NOTES ON LAGARIAS-ODLYZKO'S EFFECTIVE VERSION OF THE CHEBOTAREV DENSITY THEOREM

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Abstract. These notes describe (and in many places, simply copy) Lagarias and Odlyzko's [1] effective version of the Chebotarev density theorem. They were prepared for the benefit of the author – the original is quite clear and well deserves to be read.

Theorem and section numbers are as in [1]; equation numbers are not.

1. Introduction

Assume a number field K, and a Galois extension L/K, are fixed, with Galois group $G = \mathrm{Gal}(L/K)$. The Artin symbol is defined [as usual, but see later]..., and the **Chebotarev density theorem** asserts that

$$\pi_C(x, L/K) = \frac{|C|}{|G|}\mathrm{li}(x).$$

The purpose of [1] is to prove two versions of CDT, one under the assumption of the generalized Riemann hypothesis, and the other unconditional.

Theorem 1.1. *Assume that GRH holds for the Dedekind zeta function of L. Then, for $x \geq 2$,*

$$\left|\pi_C(x, L/K) - \frac{|C|}{|G|}\mathrm{li}(x)\right| \leq c_1\left(\frac{|C|}{|G|}x^{1/2}\log(d_L x^{n_L}) + \log d_K\right),$$

where c_1 is an effectively computable constant, d_L is the absolute value of the absolute discriminant of L, and $n_L := [L : \mathbb{Q}]$.

Remark. It would be interesting to compare this (as well as details of the proof) with the general discussion found in Chapter 5 of Iwaniec and Kowalski's book.

Corollary 1.2. *There exists an effectively computable positive absolute constant c_2 such that if GRH holds for the Dedekind zeta function of $L \neq \mathbb{Q}$, then for every conjugacy class C of G there exists an unramified prime $\mathfrak{p}$ of K with $(\mathfrak{p}, L/K) = C$ and*

$$\mathbb{N}_{K/\mathbb{Q}}\mathfrak{p} \leq c_2(\log d_L)^2(\log\log d_L)^4.$$

Remark. If $L = \mathbb{Q}$, then the Artin symbol condition is vacuous.

Proof. This follows immediately from the theorem as well as the estimate $n_L^{-1}\log d_L > 1 + \epsilon$ (for some ϵ). This follows from Minkowski's discriminant bound. □

Remark. **Minkowski's discriminant bound** states that for a number field K of degree n with $2s$ complex embeddings, we have

$$\sqrt{\Delta_K} \geq \left(\frac{4}{\pi}\right)^s \frac{n^n}{n!}.$$

The bound given then follows by taking logs and using Stirling's formula.

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We now state the unconditional result.

Theorem 1.3. *If $n_L > 1$, then $\zeta_L(s)$ has at most one zero in the region $s = \sigma + it$ defined by*
$$1 - \frac{1}{4\log d_L} \leq \sigma \leq 1, \quad |t| \leq \frac{1}{4\log d_L},$$
a so-called Siegel zero. If it exists, then it must be real and simple, and we denote it by β_0.

Moreover, there exist absolute, effectively computable constants c_3 and c_4 such that if
$$x \geq \exp(10 n_L (\log d_L)^2),$$
then
$$\left|\pi_C(x) - \frac{|C|}{|G|}\mathrm{li}(x)\right| \leq \frac{|C|}{|G|}\mathrm{li}(x^{\beta_0}) + c_3 x \exp(-c_4 n_L^{-1/2} (\log x)^{1/2}).$$
The β_0 term may be omitted if the exceptional zero does not exist.

Remark. By work of Stark, an effective bound for β_0 can be given (without which this result cannot truly be said to be effective).

There is also an interesting unconditional version of Linnik's theorem. This cannot be obtained by the theorem above, but it can be obtained via a slight modification of the proof.

Theorem 1.4. *There exist effectively computable positive absolute constants b_1, b_2, such that for every conjugacy class C of G there exists an unramified prime $\mathfrak{p}$ of K with $(\mathfrak{p}, L/K) = C$, and*
$$N_{K/\mathbb{Q}}\mathfrak{p} \leq b_1 d_L^{b_2}.$$

2. Summary

There is a very nice summary in Lagarias and Odlyzko's paper, which we give here. We define
$$\psi_C(x, L/K) := \sum_{\mathbb{N}_{K/\mathbb{Q}}\mathfrak{p}^m \leq x : (\mathfrak{p}, L/K)^m = C} \log(\mathbb{N}\mathfrak{p}),$$
and we will prove that $\psi_C(x, L/K)$ is asymptotically equal to $|C|x/|G|$, while making the error term explicit.

2.1. The inverse Mellin transform.

The first step is to write
$$I_C(x, T) = \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} F_C(s)\frac{x^s}{s} ds,$$
where $F_C(s)$ is a weighted linear sum of Artin L-functions. The integral above is a truncated inverse Mellin transform, and an application of Perron's formula (exactly as in the classical case) will show that
$$\psi_C(x) = I_C(x, T) + R_1(x, T)$$
for an error term $R_1(x, T)$ that can be bounded from above.

2.2. The abelian reduction.

In fact, we can rewrite the above weighted linear sum of Artin L-functions as a weighted linear sum of *abelian* Artin L-functions, associated to abelian subgroups of $\mathrm{Gal}(L/K)$. We will use invariance of Artin L-functions under induction, as well as a clever group theory argument.

We will then apply class field theory, which tells us that abelian Artin L-functions are in fact Hecke L-functions, which are much better understood. In particular they are holomorphic, and can be analyzed in the same way as Dirichlet L-functions.

2.3. **Shifting the contour. Cauchy's theorem.** $I_C(x,T)$ will differ from a contour integral (of the same function) by a bounded amount. In essence, we are shifting the line of integration to the left, where the integrand can be estimated. The integrand has poles at the zeroes and pole of $\zeta_L(s)$, and the main term comes from the pole of $\zeta_L(s)$ at $s=1$. In particular, we obtain the expression

$$\psi_C(x) = \frac{|C|}{|G|}x - \frac{|C|}{|G|}\sum_\chi \overline{\chi}(g)\Bigg(\sum_{\rho:|\gamma<T|}\frac{x^\rho}{\rho} - \sum_{\rho:|\rho|<1/2}\frac{1}{\rho}\Bigg)S(x,T) + E(x,T),$$

where the error term $E(x,T)$ consists of a lot of error terms which were picked up as we went along. The error term $E(x,T)$ is improved for larger T, and the sum over zeroes is of course bigger for larger T.

2.4. **The sum over zeroes.** At this point we need to count the sum over zeroes. It is not difficult to count how many zeroes there are with $|\gamma| \le T$. If we assume the Riemann hypothesis then the sum over zeroes is fairly small, and so we can choose a large value of T which will make the two error terms comparable.

If not, the error term is worse. We prove a zero-free region for the relevant L-functions, as in the classical case. In fact, we will see that all of these L-functions are factors of the Dedekind zeta function $\zeta_L(s)$, and prove a zero-free region similar to the classical case. This is in fact worked out in the paper, although nowadays quite general results can be found in Iwaniec-Kowalski and presumably elsewhere.

Unfortunately we cannot rule out an exceptional zero, and so a term corresponding to this appears in our final estimates.

3. Artin L-functions and Mellin transforms

3.1. **A little character theory.** We review a little bit of character theory, as can be found in the books by Serre, Dummit and Foote, etc.

We will be considering *representations* of the finite group $G = \mathrm{Gal}(L/K)$, and to each representation σ we associate a character ϕ by taking the trace. A character ϕ is *irreducible* if it is the character of an irreducible representation.

We will speak of "all the irreducible characters" of G. This language is justified by the following facts.

Theorem.

(i) There are only finitely many irreducible characters of any fixed finite group G.
(ii) The character of the *regular representation*, say χ_G, decomposes as

$$\chi_G = \sum_\chi \chi(1)\chi,$$

where the sum at right is over all irreducible characters of G.

We could say much more, if we liked. If χ is the character of a representation into some vector space V, then $\chi(1)$ is the degree of the character/representation, i.e., the dimension of V. In particular, by evaluating the equation above at 1 we see that

$$|G| = \sum_\chi \deg(\chi)^2.$$

Induced characters. We may define an inner product on the set of characters by

$$< \phi, \chi >:= \sum_{g \in G} \phi(g)\overline{\chi(g)}.$$

If ψ is a character on $H \subseteq G$, then there is not necessarily a way to extend it to a character on G. However, there is a natural way of *inducing* a character on G, in general of higher degree.

Theorem. If $H \subseteq G$ is a subgroup and ψ is a character of H, then there exists a unique character χ of G which satisfies

$$< \phi, \chi >_G = < \phi|_H, \psi >_H$$

for all characters ϕ on G. We call χ the character **induced** by ψ.

Moreover, χ is the character of the representation

$$Ind_G^H(V) = \{f : G \to V : f(\tau x) = \tau f(x) \ \forall \tau \in H\}, \tag{3.1}$$

where V is the representation space of H, and the action of $\sigma \in G$ is given by $(\sigma f)x = f(x\sigma)$.

We further have the formula

$$\chi(\sigma) = \sum_{\tau} \psi(\tau \sigma \tau^{-1}),$$

where τ varies over a set of coset representatives for H in G.

Remark. We check that if $H = 1$, then the trivial character induces the regular representation of G.

Remark. There is also a more highbrow definition of (3.1), in terms of tensor products.

Brauer's theorem. Every character on a finite group is a $\mathbb{Z}$-linear combination of degree 1 characters.

This theorem will let us prove that Artin L-functions are meromorphic (see below).

3.2. **Artin L-functions.**

Suppose we are given a representation

$$\rho : G = \mathrm{Gal}(L/K) \to \mathrm{GL}(V),$$

Suppose that $\mathfrak{p}$ is a prime of K, and $\mathfrak{P}$ is a prime of L lying above it. Let $G_{\mathfrak{P}}$ denote the decomposition group (the elements of the Galois group which fix $\mathfrak{p}$) and let $I_{\mathfrak{P}}$ denote the inertia group. Moreover, let $\kappa(\mathfrak{P})$ and $\kappa(\mathfrak{p})$ denote the residue class fields. Then there is a canonical isomorphism

$$G_{\mathfrak{P}}/I_{\mathfrak{P}} \longrightarrow \mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p})).$$

The latter group is generated by the *Frobenius automorphism* $x \to x^q$, and we denote its pullback by $\tau_{\mathfrak{P}}$. This is the *Frobenius automorphism* or the Artin symbol $(\mathfrak{P}, L/K)$.

Given the representation above, τ acts on the module $V^{I_{\mathfrak{P}}}$ of elements invariant by the inertia group. Note that for all unramified primes (all but finitely many), $I_{\mathfrak{p}}$ is trivial. We note further that the characteristic polynomial

$$\det(1 - \tau_{\mathfrak{P}} t; V^{I_{\mathfrak{P}}})$$

depends only on $\mathfrak{p}$ and not on $\mathfrak{P}$ – a different choice of $\mathfrak{P}$ will yield a bunch of conjugates. We may therefore make the following definition.

Definition. The **Artin L-function** associated to the Galois representation ρ is

$$L(s,\rho,L/K) := \prod_{\mathfrak{p}} [\det(1-\tau_{\mathfrak{P}}\mathbb{N}\mathfrak{p}^{-s}; V^{I_{\mathfrak{P}}})]^{-1},$$

where $\mathfrak{p}$ runs through all ideals of K.

Upon logarithmically differentiating we obtain the familiar-looking formula

$$\frac{L'}{L}(s,\rho,L/K) = \sum_{\mathfrak{p}} (\log \mathbb{N}\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s} \sum_{m\geq 1} \phi(\mathfrak{p}^m),$$

where ϕ is the character of ρ. We will hence write $\frac{L'}{L}(s,\phi,L/K)$, since this equation (and, indeed, the original Artin L-function) depend only on ρ up to ϕ.

We make a further note concerning the ramified primes. The Frobenius automorphism is only defined up to multiplication by an element of the inertia group, which is nontrivial precisely for ramified $\mathfrak{p}$. But in this case, we may write

$$\phi(\tau) = \frac{1}{e} \sum_{\alpha \in I_{\mathfrak{P}}} \phi(\tau\alpha),$$

where the right side now legitimately defines a character on G, which does not depend on τ.

We have the nice functorial property of being *invariant under induction.*

Proposition. Let L/K be a Galois extension with $K \subseteq E \subseteq L$ (so that $\mathrm{Gal}(L/E)$ is a subgroup of $\mathrm{Gal}(L/K)$. Suppose that ρ' is a representation of $\mathrm{Gal}(L/E)$, and ρ is the induced representation of $\mathrm{Gal}(L/K)$. Then

$$L(s,\rho,L/K) = L(s,\rho',L/E).$$

If L/K is *abelian* then class field theory proves the following:

Theorem: If ρ is a character of degree 1, then $L(s,\rho,L/K)$ is equal to a Hecke L-function.

Remark. We can say more about this Hecke L-function; see the discussion of the Artin conductor in Neukirch's book.

Remark. In particular if L/K is abelian, then all its characters are of degree 1.

Combining this proposition with Brauer's theorem we obtain the following

Theorem. Every Artin L-function is a quotient of Hecke L-functions.

We therefore conclude that Artin L-functions have a functional equation and meromorphic continuation to $\mathbb{C}$ - possibly with infinitely many poles. However, we have the following famous conjecture:

Artin Conjecture. Any Artin L-function $L(s,\rho,L/K)$ is entire.

The proof will be given in a subsequent paper[1], as we don't have room for it here.

[1] as soon as I figure out how to prove it.

3.3. **Back to** [1]**.** Define

$$\psi_C(x, L/K) := \sum_{\mathbb{N}_{K/\mathbb{Q}}\mathfrak{p}^m \le x : (\mathfrak{p}, L/K)^m = C} \log(\mathbb{N}\mathfrak{p}),$$

where the sum is over primes unramified in L/K. We note that this is the analogue of the usual ψ-function. For example, if $K = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_m)$, this counts the number of primes in a fixed arithmetic progression (depending on C) modulo m.

Let ϕ denote an irreducible character of $G = \mathrm{Gal}(L/K)$, and recall the definition of the Artin L-function, and its logarithmic derivative, above.

We would like to single out the $\mathfrak{p}^m$ with $(\mathfrak{p}, L/K)^m = C$. To do this, choose any $g \in C$. We then define a function $f_C : G \to \mathbb{C}$ by

$$f_C = \sum_\phi \overline{\phi}(g)\phi.$$

Then the orthogonality relations for characters imply that $f_C(\tau) = |G|/|C|$ for $\tau \in C$, and $f_C(\tau) = 0$ otherwise.

Thus, we may write

$$F_C(s) := -\frac{|C|}{|G|} \sum_\phi \overline{\phi}(g) \frac{L'}{L}(s, \phi, L/K),$$

where the sum does not depend on G, and we see that

$$F_C(s) = \sum_{\mathfrak{p}} \sum_{m \ge 1} \theta(\mathfrak{p}^m) \log(\mathbb{N}\mathfrak{p})(\mathbb{N}\mathfrak{p})^{-ms},$$

where $\theta(\mathfrak{p}^m)$ is equal to 1 whenever $(\mathfrak{p}, L/K)^m = C$, and is zero otherwise. If $\mathfrak{p}$ ramifies in L, then the result is $1/e$ times the number of $\alpha \in I_{\mathfrak{p}}$ for which $(\mathfrak{P}, L/K)\alpha \in C$. All we really care about is that $|\theta(\mathfrak{p}^m)| \le 1$.

So in other words, we have a "partial sum of the coefficients of $F_C(s)$" – a Dirichlet series whose coefficients we want to add.

Inverse Mellin transforms. Define

$$I(y) := \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} y^s \frac{ds}{s}.$$

Then, $I(y)$ is approximately equal to 1 if $y > 1$, and 0 if $0 < y < 1$. Precisely,

$$|I(y) - 1| \le y^\sigma \min(1, 1/(T|\log y|)) \qquad (y > 1),$$

$$|I(y) - \frac{1}{2}| \le \sigma/T \qquad (y = 1),$$

$$|I(y)| \le y^\sigma \min(1, 1/(T|\log y|)) \qquad (0 < y < 1).$$

Notice that the error terms go away as $T \to \infty$, i.e., when we integrate over (all of) any fixed vertical line. This is quite useful, for reasons that we will see shortly.

The proof is not too difficult; see Davenport's book for example.

We now define

$$I_C(x, T) := \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F_C(s) x^s \frac{ds}{s}.$$

By absolute convergence, we may switch the order of integration and summation, and the inverse Mellin transform above will pick off the terms with $\mathbb{N}\mathfrak{p} \leq x$. In particular, we obtain

$$\left| I_C(x,T) - \sum_{\mathfrak{p},m:\mathbb{N}\mathfrak{p}^m \leq x} \theta(\mathfrak{p}^m)\log(\mathbb{N}\mathfrak{p}) \right| \leq \sum_{\mathbb{N}\mathfrak{p}^m = x} (\log \mathbb{N}\mathfrak{p} + \sigma_0 T^{-1}) + R_C(x,T), \tag{3.2}$$

for an error term

$$R_C(x,T) := \sum_{\mathfrak{p},m:\mathbb{N}\mathfrak{p}^m \neq x} \left(\frac{x}{\mathbb{N}\mathfrak{p}^m}\right)^{\sigma_0} \min\left(1, \frac{1}{T|\log(x/\mathbb{N}\mathfrak{p}^m)|}\right) \log \mathbb{N}\mathfrak{p}.$$

Remark. In the first term, it looks like there is a mistake and this term should perhaps be $\frac{1}{2}\sum_{\mathbb{N}\mathfrak{p}^m=x}(\log \mathbb{N}\mathfrak{p})(\sigma_0 T^{-1})$. However, in either case this contribution is negligible.

We notice that the sum on the left side of (3.2) differs only from $\psi_C(x)$ in the ramified prime terms. The error for each ramified prime is at most 1, so the total error is easily (**yes, it's easy**) computed to be $\leq 2\log x \log d_L$. Also, there are at most n_K distinct pairs $\mathfrak{p}, m$ with $\mathbb{N}\mathfrak{p}^m = x$ (as there are at most n_K distinct primes lying over any given prime of $\mathbb{Q}$), and so we have

$$\psi_C(x) = I_C(x,T) + R_1(x,T),$$

for an error term

$$R_1(x,T) \leq 2\log x \log d_L + n_K \sigma_0 T^{-1} + n_K \log x + R_0(x,T).$$

We will estimate the integral $I_C(x,T)$ later, so here we will estimate the error term $R_0(x,T)$.

Proposition 3.1. *We have the estimate*

$$R_1(x,T) \ll \log x \log d_L + n_K \log x + n_K x T^{-1} (\log x)^2,$$

valid for all $x \geq 2$ and $T \geq 1$.

Proof. pp. 425-428 of [1]. The details are not very interesting, so here is a summary. We divide the contributions to R_0 from ideals with $\mathbb{N}\mathfrak{p}^m \leq \frac{3x}{4}$, $\geq \frac{5x}{4}$, or in between. On both extremes, the minimum in (**??**) is $\ll 1/T$ and the sum is

$$\ll \frac{x}{T}\left(-\frac{\zeta_K'}{\zeta}(\sigma_0)\right) \leq n_K \frac{x}{T}\left(-\frac{\zeta_K'}{\zeta}(\sigma_0)\right) \ll n_K \frac{x}{T}\frac{1}{\sigma - 1}.$$

The second step follows by an elementary argument (K has at most n_K times as many primes of a given norm as $\mathbb{Q}$), and the last step is true because $\zeta(s)$ has a simple pole at $s = 1$. (No, we **don't** care what the residue is.)

We then choose $\sigma = 1 + 1/\log x$.

We then evaluate the ideals with norm in between. If $|\mathbb{N}\mathfrak{p}^m - x| \leq 1$, there are at most $2n_K$ such $\mathfrak{p}^m$ and we can easily bound this contribution from above. Otherwise we use an estimate for $|log(x/n)|^{-1}$ and estimate the sum in an elementary way. □

Remark. If C is a union of conjugacy classes, rather than a single one, then the $\log x \log d_L$ term is unchanged. (**get a better hold on this.**)

Remark. If $L \neq \mathbb{Q}$, then $n_K \leq n_L \ll \log d_L$ and so the second error term can be absorbed into the first one.

4. Some fancy group theory: Reduction to Hecke L-functions

The nice thing about Hecke L-functions is that they are **abelian**.

In our definition of $F_C(s)$, we chose an element $g \in C$. Let $H =< g >$ be the cyclic group generated by g, E the fixed field of H, and let χ denote the irreducible characters of H. Recall that the character theory of abelian groups is blissfully simple.

Lemma 4.1 (Lemma 4.1)**.** *We have*

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) \frac{L'}{L}(s, \chi, L/E).$$

Proof. Let $\tau : H \to \mathbb{C}$ be the class function defined to be $|H|$ for $h = g$, and 0 otherwise. Then, the orthogonality relations on H imply that

$$\tau = \sum_{\chi} \overline{\chi}(g)\chi.$$

Now we let τ^* be the class function on G induced by τ. We can directly compute that $\tau^*(y) = |C_g(G)|$ for $y \in C$, and is 0 otherwise.

But as $|C_G(g)||C| = |G|$, we have $\tau^* = f_C$. In other words, our previous character f_C is the one induced by an abelian quotient. Our previous computations imply that

$$\sum_{\chi} \overline{\chi}(g)\chi^* = \sum_{\phi} \overline{\phi}(g)\phi,$$

which implies that for $\Re s > 1$,

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) \frac{L'}{L}(s, \chi^*, L/K).$$

But, by invariance under induction, $L(s, \chi^*, L/K) = L(s, \chi, L/E)$. Q.E.D. □

5. Density of zeroes of Hecke L-functions (pp. 431-439)

We have now proved that for $x \geq 2, T \geq 1$,

$$\psi_C(x) = I_C(x, T) + R_1(x, T),$$

where R_1 has been bounded already, and

$$I_C(x, T) = -\frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} x^s \frac{L'}{L}(s, \chi, L/E) \frac{ds}{s},$$

where $\sigma_0 = 1 + (\log x)^{-1}$ and χ runs through the (one-dimensional) irreducible characters of $H =< g >$. We will now evaluate each of the integrals above, and to accomplish this we will bound from above the number of poles of L'/L.

Notation: We abbreviate $L(s, \chi, L/E)$ to $L(s, \chi)$. Let $F(\chi)$ denote the conductor of χ, and write

$$A(\chi) := d_E N_{E/\mathbb{Q}}(F(\chi)).$$

We write $\delta(\chi) = 1$ if χ is the principal character, and $\delta(\chi) = 0$ otherwise.

Remark. **To do:** Determine precisely what these characters are.

We now quote some standard facts on Hecke L-functions (which are not proved here). For each χ, there exist non-negative integers $a = a(\chi)$ and $b = b(\chi)$ with

$$a(\chi) + b(\chi) = n_E,$$

and such that if we define a *gamma factor*

$$\gamma_\chi(s) := \left(\pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)\right)^b \left(\pi^{-s/2}\Gamma(s/2)\right)^a,$$

$$\xi(s,\chi) := [s(s-1)]^{\delta(\chi)} A(\chi)^{s/2}\gamma_\chi(s)L(s,\chi),$$

then $\xi(s,\chi)$ is entire of order 1, nonvanishing at $s=0$, and satisfies the functional equation

$$\xi(1-s,\overline{\chi}) = W(\chi)\xi(s,\chi)$$

for a number $W(\chi)$ (**the root number?**) of absolute value 1.

Remark. This tells us that $A(\chi)$ is "important" and is not just an analytic conductor – it appears in the functional equation!

Remark. Learn how a and b depend on χ.

The **Hadamard product theorem** of complex analysis then tells us that

$$\xi(s,\chi) = e^{B_1(\chi)+B(\chi)s}\prod_\rho \left(1-\frac{s}{\rho}\right)e^{s/\rho} \tag{5.1}$$

for some constants $B_1(\chi)$ and $B(\chi)$.

Remark. Ask Sound: Is there some nice way to "appreciate" the Hadamard theorem? I remember the proof being a pain.

In the product above, ρ runs through the zeroes of $\xi(s,\chi)$, which are the "nontrivial" zeroes of $L(s,\chi)$ – i.e., those in the critical strip.

Logarithmic differentiation of (5.1) yields the following important identity:

$$\frac{L'}{L}(s,\chi) = B(\chi) + \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) - \frac{1}{2}\log A(\chi) - \delta(\chi)\left(\frac{1}{s}+\frac{1}{s-1}\right) - \frac{\gamma_\chi'}{\gamma_\chi}(s). \tag{5.2}$$

We don't know what $B(\chi)$ is, but we can use the functional equation to prove the following result:

Lemma 5.1. *With notation as above,*

$$\Re B(\chi) = -\sum_\rho \Re\frac{1}{\rho},$$

*(***wait… that converges?!***) and*

$$\frac{L'}{L}(s,\chi) + \frac{L'}{L}(s,\overline{\chi}) = \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{s-\overline{\rho}}\right) - \log A(\chi) - 2\delta(\chi)\left(\frac{1}{s-1}+\frac{1}{s}\right) - 2\frac{\gamma_\chi'}{\gamma_\chi}(s) \tag{5.3}$$

identically in the complex variable s, where ρ runs through the nontrivial zeroes of $L(s,\chi)$.

Remark. **To do, looks easy, check the proof.** Moreover, learn whether this has anything to do with the first discriminant lower bound occuring in Odlyzko's survey, which he says wasn't discovered for 70 years. It looks similar.

This lemma will allow us to obtain estimates both of $B(\chi)$ and of the density of zeroes of $L(s,\chi)$.

Remark. It will be interesting to compare this development with Davenport on the one hand (i.e., Dirichlet L-functions) and the very general development of Iwaniec and Kowalski on the other. IK start with a list of axioms about the 'degree' of the L-function and so on, and develop general zero-free regions, density estimates, and the like.

As LO mention, an analogue of the lemma above could be proved for general Artin L-functions, but it would contain sums over the poles of such L-functions, and the pole terms would prevent us from obtaining a good estimate.

Remark. What would the proof be? Would you simply write the Artin L-function as a quotient of abelian ones, and bootstrap, or do LO have a more intrinsic approach in mind?

We start with some auxiliary results.

Lemma 5.2. *If $\sigma = \Re(s) > 1$, then*
$$\left|\frac{L'}{L}(s,\chi)\right| \ll \frac{n_E}{\sigma - 1}.$$

Proof. **To be added**, there is a reference into Chapter 3 I haven't written up yet. □

Lemma 5.3. *If $\sigma = \Re(s) > -1/2$ and $|s| \geq 1/8$, then*
$$\left|\frac{\gamma_\chi'}{\gamma_\chi}(s)\right| \ll n_E(\log|s| + 2).$$

Proof. This essentially follows from the definition of the gamma factor, and some properties of the gamma function (Stirling's formula?) Write out a proof here! □

The main result of this section is as follows. Let $n_\chi(t)$ denote the number of zeroes $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $0 < \beta < 1$, $|\gamma - t| \leq 1$.

Lemma 5.4. *For all t, we have*
$$n_\chi(t) \ll \log A(\chi) + n_E(\log|t| + 2).$$

Proof. We use (5.3), and plug in $s = 2 + it$. The previous two lemmas tell us (**yes!** - just stare at it) that
$$\sum_\rho \Re\left(\frac{1}{s-\rho} + \frac{1}{s-\overline{\rho}}\right) \ll \log A(\chi) + n_E \log(|t| + 2).$$
Now on the left, as $\Re s > 2$, all of these real parts are positive. A quick computation shows us that
$$\sum_\rho \Re\left(\frac{1}{s-\rho} + \frac{1}{s-\overline{\rho}}\right) \geq \sum_{\rho : |\gamma - t| \leq 1} \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2}.$$

Remark. We chucked a lot of the zeroes here! We only kept the ones for which $t - \gamma$, and thus the denominator, are small.

Since $1 < 2 - \beta < 2$, each summand is $\geq 1/5$, and so
$$\frac{1}{5} n_\chi(t) \leq \sum_\rho \Re\left(\frac{1}{s-\rho} + \frac{1}{s-\overline{\rho}}\right).$$
□

That's it!

Remark. Somehow this proof seems interesting. We got off pretty cheap in a sense, and proved a lot. Somehow it feels worthwhile to stare at this and think profound thoughts.

The next lemma will show that $B(\chi)$ depends mostly on the very small zeroes of $L(s,\chi)$.

Lemma 5.5. *If* $0 < \epsilon \leq 1$ *then*

$$B(\chi) + \sum_{|rho|<\epsilon} \ll \frac{1}{\epsilon}(\log A(\chi) + n_E).$$

Proof. We use the explicit formula (**??**) with $s = 2$ and Lemmas 5.2 and 5.3, and we obtain

$$B(\chi) + \sum_{\rho} \left(\frac{1}{2-\rho} + \frac{1}{\rho} \right) \ll \log A(\chi) + n_E.$$

We have

$$\left(\frac{1}{2-\rho} + \frac{1}{\rho} \right) \leq \frac{2}{|\rho|}^2,$$

We now use Lemma 5.4 to count the zeroes step by step. We obtain

$$\sum_{|\rho|\geq 1} \left| \frac{1}{2-\rho} + \frac{1}{\rho} \right| \ll \sum_{j=1}^{\infty} \frac{n_\chi(j)}{j^2} \ll \log A(\chi) + n_E,$$

(i.e., the sum over j, including the log term from Lemma 5.4, is $\ll 1$), and as $|2-\rho| \geq 1$ we have also

$$\sum_{|\rho|<1} \left| \frac{1}{2-\rho} \right| \ll \log A(\chi) + n_E.$$

This leaves

$$B(\chi) + \sum_{|\rho|<\epsilon} \frac{1}{\rho} \ll \sum_{\epsilon\leq|\rho|<1} \frac{1}{|\rho|} + \log A(\chi) + n_E,$$

and we know how many zeroes there are on the right by Lemma 5.4... so we're done. □

Remark. Once again, I like the proof. It feels instructive to stare at it. I imagine all these results are expected to be sharp (except for whatever fudging we did on the implied constants)?

Lemma 5.6. *If* $s = \sigma + it$ *with* $-1/2 \leq \sigma \leq 3$, $|s| \geq 1/8$, *then*

$$\left| \frac{L'}{L}(s,\chi) + \frac{\delta(\chi)}{s-1} - \sum_{\rho:|\gamma-t|\leq 1} \frac{1}{s-\rho} \right| \ll \log A(\chi) + n_E \log(|t|+2).$$

Proof. **To be added.** I am confused about why this lemma is interesting...? □

6. The contour integral (pp. 440-446)

To evaluate $I_C(x,T)$ we need to evaluate the integral

$$I_\chi(x,T) := \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} x^s \frac{L'}{L}(s,\chi) \frac{ds}{s}$$

for each character χ of $H =< g >$. (**To do:** Recall why the Hecke characters correspond to the characters of this group. We insisted that $T \geq 1$ before, and now we impose the additional condition that T not coincide with the ordinate of any zero of any of the $L(s,\chi)$.

We introduce another parameter U, which will be $U = j + 1/2$ for some nonnegative integer j, and we will eventually take a limit as $j \to \infty$.

We define $B_{T,U}$ to be the positively oriented rectangle with vertices at $\sigma_0 \pm iT$ and $-U \pm IT$, and define

$$I_\chi(x,T,U) := \frac{1}{2\pi i}\int_{B_{T,U}} x^s \frac{L'}{L}(s,\chi)\frac{ds}{s}.$$

This can be evaluated using Cauchy's residue theorem, and in this section we will show that the other three sides of the rectangle contribute little. In particular, we will write

$$V_\chi(x,T,U) := \frac{1}{2\pi}\int_T^{-T} \frac{x^{-U+it}}{-U+it}\frac{L'}{L}(-U+it,\chi)dt,$$

and we will separate the horizontal integrals at $\Re s = -1/4$ to write

$$H_{x,T,U} := \frac{1}{2\pi i}\int_{-U}^{-1/4}\left(\frac{x^{\sigma-iT}}{\sigma-iT}\frac{L'}{L}(\sigma-iT,\chi) + \frac{x^{\sigma-iT}}{\sigma-iT}\frac{L'}{L}(\sigma-iT,\chi)\right)d\sigma,$$

$$H^*_{x,T} := \frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\left(\frac{x^{\sigma-iT}}{\sigma-iT}\frac{L'}{L}(\sigma-iT,\chi) + \frac{x^{\sigma-iT}}{\sigma-iT}\frac{L'}{L}(\sigma-iT,\chi)\right)d\sigma.$$

Naturally the game is to show that these are all small.

Lemma 6.1. *If $|z+k| \geq 1/8$ for all nonnegative integers k, then*

$$\frac{\Gamma'}{\Gamma}(z) \ll \log(|z|+2).$$

Proof. If $\Re z \geq 1$, then this is "well-known". For $\Re z < 1$, there is a proof given by recurrence, which might seem interesting if I learn the 'well-known' part. □

This one seems interesting:

Lemma 6.2. *If $s = \sigma + it$ with $\sigma \leq -1/4$, and $|s+m| \geq 1/4$ for all nonnegative integers m, then*

$$\frac{L'}{L}(s,\chi) \ll \log A(\chi) + n_E(\log|s| + 2).$$

Remark. Is the second condition in fact necessary?

Proof. The functional equation readily implies that

$$\frac{L'}{L}(s,\chi) = -\frac{L'}{L}(1-s,\overline{\chi}) - \log A(\chi) - \frac{\gamma'_\chi}{\gamma_\chi}(1-s) - \frac{\gamma'_\chi}{\gamma_\chi}(s).$$

Since $\Re(1-s) \geq 5/4$, $(L'/L)(1-s,\overline{\chi})$ is easily bounded by Lemma 5.2. (We notice that the constant $1/4$ doesn't matter, as long as it's fixed.) We then apply Lemma 6.1, and we're done. □

Plugging in this estimate, this easily (yes, I checked it) allows us to show that

$$V_\chi(x,T,U) \ll \frac{x^{-U}}{U}T(\log A(\chi) + n_E\log(T+U)),$$

$$H_\chi(x,T,U) \ll \frac{x^{-1/4}}{T}(\log A(\chi) + n_E\log(T)),$$

although I have not bothered to reproduce the details here.

To estimate $H^*_\chi(x,T)$ we need to work a little bit harder. Applying Lemma 5.6, we quickly check that if $\sigma_0 = 1 + 1/\log x$, then

$$H^*_\chi(x,T) = \frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\left(\frac{x^{\sigma-iT}}{\sigma-iT}\sum_{\rho:|\gamma+T|\leq 1}\frac{1}{\sigma-iT-\rho} - \frac{x^{\sigma+iT}}{\sigma+iT}\sum_{\rho:|\gamma+T|\leq 1}\frac{1}{\sigma+iT-\rho}\right)$$

$$+O\left(\frac{x}{T\log x}(\log A(\chi)+n_E\log T)\right).$$

To complete our estimate we need the following lemma:

Lemma 6.3. *Let* $\rho=\beta+i\gamma$ *have* $0<\beta<1, \gamma\neq t$. *If* $|t|\geq 2, |x|\geq 2, 1<\sigma_1\leq 3$, *then*

$$\int_{-1/4}^{\sigma_1}\frac{x^{\sigma+it}}{(\sigma+it)(\sigma+it-\rho)}d\sigma \ll \frac{x^{\sigma_1}}{|t|(\sigma_1-\beta)}.$$

Proof. This is checked by moving the line of integration using Cauchy's theorem and then bounding the integrand. I didn't get a warm, fuzzy feeling from the proof but I did manage to verify it. □

This lemma shows that

$$\frac{1}{2\pi i}\int_{-1/4}^{\sigma_0}\frac{x^{\sigma-iT}}{\sigma-iT}\left(\sum_{\rho:|\gamma+T|\leq 1}\frac{1}{\sigma-iT-\rho}\right)d\sigma \ll \frac{x^{\sigma_0}}{T}(\sigma_0-1)^{-1}n_\chi(-T),$$

(*note: this corrects what I believe is a misprint in* [1]), and upon plugging in $\sigma_0=1+1/\log x$ and using Lemma 5.4, we see that this is

$$\ll \frac{x\log x}{T}(\log A(\chi)+n_E\log T).$$

This estimate holds for $x\geq 2$ and $T\geq 2$ (and we don't care about small values of x and T), and the same estimate holds for the integral involving zeroes ρ with $|\gamma-T|\leq 1$.

Remark. As [1] remark, the $\log x$ term could be removed by assuming GRH, and we could replace $\log x$ with $\log logx$ above by improving Lemma 6.3. **To do:** It would be interesting to verify these remarks.

By putting everything together, we see immediately that

$$I_\chi(x,T)-I_\chi(x,T,U)\ll \frac{x\log x}{T}(\log A(\chi)+n_E\log T)+\frac{Tx^{-U}}{U}(\log A(\chi)+n_E\log(T+U).$$

7. The explicit formula (pp. 446-452)

In this section we will combine our previous results and obtain an explicit formula for $\psi_C(x)$ in terms of the zeroes ρ. In particular, we will prove the following.

Theorem 7.1. *We have*

$$\psi_C(x)=\frac{|C|}{|G|}x-S(x,T)+O(E(x,T)),$$

where

$$S(x,T):=\frac{|C|}{|G|}\sum_\chi\overline{\chi}(g)\left(\sum_{\rho:|\gamma<T|}\frac{x^\rho}{\rho}-\sum_{\rho:|\rho|<1/2}\frac{1}{\rho}\right),$$

and the error term $E(x,T)$ *(which also depends on* $|C|, |G|$, *the discriminant and degree of* L *and* K *etc.) is defined by*

$$E(x,T):=\frac{|C|}{|G|}\left(\frac{x\log x+T}{T}\log d_L+n_L\log x+\frac{n_Lx\log x\log T}{T}\right)+\log x\log d_L+n_KxT^{-1}(\log x)^2.$$

The inner sum in $S(x,T)$ *is over the nontrivial zeroes* ρ *of* $L(s,\chi)$.

Remark. We expect $S(x,T)$ to be small, and good bounds for $S(x,T)$ will allow us to take larger values of T, which will in turn drive the error term $E(x,T)$ down. Good bounds for $S(x,T)$ depend on zero-free regions (or zero density estimates, etc.) for Hecke L-functions, and this step explains why the results are so much better conditional on GRH.

We begin by evaluating $I_\chi(x,T,U)$. This is done by Cauchy's theorem. If χ is the principal character χ_1, then L'/L has a first order pole of residue -1 at $s=1$, and we obtain a contribution of $-\delta(\chi)x$. Also, L'/L has a first order pole with residue 1 at each nontrivial ρ, and we obtain a contribution of x^ρ/ρ for each. (Note: these computations all follow easily from (**??**).)

One can show that L'/L has first order poles at $s=-(2m-1), m=1,2,\dots$ with residue $b(\chi)$, and first order poles at $s=-2m, m=0,1,2,\dots$ with residue $a(\chi)$. In particular, to do this, one does the folliowing: Write down an expression for the logarithmic derivative of the gamma factors which appear in L'/L. In particular, differentiating the Hadamard product for the gamme function, we have that

$$\frac{\Gamma'}{\Gamma}(s) = -\gamma - \frac{1}{2} + \sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{n+s}\right).$$

This shows immediately that $\Gamma'/Gamma$ has simple poles at $s=0,-1,-2,\dots$, of residue -1 each, and each of these leads to two poles of L'/L – one for the a factor and one for the b factor.

The conclusion is that the contribution of the so-called 'trivial zeroes' is

$$-b(\chi)\sum_{1\leq m\leq \frac{U+1}{2}} \frac{x^{-(2m-1)}}{2m-1} - a(\chi)\sum_{1\leq m\leq U/2}\frac{x^{-2m}}{2m}.$$

Collecting all the

There is a double pole at $s=0$, and an interesting computation (which I verified but do not reproduce here) shows that the residue is $r(\chi) + (a(\chi) - \delta(\chi))\log x$.

Collecting the residue terms, plugging in the Taylor series for log, taking a limit as $U\to\infty$, we obtain

$$I_\chi(x,T) = -\delta(\chi)x + \sum_{\rho:|\gamma|<T}\frac{x^\rho}{\rho} + r(\chi) + (a(\chi)+\delta(\chi))\log x + \frac{n_E}{2}\log(1-\frac{1}{x}) - \frac{1}{2}(b(\chi)-a(\chi))\log(1+\frac{1}{x}) + O\left(\frac{x\log x}{T}(\log A(\chi) + n_E\log T)\right). \tag{7.1}$$

This is valid for all $x\geq 2$ and all $T\geq 2$ which do not coincide with the ordinate of a zero. If we let $T\to\infty$, we would obtain an explicit formula for the inverse Mellin transform

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{x^s}{s}\frac{L'}{L}(s,\chi)ds,$$

(without the error terms!!) – but the theorem stated previously will in fact be more useful for us.

Proof of Theorem 7.1. Lemma 5.5 shows that

$$r(\chi) = \sum_{\rho:|rho|<\frac{1}{2}} + O(\log A(\chi) + n_E.$$

Equation (7.1) gives us an (approximate) explicit formula for $I_\chi(x,T)$, which forms our starting point, and then we recall that equation (**??**) defines $I_C(x,T)$ as a sum of "twisted" values of $I_\chi(x,T)$. We then recall that $\psi_C(x) = I_C(x,T) + R_1(x,T)$, for a remainder term evaluated in Section 3, and writing everything out we obtain the theorem in a straightforward way. □

Remark. We have used the *conductor-discriminant formula* to evaluate $\sum_\chi \log A(\chi)$. This formula tells us that
$$\Delta_{L/K} = \sum_\chi \mathfrak{f}(\chi)^{\chi(1)},$$
where the product is over all irreducible characters of $\mathrm{Gal}(L/K)$. The proof involves some vaguely complicated details (for example, in defining the Artin conductor $\mathfrak{f}(\chi)$), but essentially it comes from the formula $r_G = \sum_\chi \chi(1)\chi$.

Remark. In evaluating the error terms we have ignored some potential oscillation coming from the characters. Although it seems difficult to improve things, it's a possibility we shouldn't rule out.

Remark. It may be that T coincides with the ordinate of a zero. If this happens, then we replace T with $T + \epsilon$ for a very small ϵ, and we pick up $\ll \sum n_\chi T$ zeroes on the left side of the sum. Their contribution may be absorbed into the error term.

8. Zero-free regions (pp. 452-457)

In this section we will prove bounds for $S(x,T)$, and in particular for the sum
$$\sum_{\rho:|\gamma|<T} \frac{x^\gamma}{\rho}$$
which occurs in Theorem 7.1. We observe right away that if we can prove that most (or all) of the zeroes have small real part, we will obtain nontrivial bounds for this sum.

Remark. This section proceeds along very classical lines, and obtains a zero-free region similar to what you find in Davenport, etc. If one tried to prove a bound using zero-density estimates, a la Huxley (Gallagher... Fogels... others...?), what would one obtain?

If the results are not here (or in Davenport), then presumably this does not help, but it would be instructive to try.

We will prove our results for the Dedekind zeta function, which satisfies the formula
$$\zeta_L(s) = \prod_\chi L(s,\chi). \tag{8.1}$$
As these functions are holomorphic (except at $s = 1$, and for real negative s), zero-free regions for ζ_L clearly prove zero-free regions for the $L(s,\chi)$. As LO remark, we lose some by doing this, but not too bad.

Proof of (8.1). . That formula is too good to not stop and prove. But it feels like we're cheating.

We start with the formula
$$r_G = \sum_\chi \chi(1)\chi,$$
which we've used lots before, and this implies quite directly that
$$L(s,\rho_G) = \prod_\rho L(s,\rho)^{\deg\rho}.$$
(Characteristic polynomials multiply quite nicely.) However, the regular representation ρ_G is induced by the trivial representation of $\{1\} \subseteq G$. (This follows quite directly from the definition.)

By invariance under induction, $L(s,\rho_G) = L(s,1)$. And the trivial representation of the trivial group is pretty easy to understand: we just get $L(s,1) = \zeta_L(s)$.

Q.E.D. □

We will prove the following:

Lemma 8.1 (8.1)**.** *The Dedekind zeta function $\zeta_L(s)$ has no zeroes $\rho = \beta + i\gamma$ in the region*
$$|\gamma| \geq (1 + 4\log d_L)^{-1},$$
$$\beta \geq 1 - c_8(\log d_L + n_L \log(|\gamma| + 2))^{-1}.$$

Remark. Is the restriction on γ really necessary? Can't you fold it into the constraint on β, with the exception of the Siegel zero?

Remark. It would be interesting to compare this with general results found in Iwaniec and Kowalski (or elsewhere).

Proof. We start with the identity
$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0,$$
which means that this is going to be the same proof as always.

Now logarithmically differentiating the Euler product for $\zeta_L(s)$, we see that
$$-\frac{\zeta_L'}{\zeta_L}(s) = \sum_{m \geq 1} \alpha(m) m^{-s}, \tag{8.2}$$
for $\Re s > 1$, for some **nonnegative** coefficients $\alpha(m)$ that we don't really care about.

The point is that
$$\Re(-3\frac{\zeta_L'}{\zeta_L}(\sigma) - 4\frac{\zeta_L'}{\zeta_L}(\sigma + it) - \frac{\zeta_L'}{\zeta_L}(\sigma + 2it) = \sum_{m \geq 1} \alpha(m) m^{-\sigma}(3 + 4\cos(t\log m) + \cos(2t\log m) \geq 0.$$
Now we have a formula for ζ_L'/ζ_L; namely,
$$2\frac{\zeta_L'}{\zeta}(s) = \sum_\rho \left(\frac{1}{s - \rho} + \frac{1}{s - \overline{\rho}}\right) - \log d_L - \frac{2}{s} - \frac{2}{s - 1} - 2\frac{\gamma_L'}{\gamma_L}(s),$$
where the summation is over the nontrivial zeroes ρ of $\zeta_L(s)$. This was proved earlier in Lemma 5.1 – this is the trivial case of the extension L/L and the principal character.

We are interested in the real part of that. Notice that if $\Re s > 1$, then $\Re(s - \rho)^{-1} > 0$. We see that
$$-\frac{\zeta_L'}{\zeta}(\sigma) \leq \frac{1}{\sigma - 1} + \frac{1}{\sigma} + \frac{1}{2}\log d_L + \frac{\gamma_L'}{\gamma_L}(\sigma) - \sum_\rho \Re(\sigma - \rho)^{-1} \leq \frac{1}{\sigma - 1} + c_9 \log d_L + c_9 n_L,$$
and notice that we **simply threw away the sum over zeroes!** (It's the right sign, so we can.) We similarly obtain
$$-\frac{\zeta_L'}{\zeta}(\sigma + 2i\gamma) \leq c_{10}\log d_L + c_{10} n_L \log(|\gamma| + 2),$$
and
$$-\frac{\zeta_L'}{\zeta}(\sigma + i\gamma) \leq c_{11}\log d_L + c_{11} n_L \log(|\gamma| + 2) - \frac{1}{\sigma - \beta},$$
where in the last equality we kept only the contribution from the **one offending zero.** Of course, it is very near $\sigma + i\gamma$ and that is the point. Using these inequalities, we obtain that for all $\sigma > 1$,
$$\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + c_{12}\bigg(\log d_L + n_L \log(|\gamma| + 2)\bigg).$$

Now **that** is what we want! If β is really close to 1 (in terms of the stuff on the right) then we get a contradiction, hence a zero free region.

To be precise we need to choose a good value of σ. (Too big, and the error term on the right swamps everything; too small relative to $1-\beta$, and the contradiction vanishes.) In particular, upon choosing

$$\sigma = 1 + (100c_{12})^{-1}\big(\log d_L + n_L \log(|\gamma| + 2)\big)^{-1},$$

we get the stated result. □

We also have a complementary result about zeroes close to the real axis.

Lemma 8.2. *If $n_L > 1$ then $\zeta_L(s)$ has at most one zero $\rho = \beta + i\gamma$ in the region*

$$|\gamma| \leq (4 \log d_L)^{-1},$$

$$\beta \geq 1 - (4 \log d_L)^{-1}.$$

Proof. Using (**??**) and the fact that $\zeta'/\zeta \leq 0$, we compute that

$$\sum_{\rho} \frac{\sigma - \beta}{(\sigma-\beta)^2 + \gamma^2} \leq \frac{1}{\sigma - 1} + \frac{1}{2}\log d_L$$

for $1 < \sigma \leq 1 + (\log 3)^{-1}$.

Now if there is more than one zero in the region described, we obtain that

$$2\frac{\sigma - \beta}{(\sigma-\beta)^2 + \gamma^2} \leq \frac{1}{\sigma - 1} + \frac{1}{2}\log d_L,$$

which is false at $\sigma = 1 + (\log d_L)^{-1}$ (as can easily be checked). Note that if there is a nonreal zero, then its complex conjugate is a zero instead, which is why it has to be real.

We also know that if the Siegel zero exists, there exists a character χ_0 so that $L(\beta_0, \chi_0) = 0$. But then $L(\beta_0, \overline{\chi_0}) = 0$ too, so χ_0 must be real.

□

Remark. It would be fun to think about how to prove that Siegel zeroes don't exist. If I proved this, then I would be cited in every paper in the subject.

9. Final estimates (pp. 457-462)

Now we get to the punchline.

Theorem 9.1. *If $\zeta_L(s)$ satisfies the GRH, then*

$$\psi_C(x) - \frac{|C|}{|G||}(x) \ll \frac{|C|}{|G|} x^{1/2} \log x \log d_L x^{n_L} + \log x \log d_L$$

for all $x \geq 2$.

Remark. Surely in practice we don't care about the second error term.

Proof. It's beautifully simple. We need to estimate the sum

$$\Bigg| \sum_{\rho : |gamma| < T} \frac{x^\rho}{\rho} + \sum_{||rho| < 1/2} \frac{1}{\rho} \Bigg|.$$

Under GRH there are **no** zeroes with $|\rho| < 1/2$, and by Lemma 5.4 the above is

$$\leq x^{1/2} \sum_{\rho:|gamma|<T} \frac{1}{|\rho|} \ll x^{1/2} \sum_{1\leq j\leq T} \frac{n_\chi(j)}{j} \ll x^{1/2}(\log A(\chi) + n_E \log T) \log T.$$

Summing over characters and plugging into (**??**) we get

$$S(x,T) \ll \frac{|C|}{|G|} x^{1/2} (\log d_L + n_L \log T) \log T,$$

for $T \geq 2$. Choosing $T = x^{1/2}$, we're done! □

Remark. It seems like we have an extra $x^{1/2} n_K (\log x)^2$ term (outside $|C|/|G|$). Figure out how to deal with this...?

Here is the unconditional version:

Theorem 9.2. *There is an effectively computable positive absolute constant* c_{13} *so that if*

$$x \geq \exp(4 n_L (\log d_L)^2),$$

then

$$\psi_C(x) = \frac{|C|}{|G|} x - \frac{|C|}{|G|} \chi_0(g) \frac{x^{\beta_0}}{\beta_0} + O(x \exp(-c_{13} n_L^{-1/2} (\log x)^{1/2})).$$

Here the term with β_0 *only occurs if* $\zeta_L(s)$ *has an exceptional zero* β_0*, and* χ_0 *is the (real) character of* $H = \mathrm{Gal}(L/E) =< g >$ *for which* $L(s, \chi_0, L/E)$ *has a zero.*

Proof. Suppose that $\rho = \beta + i\gamma$ is a nontrivial, nonexceptional zero of one of the $L(s,\chi)$ with $|\gamma| < T$. Then, the result of Lemma 8.1 shows that

$$|x^\rho| = x^\beta \leq x \exp\left(- c_{14} \frac{\log x}{\log d_L T^{n_L}} \right).$$

Moreover, by Lemma 5.4,

$$\sum_\chi \sum_{\rho:|\rho|\geq 1/2, |\gamma|\leq T} \left| \frac{1}{\rho} \right| \ll \log T \log(d_L T^{n_L}).$$

The contribution of the nonexceptional zeroes with $|\rho| < 1/2$ is easily estimated to be $\ll x^{1/2} \log^2 d_L$ (details omitted). The functional equation gives us a zero at $1 - \beta_0$ which doesn't contribute much (again, details omitted).

Unfortunately, that Siegel zero hangs around like an obnoxious relative. We politely ask it to go away, but it doesn't. So we let it stay, and we obtain

$$S(x,T) - \frac{|C|}{|G|} \chi_0(g) \frac{x^{\beta_0}}{\beta_0} \ll \frac{|C|}{|G|} x \log T \log(d_L T^{n_L}) \exp\left(- \frac{c_{14} \log x}{\log d_L T^{n_L}} \right) + \frac{|C|}{|G|} x^{1/2} (\log d_L)^2.$$

We now choose

$$T = \exp(n_L^{-1/2} (\log x)^{1/2} - \log d_L).$$

And now this and (**??**) prove the theorem. (It's ugly, but trivial.) □

Proof of Theorems 1.1 and 1.3. Now we're basically done. We finish as in the classical case. The first step is to show that we can neglect squares and higher powers when considering the von Mangoldt function in number fields. It is easily seen that these contribute $\ll n_K x^{1/2}$, the n_K factor coming from the maximal number of prime ideals of a given norm.

Then we finish the argument off using partial summation. □

9.1. **How to improve Corollary 1.2.** This section looks ahead to the paper of Lagarias, Montgomery, and Odlyzko [**?**]. Instead of integrating

$$\frac{1}{2\pi i}\frac{x^s}{s}F_C(x),$$

we integrate against a "kernel function", and integrate

$$\frac{1}{2\pi i}\left(\frac{y^{s-1}-x^{s-1}}{s-1}\right)^2 F_C(x).$$

Basically what pops out is the Mellin transform of that kernel function; in particular, we obtain the relation

$$\frac{1}{2\pi i}\int_{(2)} -\frac{L'}{L}(s,\chi,K)k(s,x,y)ds = \sum_{\mathfrak{p},n}\chi(\mathfrak{p}^n)\Lambda(\mathfrak{p}^n)(\mathbb{N}\mathfrak{p})^{-n}\hat{k}(\mathbb{N}\mathfrak{p}^n,x,y).$$

Here k is the kernel function described above, and its Mellin transform is given by $\hat{k}(m;x,y) = \log(m/x^2)$ for $m \in [x^2, xy]$ and $\log(y^2/m)$ for $m \in [xy, y^2]$ (and is zero elsewhere). In other words, we have chosen our kernel function to isolate the contribution of primes of specified norm. When we do so, Cauchy's theorem tells us that this integral equals the contribution of the poles of the integrand, which is

$$-\sum_\rho \left(\frac{y^{\rho-1}-x^{\rho-1}}{\rho-1}\right)^2,$$

as long as χ is nonprincipal. If χ is principal, we pick up a main term of $(\log y/x)^2$.

After doing the group theory, we get that

$$\sum_{\mathfrak{p},n}\chi(\mathfrak{p}^n)\Lambda(\mathfrak{p}^n)(\mathbb{N}\mathfrak{p})^{-n}\hat{k}(\mathbb{N}\mathfrak{p}^n,x,y) = \frac{|C|}{|G|}(\log y/x)^2 - \frac{|C|}{|G|}\sum_\chi \overline{\chi}(g)\sum_\rho\left(\frac{y^{\rho-1}-x^{\rho-1}}{\rho-1}\right)^2,$$

where ρ runs through all zeroes of $L(s,\chi)$ (the trivial ones included.) We win if we can simply show that this is not zero, which is certainly the case if the main term is larger than the small one. For example, if we assume GRH, we may take $x = \log d_L$ and $y = cx$ for a suitably large constant c.

The details of this argument only work if a sufficiently good zero-free region is assumed. Of course, GRH is enough, but even if we only know that the Siegel zero does not exist, we can prove a similar result on these lines. However, we cannot prove that the exceptional zero does not exist. To prove an unconditional result, we have to use a much more complicated version of this argument.

References

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