Ten Known Integrals

By collecting the equations into certain combinations, we can integrate **some** of the new equations and solve relative 2-body motion.

Linear Momentum

If $\bar{p}$ is conserved, what happens?

If $n = 2$, what is $\dot{\bar{p}}$ in terms of $\ddot{\bar{r}}_1$ + $\ddot{\bar{r}}_2$?

$\dot{\bar{p}} =$

Thus

$\dot{\bar{p}} =$

$\bar{p} =$

$\sum_{i=1}^{n} m_i \bar{r}_i =$

Center of mass (CM)

$\bar{r}_{CM} =$

Differentiate (note m_i is constant)

$M \bar{v}_{CM} =$

Thus the CM of the system moves

Is the CM inertially fixed?

We need an inertially fixed coordinate system to find $\bar{r}_{12}$ in the 2-body system.

Define a new frame

$\bar{r}_{ICM} =$

Angular Momentum

$$\bar{H} = \sum_{i=1}^{n} (\bar{r}_i \times m_i \dot{\bar{r}}_i)$$

If $\bar{H}$ is conserved,

First, we will prove it.
For $n = 2$,

$$\frac{d}{dt}(\bar{H}) = \frac{d}{dt}(m_1 \bar{r}_1 \times \dot{\bar{r}}_1) + \frac{d}{dt}(m_2 \bar{r}_2 \times \dot{\bar{r}}_2)$$

Thus

$$\bar{H} =$$

For 2-body motion, find $\bar{H}$ in terms of CM

$$\bar{H} = m_1 \bar{r}_1 \times \dot{\bar{r}}_1 + m_2 \bar{r}_2 \times \dot{\bar{r}}_2$$

$$= m_1 \left(\frac{-m_2}{m_1+m_2} \bar{r} \times - \frac{m_2}{m_1+m_2} \dot{\bar{r}} \right)$$

$$+ m_2 \left(\frac{m_1}{m_1+m_2} \bar{r} \times \frac{m_1}{m_1+m_2} \dot{\bar{r}} \right)$$

Let

Since $\bar{h} \perp \bar{r}$ and $\bar{h} \perp \dot{\bar{r}}$ we have an

Total Energy

$T - U =$

or

$$\frac{dT}{dt} - \frac{dU}{dt} =$$

Write $T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i$ in terms of $\bar{r}$.

$$T = \frac{1}{2} m_1 \dot{\bar{r}}_1 \cdot \dot{\bar{r}}_1 + \frac{1}{2} m_2 \dot{\bar{r}}_2 \cdot \dot{\bar{r}}_2$$

$$U = \frac{1}{2} G \left(\frac{m_1 m_2}{r} + \frac{m_2 m_1}{r} \right) = \frac{G m_1 m_2}{r}$$

$T - U =$

Multiply by $\frac{m_1 + m_2}{m_1 m_2}$

Then

Now we can solve for r from $\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r}$
change dt to $d\theta$ (by h), we
can solve for $r(\theta)$ via variable
change to $1/r$.
After some mathematical manipulations,
we get the conic equation.

$$\boxed{r = \frac{p}{1 + e\cos(\theta - \omega)}}$$

where

If we derive the conic equation using just vectors, we get the eccentricity vector

$$\bar{e} = \frac{\dot{\bar{r}} \times \bar{h}}{\mu} - \frac{\bar{r}}{r}$$