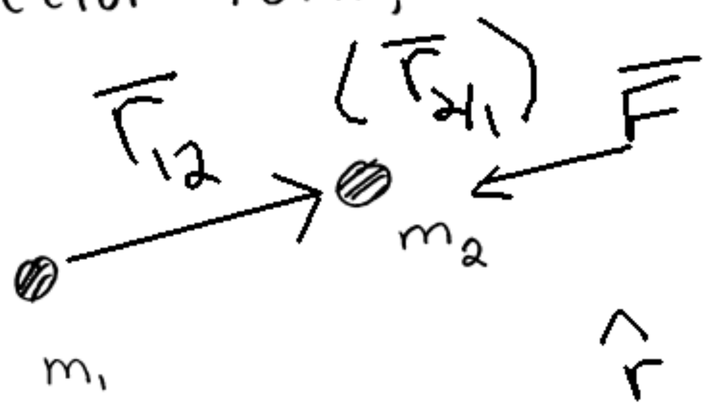
N-Body Problem

Newton's law of gravity (inverse square law)

$$F = \frac{G m_1 m_2}{r_{12}^2} \quad \leftarrow \text{Scalar not vector}$$

universal gravitational constant

In vector form,



$\bar{F}$ force on m_2 due to m_1

$$\bar{F} = -\frac{G m_1 m_2}{r_{12}^2} \overbrace{\frac{\bar{r}_{12}}{r_{12}}}^{\hat{r}}$$

negative b/c how defined $\bar{r}_{12}$

Newton's law of gravity assumes point masses

ONLY valid when a body can be modeled as a point mass.

Why does it work for planets?

We are describing a FORCE NOT shape or volume

If the gravitational force for an actual body can be written as the force for a point mass, then it is a point mass for gravitational purposes

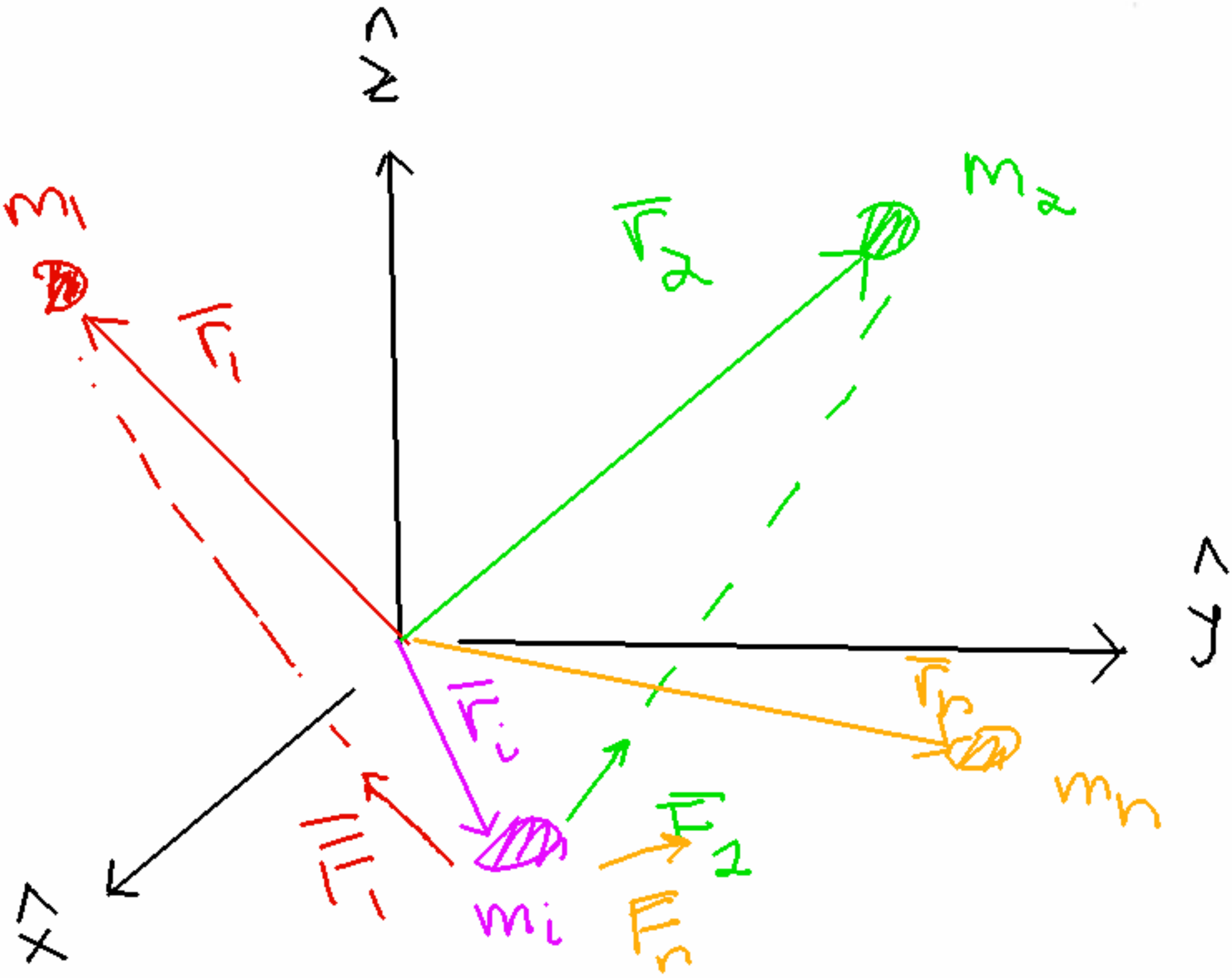
→ centrobaric body

Planets work because we can think of them as concentric shells. Not for asteroids

N-Body Problem

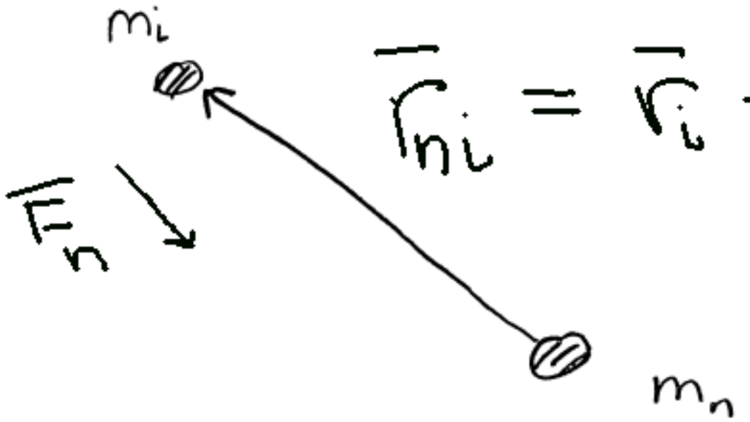
Assume

- Gravity is the only force
- System of n bodies $(m_1, m_2, \dots, m_n)$
- spherically symmetric masses



Note $\quad$ O: inertial pt

$$\bar{r}_i = \bar{r}_{Oi}$$



$$\bar{r}_{ni} = \bar{r}_i - \bar{r}_n$$

$$\bar{r}_{ni} = -\bar{r}_{in}$$

Force on m_i due to m_n is

$$\bar{F}_n = -\frac{G m_i m_n}{r_{ni}^3}\,\bar{r}_{ni}$$

Sum all forces

$$\bar{F} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \bar{F}_j = -G m_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ji}^3}\,\bar{r}_{ji}$$

Now we can write EOM from Newton's second law

$$\frac{{}^I d}{dt}(m_i \bar{v}_i) = \bar{F}$$

Assume constant mass.

$$\frac{{}^I d}{dt}(m_i \bar{v}_i) = m_i \frac{{}^I d}{dt}(\bar{v}_i) + \bar{v}_i \cancel{\frac{dm_i}{dt}}$$

$$= m_i \bar{a}_i = m_i \ddot{\bar{r}}_i$$

$$\boxed{\ddot{\bar{r}}_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_j}{r_{ji}^3} \bar{r}_{ji}}$$

N-Body motion

If we have n bodies, can we solve $\bar{r}_i(t)$?

To know $\bar{r}_i(t)$, need to find $\bar{r}_j(t)$

BUT motion m_i changes the force on m_j

→ different force

→ different acceleration

→ different position

$\bar{r}_i$ + $\bar{r}_j$ are coupled!

To solve, we need to know the vector positions + velocities of all the bodies.

We need 6n equations.

Only have 10 (integrals)

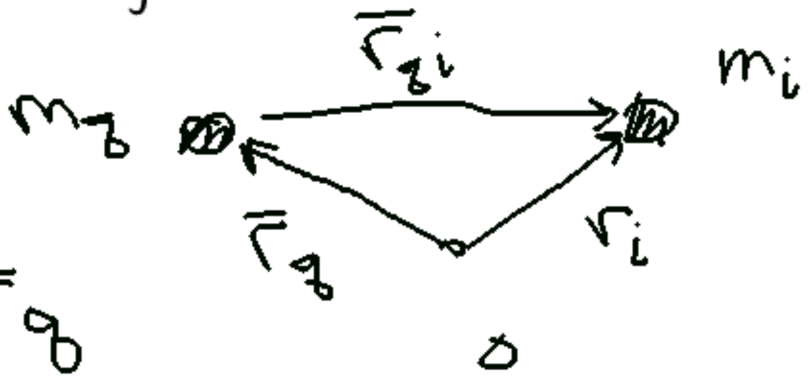
If n = 2, need 12 > 10

can't solve!

Do we really care about $\bar{r}_i, \bar{r}_q$?

What do we really care about?

$\bar{r}_{qi}$



$$\bar{r}_{qi} = \bar{r}_i - \bar{r}_q$$

$$\ddot{\bar{r}}_{qi} = \ddot{\bar{r}}_i - \ddot{\bar{r}}_q$$

We have expressions for $\ddot{\bar{r}}_i$ and $\ddot{\bar{r}}_q$

$$\ddot{\bar{r}}_{qi} = \overbrace{-G\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j}{r_{ji}^3}\bar{r}_{ji}}^{\ddot{\bar{r}}_i} + \overbrace{G\sum_{\substack{j=1\\ j\neq q}}^{n} \frac{m_j}{r_{jq}^3}\bar{r}_{jq}}^{\ddot{\bar{r}}_q}$$

dominant acceleration

remove q term

remove i term

$$\boxed{\ddot{\bar{r}}_{qi} = -\frac{G(m_i+m_q)}{r_{qi}^3}\bar{r}_{qi} + \underbrace{G\sum_{\substack{j=1\\ j\neq q,i}}^{n} m_j\left(\underbrace{\frac{\bar{r}_{ij}}{r_{ij}^3}}_{\text{direct}} - \underbrace{\frac{\bar{r}_{qj}}{r_{qj}^3}}_{\text{indirect}}\right)}_{\text{peturbing accel.}}}$$

relative n-body motion

If $n \geq 3$, still can't solve. Need

$\bar{r}_{SK\odot}, \bar{r}_{\oplus\odot}, \bar{r}_{SK☾}, \bar{r}_{\oplus☾}$ + velocities
24 eqns

What happens if $n=3$ (remove ☾)?

Still need $\bar{r}_{SK\odot}, \bar{r}_{\oplus\odot}, \dot{\bar{r}}_{SK\odot}, \dot{\bar{r}}_{\oplus\odot}$

Need 12 constants only
have 10

How about 2? 2^{nd} order DE with one unknown position!

$$\boxed{\ddot{\bar{r}}_{12} + \frac{G(m_1+m_2)}{r_{12}^3}\bar{r}_{12} = 0}$$

solvable! relative 2-body motion

$\mu = G(m_1+m_2)$ but $m_2 \ll m_1$ so ignore m_2

$$\ddot{\bar{r}}_{12} + \frac{\mu \bar{r}_{12}}{r_{12}^3} = 0$$