$f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q},\ \gcd(m,n) = 1,\ m > 0,\ n > 0 \\ 1 & \text{if } x = 0 \end{cases}$$

<u>Ex:</u> f is continuous at $x \in [0,1] \setminus \mathbb{Q}$

but is not continuous at $x \in \mathbb{Q} \cap [0,1]$.

$$f(r) > 0 \quad \forall r \in \mathbb{Q} \cap [0,1].$$

$$s_n \notin \mathbb{Q} \qquad \underbrace{f(s_n)}_{=0} \to 0$$

$s \notin \mathbb{Q}$

<u>Question:</u> ? $\exists f : \mathbb{R} \to \mathbb{R}$

s.t. f is continuous at $a \in \mathbb{Q}$

but not continuous at $a \notin \mathbb{Q}$.

$$D_f = \{ a \in \mathbb{R} : f \text{ is not continuous at } a \}.$$

$$= \bigcup_{i=1}^{\infty} E_i \leftarrow \text{closed.}$$

} HW2.

If $\exists f$ is continuous at $a \in \mathbb{Q}$

discontinuous at $a \notin \mathbb{Q}$.

$$\Rightarrow \quad D_f = \mathbb{R} \setminus \mathbb{Q}.$$

$$\mathbb{R} = \underbrace{D_f}_{\overline{\mathbb{R}\setminus\mathbb{Q}}} \cup \mathbb{Q} = \bigcup_{i=1}^{\infty} E_i \cup \{r_1, r_2, \dots\}$$

$$= \bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} \{r_j\}$$

$\overline{E_i}^{\circ} \overset{?}{=} \phi$: if not, $\exists x \in E_i^{\circ}$

$\Rightarrow \exists \varepsilon > 0$ s.t.

$(x-\varepsilon, x+\varepsilon) \subset E_i \subset \mathbb{R}\setminus\mathbb{Q}$

Baire (c) $\Rightarrow$ $\overline{\mathbb{R} - \bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty}\{r_j\}} = \mathbb{R}$

$$\left(\bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} \{r_j\}\right)^{\circ} = \phi$$

$$\mathbb{R}^{\circ} = \phi$$

Given: (X, d) complete

$\mathcal{U}_1, \mathcal{U}_2, \dots$ open in X s.t. $\overline{\mathcal{U}_i} = X$ $\forall i$

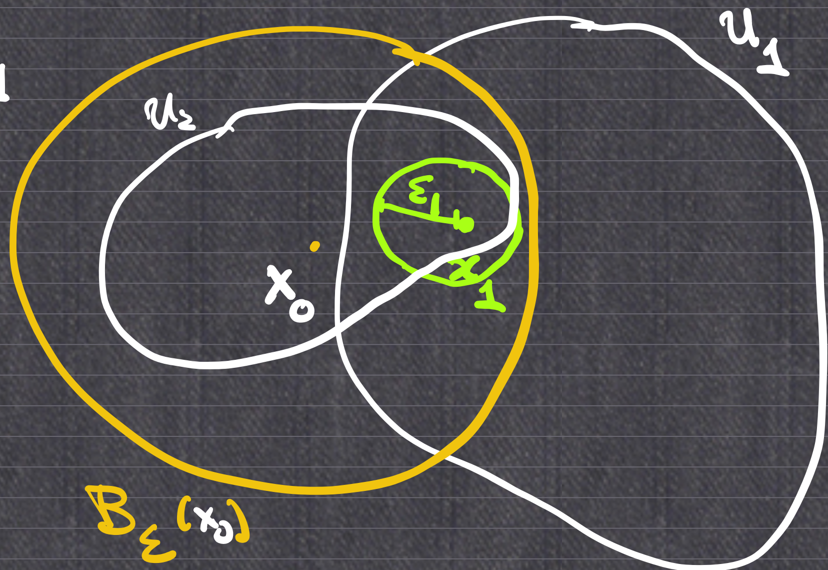
Need:

$$\overline{\bigcap_{i=1}^{\infty} \mathcal{U}_i} = X.$$

$\forall x_0 \in X, \ \forall \varepsilon > 0$.

show $\exists\, y \in B_\varepsilon(x_0) \cap \left(\bigcap_i U_i\right)$

$\exists\, \overline{B_{\varepsilon_1}(x_1)} \subset B_\varepsilon(x_0) \cap U_1$



$\exists\, \overline{B_{\varepsilon_2}(x_2)} \subset B_{\varepsilon_1}(x_1) \cap U_2$

$\exists\, \overline{B_{\varepsilon_3}(x_3)} \subset B_{\varepsilon_2}(x_2) \cap U_3$

$\vdots$

Require $\varepsilon_n < \frac{1}{2^n}$.

$d(x_n, x_{n+1}) \leq \frac{c}{2^n}$. $\Longrightarrow$ $\{x_n\}$ Cauchy.

$X \ni y$

$\forall n \in \mathbb{N}, \ x_n, x_{n+1}, x_{n+2}, \dots \in B_{\varepsilon_n}(x_n)$

$\Downarrow$

$y \in \overline{B_{\varepsilon_n}(x_n)}$

$y \in B_{\varepsilon_{n-1}}(x_n) \leftarrow \subset B_{\varepsilon_{n-1}}(x_{n-1}) \cap U_{n-1}$

$\subset \cdots \subset B_{\varepsilon_0}(x_0) \subset U_{n-1}$

$\therefore$ $y \in U_{n-1}$ $\forall n$ and $y \in B_\varepsilon(x_0)$

$$\Rightarrow \boxed{y \in \bigcap_{i=1}^{\infty} U_i \cap B_\varepsilon(x_0)}$$

∎

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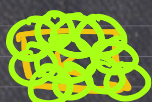
§ 2.3 : Compact set.



$\forall \varepsilon > 0$, $\exists B_\varepsilon(x_1), \dots, B_\varepsilon(x_N)$ s.t. $K \subset B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_N)$

(X, d) metric space, $K \subset X$

TFAE :

1. K is complete and totally bounded
2. K is sequentially compact
   $\forall \{x_n\} \subset K$, $\exists \{x_{n_j}\} \to y \in K$.
3. K is compact
   every open cover of K has a finite subcover.

Ex: $X = \ell_\infty(\mathbb{R}) = \{ \{x_n\}_{n=1}^{\infty} : \{x_n\} \text{ bounded} \}$.

$$\| \{x_n\}_{n=1}^{\infty} \|_\infty = \sup_{n \in \mathbb{N}} |x_n|$$

$e_i = \{0, 0, \dots, \underset{i}{1}, 0, 0, \dots\}$

$\{e_i\}_{i=1}^{\infty}$ is bounded.

$i \neq j$

$$d(e_i, e_j) = \|e_i - e_j\|_\infty = 1$$

any $\{e_{i_k}\}$ is not Cauchy

$\Rightarrow$ $\{e_{i_k}\}$ does not converge.