$f(x,y)$ is differentiable at (a,b)

$\overset{\text{def}}{\Leftrightarrow}$ $\begin{cases} ① & \frac{\partial f}{\partial x}(a,b) \text{ and } \frac{\partial f}{\partial y}(a,b) \text{ exist} \\ ② & f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) \end{cases}$

$$+ o\left(|(x,y)-(a,b)|\right) \quad \text{as } (x,y)\to(a,b)$$

$\overset{\text{HW3}}{\Longleftrightarrow}$ $\exists\, c_1, c_2 \in \mathbb{R}$ s.t.

$$f(x,y) = f(a,b) + c_1(x-a) + c_2(y-b) + o\left(|(x,y)-(a,b)|\right)$$

as $(x,y)\to(a,b)$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - c_1(x-a) - c_2(y-b)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

$F(x,y) = (u(x,y),\ v(x,y)) \quad : \mathbb{R}^2 \to \mathbb{R}^2$

$u(x,y) =$. — –

$v(x,y) =$ – – .

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} + \frac{\partial(u,v)}{\partial(x,y)}(a,b) \begin{bmatrix} x-a \\ y-b \end{bmatrix} + o\left(|(x,y)-(a,b)|\right)$$

C^1 : partial derivatives exist and are continuous.

$\Downarrow$

differentiable

$$f(x,y) = \log(1+x^2+y^2)$$

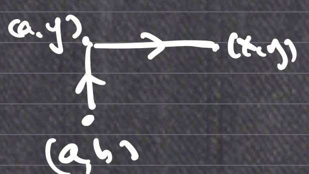
$$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{2y}{1+x^2+y^2} \qquad \text{continuous.}$$

$\Rightarrow$ f is C^1 on $\mathbb{R}^2$.

$\Rightarrow$ f is differentiable on $\mathbb{R}^2$

<u>Proof</u> ($C^1 \Rightarrow$ differentiable)

$$f(x,y) - f(a,b) = f(x,y) - f(a,y) + \underline{f(a,y) - f(a,b)}$$



$$= \frac{\partial f}{\partial x}(c_1, y)\,(x-a) + \frac{\partial f}{\partial y}(a, c_2)\cdot(y-b)$$

$\exists c_1 \in (a,x) \cup (x,a)$ $\quad$ $\exists c_2 \in (b,y) \cup (y,b)$

$$\boxed{\lim_{(x,y)\to(a,b)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(a,b)}$$

$\Leftrightarrow$ $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial x}(a,b) + o(1)$

as $(x,y) \to (a,b)$

$$= \left(\frac{\partial f}{\partial x}(a,b) + o(1)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b) + o(1)\right)(y-b)$$

$$= \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) + o(\underbrace{|x-a|}) + o(|y-b|).$$

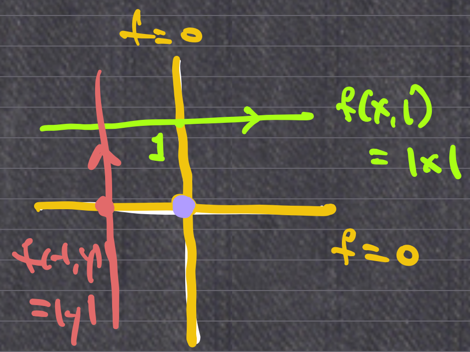
$$\le \sqrt{(x-a)^2+(y-b)^2}$$

$$= o(|(x,y)-(a,b)|).$$

differentiable $\not\Longrightarrow C^1$.

$f(x,y) = |xy|$.

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$



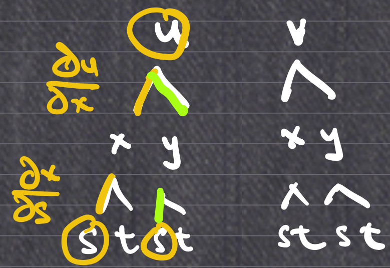
$G : \mathbb{R}^m \to \mathbb{R}^n$

$F : \mathbb{R}^n \to \mathbb{R}^k$

$F \circ G$

$(u,v) = F(x,y)$

$(x,y) = G(s,t)$



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

$$D(F\circ G)(a,b) = DF(G(a,b)) \cdot DG(a,b)$$

$$\frac{\partial(u,v)}{\partial(s,t)}(a,b) = \frac{\partial(u,v)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(s,t)}$$

$$\begin{array}{c} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \\ \begin{matrix} \uparrow \quad & \quad \uparrow \\ \frac{\partial}{\partial s} \quad & \quad \frac{\partial}{\partial t} \end{matrix} \end{array} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$F \circ G(\vec{x}) \qquad \vec{y} = G(\vec{x})$$

$$F(\vec{y}).$$

$$G(\vec{x}) = G(\vec{a}) + DG(\vec{a})(\vec{x}-\vec{a}) + o(|\vec{x}-\vec{a}|) \quad \text{as } \vec{x} \to \vec{a}$$

$$F(\vec{y}) = F(G(a)) + DF(G(a))(\vec{y} - G(a)) + o(|\vec{y} - G(a)|) \quad \text{as } \vec{y} \to G(a)$$

$$\Rightarrow F(G(x)) = F(G(a)) + DF(G(a))(G(x) - G(a)) + o(|G(x) - G(a)|)$$

$$= F(G(a)) + DF(G(a)) \cdot (DG(a) \cdot (x-a) + o(|x-a|)) + o(|G(x) - G(a)|)$$

$$= F(G(a)) + DF(G(a)) \cdot DG(a)\,(x-a) + DF(G(a)) \cdot o(|x-a|) + o(|G(x) - G(a)|)$$

$$D_{\vec{u}} f(a,b) = \lim_{t \to 0} \frac{f(a+tu_1, b+tu_2) - f(a,b)}{t}$$

$$\vec{u} = (u_1, u_2) \qquad = \lim_{t \to 0} \frac{F(t) - F(0)}{t}$$

$$F(t) := f(\underbrace{a+tu_1}_{x}, \underbrace{b+tu_2}_{y}) \qquad = F'(0).$$

$$F = f(\cdot,\cdot)$$

$$x \quad y$$

$$t \quad t$$

$$F' = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2$$

$$F'(0) = \frac{\partial f}{\partial x}(a,b)\,u_1 + \frac{\partial f}{\partial y}(a,b)\,u_2$$

$$= \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right) \cdot (u_1, u_2)$$

§ 3.2 :