State High School Mathematics Tournament

University of South Carolina

Round 1 – January 25, 2020

Tournament Round 1 – Rules

Tournament Round 1 – Rules

- You will be asked a series of questions. Each correct answer earns you a point.

Tournament Round 1 – Rules

- You will be asked a series of questions. Each correct answer earns you a point.
- You have 90 seconds to answer each question. You are racing the clock and not each other. You will be warned when only 10 seconds remain.

Tournament Round 1 – Rules

- You will be asked a series of questions. Each correct answer earns you a point.
- You have 90 seconds to answer each question. You are racing the clock and not each other. You will be warned when only 10 seconds remain.
- Write your answer in simplified form on the slip you are given, and hand it to the judge.

Tournament Round 1 – Rules

- You will be asked a series of questions. Each correct answer earns you a point.
- You have 90 seconds to answer each question. You are racing the clock and not each other. You will be warned when only 10 seconds remain.
- Write your answer in simplified form on the slip you are given, and hand it to the judge.
- If you change your mind, you may give another answer to the judge. Only the last answer you give to the judge will be counted.

Tournament Round 1 – Rules

- You will be asked a series of questions. Each correct answer earns you a point.
- You have 90 seconds to answer each question. You are racing the clock and not each other. You will be warned when only 10 seconds remain.
- Write your answer in simplified form on the slip you are given, and hand it to the judge.
- If you change your mind, you may give another answer to the judge. Only the last answer you give to the judge will be counted.
- **Two winners will be selected this round: the participants earning three points with the fewest questions.**

Tournament Round 1 – Rules

- You will be asked a series of questions. Each correct answer earns you a point.
- You have 90 seconds to answer each question. You are racing the clock and not each other. You will be warned when only 10 seconds remain.
- Write your answer in simplified form on the slip you are given, and hand it to the judge.
- If you change your mind, you may give another answer to the judge. Only the last answer you give to the judge will be counted.
- **Two winners will be selected this round: the participants earning three points with the fewest questions.**
- There will be a tiebreaker if needed.

Question 1-1

Solve for x:

$$|2x - 1| = |2x - 2|.$$

Solution 1-1

Answer. $\frac{3}{4}$.

Solution 1-1

Answer. $\frac{3}{4}$.

We have

$$2x - 1 = \pm(2x - 2),$$

and since $+$ is impossible, we have

$$2x - 1 = -(2x - 2) = -2x + 2.$$

Solution 1-1

Answer. $\frac{3}{4}$.

We have

$$2x - 1 = \pm(2x - 2),$$

and since $+$ is impossible, we have

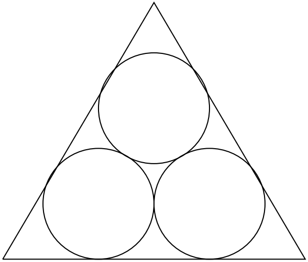
$$2x - 1 = -(2x - 2) = -2x + 2.$$

So,

$$4x = 2 + 1 = 3 \Longrightarrow x = \frac{3}{4}.$$

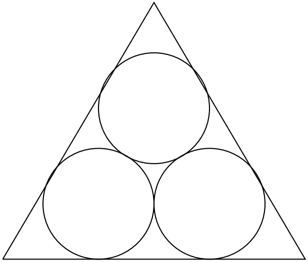
Question 1-2

In the diagram below, three circles of equal size are inscribed in an equilateral triangle, so that they are tangent to each other and to the indicated sides of the triangle.



Question 1-2

In the diagram below, three circles of equal size are inscribed in an equilateral triangle, so that they are tangent to each other and to the indicated sides of the triangle.



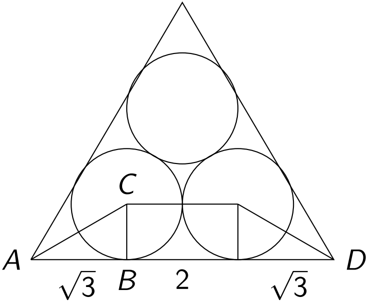
If each circle has radius 1, what is the side length of the triangle?

Solution 1-2

Answer. $2 + 2\sqrt{3}$.

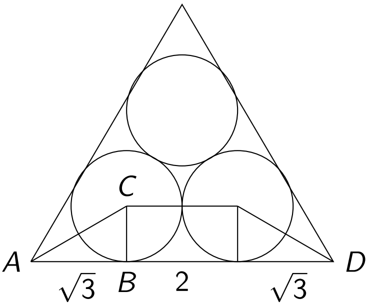
Solution 1-2

Answer. $2 + 2\sqrt{3}$.



Solution 1-2

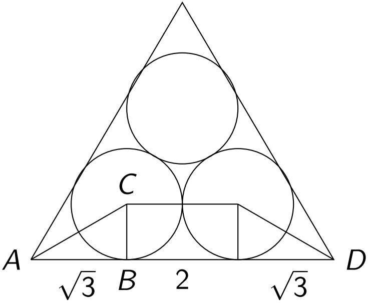
Answer. $2 + 2\sqrt{3}$.



As $\overline{AC}$ lies on an altitude, we have $\angle CAB = 30^\circ$.

Solution 1-2

Answer. $2 + 2\sqrt{3}$.



As $\overline{AC}$ lies on an altitude, we have $\angle CAB = 30^\circ$.
Hence $AB = \sqrt{3}$ and so $AD = 2 + 2\sqrt{3}$.

Question 1-3

What is the only real solution x to

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

Solution 1-3

Answer. $-\frac{11}{2}$.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

Solution 1-3

Answer. $-\frac{11}{2}$.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

Solution 1-3

Answer. $-\frac{11}{2}$.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

$$\frac{1}{(x+4)(x+5)} = \frac{1}{(x+6)(x+7)}$$

Solution 1-3

Answer. $-\frac{11}{2}$.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

$$\frac{1}{(x+4)(x+5)} = \frac{1}{(x+6)(x+7)}$$

$$(x+4)(x+5) = (x+6)(x+7)$$

Solution 1-3

Answer. $-\frac{11}{2}$.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

$$\frac{1}{(x+4)(x+5)} = \frac{1}{(x+6)(x+7)}$$

$$(x+4)(x+5) = (x+6)(x+7)$$

You can solve this for x,

Solution 1-3

Answer. $-\frac{11}{2}$.

$$\frac{x+5}{x+4} - \frac{x+6}{x+5} = \frac{x+7}{x+6} - \frac{x+8}{x+7}?$$

$$\frac{(x+5)^2 - (x+4)(x+6)}{(x+4)(x+5)} = \frac{(x+7)^2 - (x+6)(x+8)}{(x+6)(x+7)}$$

$$\frac{1}{(x+4)(x+5)} = \frac{1}{(x+6)(x+7)}$$

$$(x+4)(x+5) = (x+6)(x+7)$$

You can solve this for x, or notice that $-\frac{11}{2}$ lies on the midpoint of symmetry of the roots.

Question 1-4

$$123 \times 321 = 41703$$

Question 1-4

$$123 \times 321 = 41703$$

In what number base is this equation true?

Answer: 8.

Solution 1-4

Answer: 8.

$$(b^2 + 2b + 3) \times (3b^2 + 2b + 1) = \cdots + 8b + 3.$$

Solution 1-4

Answer: 8.

$$(b^2 + 2b + 3) \times (3b^2 + 2b + 1) = \cdots + 8b + 3.$$

So $b \mid 8$. Since the digit 7 appears in the product, $b = 8$.

Question 1-5

The circle $x^2 + y^2 = 4$ intersects the ellipse $\frac{x^2}{16} + y^2 = 1$ in exactly four points.

Question 1-5

The circle $x^2 + y^2 = 4$ intersects the ellipse $\frac{x^2}{16} + y^2 = 1$ in exactly four points.

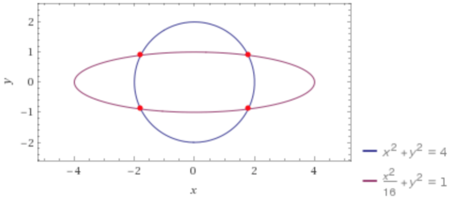
What is the area of the rectangle with these four points as vertices?

Solution 1-5

Answer. $\frac{32}{5}$.

Solution 1-5

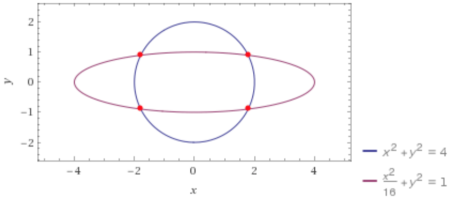
Answer. $\frac{32}{5}$.



Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$.

Solution 1-5

Answer. $\frac{32}{5}$.



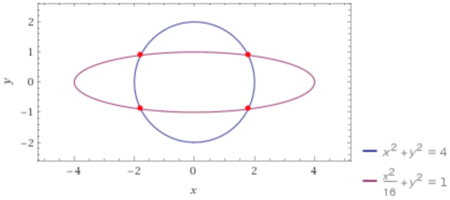
Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$.
We have $y^2 = 4 - \frac{16}{5} = \frac{4}{5}$, and so

$$x^2 \cdot y^2 = \frac{16}{5} \cdot \frac{4}{5} = \frac{64}{25},$$

and the unique solution with $x > 0$, $y > 0$ satisfies $xy = \frac{8}{5}$.

Solution 1-5

Answer. $\frac{32}{5}$.



Subtracting the two equations yields $\frac{15x^2}{16} = 3$, so $x^2 = \frac{48}{15} = \frac{16}{5}$.

We have $y^2 = 4 - \frac{16}{5} = \frac{4}{5}$, and so

$$x^2 \cdot y^2 = \frac{16}{5} \cdot \frac{4}{5} = \frac{64}{25},$$

and the unique solution with $x > 0$, $y > 0$ satisfies $xy = \frac{8}{5}$.

Since the rectangle is centered at the origin, its area is $4xy = \frac{32}{5}$.

Question 6

You eat a bunch of cupcakes. On the first day, you eat one cupcake; on the second day, you eat two cupcakes; on the third day, you eat three; and so on.

Question 6

You eat a bunch of cupcakes. On the first day, you eat one cupcake; on the second day, you eat two cupcakes; on the third day, you eat three; and so on.

On what day will you eat your five thousandth cupcake?

Solution 6

Answer. 100.

Solution 6

Answer. 100.

After n days, you will have eaten

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

cupcakes. So what is the minimal n for which

$$\frac{n(n+1)}{2} \geq 5000, \quad \text{or} \quad n(n+1) \geq 10000?$$

Solution 6

Answer. 100.

After n days, you will have eaten

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

cupcakes. So what is the minimal n for which

$$\frac{n(n+1)}{2} \geq 5000, \quad \text{or} \quad n(n+1) \geq 10000?$$

Since $10000 = 100^2$, we have $n = 100$.

Question 7

This is the logo of the Mathematical Association of America, a national organization that sponsors competitions and publishes excellent journals and books.

Question 7

This is the logo of the Mathematical Association of America, a national organization that sponsors competitions and publishes excellent journals and books.

It depicts an icosahedron: a regular solid with twenty faces, each of which is an identical equilateral triangle.

Question 7

This is the logo of the Mathematical Association of America, a national organization that sponsors competitions and publishes excellent journals and books.

It depicts an icosahedron: a regular solid with twenty faces, each of which is an identical equilateral triangle.

How many edges are not visible in the logo?

Solution 7

Answer. 12.

Solution 7

Answer. 12.

An icosahedron has 30 edges: 20 triangles times 3 edges per triangle, divided by 2 since each edge is shared between two triangles.

Solution 7

Answer. 12.

An icosahedron has 30 edges: 20 triangles times 3 edges per triangle, divided by 2 since each edge is shared between two triangles.

You can count that 18 edges are visible in the picture, and $30 - 18 = 12$.

Question 8

What is the smallest integer larger than

Question 8

What is the smallest integer larger than

$$\log_2(33) + \log_{33}(2)?$$

Solution 8

Answer: 6.

Solution 8

Answer: 6.
We have

$$2^5 = 32, \quad 2^6 = 64,$$

so that $\log_2(33)$ is slightly bigger than 5.

Solution 8

Answer: 6.

We have

$$2^5 = 32, \quad 2^6 = 64,$$

so that $\log_2(33)$ is slightly bigger than 5.

We have

$$\log_{33}(2) = \frac{1}{\log_2(33)} < \frac{1}{5}.$$

Solution 8

Answer: 6.

We have

$$2^5 = 32, \quad 2^6 = 64,$$

so that $\log_2(33)$ is slightly bigger than 5.

We have

$$\log_{33}(2) = \frac{1}{\log_2(33)} < \frac{1}{5}.$$

The sum of these numbers is less than 6.

How many real solutions x are there to the equation

Question 9

How many real solutions x are there to the equation

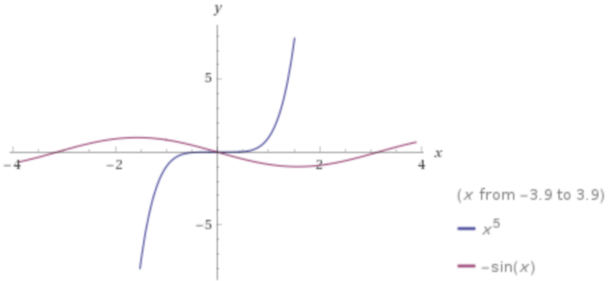
$$x^5 + \sin(x) = 0?$$

Solution 9

Answer: 1.

Solution 9

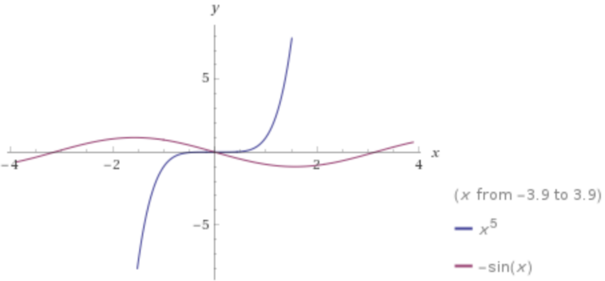
Answer: 1.



The graphs of $y = x^5$ and $y = -\sin(x)$ don't intersect in $(0, \pi)$ because of opposite signs, or in $[\pi, \infty)$ because $x^5 > 1$. Similarly, there are no intersection points with $x < 0$.

Solution 9

Answer: 1.



The graphs of $y = x^5$ and $y = -\sin(x)$ don't intersect in $(0, \pi)$ because of opposite signs, or in $[\pi, \infty)$ because $x^5 > 1$. Similarly, there are no intersection points with $x < 0$.
So $x = 0$ is the only intersection point.

You toss four coins. What is the probability that at least three of them come up heads?

Solution 10

Answer. $\frac{5}{16}$.

Solution 10

Answer. $\frac{5}{16}$.

There are $2^4 = 16$ total ways to flip four coins.
The total number with at least three heads is

$$\binom{4}{3} + \binom{4}{4} = 4 + 1 = 5,$$

Solution 10

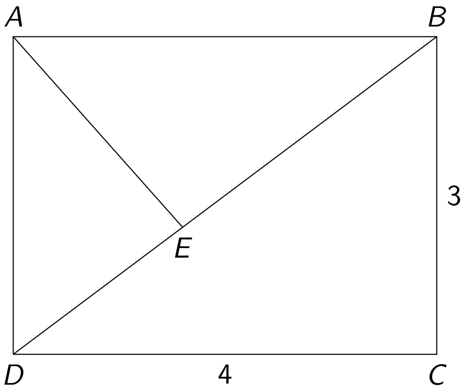
Answer. $\frac{5}{16}$.

There are $2^4 = 16$ total ways to flip four coins.
The total number with at least three heads is

$$\binom{4}{3} + \binom{4}{4} = 4 + 1 = 5,$$

$$HHHH, HHHT, HHTH, HTHH, THHH.$$

Question 11



Given rectangle $ABCD$ as above. If $\angle AEB = 90^\circ$, what is AE?

Solution 11

Answer. $\frac{12}{5}$.

Solution 11

Answer. $\frac{12}{5}$.

$BD = 5$, and $\triangle ABE \sim \triangle BDC$. So

$$\frac{AE}{AB} = \frac{BC}{BD} = \frac{3}{5}$$

and

$$AE = \frac{3}{5} \cdot AB = \frac{3}{5} \cdot 4 = \frac{12}{5}.$$

Question 12

How many pairs of positive prime numbers p, q are there with

$$p - q = 21?$$

Solution 12

Answer. 1.

Solution 12

Answer. 1.

All prime numbers other than 2 are odd. The difference of two odd numbers is even. Therefore q must be 2. Since $2 + 21 = 23$ is prime, there is one solution.

Question 13

In baseball, an **at bat** *results in either a* **hit** *or an* **out***. A player's* **batting average** *is their total number of hits divided by at bats, rounded off to the nearest thousandth.*

Question 13

In baseball, an **at bat** *results in either a* **hit** *or an* **out**. *A player's* **batting average** *is their total number of hits divided by at bats, rounded off to the nearest thousandth.*

Five games into the baseball season, Cocky Gamecock has a batting average of .435. In his sixth game, he has five at bats and gets hits in all of them.

Question 13

In baseball, an **at bat** *results in either a* **hit** *or an* **out***. A player's* **batting average** *is their total number of hits divided by at bats, rounded off to the nearest thousandth.*

Five games into the baseball season, Cocky Gamecock has a batting average of .435. In his sixth game, he has five at bats and gets hits in all of them.
If this raises his batting average to .536, how many at bats does he have through his first six games?

Solution 13

Answer. 28.

Solution 13

Answer. 28.

Let x be the number of hits through 6 games, and y the number of at bats. Within a small roundoff error,

$$\frac{x-5}{y-5} = .435, \quad \frac{x}{y} = .536.$$

Solution 13

Answer. 28.

Let x be the number of hits through 6 games, and y the number of at bats. Within a small roundoff error,

$$\frac{x-5}{y-5} = .435, \quad \frac{x}{y} = .536.$$

We have

$$x - 5 = .435(y-5), \quad x = .435y + 5 - 2.175 = .435y + 2.825.$$

Solution 13

Answer. 28.

Let x be the number of hits through 6 games, and y the number of at bats. Within a small roundoff error,

$$\frac{x-5}{y-5} = .435, \quad \frac{x}{y} = .536.$$

We have

$$x - 5 = .435(y-5), \quad x = .435y + 5 - 2.175 = .435y + 2.825.$$

We thus have

$$x = .536y, \quad .101y = 2.825.$$

Solution 13

Answer. 28.

Let x be the number of hits through 6 games, and y the number of at bats. Within a small roundoff error,

$$\frac{x-5}{y-5} = .435, \quad \frac{x}{y} = .536.$$

We have

$$x - 5 = .435(y-5), \quad x = .435y + 5 - 2.175 = .435y + 2.825.$$

We thus have

$$x = .536y, \quad .101y = 2.825.$$

Thus, we have

$$y = \frac{2.825}{.101},$$

or $y = 28$ up to the roundoff error.

Question 14

If you write $\frac{1}{2020}$ as an infinite repeating decimal,

Question 14

If you write $\frac{1}{2020}$ as an infinite repeating decimal, what is the sum of the first six digits after the decimal place?

Solution 14

Answer. 18.

$$\frac{1}{2020} = 0.00049504950\ldots$$

Solution 14

Answer. 18.

$$\frac{1}{2020} = 0.00049504950\ldots$$

Note that

$$\frac{1}{101} = .009900990099\ldots,$$

so

$$\frac{1}{1010} = .0009900990099\ldots,$$

$$\frac{1}{2020} = .0004950495049\ldots,$$