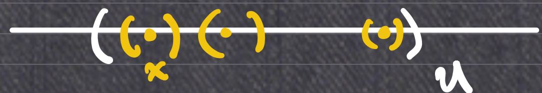
§ 2.2

$\mathcal{U} \subset \mathbb{R}$ is open

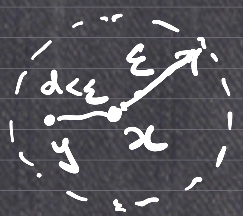
$\overset{\text{def}}{\Longleftrightarrow}$ $\forall x \in \mathcal{U}$, $\exists \varepsilon > 0$ s.t.

$\boxed{(x-\varepsilon, x+\varepsilon)} \subset \mathcal{U}$.



(X, d) metric space

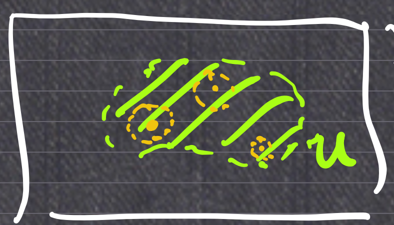
$$B_\varepsilon(x) = \{ y \in X : d(x,y) < \varepsilon \}$$



$\mathcal{U} \subset X$ is <u>open</u> in X

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall x \in \mathcal{U}$, $\exists \varepsilon > 0$ s.t.

$B_\varepsilon(x) \subset \mathcal{U}$.



E is closed in X

$\overset{\text{def}}{\Longleftrightarrow}$ $\underbrace{X \setminus E}_{E^c}$ is open.

<u>e.g.</u> (X, d) metric

$\mathcal{U} := B_r(a)$

$\forall x \in B_r(a)$

take $\varepsilon := \frac{1}{2}(r - d(x,a)) < r - d(x,a)$

Claim: $B_\varepsilon(x) \subset B_r(a)$

Proof: $\forall y \in B_\varepsilon(x), \Rightarrow d(x,y) < \varepsilon.$

$$\Rightarrow d(y,a) \le \underbrace{d(y,x)}_{<\varepsilon} + \underbrace{d(x,a)}_{d(x,a)}$$

$$< r - \cancel{d(x,a)} + \cancel{d(x,a)}$$

$$\Rightarrow y \in B_r(a).$$

$\mathbb{R}^2$, $d_1((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$

$B_1((0,0)) =$ 

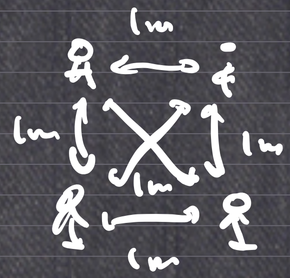
$\mathbb{R}^2$, $d_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

$B_1((0,0)) =$ 

$$\mathbb{R}^2,\ d_0(\vec{x},\vec{y}) = \begin{cases} 1 & \text{if } \vec{x} \neq \vec{y} \\ 0 & \text{if } \vec{x} = \vec{y} \end{cases}$$

$B_2((0,0)) = \mathbb{R}^2$

$B_{\frac{1}{2}}((0,0)) = \{(0,0)\}$



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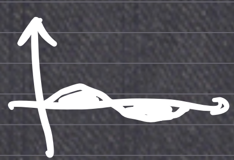
e.g. $X = C[0,1]$

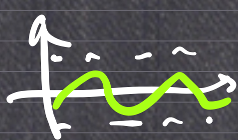
$\|f\|_1 = \int_0^1 |f(x)|\,dx \rightsquigarrow B_1^{d_1}(0)$

$\|f\|_\infty = \max_{[0,1]} |f(x)| \rightsquigarrow B_1^{d_\infty}(0)$

$B_1^{d_1}(0) = \{ f \in C[0,1] \mid \int_0^1 |f(x)|\,dx < 1 \}$

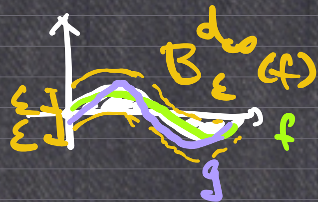


$B_1^{d_\infty}(0) = \{ f \in C[0,1] \mid \sup_{[0,1]} |f(x)| < 1 \}$



① $B_1^{d_1}(0)$ is open in (X, d_∞) ?

② $B_1^{d_\infty}(0)$ is open in (X, d_1) ?

① $f \in B_1^{d_1}(0) \longrightarrow$ 

$\int_0^1 |f(x)|\,dx < 1$

want $\underline{B_\varepsilon^{d_\infty}(f)} \subset \underline{B_1^{d_1}(0)}$

$\forall g \in B_\varepsilon^{d_\infty}(f), \quad d_\infty(f,g) < \varepsilon$

$\Rightarrow \sup_{[0,1]} |f(x)-g(x)| < \varepsilon.$

$\Rightarrow |f(x)-g(x)| < \varepsilon \quad \forall x \in [0,1].$

$$\int_0^1 |g(x)|\,dx \le \int_0^1 \underbrace{|g(x)-f(x)|}_{<\varepsilon} + |f(x)|\,dx$$

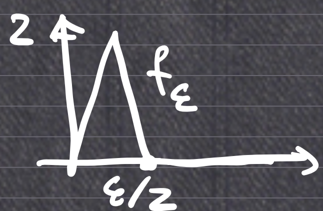
$$< \varepsilon + \underbrace{\int_0^1 |f(x)|\,dx} < 1$$

$\uparrow$

Take $\varepsilon = \underbrace{1 - \int_0^1 |f(x)|\,dx}_{>0}$.

② $B_1^{d_\infty}(0)$ ~~open?~~ in (X, d_1).

$0 \in B_1^{d_\infty}(0) = \mathcal{U}$



$\|f_\varepsilon\|_{d_1} = \frac{\varepsilon}{2} < \varepsilon$

$\Rightarrow f_\varepsilon \in B_\varepsilon^{d_1}(0).$

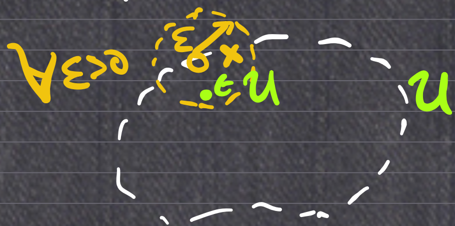
$$f_\varepsilon \notin B_1^{d_\infty}(0)$$

$\forall \varepsilon > 0, \ \exists f_\varepsilon \in B_\varepsilon^{d_1}(0)$ but $f_\varepsilon \notin B_1^{d_\infty}(0)$

$\Rightarrow \quad B_\varepsilon^{d_1}(0) \not\subset B_1^{d_\infty}(0).$

Def 2.29

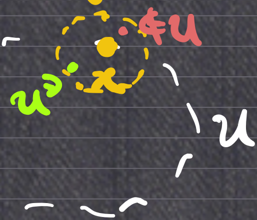
limit point



$\Rightarrow$ x is a limit point of U.

$U' :=$ the set of all limit points of U.

boundary point : $\forall \varepsilon > 0$



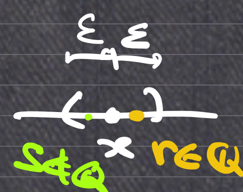
$\Rightarrow$ x is a boundary point of U

$\partial U :=$ the set of boundary points of U.

e.g. $\quad \mathbb{Q} \subset (\mathbb{R}, |x-y|)$

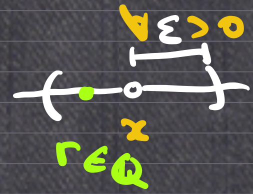
$\forall x \in \mathbb{R}, \quad B_\varepsilon(x) \cap \mathbb{Q} \neq \phi$

$B_\varepsilon(x) \cap \mathbb{Q}^c \neq \phi$



$\Rightarrow \quad x \in \partial \mathbb{Q}.$

$\mathbb{R} \subset \partial \mathbb{Q} \subset \mathbb{R} \Rightarrow \partial \mathbb{Q} = \mathbb{R}.$

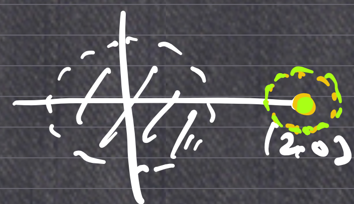


$$\left.\begin{array}{l} \forall x \in \mathbb{R},\ \forall \varepsilon > 0, \\ \mathbb{Q} \cap \big((x-\varepsilon, x+\varepsilon) \setminus \{x\} \big) \\ \neq \emptyset \\ \Rightarrow \quad x \in \mathbb{Q}' \end{array}\right\} \Rightarrow \mathbb{Q}' = \mathbb{R}.$$

eg. 2.30

$\mathbb{R}^2, \ \|\vec{x} - \vec{y}\|_2$

$S := B_1(0) \cup \{(2,0)\}$



$(2,0) \notin S'.$

$(2,0) \in \partial S.$

Prop. 2.32 : $S' \cup S = \partial S \cup S$

"$\subset$" : $\forall x \in S' \cup S$

Case ①: if $x \in S$, $x \in \partial S \cup S$.

Case ②: if $x \notin S$, then $x \in S'$.

$\Rightarrow \forall \varepsilon > 0, \ (B_\varepsilon(x) \setminus \{x\}) \cap S \neq \emptyset.$

$\Rightarrow x \in \partial S.$

$$\boxed{\overline{S} := \partial S \cup S}$$

closure of S.