NOTES ON SHINTANI'S PAPER (10/7)

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ABSTRACT. These are my personal notes on portions of Shintani's *On Dirichlet series whose coefficients are class numbers of integral binary cubic forms.* They are designed to be read alongside Shintani's paper, so we will not reproduce many of the formulas, etc. to be found there. I have also described followup work due to Datskovsky-Wright and others, as well as some research questions which seem interesting (and some of which are under current investigation). In connection with the latter I have also described a variety of other techniques which I attempted to apply to the Shintani zeta function. In most cases this was unsuccessful, for reasons I describe.

This is an extremely rough draft and a work in progress.

1. INTRODUCTION

1.1. **A few words of motivation.** A classical question in analytic number theory is to count fields ordered by discriminant. Asymptotics have been proved for degree $d = 2$ (fairly easy), 3 (Davenport-Heilbronn), and 4 and 5 (Bhargava). The result for degree 3 is that the number of fields of discriminant of absolute value $< X$ is asymptotically $X/3\zeta(3)$. (This comes from asymptotics for positive and negative discriminants.)

To prove such a statement, one counts cubic *rings*. These are orders in cubic fields, as well as some mildly degenerate cases. Providing that one can count cubic rings, one then applies a sieve method to count the fields. This degrades the error terms – which is the source of interesting questions.

So one can then count cubic rings. To do this, one uses a theorem proved by Delone and Faddeev (and also, more or less, by Davenport-Heilbronn) which states that there is a canonical, explicit, discriminant-preserving correspondence between the set of cubic rings up to isomorphism and the set of $\mathrm{GL}_2(\mathbb{Z})$- equivalence classes of integral binary quadratic forms.

So, we can count these equivalence classes. This is essentially a geometric problem: We have some nice four-dimensional space, and the discriminant is a degree four homogeneous polynomial. Our problem is to count lattice points in some fundamental domain for $\mathrm{GL}_2(Z)$. Once we figure out a suitably nice fundamental domain (i.e., the tails can't be too bad), the result for cubic rings follows.

The parameterization of such rings is a fascinating subject, pursued further by Bhargava and others. We will not pursue this further here. But we do observe that this provides substantial motivation for studying $\mathrm{GL}_2(\mathbb{Z})$ orbits on prehomogeneous vector spaces. The main subject of these notes is work of Shintani, who proved that one can naturally attach a zeta function to this data.

1.2. **The main theorem.** L denotes the lattice of integral binary cubic forms:

$$L := \{x(u, v) = x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3;\ x_1, x_2, x_3, x_4 \in \mathbb{Z}\}.$$

$\widehat{L}$ is the **dual lattice**

$$L := \{x(u, v) = x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3 \in L;\ x_2, x_3 \in 3\mathbb{Z}\}.$$

There is a natural notion of **discriminant** (the discriminant of the associated cubic polynomial), and there is an action of $\mathrm{SL}_2(\mathbb{Z})$ on L and (also, as one can check, on $\widehat{L}$) given by

$$gx(u, v) = x((u, v)g).$$

As one can check, this action preserves the discriminant.

We want to define a **class number** $h(n)$ (resp. $\widehat{h}(n)$) as the number of $\mathrm{SL}_2(\mathbb{Z})$-orbits on L (resp. $\widehat{L}$) of discriminant n. This is finite (not obvious, but not so difficult, and proved in Shintani's paper)

Our proof requires us to make a technical amendment to this. Some $\mathrm{SL}_2(\mathbb{Z})$ orbits of positive discriminant have nontrivial stablizers, always of order 3. In particular, it is proved in [Proposition 2.2, p. 151] that the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^4$ acts simply transitively on the negative part, and acts transitively with an isotropy group of order 3 on the positive part. When we go from $\mathbb{R}$ to $\mathbb{Z}$ the stabilizer group is a subgroup of the $\mathbb{R}$-stabilizer group.

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We call such orbits "of the second kind", and other orbits "of the first kind". Note that the orbits of the second kind are easily counted, see Prop. 2.12, p. 168 of [?].

We note further that by work of Davenport-Heilbronn, Delone-Faddeev, and Bhargava, there is a 1-1, discriminant-preserving correspondence between these equivalence classes and cubic rings up to isomorphism. As we will (briefly) describe later, this allows us to count cubic fields by discriminant.

The **Shintani zeta functions** are then defined by

$$\xi_+(L,s) := \sum_n \frac{1}{n^s}\left(h_1(n) + \frac{1}{3}h_2(n)\right),$$

$$\xi_-(L,s) := \sum_n \frac{1}{n^s} h(-n).$$

We further define $\xi_\pm(\widehat{L},s)$ to be the same thing, with respect to the class numbers $\widehat{L}(n)$.

Theorem 1.1 (Main Theorem. Shintani, 1972 [?])**.** *The above series converge absolutely for $\Re(s) > 1$, have meromorphic continuation to all of $\mathbb{C}$ with poles at $s = 1$ and $s = 5/6$, and satisfy the functional equation*

$$\begin{pmatrix} \xi_+(L,1-s) \\ \xi_-(L,1-s) \end{pmatrix} = \Gamma\left(s - \frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s + \frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s} \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3\sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \xi_+(\widehat{L},1-s) \\ \xi_-(\widehat{L},1-s) \end{pmatrix}. \tag{1.1}$$

Remark. As we will discuss later, this can be simplified by diagonalizing the matrix.

In these notes we will discuss Shintani's paper and the proof of the main theorem above. We will also discuss followup work by a variety of other mathematicians. and we will connect Shintani's work with more classical arithmetic questions. We will also discuss some open questions, as well as our own attempts to answer some of them.

2. Relation to counting cubic rings

We have the formula

$$\xi_\pm(s) = 2\sum_{\mathcal{O}} \frac{1}{c(\mathcal{O})}|\text{disc } \mathcal{O}|^{-s},$$

where $c(\mathcal{O}) = 6$ if $\mathcal{O}$ is a cubic order, $c(\mathcal{O}) = 2$ if $\mathcal{O}$ is a sum of a quadratic order and a multiple of $\mathbb{Z}$, and $c(\mathcal{O}) = 3$ if $\mathcal{O}$ is three multiples of $\mathbb{Z}$. This is fairly simple to prove. We use the Delone-Faddeev parameterization, except that Delone-Faddeev counted $\mathrm{GL}_2(\mathbb{Z})$ orbits and Shintani counts $\mathrm{SL}_2(\mathbb{Z})$ orbits. So, as is readily checked, the parameterization with $\mathrm{SL}_2(\mathbb{Z})$ is 2-1 to cubic orders and 1-1 to other cubic rings.

As computed by Wright, the stabilizer of a point (in $\mathrm{SL}_2(\mathbb{Z})$) is of size 1 whenever the Galois closure of the field generated by x is degree 2 or 6, and 3 otherwise. (Note that the 3 case then corresponds to the trivial case, and to cyclic cubic extensions, which are necessarily real.) So the Shintani zeta function counts cyclic extensions with weight 1/3, and non-Galois extensions with weight 1.

We also remark that if we limit $\mathcal{O} \otimes \mathbb{Q}$ to direct sums of cyclic Galois extensions (i.e., everything other than S_3 extensions), this can be understood by other means (i.e. class field theory).

To count cubic **fields**, one figures out how to distinguish the **maximal** cubic orders. This turns out to be characterized by a local condition at each prime p, and using a sieve method all this information can be put together. See Davenport-Heilbronn as well as Bhargava's paper.

This is also closely related to counting 3-torsion in quadratic fields (**to do:** elaborate).

3. Summary of Shintani's proof

Definition 3.1. *The (completed)* **zeta function** *is*

$$Z(f,L;s) := \int_{\mathrm{GL}_2^+(\mathbb{R})/SL_2(\mathbb{Z})} (\det\ g)^{6s} \sum_{x \in L'} f(gx) dg.$$

Here f is some nice test function, and L' is the lattice $\mathbb{Z}^4$, minus the zero locus.

Why the 6? Let $P(x)$ be the discriminant polynomial. Then, they key relation is

$$P(gx) = (detg)^6 x.$$

For example, a diagonal matrix α increases each of four coordinates by $\alpha^{3/2}$, since V corresponds to a cubic form, and the discriminant polynomial is homogeneous of degree 4.

The key relation is the following:

$$Z(f,L;s) = \frac{1}{4\pi}\xi_1(L,s)\int_{V_1}|P(x)|^{s-1}f(x)dx + \frac{1}{12\pi}\xi_2(L,s)\int_{V_2}|P(x)|^{s-1}f(x)dx.$$

There is a similar equation for the "dual lattice" also.

Remark. I seem to have been sloppy with notation. ξ_1 and ξ_+ mean the same thing, and ξ_2 and ξ_- mean the same thing.

Proof. We'll be vague on some of the details, while explaining what's going on. (See p. 170.) In particular, we will assume absolute convergence; this will be true for the real part of s sufficiently large. So, we switch the order of summation, and get

$$\begin{aligned} Z(f,L;s) &:= \sum_{x\in L'}\int_{\mathrm{GL}_2^+(\mathbb{R})/SL_2(\mathbb{Z})}(\det g)^{6s}f(gx)dg \\ &= \sum_{m\neq 0}\sum_{x\in L', P(x)=m}\int_{\mathrm{GL}_2^+(\mathbb{R})/SL_2(\mathbb{Z})}(\det g)^{6s}f(gx)dg. \end{aligned}$$

Now, divide it up (for each m) into the $h(m)$ orbits. Also, write $I(i,m)$ for the stabilizer of a typical point in each orbit, and $\nu(i,m)$ for its cardinality. This is

$$\sum_{m\neq 0}\sum_{i=1}^{h(m)}\sum_{\gamma\in \mathrm{SL}_2(\mathbb{Z})/I(i,m)}\int_{\mathrm{GL}_2^+(\mathbb{R})/SL_2(\mathbb{Z})}(\det g)^{6s}f(g\gamma x_i(m))dg.$$

This, in turn, is

$$\sum_{m\neq 0}\sum_{i=1}^{h(m)}\frac{1}{\nu(i,m)}\int_{\mathrm{GL}_2^+(\mathbb{R})}(\det g)^{6s}f(gx_i(m))dg.$$

Now, this is

$$\sum_{m\neq 0}\sum_{i=1}^{h(m)}\frac{1}{\nu(i,m)}\frac{1}{|m|^s}\int_{\mathrm{GL}_2^+(\mathbb{R})}|P(gx_i(m))|f(gx_i(m))dg.$$

Notice how slick that was?

We've now lost the dependence on the point x in the integral, except on its sign. In particular, we have the equation (Proposition 2.4, p. 153)

$$\int_{\mathrm{GL}_2^+(\mathbb{R})}f(gy)dg = \frac{1}{4\pi}\int_{V_i}(P(x))^{-1}f(x)dx,$$

for $y\in V_1$, and a similar formula for V_2. (We have 12π there, because of this stabilizer issue.) The proof of Proposition 2.4, in turn, depends on the exact formula for the measure on g.

What this means is that the above is

$$\frac{1}{4\pi}\left(\sum_{m>0}\sum_{i=1}^{h(m)}\frac{1}{\nu(i,m)}\frac{1}{|m|^s}\right)\left(\int_{V_1}P(x)^{-1}f(x)dx\right) + \frac{1}{12\pi}\left(\sum_{m<0}\sum_{i=1}^{h(m)}\frac{1}{\nu(i,m)}\frac{1}{|m|^s}\right)\left(\int_{V_2}P(x)^{-1}f(x)dx\right).$$

Now we just count stabilizers. For $m<0$, $\nu(i,m)$ is always 1, and hence it goes away. For $m>0$, $\nu(i,m)$ might be 1 or 3, which explains the two sets of class numbers above. $\square$

Now, let's go back the second half of the "key equation", namely,

$$F_i(s,f) := \frac{1}{\Gamma(\cdots)}\int_{V_i}f(x)|P(x)|^s dx.$$

With appropriate gamma factors (see p. 162), these are proved to be entire, holomorphic, and have a FE like what we are trying to prove. This is by what I call "general nonsense": I mean that this sort of equation is proved (see p. 142, and indeed all of the first section) for a quite general setup, where (G,V) is any "prehomogeneous vector space".

A **prehomogeneous vector space** is a vector space V, along with a group action of a group G on V so that there exists a G-orbit in V of the same dimension of V. Some equivalent definitions exist.

Remark. Note that a **homogeneous space** is similar: It is a pair (G, X) so that G acts transitively and continuously.

Let us take the "general nonsense" for granted. Its proof takes up pp. 134-149 of Shintani's paper, so it is far from trivial. We also point to p. 142, which is the main theorem of that section.

We further point out that pp. 155-165 also concern this, except specialized to the case of interest. The exact functional equation is derived, the residues are computed, and the like.

So this leaves us the zeta function. If we can understand that, then we're basically done. Write

$$Z^+(f, L; s) := \int_{\mathrm{GL}_2^+(\mathbb{R})/SL_2(\mathbb{Z}); \det g \geq 1} (\det g)^{6s} . \sum_{x \in L'} f(gx) dg.$$

Then, by Poisson summation, we have (Proposition 2,14, p. 171)

$$Z(f, L; s) = Z^+(f, L; s) + Z^+(\widehat{f}, \widehat{L}; 1 - s) - \int_{\mathrm{GL}_2^+(\mathbb{R})/SL_2(\mathbb{Z}); \det g \geq 1} (\det\ g)^{6s} \Bigg(\sum_{x \in L_0} f(gx) - \chi^{-1}(g) \sum_{x \in \widehat{L_0}} \widehat{f}(g^\iota x) \Bigg).$$

Shintani claims that this is an immediate consequence of the Poisson summation formula. Here's a proof with some fudging.

We have

$$\int_{G^+/\Gamma} \chi(g)^s \sum_{x \in L} f(gx) = \int_{G^+/\Gamma,\ \det > 1} \chi(g)^s \sum_{x \in L} f(gx) + \int_{G^+/\Gamma,\ \det < 1} \chi(g)^s \sum_{x \in L} f(gx).$$

I'm using Shintani's notation, so that $G^+ = \mathrm{GL}_2(\mathbb{R}),\ \Gamma = \mathrm{SL}_2(\mathbb{Z}),\ \chi(g) = (detg)^6$.

The first half is nice and the second half is not. So we use the Poisson summation formula, which tells us in this context that

$$\sum_{x \in L} f(gx) = \chi(g)^{-1} \sum_{x \in L'} \widehat{f}(g^\iota x).$$

I seem to have missed the minus sign.

This involution ι is defined by (2.13) on p. 154. We just have

$$g^\iota = (\det\ g)^2 g^{-1},$$

although the definition there is mysteriously complicated. See also Proposition 1.2 for some generalities. The involution satisfies nice formal properties with respect to an identification of V with its dual space, and L with its dual lattice. Note that this requires an alternating binary form; see p. 154.

Anyway, the point is that what we have covers the **whole** lattice, so we need to subtract off the zero locus to get our zeta function. The sum over the zero locus is the hard part, and that is where the Eisenstein series comes in. Let us not discuss the proof at all. Instead, we point to the Corollary to Proposition 2.16 on p. 182, which consists of an evaluation of this sum over the zero locus. (Proposition 2.16, on p. 175, is an evaluation of a meaty portion of the sum.) The end section of Shintani's paper, then, essentially puts all of this together.

The following is a description of Shintani's paper, and the pieces of the proof to be described in more detail. Probably this will be revised as I learn all of this better – !!

3.1. **Philosophical digression.** So what Shintani does, in total, is to write down a zeta function $Z(f, L; s)$ and prove it has nice properties: Shintani doesn't state it here, but it should have analytic continuation, functional equation, and the like.

He then breaks it up, via fairly elementary methods, into a product of a Dirichlet series and a real part. He proves that the real part has analytic continuation and is essentially something nice – I think that I should think of this as some analogue of the gamma function. (That is a one-dimensional Mellin transform, and you get analytic continuation by repeated integration by parts.) Moreover, he proves that the zeta function as a whole has analytic continuation. Therefore, the Dirichlet series has the desired properties.

It is interesting to compare this with the classical case. There are two proofs really. In the really classical case, the gamma function is the Mellin transform of e^{-t}, and by adding up a bunch of n^{-s} factors (i.e. by multiplying by the zeta function) we obtain the Mellin transform of the sum of $e^{-\pi n^2 t}$, i.e., by a modular form. In particular, we can use its modular transformation law, or equivalently, Poisson summation.

The role of this in Tate's thesis is played by his Poisson formula and Riemann-Roch theorem. Here we get to use any nice test function. Essentially this turns out to be the same proof if you stare at it closely.

Let's be a little bit more explicit about this. $GL(1)$ acting on $\mathbb{R}$ is a good prehomogeneous space, with $\mathbb{Z}$ being our lattice, and the zeta function is

$$Z(f, L; s) := \int_{\mathbb{R}^+} \Big(\sum_{x \in L'} f(tx) \Big) t^s d^\times t.$$

Then, this general machinery proves that

$$Z(f, L; s) = Z(\widehat{f}, \widehat{L}; 1 - s).$$

Note that $L = \widehat{L}$. If we choose the familiar choice of $f(x) = e^{-\pi x^2}$, then $f = \widehat{f}$ and this should all look familiar.

Now if we pull out the sum, **in the same way as above**, we get

$$\Big(\sum_{n \geq 1} \frac{1}{n^s} \Big) \Big(\int_{\mathbb{R}^+} f(x) x^s dx \Big).$$

The Dirichlet series on the left (i.e. the Riemann zeta function) is what we want to prove AC and FE for. So we tackle the guy on the right. In particular, (chasing down the exact notation in Shintani's paper), we end up proving that

$$\frac{1}{\Gamma(s+1)} \int_0^\infty f(x) x^s d^\times x$$

has analytic continuation (hint: integrate by parts over and over again) and is in fact entire. As an example, let $f(x) = e^{-x}$. But as a less dumb example, let $f(x) = e^{-\pi x^2}$. Note that we will pick up zeroes in this case. AC is easy to prove, but it's not a tautology.

Essentially, the Mellin transform of any nice Schwartz function behaves sort of like the gamma function. Ideas like this underlie Tate's thesis and we will be exploring them further.

One question one might ask is whether Shintani's work can be Tate's Thesis-ified. This was done by Datskovsky and Wright; see later. However there is a complication: the Dirichlet series does not have an Euler product. There are certain formulas that take the place of the product; see below.

3.2. **The rest of the paper.** Here is a description of what's in the rest of the paper.

Prehomogeneous vector spaces. He starts off with general descriptions of what a prehomogeneous vector space is. We are interested in *relative invariants*; for our case that means the discriminant. In particular, the group acts on this invariant by the quasicharacter $(\det g)^6$. We consider both V and its dual, and fix an identification between them. This, then, leads to the construction of certain differential operators, and discussion of Fourier transforms. In (1.13) on p. 138, the function

$$F_i(s, f) = \frac{1}{\gamma(s)} \int_{V_i} |P|^s f(x) dx$$

is defined, for a gamma factor described in (1.8) and an arbitrary Schwartz function f. Some lemmas and formal properties are proved, such as analytic continuation, and then on p. 142 there is a general functional equation is proved for these. It involves a matrix; there is one such function as above for each orbit of the group on the space. The proof of this extends through p. 147.

It will be worthwhile to compare this to a more well-known case. Let

$$F(s, f) := \int_0^\infty f(x) x^{s-1} dx.$$

Then for good Schwartz functions f this has meromorphic contination to all of $\mathbb{C}$. You prove it by repeated integration by parts. We also have the analytic continuation

$$F(s, f) = \rho(s) F(1 - s, \widehat{f}),$$

where

$$\rho(s) = 2^{1-s} \pi^{-s} \cos(\pi s/2) \Gamma(s),$$

which in particular is independent of s.

This is proved in Tate's thesis as follows: First of all let $f(x) = e^{-\pi x^2}$, which is its own Fourier transform, and compute that both sides are 1. Then, for another choice of f we make a formal calculation which is only valid in some region of convergence, and get the same functional equation. But by uniqueness of analytic continuation, we're good.

Dirichlet series and binary cubic forms. Shintani describes the space of binary cubic forms, the lattice in it, and their duals. He discusses the decomposition of G, and how this allows us to define a measure, and proves Proposition 2.4, which allow us to interpret the F_i as integrals over the group.

Fourier transforms of complex powers of the discriminant. There is a lot here which I need to understand better. For example, the meaning of the distributions Σ_1 and Σ_2. But the punchline seems to be a specialization of the general functional equation described above to the case of interest.

Dirichlet series and the lattice. Now he describes the action of $\mathrm{SL}_2(\mathbb{Z})$ on the lattice of integral binary cubic forms, computes the orbits, stabilizers, and so on. The computations described in detail above come next.

Computations and Eisenstein series. Shintani introduces the Eisenstein series and talks a lot about what properties they have. Note that Datskovsky's paper explains better than Shintani's why we might want to do this. In Proposition 2.16 on p. 175 and its corollary on p. 182, these are applied to evaluate the sum we are interested in.

The finish From p. 183 on, Shintani puts everything together to prove the main theorem.

4. Chapter 1: PHV's

In pp. 134-149 Shintani carries out what I described earlier as "general nonsense". Of course this is not really nonsense at all. Shintani proves analytic continuation and a functional equation for general formulas of the type

$$\int_{V_i} f(x)|P(x)|^s dx,$$

where $P(x)$ is a relative-invariant, and $f(x)$ is a general Schwartz function. Here a relative-invariant is a function P for which $P(gx) = \chi(g)P(x)$ for a quasicharacter χ.

I have not yet read this portion of the paper very carefully, so this is omitted from these notes for now.

5. Dirichlet series and cubic forms, I: (Chapter 2.1)

5.1. **Sections 1 and 2. V and G.** In this section the space of binary cubic forms is introduced, along with the action of $G = \mathrm{GL}_2$ on it.

V represents the space of binary cubic forms, and we identify it with $\mathbb{C}^4$. $G = \mathrm{GL}_2$ acts on it, and if P is the discriminant polynomial, we have

$$P(gx) = (\det g)^6 P(x).$$

We write

$$\chi(g) := (detg)^6.$$

Now G has a unique cyclic central subgroup of order 3. Call it Z. Then G/Z operates "effectively" on V and so we can identify it with an algebraic subgroup $\widehat{G}$ in $\mathrm{GL}(V)$.

[p. 151] Proposition 2.1 states that $(\widehat{G}, V)$ is a prehomogeneous vector space whose set of singular points corresponds with the set of degenerate binary cubic forms.

Let G_+ be the subgroup of GL_2 of matrices of positive determinant. Let V_1 and V_2 be the subsets of V where $P > 0$ and $P < 0$ respectively. Note then that G_+ acts separately (and transitively) on each.

In Proposition 2.2 (p. 151), Shintani proves that the isotropy group for this action is of order 3 for V_1 and trivial for V_2. Note that this is the same as the number of real roots. The proof takes a little bit of doing, but it does not seem too mysterious.

Write G^1 for $\mathrm{SL}_2(\mathbb{R})$.

In Proposition 2.3, Shintani analyzes the action of G on S (the singular set). It decomposes into three orbits (one of which is just zero) for the action of G_1. One has trivial stabilizer and the other has stabilizer similar to $\mathbb{R}$.

5.2. **Decomposing G. The alternating form.** Starting on p. 152, Shintani discusses a familiar mapping

$$KA_+N \to G^1.$$

This is an analytic diffeomorphism. Here K is SO_2 (the maximal compact), etc. This is useful because it gives us an explicit way of decomposing elements of G^1. Using this decomposition, natural invariant measures on G^1 and G^+ are defined.

[p. 153] Proposition 2.4 tells us that

$$\int_{G_+} f(gy)dg = \frac{1}{4\pi} \int_{V_1} (P(x))^{-1} f(x)dx.$$

This is very interesting, so we should stop and take a look. First of all, we notice on the left that there is no dependence on y whatsoever. Morally we are just integrating $f(g)$, but that doesn't make literal sense. But then again we need a description of our identification of G_+ with V_1.

But if we think about it, we can sort of give a proof by "pure thought", except for the constant factors out front. The idea is that, up to a constant (which results essentially from that identification mentioned above),

$$d(gx) = (detg)^6 dx.$$

So essentially, $P(x)^{-1}dx$ is the unique G_+-invariant 4-form.

to do: Do the calculation where you get the constants.

[p. 154] Shintani defines an alternating form on V, which allows an identification of V with V^* (presumably so that all the Fourier transforms will makes sense, etc.) Presumably, this is the specific version of all the generalities considered earlier.

6. 2.2. Fourier transforms of complex powers of the discriminant

In this section, Shintani constructs two distributions whose supports are contained in the singular locus. (Recall that this will be the hard part to evaluate.) Notice that these two distributions correspond exactly to the two G-orbits on the zero locus. (There is also the trivial orbit.) He then calculates their Fourier transforms. They are G^1-invariant integrals, and they will become important later on.

(Say more about this.)

6.1. **Computing the functional equation.** At this point Shintani begins a variety of computations to calculate the functional equation for the PHV of interest. (To be reread and described here.)

7. 2.3: Dirichlet series and the lattice

[p. 165] Let's get the discrete part in. We let L be the lattice of binary cubic forms. That's just $\mathbb{Z}^4$. We let $\widehat{L}$ be the dual lattice, with respect to our mystery bilinear form.

We write L_m for the set of lattice points of discriminant m, and $\Gamma := \mathrm{SL}_2(\mathbb{Z})$.

Proposition 2.10 decomposes L_0 into countably many Γ-orbits, and Proposition 2.11 decomposes L_m into finitely many. Furthermore, there is an explicit bound on the "class number". The proof does not look too hard.

[p. 168] If $m < 0$, then the isotropy subgroup of any point is trivial over $\mathbb{Z}$, since it is over $\mathbb{R}$. If $m > 0$, the isotropy subgroup over $\mathbb{R}$ is of order 3, but we don't know what it is over $\mathbb{Z}$. If trivial, we call this an "orbit of the first kind", and denote the number of orbits of this type $h_1(m)$. If order 3, we call this an "orbit of the second kind", and denote the number of orbits of this type $h_2(m)$.

[p. 168] Proposition 2.12. $2h_2(m)$ is equal to the number of integer solutions to

$$(9x^2 + 3xy + y^2)^2 = m.$$

Remark. Notice in particular that this implies there are no orbits of the "second kind" unless m is a square. Presumably if I read the proof in detail I would understand why. It is nothing too difficult or theoretical... just run some group theory and calculate what the isotropy subgroup is.

Remark. Recall that this expression occurred in our original Dirichlet series. These orbits need to be counted separately.

We also obtain an expression for $\widehat{h}_2(m)$. Pretty simple, once we remember that the definition of $\widehat{L}$ (or, rather, what we explicitly compute it to be) is a simple variant of the definition of L.

Now on p. 169, Shintani defines the same Dirichlet series again. But this time he says what they "really are". For example,

$$\xi_1(L, s) = \sum_n \frac{1}{n^s}(h_1(n) + \frac{1}{3}h_2(n)).$$

These converge absolutely for $\Re(s) > 3$ by our simple-minded "class number" bound.

This brings us to Prop. 2.13 on the bottom of p. 169. The proposition says, **morally**, that

$$Z(f, L; s) := \int_{G_+/\Gamma} (\det g)^{6s} \Bigg(\sum_{x \in L - L_0} f(gx) \Bigg) dg = *\xi_1(L, s) \int_V |P(x)|^{s-1} f(x) dx.$$

Really, the star is different for V_1 and V_2 by a factor of 3, because of this stabilizer business.

The proof was discussed in the introduction, and we don't repeat it here.

In a corollary, Shintani observes that is we restrict $Z(f, L; s)$ to those g for which the determinant is larger than 1, we get analytic continuation. It makes sense, **but** I am confused on how that is a corollary of the above. But, with this restriction on the determinant, we use the fact that f is a Schwartz-Bruhat function, and so this *will* converge.

And now, in p 171, we have the following. Define $Z+(f, L; s)$ to be the zeta function restricted to g with determinant ≥ 1. This converges absolutely. Then, Proposition 2.14 states that

$$Z(f, L; s) = Z^{+}(f, L; s) + Z^{+}(\widehat{f}, \widehat{L}; 1-s) - \int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6s} \Big(\sum_{x \in L_0} f(gx) - \chi^{-1}(g) \sum_{x \in \widehat{L}_0} \widehat{f}(g^{\iota} x) \Big) dg.$$

Digression. The classical case. It is interesting to compare this with the classical case, where the completed zeta function has the formula

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2} \int_0^{\infty} \Big(\theta(iy) - 1)\Big) y^{s/2} \frac{dy}{y}.$$

Notice that we are obliged to subtract off the zero locus! (i.e., that -1.) Else, we would have a term of $1/0^s$ in our zeta function.

Explicitly: We have

$$Z(2s) = \frac{1}{2} \int_0^{\infty} \Big(\theta(iy) - 1\Big) y^{s} \frac{dy}{y},$$

where

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

The above converges for $\Re s > 1$. Now, as usual, split the integral up. On $[0, 1]$ we can split the integral up for $\Re s > 1$. We get

$$\begin{aligned} Z(2s) &= \frac{1}{2} \int_1^{\infty} \Big(\theta(iy) - 1\Big) y^{s} \frac{dy}{y} - \int_0^1 y^s \frac{dy}{y} + \int_0^1 \sqrt{y}\theta(i/y) y^s \frac{dy}{y} \\ &= \frac{1}{2} \int_1^{\infty} \Big(\theta(iy) - 1\Big) y^{s} \frac{dy}{y} - \int_0^1 y^s \frac{dy}{y} + \int_1^{\infty} \theta(iy) y^{1/2-s} \frac{dy}{y}. \end{aligned}$$

The first integral converges everywhere, and the second everywhere except 0. For the last one we work in a left half-plane, and get

$$= \frac{1}{2} \int_1^{\infty} \Big(\theta(iy) - 1\Big) y^{s} \frac{dy}{y} - \int_0^1 y^s \frac{dy}{y} + \int_1^{\infty} \Big(\theta(iy) - 1\Big) y^{1/2-s} \frac{dy}{y} + \int_1^{\infty} y^{1/2-s} \frac{dy}{y}.$$

We evaluate the integrals, and we get the usual AC and FE.

Back to the case of interest.

As of now, the following computations are only valid for $\Re s > 3$. We don't otherwise have convergence over the zero locus. We have

$$Z(f, L; s) = Z^{+}(f, L; s) + \int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6s} \Big(\sum_{x \in L'} f(gx) \Big).$$

Now for $\Re s$ big, we can poke around and rewrite the latter integral as

$$\int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6s} \Big(\sum_{x \in L} f(gx) \Big) - \int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6s} \Big(\sum_{x \in L_0} f(gx) \Big).$$

Use Poisson summation on the first integral. Recalling formulas from earlier about the action of the group on Fourier transforms, we have

$$\int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6(s-1)} \Big(\sum_{x \in \widehat{L}} \widehat{f}(g^{\iota} x) \Big) - \int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6s} \Big(\sum_{x \in L_0} f(gx) \Big).$$

Now, separate out the zero locus; this is

$$\int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6(s-1)} \Big(\sum_{x \in \widehat{L}_0} \widehat{f}(g^{\iota} x) \Big) + \int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6(s-1)} \Big(\sum_{x \in \widehat{L}'} \widehat{f}(g^{\iota} x) \Big)$$

$$- \int_{G_{+}/\Gamma,\ \det g \leq 1} (\det g)^{6s} \Big(\sum_{x \in L_0} f(gx) \Big).$$

And now do a change of variable (g for g^{ι}) on the second integral, and we have the formula we wanted.

But then again, our final answer is pretty simple. So we ask: Can this term be "understood"? And we get involved with kind of a complicated mess. Eisenstein series and such. Was this inevitable?

7.1. **Eisenstein series and their integrals.** We wish to understand expressions itshape like

$$\int_{G^+/\Gamma} (\det g)^{6s}\Big(\sum_{x\in L_0} f(gx)\Big).$$

How will we accomplish this?

[p. 171] **Eisenstein series** are defined by

$$E(z,g) := \sum_{\gamma\in\Gamma/\Gamma\cap B} t(g\gamma)^{z+1}.$$

Here B is a Borel subgroup: the group of lower triangular matrices. The variable t is defined by the analytic diffeomorphism

$$K\times A_+\times N \to G^1,$$

where A_+ consists of diagonal matrices $(t, 1/t)$.

Remark. It bears investigating whether I can get this to look like the usual $cz+d$. I have taken it completely for granted that this is a nice Eisenstein series with the nice properties claimed.

The Eisenstein series is "nice", as described in Proposition 2.15 on p. 171: It converges for $\Re z > 1$, and has analytic continuation (in z) and a functional equation. The residue at $z = 1$ is given, and there is a Fourier expansion in u. The shape of this expansion is not particularly familiar.

Remark. I'm a little bit confused by the form of the integral. We just fix some z_0, and then the integral doesn't depend on it? (The proof of such a statement would follow from bounds on the Eisenstein series at infinity, so presumably we have this)

The basic idea. Here is the basic idea of what is going on in the proof. We wish to evaluate expressions like

$$\int_{G^1/\Gamma}\sum_{x\in L_0} f(gx).$$

We don't know how to do that. So instead throw in an Eisenstein series, and define

$$I := \int_{G^1/\Gamma}\Big(\sum_{x\in L_0} f(gx)\Big)\sum_{\gamma\in\Gamma/\Gamma\cap B} t(g\gamma)^{z+1}.$$

Subject to convergence, this is

$$I = \int_{G^1/\Gamma\cap N}\sum_{x\in L_0} f(gx)t(g\gamma)^{z+1}.$$

Since we are integrating over a larger, simpler group, this turns out to be something we know how to evaluate. On the other hand, we can pass to a limit in the expression involving Eisenstein series above, and recover the expression that we originally wanted to evaluate.

Now the above is a big lie and is not what we are actually doing. Let's dive into the legitimate details.

We define

$$\mathcal{E}(\psi, w, g) := \frac{1}{2\pi i}\int_{\Re z = x_0}\psi(z)\frac{E(z,g)}{w-z}dx.$$

This is defined for $\Re w > 1$. We have $\psi\in\Psi$, which prescribes some decay properties, and x_0 must be between 1 and $\Re w$. It should be provable that the integral does not depend on w. (As I mentioned earlier, this would depend on decay properties of the Eisenstein series.)

The interesting property, proved in Lemma 2.9(iii) (statement on p. 172) is that

$$\lim_{w\to 1^+}(w-1)\mathcal{E}(\psi, w; g) = \frac{\psi(1)}{\xi(2)}.$$

The proof of this is by the Cauchy integral theorem. We shift the contour, and pick up the residue indicated above. But then the remaining limit is zero.

I presume that the proof of the latter fact is roughly as follows: Near the origin, the integral is something fixed and bounded, and we are multiplying by $w-1$ and taking a limit as $w\to 1$. So that part goes to 0. Away from the origin, presumably the decay of the test function ψ overwhelms the Eisenstein series.

[p. 174] In Lemma 2.10, a function
$$J_L(f) := \sum_{x \in L_0} f(gx)$$
is introduced, and the same for $\widehat{L}$. Growth conditions are proved on these functions, and a sharp growth condition is proved on $J_L(f) - J_{\widehat{L}}(f)$. The proof of the latter is rather interesting, and I didn't really get the principle.

[p. 175] This brings us to Proposition 2.16, which is the crux of the matter, and whose proof takes up seven pages. We evaluate
$$\int_{G^1/\Gamma} \Big(\sum_{x \in L_0} f(gx) - \sum_{x \in \widehat{L_0}} \widehat{f}(gx) \Big) dg.$$
This is essentially the expression we wanted to evaluate in the first place, except there we need to integrate over $\int_{G/\Gamma}$ there – so we use this result and add an extra variable.

Now, by our result earlier (and our bounds on $J_L(f) - J_{\widehat{L}}(f)$), we have
$$\frac{\psi(1)}{\xi(2)} \int_{G^1/\Gamma} \Big(\sum_{x \in L_0} f(gx) - \sum_{x \in \widehat{L_0}} \widehat{f}(gx) \Big) dg = \lim_{w \to 1^+} (w-1) \int_{G^1/\Gamma} \Big(\sum_{x \in L_0} f(gx) - \sum_{x \in \widehat{L_0}} \widehat{f}(gx) \Big) \mathcal{E}(\psi, w; g) dg.$$
By arguments which I discussed earlier, we have absolute convergence for $\Re w > 3$ and so we can break up the integral.

We now recall our decomposition of L_0 into orbits. This is vaguely complicated, and so we make a little bit of a mess. In particular, we get the terms in the sublemma on p. 177, and we want to evaluate them all.

The proof of (i) corresponds to the $\{0\}$ orbit - by far the simplest. It illustrates the general principle; we simply need to evaluate
$$\int_{G^1/\Gamma} \mathcal{E}(\psi, w; g) = \frac{\psi(1)}{2(w-1)}.$$
At this point I will refrain from copying the details – the principle is fairly nice and the details are messy. See p. 178. The idea is as we described before: We get an integral over $G^1/\Gamma \cap N$, and this we know how to evaluate. In particular, previously we had a decomposition $G^1 = K \times A_+ \times N$. So our integral is over $K \times A +_\times (N \cap \Gamma)$, and we can (and very much do) break it up in this way. We reduce it to something simple, and eventually we only pick up one residue from Cauchy's theorem.

We also have parts (ii), (iii), (iv). There, we need to use the Fourier expansion for the Eisenstein series. In (ii), only the constant term ends up mattering. All of the other terms oscillate, and get integrated out over an appropriate interval $[0, 1]$, and so contribute nothing.

Note that the statement of the sublemma involves an interesting equivalence relation. The equivalence means, up to adding some function which is holomorphic in $\Re w > 0$. Indeed, we see at the bottom of p. 178 a holomorphic function which just gets ignored. Note that later, we multiply these expressions by $w - 1$ and take a limit as $w \to 1$, which means that the holomorphic part dies.

The proof of (iii) is a bit messier – the non-constant terms in the Eisenstein series come in to play; Shintani uses Parseval, etc. I have not yet read this part of the argument.

[p. 182] Now, as a corollary, we derive an expression for the thing we want! The proof makes sense. Basically, just introduce another variable, which represents the determinant, and integrate over that. The previous proposition worked for any test function, and here the new test function varies continuously in our parameter.

to do: Read it more carefully.

8. Conclusion

Finally, on page 184, the main result more or less falls out. There's some doing to it, but we just sort of put everything we've done together. I could read this more thoroughly, but it's fairly clear what's going on.

9. Datskovsky and Wright's work

Here we will describe (note: this all certainly needs to be revised) a series of three papers of Datskovsky and Wright. They give an adelic formulation of the Shintani zeta function, and as a consequence extend the definition of the Shintani zeta function to all global fields. Their work has a number of other interesting consequences as well.

9.1. **Wright's first paper.** In Wright's first paper (The adelic zeta function... I), Wright defines the adelic zeta function

$$Z(\omega, \Phi) := \int_{G_{\mathbb{A}}/G_k} \omega(\det g) \sum_{x \in V'_K} \Phi(gx) dg.$$

He proves analytic continuation with a nice functional equation, which is stated at the very end of his paper. The function has poles at 0, 2, 1/3, and 5/3, and he computes the residues.

The paper reads like a combination of Shintani's paper and Tate's thesis, and is organized as follows. In Chapter 1 he talks about functional analysis on the adeles, reviews Tate's thesis, and proves a couple of lemmas concerning seminorms with some nice upper bounds. In Chapter 2 he talks about the usual representation of GL_2 on V. Note that he does so over the global field K, in contrast to Shintani's paper, which was interested in $\mathbb{Z}$ and $\mathbb{R}$. In Chapter 3 he defines an invariant measure on the adeles, etc.

In Chapter 4, he finally defines the zeta function as above. He proves that it converges in a right half-plane (and uses the lemmas proved in Chapter 1). He also "proves" the formula

$$Z(\omega, \Phi) = \sum_{x \in G_k \backslash V'_k} |G_{k,x}|^{-1} Z_x(\omega, \Phi),$$

where

$$Z_x(\omega, \Phi) = \int_{G_{\mathbb{A}}} \omega(\det g) \Phi(gx) dg.$$

The proof is essentially formal diddling, subject to proving absolute convergence. But, note that this is where we might hope to get a Dirichlet series.

Remark. The tilde on V'_k means we restrict attention to the nonsingular part.

There is a double coset decomposition on pp. 518-519, whose significance I haven't yet figured out. Proposition 4.2 does look kind of interesting.

In Chapter 5, he talks about singular invariant distributions, i.e., eigendistributions. This is the same definition I read in Kudla's article. But he doesn't do any general theory. Instead he defines two distributions T_1 and T_2, both of which are pretty simple, and proves meromorphic continuation, etc. He defines an "averaging operator"

$$(M_\omega \Phi)(x) := \int_{U_{\mathbb{A}}} \omega(\det k) \Phi(kx) dk,$$

where $U_{\mathbb{A}}$ is the maximal compact subgroup. This has nice formal properties, and we then get a relatively invariant distribution. We define Σ_1 and Σ_2 in terms of it, and these have meromorphic continuation, etc.

Finally, in Section 6, he proves the main theorem, following Shintani closely. We get the Eisenstein series and all.

9.2. **Datskovsky and Wright, The adelic zeta function II.** The interesting fact I want to get at is that the Shintani zeta function is

$$\sum_{\mathcal{O}} |\mathrm{disc}(\mathcal{O})|^{-s} = \zeta(4s)\zeta(6s-1) \sum_K |\mathrm{disc}(K)^{-s}| \frac{\zeta_K(2s)}{\zeta_K(4s)}.$$

Of course, there are two things to explain: One, the equality above, and two, why the left side is the same thing as the Shintani zeta function.

Furthermore, we obtain a similar zeta function, with analytic continuation and the like, by considering the S-adeles. We can then derive the Davenport-Heilbronn results by a limiting process. But it is not nearly delicate enough to recover the secondary term.

Used in these results are a variety of interesting relations between discriminants over global and local fields, and "mass formulas" due to Serre.

Only fifty pages. Not that bad, right?

So what's in the paper? In Section 1 he talks about functional analysis in a local field. It is the usual familiar stuff. In Section 2 he talks about binary cubic forms.

In Section 3 he talks about orbital zeta functions. These are the same as $X_z(\omega, \Phi)$ above, except that the integral is over G_K instead of $G_{\mathbb{A}}$. (K is a local field, presumed to have characteristic not 2 or 3.) There is a discussion of differential operators, etc., which may illuminate the corresponding discussion in Shintani's paper if I read it closely.

Now Theorem 3.1 is interesting. If α is an open orbit (Zariski?), then if ω is ramified, the orbital zeta integral is 0. If it is unramified, then it is given explicitly! (This is on p. 42.) It breaks up into five cases depending on what the orbit corresponds to (e.g., a quadratic extension, a cubic extension, ...)

We also obtain a functional equation

$$Z_\alpha(\omega, g\Phi) = \omega(\det g) Z_\alpha(\omega, \Phi),$$

which doesn't leave much freedom of what this function can be.

In particular, in the same way as in Kudla's article, we obtain a functional equation for free. It comes in the form of a matrix, whose dimension is equal to the number of orbits.

In Section 4, the authors determine the matrix coefficients for local fields. $\mathbb{R}$ was done in Shintani, and $\mathbb{C}$ was also done earlier.

Remark. See the interesting comment on p. 51. Also: It would be an interesting exercise to do the computation for $\mathbb{R}$ in the same way that's done here.

Section 5 is sort of like Section 5 of the previous paper... it goes on for a long time... they do some extra stuff whose significance I don't quite understand.

The meat seems to be in Section 6. We start with the fact (which is "obvious"), that the G_k-orbits in V_k' correspond to the conjugate sets of extensions k' of k of degree not greater than three. The reason, essentially, is that points in V_k' correspond to cubic polynomials in an obvious way, which are determined by their roots. And two sets of roots generate the same field if and only if there is a $\mathrm{GL}_2(k)$ linear transformation taking one set to the other.

There is a mild complication arising from the fact that the cubic extensions might not be Galois, but suffice it to say these complications are dealt with.

One then proceeds by decomposing the global zeta function into a sum of orbital zeta functions:

$$Z(\omega, \Phi) = \sum_{x \in G_k \backslash V_k'} |G_{k,x}|^{-1} Z_x(\omega, \Phi).$$

The sum ranges over a set of representatives x of all G_k-orbits; ω is a fairly general quasicharacter. $G_{k,x}$ is the stabilizer of x under the action of G_k.

And one now sees where our formula comes from: the sum is now (basically) over cubic extensions, and each summand will be an Euler product, depending on k.

Now let's follow D+W more closely. On p. 64 there is the usual setup and notation, etc. It goes on through p. 66. Maybe I'm skipping over something important? But a lot of it looks routine.

Now, let S be any finite set of places. Minimally, this needs to at least include all infinite places – which is why the "basic" Shintani zeta function breaks up into two parts. For each $v \in S$ write A_{k_v} for the set of local orbits. Each A_{k_v} is finite. Then, write

$$A_S := \prod_{v \in S} A_{k_v}.$$

We choose "standard orbital representatives" x_{α_v} – this was described in Section 2. Define, for any $\alpha, \beta \in S$, (i.e., $\alpha = (\alpha_v)_{v \in S}$)

$$Z_\alpha := \prod_{v \in S} Z_{\alpha_v}(\omega_v, \Phi_v).$$

Here ω and Φ break up naturally (they are required to); the local zeta function was presumably defined before.

Now, for each $x \in V_k'$, choose elements $g_{x,v}$ of G_{k_v} so that $x_{\alpha_v} = gx, vx$, where α_v corresponds to the G_{k_v} orbit containing x. (I assume we only need to do this for $v \in S$?) We obtain the formula

$$Z_x(\omega, \Phi) = \tau_k \omega(\det g_x) \prod_{v \in M_k} Z_{\alpha_v}(\omega_v, \Phi_v).$$

Here M_k is the set of **all** places of k.

This brings us to

The big formula. We want to get here:

$$Z(\omega, \Phi) = \tau_k \sum_{\alpha \in A_S} \left(\prod_{v \in S} Z_{\alpha_v}(\omega_v, \Phi_v) \right) \sum_{k' \in \alpha} \frac{1}{o(k')} \omega(\sqrt{\Delta_{k'}}) \prod_{v \notin S} Z_{\alpha_v}(\omega_v, \Phi_{0,v}).$$

Now this is not how Datskovsky and Wright state it. It confused me at first. So let's try and dissect it.

The first thing to realize is that it is really just saying

$$Z(\omega,\Phi) = \tau_k \sum_{k'} \frac{1}{o(k')} \omega(\sqrt{\Delta_{k'}}) \prod_v Z_{\alpha_v}(\omega_v, \Phi_v).$$

This comes from the above and from a calculation of the determinant. But the point is that for places not in S we prescribe a standard function $\Phi_{0,v}$. Also, we break up things that depend only on the G_S-orbit of k' and put them before the sum over k'.

Moreover, the latter product (over $v \not{i}nS$) vanishes if ω is ramified outside S.

Now what is $o(k')$? It is $|G_{k,x}|$ where x is any element of V'_k corresponding to k. Also, Δ'_k is the adelic discriminant; see p. 67 of the paper for further discussion.

Now we have analytic continuation for the zeta function as a whole, and (also, as it turns out) the individual factors. In particular, write

$$\eta_{k',S} := \prod_{v \notin S} Z_{\alpha_v}(\omega_v, \Phi_{0,v}).$$

(The dependence on k' is reflected in the choice of α_v.) Then, the $\eta(k',S)$ are fairly simple and defined in terms of Hecke L-series; see p. 68 and Theorem 6.1 (as well as elsewhere). Note this works for any quasicharacter ω, generalizing what Shintani did.

Denote $\xi_\alpha(\omega)$ by

$$\xi_\alpha(\omega) := \sum_{k' \in \alpha} \frac{1}{o(k')} \omega(\sqrt{\Delta_{k'}}) \eta_{k',S}(\omega),$$

another piece of the big zeta function above. Then this is holomorphic, unless $\omega = \omega_2$ or $\omega = \chi\omega_{5/3}$, for any character χ of order 3. We obtain functional equations as well, and the formula (6.5) looks a lot like what you see in Tate's thesis for Dirichlet L-functions. There are formulas for the residues also (see pp. 69-70). See Theorem 6.2 on p. 71 for a big recap of all of this.

Datskovsky and Wright conclude by relating this to Shintani's work. As they write, recall the definition of k_S and $k_S^\times$ from earlier in the section. For the reader's convenience, they do not bother to say where in the section the definition is. Also, let $O_S = k \cap \mathbb{A}(S)$ be the ring of S-integers in k. (i.e., integrality is required outside S.) Then, write

$$Z_S(\omega_S, \Phi_S) = \int_{G_{k_S}/G_{O_S}} \omega_S(\det g) \sum_{x \in V'_{O_S}} \Phi_S(gx)|dg|_S.$$

I am guessing (the authors do not appear to say...) that if $S = \mathbb{R}$, then we are integrating over $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2(\mathbb{Z})$, so that this becomes the Shintani zeta function – and on the other hand this thing that we've been studying the whole time. This was studied in Wright's thesis. Furthermore, in Part I of this series, in Proposition 4.2, he derives a relation between the adelic zeta function and its S-adelic version.

There is a decomposition of $\mathbb{A}^\times/k^\times\mathbb{A}^\times(S)$ into h_S classes, and then D and W write

$$G_{\mathbb{A}} = \cup_{1 \le i \le h_S} G_{\mathbb{A}(S)} g_i G_k.$$

Using this decomposition, we have

$$Z_S(\omega_S, \Phi_S) = \tau_k^{-1} \frac{1}{h_S} \sum_\phi Z(\phi\omega, \Phi),$$

where ϕ runs over the characters of $\mathbb{A}^\times/k^\times\mathbb{A}^\times(S)$. There are h_S of them. (This is **really** nice.) And this forms the basis for further analysis, etc.

9.3. **D and W, Density of discriminants of cubic extensions.** They follow up this work with a generalization of Davenport and Heilbronn for any number field.

We recall the definition of the Dirichlet series

$$\xi_\alpha(s) = \sum_{k' \in \alpha} \frac{1}{o(k')} D_{k'}^{-s} \eta_{k',S}(s),$$

from the previous paper. It's changed a little bit: In particular, the general quasicharacter is gone, and the adelic discriminant has been replaced by this above, for reasons I'm not entirely clear on.

To obtain the Davenport-Heilbronn relations, the idea is to **take a limit as** $S \to M_k$. (Recall, M_k means "everywhere".) It is not too difficult to understand (at least in my opinion) why one gets a limit as one does this. The question I have is, why does one get exactly the count of fields?

Except: They say that Theorem 1.1 is deduced from Theorem 4.3 by taking S to be empty. I suppose this makes sense. We want cubic extensions in α, and we don't want to impose any conditions at all.

Working backwards, this is deduced from Theorem 4.1 by subtracting quadratic extensions. Unfortunately, Theorem 4.1 involves some notation. It counts collections **specified by local conditions at a finite number of places** k, or in their notation, by C_S.

Let's go back and figure out what the notation means.

First of all, let C denote any collection of extensions of k of degree ≤ 3. Presumably, we will be interested in all the cubic extensions. But we can look at cubic extensions with a specified Galois group, or quadratic extensions, for example.

Let A_v index the set of extensions of k_v of degree ≤ 3, up to conjugacy. (I believe this should also correspond to the G_{k_v}/G_{O_v}-orbits, or **something similar**, but I'm not completely sure.) Write A_S for products of these. Now, for any S, write

$$C_S := \cup_{\alpha \in A_S, \alpha \cap C \neq 0} \alpha.$$

This notation is confusing and I definitely want to figure it out. In particular, it is claimed that C_S **contains** C**.** Presumably, this would be obvious if I weren't confused.

We associate a sequence of Dirichlet series, adding over all fields in C_S. We define $\delta_S(C)$ to be the residue of this Dirichlet series.

In Lemma 3.1, it is proved that the $\delta_S(C)$ form a decreasing sequence of positive numbers, as S grows. Therefore, they approach a limit. This is an appeal to Lemma 2.1, and Lemma 2.1 refers to the original definition of the partial zeta functions: They are the usual Euler products, with places in S excluded. Therefore, as S grows, the Dirichlet series get smaller. So it is essentially a tautology.

But: I'm still confused about that first definition.

Now, we also have the claim (Lemma 3.2) that

$$\delta(C) = \lim_{S \to M_k} \delta(C_S).$$

This is essentially a chain of tautologies: We have, first of all,

$$\delta(C) \leq \delta(C_S).$$

This is because C_S contains C. Subject to that claim, this is indeed immediate. Then,

$$\delta(C_S) \leq \delta_S(C_S).$$

That comes from the Dirichlet series above. It's just the definition of δ_S.

And, finally,

$$\delta_S(C_S) = \delta_S(C).$$

The point (or so the authors claim) is that this is something we'll be able to use to calculate.

Let's try and figure out what is going on here. If S consists of all places (here by "all" we mean we're taking a limit), then the Euler product $\eta_{k',S}$ will approach 1, and $L_{C,S}(s)$ will just be counting fields in C without weights. If S only contains the infinite place, we have the full Euler product included, and this is the Shintani zeta function.

See also Roberts's paper for a description of this α business. And **come back to me later!!!**

Question 1. *Do these η products just count orders in certain field extensions? They must, by uniqueness of discriminants...*

9.4. **Datskovsky, The adelic zeta function... in a function field.** This is the function field version of Wright's paper above. The zeta functions turn out to be rational functions in q^{-s}, as one would hope. The proof is like that of Wright, and nicely explains some of the technical innovations in Shintani's paper.

But still, the paper is somewhat disagreeable – although on the plus side, perhaps this offers opportunities for further work. In the first place, he doesn't compute any of the polynomials which turn up. For any Schwartz function f the zeta function is given on p. 743. Actually, I'm not sure if that messy $I(\omega, f)$ is a rational function or not. But it is asserted that the first part is.

I would like to see the following theorem (so perhaps I ought to do it myself): Write down a Dirichlet series corresponding to the etale cubic extensions of $F_q[t]$. Write down more than one, if necessary. Then, this Dirichlet series has analytic continuation, functional equation, and hopefully it's the following rational function:

$$\cdots$$

10. Diagonalization

It was observed by Datskovsky and Wright in the work above (in fact, originally by Datskovsky in his thesis) that the functional equation can be diagonalized. In other words, by diagonalizing the matrix, we obtain simple functional equations for linear combinations of the positive and negative Dirichlet series.

This becomes most compelling when combined with work of Ohno and Nakagawa. Recall the dual lattices occuring in the work before. It seems that no one had computed very many of the Dirichlet coefficients. Ohno was the first to do so, and he conjectured that

$$\xi_+(\widehat{L}, s) = 3^{-3s}\xi_-(L, s),$$

$$\xi_-(\widehat{L}, s) = 3^{1-3s}\xi_+(L, s)$$

based on numerical evidence. Later, Nakagawa proved his conjecture using techniques from class field theory. He also asks for some bijection. Akshay doesn't necessarily believe it should exist.

By putting all of this together with Shintani's original theorem, one obtains the following. Write

$$\xi_{\mathrm{add}}(s) := 3^{1/2}\xi_+(L, s) + \xi_-(L, s),$$

$$\xi_{\mathrm{sub}}(s) := 3^{1/2}\xi_-(L, s) - \xi_-(L, s),$$

$$\Lambda_{\mathrm{add}}(s) := \left(\frac{432}{\pi^4}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{12}\right)\Gamma\left(\frac{s}{2}-\frac{1}{12}\right)\xi_{\mathrm{add}}(s),$$

$$\Lambda_{\mathrm{sub}}(s) := \left(\frac{432}{\pi^4}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{5}{12}\right)\Gamma\left(\frac{s}{2}+\frac{7}{12}\right)\xi_{\mathrm{add}}(s).$$

Then

$$\Lambda_{\mathrm{add}}(s) = \Lambda_{\mathrm{add}}(1-s),$$

$$\Lambda_{\mathrm{sub}}(s) = \Lambda_{\mathrm{sub}}(1-s).$$

Both have poles at $s = 1$, and Λ_{add} also has a pole at $s = 5/6$.

Note that these are now zeta functions of the usual form (i.e. in the Selberg class), except without an Euler product, and with an extra pole at $5/6$. We can now apply much of the usual analytic machinery of L-functions, and indeed we will.

Remark. It was suggested during a talk in Seoul that the gamma factor of $\Gamma(s/2 - 1/12)$ is also outside the usual Selberg class, because the constant $-1/12$ should be nonnegative in the Selberg class.

11. Interesting related work

11.1. **The relation to cubic rings and fields.** The above doesn't say anything about cubic rings and fields. But for that we appeal to work of Davenport and Heilbronn, Delone and Faddeev, and Bhargava.

Theorem 11.1. *There is a canonical bijection between the set of* $\mathrm{GL}_2(\mathbb{Z})$ *equivalence classes of binary cubic forms and the set of isomorphism classes of cubic rings.*

See, for example, Bhargava, *Higher composition laws II* for an explicit description.

This is almost what we want. Now we notice that $-1 \in \mathrm{GL}_2(\mathbb{Z})$ (i.e. the matrix -1, 1) ...

But there is another interesting bit as well. In Datskovsky-Wright's first paper, they compute the stabilizer of an arbitrary point $x \in V_K$. (K is any field, but think $K = \mathbb{Q}$.) This stabilizer has size 6, 2, 3, 1 respectively if x and its conjugates generate a field of degree 1, 2, 3, 6.

We note in particular that this provides a good "explanation" for the 1/3 coming in the definition of the Shintani zeta function. If x has stabilizer of size 3, then it generates a cyclic Galois extension. On the other hand if x has trivial stabilizer then it generates three conjugate extensions.

11.2. **Yukie's book.** There's the book by Yukie handling the degree 4 case. The details seem extraordinarily complicated.

11.3. **Taniguchi's work.** In this section we will describe some recent (and ongoing!) work of Takashi Taniguchi, Yasuo Ohno, and Satoshi Wakatsuki. Let V_Q be the vector space of binary cubic forms with rational coefficients. In [**?**], OTW compute the complete list of lattices which are invariant under $\mathrm{SL}_2(\mathbb{Z})$. It turns out there are ten of them, up to scaling, including the two (L and $\widehat{L}$) which we have considered previously. The classification is elementary, and reasonably simple, and all of them arise by imposing conditions on various sums of the coefficients modulo 2. Furthermore, they split up into five pairs of lattices and dual lattices, where in the duals the middle coefficients are required to be divisible by 3.

OTW then prove that for the lattices L_2 and L_3, a relation of the same shape as the Ohno-Nakagawa relation holds. The proof of this is elementary and reduces these lattices to linear combinations of L_1, restricted to certain congruence classes, and then appeals to the Ohno-Nakagawa result.

For L_4 and L_5 the results are more complicated and more interesting, and obtained by Ohno and Taniguchi in [**?**]. They have to dig one step deeper. The theorem proved looks messier although not particularly deeper. But for the proof they do go deeper: they investigate subsets of L_1 which are no longer $\mathrm{SL}_2(\mathbb{Z})$-invariant but invariant under congruence subgroups such as $\Gamma_0(2)$ or $\Gamma(2)$. They also obtain the same sort of diagonalized functional equations as in Nakagawa-Ohno.

The proof involves statements such as the following. Let A by the subset of L consisting of elements of discriminant congruent to 4 modulo 32. Let X_1 be a certain lattice. Then,

$$\Gamma(1) \times_{\Gamma_0(2)} X_1$$

. There is a definition of what the notation means. We are "inducing" G-sets (i.e., sets with group actions on them). But it's impossible to read the detailed definition. The real definition, I think, is the abstract nonsense one.

12. Questions about the zeroes

One can ask interesting questions about the distribution of the zeroes. Here are a few of them, as well as some of my attempts to sort of answer them. Most of these have proved unsuccessful.

Much of this work was motivated by Hejhal's ICM proceedings article on Epstein zeta functions and supercomputers.

We recall that an **Epstein zeta function** is defined by

$$\sum_{(u,v)\neq(0,0)} (au^2 + buv + cv^2)^{-s}.$$

For $a, b, c \in Z$ and $b^2 - 4ac < 0$, this is a part of a Dedekind zeta function, corresponding to one element of the class group, and can be written (using the orthogonality relations) as a finite linear combination of Hecke L-functions.

Epstein zeta functions satisfy the following properties (see Hejhal's article for more and for references to the proofs)

- At least $\gg T$ zeros are on the critical line.
- At most $O(T)$ zeros are to the right of any line $Re(s) > \sigma > 1/2$.
- At least $\gg T$ nontrivial zeros are outside the critical strip.

An open question is: Are almost all the zeros on $Re(s) = 1/2$?

We will adopt as a working hypothesis that the zeros of Shintani zeta functions ought to behave like those of Epstein zeta functions. For example, we tried to prove all of the above statements. We also performed some numerical experiements. In particular, I have determined that some of the Shintani zeroes are on the critical line, and some are not. In an ongoing calculation, I am computing the zeroes on the critical line. This uses Mike Rubinstein's L software. It is currently very slow, but Rubinstein tells me major improvements to his program are on the way.

In general I am disappointed with what I've done so far, and for "good reason". It seems that all of these techniques tend to depend on the same general facts about zeta functions: the degree and shape of the functional equation, bounds on the Dirichlet coefficients. There does not seem to be any interesting way to get distinctive or interesting information about the Shintani zeta function into the picture. So, unfortunately, this seems to mostly be a dead end.

At least for now.

12.1. **The number of zeroes.** Let $N(T)$ count the nontrivial zeroes $s = \sigma + it$ with $|t| < T$. Then σ must be close to the critical strip. Also, we have

$$N(T) = \frac{T}{\pi} \log\left(\frac{432T^4}{(2\pi e)^4}\right) + O(\log T).$$

This follows by the usual proof. Let C be a rectangle from $-2 - iT$ to $3 + iT$. Then,

$$N(T) = \int_C \frac{\Lambda'(s)}{\Lambda(s)} ds,$$

where $\Lambda(s)$ is the completed zeta function. We could have put $\xi(s)$ in here. But with $\Lambda(s)$ the functional equation works out nicely so that the left side of the contour is equal to the right side. In addition, the horizontal pieces can be estimated, so that the number of zeros is essentially equal to a vertical integral of the above on $\Re s = 3$. Now the zeta function itself is basically 1, so what we see is that the number of zeroes is controlled by the gamma factors. Use Stirling's formula.

12.2. **Infinitely many zeroes on the critical line.** One interesting result would be to prove that there are infinitely many zeroes on the critical line. Unfortunately this seems difficult because our zeta function is of degree 4. These sorts of statements seem to have been proved only for degree 1 or 2.

Let us say more about how one proves such a statement. We will sketch proofs that certain zeta functions have $\gg T$ zeroes on the critical line. We will not pay any attention to refinements of Selberg and others which show that $\gg T \log T$ (i.e., a positive proportion) do.

Start with the Riemann zeta function, and define a function $\chi(s)$ by

$$\chi(s) := \frac{\zeta(s)}{\zeta(1-s)}.$$

Basically, it is a quotient of gamma functions. Note that $|\chi(s)| = 1$ on the critical line. Define further a function $Z(t)$ by

$$Z(t) := \zeta(1/2 + it)\chi(s)^{-1/2}.$$

It is simple to check that $Z(t)$ is **real-valued** on the critical line.

Now we want to compare the two integrals

$$\int_T^{2T} Z(t)dt$$

and

$$\int_T^{2T} |Z(t)|dt.$$

If the first is smaller than the second, it follows that $Z(t)$, and thus the zeta function, has a zero between T and $2T$.

To estimate the first, we shift the contour to, say, $\Re(s) = 5/4$ where $\zeta(s) \ll 1$. The gamma factors are the main problem, and contribute $T^{7/8}$. And note that there will be more gamma factors for higher degree zeta functions. It will turn out that we can stretch this argument to cover the degree 2 case, but it seems we can't go any higher. In particular, the gamma factors are estimated using Stirling's formula, and the estimates go bad at degrees higher than 2.

To estimate the second term, we replace $Z(t)$ by $\zeta(1/2 + it)$ (they have the same absolute value!) and push the contour to $\Re(s) = 2$. There the zeta function is dominated by its constant term and we easily obtain an estimate $\gg T$. This sort of argument will work for **any** zeta function.

Note that the fact that $Z(t)$ is real-valued is quite critical (as I learned the hard way, when I - ahem - came up with a "proof" for higher degrees and gently had my error pointed out to me). In particular, one is not led to be too optimistic that the method can be adapted or tweaked.

Something I observed when I read this is that you don't actually get to use very much information about the zeta function. The functional equation (with the gamma factors) sort of give you the results. In particular, the presence or absence of an Euler product is irrelevant. So, it is probably not worth trying to prove a result like this for the Shintani zeta function in particular. However, such a result for all high degree zeta functions would be quite interesting.

One further comment: Titchmarsh's book features several other proofs of this statement for the Riemann zeta function. A casual glance seems to suggest that they depend on particular properties of the Riemann zeta function and might not generalize. However, this is worth investigating more carefully.

12.3. **Zeroes to the right of 1.** Do these zeta functions have zeroes to the right of $\Re s = 1$? The diagonalized ones do, and I've written up a proof elsewhere. For the non-diagonalized ones, I'm not sure yet. They behave kind of like the Riemann zeta function, so maybe not. Numerical experiments have not found any yet.

One question is whether these zeta functions have zeroes arbitrarily close to the critical line. I have not thought about this carefully, and it may be possible to obtain a result.

But here is another question, which I'm currently thinking about.

Question. Does **any** suitably normalized Dirichlet series have zeroes to the right of the critical line?

12.4. **Zero density results.** Here is a question. Can one prove any sort of nontrivial zero density result at all? For example, the Shintani zeta function has $O(T)$ zeroes in some region (α, β) within $[0, 1]$?

The standard methods seem to fail. The best methods are reserved for those zeta functions which have Euler products, so those are out of the question right away.

We will venture a description of other methods. One can see Chapter 9 of Titchmarsh's book for further description. Let $N(\sigma, T)$ denote the number of zeroes $\beta + it$ with $\beta > \sigma$ and $|t| < T$. We can prove that $N(\sigma_0, T) \ll T$ for any $\sigma_0 > 1/2$ as follows.

The first step is the identity

$$2\pi \int_{\sigma_0}^{1} N(\sigma, T) d\sigma = \int_0^T \log |\zeta(\sigma_0 + it)| dt + O(\log T).$$

Note the slightly curious form of the identity. This follows by some manipulation and contour integration which is not particularly difficult, but didn't particularly "speak to me".

We now continue as follows. We have (for any nonnegative continuous $f(t)$

$$\frac{1}{b-a} \int_a^b \log f(t) dt \leq \log \left(\frac{1}{b-a} \int_a^b f(t) dt \right).$$

Thus in particular,

$$\int_0^T \log |\zeta(\sigma + it)| dt = \frac{1}{2} \int_0^T \log |\zeta(\sigma + it)|^2 dt \leq \frac{T}{2} \log \left(\frac{1}{T} \int_0^T |\zeta(\sigma + it)|^2 dt \right).$$

Therefore, we conclude

Proposition 12.1. *The estimate*

$$N(\sigma, T) = O(T)$$

follows (in general) if

$$\int_0^T |\zeta(\sigma + it)|^2 dt \ll T.$$

We may further replace the 2 by any positive number we like.

Therefore we turn our attention to such an argument.

A few things are worth noting about this argument. First of all, integrals from 0 to T are basically the same as from T to 2T. Also, the choice of 2 in the above argument doesn't matter. We can choose any positive moment we like, including a fractional one. However, if we attempt to estimate the second moment, we can play the game of multiplying the zeta function by its conjugate, etc. and for all the classical examples this is usually the easiest moment to estimate.

For the Riemann zeta function we can prove the condition given above for any $\sigma > 1/2$.

Theorem 12.2. *We have, for $\sigma > 1/2$,*

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T |\zeta(\sigma + it)|^2 dt = \zeta(2\sigma).$$

Proof. We start with the estimate

$$\zeta(s) = \sum_{n < t} n^{-s} + O(t^{-\sigma}) =: Z + O(t^{-\sigma}). \tag{12.1}$$

Note that t is both the limit on n as well as the imaginary part of s. See Theorem 4.11 of Titchmarsh for a more general result.

We have

$$\int_1^T |Z|^2 dt = \int_1^T \sum_{m,n<t} m^{-\sigma-it} n^{-\sigma+it} dt = \sum_{m,n<T} (mn)^{-\sigma} \int_{max(m,n)}^T \left(\frac{n}{m}\right)^{it} dt.$$

Now the remainder of the proof is not so interesting so we will only summarize it. The main term of $\zeta(2\sigma)$ comes from the diagonal terms where $m = n$. (We can replace $\max(m,n)$ by 1 at the expense of some error term.) In the case of off-diagonal terms, there is cancellation in the integral, which we can easily evaluate. We get a sum over m, n which is readily bounded from above. We also have error terms coming from (12.1) and these are also not difficult to estimate. □

Now, let us ask what happens when we attempt to generalize this to "any zeta function". In the first place, the main term will end up looking like $\sum_n a(n)^2 n^{-2\sigma}$. If the symmetric square zeta function exists, then we win. We know that this zeta function, too, is sufficiently "nice". However, otherwise, we are in trouble. For example, with the Shintani zeta function we do not know that this converges absolutely to the right of 1. We know the bound $a(n) \ll n^{1/3+\epsilon}$ from work of Ellenberg and Venkatesh, so we do have absolute convergence to the right of 4/3. So, we can only hope for zero density estimates to the right of 2/3. Of course, this is still something.

However, this is not the only problem. Perhaps even more significant is the fact that we do not have estimates in the critical strip which are as good as the above.

We will prove the following:

Theorem 12.3. *We have the estimate above for the Shintani zeta function, for $\sigma > \sigma_0$, where σ_0 is to be determined.*

Our theorem is in fact a joke, because we will obtain $\sigma_0 = 7/6$, for which the theorem is trivial.

To prove this, we want to start with something called an approximate functional equation. This allows us to estimate the zeta function anywhere in the critical strip, and not too much is required for the proof. A functional equation of the usual shape is probably necessary, but we don't require an Euler product. And we don't really have to know where our zeta function came from or that it's "automorphic" etc.

We will have something **like**

$$\xi(s) \sim \sum_{n<T^{d/2}} \frac{a(n)}{n^s} + \chi(s) \sum_{n<T^{d/2}} \frac{a(n)}{n^{1-s}},$$

where

$$\chi(s) := \frac{\xi(s)}{\xi(1-s)}$$

is a holomorphic function (essentially, a ratio of gamma functions) of absolute value 1.

This will hold uniformly (for a single L-function; otherwise, introduce the analytic conductor) in the critical strip. $d = 4$ is the degree. This is naturally an extreme simplification. In reality formulas that look like this tend to be weighted by smooth test functions which decay around $\ll T^{d/2}$. Also I haven't seen a proof that I find acceptably PG-rated (they all seem to involve a mess of detail) so I will say nothing about how one proves such an equation.

For the zeta function, using the approximate functional equation in the above argument allows us to replace the sum $m, n < T$ by $m, n < T^{1/2}$. In our case, the degree is 4, and we need to bound from above the quantity

$$\sum_{m,n<T^2} \frac{a(m)a(n)}{(mn)^\sigma \log(n/m)}.$$

Now rewrite this by summing first over $n < T^2$, then by $r = n - m$. The main term is as above. Also, we'll be sufficiently unhappy if we assume r is positive. We approximate $\log(1+\epsilon) \sim \epsilon$. We get

$$\sum_{n<T^2} \frac{a(n)}{n^\sigma} \sum_r \frac{a(n-r)}{(n-r)^\sigma} \frac{n}{r}.$$

This looks basically like

$$\sum_{n<T^2} \frac{a(n)}{n^{\sigma-1}} T^{-2\sigma} \sum_r \frac{a(n-r)}{r}.$$

It seems hopeless to prove any better estimate for the sum over r than $T^{2/3}$, which is the trivial bound on the $a(n-r)$ for n of size around T^2. (We get a log factor from the sum over r too, but let's ignore it.)

We get
$$\sum_{n<T^2} a(n)n^{1-\sigma}T^{-2\sigma+2/3},$$
which by partial summation is essentially
$$(T^2)^{2-\sigma}T^{-2\sigma+2/3} = T^{14/3-4\sigma}.$$
We get zero density estimates for $\Re\sigma > 7/6$. Annals, here I come.

12.5. **Mean motions.** It was also suggested to me by Haseo Ki, Yoonbok Lee, and Ryotaro Okazaki that I should read Borchsenius and Jessen's paper on "mean motions" (and also Lee's paper applying the above to Epstein zeta functions). It seems that this allows one to improve $O(T)$ results such as the ones described above, to asymptotics of the form cT in vertical strips. However, this requires estimates of the form
$$\int_T^{2T} |\xi(\sigma+it)|^2 dt \ll T.$$
See comments below. So I think the moral is that if I get an $O(T)$ result, then I can apply the mean motions theory and in fact do better.

12.6. **Moments.** This question is very much related to questions posed above. (Indeed, more than a little redundant, but I wrote this first.) It would be interesting to prove a bound for the moments
$$\sum_{n\leq x} |a(n)|^2.$$
This would allow be to obtain mean value upper bounds for
$$\int_0^T |\xi(\sigma+it)|^2 dt$$
for fixed $\sigma > 0.5$, and in turn zero density estimates in $\Re(s) > \sigma$ for some fixed σ. Quite possibly I could also work backwards instead.

For the second moment we do have a trivial bound coming from $|a(n)|^{1/3+\epsilon}$, a fact proved by (double check) Jordan and Akshay. Unfortunately this does not turn out to be enough to prove any zero density estimate.

One approach is if (as mentioned previously) the symmetric square is nice. In some sense it does exist, and can sort of compute it. Using Datskovsky and Wright, the Shintani zeta function is an infinite sum of Euler products (Dedekind zeta functions and the like), we can break the Dedekind zeta functions into Artin L-functions, and sort of FOIL the whole ugly mess. The Euler products are of $2s$ and $4s$ rather than s, so we wouldn't need to assume the Artin conjecture or anything like that. However, for the same reason, we can't really expect this approach to buy us anything. To figure out how the Euler products interact upon squaring we need to estimate how many different fields can have the same discriminant, so we've returned essentially to our original question.

Of course one might try to answer this directly. For example, we could ask how many representations can have the same Artin conductor, etc.? Perhaps we will address these questions in such a way later.

Another approach to the second moment **might** come from tweaking the construction of the zeta function. Is there some way of squaring the coefficients? Nothing nice comes to mind, and this indeed would be difficult. However, we do have a great deal of latitude in choosing the test functions in Shintani's construction. Perhaps we could sqaure, and then choose different test functions to get the off-diagonal terms to cancel (????)

One might also look at methods due to Lillian Pierce and others, or look at the Davenport-Heilbronn paramterization, and see if it is implausible for a lot of different n to have lots of large coefficients. Perhaps there is some algebraic reason for this. Or, perhaps, there is some good analytic reason for this (lots of large coefficients force the zeta function to act weird).

Finally, it would be interesting to see if the Dirichlet series $\sum_n a(n)^2 n^{-s}$, or anything similar, has any nice automorphy properties. Andrew Booker's thesis has some ideas related to this, and it may be possible to perform numerical tests. However, some pessimism is called for. There doesn't seem to be any good way of predicting the functional equation for such a "symmetric square", and furthermore it would have poles, at least at $s = 1$ and possibly elsewhere in the critical strip. (And this is presuming any such functional equation exists, which is already a longshot.) His method does not seem to allow for this. I contacted him and asked if he knew of a way to work around this – his silence should probably be interpreted as lack of enthusiasm.

Finally, and this is farfetched (and quite vague), but is there any way that $\sum a(n)q^n$ is some kind of modular form? There are some formal similarities to theta functions.

13. Additional Questions

Here are some various, scattered research-type questions that occur to me after reading this work.

13.1. The test function. Here is a very general and vague sort of question. Shintani gets through his whole proof without ever constructing a canonical test function. He does choose a test function when he has to apply the general machinery (the "general nonsense") and figure out the precise form of the functional equation for the case of interest, but in the end, his zeta function is a quotient of two integrals and the particular test function doesn't matter.

Is there some canonical choice? I'm not sure what this would do for us, but it's something I wonder about.

13.2. Kable and Yukie, On the number of quintic fields. They got $\ll X^{1+\epsilon}$, and their paper is interesting to consider. In particular, one wonders if we can twist by Dirichlet characters and use the square sieve to get a good bound for A_5 extensions.

Consider a space $V_{\mathbb{Q}}^{ss}$; V is the usual space with the usual action of $GL(2)$. The prefix ss indicates that the discriminant (i.e. the invariant polynomial) should not be zero. We also have $V_{\mathbb{Z}}^{ss}$. Consider also the subset $V_{QR\mathbb{Z}}$ of $V_{\mathbb{Z}}^{ss}$ which consists of points such that the "orbit ring map" tensored over $\mathbb{Z}$ with $\mathbb{Q}$ is a quintic field. Kable and Yukie prove convergence for $V_{QR\mathbb{Z}}$ essentially for $\Re s > 1$.

What is surprising at first glance is that the QR version is easier to deal with than ss. Indeed, if one could prove convergence for the ss version then one would get $X \log^7 X$. (The $\log^7$ comes from a possible 7th order pole at $s = 1$.)

The paper is long and I haven't pretended to read all of it. Let's skip to Section 4. The authors define zeta functions associated to QR and ss, in the familiar way. They get convergence for $\Re s > 40$ but this is just a normalization issue. (Note the stars, see the bottom of p. 235.)

The main argument goes through p. 238, and although I have some idea of what is going on, I got a little confused. Where, exactly, does the $1 + \epsilon$ come from?

Proposition 9 seems interesting. If we know the convergence of $Z^\star$, which is presumably the same type of object that gets studied in Shintani's paper, we get a pole at $s = 1$ (maybe multiple) and can carry on to $s = 9/10$. Can we carry on further? Hmm.

The convergence proof is in Section 6, but a quick look at this did not prove enlightening.

13.3. Counting cubic fields. Suppose we tried to isolate cubic fields and write down some appropriate Dirichlet series. How far would we get? Is there any hope that we would be able to get something with analytic continuation and a functional equation? (i.e., something for which Poisson summation worked? – what properties would suggest that Poisson summation ought to work?) Can we relate this to other well-known zeta functions?

Here is what some other mathematicians have had to say on this topic:

Datskovsky and Wright 1988: "The series for cubic extensions seems to behave like a constant multiple of $\frac{\zeta(s)}{\zeta(3s)}$. However, a quick examination of a list of small discriminants of cubic fields will dash any hopes."

Cohen 2004: "Here it is hopeless to think that one may *prove* anything."

Let us carry on anyway.

Of course, if so, we would get a waaay better error term in Davenport-Heilbronn. Could we aim to test this numerically? i.e., how big are the discrepancies? Do we get a legitimate square root error term?

We have the relation described above. Could we hope to "invert" it, and get something along the lines of

$$\sum_K |\mathrm{disc} K|^{-s} = \sum_{\mathcal{O}} |\mathrm{disc}\mathcal{O}|^{-s} g(\mathcal{O}, s)$$

for some "reasonable" function g? This may already be asking a lot. If we did that, could we push these weights through the machinery in Shintani's paper?

OK. So I *think* I wrote down an equation along the lines of the above. But g still depends in a severe way on K. I suspect my equation is useless.

Another idea realted to this is to consider the generating function for cubic fields, take the appropriate linear combination to kill the pole at $5/6$, and multiply by $1 - 2/2^s$ to kill the pole at $s = 1$. This will then not have obvious poles. Is this analytically nice? In paarticular, how far does it have AC? There are several possible approaches, none of which I have carried very far yet. Compare it with a random Dirichlet series. Also (again) see Booker's thesis.

13.4. **Special values.** Does the Shintani zeta function take on interesting special values? How could we compute them, either numerically or theoretically? (Answer: probably we want to compute for $s = 3, 4$ and use the FE.) Would any of the techniques used for the Riemann zeta function be applicable here? There is this nice trick in Neukirch (and presumably elsewhere) to pop out the Bernoulli numbers. But it seems like it doesn't apply here.

I tried some numerical experiments (divided by the right power of π, etc.) and got nothing interesting. David Hansen has suggested to me that there is some heuristic of Deligne which predicts that I shouldn't.

13.5. **Twisting.** Could we hope to twist this by Dirichlet characters? How much harder would the methods get? If so, then this would allow us to get very good estimates for the number of rings in arithmetic progressions.

See some comments along these lines in Datskovsky and Wright's work (to be read further).

13.6. **Other parameterizations and prehomogeneous spaces.** What should we expect? Note that Yukie worked really, really, really hard to get something for the quartic case in his book.

What about other parametrizations? What about the rest of Bhargava's list? And what about the cases where the dimensions don't match up?

13.7. **Function fields.** It would be interesting to examine the function field situation. Note that Datskovsky started this, and translated Shintani's paper over to $F_q(t)$. In particular, the Shintani zeta function is a rational function. This is worked out in complete generality and seems rather messy. I want to read the paper and understand it further. Write down specific examples.

The one question that seems interesting is to look at cubic *field* extensions, and investigate the zeta function there. In particular: **should one expect a rational function?**

Of course, it would also be interesting simply to write down the generating functions for counting cubic rings.

The following would be interesting: First of all, understand Datskovsky's paper to the point where I could easily write down zeta functions in specific situations. Then, numerically compute the generating function for fields in a variety of cases, and attempt to determine if one gets a rational function.

This would seem to require a considerable amount of effort however. One would have to begin by generating the first million or billion cubic fields for $q = 5, 7$ etc. This might not be enough, but it might be.

To do this, one would have to figure out how to compute these fields. Perhaps one would follow Belabas's methods. In any case, this would already represent a substantial undertaking.

References

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