THE SECONDARY TERM IN THE COUNTING FUNCTION FOR CUBIC FIELDS

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ABSTRACT. **Work in progress, September 21, 2010.**

We prove asymptotic formulas for the number of cubic fields of positive or negative discriminant less than X. These formulas involve main terms of X and $X^{5/6}$ multiplied by appropriate constants, with error terms of $O(X^{7/9+\epsilon})$. This confirms a conjecture of Datskovsky-Wright [13] and Roberts [25]. Our results continue to hold when finitely many splitting conditions are imposed on the fields being counted.

In addition, we prove a similar result for the total amount of 3-torsion in quadratic fields $\mathbb{Q}(\sqrt{D})$, for $|D| < X$. We also prove further generalizations (to be described).

Roberts' conjecture has also been proved independently and very recently by Bhargava, Shankar, and Tsimerman [7], with an error term of $O(X^{39/48+\epsilon})$. In contrast to their work, our proof uses the analytic theory of Shintani zeta functions [27, 30, 12], with further refinements by the present authors [28].

1. INTRODUCTION

Work in progress. This paper is essentially finished, except that we are working on the proofs of several generalizations which will be added later. In addition, this paper depends on a companion paper [28] by the same authors, which is not yet available.

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Let $N_3^{\pm}(X)$ count the number of cubic fields K with $\pm\mathrm{Disc}(K) < X$. In [25], Roberts conjectured that

$$(1.1) \qquad N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + o(X^{5/6}),$$

where $C^- = 3$, $C^+ = 1$, $K^- = \sqrt{3}$, and $K^+ = 1$. This conjecture also appeared implicitly in an earlier paper of Datskovsky and Wright [13]. It was based on a combination of numerical and theoretical evidence, the latter of which will be described in due course.

In this paper we will prove the conjecture, with an additional power savings in the error term:

Theorem 1.1. *Roberts' conjecture is true. Indeed, we have*

$$(1.2) \qquad N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{7/9+\epsilon}).$$

This improves upon a result of Belabas, Bhargava, and Pomerance [3], who obtained the main term in (1.2) with an error term of $O(X^{7/8+\epsilon})$. Although their methods were not designed to extract the secondary term in (1.2), our approach nevertheless owes a great deal to theirs.

Remark. Roberts' conjecture has also been proved, with an error term of $O(X^{39/48+\epsilon})$, in independent (and very recent) work of Bhargava, Shankar, and Tsimerman [7]. Their proof also allows local specifications, as in our Theorem 1.3.

Our methods extend to the related problem of counting 3-torsion in quadratic fields. For any quadratic field with discriminant D, let $\mathrm{Cl}_3(D)$ denote the 3-torsion subgroup of the ideal class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$ of D. We will prove the following result.

Theorem 1.2. *We have*

$$(1.3) \qquad \sum_{0 < D < X} \#\mathrm{Cl}_3(D) = \frac{4}{\pi^2} X + C_1^+ X^{5/6} + O(X^{18/23+\epsilon})$$

and

$$(1.4) \qquad \sum_{0 < -D < X} \#\mathrm{Cl}_3(-D) = \frac{6}{\pi^2} X + C_1^- X^{5/6} + O(X^{18/23+\epsilon}),$$

where the sum ranges over fundamental discriminants D, and the constants $C_1^{\pm}$ are defined by

$$(1.5) \qquad C_1^+ := \frac{\sqrt{3}\zeta(1/3)\Gamma(1/3)^3}{10\pi} \prod_p \left(1 - \frac{1}{p^{5/3}} - \frac{2}{p^2} + \frac{1}{p^{8/3}} + \frac{1}{p^3}\right),$$

and $C_1^- = \sqrt{3}C_1^+$.

In this case, the error terms are slightly higher due to an additional technical complication that appears in the proof.

Remark. Extensive computational results for the 3-parts of class groups appear in [17, 2]. For smaller X it is quite practical to check (1.3) and (1.4) numerically using PARI/GP [24]. For example, for $D > 0$ and $X = 10^6$, the left side of (1.3) is 381071, and the main terms on the right sum to $381337.24\cdots$. For $D < 0$ these values are 566398 and $566448.83\cdots$ respectively.

As conjectured by Roberts, our results also extend to counting problems where various local restrictions are imposed. This is perhaps most interesting in the case of counting fields. Let $\mathcal{S} = (\mathcal{S}_p)$ be a finite set of *local specifications*. In particular, we may require that K be inert, partially ramified, totally ramified, partially split, or completely split at p. More generally, we may specify the p-adic completion K_p.

We will prove the following quantitative version of Roberts' extended conjecture:

Theorem 1.3. *With the notation above, the number of cubic fields K belonging to $\mathcal{S}$ with $\pm\mathrm{Disc}(K) < X$ is*

$$(1.6) \qquad N_3^{\pm}(X, \mathcal{S}) = C(\mathcal{S})\frac{1}{12\zeta(3)}X + K(\mathcal{S})\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O\left(X^{7/9+\epsilon}\prod_p p^8\right),$$

where the constants $C(\mathcal{S})$ and $K(\mathcal{S})$ are computed explicitly in Section 5.

We obtain a similar generalization of Theorem 1.2. In this case, we may restrict our sum to D for which finitely many primes p are inert, split, or ramified. We will prove the following result:

Theorem 1.4. *With the notation above, we have*

$$(1.7) \qquad \sideset{}{'}\sum_{0 < D < X} \#\mathrm{Cl}_3(D) = \frac{4}{\pi^2}C'(\mathcal{S})X + C_1^+ K'(\mathcal{S})X^{5/6} + O\left(X^{18/23+\epsilon}\prod_p p^8\right)$$

and

$$(1.8) \qquad \sideset{}{'}\sum_{0 < -D < X} \#\mathrm{Cl}_3(-D) = \frac{6}{\pi^2}C'(\mathcal{S})X + C_1^- K'(\mathcal{S})X^{5/6} + O\left(X^{18/23+\epsilon}\prod_p p^8\right),$$

where the sums are restricted to discriminants meeting the conditions specified by $\mathcal{S}$, and the constants $C'(\mathcal{S})$ and $K'(\mathcal{S})$ are computed explicitly in Section 5.

We will also describe an interesting generalization to arithmetic progressions (in progress).

The key ingredient of all of our proofs is the theory of the *adelic Shintani zeta function*, due to Shintani [27] and Datskovsky-Wright [30, 12, 13]. This contrasts with the geometric approach adopted by Bhargava and his collaborators [3, 5, 7].

Remark. For simplicity's sake, we will confine ourselves to describing the proof of Theorem 1.1 throughout, except as explicitly noted to the contrary (in Section 3.1 in particular). The discussion of the proofs of our generalizations is postponed to Section 5 (and is mostly removed in this preliminary version).

1.1. **The Davenport-Heilbronn and Delone-Faddeev correspondences.**

Following the original work of Davenport and Heilbronn [10], we begin by relating our problem to the more tractable problem of counting certain integral binary cubic forms. This is accomplished through the well-known correspondence of Davenport-Heilbronn and Delone-Faddeev [11]. An elegant simplified and self-contained account of this work can be found in a paper of Bhargava [5], so we will only briefly summarize it here.

Remark. For the problem of counting 3-torsion in class groups of quadratic fields, a power saving error term in the Davenport-Heilbronn theorem was also obtained by Hough [16], without any reference to binary cubic forms. Instead, Hough obtains his result as a consequence of an equidistribution result for the associated Heegner points.

In [10], Davenport and Heilbronn established the main term in (1.2) by first relating *cubic rings* to *integral binary cubic forms*, and then counting those cubic forms which correspond to maximal orders in cubic fields.

A *cubic ring* is a commutative ring which is free of rank 3 as a $\mathbb{Z}$-module. The discriminant of a cubic ring is defined to be the determinant of the trace form $\langle x, y \rangle = \mathrm{Tr}(xy)$, and the discriminant of the maximal order of a cubic field is equal to the discriminant of the field.

The lattice of *integral binary cubic forms* is defined by

$$(1.9) \qquad V_{\mathbb{Z}} := \{au^3 + bu^2v + cuv^2 + dv^3 \ : a, b, c, d \in \mathbb{Z}\},$$

and the *discriminant* of such a form is given by the usual equation

$$(1.10) \qquad \mathrm{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

There is a natural action of $\mathrm{GL}_2(\mathbb{Z})$ (and also of $\mathrm{SL}_2(\mathbb{Z})$) on $V_{\mathbb{Z}}$, given by

$$(1.11) \qquad (\gamma \cdot f)(u, v) = \frac{1}{\det \gamma} f((u, v) \cdot \gamma).$$

We call a cubic form f *irreducible* if $f(u, v)$ is irreducible as a polynomial over $\mathbb{Q}$, and *nondegenerate* if $\mathrm{Disc}(f) \neq 0$.

The Delone-Faddeev correspondence, which extends that of Davenport-Heilbronn and was further extended by Gan, Gross, and Savin [15] to include the degenerate case, is as follows:

Theorem 1.5 ([11, 15])**.** *There is a natural, discriminant-preserving bijection between the set of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral binary cubic forms and the set of isomorphism classes of cubic rings. Furthermore, under this correspondence, irreducible cubic forms correspond to orders in cubic fields.*

To count cubic fields, Davenport and Heilbronn count their maximal orders, which are exactly those orders which are maximal at all primes p:

Proposition 1.6 ([10, 5]). *Under the Delone-Faddeev correspondence, a cubic ring R is maximal if any only if its corresponding cubic form f belongs to the set $U_p \subset V_{\mathbb{Z}}$ for all p, defined by the following equivalent conditions:*

- *The ring R is not contained in any other cubic ring with index divisible by p.*
- *The cubic form f is not a multiple of p, and there is no $\mathrm{GL}_2(\mathbb{Z})$-transformation of $f(u,v) = au^3 + bu^2v + cuv^2 + dv^3$ such that a is a multiple of p^2 and b is a multiple of p.*

In particular, the condition U_p only depends on the coordinates of f modulo p^2.

The proof of the main term in (1.1) goes as follows: One obtains an asymptotic formula for the number of cubic rings of bounded discriminant by counting lattice points in fundamental domains for the action of $\mathrm{GL}_2(\mathbb{Z})$, bounded by the constraint $|\mathrm{Disc}(x)| < X$. The fundamental domains may be chosen so that almost all reducible rings correspond to forms with $a = 0$, and so these may be excluded from the count.

One then multiplies this asymptotic by the product of all the local densities of the sets U_p. This yields a heuristic argument for the main term in (1.1), and one incorporates a simple sieve to convert this heuristic into a proof.

Remark. The Davenport-Heilbronn correspondence also applies to reducible maximal cubic rings. It is readily shown that (up to isomorphism) these are the rings $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ or $\mathcal{O}_K \oplus \mathbb{Z}$, where $\mathcal{O}_K$ is the ring of integers of a quadratic field. The unit element is given by $(1,1,1)$ or $(1,1)$ respectively, and the discriminant is equal to 1 or $\mathrm{Disc}(K)$ as appropriate.

1.2. **The work of Belabas, Bhargava, and Pomerance.**

In [3], Belabas, Bhargava, and Pomerance (BBP) introduced improvements to Davenport and Heilbronn's method, and obtained an error term of $O(X^{7/8+\epsilon})$ in (1.1) (and also in (1.3) and (1.4)). They begin by observing that

$$(1.12) \qquad N_3^{\pm}(X) = \sum_{q \geq 1} \mu(q) N^{\pm}(q, X),$$

where $N^{\pm}(q, X)$ counts the number of cubic orders of discriminant $0 < \pm D < X$ which are nonmaximal at every prime dividing q. For large q, BBP prove that $N^{\pm}(q, X) \ll X3^{\omega(q)}/q^2$ using reasonably elementary methods. Therefore, one may truncate the sum in (1.12) to $q \leq Q$, with error $\ll X/Q^{1-\epsilon}$. We will use this fact in our proof as well.

For small q, BBP estimate $N^{\pm}(q, X)$ with explicit error terms using geometric methods. These error terms are good enough to allow them to take the sum in (1.12) up to $(X \log X)^{1/8}$, which yields a final error term of $O(X^{7/8+\epsilon})$.

In addition, their methods extend to counting *quartic* fields, where they obtain a main term of $C_4^{\pm}X$ with error $\ll X^{23/24+\epsilon}$.

1.3. **Shintani zeta functions and the analytic approach.**

The main idea of this paper is *essentially*[1] to count $N^{\pm}(q, X)$ using the analytic theory of Shintani zeta functions. The *Shintani zeta functions* associated to the space of binary cubic forms are defined by the Dirichlet series

$$(1.13) \qquad \xi^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_Z} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s},$$

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[1] We will never literally estimate $N^{\pm}(q, X)$, although we easily could. The technicalities will be described later.

where the lattice $V_{\mathbb{Z}}$ was defined in (1.9), and the sum is over elements of positive or negative discriminant respectively. Here $\mathrm{Stab}(x)$, the stabilizer of x in $\mathrm{SL}_2(\mathbb{Z})$, is always an abelian group of order 1 or 3.

By the Delone-Faddeev correspondence, $\xi^{\pm}(s)$ is *almost* the generating series for cubic rings. There are two differences: The Shintani zeta function counts $\mathrm{SL}_2(\mathbb{Z})$-orbits rather than $\mathrm{GL}_2(\mathbb{Z})$-orbits, and it weights some of them by a factor of $1/3$. As we will see, these discrepancies depend on the Galois group of the splitting field of the cubic form, and we will be able to adjust for them later.

These series converge absolutely for $\Re(s) > 1$, and Shintani proved [27] that that these zeta functions enjoy analytic continuation and a functional equation, to be described later. It therefore follows that we can use Perron's formula and a method of Landau [18] to estimate their partial sums. In particular, we have

(1.14)
$$\sum_{\substack{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_Z \\ \pm \mathrm{Disc}(x) < X}} \frac{1}{|\mathrm{Stab}(x)|} = \int_{2-i\infty}^{2+i\infty} \xi^{\pm}(s) \frac{X^s}{s} ds = \mathrm{Res}_{s=1} \xi^{\pm}(s) X + \frac{6}{5} \mathrm{Res}_{s=5/6} \xi^{\pm}(s) X^{5/6} + O(X^{3/5+\epsilon}).$$

One may also translate this into an estimate for the number of cubic *orders* of discriminant at most X, with main terms of order X and $X^{5/6}$ and error $\ll X^{3/5+\epsilon}$.

To count cubic fields, we introduce the *q-nonmaximal Shintani zeta function* $\xi_q^{\pm}(s)$, which counts only those cubic forms in (1.13) which correspond to orders which are nonmaximal at q. By work of Datskovsky and Wright [30, 12, 13], it follows that these zeta functions have analytic continuation and functional equations, so that their partial sums may be estimated as in (1.14). These partial sums are closely related to $N^{\pm}(q, X)$, which allows us to incorporate the sieve (1.12) and obtain our results.

The main technical difficulty is that the error terms in (1.14) now depend on q, and we must explicitly analyze this dependence. The key step is a careful analysis of certain *cubic Gauss sums* appearing in the functional equations for the q-nonmaximal Shintani zeta functions. This is carried out in (* our) companion paper [28]. These Gauss sums are small on average, so we obtain error terms in (1.14) which are not too bad in q-aspect. This fact allows us to take a large cutoff Q in (1.12) and obtain a reasonably good error term in Roberts' conjecture.

We will in fact introduce a smoothing technique to obtain better error terms, but this discussion illustrates the principle of our proof.

Notation. For the most part our choice of notation is standard. Throughout, p will denote a prime and q a squarefree integer. We have referred to $\xi^{\pm}(s)$ as "Shintani zeta functions", which has some precedent in the literature but is not universal. We also remark that the notation $\xi_1(s)$ and $\xi_2(s)$ seems to be common in place of $\xi^{\pm}(s)$, but we did not want to risk confusion with the numerical parameter q.

Throughout, ϵ will denote a positive number which may be taken to be arbitrarily small, not necessarily the same at each occurrence. Any implied constants will always be allowed to depend on ϵ.

Organization of this paper. On account of the excellent exposition in Bhargava's paper ([5]; see also [7]), we will not say any more about the Davenport-Heilbronn and Delone-Faddeev correspondences. Instead, we begin in Section 2 with the analytic theory of the q-nonmaximal zeta functions. This theory is developed in [28] and we summarize it here. We also describe the original

heuristic argument of Datskovsky-Wright and Roberts, which is quite similar to our proof. We
finish this section with a description of some related problems and ongoing work involving Shintani
zeta functions.

In Section 3 we prove bounds for certain partial sums of the duals to the q-nonmaximal Shintani
zeta functions. These will be needed in Section 4, and their proofs use corresponding bounds on
the cubic Gauss sums, proved in [28].

In Section 4 we present our proof of Roberts' conjecture. We first reduce Roberts' conjecture to
a statement about partial sums of Shintani zeta functions. We then estimate these sums using a
variation of (1.14), due essentially to Chandrasekharan and Narasimhan [8].

We conclude in Section 5 by proving our more general results, including the extension to 3-torsion
in quadratic fields. As we describe, almost all of our arguments carry over to the general case.

Acknowledgments

2. q-nonmaximal Shintani zeta functions and their duals

We recall that the *Shintani zeta functions associated to the space of binary cubic forms* are
defined by the Dirichlet series

$$(2.1) \qquad \xi^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_Z} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s},$$

where $V_{\mathbb{Z}}$ was defined in (1.9), the sum ranges over points of positive or negative discriminant
respectively, and $\mathrm{Stab}(x)$, the stabilizer of a point x in $\mathrm{SL}_2(\mathbb{Z})$, is an abelian group of order 1 or
3. We now introduce *q-nonmaximal* analogues of these zeta functions. Throughout, q will be a
squarefree integer.

Definition 2.1. *The q-nonmaximal Shintani zeta functions $\xi_q^{\pm}(s)$ are defined by the formula* (2.1),
with the sum restricted to those x not belonging to U_p (defined in Proposition 1.6) for any $p|q$.

In this section we describe the analytic theory of these functions, following Shintani [27], Datskovsky-
Wright [30, 12], and our companion paper [28].

Shintani's original theorem relates the zeta functions $\xi^{\pm}(s)$ to dual zeta functions

$$(2.2) \qquad \widehat{\xi}^{\pm}(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash \widehat{V}_{\mathbb{Z}}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s},$$

where $\widehat{V}_{\mathbb{Z}}$, the dual lattice to $V_{\mathbb{Z}}$, is defined by

$$(2.3) \qquad V_{\mathbb{Z}} := \{au^3 + bu^2v + cuv^2 + dv^3 \ : a, d \in \mathbb{Z}; \ \ b, c \in 3\mathbb{Z}\}.$$

To describe the analogue for the q-nonmaximal zeta functions $\xi_q^{\pm}(s)$, we must introduce the cubic
Gauss sums. We define $\Phi_q(x)$ to be the characteristic function of those x not in U_p for any $p|q$,
defined on either $V_{\mathbb{Z}}$ or $V_{\mathbb{Z}/q^2\mathbb{Z}}$. The *cubic Gauss sum* is the dual to $\Phi_q(x)$, defined as the following
function on $\widehat{V}_{\mathbb{Z}}$:

$$(2.4) \qquad \widehat{\Phi}_q(x) := \frac{1}{q^8} \sum_{y \in V_{\mathbb{Z}/q^2\mathbb{Z}}} \Phi_q(y) \exp(2\pi i[x, y]/q^2).$$

Here

$$(2.5) \qquad [x,y] = x_4 y_1 - \frac{1}{3} x_3 y_2 + \frac{1}{3} x_2 y_3 - x_1 y_4$$

is the alternating bilinear form used to identify V with $\widehat{V}$, where x_i and y_j are the coordinates of x and y respectively.

Remark. The dual $\widehat{\Phi}_q(x)$ might be thought of as a sum over $\frac{1}{q^2}\mathbb{Z}/\mathbb{Z}$, as it arises as a product of p-adic Fourier transforms of the function Φ_q. This integral reduces naturally to a finite sum over $V_{\frac{1}{q^2}\mathbb{Z}/\mathbb{Z}}$, which is equivalent to the sum given above.

We observe that $\widehat{\Phi}_q$ satisfies the multiplicativity property

$$(2.6) \qquad \widehat{\Phi}_q(x)\widehat{\Phi}_{q'}(x) = \widehat{\Phi}_{qq'}(x)$$

for all $(q,q') = 1$. We also note that if x corresponds to a cubic ring R under the Delone-Faddeev correspondence, then $\widehat{\Phi}_p(x)$ depends only on $R \otimes_\mathbb{Z} \mathbb{Z}_p$. This implies that $\widehat{\Phi}_p(x) = \widehat{\Phi}_p(x')$ if x and x' correspond to R and R', where R' is contained in R with index coprime to p.

We are now prepared to describe the analytic properties of $\xi_q^\pm(s)$.

Theorem 2.2 (Shintani [27]; Datskovsky and Wright [12]; [28]). *The q-nonmaximal Shintani zeta functions $\xi_q^\pm(s)$ converge absolutely for $\Re(s) > 1$, have analytic continuation to all of $\mathbb{C}$, holomorphic except for poles at $s = 1$ and $s = 5/6$, and satisfy the functional equation*

$$(2.7) \qquad \begin{pmatrix} \xi_q^+(1-s) \\ \xi_q^-(1-s) \end{pmatrix} = \Gamma\left(s - \frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s + \frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s}\begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3\sin \pi s & \sin 2\pi s \end{pmatrix}\begin{pmatrix} \widehat{\xi}_q^+(s) \\ \widehat{\xi}_q^-(s) \end{pmatrix},$$

where the dual q-nonmaximal Shintani zeta functions are given by

$$(2.8) \qquad \widehat{\xi}_q^\pm(s) := \sum_{x \in \mathrm{SL}_2(\mathbb{Z})\backslash \widehat{V}_Z} \frac{1}{|\mathrm{Stab}(x)|}\widehat{\Phi}_q(x)\left(|\mathrm{Disc}(x)|/q^8\right)^{-s}.$$

The residues are given by

$$(2.9) \qquad \mathrm{Res}_{s=1}\xi_q^\pm(s) = \mathrm{Res}_{s=1}\xi_q^\pm(s)^{\mathrm{irr}} + \mathrm{Res}_{s=1}\xi_q^\pm(s)^{\mathrm{red}},$$

with

$$(2.10) \qquad \mathrm{Res}_{s=1}\xi_q^\pm(s)^{\mathrm{irr}} = \alpha^\pm\prod_{p|q}\left(\frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^5}\right),$$

$$(2.11) \qquad \mathrm{Res}_{s=1}\xi_q^\pm(s)^{\mathrm{red}} = \beta\prod_{p|q}\left(\frac{2}{p^2} - \frac{1}{p^4}\right),$$

and

$$(2.12) \qquad \mathrm{Res}_{s=5/6}\xi_q^\pm(s) = \gamma^\pm\prod_{p|q}\left(\frac{1}{p^{5/3}} + \frac{1}{p^2} - \frac{1}{p^{11/3}}\right),$$

where $\alpha^+ = \pi^2/36$, $\alpha^- = \pi^2/12$, $\beta = \pi^2/12$, $\gamma^+ = \frac{\zeta(1/3)\Gamma(1/3)^3}{4\sqrt{3}\pi}$, and $\gamma^- = \sqrt{3}\gamma^+$.

We introduce the notation

$$\widehat{\xi}_q^{\pm}(s) =: \sum_{\mu_n} b_q^{\pm}(\mu_n)\mu_n^{-s}$$

for the dual Shintani zeta function, where $\mu_n \in \frac{1}{q^8}\mathbb{Z}$ refers to the quantity $|\mathrm{Disc}(x)|/q^8$ in (2.8). Although it would be easy to write down a functional equation relating $\xi_q^{\pm}(s)$ to a Dirichlet series with integer coefficients, we feel this choice of normalization is the most natural: The shape of the functional equation is uniform for all q, and this normalization seems to most accurately reflect the origin of the dual zeta function, as an adelic integral. Finally, this choice of notation is consistent with our analytic reference [8], which we will describe later.

We can now present the heuristic argument of Datskovsky-Wright and Roberts. For any set of primes $\mathcal{P}$, let $N_{3,\mathcal{P}}^{\pm}(X)$ denote the number of cubic orders $\mathcal{O}$ with $\pm\mathrm{Disc}(\mathcal{O}) < X$ which are maximal at all primes in $\mathcal{P}$. Assuming that we can separately estimate and subtract the contribution from reducible rings, the equations above imply that

$$(2.13) \quad N_{3,\mathcal{P}}^{\pm}(X) = \frac{1}{2}\alpha^{\pm}X\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right)+\frac{3}{5}\gamma^{\pm}X^{5/6}\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^{5/3}}\right)\left(1-\frac{1}{p^2}\right)+O_{\mathcal{P}}(X^{3/5+\epsilon}).$$

Formally taking a limit as $\mathcal{P}$ tends to the set of all primes, we obtain (1.1).

We also see that the inclusion-exclusion sieve (1.12) introduced by Belabas, Bhargava, and Pomerance allows us *in principle* to prove Roberts' conjecture. However, without an analysis of the $\mathcal{P}$-dependence of the error term in (2.13), it is unclear that we can obtain an error term smaller than $X^{5/6}$. Indeed, our initial attempts yielded "proofs" of Roberts' conjecture with error terms that were too large.

To analyze the $\mathcal{P}$-dependence of the error terms in (2.13), we must study the cubic Gauss sum in (2.4). This sum is studied in [28], and Section 3 we will use these results to prove bounds on appropriate partial sums of the dual zeta functions $\widehat{\xi}_q^{\pm}(s)$.

Before proceeding, we will simplify the functional equations by using a diagonalization argument of Datskovsky and Wright, and we will describe some related works involving Shintani zeta functions.

2.1. **Datskovsky and Wright's diagonalization.** To simplify our analysis we apply an observation of Datskovsky and Wright [12]. The functional equations above have a curious matrix form, such that the negative and positive discriminant Shintani zeta functions are interdependent. By diagonalizing this matrix, we can greatly simplify the form of the functional equation.

We define, for each q, diagonalized Shintani zeta functions

$$(2.14) \qquad\qquad\qquad \xi_q^{\mathrm{add}}(s) := 3^{1/2}\xi_q^+(s) + \xi_q^-(s),$$

$$(2.15) \qquad\qquad\qquad \xi_q^{\mathrm{sub}}(s) := 3^{1/2}\xi_q^+(s) - \xi_q^-(s).$$

We diagonalize the dual zeta functions in exactly the same way.

The diagonalizations then take the following shape. Define

$$\Lambda_q^{\mathrm{add}}(s) := \left(\frac{2^4\cdot 3^6}{\pi^4}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{12}\right)\Gamma\left(\frac{s}{2}-\frac{1}{12}\right)\xi_q^{\mathrm{add}}(s),$$

$$\Lambda_q^{\mathrm{sub}}(s) := \left(\frac{2^4\cdot 3^6}{\pi^4}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{5}{12}\right)\Gamma\left(\frac{s}{2}+\frac{7}{12}\right)\xi_q^{\mathrm{sub}}(s),$$

and define $\widehat{\Lambda}_q^{\mathrm{add}}(s)$ and $\widehat{\Lambda}_q^{\mathrm{sub}}(s)$ in the same way. Then, the functional equations take the shape

$$\Lambda_q^{\mathrm{add}}(1-s) = 3\widehat{\Lambda}_q^{\mathrm{add}}(s), \tag{2.16}$$

$$\Lambda_q^{\mathrm{sub}}(1-s) = -3\widehat{\Lambda}_q^{\mathrm{sub}}(s). \tag{2.17}$$

This is the classical shape for functional equations of zeta functions, apart from the interesting factors of ± 3, and it will be a convenient one to work with. We also note the interesting fact that only Λ_{add}, and not Λ_{sub}, retains the pole at $s = 5/6$.

2.2. **Some related work.** We conclude this section by describing some recent and ongoing related work. These results will not be needed elsewhere in this paper, but we hope that they may prove useful in addressing related problems.

We first mention a striking result, conjectured by Ohno [21] and then proved by Nakagawa [20]. They established that the dual Shintani zeta functions are related to the original Shintani zeta functions by the simple formulas

$$\widehat{\xi}^+(s) = 3^{-3s}\xi^-(s), \tag{2.18}$$

$$\widehat{\xi}^-(s) = 3^{1-3s}\xi^+(s). \tag{2.19}$$

One can incorporate these formulas into Datskovsky and Wright's diagonalization, and therefore put the classical Shintani zeta functions into a self-dual form, with functional equations related to the ones above.

More recently, Ohno, the first author, and Wakatsuki [23, 22] classified all of the $\mathrm{SL}_2(\mathbb{Z})$-invariant sublattices of $V_{\mathbb{Z}}$, and proved that the Shintani zeta functions associated to these lattices share the nice properties above.

There is also the work of Yukie [31], who has initiated the study of *quartic* Shintani zeta functions, which are associated to a certain 12-dimensional prehomogeneous vector space. These zeta functions have not yet been studied as thoroughly as their cubic analogues, but it seems that one may be able to prove estimates for quartic fields with power saving error terms, and thereby improve a result of Bhargava [6]. Moreover, if any secondary terms are present, this method seems likely to yield them, at least in principle. However, this approach comes with substantial technical difficulties, and so far it has yet to even yield the main term.

We may also study extensions of base fields other than $\mathbb{Q}$. In [13], Datskovsky and Wright proved the analogue of the Davenport-Heilbronn theorem for any global field of characteristic not equal to 2 or 3. They also suggest that secondary terms should appear in this case as well. Moreover, Morra [19] has designed and implemented an algorithm to compute cubic extensions of imaginary quadratic fields of class number 1. At present we have verified that Morra's calculations closely match the Datskovsky-Wright heuristics for extensions of $\mathbb{Q}(i)$, and (**to do:** Her work is ongoing. Describe the most recent version, compute as much as possible, and write down the conjecture for imaginary quadratic fields, before submitting!)

In principle we expect to be able to prove an analogue of Roberts' conjecture in this general setting. However, we expect that our error terms would be larger than $X^{5/6}$, even for cubic extensions of quadratic fields. However, one may be able to establish secondary terms for *smoothed* sums, such as

$$\sum_K |\mathrm{Disc}(K)| \exp^{-|\mathrm{Disc}(K)|/X},$$

where K ranges over cubic extensions of a fixed number field. We look forward to investigating this in the near future.

Finally, the methods of this paper may be used to prove statements about prime and almost-prime discriminants of cubic fields. When one replaces the q-nonmaximality condition with a divisibility condition on the discriminant, the methods of this paper yield estimates for the number of discriminants divisible by q, and combining these estimates with different sieve methods allows us to prove a variety of results. However, we were unable to improve upon results of Belabas and Fouvry [4], and so we did not pursue this further.

3. Bounds for duals of the q-nonmaximal Shintani zeta function

Let

$$\widehat{\xi}_q^{\pm}(s) =: \sum_{\mu_n} b_q^{\pm}(\mu_n) \mu_n^{-s}$$

be the dual q-nonmaximal Shintani zeta functions, defined in (2.8). Throughout, we will fix a choice of sign and drop the $\pm$ from our notation. We also recall that the sum is over $\mu_n \in \frac{1}{q^8}\mathbb{Z}$.

Our later analytic estimates will require bounds for partial sums of the $b_q(\mu_n)$. The primary goal of this section will be to prove the following bound.

Theorem 3.1. *We have the bound*

$$(3.1) \qquad \sum_{\mu_n < X} |b_q(\mu_n)| \ll q^{1+\epsilon} X,$$

uniformly for all q and X.

The proof essentially involves two steps. The first is an analysis of the Gauss sums $\widehat{\Phi}_q(x)$, carried out in [28]. This analysis shows that the Gauss sums are only supported on certain $\mathrm{GL}_2(\mathbb{Z}/q^2\mathbb{Z})$-orbits of $V_{\mathbb{Z}/q^2\mathbb{Z}}$, and in particular that cubic rings contributing to (3.1) must be either nonmaximal or totally ramified at each prime dividing q. In the second step, we bound the contribution of each orbit type, essentially following arguments of Belabas, Bhargava, and Pomerance [3].

Before presenting the details, we derive the bounds that we will need later.

Proposition 3.2. *For any z and any $\delta > 1$, we have the bound*

$$(3.2) \qquad \sum_{\mu_n > z} |b_q(\mu_n)| \mu_n^{-\delta} \ll_{\delta} q^{1+\epsilon} z^{-\delta+1}.$$

Furthermore, for any $\delta \in (0,1)$, we have the bound

$$(3.3) \qquad \sum_{\mu_n < z} |b_q(\mu_n)| \mu_n^{-\delta} \ll_{\delta} q^{1+\epsilon} z^{-\delta+1}.$$

Both bounds are uniform in q.

Proof. To prove these bounds, we divide the respective intervals into dyadic subintervals of the form $[y, 2y]$. By (3.1), the contribution of each such interval is $\ll q^{1+\epsilon} y^{-\delta+1}$. Both bounds now follow by summing over y. $\square$

To prove Theorem 3.1, we will use the following lemma, which bounds the number of nonmaximal cubic rings contained in a fixed maximal cubic ring.

Lemma 3.3. *Let R be a cubic ring which is maximal at p. Then the number of cubic rings S contained in R with index p^a is $\ll p^{\lfloor a/3 \rfloor + a\epsilon}$. (Here, as usual, $\lfloor a/3 \rfloor$ is the largest integer $\leq a/3$.)*

Moreover, the number of rings so contained with index p^{a+3} is $\ll p$ times the number of rings contained with index p^a, uniformly in a and p.

Proof. Arguments of this sort appear throughout the work of Belabas, Bhargava, and Pomerance [3]. We will argue using a formula of Datskovsky and Wright [12]. They proved that the generating series for rings of index n contained in a fixed maximal cubic ring $\mathcal{O}$ is

$$(3.4) \qquad \zeta(3s-1)\zeta(2s)R_K(s),$$

where

$$(3.5) \qquad R_K(s) := \quad \left(\frac{\zeta(s)}{\zeta(2s)}\right)^3, \quad \frac{\zeta(s)\zeta_K(s)}{\zeta(2s)\zeta_K(2s)}, \quad \frac{\zeta_K(s)}{\zeta_K(2s)}.$$

Here K is the field corresponding (uniquely) to $\mathcal{O}$, and the choice of $R_K(s)$ in (3.5) depends on whether K is of degree 1, 2, or 3 respectively. In all cases, the series (3.4) is bounded above by $\zeta(3s-1)\zeta(s)^4$, and in particular the Euler factor at p is bounded by $(1-p^{1-3s})^{-1}(1-p^{-s})^{-4}$. The coefficient of p^{as} in this expression is bounded by $p^{\lfloor a/3 \rfloor + a\epsilon}$, and the first statement follows.

To prove the second statement, write $a = 3b + c$ with $c \in \{0, 1, 2\}$; then the coefficient of p^{as} in the Euler factor at p is given by the exact expression

$$(3.6) \qquad p^b \binom{3+c}{3} + p^{b-1}\binom{6+c}{3} + \cdots + \binom{a+3}{3}.$$

The ratio of the $p^{(a+3)s}$ coefficient to the p^{as} coefficient is therefore

$$(3.7) \qquad \leq p + p^{-b}\binom{a+6}{3} \leq p + p^{-b}(3b+8)^3 \ll p.$$

$\square$

Proof of Theorem 3.1. We begin by restating the statement to be proved as

$$(3.8) \qquad \sum_{\mathrm{Disc}(x) < Y} |\widehat{\Phi}_q(x)| \ll q^{-7+\epsilon}Y,$$

where the sum is over integral binary cubic forms up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence. In light of the Delone-Faddeev correspondence, we may (and do) refer to the x as either cubic forms or cubic rings. We will find it convenient to talk about divisibility (i.e., content) in terms of forms, and maximality properties in terms of rings.

We will in fact prove the following more general statement. Suppose that r and s are squarefree, coprime integers. Then, we have

$$(3.9) \qquad \sum_{\substack{\mathrm{Disc}(x) < Y \\ r^2 | \mathrm{Disc}(x)}} |\widehat{\Phi}_s(x)| \ll r^{-2+\epsilon}s^{-7+2\epsilon}Y.$$

The proof of (3.9) will be by induction on the number of prime factors of s. The case $s = 1$ was proved by Belabas, Bhargava, and Pomerance ([3], Lemma 3.4).

Remark. Lemma 3.4 of [3] is stated for cubic orders only, but it applies equally to reducible cubic rings. One may prove this, for example, by observing that there is at most one maximal reducible cubic ring of a given discriminant, and then using (3.4).

Before our induction, we begin by separating the contributions to (3.9) according to the orbital type of x at those primes p dividing s. We distinguish six possibilities for each p. We first consider the orbital types 1^3_* and $1^3_{\max}$, which are the easiest to handle. The next two orbital types are those x which are multiples of p or p^2 (as cubic forms). For the fifth possibility, we consider those

x of type 1^3_{**} which are (as cubic rings) contained in other cubic rings with index divisible by p^3. Finally, we will consider the remaining x of type 1^3_{**}. By Theorem (* ??) of [28], $\Phi_p(x) \neq 0$ for all other orbital types.

We obtain positive contributions to (3.9) for each of $6^{\omega(s)} \ll s^\epsilon$ choices of orbital type, so it suffices to prove (3.9) for each choice individually.[2] We exclude the primes $p = 2, 3$ from our analysis, rolling the contributions from these primes into our implied constant.

By [28], the value of $|\widehat{\Phi}_s(x)|$ depends only on our choice of orbital types. It is multiplicative in s, and for each prime divisor p of s, $|\widehat{\Phi}_p(x)|$ ranges between p^{-5} and $O(p^{-2})$. Moreover, $|\widehat{\Phi}_p(x)| = 0$ whenever $p^2 \nmid \mathrm{Disc}(x)$. Throughout, we will freely refer to the computations in Theorem ?? (and later, Theorem ??) of [28].

Assume that an orbital type has been chosen for each p. We now begin the induction. Let p be any prime divisor of x, and we will consider all six of the cases listed above.

The orbital types 1^3_* and $1^3_{\max}$. In this case, $|\widehat{\Phi}_p(x)| = p^{-5}$ and $p^2 | \mathrm{Disc}(x)$. Therefore,

$$
(3.10) \qquad \sum_{\substack{\mathrm{Disc}(x) < Y \\ r^2 | \mathrm{Disc}(x)}} |\widehat{\Phi}_s(x)| \leq \sum_{\substack{\mathrm{Disc}(x) < Y \\ (pr)^2 | \mathrm{Disc}(x)}} p^{-5} |\widehat{\Phi}_{s/p}(x)| \ll (pr)^{-2+\epsilon} p^{-5} (s/p)^{-7+2\epsilon} Y,
$$

so the result follows by induction.

Rings with content divisble by p. These contribute

$$
(3.11) \qquad \sum_{\substack{\mathrm{Disc}(x) < Y/p^4 \\ r^2 | \mathrm{Disc}(x)}} |\widehat{\Phi}_s(px)|.
$$

We have $|\widehat{\Phi}_p(px)| \ll p^{-3}$ unless x remains divisible by p. Excluding such x, we see that (3.11) is

$$
(3.12) \qquad \ll p^{-3} \sum_{\substack{\mathrm{Disc}(x) < Y/p^4 \\ r^2 | \mathrm{Disc}(x)}} |\widehat{\Phi}_{s/p}(px)| = p^{-3} \sum_{\substack{\mathrm{Disc}(x) < Y/p^4 \\ r^2 | \mathrm{Disc}(x)}} |\widehat{\Phi}_{s/p}(x)|,
$$

and the latter sum is $\ll r^{-2+\epsilon} p^{-4} (s/p)^{-7+2\epsilon} Y$ by induction, so the result follows. The cubic forms divisible by p^2 contribute $\ll p^{-2}$ each, and we obtain a sum over $\mathrm{Disc}(x) < Y/p^8$ which is handled in the same manner.

Nonmaximal cubic rings with index divisible by p^3. We replace the set of all these x by the set of those x which are contained in the same p-maximal cubic rings, and have p-index smaller by a factor of exactly p^3. Lemma 3.3 implies that for each n, this is an $O(p) - 1$ correspondence between rings of discriminant n and rings of discriminant n/p^6. Moreover, [28] implies that $|\widehat{\Phi}_p(x)| \ll p^{-3}$ for each x considered.

Therefore, (3.9) is bounded above by

$$
(3.13) \qquad \ll p^{-2} \sum_{\substack{\mathrm{Disc}(x) < Y/p^6 \\ r^2 | \mathrm{Disc}(x)}} (|\widehat{\Phi}_{s/p}(x)|).
$$

By the inductive hypothesis for $\widehat{\Phi}_{s/p}(x)$, the sum above is $\ll r^{-2+\epsilon} p^{-8+\epsilon} (s/p)^{-7+\epsilon} Y$, and the result follows.

The orbital type 1^3_{**}. In this final case, each ring is p-nonmaximal with p-index p or p^2, and has content coprime to p. We again replace each ring by its p-maximal counterpart. By Lemma

---

[2]We recall our convention that ϵ is not necessarily the same at each occurrence. To be more precise here, we need to prove (3.9) for each choice, with ϵ replaced by some $\epsilon' < \epsilon$.

3.3, we have excluded any rings for which there are more than $O(p^\epsilon)$ p-nonmaximal cubic rings contained in any fixed p-maximal cubic ring.

We have $|\widehat{\Phi}_p(x)| \ll p^{-3}$ for this orbital case. By [28], the p-adic valuation of $\mathrm{Disc}(x)$ is least 4 for each x being counted. Accordingly, each x is either contained in a p-maximal ring with index p^2, or is contained in a p-maximal ring with discriminant divisible by p^2. The contribution of those x in the former case is

$$(3.14) \qquad \ll p^\epsilon \sum_{\substack{\mathrm{Disc}(x) < Y/p^4 \\ r^2 | \mathrm{Disc}(x)}} p^{-3} |\widehat{\Phi}_{s/p}(x)|,$$

and the result follows as before. The contribution of those x in the latter case is

$$(3.15) \qquad \ll p^\epsilon \sum_{\substack{\mathrm{Disc}(x) < Y/p^2 \\ (pr)^2 | \mathrm{Disc}(x)}} p^{-3} |\widehat{\Phi}_{s/p}(x)|,$$

and the result again follows by induction.

This finishes our list of cases, and so concludes the induction and the proof. $\qquad\square$

Remark. Our methods in fact yield a bound of $qN^{\omega(q)}X$ for some fixed N, but as this improvement would not help us, we have omitted the details.

3.1. Bounds for the 3-torsion problem.
For the proof of Theorem 1.2 we will require an analogue of Theorem 3.1. Unfortunately we will not be able to prove an exact analogue, and this explains why the error term in Theorem 1.2 is not as good as that in Theorem 1.1.

In this subsection (only), assume that $\Phi_p(x)$ corresponds to the set V_p instead of U_p. This set was defined in [10], and we recall the definition in Section 5.

We will prove the following analogue of Theorem 3.1:

Theorem 3.4. *If $\Phi_p(x)$ corresponds to the complement of V_p, then we have the bounds*

$$(3.16) \qquad \sum_{\mu_n < X} |b_q(\mu_n)| \ll q^{2+\epsilon} X$$

and

$$(3.17) \qquad \sum_{\mu_n < X} |b_q(\mu_n)| \ll q^{1+\epsilon} X + q^{-1+\epsilon},$$

uniformly for all q and X.

We obtain the following corollary in the same way as before.

Proposition 3.5. *For $\delta \in (0,1)$, we have the bounds*

$$(3.18) \qquad \sum_{\mu_n < z} |b_q(\mu_n)| \mu_n^{-\delta} \ll_\delta q^{2+\epsilon} z^{-\delta+1},$$

when $z \le q^{-3}$, and

$$(3.19) \qquad \sum_{q^{-3} < \mu_n < q^{-2}} |b_q(\mu_n)| \mu_n^{-\delta} \ll_\delta q^{3\delta-1+\epsilon},$$

$$(3.20) \qquad \sum_{q^{-2} < \mu_n < z} |b_q(\mu_n)| \mu_n^{-\delta} \ll_\delta q^{1+\epsilon} z^{-\delta+1},$$

when $z > q^{-2}$. We also obtain, as before, for any $\delta > 1$ and any $z > q^{-2}$,

$$(3.21) \qquad \sum_{\mu_n > z} |b_q(\mu_n)| \mu_n^{-\delta} \ll_\delta q^{1+\epsilon} z^{-\delta+1}.$$

We now come to the proof of Theorem 3.4. One expects that our main bound (3.17) should hold without the error term of $q^{-1+\epsilon}$, but we were unable to prove this. In contrast to the proof of Theorem 3.1, we need to estimate the number of cubic rings divisible by integers which are not squares, and the methods used in Lemma 3.4 of [3] do not apply to this case.

We will use the following substitute for Lemma 3.4 of [3], whose proof we postpone to the end of this section.

Proposition 3.6. *Let $d = rs^2$ be a cubefree integer. Then the number of cubic rings R with $|\mathrm{Disc}(R)| < X$ and $d|\mathrm{Disc}(R)$ is $O(X/d^{1-\epsilon} + (rs)^{2+\epsilon})$.*

Proof of Theorem 3.4. The proof of (3.16) follows immediately from the results in [28] and the proof of Theorem 3.1. In this setting, the bounds for each orbital type are at worst p times the bounds quoted in Theorem 3.1, so we immediately obtain the same result with an additional factor of $q^{1+\epsilon}$.

To prove (3.17), we again reformulate our bound in the shape

$$(3.22) \qquad \sum_{\mathrm{Disc}(x) < Y} |\widehat{\Phi}_q(x)| \ll q^{-7+\epsilon} Y + q^{-1+\epsilon},$$

and prove it separately for each orbital type. In this setting, we separate the orbital types a bit differently. We first consider rings with content divisible by p or p^2, and then consider rings whose p-index is p^3 or greater, p^2, and p. Here the p-maximal rings contribute nothing here, in contrast to the proof of Theorem 3.1.

We will again argue by induction, and in this setting we do not need an additional divisibility condition as in (3.9). We induct on the number of prime factors p of q (excluding 2 and 3) for which the p-part of the index is not exactly p. The inductive steps are essentially as in Theorem 3.1, but the base case is new, so we handle it last.

Rings with content divisible by p. The contribution to the Gauss sum is the same as in Theorem 3.1, and these rings are handled in exactly the same way. For example, the forms divisible by p but not p^2 contribute

$$(3.23) \qquad \ll p^{-3} \sum_{\mathrm{Disc}(x) < Y/p^4} |\widehat{\Phi}_{q/p}(x)| \ll p^{-3}\left(\left(\frac{q}{p}\right)^{-7+\epsilon} \frac{Y}{p^4} + \left(\frac{q}{p}\right)^{-1+\epsilon}\right) \ll q^{-7+\epsilon} Y + q^{-1+\epsilon},$$

and the forms divisible by p^2 are handled similarly.

In the remaining cases we assume that p does not divide the content.

Nonmaximal cubic rings with index divisible by p^3. We handle these rings as in Theorem 3.1. By [28], the p-contribution of each x is $\ll p^{-3}$, and so the contribution of this orbital type to (3.22) is

$$(3.24) \qquad \ll p^{-2} \sum_{\mathrm{Disc}(x') < Y/p^6} |\widehat{\Phi}_{q/p}(x')| \ll p^{-2}\left(\left(\frac{q}{p}\right)^{-7+\epsilon} \frac{Y}{p^6} + \left(\frac{q}{p}\right)^{-1+\epsilon}\right) \ll q^{-7+\epsilon} Y + q^{-1+\epsilon}.$$

Nonmaximal cubic rings with p-index p^2. These are easily handled as in (3.14). There are at most p^ϵ such rings in any fixed p-maximal ring, and each of them contributes $O(p^{-3})$. We therefore obtain p^ϵ times the contribution in (3.23).

Nonmaximal cubic rings with p-index p. This is the base case of our induction, so consider those rings with this orbital type for all p. There are at most q^ϵ rings contained in any fixed q-maximal ring. The q-nonmaximal rings may be of type 1_{**}^3 and 1_*^3 at each prime p. The corresponding overrings have discriminants divisible by p^2 and p, respectively, and the p-part of their contribution to the Gauss sum is respectively $\ll p^{-3}$ and p^{-4}.

For a choice of orbital types 1_*^3 or 1_{**}^3 for each prime divisor of q, define q' to be greatest common divisor of all of the discriminants of all of these overrings. Then the choices of orbital types correspond exactly to the integers q' which divide q^2 and are divisible by q, and there are $\ll q^\epsilon$ total choices. The contribution of each orbital type to (3.22) is

$$(3.25) \qquad \ll q^\epsilon \sum_{\substack{\mathrm{Disc}(x) < Y/q^2 \\ q' | \mathrm{Disc}(x)}} q^{-5+\epsilon} q'.$$

We now apply Proposition 3.6. This tells us that the sum is over $\ll Y/(q^2 q') + q^2$ rings, and therefore (3.25) is

$$(3.26) \qquad \ll q^{-5+\epsilon} q' \left(\frac{Y}{q^2 q'} + q^2 \right) \ll q^{-7+\epsilon} + q^{-1+\epsilon},$$

and this completes the induction and establishes Theorem 3.4. $\qquad\square$

Proof of Proposition 3.6. The proof follows the methods presented in this paper, but it is simpler. As the proof is very similar, we will only give a sketch.

We begin by defining the *d-divisible Shintani zeta functions*, which count only those discriminants divisible by d. Like the q-nonmaximal Shintani zeta functions, they enjoy analytic continuation and functional equations, so we may estimate their partial sums by contour integration.

Since we only need an O-estimate for the main term, we simplify our proof and obtain a better error term by smoothing the sum. In this proof only, let $\xi_d^{\pm}(s) := \sum_{d|n} a^{\pm}(n) n^{-s}$ denote the d-divisible Shintani zeta function, and let $\xi_d(s) := \sum_{d|n} a(n) n^{-s}$ denote either of the diagonalized zeta functions, as in (4.7). Then we have the relation

$$(3.27) \qquad \sum_{d|n} a(n) \exp(-n/X) = \int_{c-i\infty}^{c+i\infty} \xi_d(s) X^s \Gamma(s) ds$$

for any $c > 1$ (analogously to (4.11)). Shifting the contour and using the functional equation, we obtain that

$$(3.28) \quad \sum_{d|n} a(n) \exp(-n/X) = \left(\Gamma(1) \mathrm{Res}_{s=1} \xi_d(s) \right) X + \left(\Gamma(5/6) \mathrm{Res}_{s=5/6} \xi_d(s) \right) X^{5/6} +$$

$$X^{1-c} \sum_{\mu_n} \frac{b_d(\mu_n)}{\mu_n^c} \int_{c-i\infty}^{c+i\infty} \frac{\Delta(s)}{\Delta(1-s)} (X\mu_n)^{-it} \Gamma(1-s) ds,$$

where $\sum_{\mu_n \in \frac{1}{d^4}\mathbb{Z}} b_d(\mu_n) \mu_n^{-s}$ is the dual zeta function. The residues are $\ll d^{-1+\epsilon}$ and $d^{-5/6+\epsilon}$, respectively. The integral above is absolutely convergent, and we bound it by an absolute constant (which in particular does not depend on μ_n). We must therefore bound $\sum_{\mu_n} b_d(\mu_n) \mu_n^{-c}$. By an argument similar to Proposition 3.2, we see that this sum is $O((rs)^{2+\epsilon})$, provided we can show that

$$(3.29) \qquad \sum_{\mu_n < X} |b_d(\mu_n)| \ll (rs)^{2+\epsilon} X.$$

We see at this point that our argument has an odd circular flavor, but we could not find any obvious way to simplify it. The point is that (3.29) is easier to prove than the expected bound of $(rs)^{1+\epsilon}X$, and that we can then use this bound to estimate the beginnings of related Dirichlet series.

This brings us to the proof of (3.29). This again relies on the analysis of the related Gauss sums in [28]. As before, it suffices to prove

$$(3.30) \qquad \sum_{\mathrm{Disc}(x) < Y} |\widehat{\Phi}_d(x)| \ll d^{-4}(rs)^{2+\epsilon}X,$$

where $\widehat{\Phi}_d(x)$ is again multiplicative in $d = rs^2$. For prime factors of s, the definition is the same as in the 3-torsion problem (at least for $p > 3$) (double check!), so we need to extend the proof of (3.16). For prime factors p of r, the formulas in [28] establish that $|\widehat{\Phi}_p(x)| \ll p^{-2}$ for all x with content coprime to p. We therefore divide into $2^{\omega(r)}$ orbital types as before and induct on the number of prime factors of r. We use the above estimate for x with content coprime to p, and handle the remaining x by dividing through by p. This proves (3.30), and hence the proposition. $\square$

4. The proof of Roberts' conjecture

We will prove Roberts' conjecture in three steps. In Section 4.1 we discuss the relationship between the Shintani zeta coefficients and counting functions for cubic rings, and reduce Roberts' conjecture to a statement about partial sums of Shintani zeta functions. In Section 4.2 we incorporate the Datskovsky-Wright diagonalization, and transform our problem into one that can be readily addressed using a contour integration argument of Chandrasekharan and Narasimhan [8]. Finally, in Section 4.3 we do this contour integration. As it would be impractical to reproduce the entire argument in [8], we will refer to their paper for many of the details and call the reader's attention to the few changes we introduce to their argument.

4.1. **Reduction to Shintani zeta coefficients.** We want to obtain estimates for $N_3^{\pm}(X)$, the count of cubic fields of positive or negative discriminant less than X. The first step in our argument is to relate these quantities to partial sums of the coefficients of the Shintani zeta function. Define Dirichlet series $F^{\pm}(s) = \sum_n c^{\pm}(n)n^{-s}$ by

$$(4.1) \qquad F^{\pm}(s) = \sum_{n \geq 1} c^{\pm}(n)n^{-s} := \sideset{}{'}\sum_{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_Z} \frac{1}{|\mathrm{Stab}(x)|}|\mathrm{Disc}(x)|^{-s},$$

where the prime indicates that the sum is restricted to those x which are maximal at all places (i.e., contained in U_p for all p).

We define partial sums

$$(4.2) \qquad N^{\pm}(X) := \sum_{n \leq X} c^{\pm}(n).$$

We will prove the following:

Proposition 4.1. *We have*

$$(4.3) \qquad N_3^{\pm}(X) = \frac{1}{2}N^{\pm}(X) - \frac{3}{\pi^2}X + O(X^{1/2}).$$

Proof. By the Delone-Faddeev correspondence (see also Section 2 of [12]), the Dirichlet series in (4.1) counts fields of degree ≤ 3 (or, more properly, their maximal orders), with different weights for different types of fields. Non-Galois cubic fields are counted with weight 2, Galois fields are

counted with weight 2/3, quadratic fields are counted with weight 1, and $\mathbb{Q}$ is counted with weight 1/3.

The number of cyclic cubic extensions of discriminant $\leq X$ is $O(X^{1/2})$ [9], the number of quadratic extensions of either positive or negative discriminant $\leq X$ is equal to $\frac{3}{\pi^2}X + O(X^{1/2})$, and of course there is only one trivial extension of $\mathbb{Q}$. The result therefore follows by subtracting and reweighting these contributions as appropriate. $\qquad\square$

4.2. **Setup for the contour integration.** In this section we will incorporate the inclusion-exclusion sieve and Datskovsky and Wright's diagonalization, and bring our problem to a form where we can apply contour integration.

By Proposition 4.1, it suffices to count

$$(4.4) \qquad {\sum_{\substack{x \in \mathrm{SL}_2(\mathbb{Z}) \backslash V_{\mathbb{Z}} \\ \pm Disc(x) \leq X}}}' \frac{1}{\mathrm{Stab}(x)},$$

where the dash on the sum indicates that we count only those lattice points corresponding to *maximal* cubic rings. A cubic ring is maximal if and only if it is maximal at each prime. By inclusion-exclusion, this sum is equal to

$$(4.5) \qquad \sum_q \mu(q)\left(\sum_{n \leq X} a_q^{\pm}(n)\right),$$

where the $a_q^{\pm}(n)$ are the coefficients of the q-nonmaximal Shintani zeta functions. By Proposition 22 of [5], the inner sum is $\ll Xq^{-2+\epsilon}$, uniformly in q, and it follows that the total sum is

$$(4.6) \qquad \sum_{q \leq Q} \mu(q)\left(\sum_{n \leq X} a_q^{\pm}(n)\right) + O\left(\frac{X}{Q^{1-\epsilon}}\right),$$

for any choice of Q. The main term above is what we want to estimate.

Although it is not strictly necessary (see Theorem 3 of [26]), it will simplify our computations to incorporate Datskovsky and Wright's diagonalization, described in Section 2.1. We will write

$$(4.7) \qquad a_q(n) := \sqrt{3}a_q^+(n) \pm a_q^-(n),$$

such that the zeta functions $\xi_q(s) := \sum_n a_q(n)n^{-s}$ satisfy the simple functional equation (2.16) or (2.17). As we will prove our results simultaneously for both choices of sign in (4.7), we will not indicate this sign in our notation.

We write $N(X)$ for either of the analogous linear combinations of $N^{\pm}(X)$, and we will prove estimates for

$$(4.8) \qquad N_Q(X) := \sum_{q \leq Q} \mu(q)\left(\sum_{n \leq X} a_q(n)\right).$$

We then take the appropriate linear combinations to recover the analogous estimates for the original Shintani zeta function.

To evaluate (4.8), recall that Perron's formula yields the equality[3]

$$(4.9) \qquad \sum_{n \leq X} a_q(n) = \int_{c-i\infty}^{c+i\infty} \xi_q(s)\frac{X^s}{s}ds$$

---

[3]For strict equality, we must take X not equal to any value of n (any irrational number will do).

for any $c > 1$. *In principle*, one evaluates the integral by shifting the contour to the left, obtaining main terms of order X and $X^{5/6}$ from the poles of $\xi_q(s)$, along with an error term. In practice, one runs into convergence issues at infinity and must tweak the method somehow. We adopt the method of Chandrasekharan and Narasimhan [8], which has its origins in work of Landau [18]. In particular, following [8], we will smooth the sum above to obtain an integral with nice convergence properties at infinity, and then use a finite differencing method to recover the sum in (4.9) from the smoothed sum.

As we will see, we may improve our error terms by departing from [8] in one respect. We will smooth the entire sum in (4.8), estimate the smoothed sum over each q separately, and combine the contributions from all q to obtain a smoothed version of the count in (4.8). Recovering the count in (4.8) from the smoothed count involves an error term, and the error made in unsmoothing the combined count is roughly equal to the error made in unsmoothing the contribution from any individual q. Therefore, we will not actually estimate the contribution of any individual q to (4.8).

4.3. **The contour integration.** We now begin in earnest, closely following [8]. We introduce a smoothing factor $(X - n)^\rho$, and write

$$(4.10) \qquad N_Q^\rho(X) := \frac{1}{\Gamma(\rho + 1)} \sum_{q \leq Q} \mu(q) \left(\sum_{n \leq X} (X - n)^\rho a_q(n) \right).$$

Here ρ is any sufficiently large integer. We may in fact take $\rho = 3$, but to follow the notation of [8][4] we will leave the value undetermined. (Any error terms may depend on ρ.)

For each q, we have

$$(4.11) \qquad \frac{1}{\Gamma(\rho + 1)} \sum_{n \leq X} (X - n)^\rho a_q(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s(s + 1) \cdots (s + \rho)} \xi_q(s) X^{s+\rho} ds,$$

for any $c > 1$. We move the integral to the line $\sigma = 1 - c$, choosing $c < \frac{5}{4}$ so that we do not pick up any singularities of the integral left of $s = 0$, and so that the integral (4.15) converges for $\rho \geq 3$. In doing so, we pick up contributions from the residues of $\xi_q(s)$ at $s = 1$ and $s = 5/6$.

Later, we will estimate the integral on the line $\sigma = 1 - c$ using the functional equation. We first explain how $N_Q^\rho(X)$ is related to our unsmoothed count $N_Q(X)$. For a parameter y to be determined later, define a finite differencing operator Δ_y^ρ (on the space of real valued functions F) by

$$\Delta_y^\rho F(x) := \sum_{\nu=0}^{\rho} (-1)^{\rho-\nu} \binom{\rho}{\nu} F(x + \nu y).$$

It is proved in (4.14) of [8] that

$$(4.12) \qquad \Delta_y^\rho[N_Q^\rho(X) - R_Q^\rho(X)] = y^\rho[N_Q(X) - R_Q(X)] + O\left(y^{\rho+1} + y^\rho \sum_{X < n \leq X+\rho y} \sum_{q \leq Q} a_q(n) \right),$$

where

$$R_Q^\rho(x) = \sum_{q \leq Q} \mu(q) \left(\frac{1}{1(1+1)\cdots(1+\rho)} X^{1+\rho} \mathrm{Res}_{s=1} \xi_q(s) + \frac{1}{\frac{5}{6}\left(\frac{5}{6}+1\right)\cdots\left(\frac{5}{6}+\rho\right)} X^{5/6+\rho} \mathrm{Res}_{s=5/6} \xi_q(s) \right)$$

[4]In the notation of [8], we have $A = 2, q = 1, r = 1, \delta = 1$, and $N = 4$, as determined by the structure of our problem.

(with an additional residue term at $s = 0$ which we subsume into our error term), and

$$R_Q(X) = \sum_{q \leq Q} \mu(q) \left(X \operatorname{Res}_{s=1} \xi_q(s) + \frac{6}{5} X^{5/6} \operatorname{Res}_{s=5/6} \xi_q(s) \right).$$

The error term in (4.12) is $O(y^{\rho+1+\epsilon})$ if $y > X^{3/5}$; this follows by estimating

$$\sum_{X < n \leq X+\rho y} \sum_{q \leq Q} a_q(n) \ll y^\epsilon \sum_{X < n \leq X+\rho y} a(n) \ll y^{1+\epsilon},$$

the latter estimate following from partial sum estimates for the standard Shintani zeta function.

Therefore, for $y > X^{3/5}$ it follows that

$$(4.13) \qquad N(X) - R_Q(X) \ll y^{-\rho} \Delta_y^\rho[N_Q^\rho(X) - R_Q^\rho(X)] + y^{1+\epsilon} + \frac{X}{Q^{1-\epsilon}},$$

where

$$(4.14) \qquad N_Q^\rho(X) - R_Q^\rho(X) = \sum_{q \leq Q} \left(\frac{1}{2\pi i} \int_{1-c-i\infty}^{1-c+i\infty} \frac{1}{s(s+1)\cdots(s+\rho)} \xi_q(s) X^{s+\rho} ds \right).$$

We will study this integral individually for each q. To denote this, we replace the subscript Q with q throughout. Applying the functional equation (2.16) or (2.17), the integral is equal to

$$(4.15) \qquad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{(1-s)(2-s)\cdots(1+\rho-s)} \frac{\pm\Delta(s)}{3\Delta(1-s)} \widehat{\xi}_q(s) X^{1+\rho-s} ds,$$

where

$$(4.16) \qquad \Delta(s) := \left(\frac{2^4 \cdot 3^6}{\pi^4} \right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \Gamma\left(\frac{s}{2}+a_3\right) \Gamma\left(\frac{s}{2}+a_4\right).$$

Here a_3 and a_4 are equal to either $5/12$ and $7/12$ or $\pm 1/12$ as appropriate.

The integral in (4.15) is equal to

$$(4.17) \qquad \sum_{\mu_n \in \frac{1}{q^8}\mathbb{Z}} \frac{b(\mu_n)}{\mu_n^{1+\rho}} \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{(1-s)(2-s)\cdots(1+\rho-s)} \frac{\pm\Delta(s)}{3\Delta(1-s)} (\mu_n X)^{1+\rho-s} ds \right).$$

This integral and its finite difference are thoroughly analyzed in [8]. Although one might hope to play the oscillation of the $b(\mu_n)$ against oscillation in this integral, our attempts to do this were unsuccessful. However, we still obtain good error terms by taking absolute values of the $b(\mu_n)$ and using bounds for the integral proved in [8].

Recall that our error term in (4.13) consists of a sum over q of the operator Δ_y^ρ applied to this integral. Following the argument in [8], and in particular the bounds on p. 109 there, we have

$$(4.18) \qquad \Delta_y^\rho[N_q^\rho(X) - R_q^\rho(X)] \ll y^\rho X^{3/8} \sum_{\mu_n \leq z} \frac{|b_q(\mu_n)|}{\mu_n^{5/8}} + X^{3/8+3\rho/4} \sum_{\mu_n > z} \frac{|b_q(\mu_n)|}{\mu_n^{5/8+\rho/4}},$$

where z is a free parameter. We estimate the sums on the right using the bounds given in Proposition 3.2. We conclude that

$$(4.19) \qquad y^{-\rho} \Delta_y^\rho[N_q^\rho(X) - R_q^\rho(X)] \ll q^{1+\epsilon} X^{3/8+\epsilon} z^{3/8} \left(1 + \left(\frac{X^3}{y^4 z} \right)^{\rho/4} \right),$$

and therefore, adding over all q,

$$(4.20) \qquad N(X) - R_Q(X) \ll Q^{2+\epsilon} X^{3/8+\epsilon} z^{3/8} \left(1 + \left(\frac{X^3}{y^4 z} \right)^{\rho/4} \right) + y^{1+\epsilon} + \frac{X}{Q^{1-\epsilon}}.$$

We choose $y = X/Q$ and $z = X^3/y^4$ to equalize error terms. The above is then

$$(4.21) \qquad \ll Q^{7/2+\epsilon} + X^{1+\epsilon}/Q,$$

and choose $Q = X^{2/9}$ to obtain an error term of $O(X^{7/9+\epsilon})$. Note that $y > X^{3/5}$ as required for (4.13).

We now reverse our diagonalizations to obtain estimates for $N^\pm(X)$, with the same error terms. It remains to evaluate $R_Q^\pm(X)$. We see that

$$(4.22) \qquad R_Q^\pm(X) = X \sum_{q \leq Q} \mu(q) \mathrm{Res}_{s=1} \xi_q^\pm(s) + \frac{6}{5} X^{5/6} \sum_{q \leq Q} \mu(q) \mathrm{Res}_{s=5/6} \xi_q^\pm(s).$$

We now apply (*Taniguchi's / the) formulas in [28] for the residues, quoted in Theorem 2.2. We have

$$(4.23) \quad R_Q^\pm(X) = X \sum_{q \leq Q} \mu(q) \left(\alpha^\pm \prod_{p|q} \left(\frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^5} \right) + \beta \prod_{p|q} \left(\frac{2}{p^2} - \frac{1}{p^4} \right) \right)$$
$$+ \frac{6}{5} \gamma^\pm X^{5/6} \left(\sum_{q \leq Q} \mu(q) \prod_{p|q} \left(\frac{1}{p^{5/3}} + \frac{1}{p^2} - \frac{1}{p^{11/3}} \right) \right),$$

where $\alpha^+ = \pi^2/36$, $\alpha^- = \pi^2/12$, $\beta = \pi^2/12$, $\gamma^+ = \frac{\Gamma(1/3)^3 \zeta(1/3)}{4\sqrt{3}\pi}$, and $\gamma^- = \sqrt{3}\gamma^+$.

We replace the sums over $q \leq Q$ by the appropriate Euler products, with error $\ll XQ^{-1+\epsilon} + X^{5/6} Q^{-2/3+\epsilon} \ll X^{7/9+\epsilon}$, and we see that

$$(4.24) \qquad R_Q^\pm(X) = X \left(\alpha^\pm \frac{1}{\zeta(2)\zeta(3)} + \beta \frac{1}{\zeta(2)^2} \right) + X^{5/6} \left(\frac{6}{5} \gamma^\pm \cdot \frac{1}{\zeta(2)\zeta(5/3)} \right) + O(X^{7/9+\epsilon}).$$

Theorem 1.1 now follows by combining (4.20) and (4.24) with Proposition 4.1.

5. Generalizations of Roberts' conjecture

The proofs of our generalizations of Roberts' conjecture follow along very similar lines. In this section we will describe these generalizations more explicitly, and explain the new steps required in the proofs.

5.1. **3-torsion in quadratic fields.** As in [10] and [3], we use a classical result of Hasse, which is proved using class field theory. For a fundamental discriminant D, this states that $\mathrm{Cl}_3(D) - 1$ is equal to twice the number of cubic fields of discriminant D. Requiring that the discriminant of a cubic field K be fundamental is equivalent to requiring that D not be totally ramified at any prime. Therefore, one obtains estimates for the number of such fields by shrinking the set U_p, defined in Proposition 1.6, to a new set V_p, which includes this additional requirement.

One then checks that the same proof works in this case as well. Write $M_3^\pm(X)$ and $M_3^\pm(q, X)$ for the appropriate counting functions in analogy to before. We want to count

$$(5.1) \qquad M_3^\pm(X) = \sum_{q \geq 1} \mu(q) M_3^\pm(q, X).$$

Lemma 3.4 of [3] establishes that we may again truncate the sum to $q \leq Q$ with error $\ll X/Q^{1-\epsilon}$. One reformulates the problem in terms of Shintani zeta functions using Proposition 4.1; the contribution of reducible rings there is unchanged.

The analogous cubic Gauss sum is a little bit different, and we use the bounds in Proposition 3.5. In place of (4.19), we obtain

$$(5.2) \qquad y^{-\rho}\Delta_y^\rho[N_q^\rho(X) - R_q^\rho(X)] \ll q^{7/8+\epsilon}X^{3/8} + q^{1+\epsilon}X^{3/8+\epsilon}z^{3/8}\left(1 + \left(\frac{X^3}{y^4 z}\right)^{\rho/4}\right),$$

as long as $z \geq q^{-2}$.

We split the sum over $q \leq Q$ into $q \leq X^{1/18}$ and $X^{1/18} < q \leq Q$. We choose $y = X/Q$ as before. In the range $q \leq X^{1/18}$, we choose $z = 1$ for each q, and the contribution of this range to $N(X) - R_Q(X)$ is

$$(5.3) \qquad \ll (X^{1/18})^{15/8+\epsilon}X^{3/8} + (X^{1/18})^{2+\epsilon}X^{3/8+\epsilon}\left(1 + \left(\frac{X^3}{y^4}\right)^{\rho/4}\right),$$

and this is well below $X^{7/9}$ as long as $y \geq X^{3/4}$. In the range $X^{1/18} < q \leq Q$, we choose $z = X^3/y^4$ as before, and check that $z \geq q^{-2}$ for each q. We obtain a contribution to $N(X) - R_Q(X)$ of

$$(5.4) \qquad Q^{15/8+\epsilon}X^{3/8} + Q^{7/2+\epsilon} + X^{1+\epsilon}/Q.$$

Because of the new first term, the optimal choice is $Q = X^{5/23}$, which gives an error term of $X^{18/23+\epsilon}$.

The remainder of the machinery of Section 4 works unchanged. We compute the residues of the new Shintani zeta functions using the tables in [28]. We obtain, analogously to (4.23),

$$(5.5) \quad R_Q^\pm(X) = X \sum_{q \leq Q} \mu(q)\left(\alpha^\pm \prod_{p|q}\left(\frac{2}{p^2} - \frac{1}{p^4}\right) + \beta\prod_{p|q}\left(\frac{2}{p^2} - \frac{1}{p^4}\right)\right)$$
$$+ \frac{6}{5}\gamma^\pm X^{5/6}\left(\sum_{q \leq Q}\mu(q)\prod_{p|q}\left(\frac{1}{p^{5/3}} + \frac{2}{p^2} - \frac{1}{p^{8/3}} - \frac{1}{p^3}\right)\right),$$

and therefore that the number of fields of interest is

$$(5.6) \qquad \frac{1}{2\zeta(2)^2}\alpha^\pm X + \frac{3}{5}\gamma^\pm\prod_p\left(1 - \frac{1}{p^{5/3}} - \frac{2}{p^2} + \frac{1}{p^{8/3}} + \frac{1}{p^3}\right) + O(X^{18/23+\epsilon}).$$

5.2. Generalizations involving local conditions.

In this discussion we will discuss the proof of Theorem 1.3.

(TBD.)

5.3. Local conditions for 3-torsion problem in quadratic fields.

(TBD.)

5.4. Generalizations to arithmetic progressions.

(TBD.)

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