Homework 5 - Analytic number theory

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Due Friday, September 30

1. (0 points) Write down some random binary quadratic forms and some random matrices in $\mathrm{SL}_2(\mathbb{Z})$, and compute their actions on your binary quadratic forms. Ponder and admire your results before throwing them in the wastebasket.

2. (5 points) Evaluate $h(d)$ for $d = -11, -15, -24, -163$.

3. (5 points) Prove that the action of $\mathrm{SL}_2(\mathbb{Z})$ leaves the discriminant unchanged.

4. (3 points) Prove directly (i.e., without using Gauss's theory) that the quadratic forms $2x^2 + xy + 3y^2$ and $2x^2 - xy + 3y^2$ are not equivalent.

5. (3 points) Prove directly that the quadratic forms x^2+xy+y^2 and x^2-xy+y^2 are equivalent.

6. (8 points; hints to be added later) (Gauss's reduction theory) Prove that every positive definite quadratic form with $a, c > 0$ is equivalent to a unique "reduced" form satisfying $-a < b \leq a < c$ or $0 \leq b \leq a = c$.

7. (8 points) Prove that the number of automorphs of a binary quadratic form is: 6 if $d = -3$, 4 if $d = -4$, and 2 for any other negative d.

8. (5+ points) Using a computer, evaluate $h(-d)$ for $1 < d < 10^6$, and describe and explain your results.

9. (???) Prove that $h(d) > 1$ if $d < -163$.