COMPREHENSIVE EXAM IN ALGEBRAIC NUMBER THEORY (FALL 2013)

Recall that the **Minkowski bound** is

$$N(\mathfrak{a}) \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |\Delta_K|^{1/2}.$$

1. Let K be the cubic field generated by a root of $x^3 + x - 4$. Determine the following data associated to K:
   (a) its **discriminant**;
   (b) the **ring of integers**;
   (c) the number of **real and complex embeddings**;
   (d) the isomorphism class of its **unit group** (your answer should look like, e.g., $\mathbb{Z}^2 \times \mathbb{Z}/(6)$; you do not need to find the actual units);
   (e) the list of **ramified primes**;
   (f) the splitting types of the ideals $(2), (3), (5), (7)$;
   (g) whether or not K is **Galois** over $\mathbb{Q}$; and if not, the Galois group of its Galois closure (i.e., the splitting field of $x^3 + x - 4$);
   (h) the **the class group**;
   (i) the **proportion of primes** which have the splitting types you found above.

2. (a) Give two definitions of the p-adic integers $\mathbb{Z}_p$. One should be analytic (in terms of Cauchy sequences) and another should be algebraic (an inverse limit). Prove their equivalence. Define also $\mathbb{Q}_p$.
   (b) Determine (with proof) the maximal ideal $\mathfrak{m}$ of $\mathbb{Z}_p$, as well as the residue field $\mathbb{Z}_p/\mathfrak{m}$.
   (c) Let $p = 7$. Determine which of $\frac{1}{3}$, 7, $\frac{1}{7}$, $\sqrt{2}$, and $\sqrt{5}$ are 7-adic integers. For those that are, compute the 7-adic expansion to at least three decimal places.
   Note that one of the two square roots is in $\mathbb{Z}_7$; give a detailed proof of this, without quoting Hensel's lemma. For the others, a very brief explanation is enough.

3. Prove that no two p-adic fields $\mathbb{Q}_p$ are isomorphic to each other, nor to $\mathbb{R}$. (Bonus: Prove that no finite extension of $\mathbb{Q}_p$ is isomorphic to any finite extension of $\mathbb{Q}_{p'}$ for $p \neq p'$.

4. What is Hensel's lemma? State it, give an example of its use, and give a proof of it in a special case of your choosing.

5. Does $x^2 + y^2 + 7z^2 = 0$ have any rational solutions? Prove or disprove. What about $x^2 + y^2 + 7z^2 = 1$?

6. Write down a bunch of quadratic fields at random and compute their class groups.

7. Let K be a quintic field, whose Galois closure is of degree 10 over $\mathbb{Q}$ and has Galois group D_5.
   For (ordinary) primes p, determine all possibilities for how $p\mathcal{O}_K$ can decompose into prime ideals of $\mathcal{O}_K$.

8. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ for some K. Prove directly that $\mathfrak{a}$ contains an integer other than 0, and then explain the relationship of this fact to the norm of $\mathfrak{a}$.

9. Suppose that $\mathcal{O}$ an order in a quadratic field K, i.e. $\mathcal{O}$ is a subring of K, containing 1, finitely generated as a $\mathbb{Z}$-module, and containing a $\mathbb{Q}$-basis of K.
   Prove that $\mathcal{O}$ is contained in the ring of integers $\mathcal{O}_K$.

10. Let K be a field for which $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some α. Let $f(x)$ be the minimal polynomial of α.
    (a) Explain why $f(x)$ is monic.
    (b) Explain why a prime p ramifies in K if and only if $f(x)$ has a multiple root modulo p.
    (c) Prove the classical fact (in this special case) that a prime p divides the discriminant of K if and only if it ramifies in K.

11. Let K be a quadratic field, and p a prime. Compute the tensor product $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$. (There are three cases....)