The Geometry of Equidistribution

Frank Thorne

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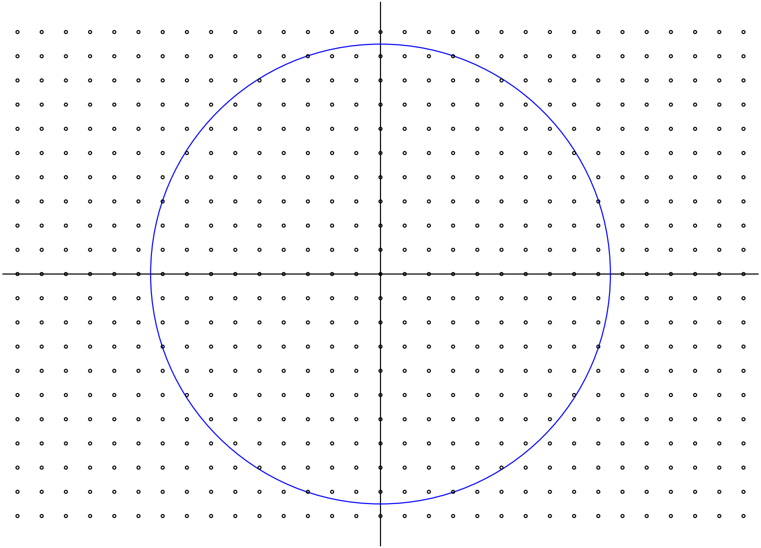
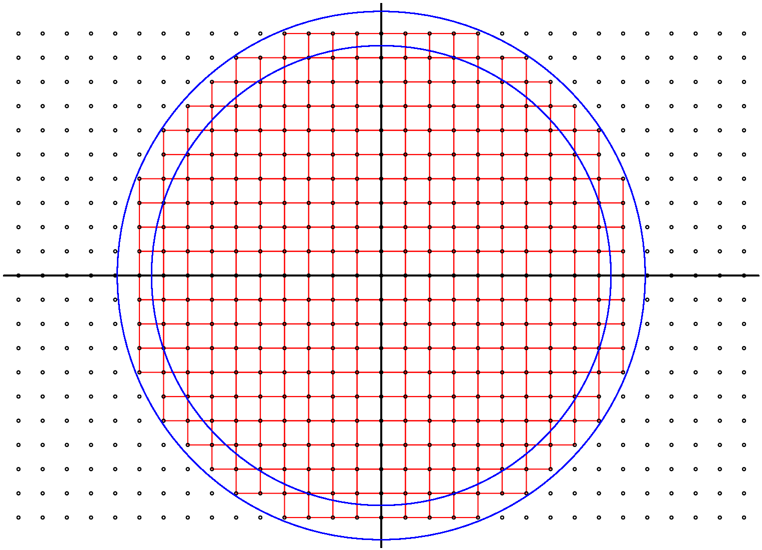
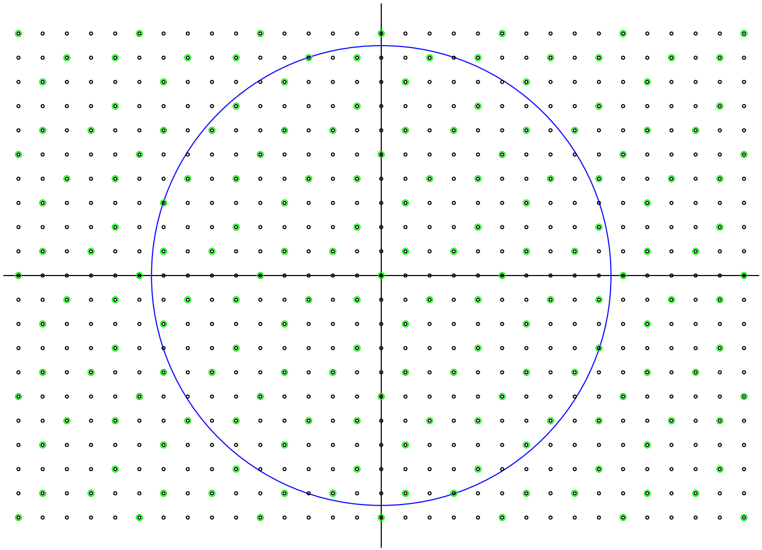
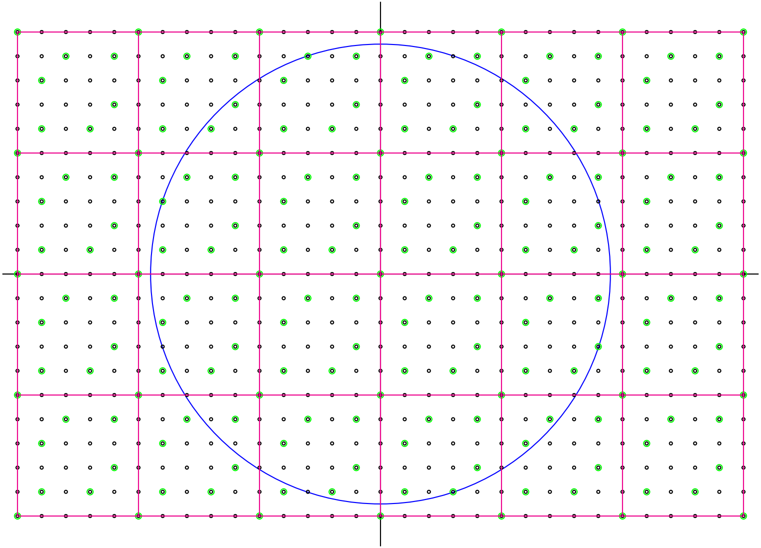
The Geometry of Arithmetic Statistics, Schloss Elmau









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Let χ be a primitive Dirichlet character $\pmod{q}$. Then we have

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Proof. By Fourier inversion, we have

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) e^{2\pi i a n/q},$$

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$$\sum_{n=M+1}^{M+N} \chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \sum_{n=M+1}^{M+N} e^{2\pi i a n/q},$$

and the innermost sum is a geometric series.

Example: Cubic fields by squarefree part

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Proposition

For any positive real numbers Y and Z, we have

$$N(Y,Z) = \Big(\sum_{f<Z} C_1(f)\Big) \cdot Y + O_\epsilon(Y^{5/6}Z^{2/3} + Y^{2/3+\epsilon}Z^{4/3});$$

$$N(Y,Z) = \Big(\sum_{\substack{|d|<Y \\ \text{fund.disc}}} \mathrm{Res}_{s=1}\Phi_d(s)\Big) \cdot Z + O_\epsilon\Big(Y^{7/6}Z^{2/3+\epsilon}\Big).$$

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- $N(X) :=$ number of $G(\mathbb{Z})$-orbits $v \in V(\mathbb{Z})$ with $0 < |\mathrm{Disc}(v)| < X$;
- $N(X, q) :=$ above, with congruence conditions $\pmod{q}$.

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A three-step proof template in arithmetic statistics:

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$$N(X, q) = \omega(q) X^a + O(X^{\alpha} q^{\beta}),$$
where $\alpha < a$ and $\beta \in \mathbb{R}$ are as small as possible.
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A three-step proof template in arithmetic statistics:

1. Bound or evaluate some Fourier transforms.
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Today: Investigate Step 1 further.

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... or any other $G(\mathbb{Z}/q\mathbb{Z})$-invariant function.

Define
$$\widehat{\Phi_q}(y) := q^{-\dim V} \sum_{x \in V(\mathbb{Z}/q\mathbb{Z})} \Phi(x) e^{-2\pi i [x,y]/q}.$$

Theorem (Poisson summation)

For a finite dimensional lattice $V(\mathbb{Z})$, we have

$$\sum_{v \in V(\mathbb{Z})} \phi(v) = \sum_{w \in \widehat{V(\mathbb{Z})}} \widehat{\phi}(w).$$

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Theorem (Poisson summation with local conditions)

For $\Phi_q : V(\mathbb{Z}/q\mathbb{Z}) \to \mathbb{C}$, we have

$$\sum_{v \in V(\mathbb{Z})} \Phi_q(v)\phi(v) = \sum_{w \in \widehat{V(\mathbb{Z})}} \widehat{\Phi_q}(w)\widehat{\phi}(w/q).$$

The Fouvry-Katz Theorem

Let Y be a (locally closed) subscheme of $\mathbb{A}^n_{\mathbb{Z}}$, of dimension d. Take $V = \mathbb{A}^n$, p prime, and Φ_p the the characteristic function of $Y(\mathbb{F}_p)$.

Theorem (Fouvry-Katz, 2001)

There exists a filtration of subschemes

$$\mathbb{A}^n_{\mathbb{Z}} \supseteq X_1 \supseteq \cdots \supseteq X_j \supseteq \cdots \supseteq X_n$$

with X_j of codimension j, so that

$$|\widehat{\Phi_p}(y)| \leq Cp^{-n+\frac{d}{2}+\frac{j-1}{2}}$$

away from $X_j(\mathbb{F}_p)$.

Example: Fouvry-Katz

Corollary (Fouvry-Katz, 2001)

There exist $\gg \frac{X}{\log X}$ *primes* $p \leq X$ *with*

$$\#\mathrm{Cl}(\mathbb{Q}(\sqrt{p+4}))[3] = 1.$$

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(Here $p + 4 \equiv 1 \pmod 4$ and squarefree.)

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$$ax^3 + bx^2y + cxy^2 + dy^3$$

with $p^2 \mid a$ and $p \mid b$.

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This is the Davenport-Heilbronn condition for nonmaximality at p.

An explicit evaluation

Theorem (Taniguchi-T., 2011)

We have

$$\widehat{\Phi_{p^2}}(v) = \begin{cases} p^{-2} + p^{-3} - p^{-5} & v/p \text{ : of type } (0), \\ p^{-3} - p^{-5} & v/p \text{ : of type } (1^3), (1^2 1), \\ -p^{-5} & v/p \text{ : of type } (111), (21), (3). \\ p^{-3} - p^{-5} & v \text{ : of type } (1^3_{**}), \\ -p^{-5} & v \text{ : of type } (1^3_{*}),\ (1^3_{\max}), \\ 0 & \text{otherwise.} \end{cases}$$

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So:

$$\frac{1}{p^8} \sum_{v \in V(\mathbb{Z}/p^2\mathbb{Z})} |\widehat{\Phi_{p^2}}(v)| \ll p^{-7}.$$

Consequence – Improved Davenport-Heilbronn

Theorem (Bhargava-Taniguchi-T.)

We have

$$N_3(X) = CX + C'X^{\frac{5}{6}} + O(X^{\frac{3}{5}+\epsilon}) + O(X^{1-\frac{1}{8-7+2}+\epsilon}).$$

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Proposition

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$$[gv, g^{-T} w] = [v, w].$$

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we have

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Obtain a G-equivariant isomorphism $\widehat{V}(\mathbb{F}_p) \to V(\mathbb{F}_p)$.

G-invariance

Proposition

If $\Phi : V(\mathbb{F}_p) \to \mathbb{C}$ is G-invariant, then so is $\widehat{\Phi}$.

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&= \sum_{v \in V(\mathbb{F}_p)} \Phi(v) \langle [v, w] \rangle .
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- The subspace method (Taniguchi-T.; Ishimoto);
- The morphism method (Ishitsuka, Ito, Taniguchi, T., Xiao).

Group decomposition (Hough, 2018)

Theorem 2. *The Fourier transform of the maximal set is supported on the mod p orbits $\mathcal{O}_0, \mathcal{O}_{D1^2}, \mathcal{O}_{D11}$ and $\mathcal{O}_{D2}$. It is given explicitly in the following tables.*

(1) *Case $\mathcal{O}_0$, $\xi = p\xi_0$.*

(6.1)

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(p\xi_0)$	*Orbit size*
$\mathcal{O}_0$	$(p-1)^4p(p+1)^2(p^5+2p^4+4p^3+4p^2+3p+1)$	1
$\mathcal{O}_{D1^2}$	$-(p-1)^3p(p+1)^4$	$(p-1)(p+1)(p^2+p+1)$
$\mathcal{O}_{D11}$	$-(p-1)^3p(2p^3+6p^2+4p+1)$	$(p-1)p(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{D2}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p(p+1)(p^2+p+1)/2$
$\mathcal{O}_{Dns}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p^2(p+1)(p^2+p+1)$
$\mathcal{O}_{Cs}$	$-p^7+5p^5-3p^4-3p^3+p^2+p$	$(p-1)^2p(p+1)^2(p^2+p+1)$
$\mathcal{O}_{Cns}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p^3(p+1)(p^2+p+1)$
$\mathcal{O}_{B11}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^2p^2(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{B2}$	$(p-1)^2p(2p^2+3p+1)$	$(p-1)^3p^2(p+1)(p^2+p+1)/2$
$\mathcal{O}_{1^4}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^2(p+1)^2(p^2+p+1)$
$\mathcal{O}_{1^31}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^3(p+1)^2(p^2+p+1)$
$\mathcal{O}_{1^21^2}$	$(p-1)^2p(3p+1)$	$(p-1)^2p^4(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{2^2}$	$-(p-1)p(p+1)^2$	$(p-1)^3p^4(p+1)(p^2+p+1)/2$
$\mathcal{O}_{1^211}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^4(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{1^22}$	$p(p^3-3p^2+p+1)$	$(p-1)^3p^4(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{1111}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/24$
$\mathcal{O}_{112}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/4$
$\mathcal{O}_{22}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/8$
$\mathcal{O}_{13}$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/3$
$\mathcal{O}_4$	$-p^3+p^2+p$	$(p-1)^4p^4(p+1)^2(p^2+p+1)/4$

Binary cubic forms – singularity

On $V = \mathrm{Sym}^3(\mathbb{F}^2)$, let Φ_p be the characteristic function of the singular locus:

$$\Phi_p(v) := \begin{cases} 1 & \text{if } \mathrm{Disc}(\mathsf{v}) = 0 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Theorem (Mori 2010)

We have

$$\widehat{\Phi_p}(v) = \begin{cases} p^{-1} + p^{-2} - p^{-3} & (v = 0), \\ p^{-2} - p^{-3} & (v \textit{ has splitting type } (1^3) \textit{ or } (1^2 1)), \\ -p^{-3} & (\textit{otherwise}). \end{cases}$$

[The above corrects a mistake pointed out in the talk.]

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Recall that $\mathrm{PGL}(2)$ acts triply transitively on $\mathbb{P}^1$.

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Therefore, the action of $\mathrm{GL}(2)$ on $\mathrm{Sym}^3(\mathbb{F}^2)$ has six orbits:

$$(0), (1^3), (1^2 1), (111), (12), (3)$$

Singularity and subspaces

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- (0) if it is G-equivalent to $(0, 0, 0, 0)$.

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- (1^3), or (0) if it is G-equivalent to some $(0, 0, 0, *)$;
- (0) if it is G-equivalent to $(0, 0, 0, 0)$.

Moral:

G-orbits of subspaces distinguish the orbits.

Sketch Proof of FT formula

Consider

$$f := \sum_{v \in (0,0,*,*)} \sum_{g \in \mathrm{GL}_2(\mathbb{F}_p)} \mathbf{1}_{gv}.$$

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a weighted version of our counting formula.

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p^4 \widehat{f}(w) &= \sum_{v \in (0,0,*,*)} \sum_{g \in G(\mathbb{F}_p)} \langle [gv, w] \rangle \\
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&= p^2 \# \{ G(\mathbb{F}_p) - \text{translates of } w \text{ in } (*, *, 0, 0) \}
\end{aligned}
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&= p^2 \sum_{g \in G(\mathbb{F}_p)} \sum_{w \in (*,*,0,0)} \mathbf{1}_{gw}.
\end{aligned}$$

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Binary quartic forms

Let V be the space of binary quartic forms, where $\mathrm{GL}(1) \times \mathrm{GL}(2)$ acts by

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Associate to $f = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4$:

$$\begin{aligned} I(f) &= 12a_0a_4 - 3a_1a_3 + a_2^2, \\ J(f) &= 72a_0a_2a_4 + 9a_1a_2a_3 - 27(a_0a_3^2 + a_1^2a_4) - 2a_2^3. \end{aligned}$$

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Let Φ_p be the characteristic function of the singular locus:

$$\Phi_p(v) := \begin{cases} 1 & \text{if } \mathrm{Disc}(v) = 0 \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Main Theorem for Quartic Forms

Theorem (Ishitsuka, Ito, Taniguchi, T., Xiao)

For a prime $p > 3$, we have

$$\widehat{\Phi_p}(v) = \begin{cases} p^{-1} + p^{-2} - p^{-3} & (v = 0), \\ p^{-2} - p^{-3} & (v \textit{ has splitting type } (1^4) \textit{ or } (1^3 1)), \\ \chi_{12}(p)(p^{-4} - p^{-3}) & (v \textit{ has splitting type } (1^2 1^2)), \\ \chi_{12}(p)(p^{-4} + p^{-3}) & (v \textit{ has splitting type } (2^2)), \\ \chi_{12}(p) p^{-4} & (v \textit{ has splitting type } (1^2 11) \textit{ or } (1^2 2)), \\ \chi_3(p) \left(\frac{I(v)}{p} \right) \cdot p^{-4} & (J(v) = 0, I(v) \neq 0), \\ a(E'_v) p^{-4} & (J(v) \neq 0, \mathrm{Disc}(v) \neq 0). \end{cases}$$

Here E'_v is the elliptic curve defined by

$$y^2 = x^3 - 3I(v)x^2 + J(v)^2,$$

with $a(E'_v) := p + 1 - \#E'_v(\mathbb{F}_p)$.

If $w \neq 0$, we have

$$\sum_{\substack{w \in \overline{w} \\ w \neq 0}} \langle [w, v] \rangle = \begin{cases} p - 1 & ([w, v] = 0) \\ -1 & ([w, v] \neq 0), \end{cases}$$

where $\overline{w}$ is the line through w and 0. So,

$$\begin{aligned} \widehat{\Phi_p}(v) &= 1 + (p-1) \sum_{\overline{w} \in \mathbb{P}(V), [\overline{w}, v] = 0} \Phi_p(\overline{w}) - \sum_{\overline{w} \in \mathbb{P}(V), [\overline{w}, v] \neq 0} \Phi_p(\overline{w}) \\ &= 1 + p \# X_v(\mathbb{F}_p) - \# X(\mathbb{F}_p), \end{aligned}$$

where

$$X := \{ w \in \mathbb{P}(V) \mid \mathrm{Disc}(w) = 0 \},$$
$$X_v := \{ w \in \mathbb{P}(V) \mid \mathrm{Disc}(w) = [w, v] = 0 \}.$$

Three morphisms

Consider projective morphisms

$$\psi_1 \colon \mathbb{P}(\mathbb{F}_p^2) \times \mathbb{P}(\mathrm{Sym}^2\mathbb{F}_p^2) \to \mathbb{P}(\mathrm{Sym}^4\mathbb{F}_p^2) = \mathbb{P}(V)$$
$$(s_0x + s_1y, t_0x^2 + t_1xy + t_2y^2) \mapsto (s_0x + s_1y)^2(t_0x^2 + t_1xy + t_2y^2).$$

$$\psi_2 \colon \mathbb{P}(\mathrm{Sym}^2\mathbb{F}_p^2) \to \mathbb{P}(\mathrm{Sym}^4\mathbb{F}_p^2) = \mathbb{P}(V)$$
$$t_0x^2 + t_1xy + t_2y^2 \mapsto (t_0x^2 + t_1xy + t_2y^2)^2$$

$$\psi_3 \colon \mathbb{P}(\mathbb{F}_p^2) \times \mathbb{P}(\mathbb{F}_p^2) \to \mathbb{P}(\mathrm{Sym}^4\mathbb{F}_p^2) = \mathbb{P}(V)$$
$$(s_0x + s_1y, t_0x + t_1y) \mapsto (s_0x + s_1y)^2(t_0x + t_1y)^2.$$

Then, the cardinalities of each $\psi_i(v)$ are:

Spitting type	$\#\psi_1^{-1}$	$\#\psi_2^{-1}$	$\#\psi_3^{-1}$
non-degenerate	0	0	0
(1^4)	1	1	1
$(1^3 1)$	1	0	0
$(1^2 1^2)$	2	1	2
(2^2)	0	1	0
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$(1^2 2)$	1	0	0

Three morphisms – inverse images

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$(1^2 2)$	1	0	0

So,

$$\Phi_p(\overline{w}) = \#\psi_1^{-1}(\overline{w}) + \#\psi_2^{-1}(\overline{w}) - \#\psi_3^{-1}(\overline{w}).$$

The elliptic curve

We have

$$\sum_{\overline{w} \in \mathbb{P}(V), [\overline{w}, v] = 0} \#\psi_3^{-1}(\overline{w}) = \# C_3(v),$$

where

$$C_3(v) = \left\{ (l_1, l_2) \in \mathbb{P}(\mathbb{F}_p^2) \times \mathbb{P}(\mathbb{F}_p^2) \mid [l_1^2 l_2^2, v] = 0 \right\}.$$

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Proposition (Bhargava-Ho)

If $\mathrm{Disc}(v) \neq 0$ and $J(v) \neq 0$, then $C_3(v)$ is of genus one, isomorphic to

$$E'_v \;:\; y^2 = x^3 - 3I(v)x^2 + J(v)^2.$$

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