Lagrangian Dynamics Project

Rigid-Body Spring Pendulum

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By:

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Submitted to:
Dr. Mark Sensmeier
In Partial Fulfillment of the Requirements of
ES204 Dynamics – Fall 2018

College of Engineering
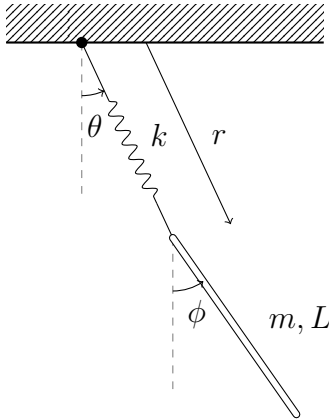Embry-Riddle Aeronautical University
Prescott, AZ

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1 Conceptualize the Problem

The pendulum system consists of a rigid bar pinned to the free end of a linear spring, which rotates about its opposite end at a fixed point; there are three degrees of freedom, since the spring and bar each have an individual angular deflection with respect to the vertical, and the radial distance the bar is from the point of rotation due to the variation in the length of the spring.

1.1 Constants and Assumptions

| Constants:		Assumptions:
Bar Mass:	$m = 1$ kg	No Losses
Bar Length:	$L = 1$ m	Released from Rest
Gravity:	$g = 9.81 \, m/s^2$	Uniform Slender Bar
Linear Spring:		Planar
Spring Coefficient:	$k = 25 \; N/m$	Rigid-Body Dynamics
Unstretched Length:	$L = 0.5$ m	

We are asked to determine the following:

1. The Equations of Motion for the system via the Lagrangian method.

2. Integrate the Equations of Motion using various initial conditions and plot the behavior of the system for 10 seconds.

   (a) $\theta_o = 0 \; rad,$ $\quad \phi_o = 0 \; rad$
   (b) $\theta_o = \pi/18 \; rad,$ $\quad \phi_o = \pi/9 \; rad$
   (c) $\theta_o = \pi/6 \; rad,$ $\quad \phi_o = \pi/3 \; rad$

3. Plot the total energy versus time for all 3 cases.

4. Repeat 2. and 3. using a 'RelTol' of 1e-6 and 'AbsTol' of 1e-9 for the `ode45` integration tolerances.
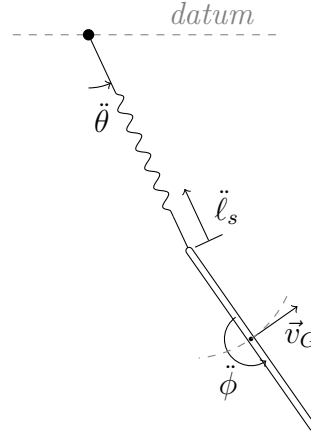
2   Free Body Diagram

Figure 1: Acceleration and Free Body Diagrams
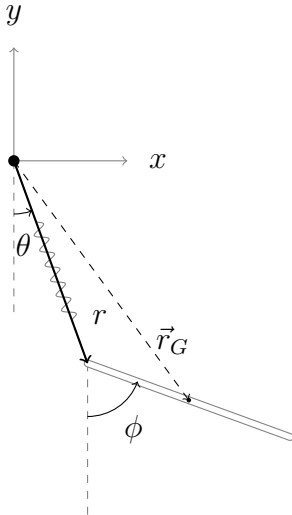
(a) Free Body Diagram
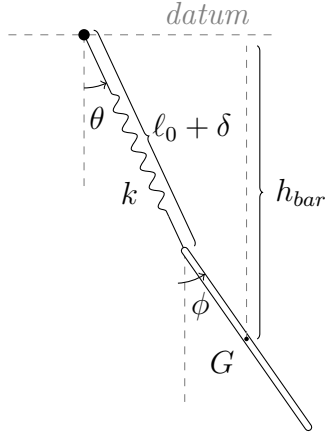
(b) Acceleration Diagram

G:	Center of gravity of the bar	
ℓ_0:	Spring unstretched length	
δ:	Spring deflection	
k:	Spring constant	
h_{bar}:	Height of bar from datum	

$\vec{v}_G$:	Velocity of bar center of gravity
$\ddot{\theta}$:	Angular velocity of spring
$\ddot{\phi}$:	Angular velocity of bar
$\ddot{\ell}_s$:	Radial acceleration of bar due to spring

3   Coordinate Frame

Motion Variables:

θ:	Angle of spring relative to vertical
ϕ:	Angle of bar relative to vertical
r:	Radial length of spring

Supplemental Variables:

$\vec{r}_G$:	Vector to bar center of mass from origin

Figure 2: Coordinate Frame

4   Sum of Forces

From Figure (2),

$$\vec{r}_G = \left[\ell_s \sin(\theta) + \frac{\ell_b}{2}\sin(\phi)\right]\hat{\imath} + \left[-\ell_s \cos(\theta) - \frac{\ell_b}{2}\cos(\phi)\right]\hat{\jmath}$$

Taking the time derivative,

$$\frac{d}{dt}\vec{r}_G = \dot{\vec{r}}_G = \vec{v}_G$$

$$\vec{v}_G = \left[\dot{\ell}_s \sin(\theta) + \frac{\ell_b\dot{\phi}\cos(\phi)}{2} + \ell_s\dot{\theta}\cos(\theta)\right]\hat{\imath} + \left[\frac{\ell_b\dot{\phi}\sin(\phi)}{2} - \dot{\ell}_s\cos(\theta) + \ell_s\dot{\theta}\sin(\theta)\right]\hat{\jmath}$$

Kinetic Energy of Spring:

$$T_1 = 0$$

Kinetic Energy of Bar, due to it's rotational and translational velocity (Figure 1b):

$$T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{2}I\omega^2$$

Since the moment of inertia I for a uniform slender bar rotating about its end is $\frac{1}{12}m\ell^2$ and $\omega = \dot{\phi}$,

$$T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{24}m_b\ell_b^2\dot{\phi}^2$$

Total Kinetic Energy:

$$T = T_1 + T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{24}m_b\ell_b^2\dot{\phi}^2 \tag{1}$$

Potential Energy of Spring due to it's stretch (Figure 1a):

$$V_1 = \frac{1}{2}k\left(\ell_s - \ell_0\right)^2$$

Potential Energy of Bar, due to it's distance below the datum, h_{bar} (Figure 1a):

$$V_2 = -m_b g\left(\ell_s \cos(\theta) + \frac{\ell_b}{2}\cos(\phi)\right)$$

Total Potential Energy:

$$V = V_1 + V_2 = \frac{1}{2}k\left(\ell_s - \ell_0\right)^2 - m_b g\left(\ell_s \cos(\theta) + \frac{\ell_b}{2}\cos(\phi)\right) \tag{2}$$

5   Constraints

The system is fully constrained, therefore no constraint equations were needed to solve the problem.

6 Solve for the Equations of Motion

The Lagrangian $\mathscr{L} \equiv T - V$. The equations of motion are a linear combination of the time derivative of the partial derivative of the Lagrangian with respect to the first derivative of the motion variable with respect to time, minus the partial derivative of the Lagrangian with respect to the variable of motion. These equations are set equal to the non-conservative forces in the system, $(Q_j)_{\text{non}}$ which in this particular case there are none, since there are no non-conservative forces acting (such as drag, applied forces, etc).

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathscr{L}}{\partial \theta} = \left(Q_\theta\right)_{\text{non}} \quad (3) \qquad \ddot{\phi} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathscr{L}}{\partial \phi} = \left(Q_\phi\right)_{\text{non}} \qquad (4)$$
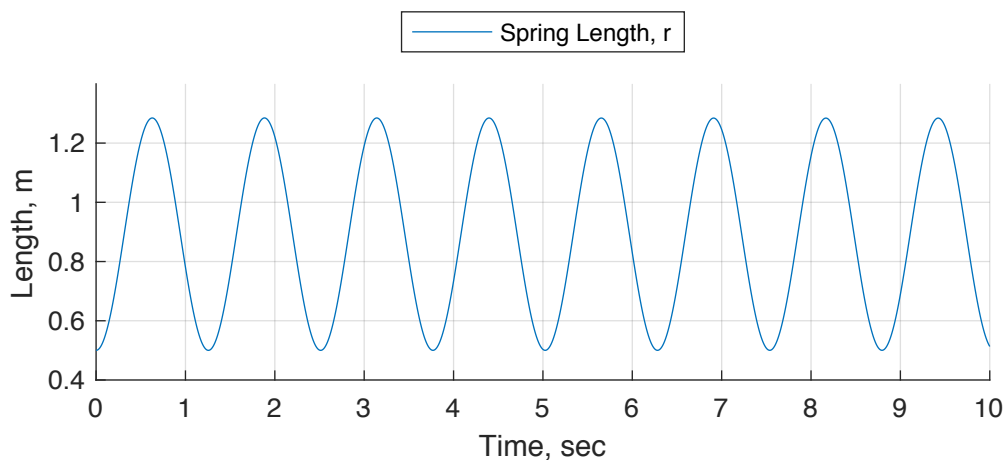
$$\ddot{\ell}_s = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\ell}_s}\right) - \frac{\partial \mathscr{L}}{\partial \ell_s} = \left(Q_{\ell_s}\right)_{\text{non}} \qquad (5)$$

$$\mathscr{L} = \frac{m_b}{2}\left[\left(\dot{\ell}_s \sin(\theta) + \frac{\ell_b \dot{\phi}\cos(\phi)}{2} + \ell_s \dot{\theta}\cos(\theta)\right)^2 + \left(\frac{\ell_b \dot{\phi}\sin(\phi)}{2} - \dot{\ell}_s \cos(\theta) + \ell_s \dot{\theta}\sin(\theta)\right)^2\right]$$

$$-\frac{k(\ell_0 - \ell_s)^2}{2} + \frac{\ell_b^2 m_b \dot{\phi}^2}{24} + gm_b\left(\frac{\ell_b \cos(\phi)}{2} + \ell_s \cos(\theta)\right) \qquad (6)$$

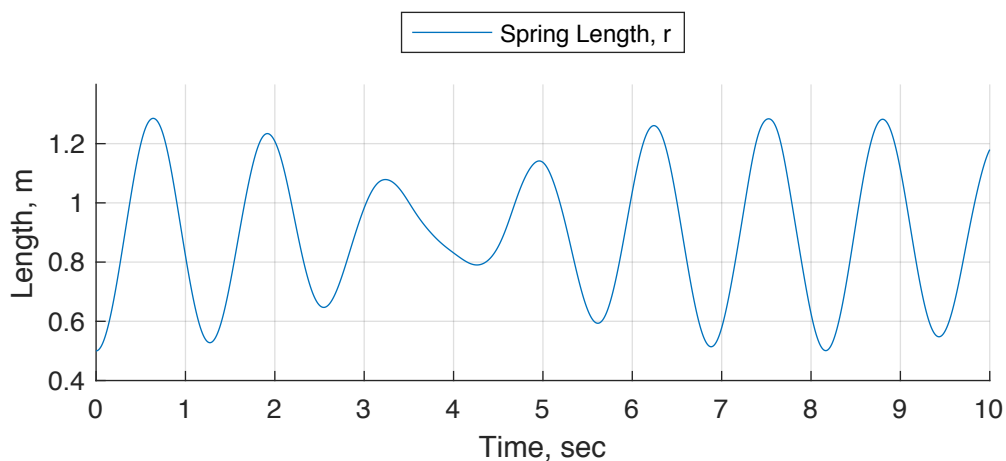Using Equations (3), (4), (5) and (6), we can solve for the three Equations of Motion.

7   Solve the Equations of Motion

Figure 3: Numerical Solution Motion Behavior Plot, $(\theta_o : 0, \ \phi_o : 0)$

(a) Spring Length vs. Time

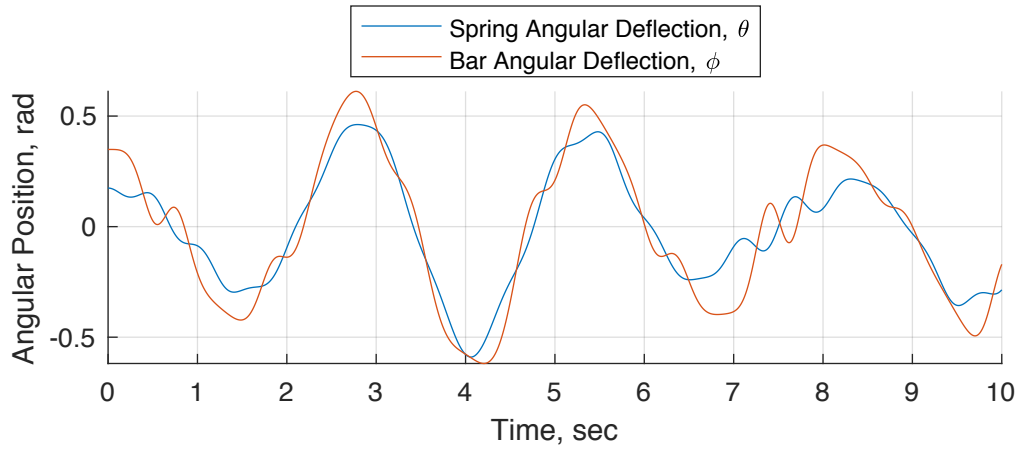Figure 4: Numerical Solution Motion Behavior Plots, $(\theta_o : \pi/18, \ \phi_o : \pi/9)$

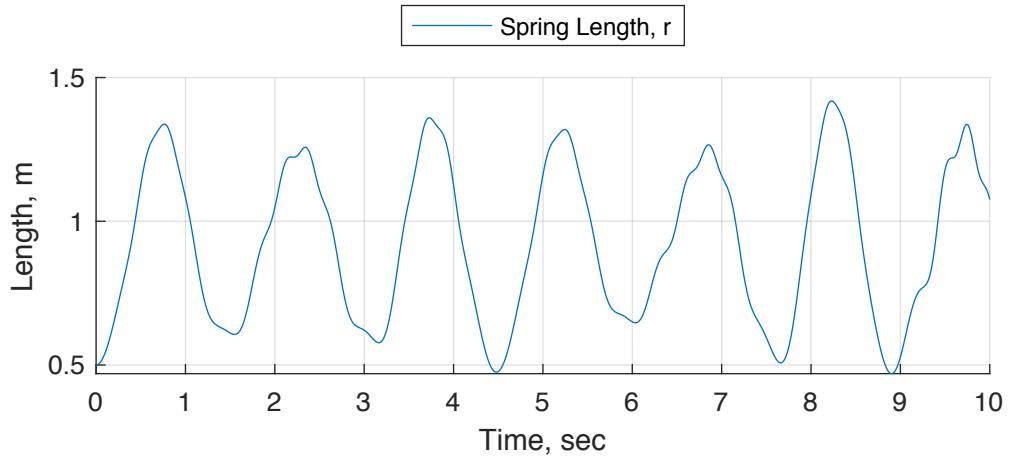(a) Spring Length vs. Time

8   Does it Make Sense?

8.1   Units

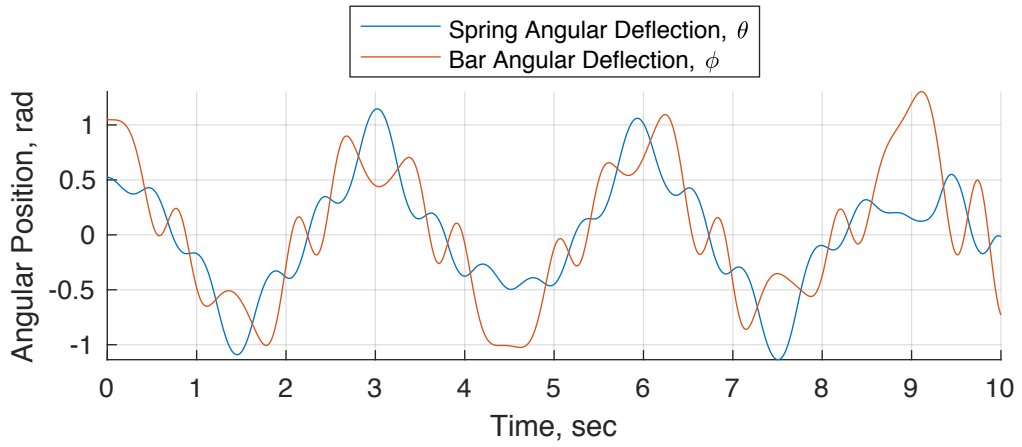Checking with the MATLAB symbolic units tool (from Section 9.3):

(b) Angular Position vs. Time

Figure 5: Numerical Solution Motion Behavior Plots, $(\theta_o : {}^\pi/_6, \ \phi_o : {}^\pi/_3)$

(a) Spring Length vs. Time

(b) Angular Position vs. Time

8.2 Magnitude

9 Appendix

9.1 Attributions

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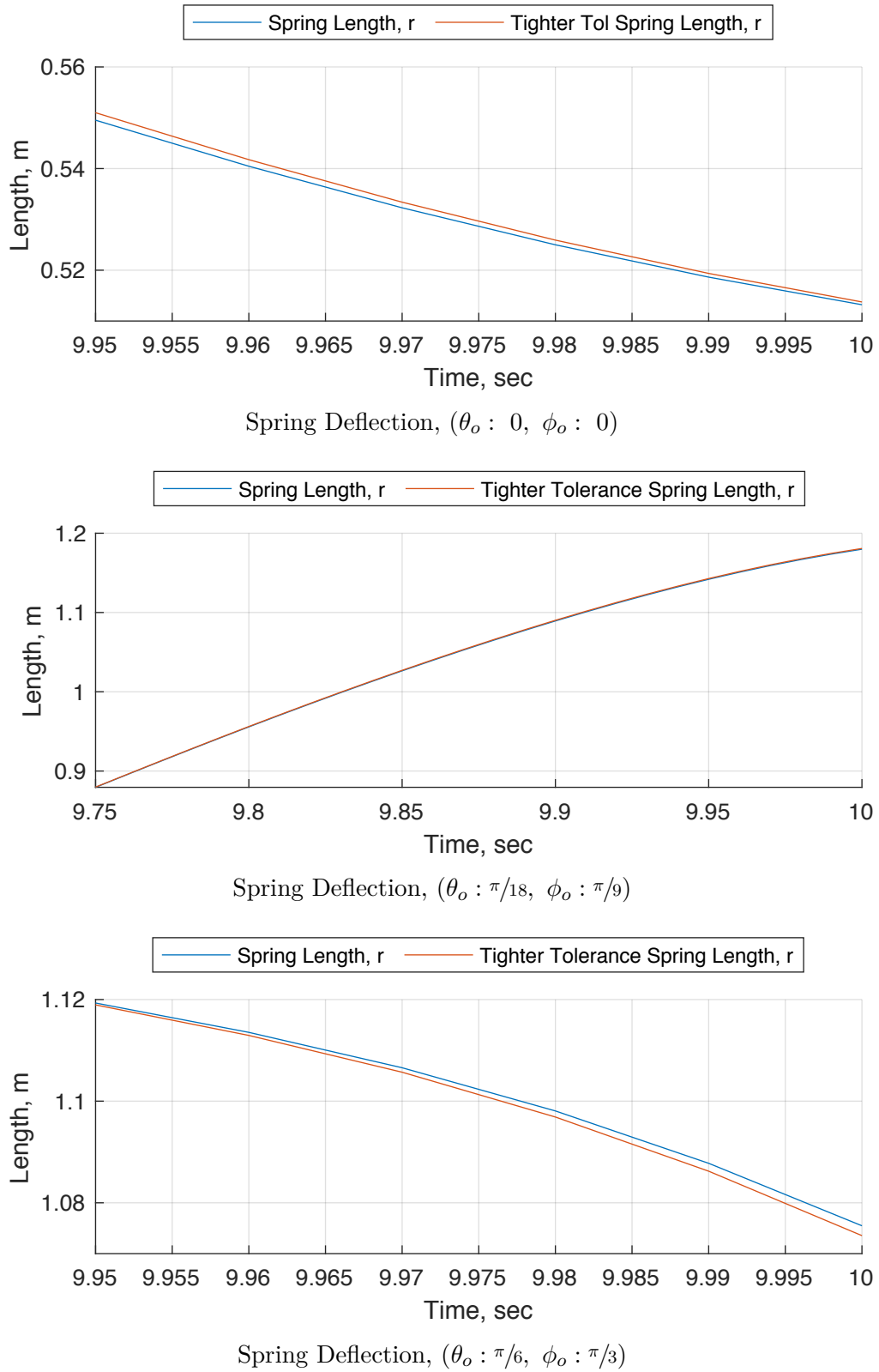Trey Dufrene
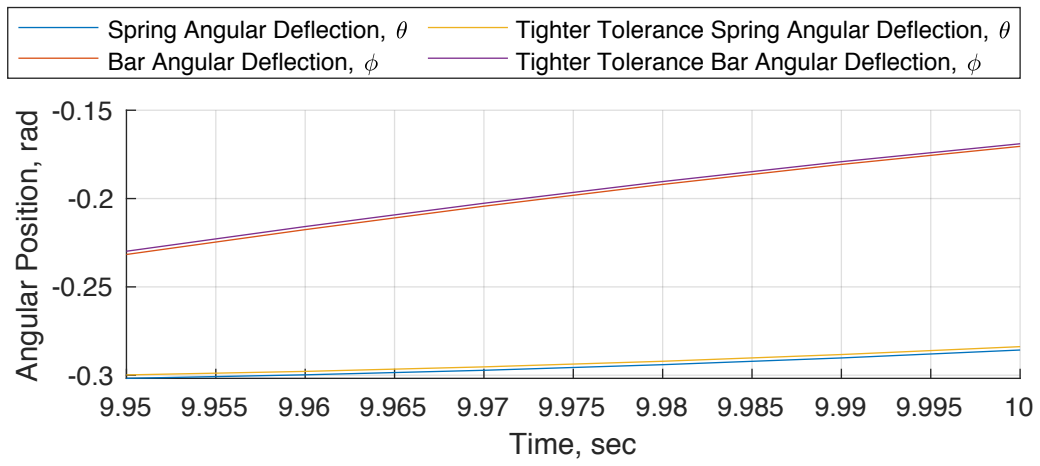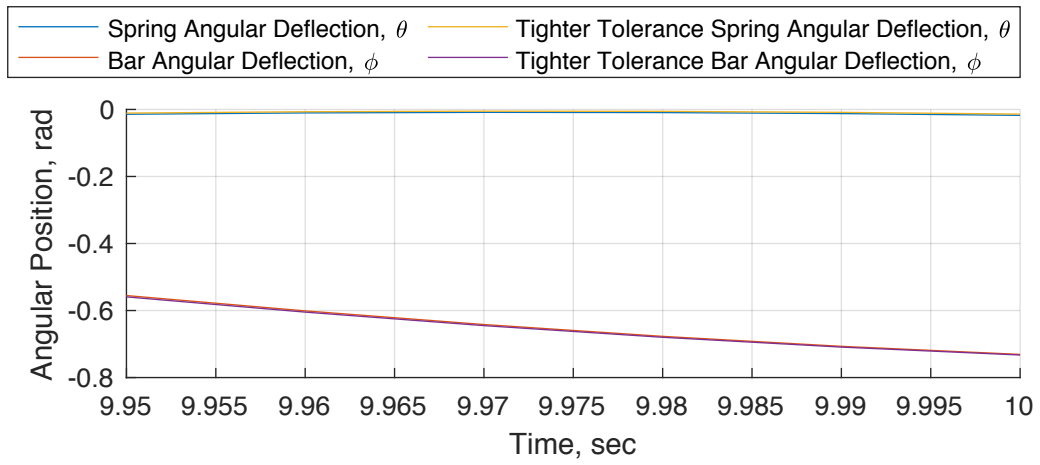Joint Effort

9.2 Analytical Solution

8

9.3 Numerical Solution

Figure 6: Comparison Plots of ode45 Tolerance Options

Spring Deflection, $(\theta_o : 0, \phi_o : 0)$

Spring Deflection, $(\theta_o : \pi/18, \phi_o : \pi/9)$

Spring Deflection, $(\theta_o : \pi/6, \phi_o : \pi/3)$

Bar and Spring Angular Deflection, $(\theta_o : \pi/18, \ \phi_o : \pi/9)$

Bar and Spring Angular Deflection, $(\theta_o : \pi/6, \ \phi_o : \pi/3)$