Lagrangian Dynamics Project

Rigid-Body Spring Pendulum

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By:

Jeffrey Chen | Thorne Wolfenbarger | Trey Dufrene

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Dr. Mark Sensmeier
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ES204 Dynamics – Fall 2018

College of Engineering
Embry-Riddle Aeronautical University
Prescott, AZ

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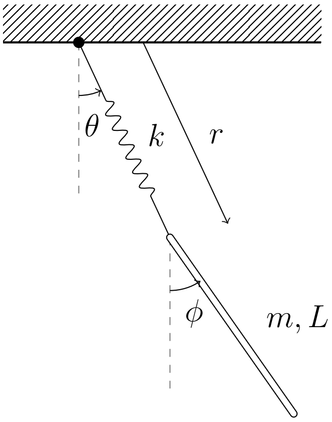
1 Conceptualize the Problem



The pendulum system consists of a rigid bar pinned to the free end of a linear spring, which rotates about its opposite end at a fixed point; there are three degrees of freedom, since the spring and bar each have an individual angular deflection with respect to the vertical, and the radial distance the bar is from the point of rotation due to the variation in the length of the spring.

1.1 Constants and Assumptions

Constants:		Assumptions:
Bar Mass:	$m_b = 1$ kg	No Losses
Bar Length:	$\ell_b = 1$ m	Released from Rest
Gravity:	$g = 9.81$ m/s²	Uniform Slender Bar
Linear Spring:		Planar
Spring Coefficient:	$k = 25$ N/m	Rigid-Body Dynamics
Unstretched Length:	$\ell_0 = 0.5$ m	

We are asked to determine the following:

1. The Equations of Motion for the system via the Lagrangian method.

2. Integrate the Equations of Motion using various initial conditions and plot the behavior of the system for 10 seconds.

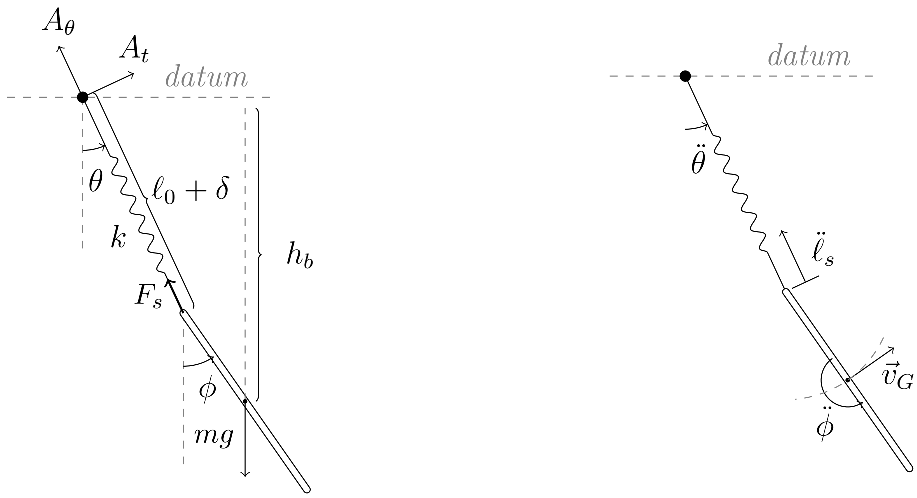
   (a) $\theta_o = 0\ rad,\quad \phi_o = 0\ rad$
   (b) $\theta_o = \pi/18\ rad,\quad \phi_o = \pi/9\ rad$
   (c) $\theta_o = \pi/6\ rad,\quad \phi_o = \pi/3\ rad$

3. Plot the total energy versus time for all 3 cases.

4. Repeat 2. and 3. using a '`RelTol`' of $1e-6$ and '`AbsTol`' of $1e-9$ for the `ode45` integration tolerances.

2 Free Body Diagram

Figure 1: Acceleration and Free Body Diagrams



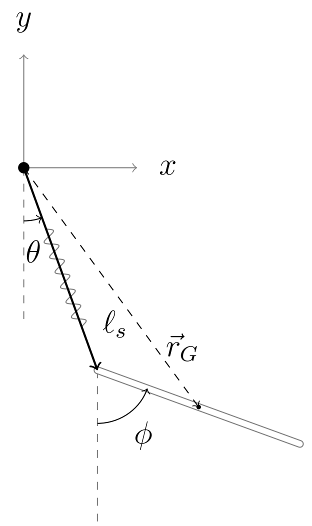
(a) Free Body Diagram

(b) Acceleration Diagram

G: Center of gravity of the bar
ℓ_0: Spring unstretched length
δ: Spring deflection
k: Spring constant
h_b: Distance to bar (G) from datum
F_s: Force onto bar due to spring
A_θ: Pin reaction in θ direction
A_t: Pin reaction in tangential direction

$\vec{v}_G$: Velocity of bar center of gravity
$\ddot{\theta}$: Angular velocity of spring
$\ddot{\phi}$: Angular velocity of bar
$\ddot{\ell}_s$: Radial acceleration of spring

3 Coordinate Frame



Motion Variables:
θ: Angle of spring relative to vertical
ϕ: Angle of bar relative to vertical
ℓ_s: Radial length of spring

Supplemental Variables:
$\vec{r}_G$: Vector to bar center of mass from origin

Figure 2: Coordinate Frame

4 Sum of Forces

From Figure (2),

$$\vec{r}_G = \left[\ell_s \sin(\theta) + \frac{\ell_b}{2}\sin(\phi)\right]\hat{\imath} + \left[-\ell_s \cos(\theta) - \frac{\ell_b}{2}\cos(\phi)\right]\hat{\jmath}$$

Taking the time derivative,

$$\frac{d}{dt}\vec{r}_G = \dot{\vec{r}}_G = \vec{v}_G$$

$$\vec{v}_G = \left[\dot{\ell}_s \sin(\theta) + \frac{\ell_b\dot{\phi}\cos(\phi)}{2} + \ell_s\dot{\theta}\cos(\theta)\right]\hat{\imath} + \left[\frac{\ell_b\dot{\phi}\sin(\phi)}{2} - \dot{\ell}_s\cos(\theta) + \ell_s\dot{\theta}\sin(\theta)\right]\hat{\jmath}$$

Kinetic Energy of Spring:

$$T_1 = 0$$

Kinetic Energy of Bar, due to it's rotational and translational velocity (Figure 1b):

$$T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{2}I\omega^2$$

Since the moment of inertia I for a uniform slender bar rotating about its end is $\frac{1}{12}m\ell^2$ and $\omega = \dot{\phi}$,

$$T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{24}m_b\ell_b^2\dot{\phi}^2$$

Total Kinetic Energy:

$$T = T_1 + T_2 = \frac{1}{2}m_b(\vec{v}_G \cdot \vec{v}_G) + \frac{1}{24}m_b\ell_b^2\dot{\phi}^2 \tag{1}$$

Potential Energy of Spring due to it's stretch (Figure 1a):

$$V_1 = \frac{1}{2}k\Big(\ell_s - \ell_0\Big)^2$$

Potential Energy of Bar, due to it's distance below the datum, h_b (Figure 1a):

$$V_2 = -m_b g\Big(\ell_s\cos(\theta) + \frac{\ell_b}{2}\cos(\phi)\Big)$$

Total Potential Energy:

$$V = V_1 + V_2 = \frac{1}{2}k\Big(\ell_s - \ell_0\Big)^2 - m_b g\Big(\ell_s\cos(\theta) + \frac{\ell_b}{2}\cos(\phi)\Big) \tag{2}$$

5 Knowns and Unknowns

Knowns:		Unknowns:	
Bar Mass:	$m_b = 0.25$ kg	Accelerations:	$\ddot{\theta},\ \ddot{\phi},\ \ddot{\ell}_s$
Bar Length:	$\ell_b = 1$ m		
Gravity:	$g = 9.81$ m/s²		
Linear Spring:			
Spring Coefficient:	$k = 25$ N/m		
Unstretched Length:	$\ell_0 = 0.5$ m		
State Variables:			
Angular & Radial Positions:	$\theta,\ \phi, \ell_s$		
Angular & Radial Velocities:	$\dot{\theta},\ \dot{\phi},\ \dot{\ell}_s$		

Since there are six equations and six unknowns, we can solve for the equations of motion analytically using Matlab.

6 Constraints

The system is fully constrained, therefore no constraint equations were needed to solve the problem.

7 Solve for the Equations of Motion

The Lagrangian $\mathscr{L} \equiv T - V$. The equations of motion are a linear combination of the time derivative of the partial derivative of the Lagrangian with respect to the first derivative of the motion variable with respect to time, minus the partial derivative of the Lagrangian with respect to the variable of motion. These equations are set equal to the non-conservative forces in the system, $(Q_j)_{\text{non}}$ which in this particular case there are none, since there are no non-conservative forces acting (such as drag, applied forces, etc).

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathscr{L}}{\partial \theta} = \left(Q_\theta\right)_{\text{non}} \quad (3) \qquad \ddot{\phi} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathscr{L}}{\partial \phi} = \left(Q_\phi\right)_{\text{non}} \quad (4)$$

$$\ddot{\ell}_s = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\ell}_s}\right) - \frac{\partial \mathscr{L}}{\partial \ell_s} = \left(Q_{\ell_s}\right)_{\text{non}} \quad (5)$$

$$\mathscr{L} = \frac{m_b}{2}\left[\left(\dot{\ell}_s \sin(\theta) + \frac{\ell_b \dot{\phi}\cos(\phi)}{2} + \ell_s \dot{\theta}\cos(\theta)\right)^2 + \left(\frac{\ell_b \dot{\phi}\sin(\phi)}{2} - \dot{\ell}_s \cos(\theta) + \ell_s \dot{\theta}\sin(\theta)\right)^2\right] - \frac{k(\ell_0 - \ell_s)^2}{2} + \frac{\ell_b^2 m_b \dot{\phi}^2}{24} + g m_b\left(\frac{\ell_b \cos(\phi)}{2} + \ell_s \cos(\theta)\right) \quad (6)$$

Using Equations (3), (4), (5) and (6), we can solve for the three Equations of Motion with Matlab. The code used to find the EOM for $\ddot{\theta}$ is shown below.

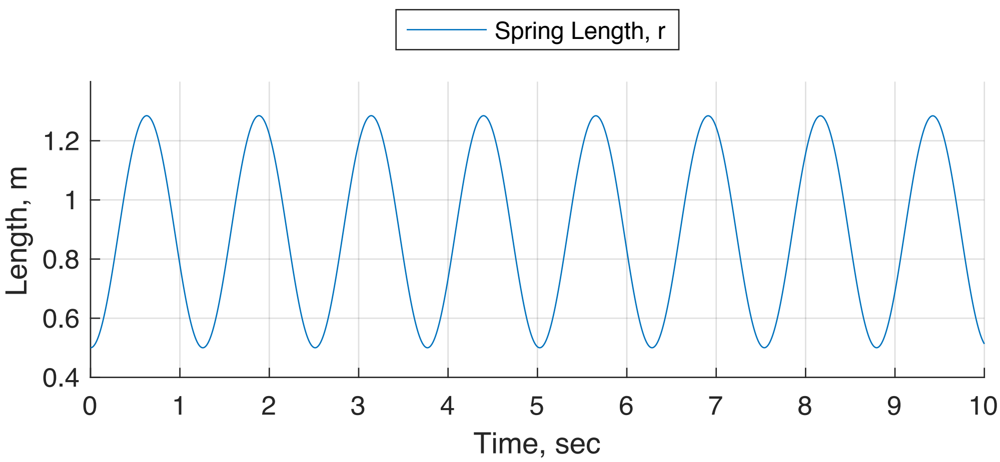
```
Lagrangian = T - V;

% Partial of Lagrange Eq. w.r.t. thetadot
dLdthetadot = diff(Lagrangian,thetadot);
dLdthetadot_subbed = subs(dLdthetadot, [thetat, thetadot, phit, phidot,
    l1t, l1dot],...
     [theta, diff(theta,t), phi, diff(phi,t), l1, diff(l1,t)]);
% Time derivative of Partial of Lagrange Eq. w.r.t. thetadot
ddLdthetadotdt = diff(dLdthetadot_subbed,t);
% Partial of Lagrange Eq. w.r.t. theta
dLdtheta = diff(Lagrangian,thetat);

% Theta EOM
eqn(1) = ddLdthetadotdt - dLdtheta == 0;
```

Similarly, the procedure used to solve for the other Equations of Motion ($\ddot{\phi}$ and $\ddot{\ell}_s$) can be found in the Appendix – Numerical Solution (10.3). Once each EOM was found, the numerical solution was created by using the `ode45` function in Matlab, as seen in Solve the Equations of Motion (8).

8 Solve the Equations of Motion

After solving for each respective EOM, we can plot the solution and depict the behavior of the system for 10 seconds. The behavior being that of the radial distance that the spring stretches as well as the spring's angular deflection from vertical, in addition to the angular deflection of the bar from vertical. The linear deflection from the stretch of the spring is plotted on it's own figure, and the angular deflection of the spring and bar from vertical are each plotted on one figure for each set of initial conditions.

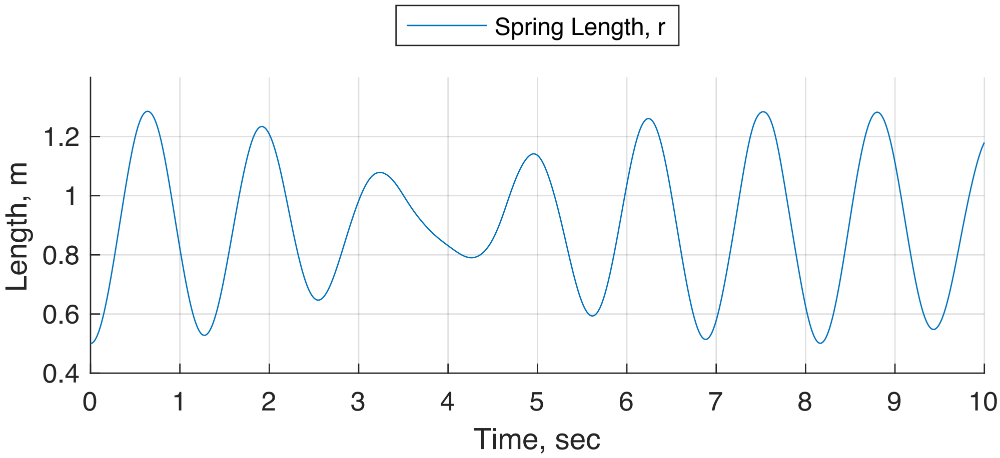
Figure 3: Numerical Solution Motion Behavior Plot, ($\theta_o :\ 0,\ \phi_o :\ 0$)



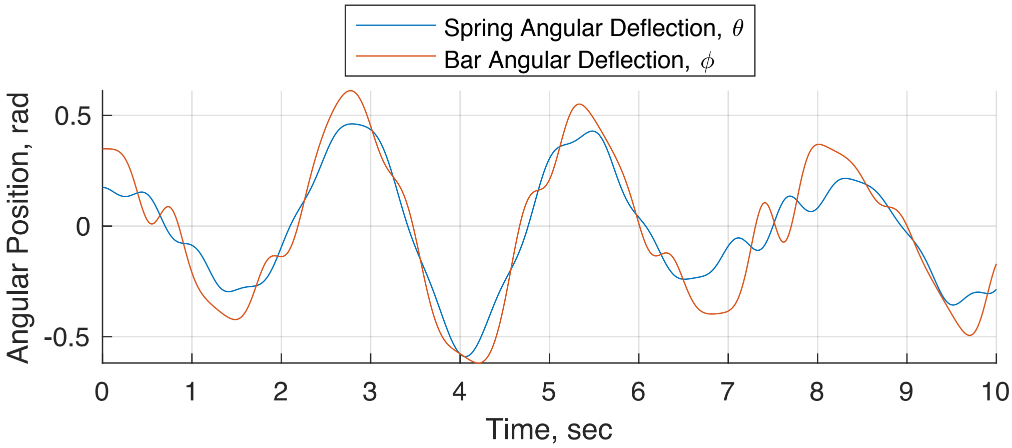
Spring Length vs. Time

The first configuration consists of the system being released from rest with no initial angular deflection, and the spring at it's unstretched initial length; this configuration therefore has no change in angular position, and simply begins to oscillate from the repeated stretching of the spring.

Figure 4: Numerical Solution Motion Behavior Plot, ($\theta_o : \, ^{\pi}/_{18}, \; \phi_o : \, ^{\pi}/_{9}$)

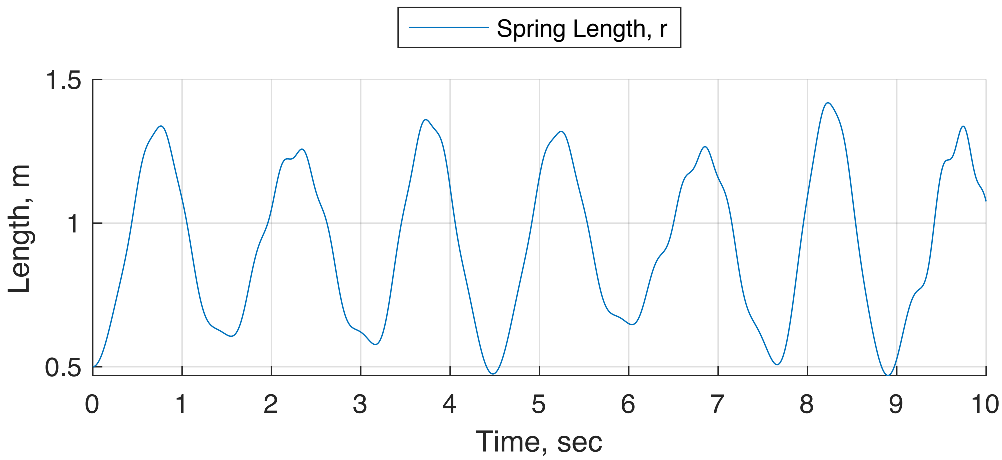


(a) Spring Length vs. Time

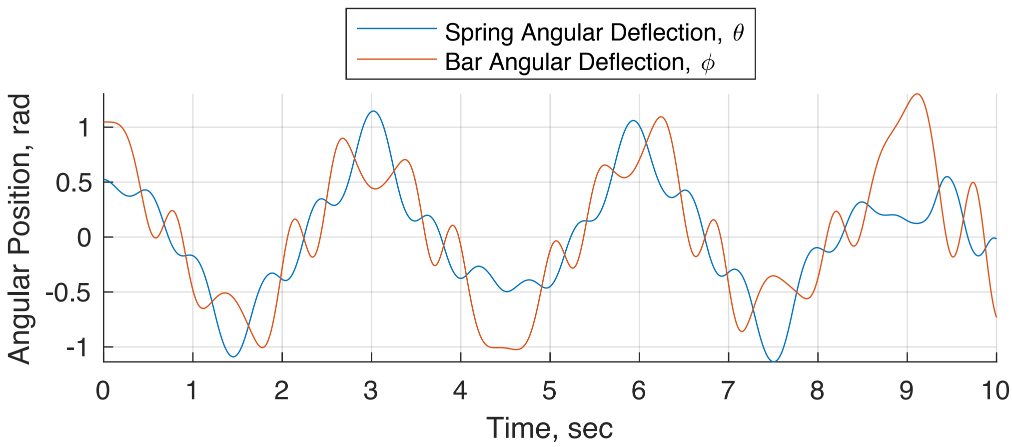


(b) Angular Position vs. Time

Figure 5: Numerical Solution Motion Behavior Plot, (θ_o : $^{\pi}/_{6}$, ϕ_o : $^{\pi}/_{3}$)

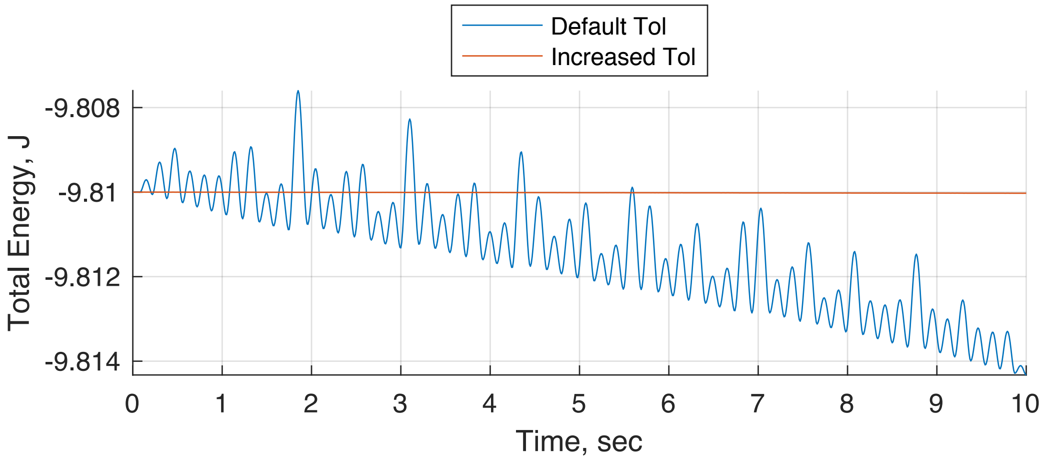


(a) Spring Length vs. Time

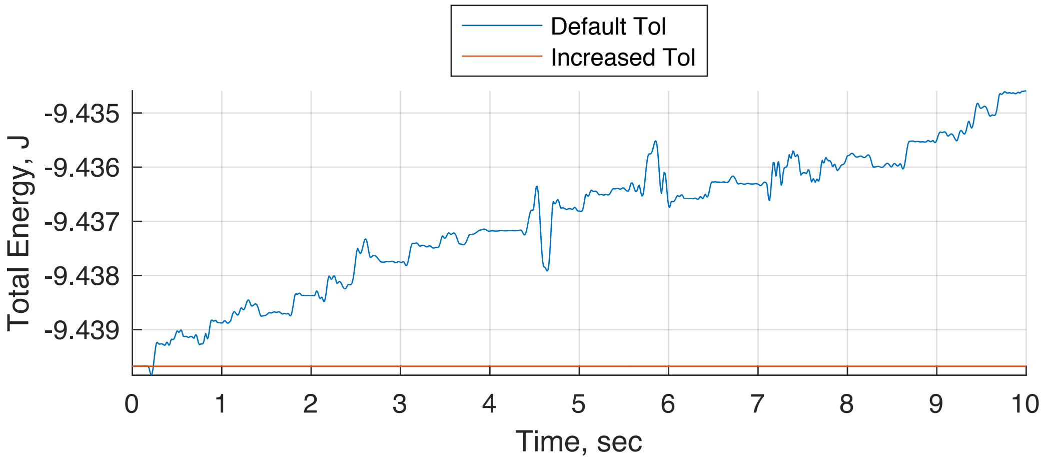


(b) Angular Position vs. Time

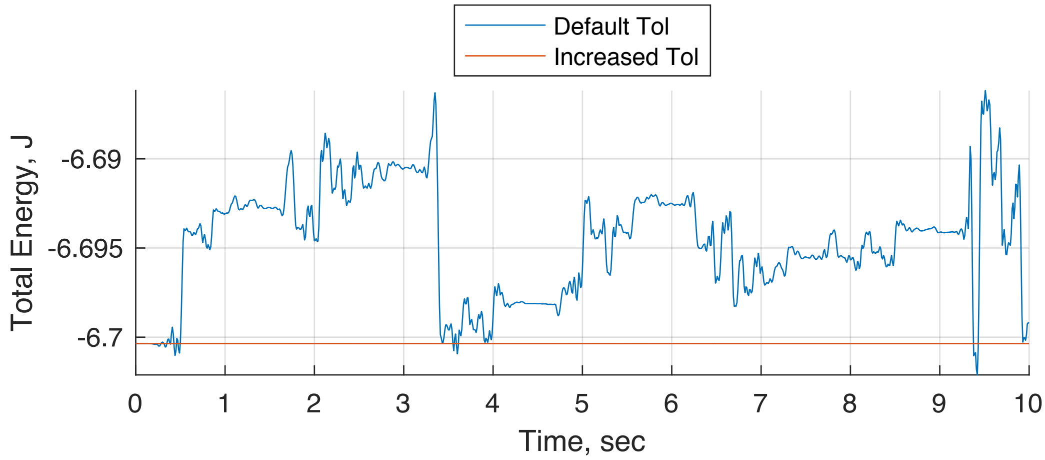
Figure 6: Total Energy Comparison Plots



(a) Total Energy in the System ($\theta_o : 0,\ \phi_o : 0$)



(b) Total Energy in the System ($\theta_o :\ ^{\pi}/_{18},\ \phi_o :\ ^{\pi}/_{9}$)



(c) Total Energy in the System (θ_o : $^{\pi}/_{6}$, ϕ_o : $^{\pi}/_{3}$)

Running the `ode45` function with a relative tolerance of $1e-5$ and an absolute tolerance of $1e-9$ we produce a total energy graph that strays significantly less than the default `ode45` implementation. The max percentage difference between the high and low tolerance integrations in Figure (6a) is -0.0245%. The max percentage difference in Figure (6b) is -0.0538%. The max percentage difference in Figure (6c) is -0.2116%.

9 Does it Make Sense?

9.1 Units

Checking with the MATLAB symbolic units tool (from Section 10.3):

9.2 Magnitude

10 Appendix

10.1 Attributions

Jeffrey Chen

Thorne Wolfenbarger

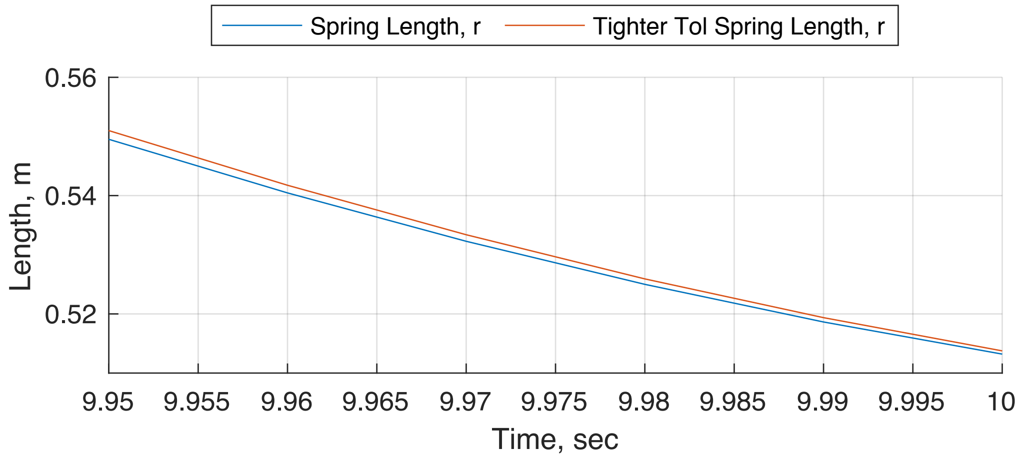
Trey Dufrene

Joint Effort

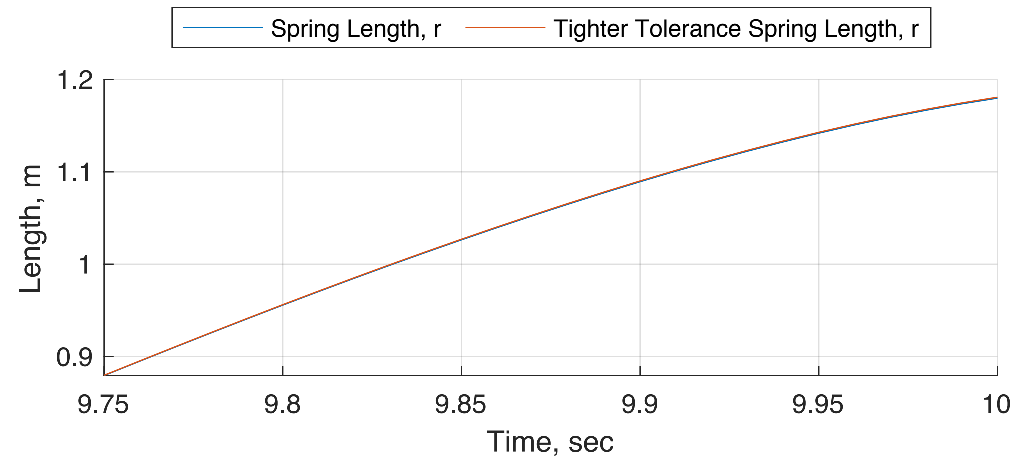
10.2 Analytical Solution

10.3 Numerical Solution

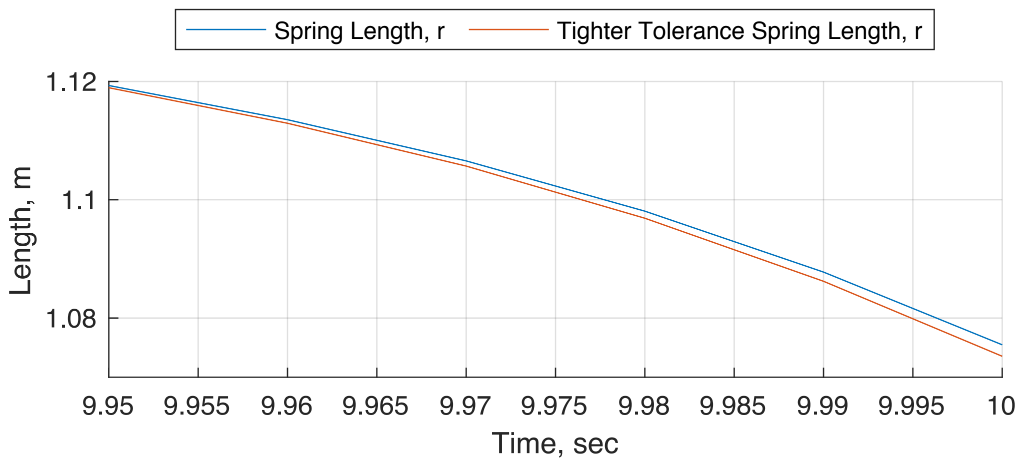
Figure 7: Comparison Plots of ode45 Tolerance Options



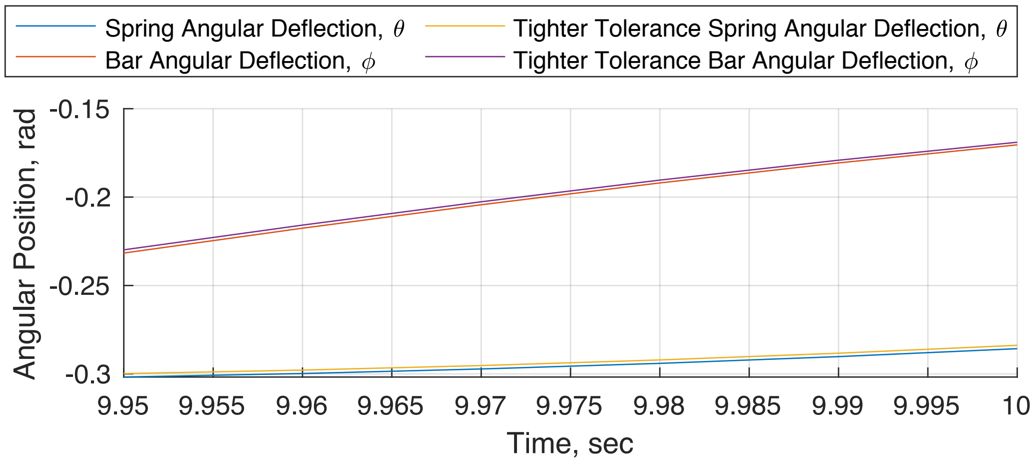
Spring Deflection, ($\theta_o:\ 0,\ \phi_o:\ 0$)



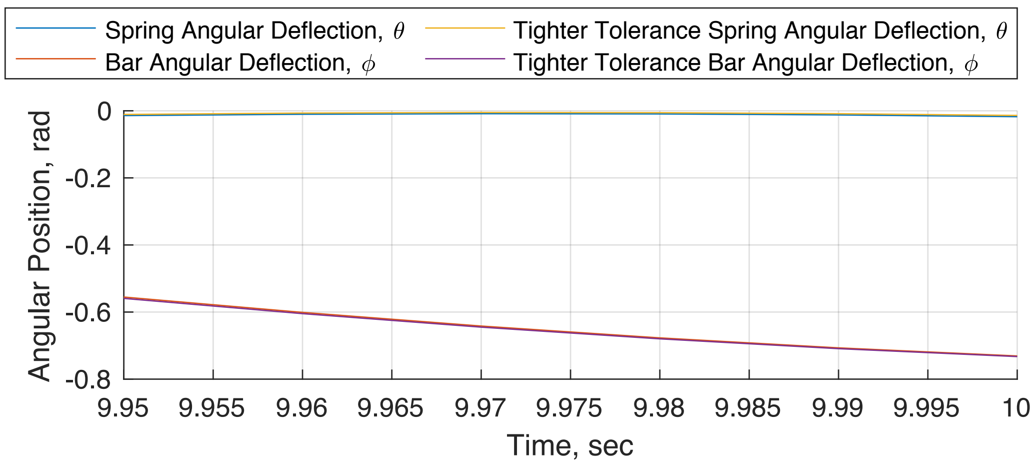
Spring Deflection, ($\theta_o : {}^{\pi}/_{18},\ \phi_o : {}^{\pi}/_{9}$)



Spring Deflection, ($\theta_o : {}^{\pi}/_{6},\ \phi_o : {}^{\pi}/_{3}$)



Bar and Spring Angular Deflection, (θ_o : $\pi/18$, ϕ_o : $\pi/9$)



Bar and Spring Angular Deflection, (θ_o : $\pi/6$, ϕ_o : $\pi/3$)