^2t^2\Big).$$

Recall that σ is the partition of unity function satisfying $\sum_{n\in\mathbb{Z}}\sigma(x+n)\equiv 1$; also, here we shorten $\mathcal{A}_{k,e,s,\ell,t} := \mathcal{A}_*$, $\Phi_{X,Y,2s_1s_2\ell t} := \Phi$.

The advantage in working this way, isolating m and n modulo t, is that solutions $m^3\equiv n^2 \bmod t$ lift 'linearly' to solutions modulo t^2, which works well with the Fourier transform. Applying Poisson summation in the variables a and b, and evaluating the Fourier transform by applying Lemma 2.1 with

$$A = (4\ell s_1s_3w)^2 + 2\ell s_1^2s_3\cdot s_2^2s_4t,\quad B = 4\ell s_1^2s_3\cdot s_2^2s_4t,$$
$$C = (4\ell s_1s_3w)^3 + 12\ell^2s_1^3s_3^2w\cdot s_2^2s_4t$$
$$D = 24\ell^2s_1^3s_3^2w\cdot s_2^2s_4t,\quad E = 4\ell^2s_1^2s_3\cdot s_2^4s_4^2t^2,$$

we arrive at

$$\begin{aligned}\mathcal{A}_* &= \frac{1}{16s_1^4s_2^6s_3^2s_4^3\ell^3t^3}\sum_{\substack{(u,v,w)\in\mathbb{Z}^3\\(w,s_2s_4t)=1}}\sigma\left(\frac{w}{s_2^2s_4t}\right)e\left(\frac{u}{2}+\frac{4\ell s_3w^2u}{s_2^2s_4t}-\frac{8s_1s_3^2\ell w^3v}{s_2^4s_4^2t^2}\right)\\ &\quad\times\Phi^{1,2}\left(\frac{u}{4s_1^2s_2^2s_3s_4\ell t}-\frac{3wv}{2s_1s_2^4s_4^2\ell t^2},\frac{v}{4s_1^2s_2^4s_3s_4^2\ell^2t^2}\right)\\ &= \frac{X^{\frac{5}{6}}}{2^{\frac{7}{3}}s_1^{\frac{7}{3}}s_2^{\frac{13}{3}}s_3^2s_4^3\ell^{\frac{4}{3}}t^{\frac{4}{3}}}\sum_{\substack{(u,v.w)\in\mathbb{Z}^3\\(w,s_2s_4t)=1}}\sigma\left(\frac{w}{s_2^2s_4t}\right)e\left(\frac{u}{2}+\frac{4\ell s_3w^2u}{s_2^2s_4t}-\frac{8s_1s_3^2\ell w^3v}{s_2^4s_4^2t^2}\right)\\ &\quad\times\Phi_M^{1,2}\left((2s_1s_2\ell t)^{\frac{2}{3}}X^{\frac{1}{3}}\left[\frac{u}{4s_1^2s_2^2s_3s_4\ell t}-\frac{3wv}{2s_1s_2^4s_4^2\ell t^2}\right],\frac{X^{\frac{1}{2}}v}{2s_1s_2^3s_3s_4^2\ell t}\right),\end{aligned}\tag{12}$$

with $M = \frac{X^{\frac{1}{3}}}{(2s_1s_2\ell t)^{\frac{4}{3}}Y^2}$.

5.3 Outline-proof of Theorem.

Recall that we've written our count for Heegner points in the Theorem as

$$\mathcal{A} = \sum_k(\mathcal{A}_{k,e}+\mathcal{A}_{k,o}) + (\text{sieving error term})$$

where $\mathcal{A}_{k,*}$ corresponds to Heegner points with imaginary part $\asymp e^k$ (see (8)) and e,o are determined by a local condition at 2. Further, we've written ((11))

$$\mathcal{A}_{k,e} = \sum_{s<Z}\mu(s)\sum_{\substack{\ell,t\\\cdots}}\mathcal{A}_{k,e,s,\ell,t}$$

with $\mathcal{A}_{k,e,s,\ell,t}$ given by (12). We now decompose $\mathcal{A}_{k,e,s,\ell,t}$ further as

$$\mathcal{A}_{k,e,s,\ell,t} = \Delta_{k,e,s,\ell,t} + E^1_{k,e,s,\ell,t} + E^2_{k,e,s,\ell,t}$$

by splitting the sum over u, v above as

$$\sum_{(u,v)\in\mathbb{Z}^2} (...) = \left\{ \sum_{(u,v)=(0,0)} + \sum_{(u,0):u\neq 0} + \sum_{(u,v):v\neq 0} \right\} (...).$$

In the next section we prove that the aggregated contributions of the volume elements Δ, given by

$$\sum_{\substack{e^k \in \left[\frac{\log^2 X}{X^{\frac{5}{6}}}, \log^3 X\right] \\ \text{or } k=\infty}} \Delta_{k,e} + \qquad \text{(corresponding odd term);} \tag{13}$$

$$\Delta_{k,e} = \sum_{\substack{s=s_1s_2s_3s_4<Z \\ \text{odd}}} \mu(s) \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2 \\ \ell\ \square\text{-free} \\ (\ell,2st)=(t,2s_1s_3s_4)=1}} \Delta_{k,e,s,\ell,t},$$

produce the two main terms of the theorem, with an error of size

$$O(X^{1+\epsilon}Z^{-1} + X^{\frac{3}{4}+\epsilon}).$$

In the following section we bound the errors introduced by E_1 and E_2. The bounds that we prove are as follows.

Proposition 5.1. *We have the bounds*

$$E^1_{k,e} = \sum_{\substack{s=s_1s_2s_3s_4<Z \\ odd}} \mu(s) \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2 \\ \ell\ \square\text{-}free \\ (\ell,2st)=(t,2s_1s_3s_4)=1}} E^1_{k,e,s,\ell,t} = O(X^{\frac{3}{4}+\epsilon}Y^{\frac{-3}{2}}) \tag{14}$$

and

$$E^2_{k,e} = \sum_{\substack{s=s_1s_2s_3s_4<Z \\ odd}} \mu(s) \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2 \\ \ell\ \square\text{-}free \\ (\ell,2st)=(t,2s_1s_3s_4)=1}} E^2_{k,e,s,\ell,t} = O(X^{\frac{13}{16}+\epsilon}). \tag{15}$$

Substituting these bounds in expression (8) we obtain the theorem.

6 Evaluation of main Δ terms

As a first step, we remove the restriction $s < Z$ in $\Delta_{k,e}$ writing (here $M := \frac{X^{\frac{1}{3}}}{(2s_1s_2\ell t)^{\frac{4}{3}}Y^2}$)

$$\Delta_{k,e} = \sum_{\substack{s=s_1s_2s_3s_4 \\ \text{odd}}} \mu(s) \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2 \\ \ell\ \square\text{-free} \\ (\ell,2st)=(t,2s_1s_3s_4)=1}} \frac{X^{\frac{5}{6}}\varphi(s_2^2s_4t)}{2^{\frac{7}{3}}s_1^{\frac{7}{3}}s_2^{\frac{13}{3}}s_3^2s_4^3\ell^{\frac{4}{3}}t^{\frac{4}{3}}}\Phi_M^{1,2}(0,0|\psi_k)$$
$$+ O\left(X^{\frac{3}{4}+\epsilon}Y^{\frac{1}{2}} \sum_{s_1s_2s_3s_4>Z} \frac{1}{s^2} \sum_{s_1s_2\ell t \ll X^{\frac{1}{4}}Y^{\frac{-3}{2}}} \frac{1}{\ell}\right)$$
$$= \Delta_{k,e}^0 + O\left(\frac{X^{1+\epsilon}}{YZ}\right).$$

In the error term we have used $\Phi_M(0,0) \ll X^\epsilon M^{\frac{-1}{4}}$; this follows from Lemma 2.2 – for $k = \infty$ we use that $M \ll X^\epsilon$.

Now let $\Psi_{-1/2}(z) = \Psi(z^{\frac{-1}{2}})$, and define truncated function $\Psi_{-1/2}^0$ on $\mathbb{R}^+$ by

$$\Psi_{-1/2}^0(z) = \Psi_{-1/2}(z) \sum_{e^k<\log^3 X} \sigma^\times\left(z^{\frac{-1}{2}}e^{-k}\right).$$

Note that $\Psi_{-1/2}^0$ satisfies the conditions on ψ in Lemma 2.4. We appeal to the following simple lemma.

Lemma 6.1. *For arbitrary $M > 0$ we have the equality*

$$\sum_{\substack{e^k\in\left[\frac{\log^2 X}{X^{\frac{1}{6}}},\log^3 X\right] \\ or\ k=\infty}} \Phi_{M/Y_k^2}^{1,2}(0,0|\psi_k) = \Phi_M^{1,2}(0,0|\Psi_{-1/2}).$$

Proof. From the definitions (9) and (10), the left hand side is

$$\int_{\mathbb{R}^2} \phi(x^3-y^2)\left[\sum_{\substack{e^k\in\left[\frac{\log^2 X}{X^{\frac{1}{6}}},\log^3 X\right] \\ \text{or } k=\infty}} \psi_k\left((x^3-y^2)\frac{M}{Y_k^2x^2}\right)\right]dx = \Phi_M^{1,2}(0,0|\Psi_{-1/2}).$$

$\square$

Thus we may write (here $M := \frac{X^{\frac13}}{(2s_1s_2\ell t)^{\frac43}}$)

$$\sum_{\substack{e^k \in \left[\frac{\log^2 X}{X^{\frac16}}, \log^3 X\right] \\ \text{or } k=\infty}} \Delta_{k,e} = O\left(\frac{X^{1+\epsilon}}{Z}\right) + \frac{X^{\frac56}}{2^{\frac73}} \sum_{\substack{s=s_1s_2s_3s_4\\ \text{odd}}} \frac{\mu(s)}{s_1^{\frac73}s_2^{\frac{13}{3}}s_3^2s_4^3}$$

$$\times \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ \ell\ \square\text{-free}\\ (\ell,2st)=(t,2s_1s_3s_4)=1}} \frac{\varphi(s_2^2s_4t)}{\ell^{\frac43}t^{\frac43}} \Phi_M^{1,2}(0,0|\Psi^0_{-1/2}).$$

Introducing the Mellin transform of function $H(z) = \Phi^{1,2}_{z^{-1}}(0,0)$, the main term becomes

$$\frac{1}{2\pi i}\int_{(2)} \tilde{H}(\alpha)\frac{X^{\frac56+\frac{\alpha}{3}}}{2^{\frac73+\frac{4\alpha}{3}}Y^{2\alpha}}$$

$$\times \sum_{\substack{s=s_1s_2s_3s_4\\ \text{odd}}} \frac{\mu(s)}{s_1^{\frac73+\frac{4\alpha}{3}}s_2^{\frac{13}{3}+\frac{4\alpha}{3}}s_3^2s_4^3} \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ \ell\ \square\text{-free}\\ (\ell,2st)=(t,2s_1s_3s_4)=1}} \frac{\varphi(s_2^2s_4t)}{\ell^{\frac43+\frac{4\alpha}{3}}t^{\frac43+\frac{4\alpha}{3}}}d\alpha$$

$$= \frac{1}{2\pi i}\int_{(2)} \frac13 \frac{\Gamma\left(\frac12\right)\Gamma\left(\frac16+\frac23\alpha\right)}{\Gamma\left(\frac23+\frac23\alpha\right)}\tilde{\phi}\left(\frac56+\frac{\alpha}{3}\right)\tilde{\Psi}^0_{-1/2}(-\alpha)\frac{X^{\frac56+\frac{\alpha}{3}}}{2^{\frac73+\frac{4\alpha}{3}}}F(\alpha)d\alpha,$$

where

$$F(\alpha) = \sum_{\substack{s=s_1s_2s_3s_4\\ \text{odd}}} \frac{\mu(s)}{s_1^{\frac73+\frac{4\alpha}{3}}s_2^{\frac{13}{3}+\frac{4\alpha}{3}}s_3^2s_4^3} \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ \ell\ \square\text{-free}\\ (\ell,2st)=(t,2s_1s_3s_4)=1}} \frac{\varphi(s_2^2s_4t)}{\ell^{\frac43+\frac{4\alpha}{3}}t^{\frac43+\frac{4\alpha}{3}}}.$$

The function F is most conveniently represented as

$$F(\alpha) = \zeta\left(\frac13+\frac{4\alpha}{3}\right)G(\alpha);$$

$$G(\alpha) = \left[1-\frac{1}{2^{\frac13+\frac{4\alpha}{3}}}\right]\prod_{p \text{ odd}}\left(1-\frac{2}{p^2}-\frac{1}{p^{\frac53+\frac{8\alpha}{3}}}+\frac{1}{p^3}+\frac{1}{p^{\frac83+\frac{8\alpha}{3}}}\right).$$

$G(\alpha)$ is readily seen to be absolutely convergent in $\Re(\alpha) > -\frac14$. Thus shifting the contour to $\Re(\alpha) = -\frac14+\epsilon$ we pass simple poles at $\frac12$, from the ζ factor, and at 0, from $\tilde{\Psi}^0_{-1/2}$, with a remaining integral of size $\ll X^{\frac34+\epsilon}$.

Now

$$G\left(\frac12\right) = \frac12\prod_{p\text{ odd}}\left(1-\frac{1}{p^2}\right)^2 = \frac{32}{\pi^4}$$

while

$$o(X^{-1}) + \tilde{\Psi}^0_{-1/2}\left(\frac{-1}{2}\right) = \tilde{\Psi}_{-1/2}\left(\frac{-1}{2}\right) = 2\tilde{\Psi}(1) = \frac{2\pi}{3}.$$

Thus the residue at $\frac{1}{2}$ is $\frac{2}{3\pi^2}\tilde{\phi}(1)X + o(1)$.

Meanwhile $G(0)$ evaluates to

$$\left[1 - \frac{1}{2^{\frac{1}{3}}}\right] \prod_{p \text{ odd}} \left[1 - \frac{1}{p^{\frac{5}{3}}} - \frac{2}{p^2} + \frac{1}{p^{\frac{8}{3}}} + \frac{1}{p^3}\right] = \frac{8}{\pi^2}\left[1 - \frac{1}{2^{\frac{1}{3}}}\right] \prod_{p \text{ odd}} \left[1 - \frac{p^{\frac{1}{3}} + 1}{p^2 + p}\right]$$

and the residue from $\tilde{\Psi}^0_{-1/2}(-\alpha)$ at $\alpha = 0$ is 1. Thus the residue term at 0 contributes

$$\tilde{\phi}\left(\frac{5}{6}\right) X^{\frac{5}{6}} \frac{\Gamma\left(\frac{1}{6}\right)}{3\pi^{\frac{3}{2}}\Gamma\left(\frac{2}{3}\right)} \zeta\left(\frac{1}{3}\right) \left[2^{\frac{2}{3}} - 2^{\frac{1}{3}}\right] \prod_{p \text{ odd}} \left[1 - \frac{p^{\frac{1}{3}} + 1}{p^2 + p}\right].$$

The analysis of the terms coming from the odd terms $\Delta_{k,o}$ is entirely analogous; one proves that this yields

$$\frac{4}{3\pi^2}\tilde{\phi}(1)X + \tilde{\phi}\left(\frac{5}{6}\right) X^{\frac{5}{6}} \frac{\Gamma\left(\frac{1}{6}\right)}{3\pi^{\frac{3}{2}}\Gamma\left(\frac{2}{3}\right)} \zeta\left(\frac{1}{3}\right) \prod_{p \text{ odd}} \left[1 - \frac{p^{\frac{1}{3}} + 1}{p^2 + p}\right],$$

again with error $\ll \frac{X^{1+\epsilon}}{Z} + X^{\frac{3}{4}+\epsilon}$. This completes the evaluation of the main terms.

7 Bounding the contributions of E_1 and E_2

By rapid decay of $\Phi_M^{1,2}$ in the first slot, we may truncate the sum over u, w by

$$\left| w - \frac{u}{6v}\frac{s_2^2 s_4 t}{s_1 s_3} \right| \ll \frac{X^\epsilon s_2^2 s_4^2}{Y^2 s_1 \ell v}, \qquad v \neq 0$$
$$|u| \ll \frac{X^\epsilon s_3 s_4}{Y^2 \ell t}, \qquad v = 0,$$

with negligible error. By rapid decay of $\Phi_M^{1,2}$ in the second slot, the sum over v may be truncated at

$$|v| \ll \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}},$$

again with negligible error. The range of u is thus bounded by

$$|u| \ll X^\epsilon \left(s_1 s_3 |v| + \frac{s_3 s_4}{Y^2 \ell t} \right).$$

Note that in particular, for $Y \gg \frac{Z^{\frac{4}{3}}}{X^{\frac{1}{6}-\epsilon}}$ the error term E_2 may be disregarded entirely.

7.1 Off-diagonal terms with $v = 0$

We first consider the simpler case of $E^1_{k,e}$, where $v = 0, u \neq 0$. Write $E^1(s, \ell, t)$ for the sum over u and w in $E^1_{k,e,s,\ell,t}$, that is, ($M := \frac{X^{\frac{1}{3}}}{(2s_1s_2\ell t)^{\frac{4}{3}}Y^2}$)

$$E^1(s,\ell,t) = \sum_{0\neq|u|\ll X^{\epsilon}\frac{s_3s_4}{Y^2\ell t}} \left[\sum_{\substack{w\in\mathbb{Z}\\(w,s_2s_4t)=1}} \sigma\left(\frac{w}{s_2^2s_4t}\right) e\left(\frac{u}{2} + \frac{4\ell s_3 w^2 u}{s_2^2 s_4 t}\right)\right] \times \Phi^{1,2}_M\left(\frac{X^{\frac{1}{3}}u}{2^{\frac{4}{3}}s_1^{\frac{4}{3}}s_2^{\frac{4}{3}}s_3s_4\ell^{\frac{2}{3}}t^{\frac{2}{3}}}, 0\right).$$

By a familiar argument, the inner sum over w is bounded by

$$\ll X^{\epsilon}(u, s_2^2s_4t)^{\frac{1}{2}}(s_2^2s_4t)^{\frac{1}{2}},$$

and therefore, (recall $\Phi^{1,2}_M \ll M^{-\frac{1}{4}}$ for $k < \infty$ and $M \ll X^{\epsilon}$ for $k = \infty$), we obtain

$$E^1(s,\ell,t) \ll \frac{s_1^{\frac{1}{3}}s_2^{\frac{4}{3}}s_3s_4^{\frac{3}{2}}}{Y^{\frac{3}{2}}X^{\frac{1}{12}-\epsilon}\ell^{\frac{2}{3}}t^{\frac{1}{6}}}.$$

Altogether, this yields the bound

$$E^1_{k,e} \ll X^{\frac{5}{6}} \sum_{\substack{s<Z\\ \text{odd}}} \frac{1}{s_1^{\frac{7}{3}}s_2^{\frac{13}{3}}s_3^2s_4^3} \sum_{\substack{(\ell,t)\in(\mathbb{Z}^+)^2\\ \ell\ \square\text{-free}\\ (\ell,2st)=(t,2s_1s_3s_4)=1}} \frac{|E_{1,k}(s,\ell,t)|}{\ell^{\frac{4}{3}}t^{\frac{4}{3}}} \ll X^{\frac{3}{4}+\epsilon}Y^{\frac{-3}{2}},$$

proving (14) of Proposition 5.1.

7.2 Off-diagonal terms with $v \neq 0$

We now turn to the more serious work of bounding the off-diagonal term $E^2_{k,e}$. As we have already remarked, the term E^2 enters only in the range

$$Y \ll \frac{Z^{\frac{4}{3}}}{X^{\frac{1}{6}-\epsilon}}$$

and so we may suppose that $k < \infty$ (recall we choose $Z = X^{\frac{3}{16}}$ while $Y_\infty > X^{\frac{1}{6}-\epsilon}$).

Performing Möbius inversion to remove the co-primality condition on w, the sums over u, v and w in $E^2_{k,e,s,\ell,t}$ are given by ($M := \frac{X^{\frac{1}{3}}}{(2s_1s_2\ell t)^{\frac{4}{3}}Y^2}$)

$$\begin{aligned}
&\sum_{d|s_2s_4t}\mu(d)\sum_{\substack{|v|\ll\frac{s_2^2s_3s_4^2}{X^{\frac{1}{4}-\epsilon}Y^{\frac{3}{2}}}\\ v\neq 0}}\ \sum_{|u|\ll X^{\epsilon}\left(|v|s_1s_3+\frac{s_3s_4}{Y^2\ell t}\right)}\\
&\quad\times\sum_{\substack{w\\ \left|dw-\frac{u}{6v}\frac{s_2^2s_4t}{s_1s_3}\right|\ll\frac{X^{\epsilon}s_2^2s_4^2}{Y^2s_1\ell v}}}\sigma\left(\frac{dw}{s_2^2s_4t}\right)e\left(\frac{4\ell s_3d^2u}{s_2^2s_4t}w^2-\frac{8\ell s_1s_3^2d^3v}{s_2^4s_4^2t^2}w^3\right)\\
&\quad\times\Phi_M^{1,2}\left((2s_1s_2\ell t)^{\frac{2}{3}}X^{\frac{1}{3}}\left[\frac{u}{4s_1^2s_2^2s_3s_4\ell t}-\frac{3dwv}{2s_1s_2^3s_4^2\ell t^2}\right],\frac{X^{\frac{1}{2}}v}{2s_1s_2^3s_3s_4^2\ell t}\right)\\
&=\sum_{d|s_2s_4t}\mu(d)\sum_{\substack{|v|\ll\frac{s_2^2s_3s_4^2}{X^{\frac{1}{4}-\epsilon}Y^{\frac{3}{2}}}\\ v\neq 0}}B_{s,\ell,t,d,v}.
\end{aligned}\tag{16}$$

We are going to prove non-trivial bounds for double sum over u and w above. We prove two different bounds depending on the relative sizes of the other parameters.

Proposition 7.1. *In the range*

$$X^{-\epsilon}\min\left(\frac{s_2^2s_4t}{s_1d},\frac{s_2s_4}{Ys_1d^{\frac{1}{2}}\ell^{\frac{1}{2}}},\frac{Y^2s_2^2t^2}{s_1s_3^2d^2}\right)<v<\frac{s_2^2s_3s_4^2}{X^{\frac{1}{4}-\epsilon}Y^{\frac{3}{2}}}$$

we have the bound

$$\begin{aligned}
|B_{s,\ell,t,d,v}|\ll X^{\epsilon}\Bigg[&\frac{s_1^{\frac{5}{6}}s_2^{\frac{4}{3}}s_3s_4t^{\frac{1}{3}}|v|^{\frac{1}{2}}}{X^{\frac{1}{12}}Y^{\frac{1}{2}}d^{\frac{1}{2}}\ell^{\frac{1}{6}}}+\frac{s_2^{\frac{7}{3}}s_4^2}{X^{\frac{1}{12}}Y^{\frac{1}{2}}d^{\frac{1}{2}}s_1^{\frac{1}{6}}\ell^{\frac{7}{6}}t^{\frac{2}{3}}|v|^{\frac{1}{2}}}+\\
&+\frac{s_1^{\frac{4}{3}}s_2^{\frac{10}{3}}s_3s_4^2\ell^{\frac{1}{3}}t^{\frac{4}{3}}}{X^{\frac{1}{3}}d}+\frac{s_1^{\frac{1}{3}}s_2^{\frac{13}{3}}s_4^3t^{\frac{1}{3}}}{X^{\frac{1}{3}}d\ell^{\frac{1}{6}}|v|}\Bigg].
\end{aligned}\tag{17}$$

In the complementary range

$$v<X^{-\epsilon}\left(\frac{s_2^2s_4t}{s_1d},\frac{s_2s_4}{Ys_1d^{\frac{1}{2}}\ell^{\frac{1}{2}}},\frac{Y^2s_2^2t^2}{s_1s_3^2d^2},\frac{s_2^2s_3s_4^2}{X^{\frac{1}{4}}Y^{\frac{3}{2}}}\right)$$

we have the bound

$$|B_{s,\ell,t,d,v}|\ll\frac{Y^{\frac{1}{2}}(s_1s_2\ell t)^{\frac{1}{3}}}{X^{\frac{1}{12}-\epsilon}}\frac{s_2^2s_3s_4^2}{Y^2d\ell}\left(\frac{(\ell d,s_1v)}{s_1v}\right)^{\frac{1}{2}}+\frac{X^{\epsilon}s_1s_3^2d^2v}{Y^2s_2^2t^2}\times|(17)|.\tag{18}$$

Assuming this Proposition for the moment, we complete our proof of the bound for $E^2_{k,e}$ given in (15) of Proposition 5.1.

Deduction of (15) of Proposition 5.1. Recall we take $Z = X^{\frac{3}{16}}$. Set

$$L = \min\left(\frac{s_2^2 s_4 t}{s_1 d}, \frac{s_2 s_4}{Y s_1 d^{\frac{1}{2}} \ell^{\frac{1}{2}}}, \frac{Y^2 s_2^2 t^2}{s_1 s_3^2 d^2}\right).$$

Evidently

$$\begin{aligned}
|(15)| \ll & X^{\frac{5}{6}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^{\frac{7}{3}} s_2^{\frac{13}{3}} s_3^2 s_4^3} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell^{\frac{4}{3}} t^{\frac{4}{3}}} \sum_{d | s_2 s_4 t} \sum_{\substack{X^{-\epsilon} L < |v| \\ < \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}}}} |(17)| \\
& + X^{\frac{5}{6}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^{\frac{7}{3}} s_2^{\frac{13}{3}} s_3^2 s_4^3} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell^{\frac{4}{3}} t^{\frac{4}{3}}} \sum_{d | s_2 s_4 t} \\
& \sum_{\substack{0 \neq |v| \\ < \min\left(L, \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}} Y^{\frac{3}{2}}}\right)}} \left(\frac{X^{\epsilon} s_1 s_3^2 d^2 |v|}{Y^2 s_2^2 t^2} \times |(17)|\right) \\
& + X^{\frac{5}{6}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^{\frac{7}{3}} s_2^{\frac{13}{3}} s_3^2 s_4^3} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell^{\frac{4}{3}} t^{\frac{4}{3}}} \sum_{d | s_2 s_4 t} \\
& \sum_{\substack{0 \neq |v| \\ < \min\left(L, \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}} Y^{\frac{3}{2}}}\right)}} \frac{Y^{\frac{1}{2}} (s_1 s_2 \ell t)^{\frac{1}{3}}}{X^{\frac{1}{12}-\epsilon}} \frac{s_2^2 s_3 s_4^2}{Y^2 d \ell} \left(\frac{(\ell d, s_1 |v|)}{s_1 |v|}\right)^{\frac{1}{2}}
\end{aligned}$$

Write these terms as $A + B + C$.

The term A itself contains four terms. The first is

$$\begin{aligned}
&\ll \frac{X^{\frac{3}{4}+\epsilon}}{Y^{\frac{1}{2}}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^{\frac{3}{2}} s_2^3 s_3 s_4^2} \sum_{\ell t \ll X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell^{\frac{3}{2}} t} \sum_{d | s_2 s_4 t} \frac{1}{d^{\frac{1}{2}}} \sum_{|v| < \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}}} |v|^{\frac{1}{2}} \\
&\ll \frac{X^{\frac{3}{8}+\epsilon}}{Y^{\frac{11}{4}}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{s_3^{\frac{1}{2}} s_4}{s_1^{\frac{3}{2}}} \ll X^{\frac{3}{8}+\epsilon} Y^{\frac{-11}{4}} Z^2 \ll \frac{X^{\frac{3}{4}+\epsilon}}{Y^{\frac{11}{4}}}.
\end{aligned}$$

The second term is

$$\begin{aligned}
&\ll \frac{X^{\frac{3}{4}+\epsilon}}{Y^{\frac{1}{2}}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^{\frac{5}{2}} s_2^2 s_3^2 s_4} \sum_{\ell t \ll X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell^{\frac{5}{2}} t^2} \sum_{d | s_2 s_4 t} \frac{1}{d^{\frac{1}{2}}} \sum_{|v| < \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}}} |v|^{\frac{-1}{2}} \\
&\ll \frac{X^{\frac{5}{8}+\epsilon}}{Y^{\frac{5}{4}}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^{\frac{5}{2}} s_2 s_3^{\frac{3}{2}}} \ll \frac{X^{\frac{5}{8}+\epsilon} Z}{Y^{\frac{5}{4}}} \ll \frac{X^{\frac{13}{16}+\epsilon}}{Y^{\frac{5}{4}}}
\end{aligned}$$

The fourth term is

$$\ll X^{\frac{1}{2}+\epsilon} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^2 s_3^2} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell^{\frac{3}{2}} t} \sum_{d | s_2 s_4 t} \frac{1}{d} \sum_{X^{-\epsilon} L < |v| < \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}}} \frac{1}{|v|}$$

$$\ll X^{\frac{1}{2}+\epsilon} Z \ll X^{\frac{11}{16}+\epsilon}$$

The third term requires a little more care. For this term we note that the summation over v is empty unless

$$X^{-\epsilon} L < \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}} \qquad \Leftrightarrow \qquad d > L'$$

where

$$L' = \min\left(\frac{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}} t}{s_1 s_3 s_4}, \frac{X^{\frac{1}{2}-\epsilon} Y}{s_1^2 s_2^2 s_3^2 s_4^2 \ell}, \frac{X^{\frac{1}{8}-\epsilon} Y^{\frac{7}{4}} t}{s_1^{\frac{1}{2}} s_3^{\frac{3}{2}} s_4} \right).$$

Thus the third term is bounded by

$$\ll X^{\frac{1}{2}+\epsilon} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1 s_2 s_3 s_4} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell} \sum_{\substack{d | s_2 s_4 t \\ d > L'}} \frac{1}{d} \sum_{|v| < \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}}} 1$$

$$\ll \frac{X^{\frac{1}{4}+\epsilon}}{Y^{\frac{3}{2}}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{s_2 s_4}{s_1} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell} \left(\frac{s_1 s_3 s_4}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}} t} + \frac{s_1^2 s_2^2 s_3^2 s_4^2 \ell}{X^{\frac{1}{2}-\epsilon} Y} + \frac{s_1^{\frac{1}{2}} s_3^{\frac{3}{2}} s_4}{X^{\frac{1}{8}-\epsilon} Y^{\frac{7}{4}} t} \right)$$

$$\ll \frac{X^{\epsilon} Z^3}{Y^3} + \frac{X^{\epsilon} Z^4}{Y^4} + \frac{X^{\frac{1}{8}+\epsilon} Z^3}{Y^{\frac{13}{4}}} \ll \frac{X^{\frac{9}{16}+\epsilon}}{Y^3} + \frac{X^{\frac{3}{4}+\epsilon}}{Y^4} + \frac{X^{\frac{11}{16}+\epsilon}}{Y^{\frac{13}{4}}}.$$

This completes the bound for A.

The term B also contains four terms. Recall

$$L = \min\left(\frac{s_2^2 s_4 t}{s_1 d}, \frac{s_2 s_4}{Y s_1 d^{\frac{1}{2}} \ell^{\frac{1}{2}}}, \frac{Y^2 s_2^2 t^2}{s_1 s_3^2 d^2} \right).$$

The first, second and fourth terms are bounded by multiplying each term by

$$\frac{1}{|v|} \frac{Y^2 s_2^2 t^2}{s_1 s_3^2 d^2} > 1$$

then proceeding exactly as in the corresponding bounds for A (the sums are identical). The third term is

$$\ll X^{\frac{1}{2}+\epsilon} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{s_3}{s_2^3 s_4} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell t^2} \sum_{d | s_2 s_4 t} d \sum_{|v| < \min(L, \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}})} |v|$$

$$\ll X^{\frac{1}{2}+\epsilon} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{s_3}{s_2^3 s_4} \sum_{\ell t < X^{\frac{1}{4}} Y^{\frac{-3}{2}}} \frac{1}{\ell t^2} \sum_{d | s_2 s_4 t} d \times \left(\frac{Y^2 s_2^2 t^2}{s_1 s_3^2 d^2} \right)^{\frac{1}{2}} \left(\frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}-\epsilon} Y^{\frac{3}{2}}} \right)^{\frac{3}{2}}$$

$$\ll \frac{X^{\frac{1}{8}+\epsilon} Z^3}{Y^{\frac{5}{4}}} \ll \frac{X^{\frac{11}{16}+\epsilon}}{Y^{\frac{5}{4}}}$$

This completes the bound for B.

Finally, the term C is bounded by

$$\ll \frac{X^{\frac{3}{4}+\epsilon}}{Y^{\frac{3}{2}}} \sum_{s_1 s_2 s_3 s_4 < Z} \frac{1}{s_1^2 s_2^2 s_3 s_4} \sum_{\ell t < \frac{X^{\frac{1}{4}}}{Y^{\frac{3}{2}}}} \frac{1}{\ell^2 t} \sum_{d | s_2 s_4 t} \frac{1}{d} \sum_{0 \neq |v| < \frac{s_2^2 s_3 s_4^2}{X^{\frac{1}{4}} Y^{\frac{3}{2}}}} \left(\frac{(\ell d, s_1 |v|)}{s_1 |v|} \right)^{\frac{1}{2}}$$

$$\ll \frac{X^{\frac{5}{8}+\epsilon} Z}{Y^{\frac{9}{4}}} \ll \frac{X^{\frac{13}{16}+\epsilon}}{Y^{\frac{9}{4}}},$$

completing the proof. □

Proof of Proposition 7.1. In this proof we adopt the convention that, inside a sum $\sum_{x_1, x_2, \ldots}$, a quantity $\theta(y_1, y_2, \ldots)$ is a complex number of absolute value 1, that does not depend on any variable $x_1, x_2, \ldots$ in the summation, other than those among $y_1, y_2, \ldots$ [but may implicitly depend on other parameters in the problem which do not appear in the specific summation].

We may simplify the two sums in $B_{s,\ell,t,d,v}$ somewhat by replacing w with $w + W$ where

$$W = \frac{u}{6dv} \frac{s_2^2 s_4 t}{s_1 s_3}.$$

This yields

$$\begin{aligned} B_{s,\ell,t,d,v} = & \sum_{|u| \ll X^{\epsilon}\left(|v| s_1 s_3 + \frac{s_3 s_4}{Y^2 \ell t}\right)} e\left(\frac{2}{27} \frac{s_2^2 s_4 \ell t u^3}{s_1^2 s_3 v^2} \right) \\ & \times \sum_{\substack{w + W \in \mathbb{Z} \\ |w| \ll \frac{X^{\epsilon} s_2^2 s_4^2}{Y^2 d s_1 \ell v}}} \sigma\left(\frac{(w+W)d}{s_2^2 s_4 t} \right) e\left(\frac{2}{3} \frac{\ell d u^2}{s_1 v} w - \frac{8 \ell s_1 s_3^2 d^3 v}{s_2^4 s_4^2 t^2} w^3 \right) \\ & \times \Phi_M^{1,2}\left(\frac{3 X^{\frac{1}{3}} v d w}{2^{\frac{1}{3}} s_1^{\frac{1}{3}} s_2^{\frac{10}{3}} s_4^2 \ell^{\frac{1}{3}} t^{\frac{4}{3}}}, \frac{X^{\frac{1}{2}} v}{2 s_1 s_2^3 s_3 s_4^2 \ell t} \right). \end{aligned}$$

Further, set $w = h - \langle W \rangle$ with h the nearest integer to w. We obtain

$$\begin{aligned} \sum_{|u| \ll X^{\epsilon}\left(|v| s_1 s_3 + \frac{s_3 s_4}{Y^2 \ell t}\right)} \theta(u) & \sum_{\substack{h \in \mathbb{Z} \\ |h| \ll \frac{X^{\epsilon} s_2^2 s_4^2}{Y^2 d s_1 \ell v}}} \sigma\left(\frac{(h + W - \langle W \rangle) d}{s_2^2 s_4 t} \right) \\ & \times e\left(\frac{2}{3} \frac{\ell d u^2}{s_1 v} h - \frac{8 \ell s_1 s_3^2 d^3 v}{s_2^4 s_4^2 t^2} (h^3 - 3h^2 \langle W \rangle + 3h \langle W \rangle^2) \right) \\ & \times \Phi_M^{1,2}\left(\frac{3 X^{\frac{1}{3}} v d (h - \langle W \rangle)}{2^{\frac{1}{3}} s_1^{\frac{1}{3}} s_2^{\frac{10}{3}} s_4^2 \ell^{\frac{1}{3}} t^{\frac{4}{3}}}, \frac{X^{\frac{1}{2}} v}{2 s_1 s_2^3 s_3 s_4^2 \ell t} \right). \end{aligned} \tag{19}$$

Now in the range

$$v > X^{-\epsilon} \min\left(\frac{s_2^2 s_4 t}{s_1 d}, \frac{s_2 s_4}{Y s_1 d^{\frac{1}{2}} \ell^{\frac{1}{2}}}, \frac{Y^2 s_2^2 t^2}{s_1 s_3^2 d^2} \right)$$

we disregard cancellation in the sign and bound simply

$$|(19)| \ll \sum_{|u|\ll X^\epsilon(s_1s_3|v|+\frac{s_2s_4}{Y^2\ell t})} \left(\frac{X^\epsilon s_2^2 s_4^2}{Y^2 d s_1 \ell |v|}\right)^{\frac12}$$
$$\times \left[\sum_{h\ll \frac{X^\epsilon s_2^2 s_4^2}{Y^2 d s_1 \ell V}} \left|\Phi^{1,2}_{\frac{X^{\frac13}}{(2s_1s_2\ell t)^{\frac43} Y^2}}\left(\frac{3X^{\frac13} vd(h-\langle W\rangle)}{2^{\frac13}s_1^{\frac13}s_2^{\frac{10}{3}}s_4^2\ell^{\frac13}t^{\frac43}}, \frac{X^{\frac12}v}{2s_1s_2^3s_3s_4^2\ell t}\right)\right|^2\right]^{\frac12}$$

by applying Cauchy-Schwarz to the inner sum. By Lemma 2.3 the bracketed term contributes

$$\ll \frac{(s_1s_2\ell t)^{\frac13}Y^{\frac12}}{X^{\frac{1}{12}-\epsilon}}\left(1+\frac{s_1s_2^4s_4^2\ell t^2 Y}{X^{\frac12}dv}\right)^{\frac12}.$$

Thus we obtain the bound

$$|(19)| \ll X^\epsilon\left[\frac{s_1^{\frac56}s_2^{\frac43}s_3s_4t^{\frac12}|v|^{\frac12}}{X^{\frac{1}{12}}Y^{\frac12}d^{\frac12}\ell^{\frac16}} + \frac{s_2^{\frac73}s_4^2}{X^{\frac{1}{12}}Y^{\frac12}d^{\frac12}s_1^{\frac16}\ell^{\frac76}t^{\frac23}|v|^{\frac12}} + \right.$$
$$\left. + \frac{s_1^{\frac43}s_2^{\frac{10}{3}}s_3s_4^2\ell^{\frac13}t^{\frac43}}{X^{\frac13}d} + \frac{s_1^{\frac13}s_2^{\frac{13}{3}}s_4^3t^{\frac13}}{X^{\frac13}d\ell^{\frac16}|v|}\right].$$

This proves the bound (17).

When

$$v < X^{-\epsilon}\min\left(\frac{s_2^2s_4t}{s_1d}, \frac{s_2s_4}{Ys_1d^{\frac12}\ell^{\frac12}}, \frac{Y^2s_2^2t^2}{s_1s_3^2d^2}\right)$$

we split the sum over h in (19) in order to control the error in phase introduced by $\langle W\rangle$. Thus we let integer $H \asymp X^\epsilon s_1 v$ be a parameter and write the sum as

$$\sum_{k\ll \frac{X^\epsilon s_2^2s_4^2}{HY^2ds_1\ell v}} \sum_{|u|\ll X^\epsilon\left(|v|s_1s_3+\frac{s_3s_4}{Y^2\ell t}\right)} \theta(k,u)\sum_h \sigma\left(\frac{h}{H}\right)\sigma\left(\frac{(h+kH+W-\langle W\rangle)d}{s_2^2s_4t}\right)$$
$$\times e\left(\frac23\frac{\ell d u^2}{s_1 v}h - \frac{8\ell s_1s_3^2d^3v}{s_2^4s_4^2t^2}(h^3+3kHh^2+3(kH)^2h+O(hkH))\right)$$
$$\times \Phi_M^{1,2}\left(\frac{3X^{\frac13}vd(h+kH-\langle W\rangle)}{2^{\frac13}s_1^{\frac13}s_2^{\frac{10}{3}}s_4^2\ell^{\frac13}t^{\frac43}}, \frac{X^{\frac12}v}{2s_1s_2^3s_3s_4^2\ell t}\right).$$

In this sum, the error term contributes

$$\ll H\cdot\frac{X^\epsilon s_3^2d^2}{Y^2s_2^2t^2} \sum_{\substack{|u|\ll \\ X^\epsilon\left(|v|s_1s_3+\frac{s_3s_4}{Y^2\ell t}\right)}} \sum_{w+W\in\mathbb{Z}} \left|\Phi_M^{1,2}\left(\frac{3X^{\frac13}vdw}{2^{\frac13}s_1^{\frac13}s_2^{\frac{10}{3}}s_4^2\ell^{\frac13}t^{\frac43}}, \frac{X^{\frac12}v}{2s_1s_2^3s_3s_4^2\ell t}\right)\right|$$
$$\ll \frac{X^\epsilon s_1s_3^2d^2v}{Y^2s_2^2t^2}\times(17) \qquad (20)$$

by again applying Cauchy-Schwarz.

In the main term, we open the integral defining the Fourier transform in the first coordinate, so that the triple sum becomes

$$\int_{x\in\mathbb{R}} \Phi_M^2\left(x, \frac{X^{\frac{1}{2}}v}{2s_1s_2^3s_3s_4^2\ell t}\right) \sum_{k\ll \frac{X^\epsilon s_2^2s_4^2}{HY^2ds_1\ell v}\wedge\frac{s_2^2s_4t}{Hd}} \sum_h \sigma\left(\frac{h}{H}\right)\theta(k,x,h) \tag{21}$$
$$\times \sum_{\substack{|u|\ll \\ X^\epsilon\left(|v|s_1s_3+\frac{s_3s_4}{Y^2\ell t}\right)}} \theta(k,x,u)\sigma\left(\frac{(h+kH+W-\langle W\rangle)d}{s_2^2s_4t}\right) e\left(\frac{2}{3}\frac{\ell du^2h}{s_1v}\right) dx.$$

By Cauchy-Schwarz, the sum over h and u is bounded by

$$H^{\frac{1}{2}}\Bigg[\sum_{\substack{|u_1|,|u_2|\ll \\ X^\epsilon\left(|v|s_1s_3+\frac{s_3s_4}{Y^2\ell t}\right)}} \theta(k,x,u_1)\overline{\theta(k,x,u_2)}$$
$$\times\sum_h \sigma\left(\frac{h}{H}\right)\sigma\left(\frac{(h+kH+W_1-\langle W_1\rangle)d}{s_2^2s_4t}\right)\sigma\left(\frac{(h+kH+W_2-\langle W_2\rangle)d}{s_2^2s_4t}\right)$$
$$\times e\left(\frac{2}{3}\frac{\ell d(u_1^2-u_2^2)h}{s_1v}\right)\Bigg]^{\frac{1}{2}}$$

In view of the restriction on H and rapid decay of the Fourier transform of σ, the sum over h is negligible unless $u_1^2 \equiv u_2^2 \bmod \frac{s_1v}{(s_1v,\ell d)}$, in which case the sum over h is $\ll H$. This yields the bound

$$|(21)| \ll \left(\frac{X^\epsilon s_2^2s_4^2}{Y^2ds_1\ell v}\wedge\frac{s_2^2s_4t}{d}\right)\left(|v|s_1s_3+\frac{s_3s_4}{Y^2\ell t}\right)\left(\frac{(\ell d, s_1v)}{s_1v}\right)^{\frac{1}{2}}$$
$$\times\int_{x\in\mathbb{R}}\left|\Phi_M^2\left(x, \frac{X^{\frac{1}{2}}v}{2s_1s_2^3s_3s_4^2\ell t}\right)\right| dx.$$

The integral is bounded by (see (6))

$$\|\Phi_M\|_1 \ll M^{-\frac{1}{4}} = \frac{Y^{\frac{1}{2}}(s_1s_2\ell t)^{\frac{1}{3}}}{X^{\frac{1}{12}}},$$

and so, combined with (20), this completes the proof of the bound (18). $\square$

8 Sieving argument

If Z is a bit less than some large X, then the probability that a randomly chosen integer of size X is divisible by q^2 for some $q > Z$ is about $\frac{1}{Z}$. For $Y < X^{\frac{1}{6}}$ our earlier work shows (roughly) that, without restricting to d squarefree, there are

$\ll \frac{X}{Y}$ Heegner points associated to $d \asymp X$ with imaginary part $> Y$. Thus we might expect that the number of such points with a divisor q^2, $q > Z$ is $\ll \frac{X}{YZ}$. We prove such a statement in this section, to within a loss of X^ϵ.

The key ingredient is the following lemma, which associates to non-squarefree $d = d_1 q^2$ and parametrization equation $\ell m^3 = \ell^2 n^2 + t^2 d$ a genuine primitive ideal in the field $\mathbb{Q}(\sqrt{-d_1})$, and of class lying in a prescribed coset of the 3-part of the class group $H(-d_1)$; the number of such cosets appearing is bounded by a divisor function (this partially explains the loss of X^ϵ in our eventual sieve bound).

Lemma 8.1. *Let $(\ell, m, n, t, q, d) \in (\mathbb{Z}^+)^5$ satisfy $\ell m^3 - \ell^2 n^2 = t^2 q^2 d$ with $q^2 d \equiv 2 \bmod 4$, d squarefree, $(\ell m n, t) = 1$ and ℓ squarefree. Set*[3]

$$q_1 = (\ell, m)\,, \qquad q_2 = \operatorname{sq}\left(\frac{m^3}{q_1}, n^2\right), \qquad q_3 = \frac{q}{q_1 q_2}.$$

Further, set also

$$q_{12} = (q_1, q_2)\,, \qquad q_{11} = \frac{q_1}{q_{12}}, \qquad \frac{q_2}{q_{12}} = q_{22} q_{23}^3, \qquad (q_{22} \text{ cube-free})\,,$$

and define

$$\ell' = \frac{\ell}{q_1}, \qquad m' = \frac{m}{q_{11} q_{12} q_{22} q_{23}^2}, \qquad n' = \frac{n}{q_2}, \qquad q' = q_{11}^2 q_{22}.$$

The congruence conditions $(m', \ell' n') = (m' n' q', q_3) = (\ell', q) = 1$ hold. Also, the ideal (q') factors in $\mathbb{Q}\left(\sqrt{-d}\right)$ as $(q') = \mathfrak{q}\bar{\mathfrak{q}}$. Moreover, there is a primitive ideal $\mathfrak{a}$ of $\mathbb{Q}\left(\sqrt{-d}\right)$ of norm $\ell' m'$ and solving $\mathfrak{q}\mathfrak{a}^3 = \ell'\left(\ell' n' + t q_3 \sqrt{-d}\right)$.

Proof. The equation $\ell m^3 - \ell^2 n^2 = t^2 q^2 d$ may be rewritten as

$$\ell' {m'}^3 q' - {\ell'}^2 {n'}^2 = t^2 q_3^2 d. \tag{22}$$

Since ℓ is square-free, $(\ell', q_1 m) = 1$ and therefore, since $q_2 | m$, $(\ell', q_2) = 1$. It follows that $(\ell', q_3) = 1$ since any prime dividing ℓ' and q_3 divides also m'. Now

$$\left({m'}^3 q', {n'}^2\right) = \left(\frac{m^3}{q_1 q_2^2}, \frac{n^2}{q_2^2}\right)$$

is square-free, and therefore $(m', n') = 1$ and also (q', n') is square-free. Then $(m', \ell' n') = 1$ implies $(m', q_3) = 1$ and $(q', \ell') = 1$ implies $(q', q_3) = 1$ since any prime factor of (q', n') divides $\ell' {m'}^3 q'$ only once. It thus follows that $(n', q_3) = 1$, so we have proven all of the congruence conditions.

Equation (22) gives a factorization of ideals

$$\left(\ell' {m'}^3 q'\right) = \left(\ell' n' + t q_3 \sqrt{-d}\right)\left(\ell' n' - t q_3 \sqrt{-d}\right)$$

[3] $\operatorname{sq}\left(p_1^{e_1} \cdots p_r^{e_r}\right) = p_1^{\lfloor e_1/2 \rfloor} \cdots p_r^{\lfloor e_r/2 \rfloor}$.

in $\mathbb{Q}\left(\sqrt{-d}\right)$. Notice $p|\ell' \Rightarrow p\|t^2q_3^2d$ so $p|d$, and therefore $\ell' = \mathfrak{h}^2$ for some $\mathfrak{h}|\mathfrak{d}$. We claim that $p|q'$ implies p is ramified or split in $\mathbb{Q}\left(\sqrt{-d}\right)$. Indeed, if p is inert then

$$(p) \mid \left(\left(\ell'n' + tq_3\sqrt{-d}\right), \left(\ell'n' - tq_3\sqrt{-d}\right)\right) \qquad \Rightarrow p|n'$$

since $p \nmid 2\ell'$. But then $(p)^2 \,|m'^3q'$, which contradicts $\left(m'^3q', n'^2\right)$ square-free. Since all primes dividing (q') are ramified or split, we obtain the factorization $(q') = \mathfrak{q}\overline{\mathfrak{q}}$ with $\mathfrak{q}| \left(\ell'n' + tq_3\sqrt{-d}\right)$.

Set $\mathfrak{b} = \left(\ell'n' + tq_3\sqrt{-d}\right)\mathfrak{h}^{-1}\mathfrak{q}^{-1}$ so that $\mathfrak{b}\overline{\mathfrak{b}} = (m')^3$. Note that

$$\left(\mathfrak{b}, \overline{\mathfrak{b}}\right) \mid (2\ell'n', m') = (1)$$

and therefore $\mathfrak{b}$ is primitive, and coprime to $\mathfrak{d}$. Therefore there exists primitive ideal $\mathfrak{c}$ satisfying $\mathfrak{c}^3 = \mathfrak{b}$, and furthermore, $\mathfrak{a} = \mathfrak{h}\mathfrak{c}$ remains primitive. Clearly $N(\mathfrak{a}) = \ell'm'$ and $\mathfrak{q}\mathfrak{a}^3 = \ell'\left(\ell'n' + tq_3\sqrt{-d}\right)$ as wanted. $\square$

Proposition 8.1. *Let ψ be a smooth function on $\mathbb{R}^+$, supported in a bounded interval $[0, C]$ for some C. Let $Z \geq 1$ and $Y > X^{-\epsilon}$ for some $\epsilon > 0$. We have the bound*

$$\sum_{q>Z} \sum_{\substack{d \asymp \frac{X}{q^2} \\ q^2d \equiv 2 \quad (4) \\ \square-free}} \sum_{\substack{(\ell,m,n,t)\in\left(\mathbb{Z}^+\right)^4 \\ (\ell mn,t)=1 \\ \ell m^3-\ell^2n^2=t^2q^2d \\ \ell\ \square-free}} \psi\left(\frac{Y\ell mt}{\sqrt{\ell m^3 - \ell^2n^2}}\right) \ll \frac{X^{1+\epsilon}}{YZ} + \frac{X^{\frac{3}{4}+\epsilon}}{Y^{\frac{3}{2}}}.$$

Proof. Keep the meaning of ℓ', m', n', q' and q_{ij} from the previous lemma. Then the sum in question is

$$\sum_{\substack{q=q_{11}q_{12}^2q_{22}q_{23}^3q_3>Z \\ \text{odd} \\ (q_{11}q_{12},q_{22}q_{23})=1 \\ q_{11}q_{12}\ \square-\text{free},\ q_{22}\ \text{cube-free} \\ q'=q_{11}^2q_{22} \\ (q',q_3)=1}} \sum_{\substack{d\asymp\frac{X}{q^2} \\ d\equiv 2 \quad (4) \\ \square-\text{free}}} \sum_{\substack{(\ell',m',n',t) \\ \ell'\ \square-\text{free} \\ m'\ \text{odd} \\ (\ell'm'n'q',tq_3)= \\ (\ell',m'q)=(m',n')=1 \\ \ell'm'^3q'-\ell'^2n'^2=t^2q_3^2d}} \psi\left(\frac{Y\ell'm'q_{11}^2q_{12}^2q_{22}q_{23}^2}{q\sqrt{d}}\right). \tag{23}$$

Split this according as $\sqrt{d} > \frac{\sqrt{X}}{Yq_{11}^2q_{12}^2q_{22}q_{23}^2}$ or not.

In the first case, at the expense of a divisor function, we may set $t' := tq_3$. Also, since $d \asymp \frac{X}{q^2}$, for a suitable[4] smooth function ψ_0 supported in a bounded

[4] We require $\psi_0(y) \gg \psi(x)$ when $y \asymp x$, the implied constants depending on those in $d \asymp \frac{X}{q^2}$.

interval of $\mathbb{R}^+$,

$$\ll X^{\epsilon} \sum_{\substack{q_0=q_{11}q_{12}^2q_{22}q_{23}^2 \\ \text{odd} \\ (q_{11}q_{12},q_{22}q_{23})=1 \\ q_{11}q_{12}\ \square-\text{free} \\ q_{22}\ \text{cube-free} \\ q'=q_{11}^2q_{22}}} \sum_{\substack{(\ell',m',n',t') \\ \ell'\ \square-\text{free}, m'\ \text{odd} \\ (\ell'm'n'q',t')=(m',\ell'n')=1 \\ \ell'm'^3q'-\ell'^2n'^2=t'^2d \\ d\equiv 2 \quad (4),\ \square-\text{free} \\ \frac{X}{Y^2q_{11}^4q_{12}^4q_{22}^2q_{23}^4}<d<\min\left(\frac{X}{Z^2},\frac{X}{q_0^2}\right)}} \psi_0\left(\frac{Yq_{11}^2q_{12}^2q_{22}q_{23}^2\ell'm'}{\sqrt{X}}\right). \tag{24}$$

Replacing $q\sqrt{d}$ with $\sqrt{X}$ in the argument of ψ_0 allows us to set apart the sum over n. To do this, we solve for d, dropping the condition that it be squarefree and 2 mod 4, and controlling it's size with a partition of unity ($d \asymp e^a$ below). Thus we find that the inner sum of (24) is bounded by ($q_1 := q_{11}q_{12}$)

$$\ll \sum_{\substack{\max\left(1,\frac{X}{Y^2q_1^4q_{22}^2q_{23}^4}\right) \\ \leq e^a<\min\left(\frac{X}{Z^2},\frac{X}{q_0^2}\right)}} \sum_{\substack{(\ell'q',t')=1 \\ \ell'\ \square-\text{free} \\ (\ell't')^2\ll\frac{q'X^{\frac{3}{2}}}{e^aY^3q_1^6q_{22}^3q_{23}^6}}} \sum_{\substack{(m',2\ell't')=1 \\ m'\ll\frac{\sqrt{X}}{Yq_1^2q_{22}q_{23}^2\ell'} \\ m'\gg\left(\frac{t^2e^a}{\ell'q'}\right)^{\frac{1}{3}}}} \psi_0\left(\frac{Yq_1^2q_{22}q_{23}^2\ell'm'}{\sqrt{X}}\right)$$

$$\sum_{\substack{(n',m't')=1 \\ \ell'm'^3q'\equiv\ell'^2n'^2 \bmod t'^2}} \sigma^{\times}\left(\frac{\ell'm'^3q'-\ell'^2n'^2}{t'^2e^a}\right)$$

Splitting the sum over n' into blocks of length t'^2, each of which contributes $O(X^{\epsilon})$ by the parametrization in Proposition 3.3, the sum over n is bounded by

$$\ll X^{\epsilon}\left(O(1)+\frac{1}{t'^2}\frac{t'^2e^a}{(\ell'm')^{\frac{3}{2}}q'^{\frac{1}{2}}}\right) \ll X^{\epsilon}\left(O(1)+\frac{e^a}{(\ell'm')^{\frac{3}{2}}q'^{\frac{1}{2}}}\right).$$

Plugging this into (24) the $O(1)$ term contributes

$$\ll \frac{X^{\frac{5}{4}+\epsilon}}{Y^{\frac{5}{2}}} \sum_{\substack{q_{11}q_{12}^2q_{22}q_{23}^3<X^{\frac{1}{2}} \\ q'=q_{11}^2q_{22}}} \frac{q'^{\frac{1}{2}}}{q_{11}^5q_{12}^5q_{22}^{\frac{5}{2}}q_{23}^5} \sum_{\max\left(1,\frac{X}{Y^2q_{11}^4q_{12}^4q_{22}^2q_{23}^4}\right)\leq e^a} \frac{1}{e^{\frac{a}{2}}}$$

$$\ll \frac{X^{\frac{3}{4}+\epsilon}}{Y^{\frac{3}{2}}} \sum_{q_{11}q_{12}^2q_{22}q_{23}^2<X^{\frac{1}{2}}} \frac{1}{q_{11}q_{12}^3q_{22}q_{23}^3} \ll \frac{X^{\frac{3}{4}+\epsilon}}{Y^{\frac{3}{2}}}.$$

Second term contributes

$$\ll X^{\epsilon} \sum_{\substack{q_{11}q_{12}^2q_{22}q_{23}^3<X^{\frac{1}{2}} \\ q'=q_{11}^2q_{22}}} \frac{1}{q'^{\frac{1}{2}}} \sum_{e^a<\frac{X}{Z^2}} e^a \sum_{\ell't'\ll\frac{q'^{\frac{1}{2}}X^{\frac{3}{4}}}{2^{\frac{a}{2}}Y^{\frac{3}{2}}q_{11}^3q_{12}^3q_{22}^{\frac{3}{2}}q_{23}^3}} \frac{1}{\ell'^{\frac{3}{2}}} \sum_{m'\gg\left(\frac{t^2 2^a}{\ell'q'}\right)^{\frac{1}{3}}} \frac{1}{m^{\frac{3}{2}}}$$

$$\ll X^{\epsilon} \sum_{\substack{q_{11}q_{12}^2q_{22}q_{23}^3<X^{\frac{1}{2}} \\ q'=q_{11}^2q_{22}}} \frac{1}{q'^{\frac{1}{3}}} \sum_{e^a<\frac{X}{Z^2}} e^{\frac{5a}{6}} \sum_{\ell't'\ll\frac{q'^{\frac{1}{2}}X^{\frac{3}{4}}}{e^{\frac{a}{2}}Y^{\frac{3}{2}}q_{11}^3q_{12}^3q_{22}^{\frac{3}{2}}q_{23}^3}} \frac{1}{t'^{\frac{1}{3}}\ell'^{\frac{4}{3}}}$$

$$\ll \frac{X^{\frac{1}{2}+\epsilon}}{Y} \sum_{q_{11}q_{12}^2q_{22}q_{23}^3<X^{\frac{1}{2}}} \frac{1}{q_{11}^2q_{12}^2q_{22}q_{23}^2} \sum_{e^a<\frac{X}{Z^2}} e^{\frac{a}{2}} \ll \frac{X^{1+\epsilon}}{YZ}.$$

We turn to the remaining case, $\sqrt{d} \le \frac{\sqrt{X}}{Yq_{11}^2q_{12}^2q_{22}q_{23}^2}$. Inserting this condition into (23) we have (recall $\mathcal{P}(-d)$ denotes the primitive ideals of $\mathbb{Q}(\sqrt{-d})$)

$$\sum_{\substack{q=q_{11}q_{12}^2q_{22}q_{23}^3q_3>Z \\ q'=q_{11}^2q_{22},(q',q_3)=1 \\ \cdots}} \sum_{\substack{d\equiv 2 \quad (4) \\ \square-\text{free} \\ d\asymp\frac{X}{q^2} \\ d<\frac{X}{Y^2q_1^4q_{22}^2q_{23}^4}}} \sum_{\ell'|d} \sum_{\substack{(q')=\mathfrak{q}\bar{\mathfrak{q}} \\ \text{in } \mathbb{Q}(\sqrt{-d})}} \sum_{\substack{\mathfrak{a}\in\mathcal{P}(-d) \\ [\mathfrak{q}\mathfrak{a}^3]=[(1)]}} \psi\left(\frac{Yq_1^2q_{22}q_{23}^2N\mathfrak{a}}{\sqrt{X}}\right)$$

$$\ll X^{\epsilon} \sum_{\substack{q_0=q_{11}q_{12}^2q_{22}q_{23}^3 \\ q_0<X^{\frac{1}{2}} \\ q'=q_{11}^2q_{22} \\ \cdots}} \sum_{\substack{d\equiv 2 \quad (4) \\ \square-\text{free} \\ d\ll\frac{X}{Z^2}\wedge\frac{X}{q_0^2}\wedge\frac{X}{Y^2q_1^4q_{22}^2q_{23}^4}}} \sum_{(q')=\mathfrak{q}\bar{\mathfrak{q}}} \sum_{\substack{\mathfrak{a}\in\mathcal{P}(-d) \\ [\mathfrak{a}^3\mathfrak{q}]=[(1)]}} \psi\left(\frac{Yq_1^2q_{22}q_{23}^2N\mathfrak{a}}{X^{\frac{1}{2}}}\right)$$

We now bound the sum over $\mathfrak{a}$ by appealing to Corollary 4.1, and the sum over $\mathfrak{q}$ by a divisor function. This yields the wanted bound

$$\ll \frac{X^{\frac{1}{2}+\epsilon}}{Y} \sum_{q_{11}q_{12}^2q_{22}q_{23}^3<X^{\frac{1}{2}}} \frac{1}{q_1^2q_{22}q_{23}^2} \sum_{d\ll\frac{X}{Z^2}} \frac{|H_3(-d)|}{\sqrt{d}} \ll \frac{X^{1+\epsilon}}{YZ}$$

by using the Davenport-Heilbronn Theorem and partial summation. This completes the proof. □

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