Project Description

Zeta Functions and the Distribution of Field Discriminants

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1 Introduction

I propose to apply the theory of *zeta functions associated to prehomogeneous vector spaces* to study the *distribution of field discriminants.*

My work is motivated by the following basic question: As a function of X, how many number fields K are there with $|\mathrm{Disc}(K)| < X$? In 1857, Hermite proved there are finitely many, and we have asymptotic formulas for fields of degree ≤ 5. The cubic case is due to Davenport and Heilbronn [19], and the quartic and quintic cases were established more recently by Bhargava [3, 5]. The key idea is that these fields are parameterized by lattice points in *prehomogeneous vector spaces.*

Oversimplifying slightly, a complex vector space is *prehomogeneous* if it has a rational representation ρ of an algebraic group G, such that the action of G is transitive, apart from a *singular locus* S defined by the vanishing of a *relative invariant* polynomial. Prehomogeneous vector spaces have a long history of study, and they are of interest in diverse contexts such as Lie theory, algebraic geometry, and invariant theory, in addition to those described here.

Integral orbits on prehomogeneous vector spaces often parameterize interesting objects. For example, Delone and Faddeev [20] proved that *cubic rings* are parameterized by $\mathrm{GL}_2(\mathbb{Z})$-orbits of *integral binary cubic forms*, and Bhargava obtained analogous quartic and quintic parameterizations. Related parameterizations have led to other interesting counting theorems as well.

Sato and Shintani [39] proved that prehomogeneous vector spaces come with naturally associated *zeta functions*, which makes them interesting to analysts as well. Zeta functions are at the heart of analytic number theory, and Riemann's original zeta function is subject of the famous Riemann Hypothesis, still unproved. Zeta functions are also of interest in physics, probability, and statistics.

Shintani [40] proved that the cubic zeta function has an anomalous pole at $s = 5/6$. This led Roberts [36] to conjecture a secondary term in the Davenport-Heilbronn theorem, and Taniguchi and I [44] proved this and obtained a variety of related results. (Roberts' conjecture was also proved independently by Bhargava, Shankar, and Tsimerman [6] using geometric methods.) Our work raises further questions about the distribution of cubic fields, and it reveals apparent connections to other objects of arithmetic interest such as elliptic curves.

Our progress on cubic fields also opens the door to a similar understanding of quartic and quintic fields. We expect secondary terms to occur in their counting functions, but so far we lack even an explicit conjecture. These and related questions will be the focus of my upcoming research.

In Section 2, I give some background and describe my previous related results [44, 45, 47], with a particular focus on my proof (with Taniguchi) of Roberts' conjecture. In Section 3, I describe the questions I will investigate if supported, many of which stem from [44]. Finally, I describe in Section 4 how I will integrate my research into educational opportunities both within and outside my department. One unique opportunity is that most contemporary research on this subject is confined to Japan, and so I will be able to bring this underappreciated work to the attention of researchers in the United States.

2 Background and Prior Results

2.1 The Davenport-Heilbronn Theorem

My postdoctoral work was motivated by the following 1971 result of Davenport and Heilbronn [19]. Let $N_3^{\pm}(0, X)$ denote the number of cubic fields K with $0 < \pm\mathrm{Disc}(K) < X$. Davenport and Heilbronn proved that

$$N_3^+(X) = \frac{1}{12\zeta(3)}X + o(X), \qquad N_3^-(X) = \frac{1}{4\zeta(3)}X + o(X). \tag{2.1}$$

Their proof can be summarized as follows. A ring is *cubic* if it is free of rank 3 as a $\mathbb{Z}$-module. The set of cubic rings includes maximal orders of cubic fields, which we hope to count, and nonmaximal, reducible, and degenerate cubic rings, which we must exclude. Cubic rings, in turn, are parameterized by the lattice of integral binary cubic forms, up to an action of $\mathrm{GL}_2(\mathbb{Z})$. These cubic forms can be counted by geometric methods, or alternatively by using the zeta function method described in this proposal.

The lattice $V_{\mathbb{Z}}$ of *integral binary cubic forms* is defined by

$$V_{\mathbb{Z}} := \{au^3 + bu^2v + cuv^2 + dv^3 \ : a, b, c, d \in \mathbb{Z}\}, \tag{2.2}$$

and the *discriminant* of a cubic form is given by the usual equation

$$\mathrm{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd. \tag{2.3}$$

There is a natural action of $\mathrm{GL}_2(\mathbb{Z})$ (or $\mathrm{SL}_2(\mathbb{Z})$) on $V_{\mathbb{Z}}$, given by

$$(g \cdot f)(u, v) = \frac{1}{\det g} f((u, v) \cdot g). \tag{2.4}$$

A cubic form f is *irreducible* if $f(u, v)$ is irreducible as a polynomial over $\mathbb{Q}$, and it is *nondegenerate* if $\mathrm{Disc}(f) \neq 0$. The pair (GL_2, V) is a typical example of a *prehomogeneous vector space*: $\mathrm{GL}_2(\mathbb{C})$ acts transitively on $V_{\mathbb{C}} - S$, where the *singular locus* S is given by the equation $\mathrm{Disc}(f) = 0$.

The action of $\mathrm{GL}_2(\mathbb{Z})$ preserves the discriminant, and counting $\mathrm{GL}_2(\mathbb{Z})$-orbits by discriminant is a *geometric* problem; it amounts to counting lattice points in a fundamental domain for the action of $\mathrm{GL}_2(\mathbb{Z})$ on a four-dimensional vector space. This problem was studied by Davenport [18]; one challenge is that conditions such as $0 < \mathrm{Disc}(f) < X$ do not cut out a compact subset of $V_{\mathbb{R}}$. These questions generalize classical problems concerning class numbers of binary quadratic forms studied by Gauss, Dirichlet, and many others.

It was proved by Delone and Faddeev [20] (and also implicitly by Davenport and Heilbronn) that $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of binary cubic forms parameterize cubic rings. The correspondence, as later extended by Gan, Gross, and Savin [23], is the following:

Theorem 2.1 (Delone-Faddeev, 1964)**.** *There is a canonical, explicit, discriminant-preserving bijection between the set of cubic rings up to isomorphism and the set of* $\mathrm{GL}_2(\mathbb{Z})$*-equivalence classes of integral binary cubic forms. Furthermore, under this correspondence, irreducible cubic forms correspond to orders in cubic fields.*

Therefore, geometric methods allow us to count cubic rings. To count cubic fields, we must exclude the reducible, degenerate, and nonmaximal rings. The nonmaximal rings are the most difficult to exclude, and for these Davenport and Heilbronn proved the following criterion:

Proposition 2.2 ([19, 6]). *Under the Delone-Faddeev correspondence, a cubic ring R is maximal if and only if, for each prime p, its corresponding cubic form f satisfies a certain explicit* $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$*-invariant condition modulo p^2.*

Putting all of these results together and sieving for maximailty, Davenport and Heilbronn obtained asymptotics for $N_3^{\pm}(X)$.

2.2 Shintani's zeta functions

The *Shintani zeta functions* associated to the space of binary cubic forms are defined by the equation

$$\xi^{\pm}(s) := \sum_{n\geq 1} a_n^{\pm} n^{-s} := \sum_{x\in \mathrm{SL}_2(\mathbb{Z})\backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}, \tag{2.5}$$

where the sum ranges over points of positive or negative discriminant respectively, and $\mathrm{Stab}(x)$, the stabilizer of any $x \in V_{\mathbb{Z}}$, is always of order 1 or 3. By the Delone-Faddeev correspondence, this Dirichlet series essentially counts cubic rings.

Shintani proved [40] that these zeta functions have analytic continuation to $\mathbb{C}$ with simple poles at $s = 1$ and $s = 5/6$. He also proved residue formulas for the poles and a functional equation. These results allow an analytic approach to studying the distribution of cubic fields. For example, together with Perron's formula, these results imply that

$$\sum_{0<\mathrm{Disc}(R)<X} \frac{2}{|\mathrm{Aut}(R)|} = \int_{2-i\infty}^{2+i\infty} \xi^{+}(s)\frac{X^s}{s}ds = \frac{\pi^2}{9}X + \frac{\sqrt{3}\zeta(1/3)\Gamma(1/3)^3}{10\pi}X^{5/6} + O(X^{3/5+\epsilon}). \tag{2.6}$$

Shintani's zeta functions were given an adelic formulation by Datskovsky and Wright [51, 16, 17]. This allowed them to incorporate the p-maximality conditions in Proposition 2.2, and they obtained a second proof of the Davenport-Heilbronn theorem, valid for relative cubic extensions of any global field.

2.3 Roberts' conjecture and the pole at $s = 5/6$

We saw that the counting function for cubic rings has a secondary term, and Roberts [36] conjectured that the same is true for the counting function of cubic *fields*. The highlight of my postdoctoral research was a proof of Roberts' conjecture [44], joint with Takashi Taniguchi:

Theorem 2.3 ([44]). *Roberts' conjecture is true. Indeed, we have*

$$N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{7/9+\epsilon}), \tag{2.7}$$

where $C^- = 3$, $C^+ = 1$, $K^- = \sqrt{3}$, and $K^+ = 1$.

Roberts' conjecture was also proved in independent work of Bhargava, Shankar, and Tsimerman [6], using geometry, with an error term of $O(X^{13/16+\epsilon})$.

To prove Theorem 2.3, we began with a sieve, which Belabas, Bhargava, and Pomerance [1] introduced to count cubic fields with an error of $O(X^{7/8+\epsilon})$. In particular, we have

$$N_3^{\pm}(X) = \sum_{q<Q} \mu(q)N^{\pm}(q, X) + O(X/Q^{1-\epsilon}), \tag{2.8}$$

where $N^{\pm}(q, X)$ counts the number of cubic orders of discriminant $0 < \pm D < X$ which are nonmaximal at every prime dividing q. We count these using *q-nonmaximal zeta functions* $\xi_q^{\pm}(s)$, whose analytic properties are established in Datskovsky and Wright's work and our followup paper [45] (described in Section 2.5). The constants occurring in Theorem 2.3 are limits of residues of adelic Shintani zeta functions.

Bhargava, Taniguchi and I [8] have since improved the error term in Theorem 2.3 to $O(X^{2/3+\epsilon})$. This is the beginning of an unfinished project which we describe in Section 3.1.

2.4 Secondary terms in other counting functions for cubic fields

The well-studied Cohen-Lenstra heuristics [14] describe the average p-ranks of the class groups of quadratic fields. They have not been proved for any $p > 3$, but class field theory implies that subgroups of $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$ of index 3 correspond to cubic fields which are *nowhere totally ramified*, and this allowed Davenport and Heilbronn to compute the average size of $\mathrm{Cl}_3(\mathbb{Q}(\sqrt{D}))$. Taniguchi and I refined their result by proving the existence of a secondary term.

Theorem 2.4. *We have*

$$\sum_{0<\pm D<X} \#\mathrm{Cl}_3(\mathbb{Q}(\sqrt{D})) = \frac{3+C^{\pm}}{\pi^2} X + K^{\pm} \frac{8\zeta(1/3)}{5\Gamma(2/3)^3} \prod_p \left(1 - \frac{p^{1/3}+1}{p(p+1)}\right) X^{5/6} + O(X^{18/23+\epsilon}), \quad (2.9)$$

where the sum ranges over fundamental discriminants D, the product is over all primes, and the constants are as in Theorem 2.3.

Theorems 2.3 and 2.4 also allow us to impose finitely many splitting conditions. For example, we may count cubic fields in which 5 is inert and 7 is totally split.

Finally, both of our results allow us to count discriminants in arithmetic progressions. Here we encounter a curious phenomenon. For example, the following table lists the number of cubic fields K with $0 < \mathrm{Disc}(K) < 2 \cdot 10^6$, arranged by the residue class of $\mathrm{Disc}(K)$ modulo 7:

Discriminant modulo 7	0	1	2	3	4	5	6
Count	15330	17229	14327	15323	17027	18058	15150

In contrast, the numerical data modulo 5 is the following:

Discriminant modulo 5	0	1	2	3	4
Count	21277	22887	22751	22748	22781

Nothing in Davenport and Heilbronn's work explains the discrepancy between these two cases.

Explaining this phenomenon, Taniguchi and I proved formulas for the number of cubic fields K with $0 < \pm\mathrm{Disc}(K) < X$ and $\mathrm{Disc}(K) \equiv a \pmod{m}$. Our formulas are complicated to state, so we refer to [44] for precise results; our formulas closely match the numerical data above.

A similar result holds for 3-torsion in class groups of quadratic fields, and we observed odd phenomena related to this lack of equidistribution. For example, there are more cubic field discriminants congruent to 3 than to 2 modulo 7 or 13, but modulo $91 = 7 \cdot 13$ the pattern is reversed. We also observed an unexplained connection to the arithmetic of elliptic curves; see Section 3.2.

2.5 Orbital L-functions

As we previously described, our proof of Theorem 2.3 relies on an understanding of the q-nonmaximal zeta functions described earlier. General results about such zeta functions were proved by Datskovsky and Wright [51, 16, 17], F. Sato [37], and others, and Taniguchi and I [45] further developed this theory to a point where it is suitable for applications. This paper was naturally motivated by [44], but it proves results beyond what is needed for [44].

In general, we consider Shintani zeta functions

$$\xi^{\pm}(s, \Phi) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} \Phi(x) |\mathrm{Disc}(x)|^{-s}, \tag{2.10}$$

where $\Phi : V_{\mathbb{Z}/m\mathbb{Z}} \to \mathbb{C}$ is a function for which $\Phi(gx) = \chi(g)\Phi(x)$ for some Dirichlet character χ modulo m. The q-nonmaximal zeta functions are a special case of this construction, with $m = q^2$. As another example, we consider the "m-divisible Shintani zeta function", counting only field discriminants modulo m. As an application, I obtained results on almost prime discriminants of cubic fields, before learning that similar results had been obtained by Belabas and Fouvry [2].

For technical reasons, we are led to study the *cubic Gauss sum*

$$\widehat{\Phi}_m(x) = \frac{1}{m^4} \sum_{y \in V_{\mathbb{Z}/m\mathbb{Z}}} \Phi(y) \exp(2\pi i [x, y]/m). \tag{2.11}$$

Sharp bounds for such Gauss sums are an essential tool for obtaining good error terms in Theorems 2.3 and 2.4, and in many cases we proved exact formulas. Some of these results hint at some kind of underlying structure, and so this question may prove interesting in the study of more complicated prehomogeneous vector spaces, such as the 12-dimensional space parameterizing quartic rings.

In addition, we proved explicit residue formulas for the functions in (2.5) for a large class of functions Φ, and extending work of Ohno [34] and Nakagawa [33], we proved a simplified functional equation for a variant of the m-divisible Shintani zeta function.

2.6 Analytic properties of Shintani zeta functions

Initially, I began my study of Shintani zeta functions by investigating their analytic properties. For example, how are the zeroes distributed? Should these zeta functions obey the Riemann hypothesis?

In hindsight, I discovered that there was little room to incorporate particular properties of the Shintani zeta functions into existing methods. I did however obtain some numerical data and general results which are described in [47]. I studied the zeroes of several versions of the zeta function, and I established that $\xi^+(s)$ has a zero outside the critical strip without actually finding one. This follows from a fairly simple criterion in terms of "twisted" sums of the Dirichlet coefficients, which reduces the problem to a long but fairly routine numerical computation. For $\xi^-(s)$ it also appeared that the method would work, but the resulting computation was infeasible.

2.7 Broader impacts of my postdoctoral work

My postdoctoral research program at Stanford was enriched by several educational opportunities, and I anticipate further such opportunities in the future.

At Stanford, I started a weekly learning seminar in analytic number theory with Akshay Venkatesh, and I was its principal organizer for all three years of my postdoctoral tenure. Graduate students gave the majority of the talks, and we also had talks by postdocs, Stanford faculty, visitors, and a Stanford undergraduate.

In March 2010, I was an assistant instructor at the Arizona Winter School in Tucson, AZ. The weeklong Winter School attracted a large audience of advanced graduate students in number theory, and I assisted students with a group project in evening sessions.

In June 2011, I was a co-organizer for the Mathematics Research Community on the Pretentious View of Analytic Number Theory in Snowbird, UT. The workshop attracted 39 graduate student, postdoc, and junior faculty participants, and my duties included reviewing program applications, planning the scientific program, lecturing, assisting with group discussions, and preparing a final report. Several of the participants proved original results during or shortly after the workshop.

I was very active in professional travel. I gave invited lectures at conferences at Yonsei University in Seoul, Kobe University, the University of Tokyo, and the Institute of Mathematical Sciences in Chennai, India, in addition to conferences in the United States and Canada. I also gave multiple seminar talks about my research. My travel to Japan in particular exposed me to interesting work on prehomogeneous vector spaces which is not well known outside Japan, and one aim of this proposal is to further study and build upon this work, and to spread it to a wider audience.

3 Proposed research

I propose to continue my work on Shintani zeta functions, and on zeta functions associated to prehomogeneous vector spaces in general. The approach Taniguchi and I developed in [44] proved to be well adapted to addressing counting problems in number theory, and we both feel that the potential of this approach is far from exhausted.

I intend to undertake much of this work in collaboration. Certainly I will continue my productive collaboration with Taniguchi, and I believe that many of these questions will also interest other researchers working in related fields.

In Sections 3.1 through 3.3 I discuss a variety of further questions involving cubic field discriminants. These range from the very accessible to the (apparently) very difficult. In Section 3.4 I outline an approach for using facts about cubic fields to study the Shintani zeta function itself, applying a Galois-theoretic approach of Cohen and Morra [15].

Section 3.5 is, in my opinion, the most important part of the proposal. The zeta function method may, at least in principle, be used to study quartic and quintic fields, and my work with Taniguchi allows us to circumvent what was previously considered to be the key technical obstacle. However, substantial difficulties remain, and I describe these there.

3.1 Counting cubic extensions of arbitrary global fields

In [17], Datskovsky and Wright extended the Davenport-Heilbronn theorem to count cubic extensions of an arbitrary global field. For any cubic extension L/K of a fixed number field K, write D_L for the absolute norm of the relative discriminant of L/K. Then, Datskovsky and Wright proved that

$$\sum_{\substack{[L:K]=3 \\ D_L<X}} 1 \sim \left(\frac{2}{3}\right)^{r_1-1} \left(\frac{1}{6}\right)^{r_2} \frac{\operatorname{Res}_{s=1} \zeta_K(s)}{\zeta_K(3)} X, \tag{3.1}$$

where r_1 and r_2 are the number of real and complex places of K, and $\zeta_K(s)$ is the Dedekind zeta function. (In this formula, conjugate fields are counted separately.) They also proved a similar result for function fields, and their results naturally allow for local specifications to be imposed on the fields L. These local specifications may include splitting behavior at the infinite places; for example, if $K = \mathbb{Q}$ this is equivalent to specifying the sign of the discriminant.

This work suggests the following natural question.

Problem 1. Prove the analogue of Roberts' conjecture for the counting function of relative cubic extensions L/K. Namely, obtain a secondary term in (3.1) and a further power savings in the error term.

The secondary term should again be of order $X^{5/6}$, as the relevant Shintani zeta functions again have poles at $s = 5/6$. Moreover, it is not difficult to write down an explicit conjecture, following Datskovsky and Wright's work. The difficulty, as in the case $K = \mathbb{Q}$, lies in bounding the error term by something smaller than $X^{5/6}$.

In collaboration with Bhargava, Taniguchi and I have made substantial progress towards a solution. Our work uses the zeta function approach developed in [44], in combination with a result in the work of Bhargava, Shankar, and Tsimerman [6], which establishes a correspondence between certain kinds of nonmaximal rings.

Using this combined approach, we have reduced the error term in Theorem 2.3 to $O(X^{2/3+\epsilon})$. Subject to some checking of details, we have also proved the conjecture when K is a quadratic number field, and we have obtained power saving error terms (larger than $X^{5/6}$) when K is a number field of higher degree. Moreover, again in preliminary work, we have proved secondary terms for any number field K for *smoothed* sums of the form $\sum_{[L:K]=3} \phi(D_L/X)$, where $\phi(t)$ is a smooth function such as e^{-t}.

Our work on arithmetic progressions has an interesting generalization in this setting. If L/K is a cubic extension, the ring of integers $\mathcal{O}_L$ is isomorphic (as an $\mathcal{O}_K$-module) to $\mathcal{O}_K \oplus \mathcal{O}_K \oplus \mathfrak{a}$ for some ideal $\mathfrak{a}$ of $\mathcal{O}_K$; although $\mathfrak{a}$ is not uniquely determined, its class in the ideal class group $\mathrm{Cl}(K)$ is, and it is called the *Steinitz class* of K.

In work prior to our collaboration, Taniguchi [43] formulated a version of the Delone-Faddeev correspondence for relative cubic extensions, and analyzed the distribution of the Steinitz class in cubic algebras extending $\mathcal{O}_K$. For quadratic fields K, he proved that the Steinitz class is not uniformly distributed in the secondary term if $\mathrm{Cl}(K)$ contains a nontrivial 3-torsion element; he proved this by twisting an appropriate zeta function by characters of the class group.

This suggests the following problem:

Problem 2. Study the distribution of the Steinitz class in relative cubic extensions L/K, and obtain generalizations of our results on cubic fields in arithmetic progressions.

We have made some progress on this problem, but some technical issues remain to be resolved. In particular, it appears that it is easier to understand the square of the Steinitz class than the Steinitz class itself.

Datskovsky and Wright's results also hold for function fields (of characteristic not equal to 2 or 3); this is a natural benefit of their adelic approach. Therefore, in this setting we may also hope to answer questions concerning secondary terms, improved error terms, and lack of equidistribution.

The arithmetic of function fields is particularly interesting because of the applicability of techniques from algebraic geometry, and I have learned that ongoing work of Zhao [54] provides an algebro-geometric explanation for a secondary term in the counting function of cubic extensions of $\mathbb{F}_q(t)$. It seems likely that this can also be proved using Shintani zeta functions. This could yield redundant results (and Bhargava, Shankar, and Tsimerman's methods may yield yet another proof), but in my opinion the existence of a variety of viewpoints on the same question lends it deeper interest.

3.2 Connections with elliptic curves

The study of cubic fields has interesting connections with the theory of elliptic curves. Recall that cubic field discriminants are not equidistributed modulo any prime $p \equiv 1 \pmod 6$. For each integer a modulo p, consider the elliptic curve $y^2 = x^3 + ma$ and count the points defined over $\mathbb{F}_7$. For $p = 7$ and $m = 2$, these numbers are strongly and positively correlated with the counting functions of cubic field discriminants $\equiv a \pmod p$. Moreover, a similar phenomenon seems to happen for every prime $\equiv 1 \pmod 6$:

p	7	13	19	31	37	43	61	67	73	79	97
m	2	6	10	6	6	3	10	7	2	2	5
r	.92	.999	.92	.95	.95	.88	.98	.87	.93	.87	.99

For each prime p, the table lists the value of m that best predicts the number of cubic field discriminants $\equiv a \pmod p$, up to multiplication by $(\mathbb{F}_p^\times)^6$, and the statistical correlation coefficient r between the two counts. (For the count of cubic fields we used our asymptotic formula rather than numerical data.)

Problem 3. Determine if, why, and how irregularity in the distribution of cubic fields is predicted by the arithmetic of elliptic curves. In particular, come up with an algorithm or heuristic which predicts the values of m above.

This question was suggested to me by Akshay Venkatesh, on the basis of a parameterization due to Soundararajan [42]. As explained previously, nowhere totally ramified cubic fields are in bijection with subgroups of class groups $\mathrm{Cl}(\mathbb{Q}(\sqrt{d}))$ of index 3. Soundararajan parameterized 3-torsion elements in these class groups by solutions to a Diophantine equation; a version of his result due to Hough [28] is as follows:

Proposition 3.1 ([42, 28]). *Let $d \equiv 2 \pmod 4$ be squarefree and positive. Then, the set*

$$\{(l, x, y, t) \in (\mathbb{Z}^+)^4 \ : lx^3 = l^2y^2 + t^2d,\ l|d,\ (x, ytd = 1)\}$$

is in bijection with primitive ideal pairs $(\mathfrak{a}, \overline{\mathfrak{a}})$ with $\mathfrak{a} \neq 1$ and $\mathfrak{a}^3$ principal in $\mathbb{Q}(\sqrt{-d})$.

Therefore, if there are a lot of solutions to the equation $y^2 = x^3 - t^2d$ for fixed d, one expects large 3-torsion in the class group of $\mathbb{Q}(\sqrt{-d})$, and if the equation $y^2 = x^3 - t^2d$ has many solutions in $\mathbb{Z}/p\mathbb{Z}$ with $d = a$, it might be expected to have many solutions in $\mathbb{Z}$ as d ranges over integers $\equiv a \pmod p$.

Therefore, it seems reasonable to expect numerical data such as that in our table, but with the constant value $m = -1$ (or perhaps $m = -4$) for each p. We do not have any good explanation for why different values of m produce good correlations for each p. Some simple explanations can be ruled out; for example, I checked that m cannot be taken to be any *fixed* divisor of $\pm 2^6 \cdot 3^6$.

Remark. Soundararajan's parameterization needs to be modified for $d \not\equiv 2 \pmod 4$, but adjusting for this does not seem to explain our numerical data. Moreover, our numerical data is similar if we replace our cubic field counts with the analogous counts of 3-torsion elements in class groups.

A study of this problem seems unlikely to lead to the proofs of any theorems. Nevertheless, our data calls for explanation, and it could lead to interesting conjectures, heuristics, or further lines of investigation.

There are further connections to elliptic curves as well. One interesting example is the Brumer-McGuinness conjecture (see [50]). Let $A^{\pm}(X)$ be the number of elliptic curves $E/\mathbb{Q}$ whose minimal discriminant is in $(0, X)$ or $(-X, 0)$; the conjecture (see [50]) states that

$$A^{\pm}(X) \asymp \frac{\sqrt{3}K^{\pm}}{10\zeta(10)} \bigg(\int_1^{\infty} \frac{dx}{\sqrt{x^3 - 1}} \bigg) X^{5/6}, \tag{3.2}$$

where $K^+ = 1$ and $K^- = \sqrt{3}$ as before. Although we don't have room to describe the details here, there is a convincing heuristic argument relating this conjecture to the cubic field secondary term. It seems that the approaches being developed by Bhargava, Shankar, and their collaborators may be more suitable for this particular problem than the zeta function approach, but it may emerge that the zeta function approach has something to contribute.

There are still more connections between cubic fields and the arithmetic of elliptic curves; we refer to the end of the introduction of [44] for further discussion and references.

3.3 Bounds for the a_n

One of the most interesting problems in the subject is the following:

Problem 4. Prove upper bounds for the individual Shintani zeta function coefficients $a_n^{\pm}$.

Up to constants, the same bound would hold for the multiplicity of cubic fields of discriminant $\pm n$, or for the size of $\mathrm{Cl}_3(\mathbb{Q}(\sqrt{\pm n}))$.

It is believed that $a_n^{\pm} \ll_{\epsilon} |n|^{\epsilon}$ for any $\epsilon > 0$, but the best known bound is $a_n^{\pm} \ll_{\epsilon} |n|^{1/3+\epsilon}$ due to Ellenberg and Venkatesh [21]. Nontrivial bounds were previously obtained by Helfgott and Venkatesh [26] and Pierce [35], using a variety of methods.

Our analytic methods suggest a different line of attack on this problem. For example, we have

$$\sum_{n<X} a_n^{\pm} = \int_{2-i\infty}^{2+i\infty} \xi^{\pm}(s) \frac{X^s}{s} ds = \mathrm{Res}_{s=1}\, \xi^{\pm}(s) X + \frac{6}{5} \mathrm{Res}_{s=5/6}\, \xi^{\pm}(s) X^{5/6} + O(X^{3/5+\epsilon}), \tag{3.3}$$

and so $a_n^{\pm} \ll |n|^{3/5+\epsilon}$. This falls short of the trivial bound of $O(|n|^{1/2+\epsilon})$ for the size of the entire class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{\pm n}))$.

If the coefficients $a_n^{\pm}$ are small, then this is reflected in the analytic properties of $\xi^{\pm}(s)$, and there may be some possibility of improving the known bounds using analytic methods. One can consider variations of (3.3), and one may also work with variants of the Shintani zeta function. For example, one may replace $\xi^{\pm}(s)$ with the n-divisible Shintani zeta function $\xi_n^{\pm}(s)$, so that we are attempting to bound the value of the *first* coefficient of a zeta function.

This problem seems surprisingly difficult: I have tried a variety of methods and none of them have yielded nontrivial results. Improving the bounds in [21] may well be out of reach. However, this question continues to motivate my work, and it is at least possible that analytic methods will yield a solution. One possible approach is suggested by the work of Hoffstein and Lockhart [27], who proved bounds for the first Fourier coefficient of a Maass form for $\Gamma_0(N)$. The methods of [27] will definitely not translate directly to our case (the Shintani zeta function does not have a "symmetric square"), but careful study of their work may suggest indirectly related approaches, or invite other questions about the Shintani zeta functions which may be more approachable.

3.4 Sums of Euler products and the Cohen-Morra formulas

In my study of analytic properties of Shintani zeta functions [47], one natural question concerned the distribution of the zeroes. As a basis for conjectures (or at least guesses), I compared the Shintani zeta functions to Epstein zeta functions associated to integral binary quadratic forms, for which there is a substantial literature. In particular, the work of Bombieri and Hejhal [9] establishes that almost all of their zeroes lie on the critical line, subject to GRH and additional unproved hypotheses. This work, however, relies on their representation as a finite sum of Euler products. The Shintani zeta function, by contrast, has no apparent such representation.

However, it is not clear that the Shintani zeta functions cannot be represented as finite sums of Euler products. Indeed, the work of Ibukiyama and Saito [29] exhibits such a representation for the zeta functions associated to a different prehomogeneous vector space, namely, that of $n \times n$ symmetric matrices. Moreover, the work of Wright [52] exhibits another such representation for the "twisted" Iwasawa-Tate zeta function, which is related to *cyclic* field extensions. These representations lead one to ask whether the Shintani zeta functions also have such representations, and Wright asks this explicitly in [52].

I conjecture that the answer is no, and have made definite progress towards a proof.

Problem 5. Prove that the Shintani zeta functions, and various related Dirichlet series, do not admit a representation as a finite sum of Euler products.

This would put the Shintani zeta functions firmly outside the framework studied by Bombieri and Hejhal and others, and would serve as further evidence of their uniqueness within the family of zeta functions.

Thus far I have proved the conjecture for $\xi^-(s)$ and laid out a strategy for $\xi^+(s)$. The key to the proof lies in work of Cohen and Morra [15]. Given an odd fundamental discriminant D, they use class field theory and Galois theory to study the distribution of cubic fields K such that the squarefree part of $\mathrm{Disc}(K)$ is equal to D. For a given such D, denote the associated Dirichlet series by $\Phi_D(s) := \sum_n b(Dn^2)n^{-s}$. The following formula is a special case of their work:

Theorem 3.2 (Cohen-Morra [15]). *If $D < 0$, $D \equiv 3 \pmod 9$, and $3 \nmid h(D)$, then we have*

$$\Phi_D(s) = -\frac{1}{2} + \frac{1}{2}\left(1 + \frac{2}{3^s}\right) \prod_{\left(\frac{-3D}{p}\right)=1} \left(1 + \frac{2}{p^s}\right). \tag{3.4}$$

Applying this formula and using a linear algebra argument, I conclude in [49] that $\xi^-(s)$ cannot be represented as a finite sum of Euler products.

The results of Cohen and Morra are *much* more general. Similar results hold for any fundamental discrminant D, but in the general case the formula for $\Phi_D(s)$ is a sum of multiple Euler products. These Euler products depend on the arithmetic of $\mathbb{Q}(\sqrt{D}, \sqrt{-3})$ (in particular, on the 3-torsion subgroup of its ideal class group) in a sophisticated way, and so it is nontrivial to prove further explicit formulas along the lines of Theorem 3.2. Nevertheless, this should be possible with a moderate amount of work, and obtaining an explicit analogue of Theorem 3.2 for a suitable infinite family of discriminants $D > 0$ will almost certainly lead to a resolution of Problem 5.

This discussion suggests a more general question:

Problem 6. Further explore the implications of Galois-theoretic approaches to the theory of prehomogeneous vector spaces, and vice versa.

Note that the Cohen-Morra approach is not limited to cubic extensions; for example we refer to related work of Cohen [13] on quartic fields.

3.5 Quartic and quintic extensions

The most natural question is to extend our methods to counting quartic and quintic fields. For example, Bhargava proved [3] that the number of quartic fields K with $0 < |\mathrm{Disc}(K)| < X$ is asymptotic to $\frac{5}{24}\prod_p(1+p^{-2}-p^{-3}-p^{-4})X$, where the product extends over all primes, and Belabas, Bhargava, and Pomerance [1] refined Bhargava's argument to prove an error term of $O(X^{23/24+\epsilon})$. Should similar results hold for relative quartic and quintic extensions? Should there be secondary terms, and if so, what?

Bhargava's results, like Davenport and Heilbronn's, rely on the fact that cubic, quartic, and quintic fields are parameterized by lattice points in prehomogeneous vector spaces. We briefly review the general theory, following Kimura's book [31]. (For the sake of simplicity, our definition incorporates several standard hypotheses and is stricter than that in [31].)

Suppose that $\rho : G \to \mathrm{GL}(V)$ is a rational representation of a connected, reductive algebraic group G on a finite-dimensional vector space V. We say that V (or the triplet (G, ρ, V)) is a *prehomogeneous vector space* if we can write $V_{\mathbb{C}} = V'_{\mathbb{C}} \cup S$, where $V'_{\mathbb{C}}$ is Zariski open and consists of a single G-orbit $\rho(G)v_0$, and the *singular set* S is defined by the vanishing of an irreducible polynomial $f(x)$. The function $f(x)$, called a *relative invariant* of (G, ρ, V), satisfies $f(\rho(g)x) = \chi(g)f(x)$ for some rational character $\chi : G \to \mathrm{GL}_1$ of G.

The space of binary cubic forms (2.2), together with the action of GL_2 given by (2.4), is a prototypical example; the relative invariant is the discriminant (2.3), and $\chi(g) = (\det g)^2$.

There is a general theory of *zeta functions* associated to prehomogeneous vector spaces, due to Sato and Shintani [39]. Suppose that $V_{\mathbb{R}} = S_{\mathbb{R}} \cup V_{\mathbb{R}}^1 \cup \cdots \cup V_{\mathbb{R}}^k$, where the $V_{\mathbb{R}}^i$ are the connected components. Then, subject to additional technical conditions, one may define zeta functions

$$\xi_i(s) := \sum_{x \in G_{\mathbb{Z}} \backslash V_{\mathbb{Z}}^i} \frac{1}{|\mathrm{Stab}_{G_{\mathbb{Z}}}(x)|} |f(x)|^{-s}. \tag{3.5}$$

Sato and Shintani proved that these Dirichlet series enjoy meromorphic continuation to $\mathbb{C}$ and satisfy functional equations. The Shintani zeta functions associated to binary cubic forms are a typical example of this construction; however, in general, the Sato-Shintani theorem does not provide an easy way of computing the location of any poles or their residues.

There is a *classification theorem* for prehomogeneous vector spaces, due to Sato and Kimura [38]. Unfortunately, it implies that sextic and higher order rings are not suitably parameterized by any prehomogeneous vector space, and so these methods will not allow us to count sextic and higher order fields.

That quartic and quintic rings *are* so parameterized was proved by Bhargava [3, 5]. Following [3], let $V_{\mathbb{Z}} = (\mathrm{Sym}^2\,\mathbb{Z}^3 \otimes \mathbb{Z}^2)^*$ be the lattice of pairs of integral ternary quadratic forms; the group $G_{\mathbb{Z}} = \mathrm{GL}_3(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})$ acts on $V_{\mathbb{Z}}$ as follows: $\mathrm{GL}_3(\mathbb{Z})$ acts on the two quadratic forms individually, and $\mathrm{GL}_2(\mathbb{Z})$ acts on the 2-dimensional lattice spanned by these quadratic forms. Writing (A, B) for a typical element of $V_{\mathbb{Z}}$, $\det(Ax + By)$ is a binary cubic form, and the unique $G_{\mathbb{Z}}$-invariant $\mathrm{Disc}(A, B)$ on $V_{\mathbb{Z}}$ is defined by $\mathrm{Disc}(4\det(Ax + By))$.

Generalizing the Delone-Faddeev correspondence, Bhargava proved that the set of orbits $G_{\mathbb{Z}} \backslash V_{\mathbb{Z}}$ is in bijection with pairs (Q, R), where Q is a quartic ring and R is a *cubic resolvent ring* of Q. He further proved that any *maximal* quartic ring has exactly one cubic resolvent.

Remark. Bhargava also proved a similar result [5] for *quintic* rings; in this case, pairs (R, S), where R is a quintic ring and S is a sextic resolvent are parameterized by $\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$-orbits on the space $\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$ of quadruples of 5×5 skew-symmetric integer matrices.

The study of quintic fields is very interesting and potentially extremely difficult; for example, Bhargava showed that the relevant fundamental domain has over a hundred cusps. I will turn my attention to quintic fields if I succeed in solving Problems 7 and 8.

By Sato and Shintani's work, one may define *quartic Shintani zeta functions*

$$\xi(s) := \sum_{x \in G_{\mathbb{Z}} \backslash V_{\mathbb{Z}}^i} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s},$$

where V^i refers to any of the three nonsingular connected components of $V_{\mathbb{R}}$; these correspond to the number of complex embeddings of the parameterized fields (zero, two, or four). These zeta functions enjoy analytic continuation to $\mathbb{C}$ and satisfy a functional equation. The quartic zeta function in particular has been extensively studied by Yukie in his monograph [53] and in subsequent unpublished work, and it should be possible to generalize my work with Taniguchi to the quartic setting.

Bhargava [3] offers the following thoughts on the zeta function approach to counting quartic fields:

> The reason that the zeta function method has required such a large amount of work, and has thus presented some related difficulties, is that intrinsic to the zeta function approach is a certain overcounting of quartic extensions. Specifically, ... reducible quartic extensions far outnumber the irreducible ones; indeed, the number of reducible quartic extensions of absolute discriminant at most X is asymptotic to $X \log X$, while we show that the number of quartic field extensions of absolute discriminant at most X is only $O(X)$. This overcount results in the Shintani zeta function having a double pole at $s = 1$ rather than a single pole. Removing this double pole, in order to obtain the desired main term, has been the primary difficulty with the zeta function method.

Prior applications of zeta functions to counting fields (namely, Datskovsky and Wright's work [17] on cubic fields, as well as unpublished work of Yukie for quartic fields) relied on a certain "filtration" process originally used by Davenport and Heilbronn; roughly speaking, this involves passing to a limit. This procedure cannot handle the double pole described by Bhargava, but with the analytic methods introduced in [45], this double pole is unlikely to present a problem. Provided the other difficulties can be handled, our methods will extract the correct contributions from each pole and establish power-saving error terms, and it will only be left to subtract the contribution of the reducible rings.

Problem 7. Extend our work on cubic fields to quartic fields, as follows:

1. Give a proof of Bhargava's theorem using zeta functions, with a better error term than in [1] if possible. Generalize the theorem to allow local specifications, to count fields in arithmetic progressions, and to handle relative quartic extensions.

2. Determine whether or not there should be secondary terms in the counting formulas for quartic fields. If these secondary terms appear to exist, formulate precise conjectures which include them.

3. Prove our conjectures. If (as I currently expect) this proves to be unrealistic, formulate and prove a version of these conjectures for smoothed counting functions, along the lines of Section 3.1.

I believe that this should be relatively straightforward, *assuming* sufficient understanding of the quartic Shintani zeta function. An analytic approach should yield error terms of the shape $O(X^{1-\delta})$ for some explicit value of $\delta > 0$. (In the case of relative extensions, δ would depend on the degree of the base field.) For unsmoothed sums, it seems unlikely that δ would be large enough to obtain secondary terms, or indeed to improve upon [1].

However, removing Bhargava's obstacle leaves another serious difficulty: our understanding of the quartic Shintani zeta function is incomplete in spite of Yukie's substantial efforts, and in particular, formulas for all the residues have not yet been proved. This, then, suggests the following problem.

Problem 8. Continuing Yukie's work, prove the complete analogue of Shintani's theorem for the quartic Shintani zeta function. In particular, determine the location of all of the poles and prove formulas for their residues. The residue formulas should be general enough to include an analogue of the 'q-nonmaximal' zeta function.

Yukie's work relies on rather lengthy computations, and so it would be desirable to simplify his proofs if at all possible. This would be especially useful in extending his results to the still more difficult quintic case, or to other prehomogeneous vector spaces.

In fact, it seems that some simplification may already be possible in the cubic case. In the interest of brevity, our discussion will be somewhat imprecise.

Essentially, the asymptotic formulas for cubic rings come from residue formulas for the Shintani zeta functions, and these residues are evaluated in terms of certain *singular integrals* associated to the singular locus S. Oversimplifying somewhat, the singular locus can be divided into an irreducible part, corresponding to rings such as $\mathbb{Z}[x]/(x^3)$, and a reducible part, corresponding to rings such as $\mathbb{Z}[x]/(x^2) \oplus \mathbb{Z}$. Also, the residue formulas break up into an irreducible part (counting the irreducible rings) and a reducible part (counting the reducible rings).

Roughly speaking, the irreducible part of the singular integrals corresponds to the irreducible residues, and the reducible part to the reducible residues. It is not *a priori* obvious that this should be the case. The papers [16, 40, 45, 51] evaluate the sum of all of the singular integrals, leading to a count in [44] of quadratic and cubic fields, so that it remains to subtract the contribution of the quadratic fields. In principle, it should be possible to count cubic fields directly by ignoring the reducible integrals. In practice, the separated integrals diverge and so one must somehow regularize them or otherwise handle the convergence issues.

Geometrically, most of the reducible cubic rings correspond to points in the cusp of a fundamental domain, and a similar phenomenon (see [4]) happens in the quartic and quintic cases. Moreover, as Bhargava explained to me, these cusps correspond to singular integrals. Therefore, if a simplified proof for cubic fields is possible, similar proofs may work for the quartic and quintic cases.

Along these lines, one interesting reference is the work of Kogiso [31], who obtains Wright's residue formulas for the adelic Shintani zeta function without using Eisenstein series, as Wright does in [51]. Kogiso's proof is simpler thanks to "some tricks", as he refers to the techniques he uses, and I intend to study his work with an eye to improving it, better understanding it, and/or extending it to the quartic case and beyond.

3.6 Further related questions

There are many other interesting questions related to counting fields and Shintani zeta functions. I have highlighted what I believe are the most promising questions, but other questions could come to the forefront of my attention, especially in the long term.

Perhaps the most interesting such questions concern other related vector spaces and parameterizations. For example, the space of binary quadratic forms was studied by Gauss and Dirichlet and given a zeta function by Shintani [41], allowing Chamizo and Iwaniec [10] to obtain asymptotic formulas for $\sum_{n \leq N} h(-n)$. More recently, Bhargava and his students have pioneered the study of *coregular spaces*, whose ring of invariants is generated by finitely many elements as opposed to just one. These spaces parameterize arithmetic invariants such as Selmer groups, and Bhargava suggested the idea of associating and studying Dirichlet series in several variables. This is an interesting long-term prospect for my research.

4 Integration of research, education, and scholarship

I have recently begun a tenure-track position at the University of South Carolina. I was attracted here by the department's tradition of organizing seminars, colloquia, and conferences, of mentoring Master's and Ph.D. students, and of reaching out to the surrounding city of Columbia.

I will continue to be extremely active in traveling to conferences and speaking about my work, and I also anticipate opportunities to organize conferences. I will assist with the organization of the Palmetto Number Theory Series (PANTS), a yearly series of two or three regional conferences, and I also intend to organize a special session at a regional meeting of the AMS, on topics (broadly) related to the research discussed in this proposal. I anticipate other opportunities to organize conferences as well.

I will have many opportunities to mentor students at USC. Previously, I was one of four instructors for Ken Ono's REU (Research Experiences for Undergraduates) program. The students produced a total of thirteen peer-reviewed research publications, one of which [46] I took principal responsibility for supervising. I am currently teaching an introductory graduate course in analytic number theory, and I anticipate opportunities to advise USC graduate students soon. I already have some ideas for projects which may interest students, and while carrying out the proposed research I anticipate developing more. Some of these are largely computational and reasonably elementary, and could be tackled by motivated undergraduates.

4.1 Mathematical communication with Japan

My proposed research offers a particularly good opportunity for cross-cultural collaboration. For historical reasons, much of the work on prehomogeneous vector spaces and their zeta functions has come from Japan. The subject started with the pioneering work of Tatsuo Kimura, Mikio Sato, and Takuro Shintani, and this inspired a younger generation of Japanese mathematicians to continue their work. Although the subject has seen meaningful contributions from outside Japan, most contemporary work on the subject continues to originate from within Japan.

My collaboration with Takashi Taniguchi began when he invited me to speak at a conference at the University of Tokyo. This collaboration itself has been very productive, and I also learned of much interesting work which is not well known in the United States.

I will continue my collaboration with Taniguchi, and I intend to seek out other Japanese collaborators and attend Japanese conferences on topics related to this proposal. My research interests mesh well with those of my Japanese colleagues, and my graduate and postdoctoral training in the United States gives me a different perspective to share. Moreover, I have lived in Japan, am well versed in Japanese culture, and can read, write, and speak the Japanese language. My budget includes a request for (partial) funding for travel to Japan, so that I can continue to contribute to this active area of Japanese mathematics and bring it to the attention of researchers in the United States.

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