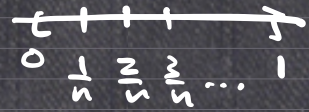
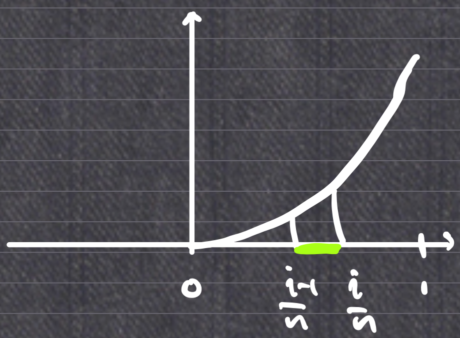
e.g. $f(x) = x^2$ on $[0,1]$

Let $P_n : 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < \frac{n}{n} = 1$ $\quad x_i = \frac{i}{n}$, $0 \le i \le n$





$$U(f,P_n) = \sum_{i=1}^{n} \underbrace{\left(\frac{i}{n}\right)^2}_{f(x_i)} \frac{1}{n}$$

$$= \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2)$$

$$= \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) \to \frac{1}{3}$$

as $n \to \infty$

$$L(f,P_n) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}(0^2 + 1^2 + \cdots + (n-1)^2)$$

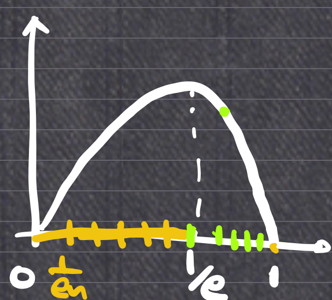
$$= \frac{1}{n^3} \cdot \frac{1}{6}(n-1)\,n\,(2n-1) \to \frac{1}{3}.$$

$\exists \{P_n\}$ s.t. $\lim_{n\to\infty} L(f,P_n) = \frac{1}{3} = \lim_{n\to\infty} U(f,P_n)$

$\therefore$ x^2 is Riemann integrable

and $\int_0^1 f(x)\,dx = \frac{1}{3}$

$$f(x) = \begin{cases} -x\log x & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$



$\left(1-\frac{1}{e}\right)\frac{1}{n}$

(ii) $\sup_P L(f,P) = \inf_P U(f,P)$

$\Downarrow$

(i) $\exists \{P_k\}$ s.t. $\lim_{k\to\infty} U(f,P_k) = \lim_{k\to\infty} L(f,P_k)$.

$\exists \{P_k\}$ s.t. $\lim_{k\to\infty} L(f,P_k) = \sup_P L(f,P)$

$\exists \{Q_k\}$ s.t. $\lim_{k\to\infty} U(f,Q_k) = \inf_P U(f,P)$

$P \subseteq Q \implies$ $U(f,P) \geq U(f,Q)$

$L(f,P) \leq L(f,Q)$

e.g. $\{x_1,\dots,x_n\}$ $\{x_1,\dots,y,\dots,x_n\}$

x_{i-1} c x_i

$R_k := P_k \cup Q_k$

$\Rightarrow$ $L(f,P_k) \leq L(f,R_k) \leq U(f,R_k) \leq U(f,Q_k)$

$\downarrow$ $\downarrow$

$\sup_P L(f,P)$ $\inf_P U(f,P)$

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

P = any partition of $[0,1]$.

$$U(\chi_{\mathbb{Q}}, P) = \sum \underbrace{\sup}_{1} \cdot \Delta x_i = 1$$

$$L(\chi_{\mathbb{Q}}, P) = \sum \underbrace{\inf}_{0} \cdot \Delta x_i = 0.$$

- Continuous on $[a,b]$ $\Longrightarrow$ Riemann integrable on $[a,b]$.

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \Big(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \Big)(x_i - x_{i-1})$$

$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon/?$

$$\leq \sum_{i=1}^{n} \varepsilon/? \,(x_i - x_{i-1}) = \varepsilon/? \,(b-a)$$