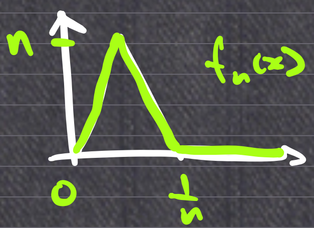
## §4 : Uniform convergence.

$$\frac{1}{2} = \lim_{n\to\infty} \underbrace{\int_0^1 f_n(x)\,dx}_{=\frac{1}{2}} \neq \int_0^1 \underbrace{\lim_{n\to\infty} f_n(x)}_{=0}\,dx = 0$$




$\xrightarrow{n\to\infty}$


if $x>0$, $f_n(x)=0$ $\forall n > \frac{1}{x}$.

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

if $x=0$, $f_n(0) = 0$ $\forall n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

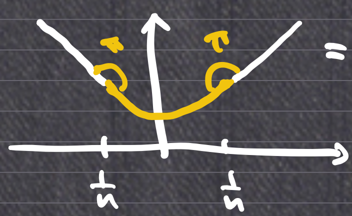
- $$\lim_{x\to a} \lim_{n\to\infty} f_n(x) \overset{?}{=} \lim_{n\to\infty} \lim_{x\to a} f_n(x)$$

Let $f_n(x) = x^n$

$$\lim_{x\to 1^-} \lim_{n\to\infty} f_n(x) = \lim_{x\to 1^-} \overset{=0}{\left( \lim_{n\to\infty} \underbrace{x}_{x<1}{}^n \right)} = 0$$

$$\lim_{n\to\infty} \lim_{x\to 1^-} f_n(x) = \lim_{n\to\infty} \underbrace{\left( \lim_{x\to 1^-} x^n \right)}_{=1} = 1$$

- $$\lim_{n\to\infty} \frac{d}{dx} f_n(x) \neq \frac{d}{dx} \lim_{n\to\infty} f_n(x)$$

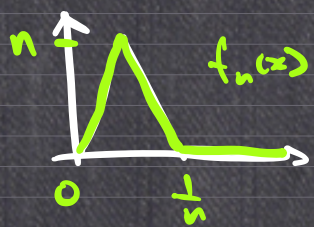



$= |x|$ if $x \approx 0$

• $f_n : D \subset \mathbb{R} \to \mathbb{R}$ is said to converge uniformly to $g$ on $E \subset D$

$\overset{\text{def}}{\iff}$ $$\boxed{\lim_{n\to\infty} \|f_n - g\|_E = 0}$$

$$\boxed{\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - g(x)| = 0}$$

• 


$E = [0,1]$

if $\|f_n - f\|_E \to 0$ as $n\to\infty$

then $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$.

Proof: $|f_n(x) - f(x)| \le \|f_n - f\|_E$ $\forall x \in E$

$$\|f_n - 0\|_{[0,1]} = \sup_{x\in[0,1]} |f_n(x)| = \sup_{x\in[0,1]} n$$

$$= n \to \infty \quad \text{as } n \to \infty.$$

$\therefore$ $f_n$ does not converges uniformly to 0 on $[0,1]$

However, $$\sup_{x\in[\frac{1}{2},1]} \underbrace{|f_n(x) - \overset{0}{\cancel{f(x)}}|}_{=0} = 0 \text{ if } n > 2$$

$f_n$ converges uniformly to 0 on $[\frac{1}{2}, 1]$

$\boxed{n \geq N}$

$f_n \rightrightarrows f$ on $E$

uniform convergence on $E$.


$$\lim_{n\to\infty} \underbrace{\sup_{x\in E} |f_n(x) - f(x)|} = 0$$

$$\Leftrightarrow \forall \varepsilon > 0,\ \exists \underbrace{N}^{=N(\varepsilon)} > 0 \text{ s.t. } n \geq N$$

$$\Rightarrow \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$$

indep. of $x$

$$\Rightarrow \forall x \in E,\ |f_n(x) - f(x)| < \varepsilon$$

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$\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$ (pointwise convergence)

$N = N(\varepsilon, x)$

$$\Longleftrightarrow \forall \varepsilon > 0,\ \forall x \in E,\ \exists N > 0 \text{ s.t. } n \geq N$$

$$|f_n(x) - f(x)| < \varepsilon.$$

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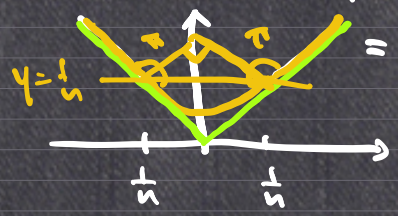


$f_n(x) \longrightarrow |x|$ as $n \to \infty$

$= |x|$ if $x \not\approx 0$

$$\sup_{x \in \mathbb{R}} \underbrace{|f_n(x) - f(x)|}_{< \frac{1}{n}} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

$$f_n \rightrightarrows f = |x| \quad \text{on } \mathbb{R}.$$

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$L(x) := \sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $E$

$\iff \lim_{k \to \infty} \underbrace{\sum_{n=1}^{k} g_n(x)}_{\text{seq. of functions}}$ converges uniformly $L(x)$ on $E$.

Weierstrass M-test :

If $\exists$ sequence of real number $\{M_n\}$ s.t.

① $|g_n(x)| \le M_n \quad \forall n \in \mathbb{N}, \forall x \in E$

② $\sum_{n=1}^{\infty} M_n$ converges (as a series of real number)

<u>Then</u> : $\sum_{n=1}^{\infty} g_n$ converges uniformly on $E$.

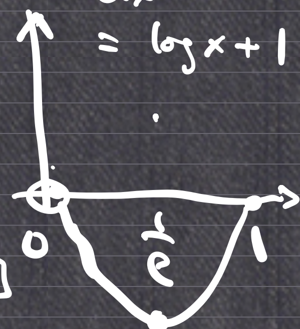
Ex:
$$\sum_{k=1}^{\infty} \underbrace{\frac{\sin(k^{|x|} x^2)}{2^k}}_{g_k(x)}$$

- $|g_k(x)| \leq \underbrace{\frac{1}{2^k}}_{M_k} \quad \forall x \in \mathbb{R}$
- $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$

$\xRightarrow{\text{Weierstrass M-test}}$ $\sum_{k=1}^{\infty} \frac{\sin(k^{|x|} x^2)}{2^k}$ converges uniformly on $\mathbb{R}$

- $\sum_{k=1}^{\infty} (x \log x)^k \qquad x \in (0, 1].$

$\frac{d}{dx} x \log x = \log x + 1$




- $|(x \log x)^k| \leq \left|\left(\frac{1}{e} \log \frac{1}{e}\right)^k\right| = \left(\frac{1}{e}\right)^k \quad \forall x \in (0,1]$
- $\sum \left(\frac{1}{e}\right)^k$ converges

$\Rightarrow \sum_{k=1}^{\infty} (x \log x)^k$ converges uniformly on $(0,1]$.

## Proof of Weierstrass's M-test

$$\begin{cases} |g_n(x)| \le M_n & \forall x \in E,\ n \in \mathbb{N}. \\ \sum M_n < \infty \end{cases}$$

- $\sum_{n=1}^{\infty} g_n(x)$ is defined $\forall x \in E$.

Proof: $\left| \sum_{n=1}^{m} g_n(x) - \sum_{n=1}^{k} g_n(x) \right|$ $\quad m > k$

$$= \left| \sum_{n=k+1}^{m} g_n(x) \right| \le \sum_{n=k+1}^{m} \underbrace{|g_n(x)|}_{\le M_n} \le \sum_{n=k+1}^{m} M_n$$

$$= \left| \underbrace{\sum_{n=1}^{m} M_n}_{} - \sum_{n=1}^{k} M_n \right|$$

- uniformly converge:

$$\sup_{x \in E} \left| \sum_{n=1}^{m} g_n(x) - \sum_{n=1}^{k} g_n(x) \right|$$

$$\le \sup_{x \in E} \cdots \le \sup_{x \in E} \left| \sum_{n=1}^{m} M_n - \sum_{n=1}^{k} M_n \right|$$

$$= \left| \sum_{n=1}^{m} M_n - \sum_{n=1}^{k} M_n \right|$$

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$f_n \rightrightarrows f$ on $[a, b]$

$$\Rightarrow \lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b \underbrace{\lim_{n \to \infty} f_n(x)}_{=f}\,dx$$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$\leq \int_a^b \underbrace{|f_n(x) - f(x)|}_{\leq \|f_n - f\|_{[a,b]}}\,dx \leq \int_a^b \|f_n - f\|_{[a,b]}\,dx = \underbrace{\|f_n - f\|_{[a,b]}(b-a)}_{\downarrow\, 0}$$

- $$\int_0^1 \frac{1}{x^x}\,dx = \int_0^1 x^{-x}\,dx = \int_0^1 e^{\log x^{-x}}\,dx$$

$$= \int_0^1 e^{-x\log x}\,dx \qquad e^y = \sum \frac{y^n}{n!}$$

$$= \int_0^1 \sum_{n=0}^{\infty} \underbrace{\frac{(-x\log x)^n}{n!}}\,dx$$

$$\left|\frac{(-x\log x)^n}{n!}\right| \leq \frac{\left(\frac{1}{e}\right)^n}{n!} \leq \frac{1}{n!}$$

$\sum \frac{1}{n!}$ converges.

$$= \sum_{n=0}^{\infty} \underbrace{\int_0^1 \frac{(-x\log x)^n}{n!}\,dx}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^n}$$