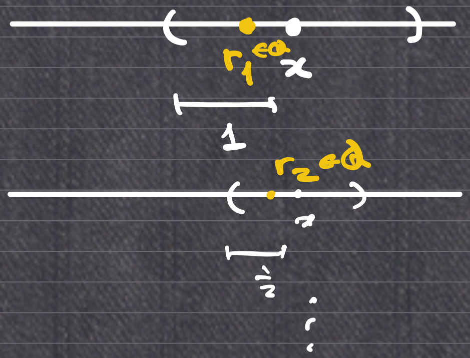
<u>Claim</u>: $\forall x \in \mathbb{R}, \quad \exists \{r_n\} \in \mathbb{Q} \quad \text{s.t.} \quad r_n \to x.$



$\forall n \in \mathbb{N}$

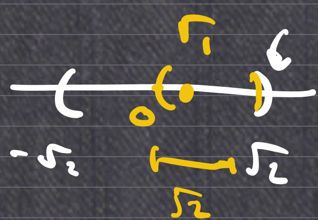
$$x - \frac{1}{n} < r_n < x + \frac{1}{n}$$

$$\begin{array}{ccccc} x - \frac{1}{n} & < & r_n & < & x + \frac{1}{n} \\ \downarrow & & \Downarrow & & \downarrow \\ x & & x & & x \end{array}$$

$S = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

① $\sqrt{2}$ is an upper bound for S

② $\exists \{r_n\} \in \mathbb{Q} \to \sqrt{2}$
$\cap$
$(-\sqrt{2}, \sqrt{2})$



$\Rightarrow \sqrt{2} = \sup S.$

- $\emptyset \neq S \subset T \subset \mathbb{R}$

$$\underbrace{\sup S}_{\text{least ub for } S} \leq \underbrace{\sup T}_{\text{upper bd for } S}$$

$$\underbrace{\inf S}_{\text{greatest lower b.}} \geq \underbrace{\inf T}_{\text{a lower bound for } S}$$

of S

- $\sup \emptyset = -\infty$
  $\inf \emptyset = +\infty$

Any real $x \in \mathbb{R}$ is an upper b. for $\emptyset$.

Proof by contradiction:

Assume $\exists x \in \mathbb{R}$ which is not an upper bound for $\emptyset$.

$\Rightarrow$ not (x is an upper bound for $\emptyset$)

$\Rightarrow$ not ($\forall y \in \emptyset,\quad x \geq y$)

$\Rightarrow$ $\exists y \in \emptyset,$ s.t. $x < y$.

---

x^y

$x^3 := x \cdot x \cdot x$ $\qquad\qquad$ $x^m \cdot x^n = x^{m+n}$.

$x^{\frac{1}{n}} = \sqrt[n]{x}$

Prop: $\forall x_0 > 1,\ n \in \mathbb{N},\ \exists!\, t > 0$ s.t. $t^n = x_0$

Proof: $S = \{t > 0 : t^n < x_0\}$

Claim:
$(\sup S)^n = x$.

Case ① : $\underline{x_0 > 1}$

$\phi \neq S \ni \{1\}$.

<u>Claim</u>: $1+x_0$ is an upper bound for S.

<u>Pf</u>: By contradiction. $\exists t \in S$ s.t.

$$t \geq 1+x_0$$

$$\Rightarrow \underbrace{x_0 > t^n}_{t \text{ in } S} \geq (1+x_0)^n > 1+x_0$$

$\phi \neq S$, bound above $\Rightarrow \sup S$

<u>Claim</u>. $(\sup S)^n = x_0$

<u>Proof</u>: $\exists \{t_k\} \in S \longrightarrow \sup S$.

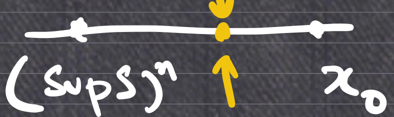
$$t_k^n < x_0$$

$$\downarrow k \to \infty$$

$$(\sup S)^n \leq x_0$$

<u>Next</u>: Assume $(\sup S)^n < x_0$. Try to derive contradiction

want to find



$(\sup S+h)^n$ $\quad \exists h > 0$ small

Need. $(\sup S+h)^n - (\sup S)^n$

$\forall h \in (0,1)$ $\quad < x_0 - (\sup S)^n$

$(\sup S + h)^n - (\sup S)^n$

$= \underbrace{(\sup S + h - \sup S)}_{h} \left(\sum_{j=0}^{n-1} \underbrace{(\sup S + h)^j}_{\le \sup S + 1} (\sup S)^{n-j-1} \right)$

$\le h \underbrace{\sum_{j=0}^{n-1} (\sup S + 1)^j (\sup S)^{n-j-1}}_{\overset{\text{"}}{C}} = Ch$

Choose $h = \min\left\{ \frac{1}{2}, \frac{x_0 - (\sup S)^n}{2C} \right\}$.

$\Rightarrow (\sup S + h)^n - (\sup S)^n$

$\overset{h<1}{\le} Ch \le C \cdot \frac{x_0 - (\sup S)^n}{2C} < x_0 - (\sup S)^n$

$\Rightarrow \sup S + h \in S$

uniqueness:

If $y_1^n = y_2^n = x_0$

$\Rightarrow 0 = y_1^n - y_2^n = (y_1 - y_2)(\underbrace{+ + + \cdots}_{+})$

Case ②: $x_0 = 1$ trivial.

Case ③: $x_0 \in (0,1) \rightsquigarrow \frac{1}{x_0} > 1 \Rightarrow \exists t > 0$ s.t. $t^n = \frac{1}{x_0} \Rightarrow (\frac{1}{t})^n = x_0$

$x>0$: $x^{\frac{1}{n}}$ ✓ $\longrightarrow$ $\underbrace{x^{\frac{m}{n}} := \left(x^{\frac{1}{n}}\right)^m}$

check: $\frac{m}{n} = \frac{p}{q} \Rightarrow \left(x^{\frac{1}{n}}\right)^m = \left(x^{\frac{1}{q}}\right)^p$

$p, q, m, n \in \mathbb{N}$

$\boxed{x>1}$, $\boxed{y>0}$

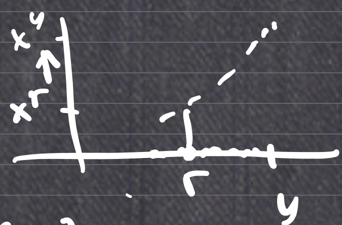
$x^y = ?$

$x^{\sqrt{2}}$

$S(x,\sqrt{2}) = \{x^1, x^{1.4}, x^{1.41}, \ldots\}$

$$S(x,y) := \{x^r \mid 0<r<y,\ r\in\mathbb{Q}\}.$$

$$x^y := \sup S(x,y)$$



Check

$r\in\mathbb{Q}_+$, $x^r = \sup S(x,r)$

$= \sup\{x^s \mid 0<s<r,\ s\in\mathbb{Q}\}$

$\forall s\in(0,r)\cap\mathbb{Q}$, $x^s < \underbrace{x^r}_{\text{upper bd for } S(x,r)}$ $\Rightarrow$ $\sup(x,r) \leq x^r$.

$s_j\in\mathbb{Q} \nearrow r$

$s_j - r \to 0$

$\underbrace{x^{s_j}}_{\in S(x,r)} \overset{?}{\longrightarrow} x^r$

$$|x^{s_j} - x^r| = |x^r(x^{s_j-r}-1)| = |x^r|\underbrace{|x^{s_j-r}-1|}_{\downarrow\, 0}$$