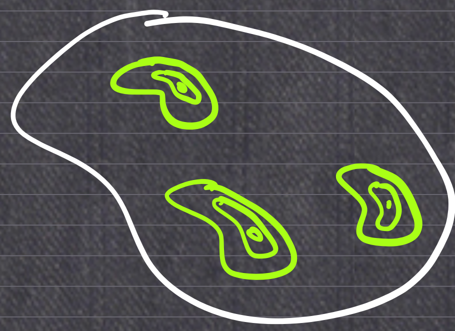
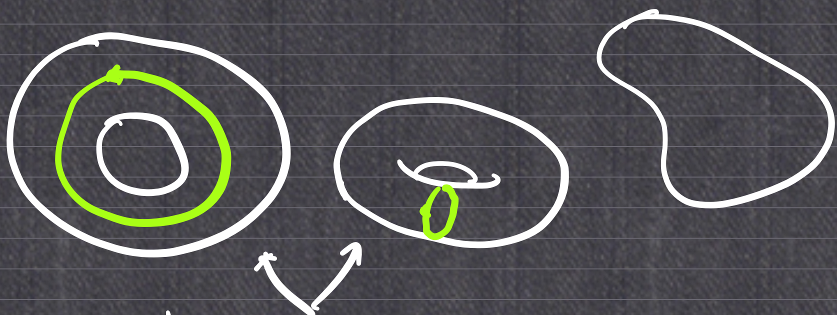
S is Simply-connected $\overset{def}{\iff}$ $\forall$ loop in S can contract to a point without leaving S.





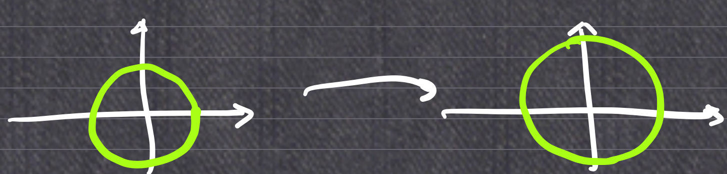
not simply-connected

Fundamental Theorem of Algebra

$$p(z) = \sum_{n=0}^{k} a_n z^n \ , \quad a_n \in \mathbb{C} \qquad (a_0 \neq 0)$$

must a complex root $\alpha \in \mathbb{C}$.

$$p : \mathbb{C} \to \mathbb{C}$$

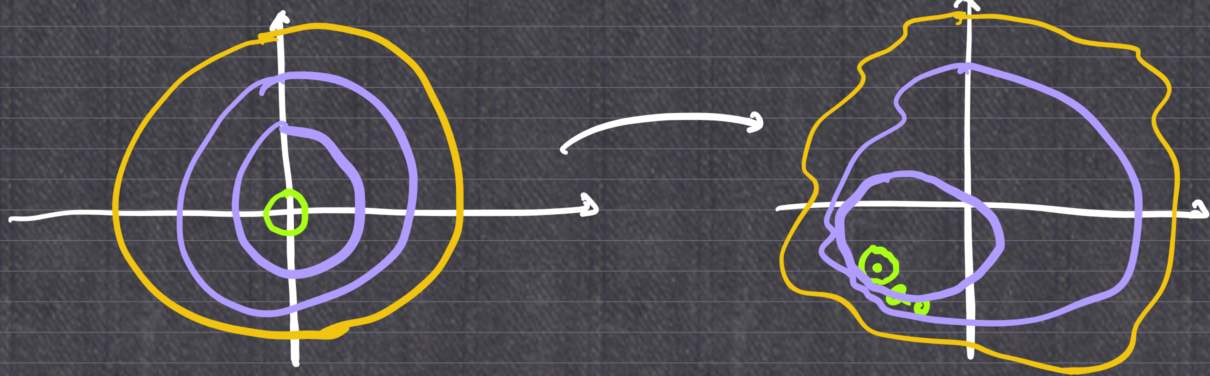


$f(z) = z^n$

$$p(z) = a_n z^n + \cdots + a_1 z + a_0$$

$|z| \sim$ small , $p(z) \approx a_0 \neq 0$

$|z|$ large , $p(z) \approx a_n z^n$



Ch. 3 $\boxed{F : \mathbb{R}^m \to \mathbb{R}^n}$

$f : \mathbb{R}^n \to \mathbb{R}$.
$(x_1, \ldots, x_n)$

$e_i = (0, \ldots, \underset{i}{1}, \ldots, 0)$

$$\frac{\partial f}{\partial x_i}\Big|_{\underbrace{(a_1, \ldots, a_n)}_{\vec{a}}} := \lim_{t \to 0} \frac{f(a_1, \ldots, a_i + t, \ldots a_n) - f(a_1, \ldots, a_n)}{t}$$

$$= \lim_{t \to 0} \frac{f(\vec{a} + t\vec{e}_i) - f(\vec{a})}{t}$$

$$(D_{\vec{u}} f)(a_1, \ldots, a_n) = \lim_{t \to 0} \frac{f(a_1 + tu_1, \ldots, a_n + tu_n) - f(a_1, \ldots, a_n)}{t}$$

$\vec{u} = (u_1, \ldots, u_n)$ $\quad = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f(0+t,0) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^2 \cdot 0}{t^4+0^2} - 0}{t} = \lim_{t \to 0} \frac{0}{t} = 0.$$

$\vec{u} = (u_1, u_2)$

$$D_{\vec{u}} f(0,0) = \lim_{t \to 0} \frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \left(\frac{(tu_1)^2 (tu_2)}{(tu_1)^4 + (tu_2)^2} - 0 \right) \Big/ t$$

$$= \lim_{t \to 0} \frac{t u_1 u_2}{t^2 u_1^4 + u_2^2} = 0.$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$$r(t) = (t, t^2) \longrightarrow 0 \text{ as } t \to 0.$$

$$\lim_{t \to 0} f(r(t)) = \lim_{t \to 0} \frac{t^2(t^2)}{t^4 + (t^2)^2} = \frac{1}{2}.$$

$$\lim_{x \to 0, y=0} f(x,y) = \lim_{x \to 0, y=0} 0 = 0 \quad \times$$

$f: \mathbb{R} \to \mathbb{R}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\iff f(x) = f(a) + f'(a)(x-a) + \underbrace{o((x-a))}_{= h(x)}$$

s.t. $\lim_{x \to a} \frac{h(x)}{x-a} = 0.$

$F: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$

$\overset{\text{def}}{\iff}$ ① $\frac{\partial F}{\partial x_i}(\vec{a})$ exists $\forall i = 1, 2, \ldots, n$

② $F(\vec{x}) = \boxed{F(\vec{a}) + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\vec{a})(x_i - a_i)}$

$$+ \underbrace{o(|\vec{x} - \vec{a}|)}_{= h(\vec{x})} \quad \text{s.t.} \lim_{\vec{x} \to \vec{a}} \frac{h(\vec{x})}{|\vec{x} - \vec{a}|} = 0$$

$z = F(x,y)$

$$F(\vec{a}) + \nabla F(\vec{a}) \cdot (\vec{x} - \vec{a})$$

e.g. $f(x,y) = |xy|$ differentiable at $(0,0)$?

$$\frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \left(f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)\cdot(y-0)\right)}{|(x,y)-(0,0)|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{|xy| - 0}{\sqrt{x^2+y^2}} = \lim_{\substack{r\to 0^+ \\ \theta \text{ anything}}} \frac{r^2|\sin\theta\cos\theta|}{r}$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \lim_{\substack{r\to 0^+ \\ \theta \text{ anything}}} r|\sin\theta\cos\theta|$$

$$\underset{\downarrow\ 0}{0} \le \underbrace{r|\sin\theta\cos\theta|}_{\downarrow\ 0} \le \underset{\downarrow\ 0}{r}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{y} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r\sin\theta} = \lim_{r\to 0} r\frac{\cos^2\theta}{\sin\theta}$$

$$\frac{|xy|}{\sqrt{x^2+y^2}} \le \underbrace{|x|\cdot 1}_{\downarrow\ 0}$$

$$\frac{|y|}{\sqrt{x^2+y^2}} \le \underbrace{\frac{|y|}{\sqrt{y^2}}}_{=1}$$