An Analytic View of Arithmetic Statistics

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Carl Friedrich Gauss (1777-1855)

```
?
? exp(Pi*sqrt(163))
%1 = 262537412640768743.99999999999925007259
?
? exp(Pi*sqrt(67))
%2 = 147197952743.99999866245422450682926131
? exp(Pi*sqrt(43))
%3 = 884736743.99977746603490666193746207861
? 
```

(Computation done in PARI/GP)

```
.
? f(n) = n^2 + n + 41
%11 = (n)->n^2+n+41
? vector(15, n, f(n))
%12 = [43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281]
?
```

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Binary cubic forms I: Definitions

The lattice of *binary cubic forms* is

$$V(\mathbb{Z}) := \{au^3 + bu^2v + cuv^2 + dv^3 \ : \ a, b, c, d \in \mathbb{Z}\}.$$

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$$(g \circ f)(u, v) = \frac{1}{\det g} f((u, v)g),$$

which satisfies

$$\mathrm{Disc}(g \circ f) = (\det g)^2 \mathrm{Disc}(f),$$

$$\mathrm{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

Binary cubic forms II: Parametrization

Theorem (Levi 1914, Delone-Faddeev 1940, Gan-Gross-Savin 2002)

There is an explicit, discriminant-preserving bijection between the set of $\mathrm{GL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ and the set of **cubic rings**.

Binary cubic forms III: A Counting Theorem

Theorem (Davenport-Heilbronn 1971)

Let $N_3(X)$ count cubic fields K with $|\mathrm{Disc}(K)| < X$. Then,

$$N_3(X) = \frac{1}{3\zeta(3)} X + o(X).$$

Table 1: Summary of Higher Composition Laws

#	Lattice ($V_{\mathbb{Z}}$)	Group acting ($G_{\mathbb{Z}}$)	Parametrizes ($\mathcal{C}$)	(k)	(n)	(H)
1.	$\{0\}$	-	Linear rings	0	0	A_0
2.	$\widetilde{\mathbb{Z}}$	$\mathrm{SL}_1(\mathbb{Z})$	Quadratic rings	1	1	A_1
3.	$(\mathrm{Sym}^2\mathbb{Z}^2)^*$ (GAUSS'S LAW)	$\mathrm{SL}_2(\mathbb{Z})$	Ideal classes in quadratic rings	2	3	B_2
4.	$\mathrm{Sym}^3\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})$	Order 3 ideal classes in quadratic rings	4	4	G_2
5.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^2$	Ideal classes in quadratic rings	4	6	B_3
6.	$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^3$	Pairs of ideal classes in quadratic rings	4	8	D_4
7.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^4$	$\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_4(\mathbb{Z})$	Ideal classes in quadratic rings	4	12	D_5
8.	$\wedge^3\mathbb{Z}^6$	$\mathrm{SL}_6(\mathbb{Z})$	Quadratic rings	4	20	E_6
9.	$(\mathrm{Sym}^3\mathbb{Z}^2)^*$	$\mathrm{GL}_2(\mathbb{Z})$	Cubic rings	4	4	G_2
10.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Order 2 ideal classes in cubic rings	12	12	F_4
11.	$\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})^2$	Ideal classes in cubic rings	12	18	E_6
12.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^6$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_6(\mathbb{Z})$	Cubic rings	12	30	E_7
13.	$(\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3)^*$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Quartic rings	12	12	F_4
14.	$\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$	$\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$	Quintic rings	40	40	E_8

(M. Bhargava, Higher composition laws IV: The parametrization of quintic rings, Ann. Math., 2008.)

	Group (s.s.)	Representation	Geometric Data	Invariants	Dynkin	§
1.	SL_2	$\mathrm{Sym}^4(2)$	(C, L_2)	$2, 3$	$A_2^{(2)}$	4.1
2.	SL_2^2	$\mathrm{Sym}^2(2) \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_3^{(2)}$	6.1
3.	SL_2^4	$2 \otimes 2 \otimes 2 \otimes 2$	$(C, L_2, L_2', L_2'') \sim (C, L_2, P, P')$	$2, 4, 4, 6$	$D_4^{(1)}$	6.2
4.	SL_2^3	$2 \otimes 2 \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 4, 6$	$B_3^{(1)}$	6.3.1
5.	SL_2^2	$\mathrm{Sym}^2(2) \otimes \mathrm{Sym}^2(2)$	$(C, L_2, L_2') \sim (C, L_2, P)$	$2, 3, 4$	$D_3^{(2)}$	6.3.3
6.	SL_2^2	$2 \otimes \mathrm{Sym}^3(2)$	(C, L_2, P_3)	$2, 6$	$G_2^{(1)}$	6.3.2
7.	SL_2	$\mathrm{Sym}^4(2)$	(C, L_2, P_3)	$2, 3$	$A_2^{(2)}$	6.3.4
8.	$\mathrm{SL}_2^2 \times \mathrm{GL}_4$	$2 \otimes 2 \otimes \wedge^2(4)$	$(C, L_2, M_{2,6})$	$2, 4, 6, 8$	$D_5^{(1)}$	6.6.1
9.	$\mathrm{SL}_2 \times \mathrm{SL}_6$	$2 \otimes \wedge^3(6)$	$(C, L_2, M_{3,6})$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(1)}$	6.6.2
10.	$\mathrm{SL}_2 \times \mathrm{Sp}_6$	$2 \otimes \wedge_0^3(6)$	$(C, L_2, (M_{3,6}, \varphi))$ with $L^{\otimes 3} \cong \det M$	$2, 6, 8, 12$	$E_6^{(2)}$	6.6.3
11.	$\mathrm{SL}_2 \times \mathrm{Spin}_{12}$	$2 \otimes S^+(32)$	$(C \to \mathbb{P}^1(\mathscr{H}_3(\mathbb{H})), L_2)$	$2, 6, 8, 12$	$E_7^{(1)}$	6.6.3
12.	$\mathrm{SL}_2 \times E_7$	$2 \otimes 56$	$(C \to \mathbb{P}^1(\mathscr{H}_3(\mathbb{O})), L_2)$	$2, 6, 8, 12$	$E_8^{(1)}$	6.6.3
13.	SL_3	$\mathrm{Sym}^3(3)$	(C, L_3)	$4, 6$	$D_4^{(3)}$	4.2
14.	SL_3^3	$3 \otimes 3 \otimes 3$	$(C, L_3, L_3') \sim (C, L_3, P)$	$6, 9, 12$	$E_6^{(1)}$	5.1
15.	SL_3^2	$3 \otimes \mathrm{Sym}^2(3)$	(C, L_3, P_2)	$6, 12$	$F_4^{(1)}$	5.2.1
16.	SL_3	$\mathrm{Sym}^3(3)$	(C, L_3, P_2)	$4, 6$	$D_4^{(3)}$	5.2.2
17.	$\mathrm{SL}_3 \times \mathrm{SL}_6$	$3 \otimes \wedge^2(6)$	$(C, L_3, M_{2,6})$ with $L^{\otimes 2} \cong \det M$	$6, 12, 18$	$E_7^{(1)}$	5.5
18.	$\mathrm{SL}_3 \times E_6$	$3 \otimes 27$	$(C \hookrightarrow \mathbb{P}^2(\mathbb{O}), L_3)$	$6, 12, 18$	$E_8^{(1)}$	5.4
19.	$\mathrm{SL}_2 \times \mathrm{SL}_4$	$2 \otimes \mathrm{Sym}^2(4)$	(C, L_4)	$8, 12$	$E_6^{(2)}$	4.3
20.	$\mathrm{SL}_5 \times \mathrm{SL}_5$	$\wedge^2(5) \otimes 5$	(C, L_5)	$20, 30$	$E_8^{(1)}$	4.4

Table 1: Table of coregular representations and their moduli interpretations

(M. Bhargava and W. Ho, Coregular spaces and genus one curves, Cambridge J. Math., to appear)

Application 1: Counting cubic fields

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Theorem (BBDHPSTTT*)

We have

$$N_3(X) = C^{\pm} \frac{1}{3\zeta(3)} X + \frac{4(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + O(X^{2/3+\epsilon}).$$

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*: Davenport, Heilbronn (1971), Belabas (1999), Belabas, Bhargava, Pomerance (∼2006), Bhargava, Shankar, Tsimerman (2010), Taniguchi, T. (2011), Bhargava, Taniguchi, T. (2017?)

Application 2: 3-torsion in class groups of quadratic fields

Theorem (BBDHPSTTT)

We have

$$\sum_{0<\pm|D|<X} \#\mathrm{Cl}(\mathbb{Q}(\sqrt{D})[3]) = \frac{6}{\pi^2}X + \frac{8(1+\sqrt{3})\zeta(1/3)}{5\Gamma(2/3)^3}\prod_p \left(1 - \frac{p^{1/3}+1}{p(p+1)}\right)X^{5/6} + O(X^{2/3+\epsilon}).$$

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Also: Similar results for 2-torsion in cubic fields (Bhargava, Belabas-Bhargava-Pomerance)

Application 3: Quartic and quintic fields

Theorem (Bhargava (2002), Belabas-Bhargava-Pomerance (2006))

*We have**

$$N_4(X) = \frac{5}{24} \prod_p \left(1 + p^{-2} - p^{-3} - p^{-4}\right) \cdot X + O(X^{23/24+\epsilon}).$$

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Theorem (Bhargava (2005), Shankar-Tsimerman (2013))

We have

$$N_5(X) = \frac{13}{120} \prod_p \left(1 + p^{-2} - p^{-4} - p^{-5}\right) \cdot X + O(X^{199/200+\epsilon}).$$

Application 4: Erdős-Kac for number fields

Let K be a 'random' cubic, quartic (non-D_4), or quintic number field with $|\mathrm{Disc}(K)| \asymp X$.

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Theorem (Lemke Oliver-T. (2014))

The average number of primes ramified in K is normally distributed with mean and variance $\log \log X$*.*

Application 5: The least prime to not split completely

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Theorem (Martin-Pollack (2011))

Over all cubic fields, the average of $p(K)$ is

$$\sum_{\ell} \frac{\ell(5/6 + 1/\ell + 1/\ell^2)}{1 + 1/\ell + 1/\ell^2} \prod_{p<\ell} \frac{1/6}{1 + 1/p + 1/p^2} = 2.1211\cdots$$

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Cho-Kim (2016): extensions to other splitting types, quartic and quintic fields.

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- For each K, the associated 2-dimensional representation $\rho = \rho_K$ of $\mathrm{Gal}(\widehat{K}/\mathbb{Q})$ with associated Artin L-function $L(s, \rho)$.

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Theorem (Yang '09, Cho-Kim '15, Shankar-Södergren-Templier '15)

We have

$$\lim_{X \to \infty} \frac{1}{\#\mathcal{F}(X)} \sum_{K \in \mathcal{F}(X)} \sum_{L(\frac{1}{2}+i\gamma, \rho_K)=0} \phi\left(\frac{\gamma_j}{2\pi} \log(|\mathrm{Disc}(K)|)\right) = \widehat{\phi}(0) - \frac{\phi(0)}{2}.$$

Application 7: The shape of number fields

Theorem (Terr 1997, Bhargava-Harron 2016)

The 'shape' of fields of degree $n \in \{3, 4, 5\}$ is equidistributed in

$$\mathrm{GL}_{n-1}(\mathbb{Z})\backslash\mathrm{GL}_{n-1}(\mathbb{R})/\mathrm{GO}_{n-1}(\mathbb{R}).$$

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Theorem (Hough 2017+)

Let ϕ be a [nice] cuspidal automorphic form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$, and $F : (0, \infty) \to (0, \infty)$ a smooth test function. Then,

$$\sum_{[K:\mathbb{Q}]=3} \phi(\Lambda_K) F\left(\frac{\pm\mathrm{Disc}(K)}{X}\right) \ll_\phi X^{3/4+\epsilon}.$$

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A version for quartic fields too.

Application 8: Almost prime field discriminants

Theorem (Belabas-Fouvry 1999)

There are $\gg \frac{X}{\log X}$ *cubic fields* K *with* $|\mathrm{Disc}(K)| < X$*, such that* $\mathrm{Disc}(K)$ *is fundamental and has at most 7 prime factors.*

Analytic Number Theory Principle 1: Zeta Functions

The *Riemann zeta function* is

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

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Theorem (Riemann, 1859)

The function $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Let $\Lambda(n) := \log(p)$ if $n = p^a$ and zero otherwise. Then

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and

$$\sum_{n \leq X} \Lambda(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'(s)}{\zeta(s)} X^s \frac{ds}{s}.$$

Definition

The *Shintani zeta function* is

$$\xi^{\pm}(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}$$

$$\text{“} = \text{''} \sum_{\pm \mathrm{Disc}(\mathcal{O}) > 0} |\mathrm{Disc}(\mathcal{O})|^{-s},$$

where $\mathcal{O}$ ranges over orders in cubic fields.

Shintani's zeta function

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$$\text{“} = \text{”} \sum_{\pm \mathrm{Disc}(\mathcal{O}) > 0} |\mathrm{Disc}(\mathcal{O})|^{-s},$$

where $\mathcal{O}$ ranges over orders in étale cubic algebras.

Don't worry about this, UNLESS you would like to generalize these results to quartic fields.

The functional equation

Theorem (Shintani, 1971)

The Shintani zeta functions converge absolutely for $\Re(s) > 1$. They continue to functions holomorphic in the plane except for simple poles at 1 and $5/6$, and satisfy the functional equation

$$\begin{pmatrix} \xi^+(1-s) \\ \xi^-(1-s) \end{pmatrix} = \Gamma\left(s - \frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s + \frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s} \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3 \sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \widehat{\xi}^+(s) \\ \widehat{\xi}^-(s) \end{pmatrix}.$$

Moreover,

$$\mathrm{Res}_{s=1}\xi^{\pm}(s) = \frac{\pi^2(3 + C^{\pm})}{36}, \quad \mathrm{Res}_{s=\frac{5}{6}}\xi^{\pm}(s) = K^{\pm}\frac{\zeta(1/3)\Gamma(1/3)^3}{4\sqrt{3}\pi}.$$

Analytic Number Theory Principle 2: Sieve Methods

Problem: Estimate the number of squarefree integers $\leq X$.

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$$\begin{aligned}
\#\{n \leq X, \ n \text{ squarefree}\} &= \sum_{n=1}^{X} \left(\sum_{d^2 | n} \mu(d) \right) \\
&= \sum_{n=1}^{X} \left(\sum_{d^2 | n, \ d \leq \sqrt{X}} \mu(d) \right) \\
&= \sum_{d \leq \sqrt{X}} \mu(d) \left(\sum_{d^2 | n \leq X} 1 \right).
\end{aligned}$$

$$\#\{n \leq X,\ n \text{ squarefree}\} = \sum_{d \leq \sqrt{X}} \mu(d) \left(\sum_{d^2 | n \leq X} 1 \right)$$

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$$= \sum_{d \leq \sqrt{X}} \mu(d) \left(\frac{X}{d^2} + O(1) \right)$$

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&= \sum_{d \leq \sqrt{X}} \mu(d) \left(\frac{X}{d^2} + O(1) \right) \\
&= X \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(\sqrt{X})
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&= \frac{6}{\pi^2} X + O(\sqrt{X}).
\end{aligned}
$$

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\end{aligned}$$

The red $\sqrt{X}$ is a level of distribution.

Analytic Number Theory Principle 3: Fourier transforms

Inserting an 'arithmetic kernel' function $\Phi_d : V(\mathbb{Z}/d\mathbb{Z}) \to \mathbb{C}$,

$$\xi^{\pm}(s, \Phi_d) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} \Phi_d(x) |\mathrm{Disc}(x)|^{-s}$$

is also a good zeta function.

Analytic Number Theory Principle 3: Fourier transforms

Inserting an 'arithmetic kernel' function $\Phi_d : V(\mathbb{Z}/d\mathbb{Z}) \to \mathbb{C}$,

$$\xi^{\pm}(s, \Phi_d) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} \Phi_d(x) |\mathrm{Disc}(x)|^{-s}$$

is also a good zeta function.

Its functional equation involves the Fourier transform

$$\widehat{\Phi_d}(x) = \frac{1}{d^4} \sum_{y \in V(\mathbb{Z}/d\mathbb{Z})} \Phi_d(y) e^{2\pi i [x,y]}.$$

Unspecialized tools yield inadequate bounds for $|\widehat{\Phi_d}(x)|$.

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Let $\Phi_q : V(\mathbb{Z}/q^2) \to \{0, 1\}$ be the characteristic function of cubic forms 'nonmaximal at q'.

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Theorem (Taniguchi-T.)

The Fourier transform $\widehat{\Phi_q}$ is multiplicative in q, and satisfies

$$\widehat{\Phi_{p^2}}(x) = \begin{cases} p^{-2} + p^{-3} - p^{-5} & x = 0, \\ p^{-3} - p^{-5} & x : \textit{of type } p \cdot (1^3), p \cdot (1^2 1), \\ -p^{-5} & x : \textit{of type } p \cdot (111), p \cdot (21), p \cdot (3). \\ p^{-3} - p^{-5} & x : \textit{of type } (1^3_{**}), \\ -p^{-5} & x : \textit{of type } (1^3_{*}),\ (1^3_{\max}), \\ 0 & \textit{otherwise.} \end{cases}$$

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Theorem (Taniguchi-T.)
If char$(\mathbb{F}_q) \neq 3$, *then we have*

$$\widehat{\Psi_q}(x) = \begin{cases} q^{-3} - q^{-4} - 2q^{-5} + 2q^{-6} + q^{-7} - 1 & x \in \mathcal{O}_{D1}, \\ 2q^{-4} - 5q^{-5} + 3q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_{D2}, \\ -q^{-5} + q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_{D3}, \mathcal{O}_{D4}, \mathcal{O}_{C2}, \mathcal{O}_{1^4}^{(2a)}, \mathcal{O}_{1^4}^{(2b)}, \\ q^{-4} - 3q^{-5} + 2q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_{C1}, \\ q^{-7} - q^{-8} & x \in \mathcal{O}_{1^4}^{(1)}, \mathcal{O}_{1^3 1}, \mathcal{O}_{1^2 11}, \mathcal{O}_{1^2 2}, \\ -q^{-6} + 2q^{-7} - q^{-8} & x \in \mathcal{O}_{1^2 1^2}, \\ q^{-6} - q^{-8} & x \in \mathcal{O}_{2^2}, \\ q^{-8} & x \in \mathcal{O}_{22}, \mathcal{O}_{112}, \mathcal{O}_{1111}, \mathcal{O}_{4}, \mathcal{O}_{13}, \\ q^{-1} + 2q^{-2} - q^{-3} - 2q^{-4} - q^{-5} + 2q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_{0}. \end{cases}$$

Thank you!