$$\limsup_{n\to\infty} x_n = \sup \mathrm{LIM}\{x_n\}$$

$$\liminf_{n\to\infty} x_n = \inf \mathrm{LIM}\{x_n\}.$$

Ex. 1.27

$$x_n = \begin{cases} 1/p_1 & \text{if } n = p_1^{\alpha_1} \\ \dfrac{p_{m_1}}{p_{m_k}} & \text{if otherwise} \end{cases}$$

$p_{m_1}^{\alpha_1} \cdots p_{m_k}^{\alpha_k}$

$p_{m_1} < p_{m_2} < \cdots < p_{m_k}$

$k \geq 2.$

$$0 < x_n < 1$$

$\mathrm{LIM}\{x_n\} \subset [0,1]$

list of primes:

$\{p_1, p_2, p_3, p_4, \ldots\}$

$\limsup \leq 1$

$\liminf \geq 0.$

$\{x_{p_1}, x_{p_2}, x_{p_3}, \ldots\}$

$= \{\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \ldots\}$

$\liminf = 0. \leftarrow \to 0$

$x_6 = x_{2\cdot 3} = \frac{2}{3}, \quad x_{12} = x_{2^2\cdot 3} = \frac{2}{3}$

$x_{p_n p_{n+1}} = \frac{p_n}{p_{n+1}} \to ?$

$$\boxed{\lim_{n\to\infty} \frac{p_n}{n \log n} = 1}$$

Prime Number Theorem.

$$x_{P_nP_{n+1}} = \frac{P_n}{P_{n+1}} = \underbrace{\frac{P_n}{n\log n}}_{\downarrow\, 1} \cdot \underbrace{\frac{(n+1)\log(n+1)}{P_{n+1}}}_{\downarrow\, 1} \cdot \underbrace{\frac{n\log n}{(n+1)\log(n+1)}}_{\downarrow\, 1}$$

$$\boxed{\limsup x_n = 1}$$

$$x_{P_{2n}P_{3n}} = \frac{P_{2n}}{P_{3n}} \sim \frac{2n\log(2n)}{3n\log(3n)} \to \frac{2}{3}$$

$$\left|\frac{P}{n\log n} - 1\right| < ?$$

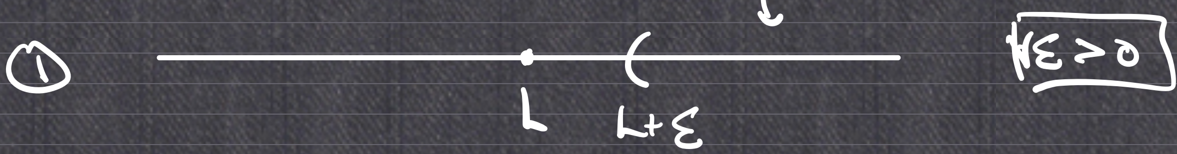
$x_n = n \to +\infty$

$LIM\{x_n\} = \{+\infty\}$

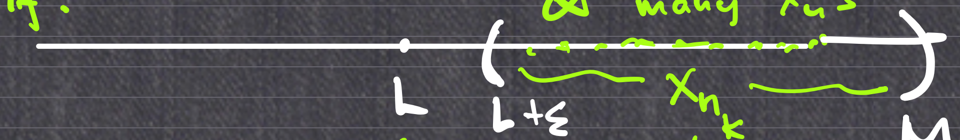
$\limsup x_n = +\infty$.

$L = \limsup x_n \in \mathbb{R}$.

finitely many x_n's.



if:



x_{n_k}

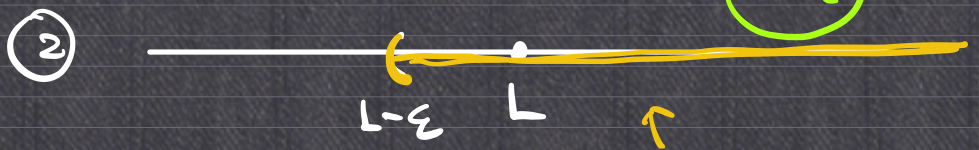
$\downarrow$

$x_{n_{k_l}} \to \alpha \in [L+\varepsilon, M]$

$\cap$

LIM

sup LIM.



infinitely many x_n's

$x_n \in \longrightarrow\,] \ \forall n >> N, \implies LIM \subset \longrightarrow\,]$

$L-\varepsilon$

if: $L-\varepsilon$ $\quad L$ $\quad$ at most finitely x_n's

sup LIM $\longrightarrow \implies \limsup \le L-\varepsilon$

$$\boxed{\limsup x_n \in LIM}$$

$\forall \varepsilon \quad \varepsilon > 0$

$\limsup \{x_n\} =: L$

at most finitely many x_n's

Example 1.30 $\quad \{x_n\}$

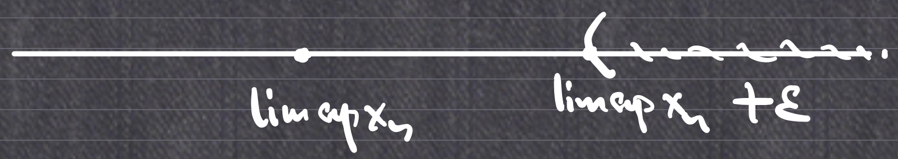
$$\liminf x_n \le \liminf \frac{x_1+\cdots+x_n}{n} \le \underbrace{\limsup \frac{x_1+\cdots+x_n}{n} \le \limsup x_n}$$

WTP:

$\forall \varepsilon > 0$

$$\limsup \frac{x_1+\cdots+x_n}{n} \le \limsup x_n + \varepsilon$$

let $\varepsilon \to 0^+$ $\implies \ldots \le \ldots$



at most finitely many x_n.

$\Leftarrow$ $\exists N > 0$ s.t.

$n > N \Rightarrow x_n \le \limsup x_k + \varepsilon$

$$\frac{x_1 + \cdots + x_n}{n} = \frac{x_1 + \cdots + x_N}{n} + \frac{x_{N+1} + \cdots + x_n}{n}$$

$$\frac{x_1 + \cdots + x_n}{n} \le \frac{x_1 + \cdots + x_N}{n} + \frac{(n-N)(\limsup + \varepsilon)}{n}$$

$$\limsup \frac{x_1 + \cdots + x_n}{n} \le \limsup \Big(\underbrace{\frac{x_1 + \cdots + x_n}{n} + \frac{(n-N)}{n}(\limsup x_k + \varepsilon)}_{\text{conv.}} \Big)$$

$\Vert$ lim.

$$= 0 + 1 \cdot (\limsup x_k + \varepsilon)$$

$$= \limsup x_k + \varepsilon$$