

# Fourier Analysis in Arithmetic Statistics

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IIT Bombay, December 3, 2025  
[thornef.github.io/iitb-2025.pdf](https://thornef.github.io/iitb-2025.pdf)



```
frankthorne — gp — 80x24
Last login: Wed Dec  3 09:31:01 on ttys000
Franks-MacBook-Pro-2:~ frankthorne$ gp
      GP/PARI CALCULATOR Version 2.8.0 (development 18205-1c269ec)
          i386 running darwin (x86-64 kernel) 64-bit version
compiled: Nov 11 2015, Apple LLVM version 6.0 (clang-600.0.54) (based on LLVM 3.
5svn)
                  threading engine: single
                  (readline not compiled in, extended help enabled)

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Type ? for help, \q to quit.
Type ?15 for how to get moral (and possibly technical) support.

parisize = 8000000, primelimit = 500000
? default(realprecision, 50)
? exp(Pi*sqrt(163))
%2 = 262537412640768743.99999999999925007259719818568888
?
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For any integer  $d \geq 1$ , write

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Theorem (Davenport-Heilbronn)

We have

$$N_3(X) = \frac{1}{3\zeta(3)}X + o(X).$$

## Sample Theorem 2: Counting Quartic and Quintic Fields

Theorem (Bhargava)

We have

$$N_4(X, S_4) \sim \frac{5}{24} \prod_p (1 + p^{-2} - p^{-3} - p^{-4})X,$$

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## Sample Theorem 3: 3-torsion in Quadratic Class Groups

Theorem (Davenport-Heilbronn)

We have

$$\sum_{|D| < X} \#|\text{Cl}(\mathbb{Q}(\sqrt{D}))[3]| = \frac{10}{\pi^2}X + o(X).$$

# Sample Theorem 4: 2-Selmer Groups in Elliptic Curves

## Theorem (Bhargava-Shankar)

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When elliptic curves  $E$  are ordered by *height*, the average size of their 2-Selmer groups is 3.

Corollary

Their average *rank* is at most 1.5.

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- ▶ Parametrization and **counting** theorems involving **lattice points**;

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# Parametrization: The Basic Metatheorem

## Theorem

*There exists an explicit, “nice” bijection*

$$\{ \text{Something nice} \} \longleftrightarrow G(\mathbb{Z}) \backslash V(\mathbb{Z})$$

*where  $V$  is a f.d. representation of an algebraic group  $G$ .*

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Moreover, certain arithmetic properties on the left correspond to congruence conditions on the right.

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- ▶  $\mathrm{Disc}(gx) = (\det g)^2 \mathrm{Disc}(x);$
- ▶  $\mathrm{Disc}(x) = 0$  if and only if  $x(u, v)$  has a repeated root.

## Example: Binary Cubic Forms (2)

Theorem (Levi, Delone-Faddeev, Gan-Gross-Savin)

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- ▶  $\text{Disc}(v) = \text{Disc}(R)$ ;
- ▶ (Davenport-Heilbronn)  $R$  is *maximal* iff, for all primes  $p$ ,  $v$  satisfies a certain congruence condition  $(\bmod p^2)$ .

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- ▶ ... and more!  
(Bhargava, Ho, Shankar, Varma, X. Wang, Wood, ....)

# More Interesting Parametrizations

Table 1: Summary of Higher Composition Laws

#	Lattice ( $V_{\mathbb{Z}}$ )	Group acting ( $G_{\mathbb{Z}}$ )	Parametrizes ( $\mathcal{C}$ )	( $k$ )	( $n$ )	( $H$ )
1.	$\{0\}$	-	Linear rings	0	0	$A_0$
2.	$\tilde{\mathbb{Z}}$	$SL_1(\mathbb{Z})$	Quadratic rings	1	1	$A_1$
3.	$(Sym^2 \mathbb{Z}^2)^*$ (GAUSS'S LAW)	$SL_2(\mathbb{Z})$	Ideal classes in quadratic rings	2	3	$B_2$
4.	$Sym^3 \mathbb{Z}^2$	$SL_2(\mathbb{Z})$	Order 3 ideal classes in quadratic rings	4	4	$G_2$
5.	$\mathbb{Z}^2 \otimes Sym^2 \mathbb{Z}^2$	$SL_2(\mathbb{Z})^2$	Ideal classes in quadratic rings	4	6	$B_3$
6.	$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$	$SL_2(\mathbb{Z})^3$	Pairs of ideal classes in quadratic rings	4	8	$D_4$
7.	$\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4$	$SL_2(\mathbb{Z}) \times SL_4(\mathbb{Z})$	Ideal classes in quadratic rings	4	12	$D_5$
8.	$\wedge^3 \mathbb{Z}^6$	$SL_6(\mathbb{Z})$	Quadratic rings	4	20	$E_6$
9.	$(Sym^3 \mathbb{Z}^2)^*$	$GL_2(\mathbb{Z})$	Cubic rings	4	4	$G_2$
10.	$\mathbb{Z}^2 \otimes Sym^2 \mathbb{Z}^3$	$GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$	Order 2 ideal classes in cubic rings	12	12	$F_4$
11.	$\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$	$GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})^2$	Ideal classes in cubic rings	12	18	$E_6$
12.	$\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^6$	$GL_2(\mathbb{Z}) \times SL_6(\mathbb{Z})$	Cubic rings	12	30	$E_7$
13.	$(\mathbb{Z}^2 \otimes Sym^2 \mathbb{Z}^3)^*$	$GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$	Quartic rings	12	12	$F_4$
14.	$\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$	$GL_4(\mathbb{Z}) \times SL_5(\mathbb{Z})$	Quintic rings	40	40	$E_8$

# Still More Interesting Parametrizations

Group (s.s.)	Representation	Geometric Data	Invariants	Dynkin	$\S$	
1.	$SL_2$	$Sym^4(2)$	$(C, L_2)$	2, 3	$A_2^{(2)}$	4.1
2.	$SL_2^2$	$Sym^2(2) \otimes Sym^2(2)$	$(C, L_2, L'_2) \sim (C, L_2, P)$	2, 3, 4	$D_3^{(2)}$	6.1
3.	$SL_2^4$	$2 \otimes 2 \otimes 2 \otimes 2$	$(C, L_2, L'_2, L''_2) \sim (C, L_2, P, P')$	2, 4, 4, 6	$D_4^{(1)}$	6.2
4.	$SL_2^3$	$2 \otimes 2 \otimes Sym^2(2)$	$(C, L_2, L'_2) \sim (C, L_2, P)$	2, 4, 6	$B_3^{(1)}$	6.3.1
5.	$SL_2^2$	$Sym^2(2) \otimes Sym^2(2)$	$(C, L_2, L'_2) \sim (C, L_2, P)$	2, 3, 4	$D_3^{(2)}$	6.3.3
6.	$SL_2^2$	$2 \otimes Sym^3(2)$	$(C, L_2, P_3)$	2, 6	$G_2^{(1)}$	6.3.2
7.	$SL_2$	$Sym^4(2)$	$(C, L_2, P_3)$	2, 3	$A_2^{(2)}$	6.3.4
8.	$SL_2^2 \times GL_4$	$2 \otimes 2 \otimes \wedge^2(4)$	$(C, L_2, M_{2,6})$	2, 4, 6, 8	$D_5^{(1)}$	6.6.1
9.	$SL_2 \times SL_6$	$2 \otimes \wedge^3(6)$	$(C, L_2, M_{3,6})$ with $L^{\otimes 3} \cong \det M$	2, 6, 8, 12	$E_6^{(1)}$	6.6.2
10.	$SL_2 \times Sp_6$	$2 \otimes \wedge_0^3(6)$	$(C, L_2, (M_{3,6}, \varphi))$ with $L^{\otimes 3} \cong \det M$	2, 6, 8, 12	$E_6^{(2)}$	6.6.3
11.	$SL_2 \times Spin_{12}$	$2 \otimes S^+(32)$	$(C \rightarrow \mathbb{P}^1(\mathcal{H}_3(\mathbb{H})), L_2)$	2, 6, 8, 12	$E_7^{(1)}$	6.6.3
12.	$SL_2 \times E_7$	$2 \otimes 56$	$(C \rightarrow \mathbb{P}^1(\mathcal{H}_3(\mathbb{O})), L_2)$	2, 6, 8, 12	$E_8^{(1)}$	6.6.3
13.	$SL_3$	$Sym^3(3)$	$(C, L_3)$	4, 6	$D_4^{(3)}$	4.2
14.	$SL_3^3$	$3 \otimes 3 \otimes 3$	$(C, L_3, L'_3) \sim (C, L_3, P)$	6, 9, 12	$E_6^{(1)}$	5.1
15.	$SL_3^2$	$3 \otimes Sym^2(3)$	$(C, L_3, P_2)$	6, 12	$F_4^{(1)}$	5.2.1
16.	$SL_3$	$Sym^3(3)$	$(C, L_3, P_2)$	4, 6	$D_4^{(3)}$	5.2.2
17.	$SL_3 \times SL_6$	$3 \otimes \wedge^2(6)$	$(C, L_3, M_{2,6})$ with $L^{\otimes 2} \cong \det M$	6, 12, 18	$E_7^{(1)}$	5.5
18.	$SL_3 \times E_6$	$3 \otimes 27$	$(C \hookrightarrow \mathbb{P}^2(\mathbb{O}), L_3)$	6, 12, 18	$E_8^{(1)}$	5.4
19.	$SL_2 \times SL_4$	$2 \otimes Sym^2(4)$	$(C, L_4)$	8, 12	$E_6^{(2)}$	4.3
20.	$SL_5 \times SL_5$	$\wedge^2(5) \otimes 5$	$(C, L_5)$	20, 30	$E_8^{(1)}$	4.4

Table 1: Table of coregular representations and their moduli interpretations

Bhargava and Ho, *Coregular spaces and genus one curves*, Camb. J. Math.

# Improved Davenport-Heilbronn

Theorem (DHBBPBSTTTBTT\*)

We have

$$N_3(X) = \frac{1}{3\zeta(3)}X + \frac{4(1 + \sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{\frac{2}{3}}(\log X)^3).$$

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\*: Davenport-Heilbronn, Belabas, Belabas-Bhargava-Pomerance, Bhargava-Shankar-Tsimerman, Taniguchi-T., Bhargava-Taniguchi-T.

A fundamental task in this subject is to give some quantitative measures of additive structure in a set, and then investigate to what extent these measures are equivalent to each other. For example, one could try to quantify each of the following informal statements as being some version of the assertion “ $A$  has additive structure”:

- $A + A$  is small;
- $A - A$  is small;
- $A - A$  can be covered by a small number of translates of  $A$ ;
- $kA$  is small for any fixed  $k$ ;
- there are many quadruples  $(a_1, a_2, a_3, a_4) \in A \times A \times A \times A$  such that  $a_1 + a_2 = a_3 + a_4$ ;
- there are many quadruples  $(a_1, a_2, a_3, a_4) \in A \times A \times A \times A$  such that  $a_1 - a_2 = a_3 - a_4$ ;
- the convolution  $1_A * 1_A$  is highly concentrated;
- the subset sums  $FS(A) := \{\sum_{a \in B} a : B \subseteq A\}$  have high multiplicity;
- the Fourier transform  $\widehat{1_A}$  is highly concentrated;
- the Fourier transform  $\widehat{1_A}$  is highly concentrated in a cube;
- $A$  has a large intersection with a generalized arithmetic progression, of size comparable to  $A$ ;
- $A$  is contained in a generalized arithmetic progression, of size comparable to  $A$ ;
- $A$  (or perhaps  $A - A$ , or  $2A - 2A$ ) contains a large generalized arithmetic progression.

The reader is invited to investigate to what extent these informal statements are true for sets such as progressions and cubes, and false for sets such as random sets.

As it turns out, once one makes the above assertions more quantitative, there are

a number of deep and important equivalences between them; indeed, to oversimplify tremendously all of the above criteria for additive structure are “essentially”

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- ▶ Show that  $\Phi_{p^2}$  is nice in some way we can quantify;
- ▶ Develop lattice point counting methods which use the niceness.

# An explicit evaluation

Theorem (Taniguchi-T., 2011)

We have

$$\widehat{\Phi_{p^2}}(\nu) = \begin{cases} p^{-2} + p^{-3} - p^{-5} & \nu/p : \text{of type } (0), \\ p^{-3} - p^{-5} & \nu/p : \text{of type } (1^3), (1^21), \\ -p^{-5} & \nu/p : \text{of type } (111), (21), (3). \\ p^{-3} - p^{-5} & \nu : \text{of type } (1^3_{**}), \\ -p^{-5} & \nu : \text{of type } (1^3_*), (1^3_{\max}), \\ 0 & \text{otherwise.} \end{cases}$$

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So:

$$\frac{1}{p^8} \sum_{\nu \in V(\mathbb{Z}/p^2\mathbb{Z})} |\widehat{\Phi_{p^2}}(\nu)| \ll p^{-7}.$$

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We have

$$N_3(X) = \frac{1}{3\zeta(3)}X + \frac{4(1 + \sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{\frac{3}{5} + \epsilon}) + O(X^{1 - \frac{1}{8-7+2} + \epsilon}).$$

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- ▶ The **subspace method** (Taniguchi-T.; Ishimoto);
- ▶ The **morphism method** (Ishitsuka, Taniguchi, T., Xiao).

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- ▶ The Averaging Technique (Bhargava)

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Thank you!  
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