42.1 (Math 702: Spring semester) [MWF 12:00, LC 310.]

Announcements: - Seminor schedule in flux
Noth Contest 2-3

GAGS, 2!23-25, Oa Teeli

SERMON, 3:10-11

Algebraic Curves, 4.6-8, Medison

Additive combinatorics, 5:21-25

Class time adjistment -TA Sessions

Any algebra class requests for next year (talk to Andy)

The Tensor Product Construction.

Idea: Given a ring & (with 1, not nec. commutative)

Two (left) R-modules M and N

Will cook up a "product", called M@N or M@N.

This will be quite different than the direct sum (or cortesian product).

Example. (1) Suppose R = M = N = C.

Then  $C \propto C$  should be C.

If you multiply complex numbers, get complex numbers.

(2) Suppose M = C and N = IR[x,,..., xn].

Then M@N (really, M@N) should be

C[x,,..., xn].

Let's generalize this and see what we demand.

Suppose that RES are rings and N is a left Suppose that RES are rings and N is a left.

Then N is also a left P-module, and  $(s_1+s_2)_n = s_1 n + s_2 n$   $s(u_1+u_2) = su_1 + su_2$   $s(u_1+u_2) = su_1 + su_2$   $s(u_1,u_2) = su_1 + su_2$   $s(u_1,u_2) = su_2$   $s(u_1,u_2) = su_1 + su_2$ 

(in particular) (Sr)n = s(rn)

(r + R)

Can we reverse this? RES and N is an R-madule. Can we make it on S-module?

In general, no. e.g. ReQZED, Zis a Z-module. Claim. There is no Q-module structure on Z.

Proof, Suppose there was, and write \frac{1}{2} = 7 for 7 \in \bar{2}.

Then  $\left(\frac{1}{2} + \frac{1}{2}\right) \circ 1 = 1 \circ 1 = 1$  (demand 1m = m for all  $m \in M$ )

 $\frac{1}{2}$  o  $\left( + \frac{1}{2} \right)$  o  $\left( = 7 + 7 \right)$ .

You cannot solve 2+7=1 in Z!

Revised question. If RES and N is an R-module, can we embed N as an R-submodule of en S-module?

Let's try to do this. Z = Q again, and N = Z/2 as a Z-module.

Is there a Q-module containing Z/2 as a Z-submodule?

No: Q-modules one Q-vector spaces and they never have torsion: k·x = 0 => k or x = 0.

Another vey to think about this. I has lots et ideals a doesn't.

But: What the hell, let's try anyway.

Wort to try to define su for ses (RES with NEN. Non R-medule)

Start with the free abelian group on S x N.

(free 70-module)

Formal products (s, n) with no relations.

Now we make it into an S-module.

Now we make it into an S-module. We want:

$$(s_1 + s_2, n) = (s_1, n) + (s_2, n)$$
  
 $(s_1, n_1 + n_2) = (s_1, n_1) + (s_2, n_2)$   
 $(s_1, n_1 + n_2) = (s_1, n_1) + (s_2, n_2)$   
 $(s_1, n_1 + n_2) = (s_1, n_1) + (s_2, n_2)$   
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 $(s_1, n_1 + n_2) = (s_1, n_1) + (s_2, n_2)$   
 $(s_1, n_1 + n_2) = (s_1, n_1) + (s_2, n_2)$   
 $(s_1, n_1, n_2 \in \mathbb{N})$ 

Definition #1. If RES are rings and N is an R-module, then the tensor product Soop N is the following quotient group:

{Free abelian group generated by symbols}
(s,n) with seS and nEN)

 $\begin{cases} Subgroup generated by \\ (s, +s_2, n) - (s, n) - (s_2, n) & for all \\ (s, +s_2, n) - (s, n) - (s, n_2) & s_1 s_1 s_2 e s_1 \\ (s, +s_2, n) - (s, +s_2) & u_1 n_1 n_2 e N \\ (sr, n) & - (s, rn) \end{cases}$ 

We write soon for the coset containing (s,n).

Note: (1) It is possible that son = s'oon'

even if s #s', n # n'

(2) Not necessarily true that every elt. looks like some ses and ne M.

Proposition. Soop N is indeed a left 5-module, with action

s(\(\begin{array}{c} \Sigma \mu'\) = \(\begin{array}{c} (ss;) \varphi \mu'\).

Sketch proof.

(1) Must show this is well - defined, i.e. if

\[ \subseteq \text{solis} \text{ is in the subgroup we're quotienting out} \]

\[ \bullet\_1 \text{so is} \frac{2}{5} \text{ss}; \text{ \

(2) Mist show it satisfies the S-module axioms.

These are tedious, but useful for newcomers.

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42.5
 Proposition. There is a natural map
             (: N -> S & N
                N -> 1 06 N.
 It is an R-module homomorphism because
    r(100n) = roon (by definition)
               = 1. r @ n
               = 10 rn (by construction)
   and (100 n) + (100 n') = 100 (n+n')
It might not be injective, e.g.
           1: Z/2 -> Q 0 Z/2
 because Q = 2/2 = 0.
Exercise. Prove this. Namely, as abelien groups let
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Exercise. Prove this. Namely, as abelian groups let  $A = \text{free abelian group generated by } (s,n) \text{ seQ}, n \in \mathbb{Z}/2$   $B = \text{subgroup gen by } (s, +s_2, n) - (s_1, n) = (s_2, n)$   $(s, +s_2, n) - (s, n) - (s, n_2)$  (sr, n) - (s, rn)

Then A=B.

42.6.

But Sop N is the "best possible" S-module to serve as the target of an R-module homomorphism from N.

Proposition (the universal property).

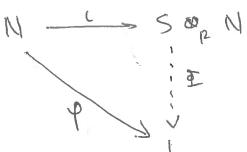
Let RES be rings, N a left R-module, and

(:N -> S@RN

n -> 100 n.

an R-module homorphism N 4 > L.

Then there exists a unique S-module hom SORN \$ L



Conversely, given such a \$, 9 = \$00 is an P-mod hom N-> L.

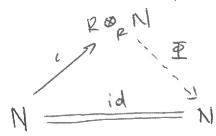
Will be proved later in more generality.

42.7.

Examples.

1. If N is any left P-nodule, R&NZN.

Proof. Use the UP with y = id.



L is injective because id is.

Surjective because ron = 1 orn = 1m (i).

2. R= 72, S= Q, A any finite obelien group. Check: QQQ A = 0.

3. (To be proved shortly)

If N=R' is a free ronk n R-module, then

S & R N = S' is a free ronk n S-module.

So, e.g. if N = F" is a vector space, and K/F is an extension field, then K & N is a K-vector space with the same basis.

4. If A is any f.g. abelien group, then
A = Z" & T for some nounez integer u, finite abelien
group T.

Then

 $Q \otimes_{\mathcal{I}} A \cong Q^{n}$ .

(Will see tensor products commute ul direct suns)

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42.8
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Tensor products in general:

Let N be a left P-modile. M be a right P-modile.

Then M & N consists of the quotient

Efree stagroup on M×N?

 $\begin{cases} subgroup & (m_1 + m_2, n) - (m_1, n) - (m_2, n) \\ (m_1, n_1 + m_2) - (m_1, n_1) - (m_1, n_2) \end{cases}$   $\begin{cases} qen by & (mr, n) - (m_1, rn) \end{cases}$ 

Write elements as  $m \otimes n$ , with  $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$   $(m_1 + m_2) \otimes n = m_1 \otimes n + m \otimes n_2$   $m \otimes (n_1 + n_2) = m \otimes n.$   $mr \otimes n = m \otimes rn.$ 

43.1. Last time.

Tensor products Sop N where PES ore rings.

Pairs (s,n) subject to bilinearity relations.

Write son

Satisfied a universal property

N

Satisfied a universal property

L

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Today: Take the tensor product of two modules. But: Assume the ring R is commutative.

(DF do the general case. But usually R is commutative.) Then we need only consider left R-modules.

Definition 1. Let M and N be P-modeles.

The tensor product M@R N consists of the R-module generated by symbols mon, with relations

(m+m') & n = m & n + m & n m @ (n + n') = m @ n + m @ n' LMRN = MQLN

and R-action r(mon) = rmon.

(Formally: Free abelian group on (m,n), quotient out by relations.)

43.2. Definition. Let M, N, L be P-modules.

A function  $\varphi: M \times N \longrightarrow L$  is R-bilinear if

it is an R-module hom in each variable separately: q(m+m',n) = q(m,n) + q(m',n) q(m,n+n') = q(m,n) + q(m,n')  $q(rm,n) = q(m,rn) = r \cdot q(m,n).$ 

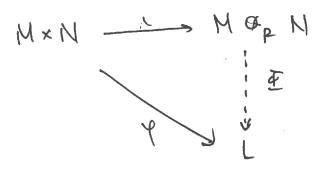
Theorem, (the universal property)

The tensor product Map N satisfies the following:

(1) The map M×N --> Mop N is P-bilinear.

(immediate)

(2) If &MXN Is L is any other P-bilinear map, then there is a unique  $\bar{\Psi}: M \otimes_{p} N \longrightarrow L$  making the diagram commute.



43.7.

Some computations (exercises):

Q/Z & Q/Z = 0 (do by gens + rel'ns)

A & B = 0 if { A is divisible (given a f A, n f 76-10), a/n f A)

B is torsion (every elt. has finite order)

QQ D = Q Q Q Q Q Q

CORC = C (think of these as 19-vector spaces...)

Some basic properties:

Associativity: (M&RN) & = M&R(N&P)

Commetativity: Map N = N &p M

Distributive Laws: (M & M') & N = (M & N) & (M' & N) (and in the second variable too).

Proof of associative law.

The map (m,n) -> m @ (n @ l), for each fixed l, is R-bilinear, so the UP yields a hom

MOEN -> MOE (NOE L)

sotisfying mon - mo (nod).

So we get a WD map (MøRN) × L ---> MøR(NøRL)

 $(m \otimes n, l)$   $\longrightarrow$   $m \otimes (n \otimes l)$ 

which is R - bilinear.

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43.8
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Use the UP again, induces a map  $(M \otimes_{p} N) \otimes_{p} L \longrightarrow M \otimes_{p} (N \otimes_{p} L)$   $(m \otimes n) \otimes l \longrightarrow m \otimes (u \otimes l)$ 

Similarly get a morp in the opposite direction. So these are mutually inverse isomorphisms.

Distributive Law: (M & M') op N = (M op N) & (M' op N)

Consider the map (M & M') × N -> RHS

((m, m'), n) -> (m @ n, m' @ n)

It is R-bilinear, so get by the UP again

('M @ M') @ N -> RHS

(m,m') & n -> (m @ n, m' @ n)

In the other dir. consider M×N -> (M⊕M') ® N M'×N -> (M⊕M') ® N

> (m, n) -> (m, o) & n (m', n) -> (o, m') & n resp.

Give homs M&N \_\_\_ > (M&M') &N

and hence a hom from the direct sum with

(mon, m'onz) - (m, o) on, + (o, m') onz.

Easily checked: commutes with the previous one.

44.1. \* Review aonstruction and UP of tensor products.

ACD laws + prove dist. Cor. (Extension of scalors for free modules) Given RES rings, SøR R" = 5". [Use distributive law and Sop P = 5.] Cor. RS & R+ = RS+ Moreover, it bases for RS, Rt ere m, ... ms then missuj form a basis. Note that vector spaces one free modules over the field. Another way to think about the wareepointenanup! Proposition. For each R-module L, there is a bijection SP-bilineer maps } P-bilineer Proof. Recoll: MXN \_\_\_\_ NOR N 9 3 11 Any & induces a unique F. Conversely, too &= for.

[10.5 important but advanced, will skip for now]

99.3. Definition. If R is a commetative ring, an R-algebra is, vivolently: equivolently: (1) a ring A with a ring how  $f: R \longrightarrow A$ , such that  $f(R) \subseteq A$  is in the center of A. (Dumnit-Foote)

(2) an R-module which is simultaneously a ring (Wikipedia)  $r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$  in A. (3) on R-module A with an R-bilinear map A x A -> A (multiplication) satisfying the associative law. Exercise. Convince yourself that these are all the same. Anyway, for any commutative ring R, T(M) is a R-algebra, which multiplication given by the teusor product. It is not in general commutative, because x or y + y ox in general. [Note that M&N = N&M, but this is not the same.]

Theorem. The tensor algebra satisfies the following UP:

If  $q: M \to A$  is an R-module homomorphism

into any R-algebra A, get a map  $P:T(M) \to A$ with  $M \longrightarrow T(M)$ 

する

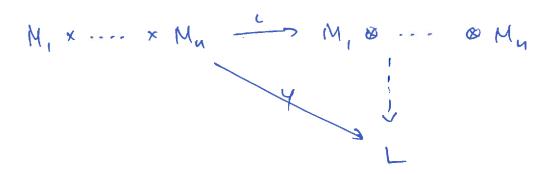
Proof: Exercise! (Or reed DF.) 44.2.

Note also we can define

MINEMZ DE M3 & ... & MK

= (((M, & Mz) & M3) & ...
Put the parentheses any way

We could have defined this directly pope (gens + rel'ns) or via a UP



where y is R-miltilinear: an R-modile how in each variable separately when other variables are constant.

Tensor, Symmetric, and Exterior Algebras (DF, 11.5)

Define T°(M) = R

Tk(M) = M @ M @ ... M for each k = 1.

T(M) = D T'(M). (Identify M with T(M).)

This is an R-module containing M as a submodule.

44.4. This is an example of a graded ring: Def. A ring S is graded if it is the direct sum of additive subgroups S = S, & S, & S<sub>2</sub> & ··· @ with Sisi & Siti for all i, j = 0. The elements of Si one the homogeneous of degree i. Examples. \* The tensor algebra, with T'(M) being the homogeneous component of degree i. \* Polynomial rings R[X] or R[X1,..., Xn]. Degree is the total poly degree. Note that polynomial rings are also 12-algebras (commutative ring extensions always one). But wait. PEXTERN ST All as R-modules. we get a homomorphism!

[2[x] T(P) = T°(P) O T'(P) O T2(P) O T3(P) O ....

115 115 P ROPPER POPORZE etc. but the "obvious" R-module hom R -> P[x] just gives the multiplication map on T(R). Yeah, we can map R -> P[x] by r->rx, but that's less intitive

44.5 = 45.1

The symmetric algebra.

Remember that T(N) is not commutative in general, because  $x \otimes y \neq y \otimes x$ .

What if I don't like this?

Definition. The symmetric algebra S(M) is T(M)/C(M), where CSM is the ideal generated by all elements of the form  $m_1 \otimes m_2 - m_2 \otimes m_1$ .

Remember: Quotienting out by ideals like this luposing additional relations.

Some properties.

(1) This ring is commodative.

Why? T(M) is generated as a ring by  $P = T^{\circ}(M)$  and M = T'(M).

These commute is S(M) by construction.

(2) The ring is graded, it inherits the grading from T(M).

There is a bit of machinery you can develop here.

C(M) is a graded ideal, C(M) = O (C(M) n Ti(M))

Quatient a graded ring by a graded ideal,

again get a graded ring.

44.6, 3. Once again have M -> S(M). 4. With the grading T(M) = @ T'(M) have n...

and  $S^{k}(M) = \frac{M \otimes \cdots \otimes M}{\{\text{submodule gen. by the } m_{1} \otimes \cdots \otimes m_{r(k)}\}}$   $- m_{r(1)} \otimes \cdots \otimes m_{r(k)}\}.$ (for  $\tau \in S^{k}(M)$ .) Sk(M) = M 8 .... & M Ck(M)

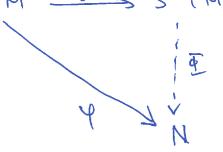
where  $C^k(M)$  are finite sums of elts, of the form  $m_1 \otimes \cdots \otimes m_{i-1} \otimes (m_i \otimes m_{i+1} - m_{i+1} \otimes m_i) \otimes m_{i+2} \otimes \cdots \otimes m_{je}$ .

C'(M) is clearly contained in the enbmodule above.

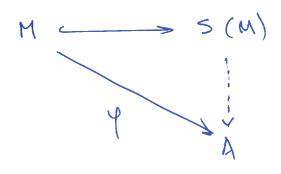
Conversely, they've equal because transpositions generate suppositions

94.7. Universal properties.

S. If  $\varphi: M \times \cdots \times M \longrightarrow N$  is a symmetric k-multilinear mop over R then  $\exists$  a unique R-mod hom  $\Phi: S^k(M) \xrightarrow{k} \longrightarrow N$  with  $\varphi = \Phi \circ \iota:$   $\iota: (m_1, \dots, m_k) \longrightarrow m_1 & \cdots & m_k$   $M \times M \times \cdots \times M \longrightarrow S^k(M)$ 



6. If A is any commutative R-algebra and  $\psi: M \longrightarrow A$  is an R-mod how, get a unique R-alg. hom  $\bar{\mathfrak{T}}: S(M) \rightarrow A$  with  $\bar{\mathfrak{T}}|_{M} = \psi$ .



44.8. = 45.4 Do we know any commutative P-algebras? Suppose M = P is the free rank on P-module. Write a basis for R X, ...., Xx. (1,0,...,0) (0,0,...,0,1) Then what is Sk(M)? Consists of linear sums of expressions of the form CONTRACTOR NOTE OF THE PARTY OF (L'M') Q (L3 MS) Q ... Q (Lk MF)

modulo rearranging, i.e. r, rz...rk m, o mz o ... mk

or just rm, & --- & mk.

Write each mi in terms of the basis and recreange.

S'(M) consists of P-linear combinations of expressions Xi, & .... & Xix , up to reordering.

Bet this can be regarded as degree k polynomials. So,  $S(R^n) \subseteq P(x_1, ..., x_n)$ .

The exterior algebra.

Suitch around this idea. Demand our algebra anticommete.

Def. Let M be an R-module. The exterior algebra of M is  $\Lambda(M) := T(M) / \Lambda(M)$ , where A(M) is the ideal gen by mom for m & M.

Again a graded algebra.

The image of m, a m2 0 ... o mk in N(M) is written m, 1 m, 1 ... 1 mx (a wedge product)

Basic properties.

 $(1) \quad m \wedge m' = -m' \wedge m.$ 

This is because

 $0 = (m + m') \wedge (m + m') = m \wedge m + m' \wedge m + m \wedge m'$ + m / m 0

- (2) M, N W2 N... NMK = 0 if any M; = M, (Poke oround)
- (3) It you switch the places of a wi and will you negate the wedge product. (Prove using above)

(4) N'(M) equals T'(M) modulo the submodule gen. by elts. of the form m, & mz & ... & mk with mi = mi for some i, j. (5) 1 (M) satisfies the UP (m1,..., mk) -> m, 1... 1 mk Mx ... × M \_\_\_\_\_ NE (M) P 3: with respect to alternating multilineer maps 4: Wx ... x M - N: 6 (m1 ... mk) =0 whenever any mi = any mi. (6) Let 11 be au n-dimensional F-vector space. Then  $\Lambda(V)$  is finite dimensional, and  $N^{k}(V) = 0$  for  $k \geq N$ . why? consider a generating element in 1'(V), of the form Y, A .... AVE. Write the Vi in terms of a fixed basis william of V. Get a finite linear sum of terms Wi, n...., ik & \{1, ..., ik & \{1, ..., u\}. Must have repeats if t > n.

Again, if V is an u-dim F-VS with basis  $w_1 - w_{R}$ , a basis of  $\Lambda^{k}(V)$  is consists of the vectors  $W_{i_1} \wedge W_{i_2} \wedge \cdots \wedge W_{i_{R}} \qquad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{K} \leq u_1$ 

Wedge Products and Geometry.

so that dimp  $\Lambda^{k}(V) = {n \choose k}$ .

Let V be an n-dimensional F-vector space.

P: V -> V an @ endomorphism.

Then  $\varphi$  also induces an endousorphism on each  $\Lambda^{k}(V)$ , by

6(11 V .... V XK) = 6(11) V ... V A(1/16).

 $\Lambda^{k}(V)$  is also a vector space and this mop is linear. The space  $\Lambda^{M}(V)$  is one-dimensional, spanned by  $W_{1} \wedge \cdots \wedge W_{K}$  for any basis elements  $W_{1} - W_{1} \in \cdots \cap W_{K}$ . If  $\psi$  sends the basis  $\{w_{i}\} \longrightarrow \{x_{i}\}_{\emptyset}$  (i.e. if  $\psi$  is invertible) then  $\psi(w_{1} \wedge \cdots \wedge w_{K}) = (x_{1} \wedge \cdots \wedge x_{K})$ 

for some scalor Ny)depending on y.

Moreover, since y induces lineer endomorphisms on

Non(V), get a homomorphism

GL (m) -> GL (N" (V)) = GL, (R).

What is this homomorphism? Multilinear, alternating, must be the determinant.

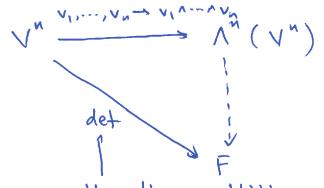
Can construct the determinant in a slightly different (but equivolent) way.

Regard det as a map V" -> F

by det (v,, v2, ..., vn) = det [v, v2 ... vn].

This is the determinant of the LT sending the standard basis e,,..., en to V,.... In. (Need to choose a basis)

Have



alternating, multilinear.

The UP asserts that the determinant factors through

Here v, .... vu may be considered a (signed) volume element on V.

Also used to construct differential forms.

Start with the vector space consisting of symbols dv, vell Take wedge products.

45.9. Some more grametry. Let V= IR3 ul std. basis e, lez, ez. If u= u, e, + u2 e2 + u3 e3 V = V, e, + V2 e2 + V3 e3 unv = (u, v2 - u2 v1) e1 ^ e2 + (u, vs - u3 v1) e1 ^e3 + (uz V3 - V3 U2) ez re3. Prop. The map  $\Lambda^2(\mathbb{R}^3) \longrightarrow (\mathbb{R}^3)^* = \text{Hom}(\mathbb{R}^3, \mathbb{R})$  $V \wedge W \longrightarrow \{ \times \rightarrow (V \wedge W) \wedge X \}$ is an isomorphism. Proof. Both sides one IR-vector spaces of dim 3, so enough to prove injective. But it VIW #0, then V, w one nonzero and linearly independent. Let  $x \in \mathbb{R}^3$  be such that  $\{v, w, x\}$  is a basis. Then v ~ w ~ x = det [v w x] e, 1e2 ^e3 We can check that e, rez - e3 e, ^e3 - e2\* er res - et.

If we now identify R3 with its dual, we see that the wedge coincides with the cross product.