MALLE'S CONJECTURE FOR FROBENIUS GROUPS

rewrite.

Abstract. We attain upper bounds for the conjecture of Malle for groups of the form $G = C_m \rtimes C_t$ where $1 \neq t \mid (m-1)$ with m odd and square free, and all subgroups of C_m are normal in G. In certain cases above, we show that under the assumption of the Cohen Lenstra heuristic, Malle's conjecture holds.

Backwards.

1. Introduction

Let k be a number field and let K/k be a finite extension. If the Galois closure $\tilde{K}$ of K/k has Galois group isomorphic to G, we say (by abuse of notation), $\mathrm{Gal}(K/k) \cong G$. We assume that $1 \neq G \leq S_n$ is a transitive subgroup of a permutation group. We are interested in the asymptotics of the following quantity as $x \to \infty$

what perm group?

$$N_d(k, G; x) = |\{N/k : \mathrm{Gal}(N/k) \cong G, [N : k] = d, \text{and } \mathcal{N}_{k/\mathbb{Q}}(d_{N/k}) \leq x\}| \tag{1.1}$$

where $\mathcal{N}_{k/\mathbb{Q}}$ is the relative norm of $k/\mathbb{Q}$ and all the extensions N/k lie in a fixed algebraic closure of $\mathbb{Q}$.

Gunter Malle made a conjecture [5] on what the asymptotics of (1.1) should be, and in order to state it, we need to introduce notation. Note, when computing $N_d(k, G; x)$, we view G as a subgroup of S_d. above.

Definition 1.1. Let G be a non-trivial, transitive subgroup of S_d that acts on the set of d elements, $[d] := \{1, 2, \ldots, d\}$. Let $g \in G$, then,

again redundant

1. The index of g, is $\mathrm{ind}(g) := d-$ the number of orbits of g on $[d]$.
2. $\mathrm{ind}(G) := \min\{\mathrm{ind}(g) : 1 \neq g \in G\}$.
3. $a(G) := 1/\mathrm{ind}(G)$.

Notation 1.2. We say $f(x) \ll g(x)$ when there exist positive constants C, N such that for all $x > N$, $|f(x)| \leq C|G(x)|$. Also, we say that $f(x) = O(g(x))$ if and only if $f(x) \ll g(x)$.

Conjecture 1.3. *(Malle's conjecture)*
For any non-trivial permutation group G acting on $[d]$, and any number field k, the bounds

$$x^{a(G)} \ll N_d(k, G; x) \ll x^{a(G)+\epsilon} \tag{1.2}$$

holds for all $\epsilon > 0$ as $x \to \infty$.

For a more precise formulation of this conjecture please refer to [6].

Example 1.1. Let $G = D_\ell$ be the dihedral group of size 2ℓ, with ℓ being an odd prime. If G acts on $[\ell] = \{1, \ldots, \ell\}$ then the rotations r have one orbit, hence $\mathrm{ind}(r) = \ell - 1$. The reflections s fix one point and other orbits have 2 elements, implying there are a total of $1 + (\ell-1)/2$ orbits. Hence $a(G) = 2/(\ell-1)$. If $G = D_\ell$ acts on $[2\ell]$, the action is the same as the action of G on itself, hence $a(G) = 1/\ell$.

by ? how??

Jurgen Klüners attained upper bounds for $N_\ell(k, D_\ell; x)$ and $N_{2\ell}(k, D_\ell; x)$ in [7] where ℓ is an odd prime. Under the assumption of the conjecture of Cohen and Lenstra [9], he was able to show that his upper bounds are exactly the asymptotic predicted by Malle. Here we generalize his result to a larger set of groups. The set of groups we look at is described below.

Definition 1.4. Let $\mathcal{F}$ be a the set of Frobenius groups of the form $C_m \rtimes C_t$ such that

refer ahead.

1. m is odd and square free, ?
2. Every subgroup of C_m is normal in $C_m \rtimes C_t$,
3. $t \mid (m-1)$.

Example 1.2. The set contains dihedral groups of the form D_ℓ where ℓ is an odd prime. It contains groups of the form $C_\ell \rtimes C_{\ell-1}$, and more generally, any group of the form $C_\ell \rtimes C_t$ where t is any divisor of $\ell - 1$. This set also contains groups where m is not prime, for instance $C_{77} \rtimes C_4$ and $C_{133} \rtimes C_6$.

Notation 1.5. Let M/k be a Galois extension with Galois group C_t. Let $\mathrm{Cl}_M[m]$ the m torsion elements of the ideal class group of M. We let δ be the smallest constant such that $|\mathrm{Cl}_M[m]| \leq d_{M/\mathbb{Q}}^{\delta}$.

weird. wait, Huh?

We attain upper bounds for $N_m(k, G; x)$ and $N_{mt}(k, G; x)$ where k is any finite extension of $\mathbb{Q}$ and $G \in \mathcal{F}$. In particular we show the following:

Explain what these are.

Theorem 1.6. *Let k be a number field, and $G = C_m \rtimes C_t$ be a group in $\mathcal{F}$. Let Q be the smallest prime divisor of t and let p be the smallest prime divisor of m. Then we have*

$$N_m(k, C_m \rtimes C_t; x) \ll (\log(x))^{Q-2} x^J \tag{1.3}$$

where

$$J = \max\left(\frac{p}{m(p-1)} + \epsilon, \frac{Q}{(m-1)(Q-1)} + \frac{t}{m-1}\mathfrak{H}\right) \tag{1.4}$$

What are the variables?

?

where ϵ is any constant greater than 0. The constant here depends on $d_{K/k}, t, \epsilon$ and m.

?

Theorem 1.7. *Let k be a number field, and $G = C_m \rtimes C_t \in \mathcal{F}$. Let Q be the smallest prime divisor of t and let p be the smallest prime divisor of m. Then we have*

$$N_{mt}(k, C_m \rtimes C_t; x) \ll x^H \tag{1.5}$$

where

$$H = \max\left(\frac{p}{mt(p-1)} + \epsilon, \frac{\mathfrak{H}}{m} + \frac{Q}{tm(Q-1)}\right) \tag{1.6}$$

with $\epsilon > 0$ being an arbitrarily small constant. The constant here depends on $d_{K/k}, t, \epsilon$ and m.

Corollary 1.8. *Let $m = \ell$ be an odd prime and t is any divisor of $\ell - 1$. Let Q be the smallest prime divisor of t, we have*

$$N_\ell(k, C_\ell \rtimes C_t(\ell); x) \ll x^J$$

Usually $x^{J+\varepsilon}$

leave out ε from definition.

where, for any real $\epsilon > 0$,

$$J = \frac{Q}{(\ell-1)(Q-1)} + \frac{t}{\ell-1}\left(\frac{1}{2} + \epsilon\right).$$

Additionally, if $t = Q$ and $k = \mathbb{Q}$, we have

$$J = \frac{Q}{(\ell-1)(Q-1)} + \frac{Q}{\ell-1}\left(\frac{1}{2} - \frac{1}{2Q(\ell-1)} + \epsilon\right).$$

Similarly we have,

$$N_{\ell t}(k, C_\ell \rtimes C_t; x) \ll x^H$$

where

$$H = \frac{Q}{\ell t(Q-1)} + \frac{1}{2\ell} + \epsilon.$$

When $t = Q$ and $k = \mathbb{Q}$, we have

$$H = \frac{1}{\ell(Q-1)} + \frac{1}{2\ell} - \frac{1}{2Q\ell(\ell-1)} + \epsilon.$$

?

Under the assumption of the conjecture of Cohen and Lenstra [9], the upper bounds we achieve are exactly those conjectured by Malle. (get the "weak Malle conj" ~)

Corollary 1.9. *Assume the Cohen-Lenstra hueristic. Let k be a number field, and $G = C_m \rtimes C_t \in \mathcal{F}$. Under the assumption, both $N_m(k, G; x)$ and $N_{mt}(k, G; x)$ have the upper bound that is predicted by Malle's conjecture.*

specific.

Example 1.3. If ℓ is an odd prime, we have

$$N_\ell(k, C_\ell \rtimes C_{\ell-1}; x) \ll x^{1/2+2/(\ell-1)+\epsilon}$$
$$N_{\ell^2-\ell}(k, C_\ell \rtimes C_{\ell-1}; x) \ll x^{1/2\ell+2/(\ell(\ell-1))+\epsilon}.$$

If ℓ is an odd prime with $\ell \equiv 1 \pmod 3$ then we have

Do $\frac{1}{2(\ell-1)^2}$ etc.

$$N_\ell(k, C_\ell \rtimes C_3; x) \ll x^{3/(\ell-1)-1/(2(\ell-1)^2)+\epsilon}$$
$$N_{3\ell}(k, C_\ell \rtimes C_3; x) \ll x^{1/\ell-1/(6\ell(\ell-1))+\epsilon}.$$

Under the assumption of the Cohen-Lenstra-heuristic we have

$$N_{288}(k, C_{77} \rtimes C_4; x) \ll x^{1/77+\epsilon}$$

and

$$N_{35}(k, C_{35} \rtimes C_{17}; x) \ll x^{1/28+\epsilon}.$$

Do both with and without

2. Preliminaries

One of the conjectures in the direction of Malle's conjecture that seems to hold based on limited computational evidence is that

$$N_d(k;x) = O(x).$$

??! [-] ?

~~The upper bounds in this conjecture are independent of the Galois group~~. The best upper bound we have in this direction is due to ~~Ellenberg and Venkatesh~~ [1], where they show for $d > 3$ and a positive constant C,

general. Quote DH, Bhargava

$$N_d(k;x) \ll x^{\exp(C\sqrt{\log d})}.$$

For the groups we study in $\mathcal{F}$, upper bounds for $N_\ell(k, D_\ell; x)$, with ℓ an odd prime, were first attained by Klüners. These were improved upon by Cohen and Thorne (Theorem 1.1, [2]) in the case that $k = \mathbb{Q}$. Cohen and Thorne improved the result of Klüners by attaining non trivial bounds for averages of ℓ torsion of the class group for quadratic extensions. The result of (1.6) does not improve the upper bounds in these cases, it only matches them. Here we use the non trivial upper bound on the size of [-] certain class groups to match the result of Cohen and Thorne. In particular, Pierce, Turnage-Butterbaugh and Wood ~~have~~ recently shown that for almost all Galois extensions $M/\mathbb{Q}$ with Galois group C_p, where p is a prime, for any $n \in \mathbb{N}$, any $\epsilon > 0$, Keep!

(-?) Cite the most important paper first not us →

$$|\mathrm{Cl}_M[n]| \ll_{p,n,\epsilon} d_{M/\mathbb{Q}}^{\frac{1}{2} - \frac{1}{2n(p-1)} + \epsilon}.$$

The case of $C_5 \rtimes C_4$ has been looked at in a paper by Bhargava, Cojocaru and Thorne, [3] and they show that

$$N_5(k, C_5 \rtimes C_4(5); x) \ll x^{39/40+\epsilon}.$$

=?

This is a better bound then the one established here.

Conditionally, for F_ℓ, Klüners assumed a weak form of the Cohen-Lenstra heuristic, to show that the upper bounds are those predicted by Malle's conjecture. We use a slightly stronger implication of the Cohen-Lenstra heuristic here to show the result in (1.9), namely we assume: from where?

Conjecture 2.1. *($\ell-$torsian conjecture) Let $K/\mathbb{Q}$ be a number field of degree n. Then for every $m \in \mathbb{N}$*

$$|Cl_K[m]| \ll_{n,m,\epsilon} d_{K/\mathbb{Q}}^{\epsilon} \tag{2.1}$$

for every $\epsilon > 0$.

[-] -

2.1. Frobenius group.

A Frobenius group $G \leq S_n$ is a transitive permutation group on a finite set, such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. Frobenius groups have form $G = F \rtimes H$ where F is a normal subgroup of G and is known as the *Frobenius Kernel*, and H is the *Frobenius complement.*

We now compute $a(G)$ when $G = C_m \rtimes C_t$ acts on the set of m elements. The elements T_j that have order $j \neq 1$ such that $j|t$ fix one point, and all other orbits have j elements in them. Hence how?

$$\mathrm{ind}(T_j) = m - \left(1 + \frac{m-1}{j}\right).$$

If an element $M_{j'}$ has order $j' \neq 1$ where $j'|m$, then when it acts on the set of m elements, it does not fix any point and hence has orbits of length j', and

$$\mathrm{ind}(M_{j'}) = m - \frac{m}{j'}.$$

Hence, if Q is the smallest prime divisor of t and p is the smallest prime divisor of m,

$$a(C_m \rtimes C_t) = \frac{1}{m - \max\left(\frac{m}{p}, 1 + \frac{m-1}{Q}\right)}. \tag{2.2}$$

Similarly, when G acts on $[tm]$,

$$a(C_m \rtimes C_t) = \frac{1}{tm}\max\left(\frac{Q}{Q-1}, \frac{p}{p-1}\right). \tag{2.3}$$

what is?

One of the main tools in attaining upper bounds to $N_m(k, G; x)$ is making use of a Brauer relation. By a result of Klüners and Fieker [8], Theorem 4:

Theorem 2.2. *Fix an algebraic number field k. Let G be a Frobenius group with $G = F \rtimes H$. Let N/k be a normal extension with $Gal(N/k) = G$. Let K be the fixed field of H and M be the fixed field of F. Then*

$$d_{K/k} = d_{M/k}^{(|F|-1)/|H|} N_{M/k}(d_{N/M})^{1/|H|}. \tag{2.4}$$

Corresponding to the above theorem, in the case when $G = C_m \rtimes C_t$, the following is the diagram of field extensions. The numbers on the arrow indicate the degree of the corresponding extension.

$$\begin{tikzcd}
& N & \\
K = \mathrm{Fix}(C_t) \arrow[ur, "t"] & & M = \mathrm{Fix}(C_m) \arrow[ul, "m"'] \\
& k \arrow[ul, "m"] \arrow[ur, "t"'] \arrow[uu, "tm"] &
\end{tikzcd}$$

If we fix K/k, N is unique up to isomorphism similarly if we fix N/k, K is unique up to isomorphism.

3. Upper bounds

Our strategy to attain upper bounds is to make use of (2.4) and

$$d_{N/k} = d_{M/k}^{[N:M]} \mathcal{N}_{M/k}(d_{N/M}). \tag{3.1}$$

We now establish some notation.

Notation 3.1. We fix the use of N, M, k and K for the fields as described in the diagram above. That is $mt = |C_m \rtimes C_t| = [N:k] = |\mathrm{Gal}(N/k)|$ with $K = \mathrm{Fix}(C_t)$ and $M = \mathrm{Fix}(C_m)$. We will always assume that $G = C_m \rtimes C_t \in \mathcal{F}$ and ϵ is a positive constant that is appropriately small. We let Q be the smallest prime divisor of t and p be the smallest prime divisor of m. We will denote $d_{M/k}$ by $\mathfrak{a}$ and $\mathcal{N}_{M/k}(d_{N/M})$ by $\mathfrak{b}$. Under this notation, the equations (2.4) and (3.1) state

$$d_{K/k} = \mathfrak{a}^{(m-1)/t}\mathfrak{b}^{1/t} \qquad d_{N/k} = \mathfrak{a}^m \mathfrak{b}. \tag{3.2}$$

The strategy to compute $N_m(k, G; x)$ is as follows. First, we fix a finite algebraic number field $k/\mathbb{Q}$. Then, given k, we count the number of extensions M/k in the diagram above that are Galois with some fixed discriminant $d_{M/k}$. Then with M/k and its discriminant fixed, we count the number of abelian extensions N/M with appropriate discriminant conditions.

Theorem 3.2. (*Wright-Theorem 1.2 [4]*) bound, yes?
If $k/\mathbb{Q}$ is a number field, then as $x \to \infty$,

$$N_t(k, C_t; x) = (1 + o(1))C(k, G)x^{Q/(t(Q-1))}(\log(x))^{Q-2} \tag{3.3}$$

for an explicitly defined constant C that depends on k and C_t. ?

Let

$$m = p_1 \dots p_j \tag{3.4}$$

? what does this mean

where ~~$2 < p_1 <$~~ $\cdots < p_j$. Here $p = p_1$.

Lemma 3.3. *Given that $G = C_m \rtimes C_t \in \mathcal{F}$ with m as in (3.4), ~~we can show~~ that $\mathcal{N}_{M/\mathbb{Q}}(d_{N/M})$ is at least a $mt(1-p^{-1})$th power in $\mathbb{N}$ for all but finitely many $d_{N/M}$.*

Subgroups of C_m. Turned.

Proof. Every subfield $M = J_0 \subseteq J_i \subseteq= J_jN$ is Galois over M since by assumption every subgroup of C_m is ~~normal in G.~~ Let $m = p_1p_2\dots p_j$ ~~where $2 < p_1 < \cdots < p_j$ and let $J_i = \mathrm{Fix}(C_{m/(p_1\dots p_i)})$~~, be the fixed field of $C_{m/(p_1\dots p_i)}$ ~~acting on N. This~~ implies that J_{i+1}/J_i is a Galois extension of degree p_{i+1}. We have

Mess.
Use inertia group

$$\begin{aligned}
d_{N/M} =& d_{J_1/M}^{m/p_1}\mathcal{N}_{J_1/M}(d_{N/J_1}) \\
=& d_{J_1/M}^{m/p_1}\mathcal{N}_{J_1/M}(d_{J_2/J_1}^{m/(p_1p_2)}\mathcal{N}_{J_2/J_1}(d_{N/J_2})) \\
=& d_{J_1/M}^{m/p_1}\mathcal{N}_{J_1/M}(d_{J_2/J_1}^{m/(p_1p_2)}\mathcal{N}_{J_2/J_1}(d_{J_3/J_2}^{m/(p_1p_2p_3)}\mathcal{N}_{J_3/J_2}(\dots))).
\end{aligned}$$

If there is a prime $\mathfrak{p} \subset \mathcal{O}_k$ that divides $\mathcal{N}_{M/k}(d_{N/M})$ then it divides at least one term of the form

$$\mathcal{N}_{M/k}(\mathcal{N}_{J_1/M}(\dots\mathcal{N}_{J_i/J_{i-1}}(d_{J_{i+1}/J_i}^{m/(p_1\dots p_{i+1})})\dots) = \mathcal{N}_{J_i/k}(d_{J_{i+1}/J_i}^{m/(p_1\dots p_{i+1})})$$

Since J_{i+1}/J_i is a degree p_{i+1} extension, $d_{J_{i+1}/J_i} = c_{J_{i+1}/J_i}^{p_{i+1}-1}$ where c_{J_{i+1}/J_i} is the conductor of the extension. Since J_i/k and J_{i+1}/k are Galois extensions, let $G_i = \mathrm{Gal}(J_{i+1}/k)$. G_i preserves the extension $J_{i+1}/J_i/k$ and hence it fixes c_{J_{i+1}/J_i}. If $\mathfrak{B} \subset \mathcal{O}_{J_i}$ lies above $\mathfrak{p}$ and $\mathfrak{p}|\mathcal{N}_{J_i/k}(c_{J_{i+1}/J_i})$, then since G_i acts transitively over the primes that lie above $\mathfrak{p}$, all of the

?

First: what divides $d_{N/M}$?

conjugates of $\mathfrak{B}$ divide c_{J_{i+1}/J_i}. In particular this implies that, when $\mathfrak{p}$ is unramified, $\mathfrak{p}^{[J_i:k]} | \mathcal{N}_{J_i/k}(c_{J_{i+1}/J_i})$. Since the only rational primes that are wildly ramified in M divide mt, they divide at most finitely many $\mathcal{N}_{M/\mathbb{Q}}(d_{N/M})$. This implies that for all but finitely many primes $\mathfrak{p}$, we have

$$\nu_{\mathfrak{p}}(\mathcal{N}_{M/k}(d_{N/M})) \geq \frac{m}{p_1 \cdots p_{i+1}}(p_{i+1}-1)[J_i:k] = \frac{m(p_{i+1}-1)t}{p_{i+1}}.$$

Note we have the $\geq$ sign since it is possible that $\mathfrak{p}$ divides more than one term of the form

$$\mathcal{N}_{J_i/k}\left(\left(c_{J_{i+1}/J_i}^{p_{i+1}-1}\right)^{m/(p_1 \ldots p_{i+1})}\right).$$

Since $\mathcal{N}_{M/k}(d_{N/M})$ is at least a $mt(1-p_1^{-1})$-th power in k, $\mathcal{N}_{M/\mathbb{Q}}(d_{N/M}) = \mathcal{N}_{k/\mathbb{Q}}(\mathcal{N}_{M/k}(d_{N/M}))$ is also at least a $mt(1-p_1^{-1})$-th power in $\mathbb{Q}$.

□

We now make more precise the definition of $\mathfrak{H}$ that was defined in Notation(1.5).

I think you want room for a ≪ here

$$\mathfrak{H} := \sup_{d_{M/\mathbb{Q}}} \log_{d_{M/\mathbb{Q}}}(|\operatorname{Cl}_M[m]|) \tag{3.5}$$

Lemma 3.4. *With the field extensions $N/M/k$ (and when $k \neq \mathbb{Q}$, $N/M/k/\mathbb{Q}$) as defined earlier, the number of abelian extensions N/M in this extension tower with a fixed discriminant $d_{N/M}$ is bounded above by*

$$O_{k,t,C}\left(C^{\omega(\mathcal{N}_{M/\mathbb{Q}}(d_{N/M}))}\mathcal{N}_{k/\mathbb{Q}}(d_{M/k})^{\mathfrak{H}}\right).$$

Here $\omega(j)$ is the number of prime divisors of j in $\mathbb{N}$, C is some positive constant.

Proof. Every abelian extension N/M with discriminant supported on a modulus $\mathfrak{m}$ is a subfield of the ray class field of $\mathfrak{m}$. The number of ray class fields is bounded above by the size of the ray class group of modulus $\mathfrak{m}$. By the exact sequence for ray class groups, we have that the size of the ray class group in question is bounded above by

write down

$$(\mathcal{O}_M/\mathfrak{m})^\times \times |\operatorname{Cl}_M|.$$

Since N/M is a degree m extension, the possible extensions N/M correspond to subfields of the m torsion of the ray class group of $\mathfrak{m}$, hence the number of such fields is bounded above by the size of

$$(\mathcal{O}_M/\mathfrak{m})^\times \times |\operatorname{Cl}_M[m]|.$$

And hence by equation(3.5)

$$|\operatorname{Cl}_M[m]| \ll d_{M/\mathbb{Q}}^{\mathfrak{H}}$$
$$|\operatorname{Cl}_M[m]| \ll (d_{k/\mathbb{Q}}^t \mathcal{N}_{k/\mathbb{Q}}(d_{M/k}))^{\mathfrak{H}}$$
$$|\operatorname{Cl}_M[m]| \ll_{k,t} \mathcal{N}_{M/\mathbb{Q}}(d_{N/M})^{\mathfrak{H}}$$

???

False

Let M be a dimension C extension over $\mathbb{Q}$, then $(\mathcal{O}_M/\mathfrak{m})^\times \ll C^{\omega_M(\mathfrak{m})}$. Here $\omega_M(\mathfrak{m})$ represents the number of prime divisors of $\mathfrak{m}$ in $\mathcal{O}_M$ and clearly $\omega_M(\mathfrak{m}) = \omega_M(d_{N/M})$. We know that $\omega_M(\mathfrak{m})$ is bounded above by $[M:\mathbb{Q}]\omega(\mathcal{N}_{M/\mathbb{Q}}(d_{N/M}))$ where ω now counts the number of prime divisors in $\mathbb{Z}$. This holds because each rational prime splits into at most $[M:\mathbb{Q}]$ primes in $\mathcal{O}_M$. Hence we have

$$C^{\omega_M(\mathfrak{m})} \ll C^{[M:\mathbb{Q}]\omega(\mathcal{N}_{M/\mathbb{Q}}(d_{N/M}))} \ll_{C,t,k} C_1^{\mathcal{N}_{M/\mathbb{Q}}(d_{N/M})}$$

for some other positive constant $C_1 = C^{[M:\mathbb{Q}]}$ □

Let w_A be the number of extensions M/k that are Galois with $\mathrm{Gal}(M/k) \cong C_t$, and with $\mathcal{N}_{k/\mathbb{Q}}(d_{M/k}) = A$. Let $w_B = 1$ if it is possible for $\mathcal{N}_{M/\mathbb{Q}}(d_{N/M})$ to be B, else let $w_B = 0$. We want to study how $\mathcal{N}_{k/\mathbb{Q}}(d_{K/k})$ grows and by equation(3.2) and the fact that norms are totally multiplicative, we have

$$\begin{aligned}\mathcal{N}_{k/\mathbb{Q}}(d_{K/k}) &= \mathcal{N}_{k/\mathbb{Q}}(d_{M/k})^{(m-1)/t}\mathcal{N}_{M/\mathbb{Q}}(d_{N/M})^{1/t} \\ &= A^{(m-1)/t}B^{1/t} \\ \mathcal{N}_{k/\mathbb{Q}}(d_{N/k}) &= \mathcal{N}_{k/\mathbb{Q}}(d_{M/k})^m \mathcal{N}_{M/\mathbb{Q}}(d_{N/M}) \\ &= A^m B.\end{aligned} \tag{3.6}$$

Now using the above results, we have

$$\begin{aligned}N_m(k, C_m \rtimes C_t; x) &\ll_{C,k,t,\mathfrak{H}} \sum_{A^{(m-1)/t}B^{1/t} \leq x} w_A w_B A^{\mathfrak{H}} C^{\omega(B)} \\ N_{mt}(k, C_m \rtimes C_t; x) &\ll_{C,k,t,\mathfrak{H}} \sum_{A^m B \leq x} w_A w_B A^{\mathfrak{H}} C^{\omega(B)}\end{aligned} \tag{3.7}$$

We know work only on $N_m(k, C_m \rtimes C_t; x)$.
We have from above

$$N_m(k, C_m \rtimes C_t; x) \ll \sum_{B^{1/t} \leq x} w_B C^{\omega(B)} \sum_{A \leq x^{t/(m-1)} B^{-1/(m-1)}} W_A A^{\mathfrak{H}}.$$

We know from Lemma (3.2) that

$$\sum_{A \leq x} w_A = (1 + o(1)) C(k, G) x^{Q/(t(Q-1))} (\log(x))^{Q-2}.$$

By partial summation,

~~We will repeatedly make use of partial summation of Abel summation, we state this precisely below.~~

~~Proposition 3.5.~~ ~~*Given that we know $\sum_{n \leq x} f(n) = M_f(x)$, and $g(x)$ is a smooth function which does not oscillate, we have*~~

$$\cancel{\sum_{n \leq x} f(n) g(n) = M_f(x) g(x) - \int_1^x M_f(t) g'(t) dt} \tag{3.8}$$

~~Using the above and the fact that $\mathfrak{H} > 0$ we have~~

where did O(1) go?

$$\begin{aligned} \sum_{A \leq x} W_A A^{\mathfrak{H}} &= C x^{\mathfrak{H} + Q/(t(Q-1))} (\log(x))^{Q-2} - C\mathfrak{H} \int_1^x y^{\mathfrak{H} - 1 + Q/(t(Q-1))} (\log(y))^{Q-2} dy \\ &= (1 + o(1)) \left(C - \frac{C\mathfrak{H}}{Q/(t(Q-1)) + \mathfrak{H} - 1} \right) x^{\mathfrak{H} + Q/(t(Q-1))} (\log(x))^{Q-2}. \end{aligned}$$

and hence

$$N_m(k, C_m \rtimes C_t; x) \ll (1 + o(1)) x^{\frac{t\mathfrak{H}}{m-1} + \frac{Q}{(m-1)(Q-1)}} (\log(x))^{Q-2} \sum_{B^{1/t} \leq x} \frac{w_B C^{\omega(B)}}{B^{\mathfrak{H}/(m-1) + Q/(t(Q-1)(m-1))}}$$

Now we look at the inner sum. We know that B is always at least a $mt(1 - p^{-1})$th power and we know that for all B

$$\omega(B) \leq \frac{\log B}{\log \log B}.$$

cite

Hence

$$\begin{aligned} &\sum_{B^{1/t} \leq x} \frac{w_B C^{\omega(B)}}{B^{\mathfrak{H}/(m-1) + Q/(t(Q-1)(m-1))}} \\ \leq &\sum_{B^{m(1-p^{-1})} \leq x} \frac{C^{\log(B)/\log\log(B)}}{B^{mt(1-p^{-1})(\mathfrak{H}/(m-1) + Q/(t(Q-1)(m-1)))}} \\ \leq &\sum_{B \leq x^{\frac{p}{m(p-1)}}} \frac{1}{B^{mt(1-p^{-1})(\mathfrak{H}/(m-1) + Q/(t(Q-1)(m-1))) - \log_{\log(B)}(C)}} \end{aligned} \tag{3.9}$$

Since C is a fixed positive constant, $\log_{\log(B)}(C)$ tends to zero as B tends to infinity. Hence we may fix some small constant $\epsilon > 0$ such that for all $B > N(\epsilon, C)$, $\epsilon > \log_{\log(B)}(C)$. Since we let x tend to infinity, we may break the series above into a finite part where $B < N(\epsilon, C)$ and an infinite part where the finite part has an at most finite contribution. Hence the above can be upper bounded by

$$\begin{aligned} &O(1) + \sum_{N < B \leq x^{\frac{p}{m(p-1)}}} \frac{1}{B^{mt(1-p^{-1})(\mathfrak{H}/(m-1) + Q/(t(Q-1)(m-1))) - \epsilon}} \\ \ll &O(1) + O\left(x^{\frac{p}{m(p-1)} - \frac{t\mathfrak{H}}{m-1} - \frac{Q}{(Q-1)(M-1)} + \epsilon} \right) \end{aligned} \tag{3.10}$$

Hence

$$N_m(k, C_m \rtimes C_t; x) \ll (\log(x))^{Q-2} \left(x^{\frac{p}{m(p-1)} + \epsilon} + x^{\frac{t\mathfrak{H}}{m-1} + \frac{Q}{(m-1)(Q-1)}} \right). \tag{3.11}$$

Using he same method as above, we have

$$N_{tm}(k, C_m \rtimes C_t; x) \ll (\log(x))^{Q-2} \left(x^{\frac{p}{mt(p-1)} + \epsilon} + x^{\frac{\mathfrak{H}}{m} + \frac{Q}{tm(Q-1)}} \right). \tag{3.12}$$

3.1. **Results on $\mathfrak{H}$.** Unconditionally, we always have that $\mathfrak{H} = 1/2 + \epsilon$, though in some cases, we can do better. In the case that t is a prime number, ie, $t = Q$, we have non-trivial bounds on the $\mathfrak{H}$. This section addresses the precise statements and consequences of these bounds. First we need to establish terminology.

Definition 3.6. ($\delta-$exceptional field)
A field $K \in Z_n(\mathbb{Q}, G)$ is called a $\delta-$exceptional field for $0 < \delta < 1/2$ precisely when the Dedekind zeta function of the Galois closure $\overline{K}$ of K over $\mathbb{Q}$ has the property that $\zeta_{\overline{K}}(s)/\zeta(s)$ has a zero in the region

$$[1-\delta] \times [-(\log d_{\overline{K}/\mathbb{Q}})^{2/\delta}, (\log d_{\overline{K}/\mathbb{Q}})^{2/\delta}].$$

Under GRH, no field is δ exceptional, however, we do not assume GRH. Recently, M. Wood, C. Turnage-Butterbaugh and L. Pierce showed non-trivial bounds for $|\mathrm{Cl}_K[\ell]|$ for field extensions with cyclic Galois groups of prime degree (and many other families of fields), under certain conditions of the zero free regions of the concerning Dedekind zeta function. To be precise, they show that

Theorem 3.7. *(Theorem 1.19) Fix a group C_p where p is prime and fix $0 < \epsilon_0 < 1/(4(p-1))$. Define*

$$\delta = \frac{\epsilon_0}{5p + 2/(p-1) + 4\epsilon_0}.$$

Then we have that there are at most $O_{p,\epsilon_0}(x^{\epsilon_0})$ $\delta-$exceptional fields $M/\mathbb{Q}$ with $[M : \mathbb{Q}] = p$, $Gal(M/\mathbb{Q}) = C_p$ and $d_{M/\mathbb{Q}} \leq x$. Aside from the $\delta-$exceptional fields, every field in $N_p(\mathbb{Q}, C_p, x)$ satisfies the following, for every $\ell \in \mathbb{N}$:

$$|\ Cl_M[\ell]| \ll_{p,\ell,\epsilon} d_{M/\mathbb{Q}}^{\frac{1}{2} - \frac{1}{2\ell(p-1)} + \epsilon} \tag{3.13}$$

This implies that $\mathfrak{H} = 1/2 - 1/(2mQ - 2m)$ when $t = Q$ for almost all field extensions $M/\mathbb{Q}$ that are Galois with Galois group C_p. We can make use of this as follows. Let $\Delta_{\epsilon_0}(x)$ be the set of δ-exceptional fields $M/\mathbb{Q}$, with $M/\mathbb{Q}$ being a Galois extension with Galois group C_Q where Q is some prime and $d_{M/\mathbb{Q}} \leq x$. Then, in equation(3.7) we have

$$\begin{aligned} N_m(\mathbb{Q}, C_M \rtimes C_p; x) &\ll \sum_{B^{1/Q} \leq x} w_B C^{\omega(B)} \sum_{A \leq x^{Q/(m-1)} B^{-1/(m-1)}} w_A A^{\mathfrak{H}} \\ &\ll \sum_{B^{1/Q} \leq x} w_B C^{\omega(B)} \left(\sum_{\substack{A \leq x^{Q/(m-1)} B^{-1/(m-1)} \\ M \notin \Delta(x^{Q/(m-1)} B^{-1/(m-1)})}} w_A A^{\mathfrak{H}} + \sum_{\substack{A \leq x^{Q/(m-1)} B^{-1/(m-1)} \\ M \in \Delta(x^{Q/(m-1)} B^{-1/(m-1)})}} w_A A^{\mathfrak{H}} \right). \end{aligned}$$

In the second inner sum above, we have $\mathfrak{H} = 1/2 + \epsilon$ and $\sum_{A \leq x} w_A = O(x^{\epsilon_0})$ and hence will contribute

$$(1 + o(1)) \frac{x^{\frac{t}{2m-2} + \frac{Q\epsilon_0}{(m-1)(Q-1)}}}{B^{\frac{1}{2m-2} + \frac{Q\epsilon_0}{(m-1)(Q-1)}}} (\log(x))^{Q-2}$$

and this will not contribute to the mail term because we can take ϵ_0 to be quite small. Going through the rest of the preocedure similarly as above, we get results like those in the cases $t = Q$ in Corollary(1.8).

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