Lagrangian Dynamics Project

Rigid-Body Spring Pendulum

December 5, 2018

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Submitted to:
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In Partial Fulfillment of the Requirements of
ES204 Dynamics – Fall 2018

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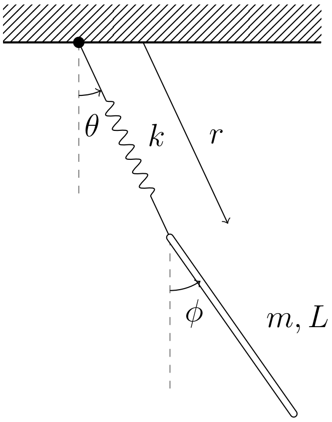
1 Conceptualize the Problem



The pendulum system consists of a rigid bar pinned to the free end of a linear spring, which rotates about its opposite end at a fixed point; there are three degrees of freedom, since the spring and bar each have an individual angular deflection with respect to the vertical, and the radial distance the bar is from the point of rotation due to the variation in the length of the spring.

1.1 Constants and Assumptions

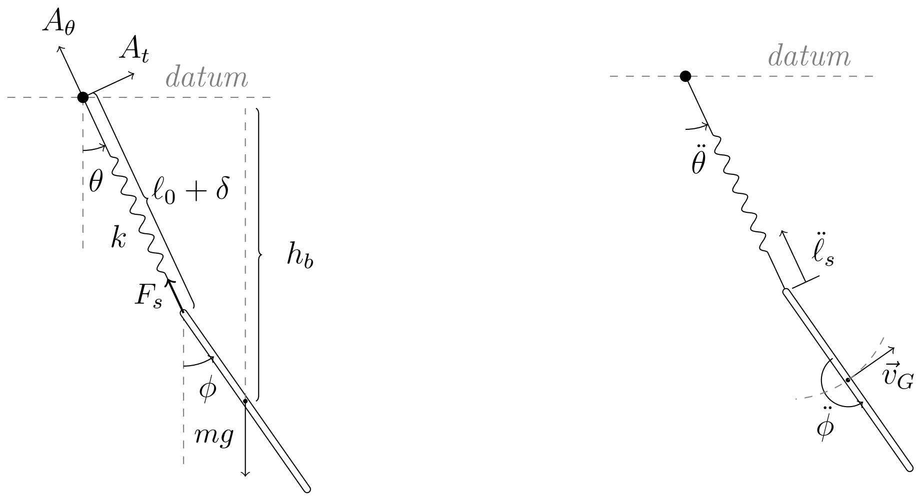
Constants:		Assumptions:
Bar Mass:	$m_b = 1$ kg	No Losses
Bar Length:	$\ell_b = 1$ m	Released from Rest
Gravity:	$g = 9.81$ m/s²	Uniform Slender Bar
Linear Spring:		Planar
Spring Coefficient:	$k = 25$ N/m	Rigid-Body Dynamics
Unstretched Length:	$\ell_0 = 0.5$ m	

We are asked to determine the following:

1. The Equations of Motion for the system via the Lagrangian method.
2. Integrate the Equations of Motion using various initial conditions and plot the behavior of the system for 10 seconds.
   (a) $\theta_o = 0\ rad,\quad \phi_o = 0\ rad$
   (b) $\theta_o = {}^{\pi}/_{18}\ rad,\quad \phi_o = {}^{\pi}/_{9}\ rad$
   (c) $\theta_o = {}^{\pi}/_{6}\ rad,\quad \phi_o = {}^{\pi}/_{3}\ rad$
3. Plot the total energy versus time for all 3 cases.
4. Repeat 2. and 3. using a '`RelTol`' of $1e-6$ and '`AbsTol`' of $1e-9$ for the `ode45` integration tolerances.

2 Free Body Diagram

Figure 1: Acceleration and Free Body Diagrams



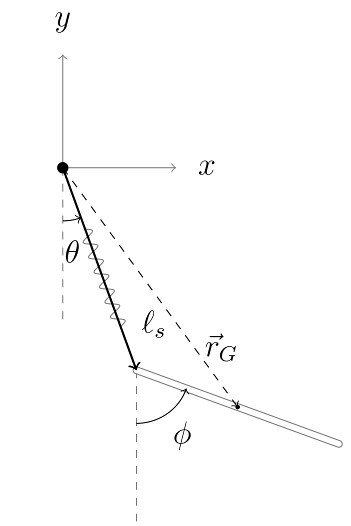
(a) Free Body Diagram

(b) Acceleration Diagram

G: Center of gravity of the bar
ℓ_0: Spring unstretched length
δ: Spring deflection
k: Spring constant
h_b: Distance to bar (G) from datum
F_s: Force onto bar due to spring
A_θ: Pin reaction in θ direction
A_t: Pin reaction in tangential direction

$\vec{v}_G$: Velocity of bar center of gravity
$\ddot{\theta}$: Angular velocity of spring
$\ddot{\phi}$: Angular velocity of bar
$\ddot{\ell}_s$: Radial acceleration of spring

3 Coordinate Frame



Motion Variables:
θ: Angle of spring relative to vertical
ϕ: Angle of bar relative to vertical
ℓ_s: Radial length of spring

Supplemental Variables:
$\vec{r}_G$: Vector to bar center of mass from origin

Figure 2: Coordinate Frame

4 Sum of Forces

From Figure (2),

$$\vec{r}_G = \left[\ell_s \sin(\theta) + \frac{\ell_b}{2}\sin(\phi)\right]\hat{\imath} + \left[-\ell_s\cos(\theta) - \frac{\ell_b}{2}\cos(\phi)\right]\hat{\jmath}$$

Taking the time derivative,

$$\frac{d}{dt}\vec{r}_G = \dot{\vec{r}}_G = \vec{v}_G$$

$$\vec{v}_G = \left[\dot{\ell}_s\sin(\theta) + \frac{\ell_b\dot{\phi}\cos(\phi)}{2} + \ell_s\dot{\theta}\cos(\theta)\right]\hat{\imath} + \left[\frac{\ell_b\dot{\phi}\sin(\phi)}{2} - \dot{\ell}_s\cos(\theta) + \ell_s\dot{\theta}\sin(\theta)\right]\hat{\jmath}$$

Kinetic Energy of Spring:

$$T_1 = 0$$

Kinetic Energy of Bar, due to it's rotational and translational velocity (Figure 1b):

$$T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{2}I\omega^2$$

Since the moment of inertia I for a uniform slender bar rotating about its end is $\frac{1}{12}m\ell^2$ and $\omega = \dot{\phi}$,

$$T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{24}m_b\ell_b^2\dot{\phi}^2$$

Total Kinetic Energy:

$$T = T_1 + T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{24}m_b\ell_b^2\dot{\phi}^2 \tag{1}$$

Potential Energy of Spring due to it's stretch (Figure 1a):

$$V_1 = \frac{1}{2}k\Big(\ell_s - \ell_0\Big)^2$$

Potential Energy of Bar, due to it's distance below the datum, h_b (Figure 1a):

$$V_2 = -m_b g\Big(\ell_s\cos(\theta) + \frac{\ell_b}{2}\cos(\phi)\Big)$$

Total Potential Energy:

$$V = V_1 + V_2 = \frac{1}{2}k\Big(\ell_s - \ell_0\Big)^2 - m_b g\Big(\ell_s\cos(\theta) + \frac{\ell_b}{2}\cos(\phi)\Big) \tag{2}$$

5 Knowns and Unknowns

Knowns:		Unknowns:	
Bar Mass:	$m_b = 0.25$ kg	Accelerations:	$\ddot{\theta},\ \ddot{\phi},\ \ddot{\ell}_s$
Bar Length:	$\ell_b = 1$ m		
Gravity:	$g = 9.81\ ^{\mathrm{m}}/_{\mathrm{s}^2}$		
Linear Spring:			
Spring Coefficient:	$k = 25\ ^{\mathrm{N}}/_{\mathrm{m}}$		
Unstretched Length:	$\ell_0 = 0.5$ m		
State Variables:			
Angular & Radial Positions:	$\theta,\ \phi, \ell_s$		
Angular & Radial Velocities:	$\dot{\theta},\ \dot{\phi},\ \dot{\ell}_s$		

Since there are six equations and six unknowns, we can solve for the equations of motion analytically using Matlab.

6 Constraints

The system is fully constrained, therefore no constraint equations were needed to solve the problem.

7 Solve for the Equations of Motion

The Lagrangian $\mathscr{L} \equiv T - V$. The equations of motion are a linear combination of the time derivative of the partial derivative of the Lagrangian with respect to the first derivative of the motion variable with respect to time, minus the partial derivative of the Lagrangian with respect to the variable of motion. These equations are set equal to the non-conservative forces in the system, $(Q_j)_{\text{non}}$ which in this particular case there are none, since there are no non-conservative forces acting (such as drag, applied forces, etc).

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathscr{L}}{\partial \theta} = \Big(Q_\theta\Big)_{\text{non}} \quad (3) \qquad \ddot{\phi} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathscr{L}}{\partial \phi} = \Big(Q_\phi\Big)_{\text{non}} \quad (4)$$

$$\ddot{\ell}_s = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\ell}_s}\right) - \frac{\partial \mathscr{L}}{\partial \ell_s} = \Big(Q_{\ell_s}\Big)_{\text{non}} \tag{5}$$

$$\mathscr{L} = \frac{m_b}{2}\left[\left(\dot{\ell}_s \sin(\theta) + \frac{\ell_b \dot{\phi}\cos(\phi)}{2} + \ell_s \dot{\theta}\cos(\theta)\right)^2 + \left(\frac{\ell_b \dot{\phi}\sin(\phi)}{2} - \dot{\ell}_s \cos(\theta) + \ell_s \dot{\theta}\sin(\theta)\right)^2\right] - \frac{k(\ell_0 - \ell_s)^2}{2} + \frac{\ell_b^2 m_b \dot{\phi}^2}{24} + g m_b\left(\frac{\ell_b \cos(\phi)}{2} + \ell_s \cos(\theta)\right) \tag{6}$$

Using Equations (3), (4), (5) and (6), we can solve for the three Equations of Motion with Matlab. The code used to find the EOM for $\ddot{\theta}$ is shown below.

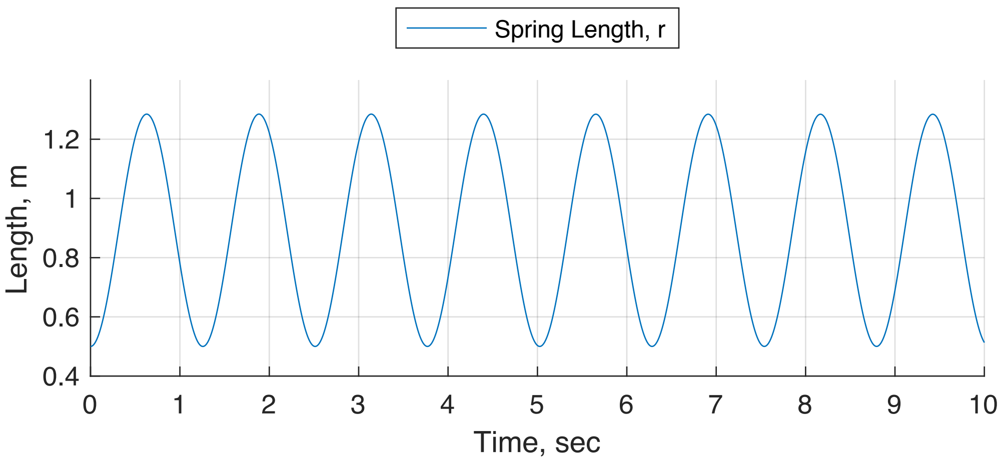
```
Lagrangian = T - V;

% Partial of Lagrange Eq. w.r.t. thetadot
dLdthetadot = diff(Lagrangian,thetadot);
dLdthetadot_subbed = subs(dLdthetadot, [thetat, thetadot, phit, phidot,
    l1t, l1dot],...
     [theta, diff(theta,t), phi, diff(phi,t), l1, diff(l1,t)]);
% Time derivative of Partial of Lagrange Eq. w.r.t. thetadot
ddLdthetadotdt = diff(dLdthetadot_subbed,t);
% Partial of Lagrange Eq. w.r.t. theta
dLdtheta = diff(Lagrangian,thetat);

% Theta EOM
eqn(1) = ddLdthetadotdt - dLdtheta == 0;
```

Similarly, the procedure used to solve for the other Equations of Motion ($\ddot{\phi}$ and $\ddot{\ell}_s$) can be found in the Appendix, in both the Numerical and Analytic Solution (10.2). Once each EOM was found, the numerical solution was created by using the `ode45` function in Matlab, as seen in Solve the Equations of Motion (8).

8 Solve the Equations of Motion

After solving for each respective EOM, we can plot the solution and depict the behavior of the system for 10 seconds. The behavior being that of the radial distance that the spring stretches as well as the spring's angular deflection from vertical. The linear deflection from the stretch of the spring is plotted on it's own figure, and the angular deflection of the spring and bar from vertical are each plotted on one figure for each set of initial conditions.

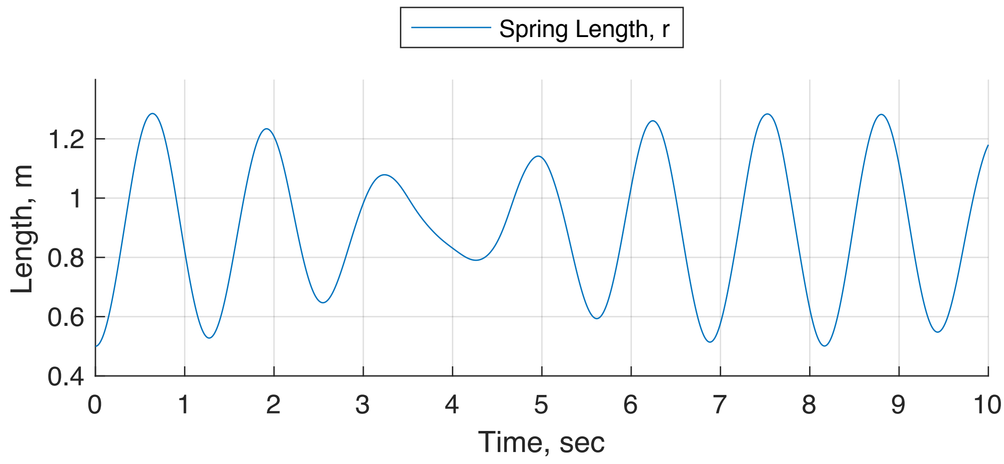
Figure 3: Numerical Solution Motion Behavior Plot, ($\theta_o:\ 0,\ \phi_o:\ 0$)



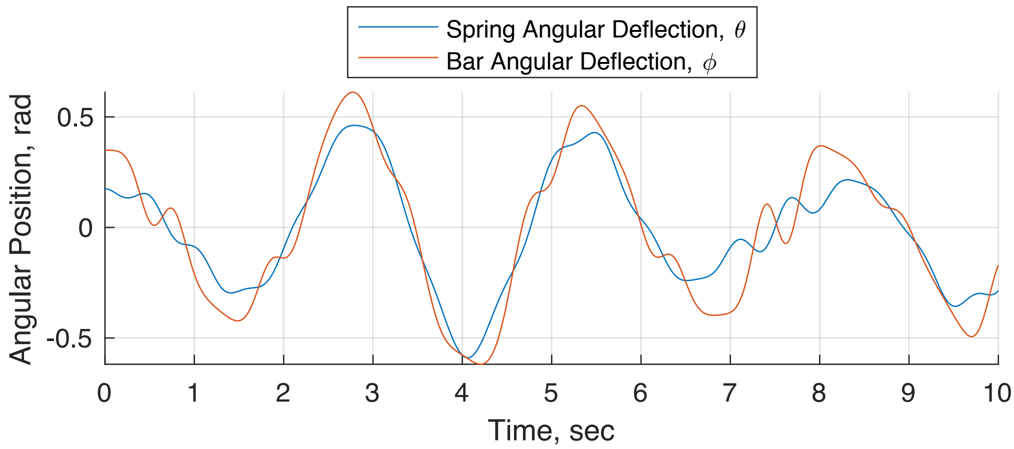
Spring Length vs. Time

The first configuration consists of the system being released from rest with no initial angular deflection, and the spring at it's unstretched initial length; therefore this configuration has no change in angular position, and simply begins to oscillate from the repeated stretching of the spring.

Figure 4: Numerical Solution Motion Behavior Plot, (θ_o : $^{\pi}/_{18}$, ϕ_o : $^{\pi}/_{9}$)



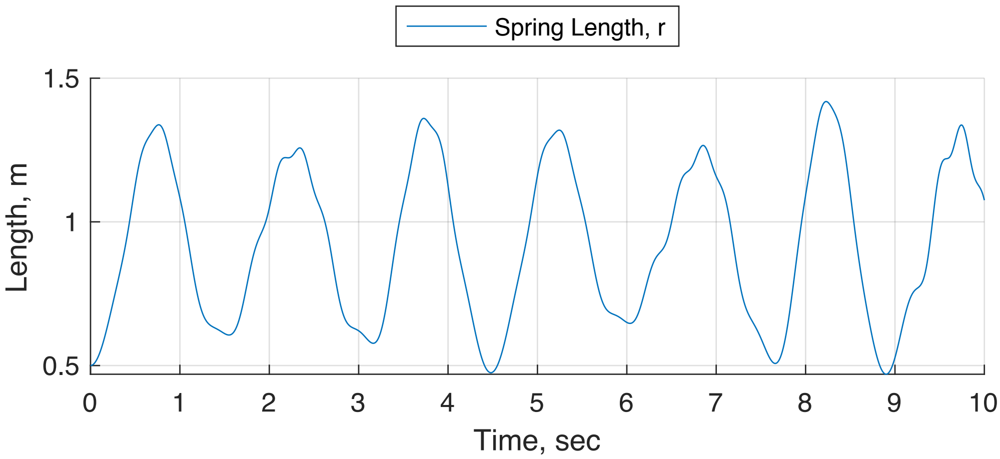
(a) Spring Length vs. Time



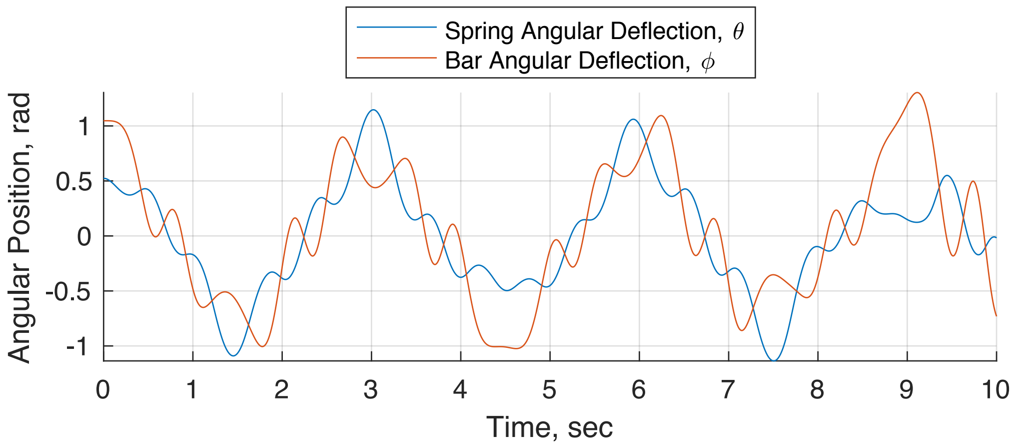
(b) Angular Position vs. Time

Figures (4a) and (4b) depict the behavior of the spring length as well as angular deflection of the spring and bar, given the initial conditions shown. Since the overall behavior in Figure (4b) is relatively consistent (compared to that of Figure (5b) below), it makes sense that the spring deflection plot depicts mostly consistent oscillatory motion – since the amplitude in the oscillations of the pendulum are smaller, there is less energy being "transferred," so-to-speak, from the kinetic energy of the bar (force outward from the spring, dependent on it's velocity and acceleration) to or from the energy in the spring. This phenomena can also be seen in Figure (5a), where the amplitudes of oscillation in the spring deflection are larger relative to that of this situation, due to the fact that the inital conditions in the case of Figure (5a) are of a greater angular deflection. Given that the inital release angle of the spring and bar are small (less than 45^o), the overall inital potential energy of the system is lower, subsequently confirming the forementioned idea that there is less overall energy in the system, as represented by the generally smaller amplitude in oscillations.

Figure 5: Numerical Solution Motion Behavior Plot, (θ_o : $^{\pi}/_{6}$, ϕ_o : $^{\pi}/_{3}$)



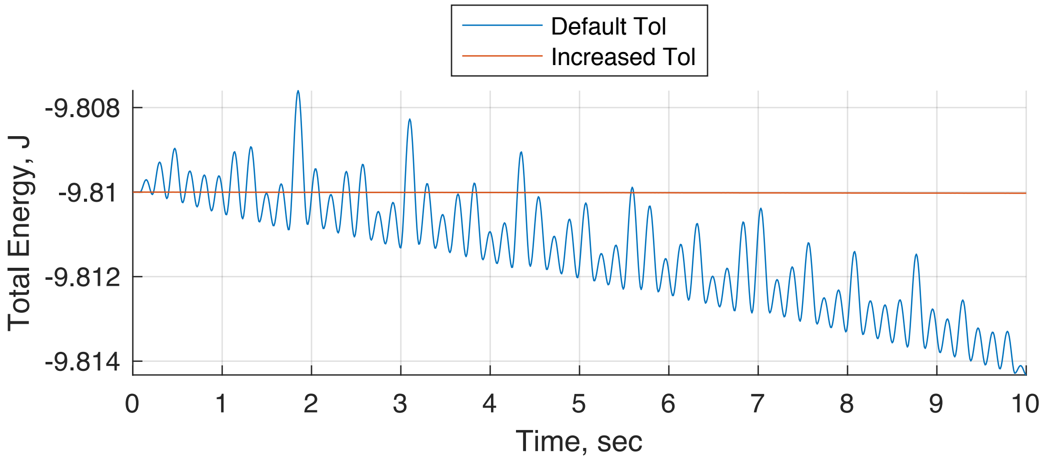
(a) Spring Length vs. Time



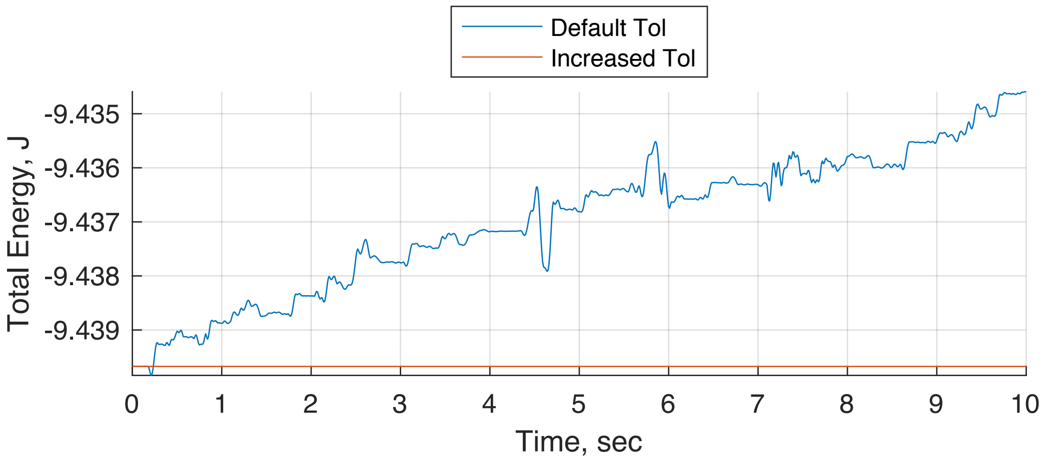
(b) Angular Position vs. Time

Figures (5a) and (5b) depict the behavior of the system when released from larger inital angular deflections, as discussed previously, introducing a larger initial potential energy, resulting in larger amplitudes of oscillation – 1 radian peak-to-peak in Figure (4b) compared to about 2 radians peak-to-peak in Figure (5b); similarly, the magnitude of the spring deflection is larger, going from a consistent 1.3 meters in Figure (4a) to almost 1.5 meters in Figure (5a).

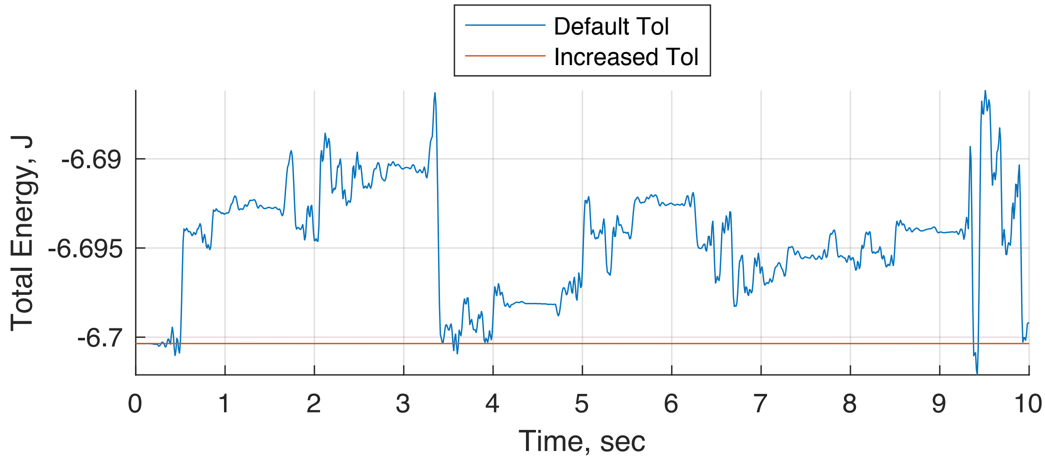
Figure 6: Total Energy Comparison Plots



(a) Total Energy in the System ($\theta_o : 0,\ \phi_o : 0$)



(b) Total Energy in the System ($\theta_o :\ ^{\pi}/_{18},\ \phi_o :\ ^{\pi}/_{9}$)



(c) Total Energy in the System (θ_o : $^{\pi}/_{6}$, ϕ_o : $^{\pi}/_{3}$)

Running the `ode45` function with a relative tolerance of $1 * 10^{-6}$ and an absolute tolerance of $1 * 10^{-9}$ we produce a total energy graph that strays significantly less than the default `ode45` implementation. The max percentage difference between the high and low tolerance integrations in Figure (6a) is -0.0245%. The max percentage difference in Figure (6b) is -0.0538%. The max percentage difference in Figure (6c) is -0.2116%.

9 Does it Make Sense?

9.1 Units

Checking with the MATLAB symbolic units tool (from Section 10.2):

```
% Checking EOM Units
u = symunit;
m2 = m2*u.kg;
k = k*u.N/u.m;
L0 = L0*u.m;
l2 = l2*u.m;
g = g*u.m/u.s^2;
thetat = 'thetat';
thetadot = 'thetadot'/u.s;
thetaddot = 'thetaddot'/u.s^2;
phit = 'phit';
phidot = 'phidot'/u.s;
phiddot = 'phiddot'/u.s^2;
l1t = 'l1t'*u.m;
l1dot = 'l1dot'*u.m/u.s;
l1ddot = 'l1ddot'*u.m/u.s^2;

eqn = subs(eqn)
unitCheck = checkUnits(eqn)
```

```
unitCheck =
  struct with fields:
    Consistent: [1 1 1]
    Compatible: [1 1 1]
```

9.2 Magnitude

Examining the figures we can see that the magnitude of spring length never exceeds the max length achieved in the first period. This indicates that no energy is added in the spring oscillation, which is expected due to the fact that we have no external forces acting on the spring-bar system. Similarly, figures (6a), (6b), and (6c) all indicate that the total energy within the system remains essentially constant. There is a minor drift in the total energy on the magnitude of $10^{-3}J$ due to the loose tolerances in the `ode45` integration function. With the more highly toleranced integrations the energy drift is on the magnitude of 10^{-6}. This low change in energy relative to the total energy of the system indicates that the conservation of energy principle holds true in our system that lacks non-conservative forces.

10 Appendix

10.1 Analytical Solution

Project 3

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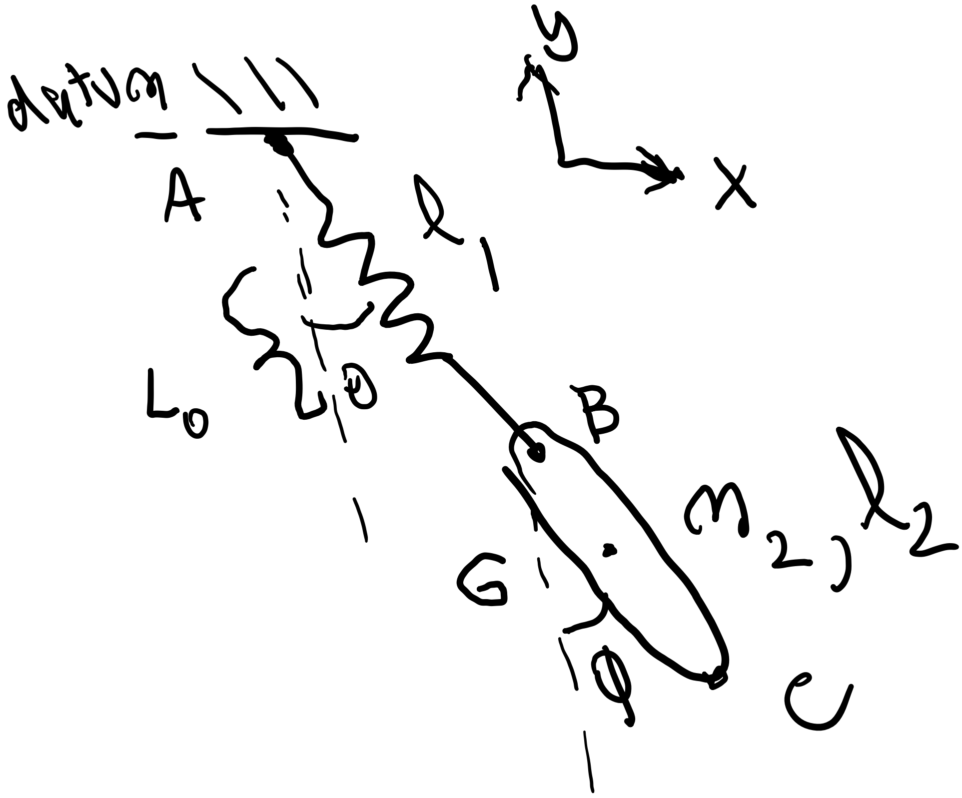


$\underline{KE = T}$

$T_1 = 0$

$T_2 = \frac{1}{2} m V_G^2 + \frac{1}{2} I_{G_2} \dot{\phi}^2 = \frac{1}{2} m V_{G_2}^2 + \frac{1}{24} m_2 l_2 \dot{\phi}^2$

$\vec{r}_{G_2} = \vec{r}_{AG} = (l_1 S_\theta + \frac{l_2}{2} S_\phi)\hat{i} + (-l_1 C_\theta - \frac{l_2}{2} C_\phi)\hat{j}$

$\vec{V}_{G_2} = \dot{\vec{r}}_G = [l_1 C_\theta \dot{\theta} + \frac{l_2}{2} C_\phi \dot{\phi}]\hat{i} + (l_1 S_\theta \dot{\theta} + \frac{l}{2} S_\phi \dot{\phi})\hat{j}$

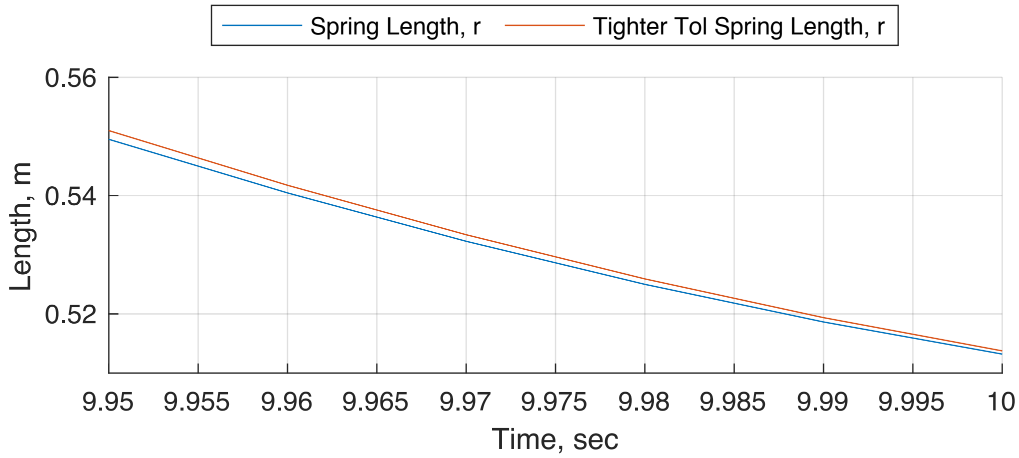
$\vec{V}_{G_2}^2 = l_1^2 \dot{\theta}^2 + \frac{1}{4} l_2^2 \dot{\phi}^2 + l_1 l_2 \dot{\theta} \dot{\phi} C_{(\theta-\phi)}$

$T_2 = \frac{1}{2} m_2 l_1^2 \dot{\theta}^2 + \frac{1}{6} m_2 l_2^2 \dot{\phi}^2 + \frac{1}{2} m_2 l_1 l_2 C_{(\theta-\phi)} \dot{\theta} \dot{\phi}$

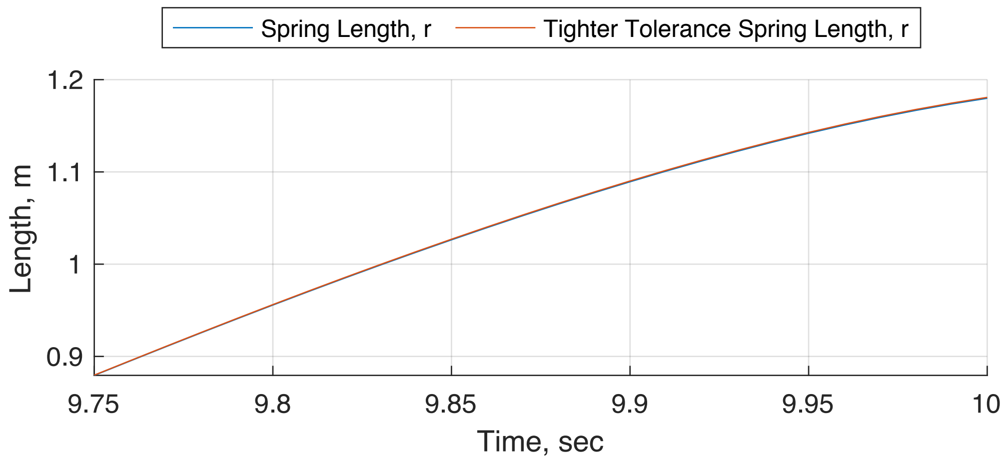
$T = T_1 + T_2 = 0 + T_2 = T_2$

10.2 Numerical Solution

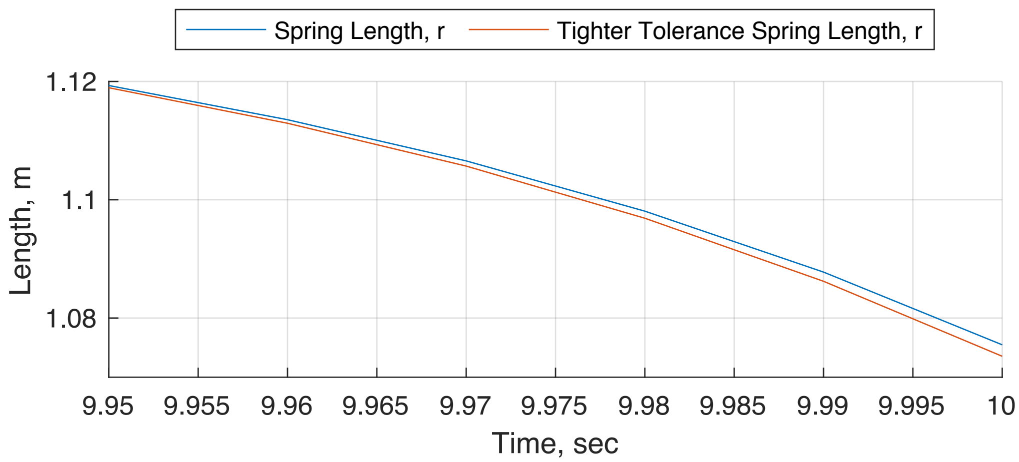
Figure 7: Comparison Plots of ode45 Tolerance Options



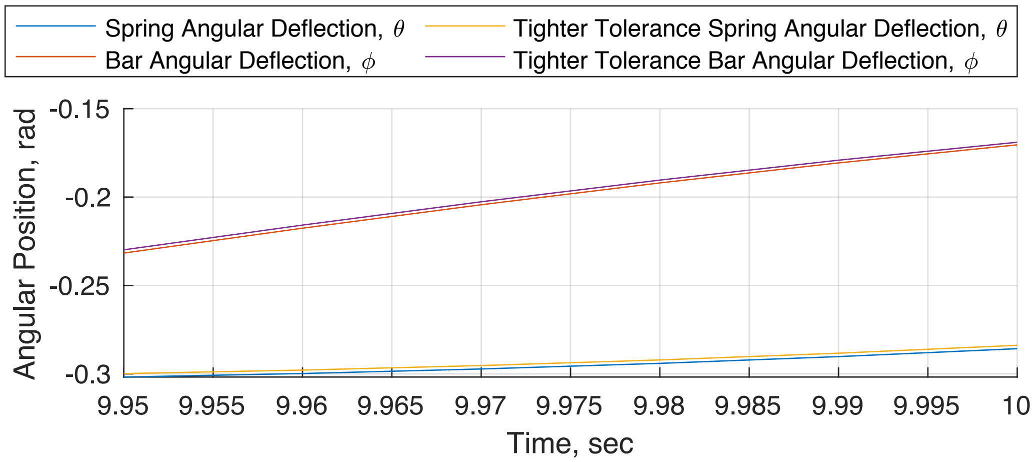
Spring Deflection, $(\theta_o :\ 0,\ \phi_o :\ 0)$



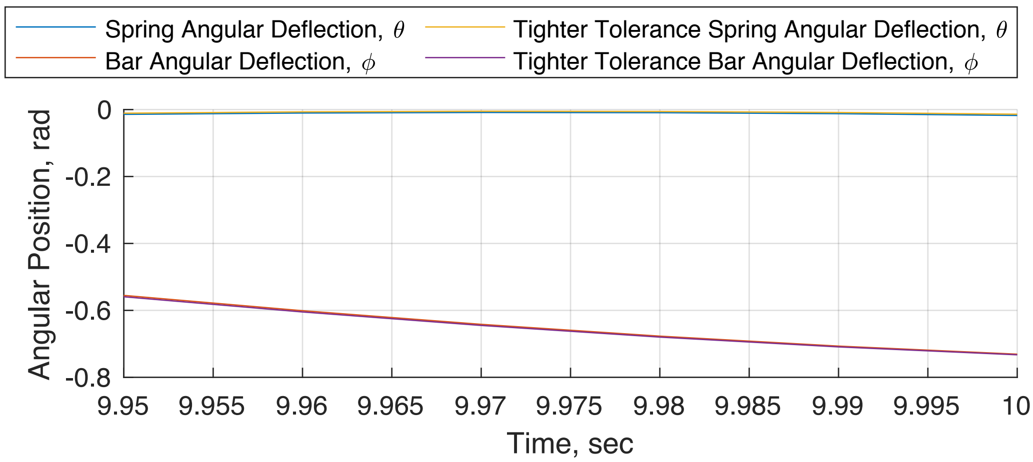
Spring Deflection, $(\theta_o : \pi/18,\ \phi_o : \pi/9)$



Spring Deflection, $(\theta_o : \pi/6,\ \phi_o : \pi/3)$



Bar and Spring Angular Deflection, $(\theta_o : {}^{\pi}/_{18},\ \phi_o : {}^{\pi}/_{9})$



Bar and Spring Angular Deflection, $(\theta_o : {}^{\pi}/_{6},\ \phi_o : {}^{\pi}/_{3})$