Particle Dynamics Project

Vibration Analysis

March 18, 2018

By:

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Name 1 Name 2 Name 3

Submitted to Dr. Bradley Wall
Department of Aerospace Engineering
College of Engineering
In Partial Fulfillment
Of the Requirements
Of
ES 204
Dynamics
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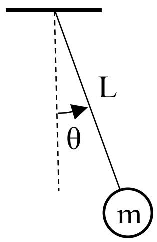
Embry-Riddle Aeronautical University
Prescott, AZ

1. CONCEPTUALIZE THE PROBLEM



This particle dynamics project asks us to model a simple pendulum with and without drag.

Without drag:

- Determine the natural frequency of the system for the following initial conditions: 5°, 10°, 15°, 30°, 60°, and 90°.
- Analytically derive the natural frequency using small angle approximation, $\sin(\theta)\approx\theta$.

With Drag:

- Integrate the EOM with including a drag term: $\bar{F}_{drag}$= 1.635e-3$\dot{\theta}\cdot|\dot{\theta}|(-\hat{e}_T)$
- Model the nonlinear EOM as a linear 2nd order ODE:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$
$$\theta(t) = e^{-\zeta\omega_n t}(A\sin(\omega_d t) + B\cos(\omega_d t))$$

That is, determine the damped natural frequency ω_d, damping ratio ζ, and the undamped natural frequency ω_n.

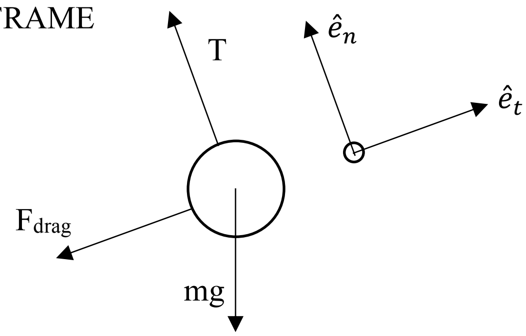
Givens: m = 0.142 kg, L = 0.5 m, g = 9.81 m/s^2

1 DOF $\theta \rightarrow$ 1 EOM $\ddot{\theta}$

Circular motion: Polar coordinates, normal-tangential

Ignore friction at the pin, assume planar motion, particle dynamics.

2. FREE-BODY DIAGRAM
3. COORDINATE FRAME



F

F_{drag}: Drag force, 1.635e-3$\dot{\theta}\cdot|\dot{\theta}|$, positive force in the negative tangential direction for assumed positive angular velocity, $\dot{\theta}$, but a negative force in the negative tangential direction, which is a positive force in the positive tangential direction for negative angular velocity, $\dot{\theta}$.

mg: mass times gravity, weight, acting straight down from the center of mass.

T: Tension of the string, pointing towards the fixed point for this system, the pin, assuming small positive deflection, θ.

Normal-Tangential polar coordinates chosen since the mass moves in a circular path. The positive normal direction points to the center of the circular path, the pin that is the fixed point for this problem, and positive tangential direction points counter-clockwise about the pin. The coordinate frame is rotated counter-clockwise by a small positive angle, θ.

4. SUM OF FORCES

Variables listed in alphabetical order:
g: gravity
L: Length of the string
m: mass of the bob
T: Tension of the string
θ: Angular position of the mass
$\dot{\theta}$: Angular velocity of the mass
$\ddot{\theta}$: Angular acceleration of the mass

No Drag:
Sum of forces in the normal direction is mass times the normal acceleration. Forces include the tension, *T*, and the cosine component of the weight, *mg*.

$$\sum F_n = m(L\dot{\theta}^2) = T - mg\cdot\cos(\theta) \tag{1}$$

Sum of forces in the tangential direction is mass times the tangential acceleration. The only force is the sine component of the weight, *mg*.

$$\sum F_t = m(L\ddot{\theta}) = -mg\cdot\sin(\theta) \tag{2}$$

With Drag:
Sum of forces in the normal direction is mass times the normal acceleration. Forces include the tension, *T*, and the cosine component of the weight, *mg*.

$$\sum F_n = m(L\dot{\theta}^2) = T - mg\cdot\cos(\theta) \tag{3}$$

Sum of forces in the tangential direction is mass times the tangential acceleration. Forces include the drag force and the sine component of the weight, *mg*.

$$\sum F_t = m(L\ddot{\theta}) = -mg\cdot\sin(\theta) - 1.635e\text{-}3\dot{\theta}\cdot|\dot{\theta}| \tag{4}$$

5. KNOWNS AND UNKNOWNS
Knowns: mass of the bob (*m*, given), Length of the string (*L*, given), angular velocity of the mass ($\dot{\theta}$, state variable), gravity (*g*, constant), angular position of the mass (θ, state variable)
Unknowns: Tension of the string (*T*), angular acceleration of the mass ($\ddot{\theta}$, the equation of motion we want to find)
This results in a 2 equations, 2 unknowns system. Skip to Step 7: Solve for the Equation of Motion

6. CONSTRAINTS
Not applicable

7. SOLVE FOR THE EQUATION(S) OF MOTION

No Drag:

Rearranging Equation (2) yields:

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) \tag{5}$$

Using the small-angle approximation that $\sin(\theta) \approx \theta$ yields the linear equation of motion:

$$\ddot{\theta} = -\frac{g}{L}\theta \tag{6}$$

With Drag:

Rearranging Equation (4) yields:

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \frac{1.635e\text{-}3\dot{\theta}|\dot{\theta}|}{mL} \tag{7}$$

8. SOLVE THE EQUATION OF MOTION, SOLVE THE PROBLEM

Step 1:

The equation of motion for the simple pendulum with no drag, Equation (5), cannot be solve analytically and thus was integrated numerically using MATLAB. 6 different initial angles or release were used and their time responses shown below.

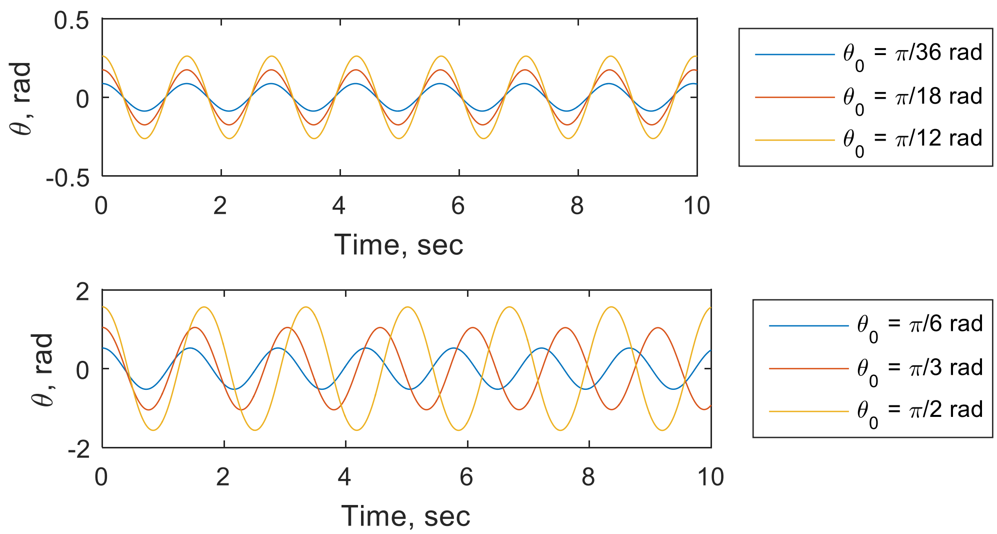


Figure 1: Time histories of the simple pendulum without drag for various initial conditions.

The top graph shows that although the initial angle of release is changed it has very little effect on the period of oscillation. Contrarily, the bottom graph shows that as the initial angle of release is varied from π/6 radians to π/2 radians, the period of oscillation varies significantly. Below is a table of data the corroborate this analysis:

Table 1: Natural frequency versus angle of release

Angle of Release (rad)	Period (s)	Natural Frequency (rad/sec)
π/36	1.419	4.428
π/18	1.421	4.421
π/12	1.424	4.411
π/6	1.443	4.355
π/3	1.521	4.131
π/2	1.672	3.758

Step 2:
If we assume that θ starts small and stays small then sin(θ)≈θ. Now the equation of motion, Equation (6), can be solved analytically. Let $\theta(t)=A\cdot e^{st}$ then $\dot{\theta}(t)=s\cdot A\cdot e^{st}$ and $\ddot{\theta}(t)=s^2\cdot A\cdot e^{st}$. Substituting these expressions into the equation of motion yields:

$$A\cdot e^{st}(L\cdot s^2+g)=0 \tag{8}$$

Thus,

$$s = \pm i\sqrt{\frac{g}{L}} \tag{9}$$

and our solution for θ(t) is a pure sinusoid:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}t\right) \tag{10}$$

where $\sqrt{\frac{g}{L}}$ is the natural frequency of the system, 4.429 rad/sec. This is extremely close, less than 1% different, for the angles of release less than π/12 radians and only 1.7% different than the π/6 radians angle of release case. A figure depicting the changing natural frequency for the various angles of release is shown in Figure 2.

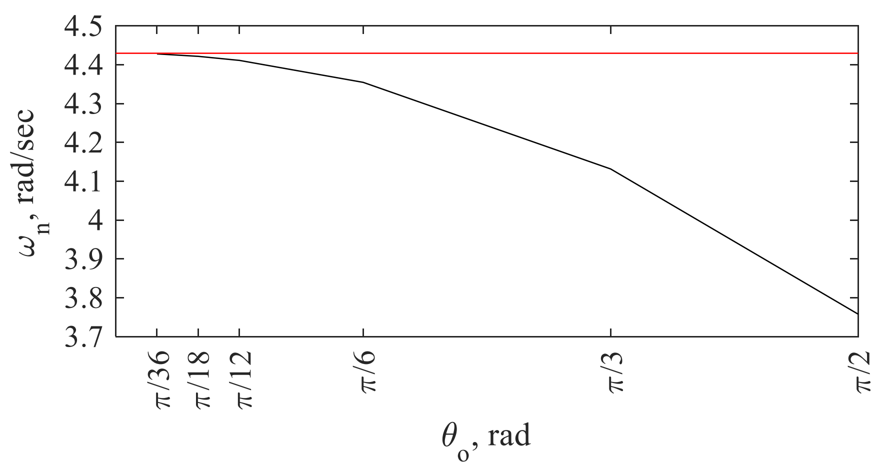


Figure 2: Natural frequency, ω_n, versus angle of release, θ_o.

This means that we can safely model the simple pendulum with no drag up to π/12 radians using small angle approximation, and thus using a linear system equation of motion, and maintain accuracy of less than 1% error.

Step 3:
In Step 3 we include the drag term and again integrate the equation of motion, Equation (7), numerically using MATLAB. The θ time history is shown below in Figure 3.

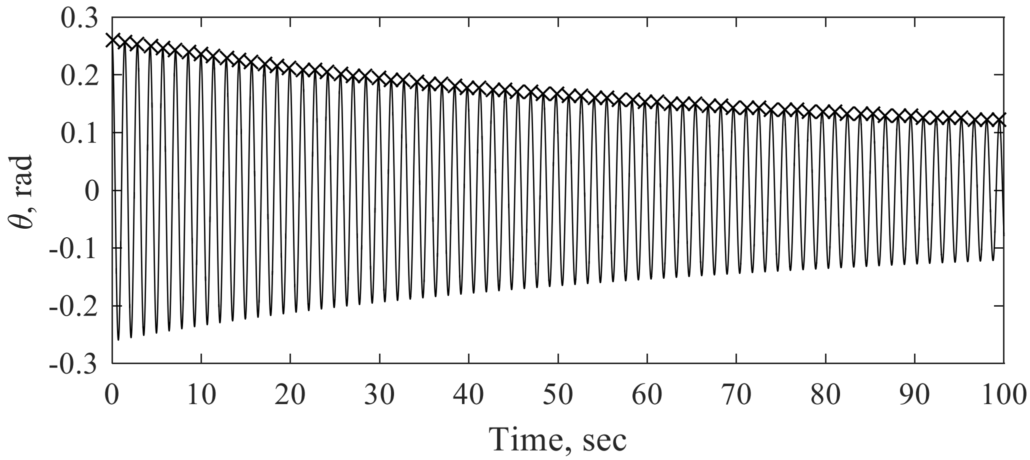


Figure 3: θ time history for system with drag

As expected, we see the amplitude of oscillations decrease over time. The markers outline the decreasing peak amplitudes over the 100 second time frame. Starting at rest with an initial angular deflection of π/12 radians (0.2618 rad) the last peak value is 0.1213 radians. Given both the initial peak and the final peak have zero velocity the energy in the system can be analyzed as potential energy only. If the pin is defined as the fixed zero reference then the potential energy function is simply:

$$PE = -mg\cdot\cos(\theta) \quad (11)$$

Then the initial energy in the system is -1.34 Joules and the energy in the system at the last peak is -1.38 Joules for a loss of only 0.04 Joules over 99.5 seconds, the time of the last peak.

Step 4:
In Step 4 we must analyze the output from Step 3 to model the system as a linear 2nd order differential equation:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \quad (12)$$

with solution:

$$\theta(t) = e^{-\zeta\omega_n t}(A\sin(\omega_d t) + B\cos(\omega_d t)) \quad (13)$$

where ζ is the damping ratio, ω_n is the (undamped) natural frequency and ω_d is the damped natural frequency. First the constants A and B are determined from the initial conditions:

$$\theta(0) = \pi/12 \text{ rad and } \dot{\theta} = 0 \text{ rad/sec} \quad (14)$$

Then, inputting t = 0 into the θ(t) function yields:

$$\theta(0) = B = \pi/12 \text{ rad} \quad (15)$$

Taking the derivative of the θ(t) function yields:

$$\dot{\theta}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(A\sin(\omega_d t) + B\cos(\omega_d t)) + \omega_d e^{-\zeta\omega_n t}(A\cos(\omega_d t) - B\sin(\omega_d t)) \quad (16)$$

Inputting t = 0 into the $\dot{\theta}(t)$ equation yields:

$$\dot{\theta}(0) = 0 = -\zeta\omega_n B + \omega_d A \quad (17)$$

Therefore,

$$A = \frac{\zeta\omega_n B}{\omega_d} \quad (18)$$

Thus, $\theta(t)$ is now:

$$\theta(t) = \frac{\pi}{12} e^{-\zeta\omega_n t}\left(\frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t) + \cos(\omega_d t)\right) \tag{19}$$

The damped natural frequency was found using the average time between successive peaks, the period τ.

$$\omega_d = 2\pi/\tau \tag{20}$$

$$\omega_d = 4.4211015 \text{ rad/sec}$$

Next we note that at the peak times, $t_p = N\cdot\tau$, where N is the N^{th} peak, and using the definition of the damped natural frequency, the product of ω_d and t_p is simply $2\pi\cdot N$. Thus; $\cos(\omega_d\cdot t_p) = \cos(2\pi\cdot N) = 1$ and $\sin(\omega_d\cdot t_p) = \sin(2\pi\cdot N) = 0$. This yields θ as a function of the peak times:

$$\theta(t_p) = \frac{\pi}{12} e^{-\zeta\omega_n t_p} \tag{21}$$

Since the natural frequency, the damped natural frequency, and the damping ratio are all related through the following equation:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{22}$$

The equation for $\theta(t_p)$ can be written as only a function of the damping ratio, ζ.

$$\theta(t_p) = \frac{\pi}{12} e^{-\zeta\left(\frac{\omega_d}{\sqrt{1-\zeta^2}}\right)(N\cdot\tau)} = \frac{\pi}{12} e^{-\zeta\left(\frac{2\pi/\tau}{\sqrt{1-\zeta^2}}\right)(N\cdot\tau)} = \theta_0 e^{-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)(2\pi N)} \tag{23}$$

Algebraically solving for ζ yields:

$$\zeta = \sqrt{\frac{\ln^2\left(\frac{\theta(t_p)}{\theta_0}\right)}{\ln^2\left(\frac{\theta(t_p)}{\theta_0}\right) + (2\pi N)^2}} \tag{24}$$

This equation was evaluated at each of the 70 peaks and then averaged. This yields:

$$\zeta = 2.0738970\text{e-}03$$

Now that ω_d and ζ are known, ω_n can be determined from the equation relating the damped natural frequency, the damping ratio, and the natural frequency:

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 4.4211110 \text{ rad/sec}$$

This value, although very close to, is slightly greater than the value of ω_d. This must be true for an underdamped system, where the damping ratio is less than 1.

The angular deflection output for the numerically integrated solution, Figure 3, and for the linear-approximated system, Equation (19), is shown on the following page:

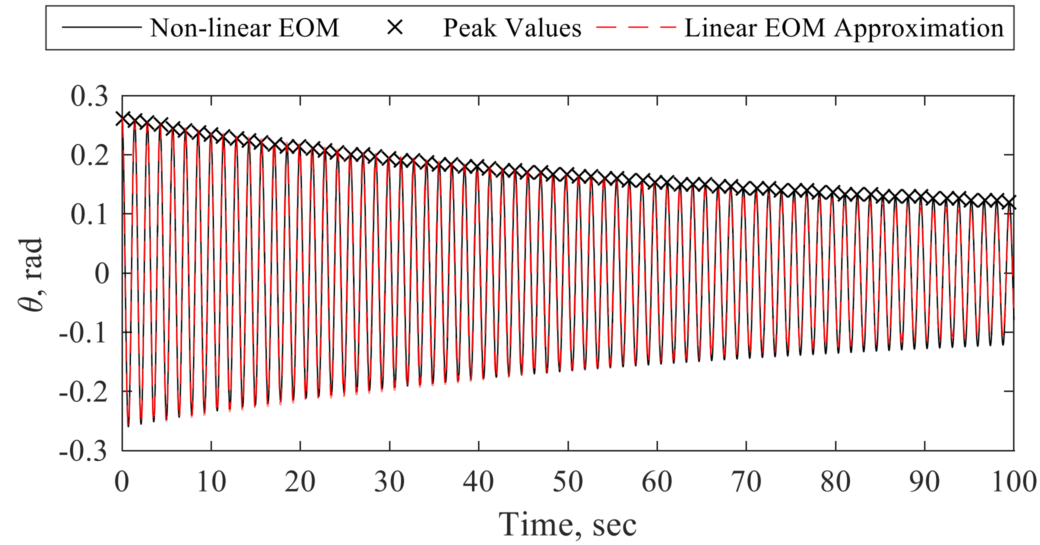


Figure 4: Comparison of nonlinear equation of motion output and modeled linear equation of motion for 100 seconds.

At this scale the difference of the nonlinear equation of motion to the linear 2nd order equation of motion is imperceptible. To see the difference, we must zoom in to near the end of the 100 seconds time frame. Showing the last 5 seconds is Figure 5 below:

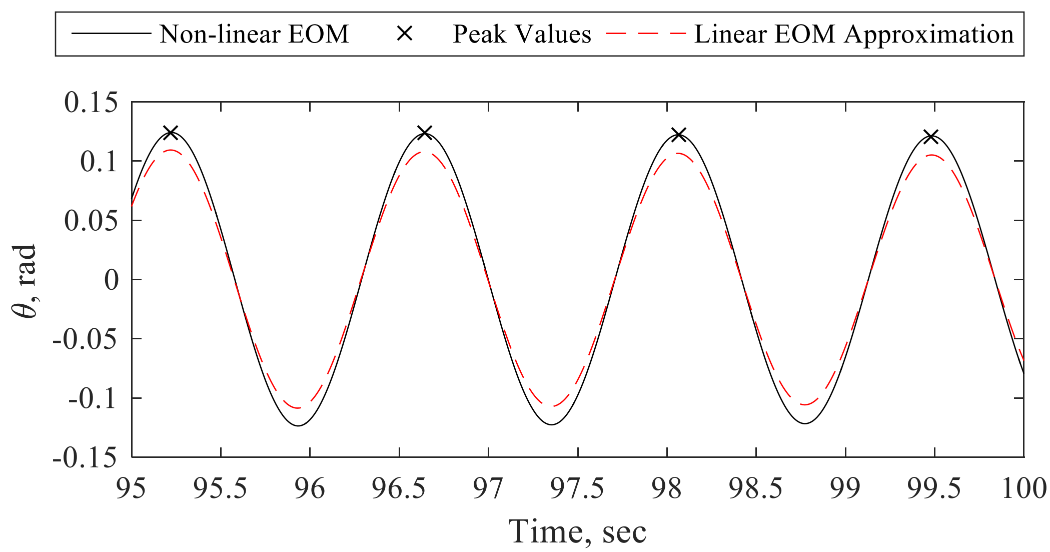


Figure 5: Comparison of nonlinear equation of motion output and modeled linear equation of motion for 95 to 100 seconds.

In Figure 5 we can see that the peak times are still accurately modeled over 100 seconds. This proves that our value for ω_d models the system well. The peak value itself at 99.5 seconds, the last peak, is 0.1051 radians for the linear system versus 0.1213 radians for the true nonlinear equation of motion. This is an error of 13.4%. Although this may seem like a significant amount of error, if we compare the energies at this last peak, -1.381 J for the linear system versus -1.379 J for the true nonlinear system, it is only a difference of 0.2%!

9. DOES IT MAKE SENSE

Units:

Step 1 EOM:

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) = \frac{\frac{m}{s^2}}{m} = \frac{rad}{s^2}$$

Checks

Step 2 EOM:

$$\ddot{\theta} = -\frac{g}{L}\theta = \frac{\frac{m}{s^2}}{m} = \frac{rad}{s^2}$$

Checks

Step 3 EOM:

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \frac{1.635e\text{-}3\dot{\theta}|\dot{\theta}|}{mL}$$

First term checks as part of Step 1. The drag term must also have units of rad/s^2. Given the current units of 1/(kg·m·s^2) that means the constant, 1.635e-3, must have units of kg·m. Using the drag equation: $\frac{1}{2}\rho C_D A v^2(\text{-}\hat{e}_t) = \frac{1}{2}\rho C_D A L^2\dot{\theta}|\dot{\theta}|$ the units on the constant should be:

$$\frac{1}{2}\rho C_D A L^2 \rightarrow \frac{kg}{m^3}\cdot m^2 \cdot m^2 = kg \cdot m$$

Checks

Step 4 equation:

$$\theta(t) = \theta_o e^{-\zeta\omega_n t}\left(\frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t) + \cos(\omega_d t)\right)$$

The damping ratio ζ is unit-less. The natural frequency, ω_n, and the damped natural frequency, ω_d, are in rad/sec. The exponent term and the term inside of the parenthesis are thus unit-less. The only units on $\theta(t)$ are the units on θ_o, radians.
Checks.

Magnitude:

Step 1:

The amplitude of the oscillations, shown in Figure 1, are consistent with a system with no change in energy; the amplitude is constant.

Step 1 and 2:

The natural frequency of the system decreases, as the initial angle of release and the period of oscillations increases. This makes physical sense. Also, the natural frequency for small angles approaches the value of $\sqrt{\frac{g}{L}}$ obtained analytically by solving the equation of motion assuming the small angle approximation, ($\sin(\theta) \approx \theta$).

Step 3:

The amplitude of the oscillations, shown in Figure 3, are consistent with a system with drag, a loss in energy; the amplitude of oscillation decreases over time.

Step 3 and 4:

The natural frequency, ω_n, and the damped natural frequency, ω_d, are very close to the natural frequency from Step 1 using the same initial condition. The damping ratio is small, thus very low energy loss, which makes sense for the small pendulum bob used in the physical

demonstration in class.
Signs:
Step 1, 2, and 3 EOM:
For $\theta_o > 0$ the initial angular acceleration is negative, i.e., it swings back towards $\theta = 0$.
For $\theta_o < 0$ the initial angular acceleration is positive, i.e., it swings back towards $\theta = 0$.
Both cases make physical sense.
Step 4:
The damping ratio, ζ, the natural frequency, ω_n, and the damped natural frequency, ω_d, are in rad/sec

APPENDICES:

A. ATTRIBUTIONS
Who did what? List each team member and how they contributed to the project. This can be in a simple table.

B. ANALYTICAL SOLUTION
Scan the full analytical solution into this appendix.

C. NUMERICAL SOLUTION
Publish your MATLAB code so that your code, command window output, and any figures are present in this appendix.