Homework 3 - Analytic number theory

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Due Friday, September 16

1. (8 points) Let $d(n)$ be the divisor function. Prove that $d(n) \ll n^\epsilon$ for any $\epsilon > 0$, with the implied constant depending on ϵ.

2. (12- points) Find, with proof, the value of $n < 10^{50}$ such that $d(n)$ is maximized.

   (Partial credit for partial results: $d(n)$ is within a factor of two of the maximum possible value, etc.)

3. (3 points) Let $d_k(n)$ the number of ways of writing n as a product of k factors. Explain why

$$\sum_{n \geq 1} \frac{d_k(n)}{n^s} = \zeta(s)^k \tag{1}$$

   for $\Re(s) > 1$.

4. (5-12 points) For fixed k, determine an asymptotic formula for $\sum_{n \leq x} d_k(n)$. (Five points for an asymptotic formula; more points for secondary terms (you can be a little bit vague here) and good error terms.

5. (5 points) Determine a formula for $\sum_{n \leq x} \phi(n)$. You should be able to come up with a very good error term.

6. (5+ points) Using a computer, compute $\sum_{n \leq x} \mu(n)$ for a good range of x (You should be able to do it at least to $x = 10^6$). Describe and/or graph your results. (Conjectures encouraged!)