Rigid-Body Dynamics Project

Rigid-Body Spring Pendulum

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Dr. Mark Sensmeier
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ES204 Dynamics - Fall 2018

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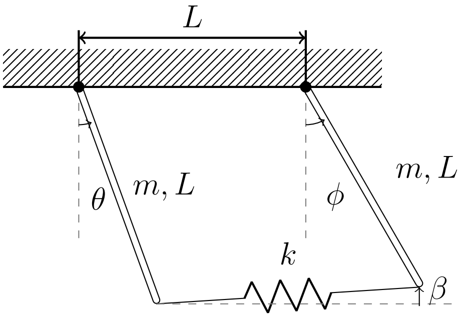
1 Conceptualize the Problem



The pendulumn system consists of two rigid bars rotating about one end, attached at the opposite ends by a linear spring.

1.1 Constants and Assumptions

Constants:		Assumptions:
Bar Mass:	$m = 0.25$kg	No Losses
Bar Length:	$L = 0.5$m	Released from Rest
Gravity:	$g = 9.81 {}^{m}/_{s^2}$	Slender Bars
Linear Spring:		Planar
Spring Coefficient:	$k = 25\ {}^{N}/_{m}$	Rigid-Body Dynamics
Unstretched Length:	L	

We are asked to determine the following:

1. The 6 Equations / 6 Unknowns for the system to solve for the Equations of Motion.

2. Integrate the Equations of Motion using various initial conditions.

   (a) $\theta_o = {}^{\pi}/_{12}\ rad, \quad \phi_o = {}^{\pi}/_{12}\ rad$

   (b) $\theta_o = {}^{-\pi}/_{12}\ rad, \quad \phi_o = {}^{\pi}/_{12}\ rad$

   (c) $\theta_o = {}^{\pi}/_{36}\ rad, \quad \phi_o = {}^{\pi}/_{12}\ rad$

3. Linearize the Equations of Motion assuming small angular positions and velocities (i.e. small angle approximation $\sin(x) \approx x,\ \cos(x) \approx 1$)

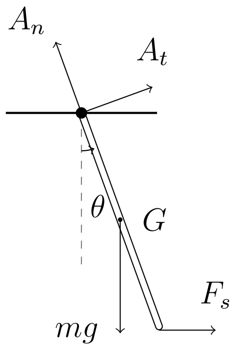
   - Determine the A matrix below.

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \theta \\ \phi \end{bmatrix}$$

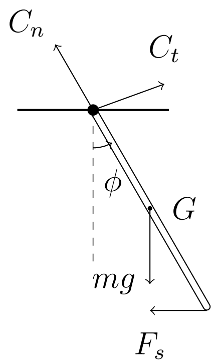
4. Find the natural frequencies of the system and their respective eigenvectors using the eigenvalues and eigenvectors of [A].

5. Using information from (5), solve for the analytical solution to the linearized Equations of Motion and plot them for the initial conditions defined in (2).

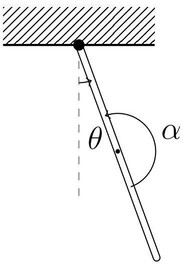
2 Free Body Diagram



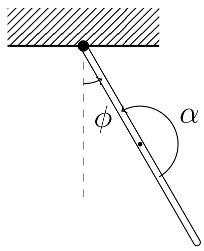
F_s:	Force onto bar by the spring
mg:	Mass · gravity, weight of each bar
G:	Center of gravity of each bar
$\theta,\ \phi$:	Angle of bar relative to vertical
$A_n,\ C_n$:	Reaction forces in the normal direction at pin
$A_t,\ C_t$:	Reaction forces in the tangential direction at pin



Acceleration Diagram

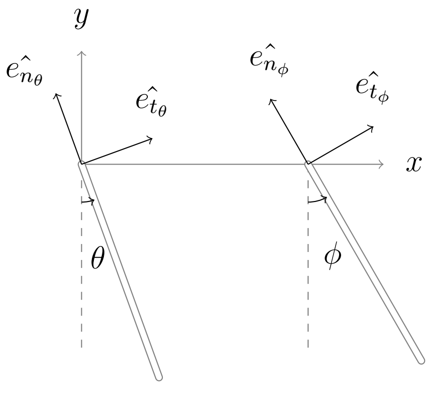


α: $\ddot{\theta},\ \ddot{\phi}$ respectively

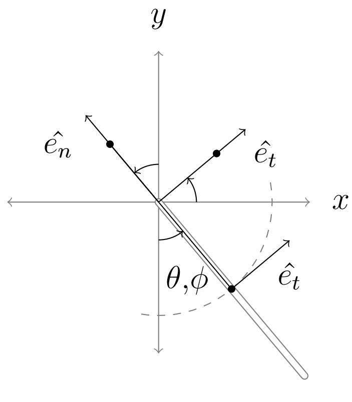


3 Coordinate Frame

(a) Separate Coordinate Frames



(b) Unit Vector Calculations



$\hat{e_n}$: [- $\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}$]
$\hat{e_t}$: [$\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$]

Figure 1: Visual Representation of Coordinate Frame Unit Vectors

Due to the complexity of the sytem if it were to be defined in cartesian coordinates, we determined that the motion of the system would most effectively be represented by two separate normal-tangential coordinate frames because the motion of each bar is purely rotational, where the center of mass of each bar is following a curve. As seen in Figure (1a), two coordinate frames were used, one for each bar. Similarly, Figure (1b) shows how the unit vectors are calculated as well as the path the center of mass travels.

4 Sum of Forces

The six equations that represent the forces on the system are comprised of four force summations and two moment equations,

$$\sum F_n = m\frac{L}{2}\dot{\theta}^2 = F_{s_n} + A_n - mg\cos(\theta) \tag{1}$$

Sum of Normal Forces on the left bar (θ)

$$\sum F_t = m\frac{L}{2}\ddot{\theta} = F_{s_n} + A_t - mg\sin(\theta) \tag{2}$$

Sum of Tangential Forces on the left bar (θ)

$$\sum F_n = m\frac{L}{2}\dot{\phi}^2 = F_{s_n} + C_n - mg\cos(\phi) \tag{3}$$

Sum of Normal Forces on the right bar (ϕ)

$$\sum F_t = m\frac{L}{2}\ddot{\phi} = F_{s_n} + C_t - mg\sin(\phi) \tag{4}$$

Sum of Tangential Forces on the right bar (ϕ)

$$\sum M_A = -\left(\frac{L}{2} \cdot mg\sin(\theta)\right) + (F_{st} \cdot L) = \frac{1}{3}mL^2\ddot{\theta} \tag{5}$$

Sum of Moments about A (θ)

$$\sum M_C = -\left(\frac{L}{2} \cdot mg\sin(\phi)\right) + (F_{st} \cdot L) = \frac{1}{3}mL^2\ddot{\phi} \tag{6}$$

Sum of Moments about C (ϕ)

Where:

θ: Position of the left bar.
ϕ: Position of the right bar.
$\dot{\theta}$: Angular velocity of the left bar.
$\dot{\phi}$: Angular velocity of the right bar.
$\ddot{\theta}$: Angular acceleration of the left bar.
$\ddot{\phi}$: Angular acceleration of the right bar.
A: Reaction force in the normal or tangential direction on the left bar.
C: Reaction force in the normal or tangential direction on the right bar.
F_s: Force due to the spring in either the normal or tangential direction.
$m,\ L,\ g$: Mass, length of each bar, and gravity, respectively.

5 Knowns and Unknowns

Knowns:	
Mass:	$m = 0.25\text{kg}$
String Length:	$L = 0.5\text{m}$
Gravity:	$g = 9.81 {}^{m}/_{s^2}$
Linear Spring:	
Spring Coefficient:	$k = 25\ {}^{N}/_{m}$
Unstretched Length:	L
State Variables:	
Angular Position:	$\theta,\ \phi$
Angular Velocity:	$\dot{\theta},\ \dot{\phi}$

Unknowns:	
Reaction Forces:	$A_n,\ A_t$
	$C_n,\ C_t$
Angular Accelerations:	$\ddot{\theta},\ \ddot{\phi}$

Since there are six equations and six unknowns, we can solve for the equations of motion analytically using Matlab.

6 Constraints

No constraint equations were needed to find a solution to the system.

7 Solve for the Equations of Motion

The equations of motion for this system are (from Eqs. (1-6))

$$\ddot{\theta} = -3\frac{gm\sin\theta - 2kL\cos\theta - 2kL\sin\phi - \theta + \frac{2kL^2\cos\theta}{\sqrt{L^2(\sin\phi - \sin\theta + 1)^2 + L^2(\cos\phi - \cos\theta)}} + \frac{2kL^2\sin\phi - \theta}{\sqrt{L^2(\sin\phi - \sin\theta)^2 + L^2(\cos\phi - \cos\theta)^2 + L^2(\cos\phi - \cos\theta)^2)}}}{2Lm}$$

$$\ddot{\phi} = -3\frac{gm\sin\phi - 2kL\cos\phi - 2kL\sin\phi - \theta + \frac{2kL^2\cos\phi}{\sqrt{L^2(\sin\phi - \sin\theta + 1)^2 + L^2(\cos\phi - \cos\theta)}} + \frac{2kL^2\sin\phi - \theta}{\sqrt{L^2(\sin\phi - \sin\theta)^2 + L^2(\cos\phi - \cos\theta)^2 + L^2(\cos\phi - \cos\theta)^2)}}}{2Lm}$$

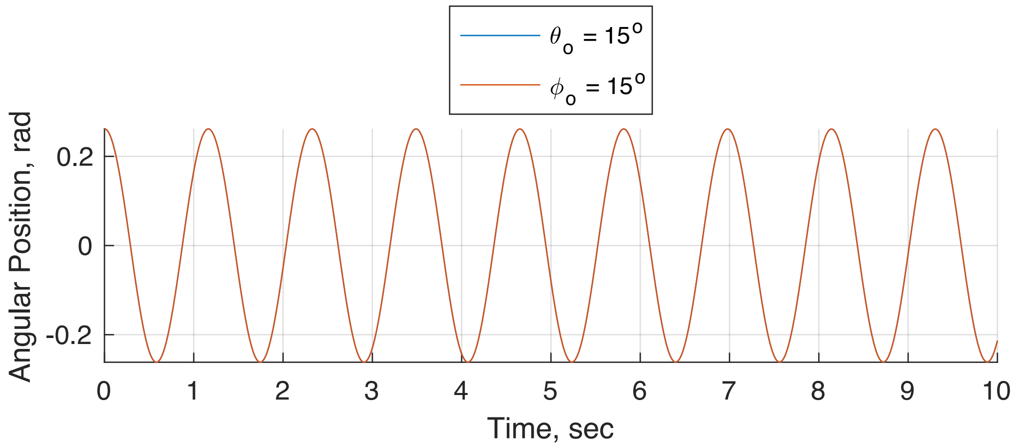


Figure 2: Meow.

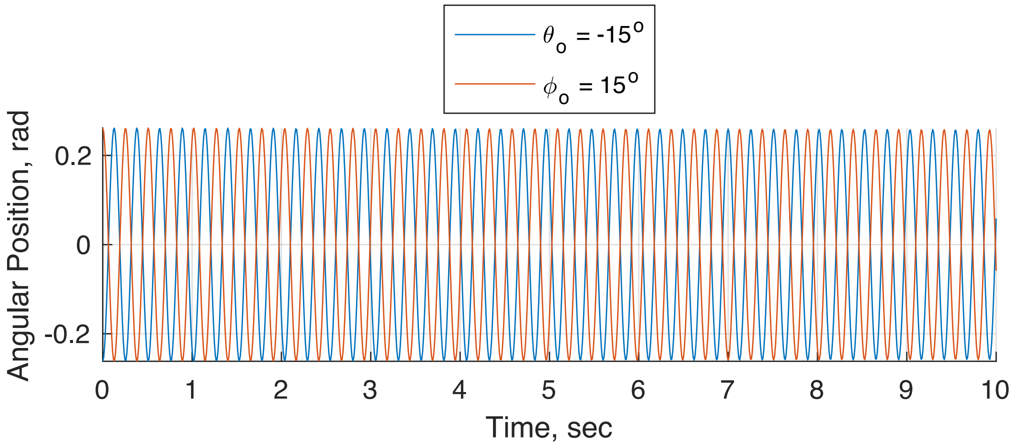


Figure 3: Meow.

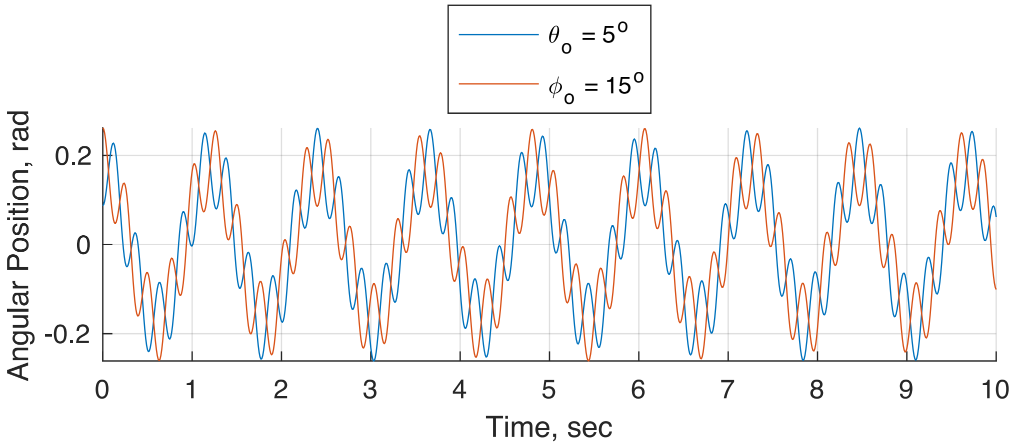


Figure 4: Meow.

8 Solve the Equations of Motion

9 Does it Make Sense?

9.1 Units

Checking with the MATLAB symbolic units tool (from Section 10.3):

```
% Checking EOM Units
u = symunit;
m = m*u.kg;
g = g*u.m/u.s^2;
l = l*u.m;
k = k*u.N/u.m;
An = An*u.N;
At = At*u.N;
Cn = Cn*u.N;
Ct = Ct*u.N;
theta = 'theta';
```

```
thetadot = 'thetadot'/u.s;
thetaddot = 'thetaddot'/u.s^2;
phi = 'phi';
phidot = 'phidot'/u.s;
phiddot = 'phiddot'/u.s^2;

eqn = subs(eqn);
unitCheck = checkUnits(eqn)
```

```
unitCheck =
  struct with fields:

    Consistent: [1 1 1 1 1 1]
    Compatible: [1 1 1 1 1 1]
```

9.2 Magnitude

Looking at Figure (2), the amplitude and perod of both phi and theta perfectly overlap with eachother. This is expected due to the fact that two pendulums that begin oscillating when released from the same state will never have a change in their relative positions. This means that the spring connecting the two is neither streched nor compressed throughout the system's oscillations. This is valid because the amplitude and period of oscillation never changes in this lossless system.
Looking at the generic linearization equation,

$$[m]\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix}+[k]\begin{bmatrix}\theta\\ \phi\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

We can see that the solution to the system when $\theta_0 = \phi_0$ will be of the form,

$$-\omega^2[m]\begin{bmatrix}\theta\\ \phi\end{bmatrix}\cos(\omega t+\Psi)$$

instead of the form,

$$-\omega^2[m]\begin{bmatrix}\theta\\ \phi\end{bmatrix}\cos(\omega t+\Psi)+[k]\begin{bmatrix}\theta\\ \phi\end{bmatrix}\cos(\omega t+\Psi)$$

The fact that the equation takes the former's form supports our idea that this behavior makes physical sense when $\theta_0 = \phi_0$. For initial conditions where $\theta_0! = \phi_0$ or $\ddot{\theta}_0! = \ddot{\phi}_0$ we find a non-zero $[k]$, which lines up with the physical concept that the bars will inconsistently oscillate.

10 Appendix

10.1 Attributions

Jeffrey Chen	
Thorne Wolfenbarger	Is just really really cool
Trey Dufrene	
Joint Effort	MATLAB Coding, Solving EOM's

10.2 Analytical Solution

10.3 Numerical Solution

Non-Linearized

```
clear;close all;clc

% Constants
c.m = 0.25; % kg
c.k = 25; % N/m
c.l = 0.5; % m
c.I = (1/3)*c.m*c.l^2; %kg.m^2
c.g = 9.81; % m/s^2

syms l g m k I theta phi thetadot thetaddot phidot phiddot...
    An At Cn Ct

% Cartesian locations of bar ends
b = l*[sin(theta);-cos(theta)];
d = l*[sin(phi)+1;-cos(phi)];
rbd = d - b;
rbdSquared = rbd.^2;
% Spring Length (mag)
ls = simplify(rbdSquared(1) + rbdSquared(2))^(1/2);
% Spring Direction (Left and Right)
ehatsL = (d - b) / ls;
ehatsR = (b - d) / ls;
% Normal / Tangential for Theta
ehatnT = [-sin(theta);cos(theta)];
ehattT = [cos(theta);sin(theta)];
% Normal / Tangential for Phi
ehatnP = [-sin(phi);cos(phi)];
ehattP = [cos(phi);sin(phi)];
% Spring Force (Left and Right)
FsL = k*(ls-l)*ehatsL;
FsR = k*(ls-l)*ehatsR;

% Spring force in normal direction (Theta)
Fs_nT = simplify(FsL(1)*ehatnT(1)+FsL(2)*ehatnT(2));
% Spring force in tangential direction (Theta)
Fs_tT = simplify(FsL(1)*ehattT(1)+FsL(2)*ehattT(2));

% Spring force in normal direction (Phi)
Fs_nP = simplify(FsR(1)*ehatnP(1)+FsR(2)*ehatnP(2));
% Spring force in tangential direction (Phi)
Fs_tP = simplify(FsR(1)*ehattP(1)+FsR(2)*ehattP(2));

% Sum of Normal Forces, left bar (Theta)
eqn(1) = m*(l/2)*thetadot^2 == Fs_nT + An - m*g*cos(theta);
% Sum of Tangential Forces, left bar (Theta)
```

```
eqn(2) = m*(l/2)*thetaddot == Fs_tT + At - m*g*sin(theta);
% Sum of Moments, left bar (Theta)
eqn(3) = -(l/2)*(m*g*sin(theta)) + (Fs_tT)*(l) == (1/3)*m*(l
    ^2)*thetaddot;

% Sum of Normal Forces, right bar (Phi)
eqn(4) = m*(l/2)*phidot^2 == Fs_nP + Cn - m*g*cos(phi);
% Sum of Tangential Forces, right bar (Phi)
eqn(5) = m*(l/2)*phiddot == Fs_tP + Ct - m*g*sin(phi);
% Sum of Moments, right bar (Phi)
eqn(6) = -(l/2)*(m*g*sin(phi)) + (Fs_tP)*(l) == (1/3)*m*(l^2)
    *phiddot;

x = solve(eqn,[thetaddot,phiddot,An,At,Cn,Ct]);

syms theta(t) thetadot(t) phi(t) phidot(t)

thetaEOM = subs(x.thetaddot, {'theta', 'thetadot'}, {theta,
    thetadot});
phiEOM = subs(x.phiddot, {'phi', 'phidot'}, {phi, phidot});

eom = odeFunction([thetadot; thetaEOM; phidot; phiEOM],...
    [theta; thetadot; phi; phidot],m,k,l,I,g);

theta_o = [pi/12 -pi/12 pi/36];

for i = 1:3
    [T,S] = ode45(@(t,s)eom(t,s,c.m,c.k,c.l,c.I,c.g),linspace
        (0,10,1001),...
         [theta_o(i),0,pi/12,0]);
    figure
    hold on
    grid on
    xlabel('Time, sec')
    ylabel('Angular Position, rad')
    plot(T,S(:,1),'DisplayName', ['\theta_o = ' num2str(
        rad2deg(theta_o(i))) '^o'])
    plot(T,S(:,3),'DisplayName', '\phi_o = 15^o')
    legend('show')
end

% Checking EOM Units
u = symunit;
m = m*u.kg;
g = g*u.m/u.s^2;
l = l*u.m;
k = k*u.N/u.m;
An = An*u.N;
```

```
At = At*u.N;
Cn = Cn*u.N;
Ct = Ct*u.N;
theta = 'theta';
thetadot = 'thetadot'/u.s;
thetaddot = 'thetaddot'/u.s^2;
phi = 'phi';
phidot = 'phidot'/u.s;
phiddot = 'phiddot'/u.s^2;

eqn = subs(eqn);
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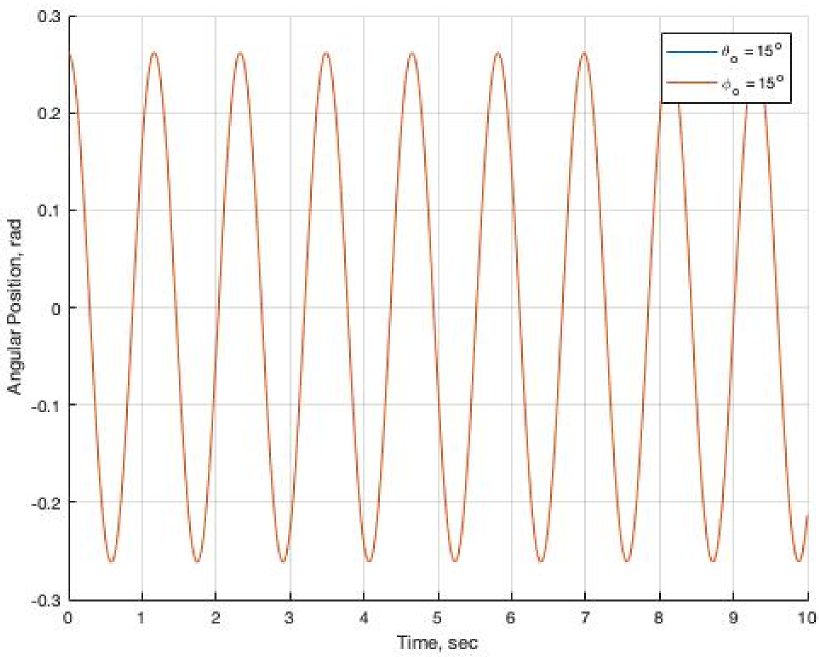
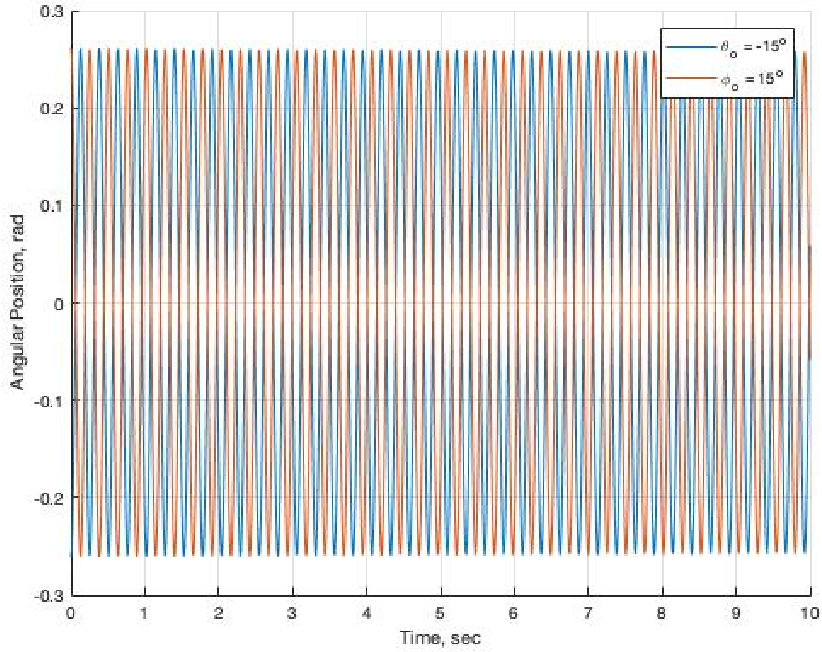
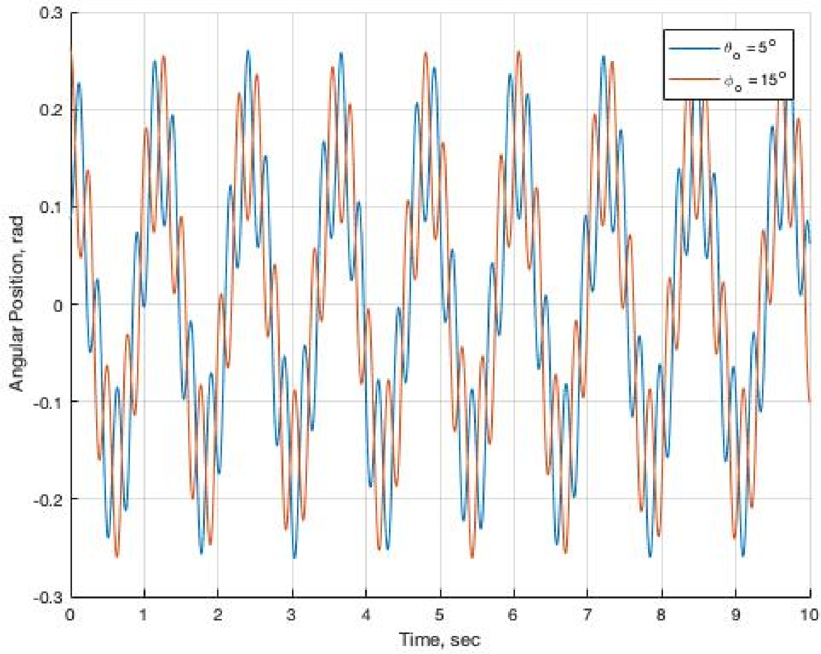




θ_o = -15°
ϕ_o = 15°
Angular Position, rad
Time, sec



θ_o = 5°
ϕ_o = 15°
Angular Position, rad
Time, sec