If $\quad F\circ G = id \quad$ and $\quad G\circ F = id$

then $\quad \underbrace{D(F\circ G)}_{DF\cdot DG} = D(id) = I$

$$\underbrace{D(G\circ F)}_{DG\cdot DF} = D(id) = I \qquad \frac{\partial(x_1,\dots,x_n)}{\partial(x_1,\dots,x_n)} = \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$\Rightarrow \quad DG = (DF)^{-1}$$

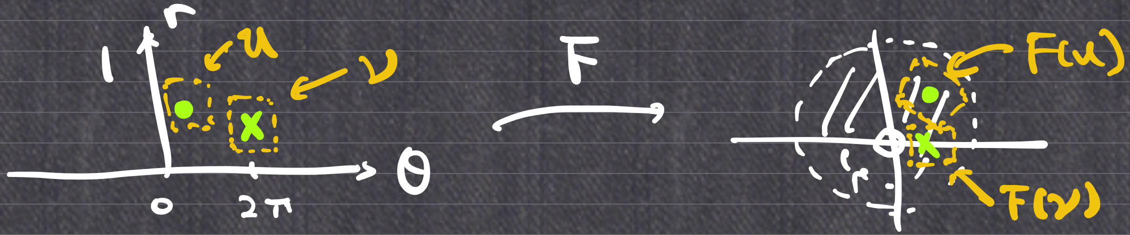
$\therefore$ F bijective $\Longrightarrow$ DF is invertible and $D(F^{-1}) = (DF)^{-1}$.

$$F : (0,\infty)\times\mathbb{R} \longrightarrow \mathbb{R}^2\setminus\{(0,0)\}$$
$$(r,\theta) \mapsto (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$$

$$DF = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{array}{c} \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} \end{array} \end{array} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det DF = r > 0$

but F is not injective.



e.g. 3.25

$$(*)\begin{cases} x-y^2=a \\ x^2+y+y^3=b \end{cases}$$

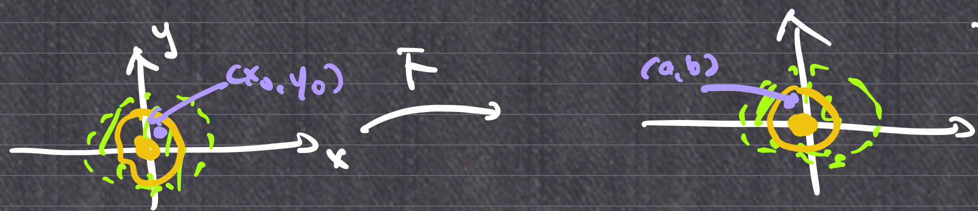
Claim: when (a,b) small then $(*)$ has a solution.

$F:\mathbb{R}^2\to\mathbb{R}^2$

$F(x,y)=(x-y^2,\ x^2+y+y^3)$

$F(0,0)=(0,0)$, $DF=\begin{bmatrix}1 & -2y\\ 2x & 1+3y^2\end{bmatrix}$

$\Rightarrow DF(0,0)=\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\text{invertible}}$



e.g. 3.26

$F:(0,\infty)\times(0,\pi)\times(0,2\pi)\to\mathbb{R}^3$

$(\rho\ ,\ \phi\ ,\ \theta)\longmapsto(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi\ )$

$$\det \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi \neq 0$$

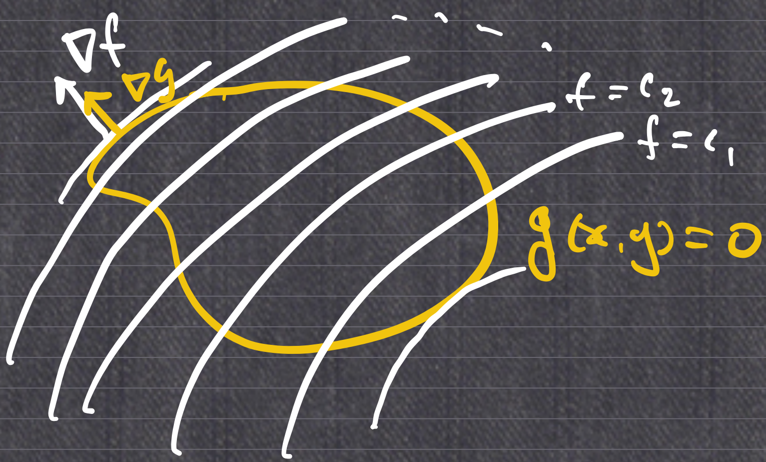
$$\Rightarrow F^{-1} \text{ locally exists and } \underbrace{C^\infty}_{(\because\, F \text{ is } C^\infty)}$$

3.3.4 Lagrange's multiplier.

optimize: $f(x,y)$

subject to: $g(x,y) = 0$

$$\text{Solve } \begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0. \end{cases} \Longleftarrow \{\nabla f, \nabla g\} \text{ are linearly dependent.}$$



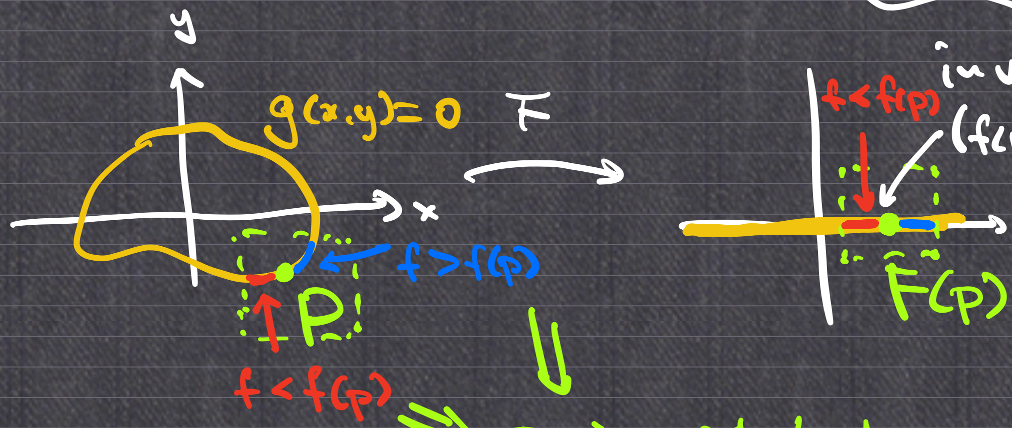
Claim: $\{\nabla f(p), \nabla g(p)\}$ are linearly indep.

$\Rightarrow$ p is not max/min for f under constraint $\{g=0\}$.

Proof:

$$F(x,y) = (f(x,y), g(x,y))$$

$$DF(p) \begin{bmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \\ \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y}(p) \end{bmatrix} = \underbrace{\begin{bmatrix} - & \nabla f(p) & - \\ - & \nabla g(p) & - \end{bmatrix}}_{\text{invertible}}$$



$\Downarrow$

$\Rightarrow$ p is not local max/min for f under $\{g=0\}$.

Banach Contraction Mapping

(X, d) complete metric space.

$f: X \to X$. given $\exists\, \alpha \in (0,1)$

s.t. $d(f(x), f(y)) \le \alpha\, d(x,y)$.

$\forall x, y \in X$.

$\Longrightarrow$ then $\exists!\, x_0 \in X$ s.t. $f(x_0) = x_0$.

UST

$f \circ f$

<u>Proof:</u> Let $x_1 \in X$ any point.

$x_n := f(x_{n-1}) \quad \forall n \ge 2$

$$\underline{d(x_{n+1}, x_n)} = d(f(x_n), f(x_{n-1}))$$
$$\le \alpha\, d(x_n, x_{n-1})$$
$$\le \alpha^2\, d(x_{n-1}, x_{n-2})$$
$$\le \cdots \le \alpha^{n-1}\, \underline{d(x_2, x_1)}$$

$\alpha \in (0,1)$ assume $\neq 0$

$$\underset{m \ge n \ge N}{d(x_m, x_n)} \le \cdots \le O(\alpha^N)_{\downarrow 0}$$

$\{x_n\}$ is Cauchy $\xRightarrow{\text{complete}}$ $\boxed{x_n \to x_0}$

$\exists\, x_0 \in X$

$$x_n = \underbrace{f(x_{n-1})}$$

$\downarrow \quad \downarrow$ f cts

$$x_0 = f(x_0).$$

$$(*)\begin{cases} y'(x) = F(y(x)) \\ y(0) = 1 \end{cases}$$

$$\begin{cases} y'(x) = \underline{\sin(y^2(x)+1)} \\ y(0) = 1 \end{cases}$$

$F(y) = \sin(y^2+1)$

$$\Leftrightarrow \quad y(x) = 1 + \int_0^x F(y(t))\,dt$$

chosen later

$$\Phi : C[0,s] \longrightarrow C[0,s]$$

$$\Phi(f) := 1 + \int_0^x F(f(t))\,dt$$

$$\|\Phi(f) - \Phi(g)\|_\infty$$

$$= \left\| \int_0^x F(f(t))\,dt - \int_0^x F(g(t))\,dt \right\|_\infty$$

$$= \sup_{x \in [0,s]} \left| \int_0^x F(f(t)) - F(g(t)) \, dt \right|$$

$$\leq \sup_{x \in [0,s]} \int_0^x | F(f(t)) - F(g(t)) | \, dt$$

$$\leq \sup_{x \in [0,s]} \int_0^x L \underbrace{|f(t) - g(t)|} dt$$

Assume $|F(y_1) - F(y_2)| \leq L |y_1 - y_2|$

$$\leq L \sup_{x \in [0,s]} \int_0^x \|f - g\|_\infty \, dt$$

$$= \sup_{[0,s]} L \|f - g\|_\infty \cdot x \leq Ls \|f - g\|_\infty$$