$1 + 2 + 3 + 4 + \cdots$

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Math 142

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Introduction.

Srinivasa Ramanujan (1887-1920)

The Man Who Knew Infinity

A fellow Ken Ono student

Ramanujan's second letter to Hardy

"Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. I told him that the sum of an infinite number of terms of the series: $1+2+3+4+\cdots=-1/12$ *under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter. . . ."*

(S. Ramanujan, 27 February 1913)

Warmup.

Ramanujan's proof

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E.G. The constant of the series $1+1+1+\&c = -\frac{1}{2}$

the Sum to x terms $= x = c + \int 1\,dx + \frac{1}{2}$ ∴

We may also find the Constant thus:—

$$c = 1+2+3+4+\&c$$

$$\therefore 4c = \quad 4 \quad + 8 + \&c$$

$$\therefore -3c = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$\therefore c = -\frac{1}{12}$$

2. $\phi(x) + \sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi n}{2} = 0$

Sol. Let $\frac{B_n}{\lfloor n}\psi(n)$ be the coeff^t. of $f^{n-1}(x)$, then

Q.E.D.

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"The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever."

(N. Abel, 1832)

The Riemann zeta function.

Analytic continuation

Theorem (Riemann, 1859)

The zeta function has analytic continuation to all complex numbers $s \neq 1$, with

$$\zeta(s) = \zeta(1-s)\frac{\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}}{\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}}.$$

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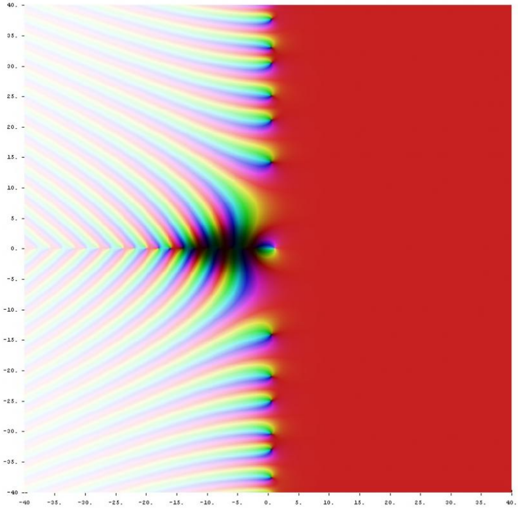
Therefore,

$$\zeta(-1) = \zeta(2)\frac{\Gamma(1)\pi^{-1}}{\Gamma(-\frac{1}{2})\pi^{1/2}} = \frac{\pi^2}{6} \cdot \frac{1 \times \pi^{-1}}{(-2\sqrt{\pi})\pi^{1/2}} = -\frac{1}{12}.$$

Graph of the Riemann zeta function



$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} = \frac{1}{\Gamma(z)} \int_0^\infty dt \frac{t^{z-1}}{e^t - 1}$$

$$\hat{H} = \frac{\mathbb{1}}{\mathbb{1} - e^{-i\hat{p}}} (\hat{x}\,\hat{p} + \hat{p}\,\hat{x})(\mathbb{1} - e^{-i\hat{p}})$$

Poisson summation

The usual proof is by **Poisson summation.**

When you first see it, it looks like a piece of magic.

(anonymous, MathOverflow comment)

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Can compute $\zeta(-1) = -\frac{1}{12}$ using elementary methods?

Integration by parts.

It keeps going...

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$$\int_0^1 P_3(t)dt = 0,$$

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similarly

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so

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)(s+2)}{720} - s(s+1)(s+2)\int_1^\infty \frac{P_4(t)}{t^{s+4}}.$$

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$1 + 8 + 27 + 64 + \ldots$

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and we can compute any value of $\zeta(-n)$ similarly.

Some standard terminology

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- If n is odd, then $B_n = 0$ (except $B_1 = -\frac{1}{2}$).

$$B_{22} = \frac{11(57183 + 20500)}{138}$$

$$B_{24} = \frac{236364091}{2730} = \frac{19.1617^2 + 10.4200^2 + 34.550^2}{2730}$$

$$B_{26} = \frac{8553103}{6} = \frac{13(392931 + 265000)}{6}.$$

$$236364091 + 131040\left(\frac{1^{23}x}{1-x} + \frac{2^{23}x^2}{1-x^2} + \&c\right)$$

$$= 49679091\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^6$$

$$+ 176400000\left\{1 + 240\left(\frac{1^3x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&c\right)\right\}^2$$

$$+ 10285000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^4.$$

$$B_{28} = \frac{23749461029}{870} = \frac{7}{870}\left(19.23.11^2.21^3 + 2.5\,25^2.4549 + 55.10^4.719\right)$$

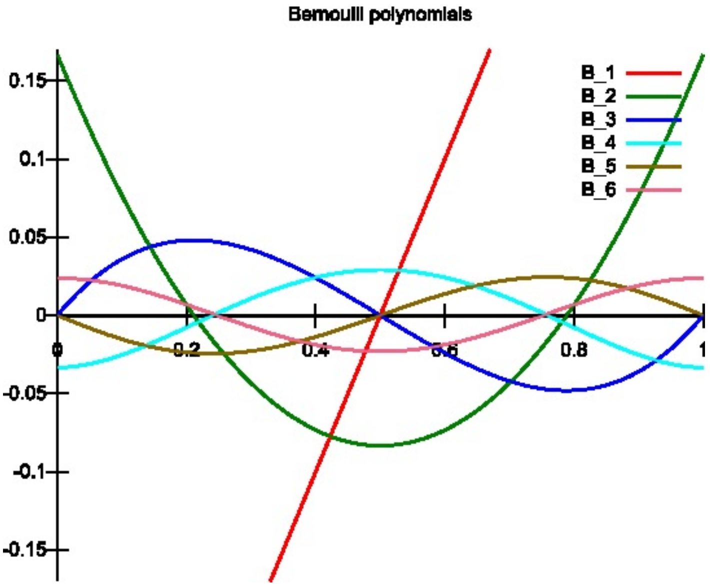
$$B_{30} = \frac{8615841276005}{14322}$$

$$B_{32} = \frac{7709321041217}{510}$$

$$B_{34} = \frac{2577687858367}{6}$$

$$B_{36} = \frac{26315271553053477373}{1919190}$$

Some Bernoulli polynomials



The Euler-Maclaurin sum formula

Theorem

If $f \in C^\infty[0,\infty)$, then for all integers $a,\ b,\ k$ we have

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \frac{1}{2}\Big(f(a) + f(b)\Big)$$

$$+ \sum_{\ell=2}^{k} \frac{B_k}{k!}\Big(f^{(k-1)}(b) - f^{(k-1)}(a)\Big)$$

$$+ \frac{1}{k!}\int_a^b B_k(x) f^{(k)}(t)dt.$$

Example

Stirling's formula: Take $f(n) = \log(n)$:

$$\log(x!) = \sum_{n=1}^{x} \log n \to \int_1^x \log t \; dt + C + \frac{1}{2} \log x.$$

CHAPTER VI

Let $f(1)+f(2)+f(3)+f(4)+\cdots+f(x)=\phi(x)$ then

$$\phi(x) = c + \int f(x)\,dx + \tfrac{1}{2}f(x) + \frac{B_2}{\underline{|2}}f'(x) - \frac{B_4}{\underline{|4}}f'''(x) + \frac{B_6}{\underline{|6}}f^{v}(x) - \frac{B_8}{\underline{|8}}f^{vii}(x) + \&c$$

Sol. $\phi(x) - \phi(x-1) = f(x)$; apply VI.

N.B. By giving any value to x, c can be found. R.S. is not a terminating series except in some special cases. Consequently no constant can be found in $\tfrac{1}{2}f(x) + \frac{B_2}{\underline{|2}}f'(x) - \frac{B_4}{\underline{|4}}f'''(x) + \&c$ except in those special cases. If R S be a terminating series it must be some integral function of

The Ramanujan constant C_R

We have

$$\sum_{n=1}^{x} f(n) = \int_0^x f(t)dt + C_R + \frac{1}{2}f(x) + \sum_{k=2}^{\infty} \frac{B_k}{k!} f^{(k-1)}(x),$$

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where the *Ramanujan constant* C_R is defined by

$$C_R = -\frac{1}{2}f(0) - \sum_{k=2}^{\infty} \frac{B_k}{k!} f^{(k-1)}(0).$$

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The Ramanujan constant of $\sum_{n=1}^{\infty} 1$ is $-\frac{1}{2}$, because

$$\sum_{n=1}^{x} 1 = \int_0^x 1 \; dt + C_R + \frac{1}{2} \cdot 1.$$

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The Ramanujan constant of $\sum_{n=1}^{\infty} n$ is $-\frac{1}{12}$, because

$$\sum_{n=1}^{x} n = \int_0^x t \ dt + C_R + \frac{1}{2}n + \frac{1}{12}.$$

A broader definition of Ramanujan sums?

Definition

(???) We define the value of *any* infinite sum $\sum_{n=1}^{\infty} f(n)$ to be the Ramanujan constant C_R.

A convergent sum

Consider

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - 1.$$

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$$C_R = -\frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot (2k)!$$

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Can we speed up the convergence?

Warning: I am lying on this slide.

A convergent sum (cont.)

Consider instead

$$\sum_{n=1}^{\infty} \frac{1}{(n+5)^2} = \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\right).$$

A convergent sum (cont.)

Consider instead

$$\sum_{n=1}^{\infty} \frac{1}{(n+5)^2} = \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\right).$$

Now the Ramanujan constant is

$$\begin{aligned} C_R &= -\frac{1}{2} \cdot \frac{1}{25} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot \frac{(2k)!}{5^{2k+1}} \\ &= -\frac{1}{50} + \frac{1}{750} - \frac{1}{93750} + \frac{1}{3281250} - \cdots \\ &= -0.018677028\ldots \end{aligned}$$

A convergent sum (cont.)

$\frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \cdots$ is **not** -0.018677028.

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$\frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \cdots$ is **not** -0.018677028. However,

$$\sum_{n=1}^{\infty} \frac{1}{(n+5)^2} \approx -0.018677028 + \int_0^{\infty} \frac{1}{(t+5)^2} dt = 0.181322971\ldots$$

A convergent sum (cont.)

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and

$$\begin{aligned} \sum_{n=1} \frac{1}{(n+5)^2} &= \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\right) \\ &= 0.181322955. \end{aligned}$$

A convergent sum (cont.)

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$$= 0.181322955.$$

This calculation convinced Euler that $\zeta(2) = \frac{\pi^2}{6}$.

Rate of convergence

Question: Does the infinite series

$$C_R = -\frac{1}{2} \cdot \frac{1}{25} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot \frac{(2k)!}{5^{2k+1}}$$

converge?

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No,

$$|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}},$$

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converge?

No,

$$|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}},$$

but this can be fixed rigorously.

How to get the correct constant?

The 'Ramanujan sum' $\sum_{n=1}^{\infty}$ is not equal to $\frac{\pi^2}{6} - 1$.

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The 'Ramanujan sum' $\sum_{n=1}^{\infty}$ is not equal to $\frac{\pi^2}{6} - 1$.

Hardy: Introduce another parameter a.

> *"The introduction of the parameter a allows more flexibility and enables one to always obtain the "correct" constant; usually, there is a certain value of a which is more natural than other values. If* $\sum f(k)$ *converges, then normally we would take* $a = \infty$. *Although the concept of the constant of a series has been made precise, Ramanujan's concomitant theory cannot always be made rigorous."*

(B. Berndt)

Table 1: Summary of Higher Composition Laws

#	Lattice ($V_{\mathbb{Z}}$)	Group acting ($G_{\mathbb{Z}}$)	Parametrizes ($\mathcal{C}$)	(k)	(n)	(H)
1.	$\{0\}$	-	Linear rings	0	0	A_0
2.	$\widetilde{\mathbb{Z}}$	$\mathrm{SL}_1(\mathbb{Z})$	Quadratic rings	1	1	A_1
3.	$(\mathrm{Sym}^2\mathbb{Z}^2)^*$ (GAUSS'S LAW)	$\mathrm{SL}_2(\mathbb{Z})$	Ideal classes in quadratic rings	2	3	B_2
4.	$\mathrm{Sym}^3\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})$	Order 3 ideal classes in quadratic rings	4	4	G_2
5.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^2$	Ideal classes in quadratic rings	4	6	B_3
6.	$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$	$\mathrm{SL}_2(\mathbb{Z})^3$	Pairs of ideal classes in quadratic rings	4	8	D_4
7.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^4$	$\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_4(\mathbb{Z})$	Ideal classes in quadratic rings	4	12	D_5
8.	$\wedge^3\mathbb{Z}^6$	$\mathrm{SL}_6(\mathbb{Z})$	Quadratic rings	4	20	E_6
9.	$(\mathrm{Sym}^3\mathbb{Z}^2)^*$	$\mathrm{GL}_2(\mathbb{Z})$	Cubic rings	4	4	G_2
10.	$\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Order 2 ideal classes in cubic rings	12	12	F_4
11.	$\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})^2$	Ideal classes in cubic rings	12	18	E_6
12.	$\mathbb{Z}^2 \otimes \wedge^2\mathbb{Z}^6$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_6(\mathbb{Z})$	Cubic rings	12	30	E_7
13.	$(\mathbb{Z}^2 \otimes \mathrm{Sym}^2\mathbb{Z}^3)^*$	$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$	Quartic rings	12	12	F_4
14.	$\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$	$\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$	Quintic rings	40	40	E_8