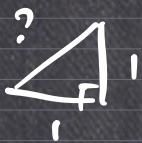


$$\mathbb{N} < \mathbb{Z} < \mathbb{Q} \subset \mathbb{R}$$



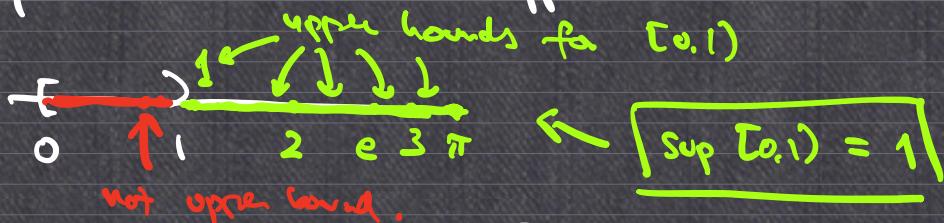
Supremum and infimum.

Generalization of:

maximum and minimum.

$$\begin{array}{c} \text{maximum} \\ \text{o} \\ \text{minimum} \end{array} \quad y = \text{maximum of } S \quad \begin{array}{l} \xleftarrow{\text{def}} \begin{cases} y \geq x & \forall x \in S \\ y \in S \end{cases} \end{array}$$

Supremum = least upper bound



y = "The" supremum of S

$\xleftarrow{\text{def}}$ $\begin{cases} y \text{ is an upper bound for } S. \\ \text{if } x \text{ is an upper bound for } S, \text{ then } y \leq x. \end{cases}$

if $y > x$, then x is not an upper bound for S .

$$y =: \underline{\sup S}$$

infimum := greatest lower bound.

Claim: Supremum of a set $S \neq \emptyset$ is unique.

Proof: let L and M be supremums of S .

$L \geq M$

\nearrow is a sup \nwarrow is sup
 \Rightarrow is an upper bound \rightarrow least upper bound

$L \leq M$

left as an exercise.

$$\therefore \boxed{L = M}$$

COMPLETENESS AXIOM:

If $\phi \neq S \subset \mathbb{R}$ is bounded from above.

$\exists y \in \mathbb{R}$ s.t. $y \geq x$

then $\sup S$ exists $\in \mathbb{R}$

$\sup S$ $\in \mathbb{R}$ $\forall x \in S$.

Dedekind cut

Cauchy completion.

$$S := \mathbb{Q} \cap (-\sqrt{2}, \sqrt{2})$$

$\mathbb{Q} \hookrightarrow S$

$$\sup S = \sqrt{2} \notin \mathbb{Q}$$



Prop 1.5 $\phi \neq S \subset \mathbb{R}$ bounded above.

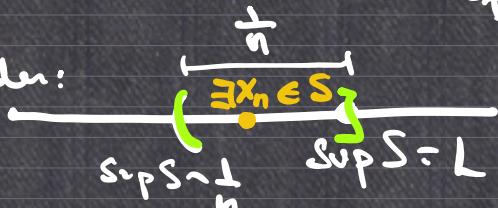
$$L = \sup S$$

$\Leftrightarrow \begin{cases} \textcircled{1} & L \text{ is an upper bound for } S. \\ \textcircled{2} & \exists \{x_n\} \in S \text{ s.t. } x_n \rightarrow L. \end{cases}$

(\Rightarrow) Given $L = \sup S$

$\Rightarrow \textcircled{1}$ is true (by definition of $\sup S$)

$\forall n \in \mathbb{N}$, consider:



$\Rightarrow \sup S - \frac{1}{n} < x_n \leq \sup S \quad \forall n \in \mathbb{N}$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ \sup S & \underline{\sup S} & \sup S \end{matrix}$$

(\Leftarrow) Exercise.

Ex: $S = \left\{ \frac{1}{n} + \frac{1}{2^m} : m, n \in \mathbb{N} \right\}.$

WTS: $\inf S = 0$ ~~exists~~

$\textcircled{1} \quad 0 < \frac{1}{n} + \frac{1}{2^m} \quad \forall m, n \in \mathbb{N}$

$\Rightarrow 0$ is a lower bound for S .

$\textcircled{2} \quad \exists \left\{ \frac{1}{n} + \frac{1}{2^m} \right\} \rightarrow 0$

$$\overbrace{\in S}$$

$$\therefore 0 = \inf S.$$

Ex: $\phi \neq S < (0, \infty)$

$$T := \left\{ \frac{1}{x} : x \in S \right\}.$$

Claim: $\sup S = \frac{1}{\inf T}$

Proof: ① $\forall x \in S$,

WTS: $\exists ? \in S \rightarrow \frac{1}{\inf T}$

$$x = \frac{1}{\frac{1}{x}} \geq \inf T \Leftrightarrow \frac{1}{\inf T}$$

an upper bound for S .

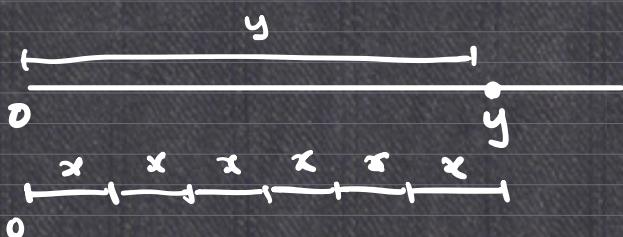
$$\exists \{y_n\} \subset T \text{ s.t. } y_n \rightarrow \inf T.$$

$$\frac{1}{y_n} \in S$$

$$\rightarrow \frac{1}{\inf T}$$

$$\sup S = \frac{1}{\inf T}$$

Theorem: (Archimede's Principle)



Given any $x, y > 0$, $(\exists n \in \mathbb{N} \text{ s.t. } nx > y)$ *

Proof: $S := \{nx : n \in \mathbb{N}\} \neq \emptyset$

Assume (*) is false:

$\forall n \in \mathbb{N} \text{ s.t. } nx \leq y.$

y is an upper bound for S .

$\Rightarrow \sup S$ exists in \mathbb{R}

compl. nos.

$$\begin{array}{c} \frac{x}{2} \\ \hline (\bullet) \end{array} \quad \sup S - \frac{x}{2} < \sup S \quad m \geq x = (m+1)x \quad \in S$$

$\exists t \in S$ 
" "
 $mx, \exists m \in \mathbb{N}.$

$$\sup - \frac{x}{2} < mx < \sup$$

$$(\sup - \frac{x}{2}) + \frac{x}{2} < mx + x < \sup + 1$$

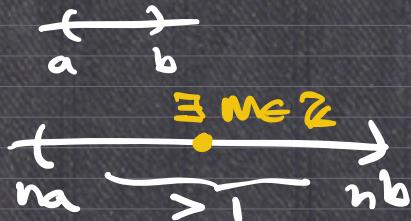
Prop: (Density of \mathbb{Q})

$\forall a < b$, then $\underbrace{(a, b) \cap \mathbb{Q}}_{\subset \mathbb{R}} \neq \emptyset$.

$\exists r \in \mathbb{Q} \text{ s.t. } a < r < b.$

Proof:

$$\text{AP} \quad \left\{ \exists n \in \mathbb{N} \right.$$



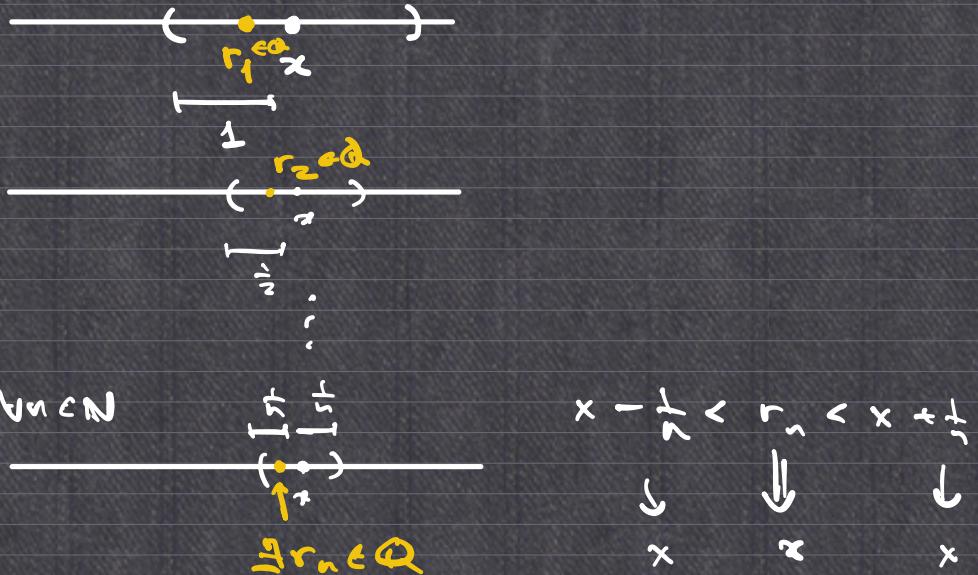
$$\left\{ \begin{array}{l} \exists n \in \mathbb{N} \\ \text{s.t.} \\ n(b-a) > 1 \end{array} \right.$$

$$na < m < nb$$

$$\Rightarrow a < \frac{m}{n} < b$$

EQ.

Claim: $\forall x \in \mathbb{R}, \exists \{r_n\} \subset \mathbb{Q}$ s.t. $r_n \rightarrow x$.



$$S = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$$

$\left\{ \begin{array}{l} \textcircled{1} \quad \sqrt{2} \text{ is an upper bound for } S \\ \textcircled{2} \quad \exists \{r_n\} \subset \mathbb{Q} \rightarrow \sqrt{2} \\ \qquad (\sqrt{2}, \sqrt{2}) \end{array} \right.$

$\Rightarrow \sqrt{2} = \sup S.$

$$\cdot \phi \neq S \subset T \subset \mathbb{R}$$

$$\underbrace{\sup S}_{\text{least ub for } S} \leq \underbrace{\sup T}_{\text{upper ub for } S}$$

$$\underbrace{\inf S}_{\text{greatest lnb.}} \geq \underbrace{\inf T}_{\text{a lower bound for } S}$$

of \mathcal{S}

$$\begin{aligned}\sup \phi &= -\infty \\ \inf \phi &= +\infty\end{aligned}$$

Any real $x \in \mathbb{R}$ is an upper b. for ϕ .

Proof by contradiction :

Assume $\exists x \in \mathbb{R}$ which is not an upper bound for ϕ .

\Rightarrow not (x is an upper bound for ϕ)

\Rightarrow not ($\forall y \in \phi, x \geq y$)

\Rightarrow $\exists y \in \phi, \text{ s.t. } x < y.$

$\cancel{x > y},$

x^y

$$x^3 := x \cdot x \cdot x$$

$$x^m \cdot x^n = x^{m+n}.$$

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

Prop: $\forall x_0 > 1, n \in \mathbb{N}, \exists! t > 0 \text{ s.t. } t^n = x_0$

Proof: $S = \{t > 0 : t^n < x_0\}$
Claim: $(\sup S)^n = x_0.$



Case ①: $\underline{x_0 > 1}$

$\phi \neq S \Rightarrow \{1\}.$

Claim: $1+x_0$ is an upper bound for S .

Pf. By contradiction. $\exists t \in S$ s.t.

$$t \geq 1+x_0$$

$$\Rightarrow \underbrace{x_0}_{t \text{ in } S} > t^n \geq (1+x_0)^n > 1+x_0$$

$\phi \neq S$, bound above $\Rightarrow \sup S$

Claim: $(\sup S)^n = x_0$

Proof: $\exists \{t_k\} \in S \rightarrow \sup S$.

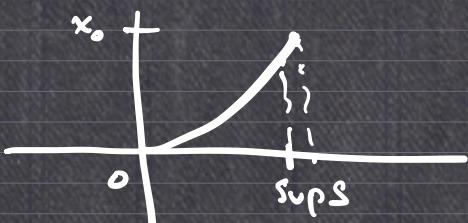
$$t_k^n < x_0$$

\downarrow $k \rightarrow \infty$

$$(\sup S)^n \leq x_0$$

Next: Assume $(\sup S)^n < x_0$,

Try to derive contradiction



Want to find
 \downarrow
 $(\sup S)^n \uparrow x_0$

$$(\sup S + h)^n \quad \exists h > 0 \text{ small}$$

Need:
 $(\sup S + h)^n - (\sup S)^n$

$\forall h \in (0, 1)$

$$< x_0 - (\sup S)^n$$

$$(\sup S + h)^n - (\sup S)^n$$

$$= \underbrace{(\sup S + h - \sup S)}_h \left(\sum_{j=0}^{n-1} (\sup S + h)^j (\sup S)^{n-j-1} \right) \leq \sup S + 1$$

$$\leq h \underbrace{\sum_{j=0}^{n-1} (\sup S + 1)^j}_{\substack{\text{if } \\ C}} (\sup S)^{n-j-1} = Ch$$

$$\text{Choose } h = \min \left\{ \frac{1}{2}, \frac{x_0 - (\sup S)^n}{2C} \right\}.$$

$$\Rightarrow (\sup S + h)^n - (\sup S)^n$$

$$h < 1 \Rightarrow ch \leq C \frac{x_0 - (\sup S)^n}{2C} < x_0 - (\sup S)^n$$

$$\Rightarrow \sup S + h \in S$$



uniqueness:

$$\text{If } y_1^n = y_2^n = x_0$$

$$\Rightarrow 0 = y_1^n - y_2^n = (y_1 - y_2) \underbrace{(+ + - -)}_{(+)}$$

Case ②: $x_0 = 1$ trivial.

Case ③: $x_0 \in (0, 1) \rightsquigarrow \frac{1}{x_0} > 1 \Rightarrow \exists t > 0 \text{ s.t. } t^n = \frac{1}{x_0} \Rightarrow \left(\frac{1}{t}\right)^n = x_0$

$$\underline{x > 0}: \quad x^{\frac{1}{n}} \quad \rightsquigarrow \quad \underbrace{x^{\frac{m}{n}} := (x^{\frac{1}{n}})^m}$$

$$\text{check: } \frac{m}{n} = \frac{p}{q} \Rightarrow (x^{\frac{1}{n}})^m = (x^{\frac{1}{q}})^p$$

$p, q, m, n \in \mathbb{N}$

$$\boxed{x > 1}, \quad \boxed{y > 0}$$

$$x^y = ?$$

$$S(x, \sqrt{2}) = \{x^1, x^{1.4}, x^{1.41}, \dots\}$$

$$S(x, y) := \{x^r \mid 0 < r < y, r \in \mathbb{Q}\}$$

$$x^y := \sup_{r \in \mathbb{Q}_+} S(x, r)$$



$$\begin{aligned} \text{Check } r \in \mathbb{Q}_+, \quad x^r &= \sup_{s \in \mathbb{Q}} S(x, s) \\ &= \sup \{ x^s \mid 0 < s < r, s \in \mathbb{Q} \} \end{aligned}$$

$$\forall s \in (0, r) \cap \mathbb{Q}, \quad x^s < \underbrace{x^r}_{\substack{\text{upper} \\ \text{bd for } S(x, r)}} \Rightarrow \sup_{s \in \mathbb{Q}} S(x, s) \leq x^r.$$

$$\begin{array}{ccc} s_j \in \mathbb{Q} & \xrightarrow{s_j \rightarrow r} & s_j - r \rightarrow 0 \\ & \xrightarrow{x^{s_j} \xrightarrow{s_j \rightarrow r} x^r} & \end{array}$$

$S(x, r)$

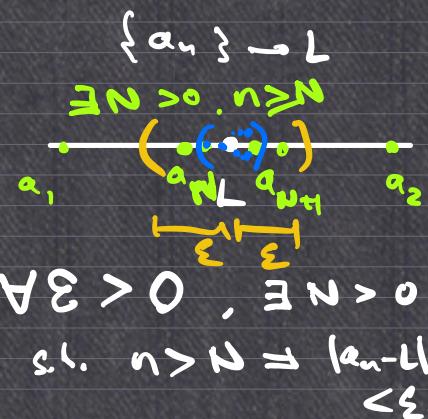
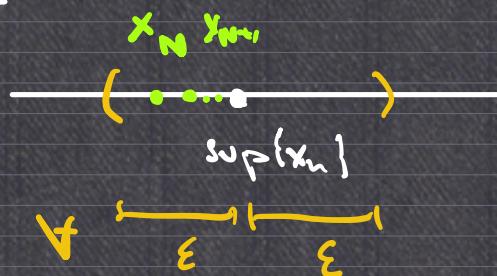
$$|x^{s_j} - x^r| = |x^r (x^{s_j - r} - 1)| = |x^r| \underbrace{|x^{s_j - r} - 1|}_{\downarrow 0}$$

Thm:

$\{x_n\}$ monotone increasing \Rightarrow and bounded above,

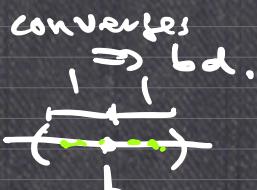
then $\lim_{n \rightarrow \infty} x_n = \sup \{x_n\}$.

Proof:



Thm Bolzano - Weierstrass.

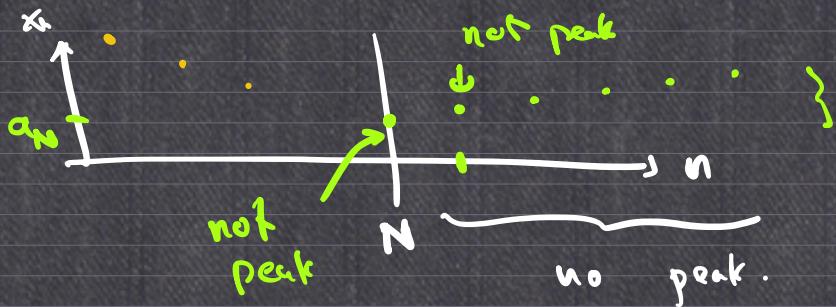
Any bounded sequence $\{x_n\}$
must have a converging
subsequence.



Proof: Case ① \exists infinitely peaks



Case ② only finitely many peaks.



Thm Cauchy seq. must converge.

Cauchy seq. $\left(\forall \varepsilon > 0, \exists N > 0 \text{ s.t. } m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon \right)$

$$\xrightarrow{x_m \ x_n} (\dots) \quad (\infty)$$

Outline: ①

$$\xrightarrow{\substack{x_N \ x_n \\ \dots}} \quad \forall n > N \quad |x_n - x_N| < 1$$

Cauchy \Rightarrow bounded.

② BW \Rightarrow $\{x_n\}$ has a converging subseq. $\{x_{n_k}\} \xrightarrow{\mathbb{R}} L$

$$|x_n - L| = |(x_n - x_{n_k}) + (x_{n_k} - L)|$$

$$\leq |x_n - x_{n_k}| + |x_{n_k} - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

§ 1.2 \limsup \liminf .

$\{x_n\}$ bounded seq.

$\underline{\text{LIM}} \{x_n\} := \left\{ \begin{array}{l} \text{set of limits of converging} \\ \text{subseq. of } \{x_n\} \end{array} \right\}$

limit set
of $\{x_n\}$



$$\{(-1)^n\} = \{(-1), 1, (-1), 1, (-1), 1, \dots\}$$

$$\underline{\text{LIM}} \{(-1)^n\} = \{-1, 1\}$$

$$\limsup_{n \rightarrow \infty} x_n = \sup \underline{\text{LIM}} \{x_n\}$$

$$\liminf_{n \rightarrow \infty} x_n = \inf \underline{\text{LIM}} \{x_n\}$$

$$\limsup (-1)^n = 1$$

$$\liminf (-1)^n = -1$$

Def: L is a limit point
of $\{x_n\} \iff \exists \{x_{n_k}\} \rightarrow L$.

Prop 1.24: $\{x_n\} \subset \mathbb{R}, L \in \mathbb{R}$

$$\textcircled{1} \quad L \in \underline{\text{LIM}} \{x_n\}$$

$$\hookrightarrow \textcircled{2} \quad \forall \varepsilon > 0, \exists \text{ infinitely } n's \text{ s.t.}$$

$$x_n \in (L-\varepsilon, L+\varepsilon)$$



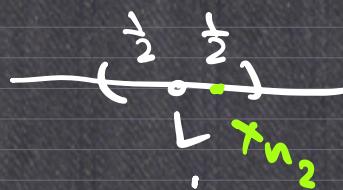
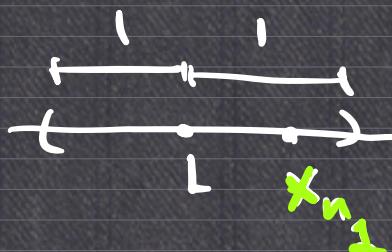
∞ many.

$\textcircled{1} \Rightarrow \textcircled{2} : \exists x_{n_k} \rightarrow L$



x_{n_k} $k \gg 1.$

$\textcircled{2} \Rightarrow \textcircled{1} :$



$n_2 > n_1$

$\exists x_{n_k} \text{ s.t. } x_{n_k} \in \left(L - \frac{1}{k}, L + \frac{1}{k}\right)$

$$\lim \{-1\}^n = \{-1, 1\}$$



e.g.

$$\{x_n\} = \left\{ 1 - \frac{1}{n}, -1 + \frac{1}{n} \right\}$$

$1, -1 \in \lim.$



$$\limsup_{n \rightarrow \infty} x_n = \sup L(M \{x_n\})$$

$$\liminf_{n \rightarrow \infty} x_n = \inf L(M \{x_n\}).$$

Ex. 1.27

$$x_n = \begin{cases} 1/P_2 & \text{if } n = P_1^\alpha \\ \frac{P_{m_1}}{P_{m_k}} & \text{if otherwise} \end{cases}$$

$$0 < x_n < 1$$

$$\begin{aligned} P_{m_1}^{\alpha_1} \cdots P_{m_k}^{\alpha_k} \\ P_{m_1} < P_{m_2} < \cdots < P_{m_k} \\ k \geq 2. \end{aligned}$$

$$L(M \{x_n\}) \subset [0, 1]$$

list of primes:

$$\{P_1, P_2, P_3, P_4, \dots\}$$

$$\limsup \leq 1$$

$$\liminf \geq 0.$$

$$\begin{aligned} & \{x_{P_1}, x_{P_2}, x_{P_3}, \dots\} \\ &= \left\{ \frac{1}{P_1}, \frac{1}{P_2}, \frac{1}{P_3}, \dots \right\} \end{aligned}$$

$$\liminf = 0. \quad \overbrace{\longrightarrow}^{\text{?}} \quad \underline{\text{?}}$$

$$x_b = x_{2 \cdot 3} = \frac{2}{3}, \quad x_{12} = x_{2 \cdot 3} = \frac{2}{3}$$

$$x_{P_n P_{n+1}} = \frac{P_n}{P_{n+1}} \rightarrow ?$$

$$\underbrace{\lim_{n \rightarrow \infty} \frac{P_n}{n \log n} = 1}_{\text{Prime Number Theorem.}}$$

$$X = \frac{P_n}{P_{n+1}} = \underbrace{\frac{P_n}{n \log n}}_1 \cdot \underbrace{\frac{(n+1) \log(n+1)}{P_{n+1}}}_1 \cdot \underbrace{\frac{n \log n}{(n+1) \log(n+1)}}_1$$

$\limsup x_n = 1$

(1)

$$\frac{P_{2n}}{P_{3n}} = \frac{P_{2n}}{P_{3n}} \sim \frac{2n \log(2n)}{3n \log(3n)} \rightarrow \frac{2}{3}$$

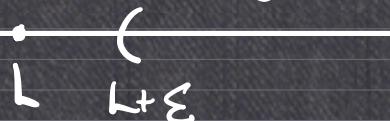
$$\left| \frac{P}{n \log n} - 1 \right| < ?$$

$x_n = n \rightarrow +\infty$
 $\{x_n\} = \{1, 2, \dots\}$
 $\limsup x_n = +\infty$.

$L = \limsup x_n \in \mathbb{R}$.

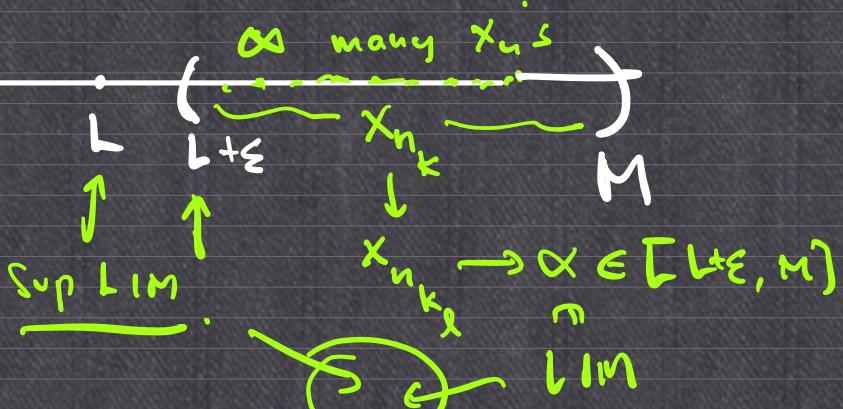
finitely many x_n 's.

(1)



$\forall \varepsilon > 0$

if:

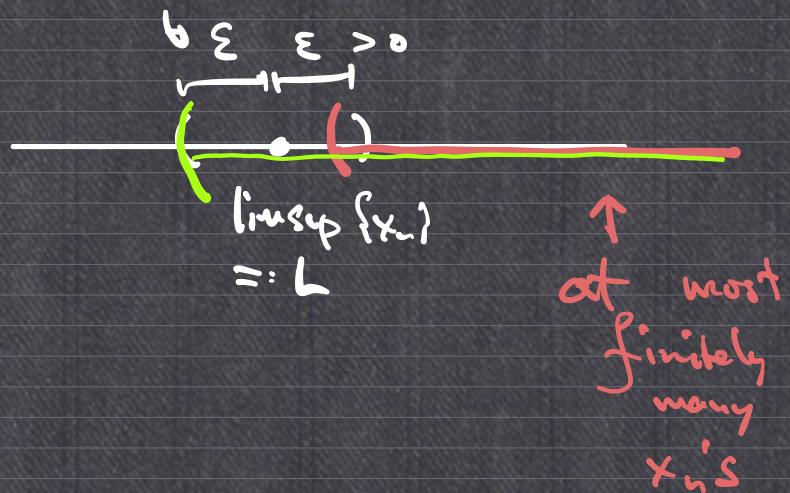


(2)



$x_n \in \sum_{n>N}$ infinitely many x_n 's $\Rightarrow \lim < \sum_{n \geq N}$
 if: $L - \varepsilon < \sup \{x_n\}$ at most finitely x_n 's $\limsup \leq L - \varepsilon$

$\boxed{\limsup x_n \in \text{LIM}}$



Example 1.30 $\{x_n\}$

$$\liminf x_n \leq \liminf \frac{x_1 + \dots + x_n}{n} \leq \limsup \frac{x_1 + \dots + x_n}{n} \leq \limsup x_n$$

WTP: $\forall \varepsilon > 0$
 $\limsup \frac{x_1 + \dots + x_n}{n} \leq \limsup x_n + \varepsilon$ | let $\varepsilon \rightarrow 0^+$ $\Rightarrow \dots \leq \dots$

$\limsup x_n$ $\limsup x_n + \varepsilon$
 at most finitely
 many x_n .

$\exists N > 0$ s.t.

$$n > N \Rightarrow x_n \leq \limsup x_k + \varepsilon$$

$$\frac{x_1 + \dots + x_n}{n} = \frac{x_1 + \dots + x_N}{n} + \frac{x_{N+1} + \dots + x_n}{n}$$

$$\frac{x_1 + \dots + x_n}{n} \leq \frac{x_1 + \dots + x_N}{n} + \frac{(n-N)(\limsup + \varepsilon)}{n}$$

$$\limsup \frac{x_1 + \dots + x_n}{n} \stackrel{1}{\leq} \limsup \underbrace{\frac{x_1 + \dots + x_N}{n} + \frac{(n-N)(\limsup x_k + \varepsilon)}{n}}_{\text{lim.} \quad \text{c. c. v.}}$$

$$= 0 + 1 \cdot (\limsup x_k + \varepsilon)$$

$$= \limsup x_k + \varepsilon$$

$$\liminf_{n \rightarrow \infty} x_n = \inf \lim\{x_n\}$$

$$\limsup_{n \rightarrow \infty} x_n = \sup \lim\{x_n\}.$$

$$M_n := \sup \{x_n, x_{n+1}, x_{n+2}, \dots\} \quad \text{→ } \{x_n\} \text{ bounded}$$

$$\lim_{n \rightarrow \infty} M_n = \lim_{n \rightarrow \infty} \sup \{x_n, x_{n+1}, \dots\}.$$

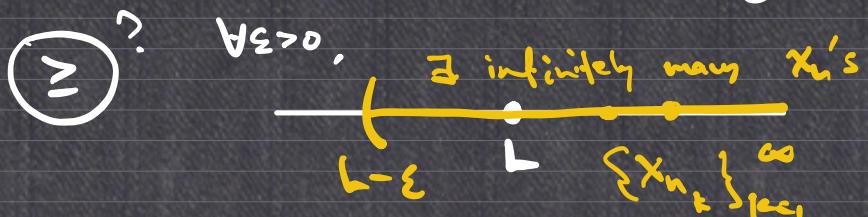
Claim: $\lim_{n \rightarrow \infty} M_n = \limsup_{n \rightarrow \infty} x_n =: L$

Proof: $\forall \varepsilon > 0$, $\exists N$ s.t. $x_n \in [L, L+\varepsilon]$ finitely many x_i 's.

$$\Rightarrow n \geq N \Rightarrow M_n = \sup \{x_n, x_{n+1}, x_{n+2}, \dots\} \leq L + \varepsilon$$

$$\lim_{n \rightarrow \infty} M_n \leq L + \varepsilon. \quad \Rightarrow \quad \boxed{\lim_{n \rightarrow \infty} M_n \leq L} -$$

let $\varepsilon \rightarrow 0^+$



$$M_n = \sup \{x_n, x_{n+1}, x_{n+2}, \dots\} \geq x_{n_k} > L - \varepsilon$$

\downarrow $\exists x_{n_k} \exists k.$

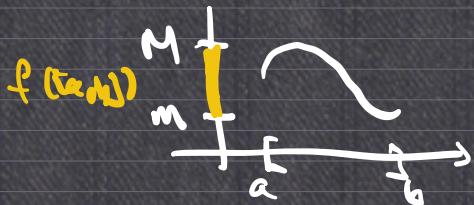
$$\lim_{n \rightarrow \infty} M_n \geq L - \varepsilon \quad \text{let } \varepsilon \rightarrow 0^+ \quad \boxed{\lim_{n \rightarrow \infty} M_n \geq L}$$

EUT . INT

EUT: $f: [a,b] \rightarrow \mathbb{R}$ max. min. point exists.

cts

$f([a,b])$ is closed and bounded



INT: $f: [a,b] \rightarrow \mathbb{R}$ cts

$$\boxed{f(a) < 0 < f(b) \Rightarrow \exists c \in (a,b) \text{ s.t. } f(c) = 0.}$$

$f(\text{interval})$ = interval.



$f(\text{connected set})$ = connected set.

§ 1.3

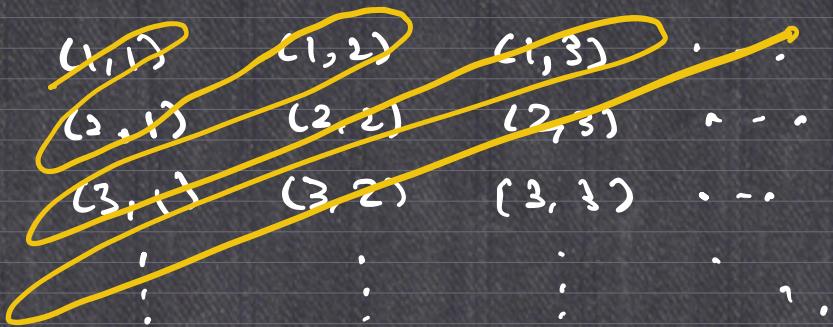
$\left\{ \begin{array}{l} \emptyset \neq S \text{ is countable} \\ \Leftrightarrow \exists f: S \rightarrow \mathbb{N} \text{ injective.} \end{array} \right.$

- $S = \{ \text{FF, Ivan, Yan Min, Henry Cheung} \}$

$$f \downarrow \quad \begin{matrix} \downarrow & \downarrow & \downarrow & \overline{\downarrow} & \overline{\downarrow} \\ \{380, 1, 2, 3\} & \subset \mathbb{N}. \end{matrix}$$

- $S = \{2, 4, 6, 8, 10, \dots\}$
- $\begin{matrix} f \\ \downarrow \\ \mathbb{N} \end{matrix}$
 ↓ ↓ ↓ ↓ ↓ ...
 1 2 3 4 5 ...
- $f(2) = 1$
 $f(4) = 2$
 $f(6) = 3$
- $f(2k) = k \quad \forall k \in \mathbb{N}.$

- $\mathbb{N} \times \mathbb{N} = \{(m, n) : m, n \in \mathbb{N}\}.$



- $S \subset T$
 \hookrightarrow countable $\Rightarrow S$ countable.

$\exists f: T \rightarrow \mathbb{N}$ injective.

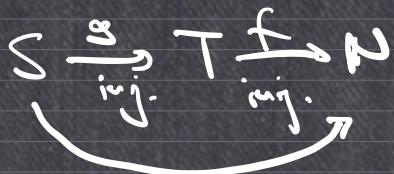
$f|_S: S \rightarrow \mathbb{N}, \quad f|_S(x) = f(x)$

$f|_S(x) = f|_S(y) \quad \exists x, y \in S$

$\Rightarrow f(x) = f(y)$

$\Rightarrow x = y. \quad (f \text{ is injective})$

- $S \xrightarrow[\text{injective}]{} T \hookrightarrow$ countable $\Rightarrow S$ countable.



$f \circ g$
l circ

$$f(g(x))$$

- \mathbb{Q} is countable

$$\{\dots, -3, -2, -1, 0, 1, 2, \dots\}$$

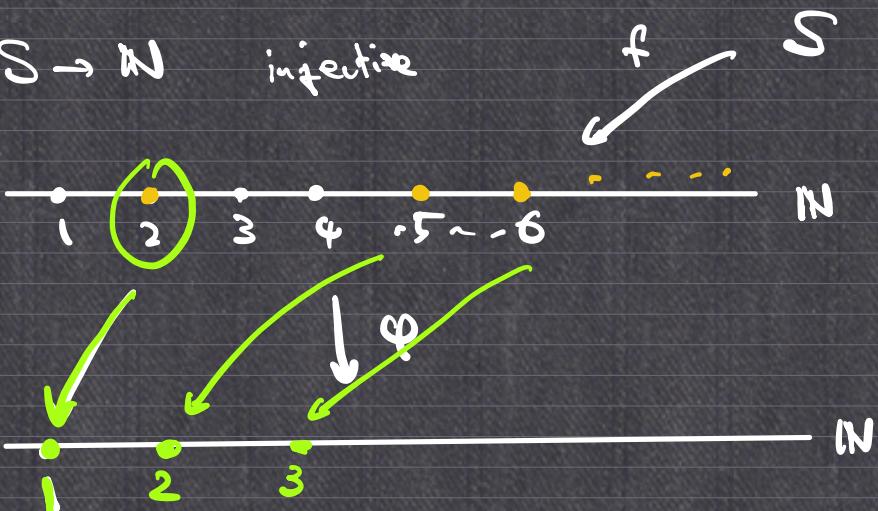
$$f: \mathbb{Q} \rightarrow \mathbb{Z} \times \mathbb{Z}$$

$$f\left(\frac{m}{n}\right) = \begin{cases} (m, n) & \text{if } m \neq 0 \\ (0, 0) & \text{if } m = 0. \end{cases}$$

in simplest form, $n > 0$
 \uparrow
 $\gcd(m, n) = 1$

S infinite set, countable.

$$f: S \rightarrow \mathbb{N} \quad \text{injective}$$



$$g := \varphi \circ f : S \rightarrow \mathbb{N} \quad \text{bijection.}$$

$$g^{-1} : \mathbb{N} \rightarrow S$$

e.g. $(0,1)$ is uncountable.

Proof: Assume $\overbrace{(0,1)}^{\text{infinite set}}$ is countable

$f: \mathbb{N} \rightarrow (0,1)$ bijection

$$f(1) = 0.a_1 a_{11} a_{12} a_{13} a_{14} \dots$$

$$f(2) = 0.a_{21} a_{22} a_{23} a_{24} \dots$$

$$f(3) = 0.a_{31} a_{32} a_{33} a_{34} \dots$$

⋮

$$x := 0.b_1 b_2 b_3 b_4 \dots \neq f(n)$$

$\neq \neq \neq \neq$

$\forall n.$

• S, T countable $\Rightarrow S \cup T$ is Countable.

$$\mathbb{Q} \cup \underbrace{(\mathbb{R} \setminus \mathbb{Q})}_{\text{uncountable}} = \mathbb{R} \supset (0,1)$$

open, closed.



• $S \subset \mathbb{R}$ is open if $\forall x \in S, \exists \varepsilon > 0$
s.t. $(x - \varepsilon, x + \varepsilon) \subset S$



$\cdot S \subset \mathbb{R}$ is closed \Leftrightarrow def $\mathbb{R} \setminus S$ is open.

$$S = \overline{\text{---}} \quad \text{closed}$$

$$\mathbb{R} \setminus S = (\text{---}) \quad \text{open}$$

$$T = \overline{\text{---}}) \quad \text{not closed.}$$

$$\mathbb{R} \setminus T = \text{---} \quad \text{not open}$$

$\cdot \emptyset, \mathbb{R}$ both open and closed

$\mathbb{R} / \mathbb{R} \setminus \emptyset$ is closed $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is closed
open: $\forall x \in \emptyset, \exists \epsilon > 0$, s.t. $(x-\epsilon, x+\epsilon) \subset \emptyset$

(~~not open~~: $\exists x \in \emptyset, \forall \epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \not\subset \emptyset$).

Midterm date:

22 March 2pm - 5pm
(Saturday)

Open : 
 $\forall x \in U, \exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset U$

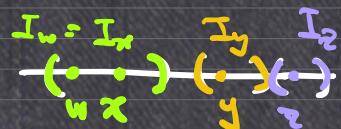
closed : E is closed $\Leftrightarrow \mathbb{R} \setminus E$ is open.

Prop 1.42: Any non-empty open set $S \subset \mathbb{R}$ is a disjoint union of countably many open intervals

$$\text{i.e. } S = \bigsqcup_{j=1}^{\infty} (a_j, b_j)$$

Proof:

$$\xrightarrow{x \in S} \underset{x \in S}{\overbrace{(\bullet, \bullet)}}$$

$$I_x = I_x \quad I_y \quad I_z$$


$I_x = \text{largest open interval } \ni x \subset S$

$$S = \bigcup_{x \in S} \underbrace{\{x\}}_{I_x} \subset \overbrace{\bigcup_{x \in S} \underbrace{I_x}_{\subset S} \subset S}^{\text{def}}$$

$$= =$$

$$S = \bigcup_{x \in S} I_x$$

Given $x, y \in S$. $x \neq y$

$$I_x = I_y \quad \text{or} \quad I_x \cap I_y = \emptyset$$

Otherwise

$$\begin{array}{c} I_x \\ \sqsubset (\dots) \\ I_y \end{array}$$

$$\begin{array}{c} I_x \\ \sqsubset (\dots) \\ I_y \end{array}$$

$$S = \bigsqcup_{x \in T} I_x \neq \emptyset$$

\leftarrow T open interval

$$\begin{array}{c} T \subset S \\ \xrightarrow{\Psi} \\ x \\ \uparrow \\ I_x \ni q_x \in \mathbb{Q} \end{array}$$

$$\begin{array}{c} I_x \quad I_y \\ \sqsubset (\dots) \quad \sqsubset (\dots) \\ x \uparrow \quad y \uparrow \\ q_x \quad q_y \end{array}$$

$$\begin{array}{c} \varphi: T \rightarrow \mathbb{Q} \\ \text{injective} \\ x \mapsto q_x \end{array}$$

← countable

- any union of open sets is open.

$$\boxed{\bigcup_{\alpha \in A} \bigcup_{\beta \in B}}$$

$$\forall x \in \bigcup_{\alpha \in A} U_\alpha \Rightarrow x \in \bigcup_{\substack{\text{open} \\ \alpha \in A}} U_\alpha$$

↑
T $\Rightarrow \exists \varepsilon > 0 \text{ s.t. } (x - \varepsilon, x + \varepsilon) \subset U_\alpha \subset T$

- finite intersection of open sets is open.

$$\leftarrow (\text{---}) \rightarrow \quad \frac{U_1, U_2}{\text{open}}$$

$$\forall x \in U_1 \cap U_2, \Rightarrow x \in U_1, \quad x \in U_2$$

$$\exists \varepsilon_1 > 0 \text{ s.t. } (x - \varepsilon_1, x + \varepsilon_1) \subset U_1$$

$$\exists \varepsilon_2 > 0 \text{ s.t. } (x - \varepsilon_2, x + \varepsilon_2) \subset U_2$$

$$\leftarrow (\text{---}) \overset{\pi}{\rightarrow} \rightarrow$$

$$\varepsilon := \frac{1}{2} \min(\varepsilon_1, \varepsilon_2)$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right)$$

$$\leftarrow \left(\text{---} \overset{\pi}{\rightarrow} \text{---} \right) \rightarrow \quad = \{0\}$$

$-\frac{1}{n}$ $\frac{1}{n}$

Prop 1.48 :

$$S \subset \mathbb{R}$$

① S is closed ($\mathbb{R} \setminus S$ is open)

\Leftrightarrow ② any convergent seq. $\{x_n\}$ in S
has limit in S .

$$\boxed{\begin{array}{l} \text{Given} \\ x_n \rightarrow L \end{array}} \quad a \leq x_n \leq b \quad \forall n \quad \left| \begin{array}{l} x_n \in [a, b] \quad \forall n \\ \downarrow \\ a \leq L \leq b \end{array} \right. \quad \left| \begin{array}{l} \downarrow \\ L \in [a, b] \end{array} \right.$$

① \Rightarrow ② Given S is closed.

$\Leftrightarrow \mathbb{R} \setminus S$ is open.

Assume ② is not true.

\exists conv. seq. $\{x_n\}$ in S $\cup \mathbb{R}$
 $\rightarrow L \notin S \Rightarrow L \in \mathbb{R} \setminus S$

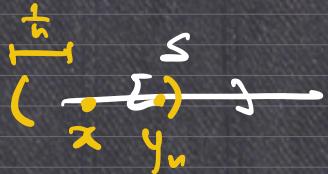
$\exists \varepsilon > 0$ s.t. $(L - \varepsilon, L + \varepsilon) \subset \mathbb{R} \setminus S \Leftarrow$

$L \in S$ $\exists \varepsilon > 0$ $\forall n > N$.
 $x_n \notin (L - \varepsilon, L + \varepsilon)$

$\textcircled{2} \Rightarrow \textcircled{1}$ Consider $x \in \text{RIS}$

Want: $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset \text{RIS}$.

If not: $\forall \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \notin \text{RIS}$



$\Rightarrow \exists y \in (x-\varepsilon, x+\varepsilon)$

w.t $y \notin \text{RIS}$.

$\Leftrightarrow y \in S$.

$\forall n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$, $\exists y_n \in (x-\frac{1}{n}, x+\frac{1}{n})$

$$x \leftarrow \in \overset{\uparrow}{S}$$

e.g. $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

$x_n = \frac{1}{n} \in S \quad \text{but}$
 \downarrow
 $0 \notin S$

$\Rightarrow S$ is not closed.

e.g. $\{x_n\} \subset S$ ^{closed} bounded sequence.

BW $\Rightarrow \exists \{x_{n_k}\} \xrightarrow{\substack{\uparrow \\ S \text{ closed.}}} L \in S$

\uparrow

Definition:

$S \subset \mathbb{R}$ is sequentially compact

\Leftrightarrow any subseq. $\{x_{n_k}\} \subset S$ has a convergent

[On \mathbb{R}]

S closed and bounded $\iff S$ is sequentially compact
 $\iff S$ is compact

Definition $S \subset \mathbb{R}$ is compact

def any open cover of S has a finite subcover.

$S \subset \bigcup_{\alpha \in A} U_\alpha$ \Rightarrow say $\{U_\alpha\}$ is an open cover of S .

$$\left(\overset{\text{---}}{0} \overset{\text{---}}{1} \right)^{U_1} \{U_1\} \quad \left(\overset{\text{---}}{0} \overset{\text{---}}{1} \right) \underset{-\frac{1}{n}}{\overset{1-\frac{1}{n}}{\substack{\uparrow \\ \downarrow}}} \quad \text{---}$$

$$[0, 1] \subset \left(\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1 - \frac{1}{n} \right) \right) \cup \left(\frac{1}{2}, \frac{3}{2} \right)$$
$$\bigcup_{n=1}^{\infty} \left\{ \left(-\frac{1}{n}, 1 - \frac{1}{n} \right) \right\} \cup \left\{ \left(\frac{1}{2}, \frac{3}{2} \right) \right\}.$$

open cover of $[0, 1]$.

$\{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{3}{2})\}$
is a finite subcover
of $[0, 1]$.

$$\left(\left(\frac{1}{n}, \frac{3}{2} \right) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, \frac{3}{2} \right) \right) \subset (0, 1] \subset (0, \frac{3}{2})$$

$\left\{ \left(\frac{1}{n}, \frac{3}{2} \right) : n \in \mathbb{N} \right\}$ is open cover
of $[0, 1]$.

Ex: Prove $[0, 1]$ is compact.

Proof: Assume not.

compact: any open cover of $[0, 1]$ has finite subcover.

not compact: \exists open cover of $[0, 1]$ has no finite subcover.

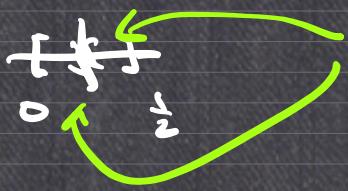
$\therefore \exists \{U_\alpha\}_{\alpha \in A}$ s.t. $[0, 1] \subset \bigcup_{\alpha \in A} U_\alpha$

\uparrow open

but $\{U_\alpha\}$ has no finite subcover.



at least one cannot be covered by



finitely many U_α 's.

$$[a_k, b_k]$$

$$\exists \quad I_1 \supset I_2 \supset I_3 \supset \dots \text{ s.t. } \\ \cup_{k=1}^{\infty} I_k = [0, 1]$$

I_k cannot be covered by
finitely many U_α 's.

$$\bigcap_{k \in \mathbb{N}} I_k = \{x\}$$

$\xrightarrow{\text{[a}_k, b_k]}$

$$x \in [0, 1] \subset \bigcup_{\alpha \in A} U_\alpha$$

↓

$\exists \alpha \in A$ s.t.

$$x \in U_\alpha.$$

$$\exists \epsilon > 0$$

$\xrightarrow{\text{[a}_k, b_k)}$

$$-(\underbrace{(x - \frac{\epsilon}{2})}_{a_k} \underbrace{x}_{b_k} \underbrace{(x + \frac{\epsilon}{2})}_{b_k}) \quad U_\alpha$$

Heine - Borel

$K \subset \mathbb{R}$, TFAE:

- ① K is closed and bounded
- ② K is sequentially compact
(\forall seq. $\{x_n\}$ in K , $\exists \{x_{n_k}\} \rightarrow L \in K.$)
- ③ K is compact
(every open cover of K has a finite subcover)

① \Rightarrow ② : Bolzano - Weierstrass, $\because K$ is closed.

② \Rightarrow ③ : Given K is sequentially compact.

Need: any open cover $\{U_\alpha\}_{\alpha \in A}$ of K has a finite subcover.

Step 1: \therefore countable \downarrow

$$U_\alpha = \bigcup_{i=1}^{\infty} (c_{\alpha,i}, d_{\alpha,i})$$

$$= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (p_{\alpha,i,j}, q_{\alpha,i,j})$$

$$K \subset \bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (p_{\alpha,i,j}, q_{\alpha,i,j})$$

$$= \bigcup_{k=1}^{\infty} (p_k, q_k)$$

$$\subset \bigcup_{n=1}^{\infty} U_{\alpha_n}.$$

Step 2: Assume opposite

K cannot be covered by finitely many U_{α_n} 's.

$$\exists x_1 \in K, \quad x_1 \notin U_{\alpha_1}$$



$$\exists x_2 \in K, \quad x_2 \notin U_{\alpha_1} \cup U_{\alpha_2}$$



$$\exists x_3 \in K, \quad x_3 \notin U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3}$$

$$\exists x_4 \in K, \quad x_4 \notin U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3} \cup U_{\alpha_4}$$

:

:

Seq.

compact.

$x_{n_j} \in K$

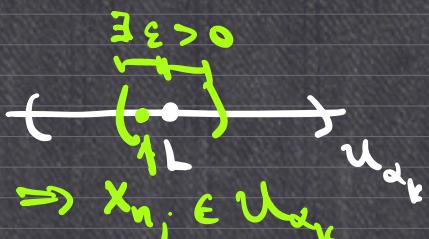
$$x_{n_j} \notin U_{\alpha_1} \cup \dots \cup U_{\alpha_{n_j}}$$

$L \in K$

$$L \in K \subset \bigcup_{k=1}^{\infty} U_{\alpha_k}$$

$\Rightarrow \exists k \text{ s.t.}$

$$L \in U_{\alpha_k}$$



$$\exists \varepsilon > 0$$

$$\text{---} (L - \varepsilon, L + \varepsilon) \text{ ---}$$

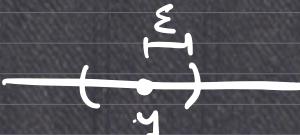
$$U_{\alpha_k}$$

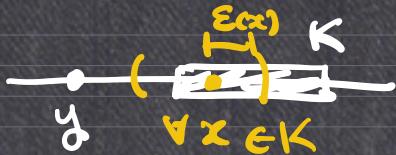
$$j \gg i \Rightarrow x_{n_j} \in U_{\alpha_k}$$

③ \Rightarrow ① : Given K is compact.

- Prove K is closed.

Part: $y \in \mathbb{R} \setminus K$





$\forall x \in K$, choose $\varepsilon(x) > 0$ s.t.
 $y \notin (x - \varepsilon(x), x + \varepsilon(x))$

$$K = \bigcup_{x \in K} \{x\} \subset \bigcup_{x \in K} (x - \varepsilon(x), x + \varepsilon(x))$$

K compact $\Rightarrow \exists x_1, \dots, x_n \in K$ s.t.

$$K \subset \bigcup_{i=1}^n (x_i - \varepsilon(x_i), x_i + \varepsilon(x_i))$$



- Prove K is bounded.

Proof: $\forall x \in K$, consider $(x-1, x+1)$



$$K = \bigcup_{x \in K} \{x\} \subset \bigcup_{x \in K} (x-1, x+1)$$

compact

$\exists x_1, \dots, x_N \in K$ s.t.

$$K \subset \bigcup_{i=1}^N (x_i - 1, x_i + 1)$$

$$K \subset \bigcup_{r>0} (-r, r) = \mathbb{R}$$



Extreme Value Theorem.

$$\left\{ \begin{array}{l} f: [a,b] \rightarrow \mathbb{R} \text{ continuous.} \\ \exists x_0, x_1 \in [a,b] \text{ s.t.} \\ f(x_0) = \inf_{[a,b]} f \in \mathbb{R} \\ f(x_1) = \sup_{[a,b]} f \in \mathbb{R}. \end{array} \right.$$

Goal: $f([a,b])$ is closed and bounded.

Proof (1): Show $f([a,b])$ is sequentially compact.

Take sequence $\{y_n\} \subset f([a,b])$

$$y_n = f(x_n) \quad \exists x_n \in [a,b].$$

$$\tilde{y}_n = \underbrace{f(\underbrace{x_n}_{\substack{\text{sequentially} \\ \text{compact}}})}_{f(L)} \quad \exists x_n \rightarrow L \in [a,b].$$

Proof (2): Show $f([a,b])$ is compact.

Take any open cover $\{U_\alpha\}_{\alpha \in A}$ of $f([a,b])$

$$\begin{aligned} f([a,b]) &\subset \bigcup_{\alpha \in A} U_\alpha. & f \text{ continuous} \\ \Rightarrow [a,b] &\subset f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) & \Leftrightarrow f^{-1}(\text{open}) \\ &= \bigcup_{\alpha \in A} f^{-1}(U_\alpha). & \text{is open.} \end{aligned}$$

Open.

$[a, b]$ is compact

$$\Rightarrow \exists \alpha_1, \dots, \alpha_n \text{ s.t. } [a, b] \subset \bigcup_{i=1}^n f^{-1}(U_{\alpha_i})$$

$$\Rightarrow f([a, b]) \subset \bigcup_{i=1}^n U_{\alpha_i}$$

2 : Metric Spaces.

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \\ \Rightarrow |f(x) - L| < \varepsilon. \end{aligned}$$

$d(x, a)$
 $D(f(x), L)$

let $X \neq \emptyset$.

$d: X \times X \rightarrow [0, \infty)$ is a metric
(distance function)
if d satisfies :

(1) $d(x, y) \geq 0 \quad \forall x, y \in X,$
and equality holds $\Leftrightarrow x = y$.

(2) $d(x, y) = d(y, x) \quad \forall x, y \in X$

(3) $d(x, z) \leq d(x, y) + d(y, z)$

$\forall x, y, z \in X$.



then call (X, d) is
called a metric space.

$(\mathbb{R}, |x-y|)$ ✓

$$\vec{x} = (x_1, \dots, x_n)$$

$$(\mathbb{R}^n, d(\vec{x}, \vec{y})) := \|\vec{x} - \vec{y}\| = \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}.$$

• $X \neq \emptyset$

discrete. $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

① ✓
② ✓

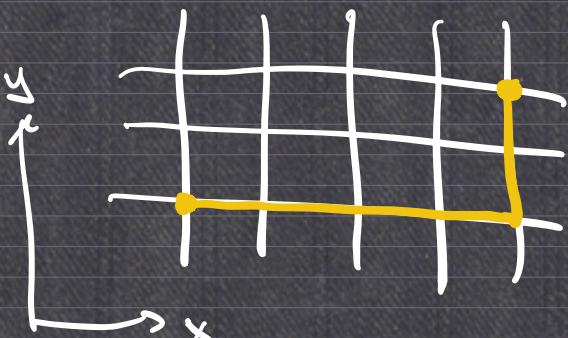
③

x, y, z

$$x=y, \quad x \neq y \neq z$$

• \mathbb{R}^2 :

$$d(\vec{a}, \vec{b}) = |x_1 - x_2| + |y_1 - y_2|$$



$$\vec{a} = (x_1, y_1)$$

$$\vec{b} = (x_2, y_2)$$

(X, d) metric

$$x_n \rightarrow y \quad \stackrel{\text{def}}{\iff} \quad \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } n \geq N \Rightarrow d(x_n, y) < \varepsilon$$

$\uparrow \quad \uparrow$

Cauchy sequence:

$$\{x_n\} \in X \text{ is Cauchy} \quad \stackrel{\text{def}}{\iff} \quad \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon.$$

e.g. $X = C[0, 1]$

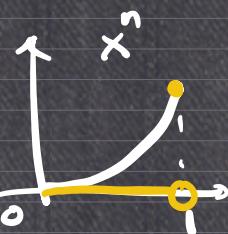
$$\|f\|_p := \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}$$

$$\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|$$

$$\boxed{f_n(x) = x^n}$$

$$\|f_n - 0\|_p = \left(\int_0^1 |x^n|^p dx \right)^{\frac{1}{p}}$$

$$\begin{aligned} &= \left(\int_0^1 x^{pn} dx \right)^{\frac{1}{p}} = \left(\left[\frac{x^{pn+1}}{pn+1} \right]_{x=0}^{x=1} \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{pn+1} \right)^{\frac{1}{p}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$



$$|f_n(y) - g(y)| \leq \|f_n - g\|_\infty := \sup_{x \in [0,1]} |f_n(x) - g(x)|$$

↑
 $\forall y \in [0,1]$

$$\|f_n - g\|_\infty \rightarrow 0 \implies f_n(y) \rightarrow g(y)$$

$\forall y \in [0,1]$

(X, d) is a complete metric space

\hookleftarrow & Cauchy sequence in (X, d)

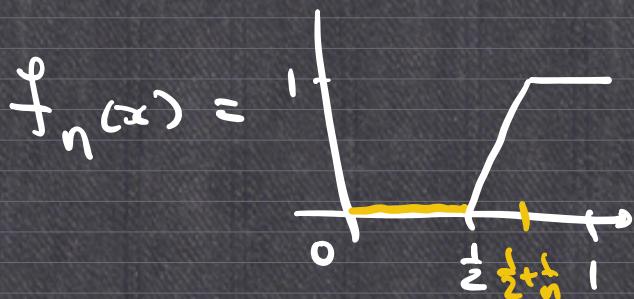
converges to a limit in X .

$(V, \|\cdot\|)$ is said to be a Banach space

\hookleftarrow & Cauchy sequence in $(V, \|\cdot\|)$

converges to a limit in V .

e.g. $X = C[0,1]$, $\|f\|_1 = \int_0^1 |f(x)| dx$



$$\forall \varepsilon > 0, \exists N > \frac{2}{\varepsilon} \quad \|f_m - f_n\|_1 = \int_0^1 |f_m(x) - f_n(x)| dx \quad \underline{m > n \geq N}$$



$\{f_n\}$ is Cauchy wrt $\|\cdot\|_1$.

WTS: $f_n \not\rightarrow \text{Vg}_{\mathcal{C}([0,1])}$

Assume $f_n \rightarrow g \in \mathcal{C}([0,1], \|\cdot\|_1)$



$$0 \leftarrow \|f_n - g\|_1 = \int_0^1 |f_n(x) - g(x)| dx$$

as
 $n \rightarrow \infty$

$$= \int_0^{\frac{1}{2}} |f_n(x) - g(x)| dx + \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} ..$$

$$+ \int_{\frac{1}{2} + \frac{1}{n}}^1 .. dx$$

$$= \int_0^{\frac{1}{2}} |g(x)| dx + \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} |f_n(x) - g(x)| dx$$

$$+ \int_{\frac{1}{2} + \frac{1}{n}}^1 |1 - g(x)| dx$$

$$\Rightarrow \lim_{n \rightarrow \infty} \underbrace{\int_0^{\frac{1}{2}} |g(x)| dx}_{\longrightarrow} = 0 \Rightarrow \int_0^{\frac{1}{2}} (g(x)) dx \Big|_{x=0}$$

$$\Rightarrow \underline{g(x) = 0 \text{ on } [0, \frac{1}{2}]}$$

$$\lim_{n \rightarrow \infty} \int_{\frac{1}{2}}^{1+\frac{1}{n}} |f_n(x) - g(x)| dx = 0$$

$$\lim_{n \rightarrow \infty} \int_{\frac{1}{2} + \frac{1}{n}}^1 |1 - g(x)| dx = 0$$

$$\Rightarrow \int_{\lim_{n \rightarrow \infty} (\frac{1}{2} + \frac{1}{n})}^1 |1 - g(x)| dx \Rightarrow \int_{\frac{1}{2}}^1 |1 - g(x)| dx$$

$$\Rightarrow g(x) = 1 \text{ on } [\frac{1}{2}, 1]$$

$g \notin C[0,1]$

Conclusion: $(C[0,1], \|\cdot\|_1)$

is not a Banach space.

$$C[0,1], \quad \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

$$\|f_n - g\|_\infty = \sup_{x \in [0,1]} |f_n(x) - g(x)|$$

$$f_n \rightarrow g \text{ in } \|\cdot\|_\infty$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } n \geq N$$

$$\Rightarrow \sup_{x \in [0,1]} |f_n(x) - g(x)| < \varepsilon.$$

$$\Rightarrow \forall x \in [0,1], |f_n(x) - g(x)| < \varepsilon.$$

$$\forall \varepsilon > 0, \exists N > 0 \text{ s.t. } \forall n \geq N, \forall x \in [0,1] \\ |f_n(x) - g(x)| < \varepsilon.$$

Take $\{f_n\}$ Cauchy seq in $(C[0,1], \| \cdot \|_\infty)$.

$$|f_n(x) - f_m(x)| \leq \sup_{x \in [0,1]} |f_n - f_m| < \varepsilon/2$$

$\forall x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in \mathbb{R} .

$$\Rightarrow f_n(x) \rightarrow g(x).$$

$$\|f_n - g\|_\infty = \sup_{x \in [0,1]} |f_n - g|$$

$$|f_n(x) - g(x)| \leq \underbrace{|f_n(x) - f_N(x)|}_{< \varepsilon/2} + \underbrace{|f_N(x) - g(x)|}_{< \varepsilon/2}$$

§ 2.2

$U \subset \mathbb{R}$ is open

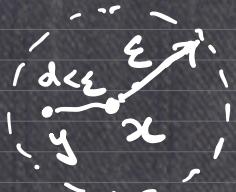
$\Leftrightarrow \forall x \in U, \exists \varepsilon > 0$ s.t.

$$\boxed{(x-\varepsilon, x+\varepsilon) \subset U.}$$

$$-\underset{U}{\overset{((\cdot)-(\cdot))}{(x)}}-\underset{U}{\overset{(\cdot)}{}}$$

(X, d) metric space

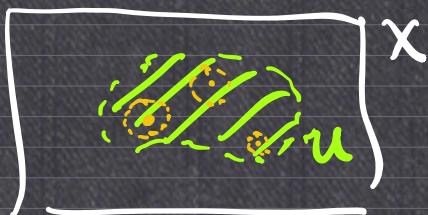
$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$



$U \subset X$ is open in X

$\Leftrightarrow \forall x \in U, \exists \varepsilon > 0$ s.t.

$$B_\varepsilon(x) \subset U.$$



E is closed in X

$\Leftrightarrow \overline{X \setminus E}$ is open.
 E^c

e.g.: (X, d) metric

$$U := B_r(a)$$

$$\forall x \in B_r(a)$$



take $\varepsilon := \frac{1}{2}(r - d(x, a)) < r - d(x, a)$

Claim: $B_\varepsilon(x) \subset B_r(a)$

Proof: $\forall y \in B_\varepsilon(x), \Rightarrow d(x, y) < \varepsilon$.

$$\Rightarrow d(y, a) \leq \underbrace{d(y, x)}_{<\varepsilon} + \underbrace{d(x, a)}_{d(x, a)}$$

$$< r - \cancel{\frac{1}{2}d(x, a)} + \cancel{\frac{1}{2}d(x, a)}$$

$$\Rightarrow y \in B_r(a).$$

$$\mathbb{R}^2, d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

$$B_1(0, 0) = \text{A square centered at } (0, 0) \text{ with side length } 2.$$

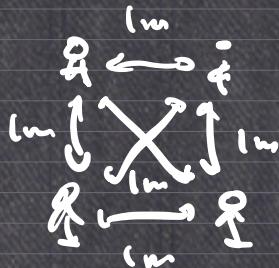
$$\mathbb{R}^2, d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

$$B_1(0, 0) = \text{A diamond centered at } (0, 0) \text{ with side length } 2.$$

$$\mathbb{R}^2, d_0(\vec{x}, \vec{y}) = \begin{cases} 1 & \text{if } \vec{x} \neq \vec{y} \\ 0 & \text{if } \vec{x} = \vec{y} \end{cases}$$

$$B_2((0,0)) = \mathbb{R}^2$$

$$B_{\frac{1}{2}}((0,0)) = \{(0,0)\}$$



e.g. $X = C[0,1]$

$$\|f\|_1 = \int_0^1 |f(x)| dx \sim B_1^{d_1}(0)$$

$$\|f\|_\infty = \max_{[0,1]} |f(x)| \sim B_1^{d_\infty}(0)$$

$$B_1^{d_1}(0) = \{ f \in C[0,1] \mid \int_0^1 |f(x)| dx < 1 \}$$



$$B_1^{d_\infty}(0) = \{ f \in C[0,1] \mid \sup_{[0,1]} |f(x)| < 1 \}$$



① $B_1^{d_1}(0)$ is open in (X, d_∞) ?

② $B_1^{d_\infty}(0)$ is open in (X, d_1) ?

(1)

$$f \in B_1^{d_\omega(\omega)} \rightarrow$$

want $B_\epsilon^{d_\omega(f)} \subset B_1^{d_\omega(\omega)}$

$$\int_0^1 |f(x)| dx < 1$$

$$\forall g \in B_\epsilon^{d_\omega(f)}, \quad d_\omega(f, g) < \epsilon$$

$$\Rightarrow \sup_{[0,1]} |f(x) - g(x)| < \epsilon.$$

$$\Rightarrow |f(x) - g(x)| < \epsilon \quad \forall x \in [0,1].$$

$$\int_0^1 |g(x)| dx \leq \int_0^1 \underbrace{|g(x) - f(x)| + |f(x)|}_{\leq \epsilon} dx$$

$$< \epsilon + \underbrace{\int_0^1 |f(x)| dx}_{\uparrow} < 1$$

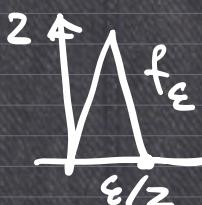
Take $\epsilon = 1 - \underbrace{\int_0^1 |f(x)| dx}_{> 0}$

(2)

$$B_1^{d_\omega(\omega)} \quad \text{in } (\mathcal{X}, d_1).$$

~~open~~

$$0 \in B_1^{d_\omega(\omega)} = \mathcal{U}$$



$$\|f_\epsilon\|_{d_1} = \frac{\epsilon}{2} < \epsilon$$

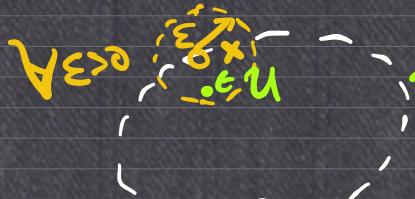
$$\Rightarrow f_\epsilon \in B_\epsilon^{d_1(0)}.$$

$$f_\varepsilon \notin B_{\frac{1}{1}}^{d_\infty}(o)$$

$$\forall \varepsilon > 0, \exists f_\varepsilon \in B_\varepsilon^{d_1}(o) \text{ but } f_\varepsilon \notin B_1^{d_\infty}(o) \\ \Rightarrow B_\varepsilon^{d_1}(o) \not\subset B_1^{d_\infty}(o).$$

limit point

Def 2.29

$\forall \varepsilon > 0$  $\Rightarrow x$ is a limit point of U .

$U' :=$ the set of all limit points of U .

boundary point : $\forall \varepsilon > 0$

 $\Rightarrow x$ is a boundary point of U .

$\partial U :=$ the set of boundary points of U .

e.g.

$$\mathbb{Q} \subset (\mathbb{R}, |x-y|)$$

$$\frac{\varepsilon}{q}, \varepsilon$$

$$\forall x \in \mathbb{R}, B_\varepsilon(x) \cap \mathbb{Q} \neq \emptyset$$

$$\xleftarrow{x \in \mathbb{Q}} \quad \xleftarrow{x \in \mathbb{R}}$$

$$B_\varepsilon(x) \cap \mathbb{Q} \neq \emptyset$$

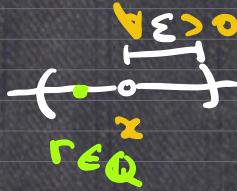
$$x \in \mathbb{Q}$$

$$\Rightarrow x \in \partial Q.$$

$$R \subset \partial Q \subset R \Rightarrow \partial Q = R.$$

$$\forall x \in R, \forall \varepsilon > 0, \\ Q \cap ((x-\varepsilon, x+\varepsilon) \setminus \{x\})$$

$$\neq \emptyset \\ \Rightarrow x \in Q'$$

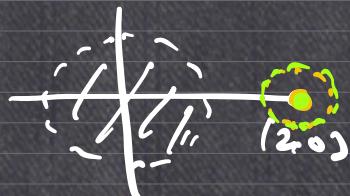


$$\left. \right\} \Rightarrow Q' = R.$$

ex. 2.30

$$R^2, \|\vec{x} - \vec{y}\|_2$$

$$S := B_1(0) \cup \{(2,0)\}$$



$$(2,0) \notin S'.$$

$$(2,0) \in \partial S.$$

Prop. 2.32 : $S' \cup S = \partial S \cup S$

" \subset " : $\forall x \in S' \cup S$

Case ① : if $x \in S, x \in \partial S \cup S$.

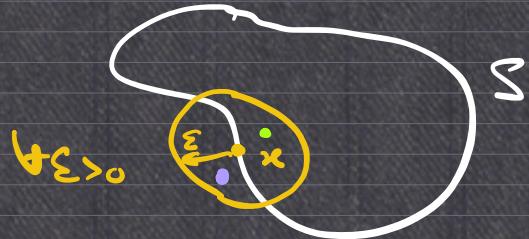
Case ② : if $x \notin S$, then $x \in S'$.

$\therefore x \in S' \cup S \Rightarrow \forall \varepsilon > 0, (B_\varepsilon(x) \setminus \{x\}) \cap S \neq \emptyset.$

$$\overline{S} := \partial S \cup S$$

closure of S .



$x \in S'$  $x \in \partial S$ 

$$\partial S \cup S = S' \cup S =: \bar{S}.$$

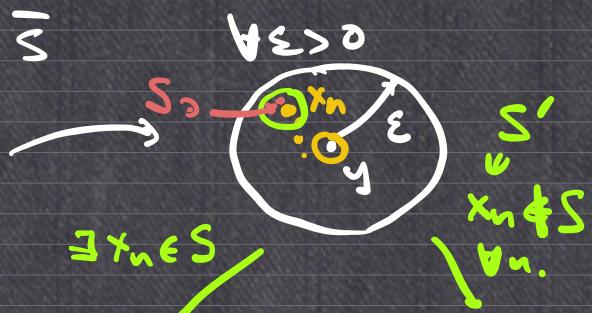
Claim: \bar{S} is closed. (X, d)

Proof: Let $\{x_n\} \rightarrow y \in X$

$\overset{\text{↑}}{\bar{S}}$ \leftarrow need: $y \in \bar{S}$.

Case ①: $y \in S \subset \bar{S}$

Case ②: $y \notin S$



$\partial S \subset S \Leftrightarrow S' \subset S$

Prop 2.34 $\bar{S} = S \Leftrightarrow S$ is closed.

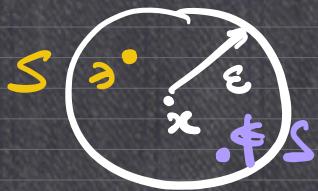
(\Rightarrow) trivial

\Leftrightarrow Assume S is closed.

Goal: $\partial S \subset S$

$\forall x \in \partial S,$

$\forall \epsilon > 0$



$\forall n \in \mathbb{N}, \exists x_n \in S \cap B_{\frac{1}{n}}(x), d(x_n, x) < \frac{1}{n}$

$\downarrow \quad \uparrow \text{closed} \quad \downarrow$

$x \in S \quad x$

Alternately: $\forall x \in \partial S, \text{ assume } x \notin S$

$\Rightarrow x \in X \setminus S$

$\exists \epsilon > 0$

$x \in B_\epsilon(x) \subset X \setminus S$

$$\overline{S} = \bigcap_{\substack{S \subset E \\ \text{closed}}} E$$

$$\overline{B_\epsilon(x)} = \{y \in X \mid d(x, y) \leq \epsilon\}$$

(X, d) discrete metric

$$B_1(x) = \{y \in X : d(x, y) < 1\} = \{x\}.$$

$$C_1(x) = \{y \in X : d(x, y) \leq 1\} = X.$$

$$\overline{B_1(x)} = B_1(x).$$

dense $S \subset (X, d)$ is dense $\overset{\text{def}}{\iff} \overline{S} = X.$



- $\overline{\mathbb{Q}} = \mathbb{R}$, \mathbb{Q} is dense in \mathbb{R} .
- $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$. . .

$$f(x+y) = f(x)f(y) \quad \forall x, y \in \mathbb{R}.$$

$$S^\circ := S \setminus \partial S.$$

$$f: (X, d_X) \rightarrow (Y, d_Y)$$

$x \in B_\delta^X(a) \setminus \{a\}.$

$\underset{\text{def}}{\lim}_{x \rightarrow a} f(x) = p$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < d_X(x, a) < \delta \Rightarrow d_Y(f(x), p) < \varepsilon.$



$$S \subset X$$

$$a \in S'$$

$$\lim_{\substack{x \rightarrow a \\ x \in S}} f(x) = p \quad \leftarrow$$

$$\stackrel{\text{def}}{\iff} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < d_x(x, a) < \delta, x \in S \Rightarrow d_y(f(x), p) < \varepsilon.$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\lim_{\substack{x \rightarrow a \\ x \in (a, a+1)}} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = L$$

$$1_Q : \mathbb{R} \rightarrow \mathbb{R}, \quad 1_Q(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

$$\lim_{\substack{x \rightarrow a \\ x \in \mathbb{R}}} 1_Q(x) \text{ does not exist}$$

$$\lim_{\substack{x \rightarrow a \\ x \in Q}} \underbrace{1_Q(x)}_1 = 1 \quad \mid \quad \lim_{\substack{x \rightarrow a \\ x \in \mathbb{R} \setminus Q}} 1_Q(x) = 0.$$

$f: (X, d_X) \rightarrow (Y, d_Y)$ is continuous

at $x=a$

$$\stackrel{\text{def}}{\iff} \lim_{x \rightarrow a} f(x) = f(a)$$

e.g. 2.44:

$$A \subset (X, d)$$



$$d_A : X \rightarrow \mathbb{R}$$

$$d_A(x) := \inf \{ d(x, a) : a \in A \}$$

Claim: d_A is continuous on X .

Proof: Want: $|d_A(x) - d_A(y)| \leq d(x, y)$.

$$d(x, a) \leq d(x, y) + d(y, a) \quad \forall a \in A.$$

$$\begin{aligned} \inf_{a \in A} & \Rightarrow d_A(x) \leq \inf \{ d(x, y) + d(y, a) : a \in A \} \\ & = d(x, y) + \underbrace{\inf \{ d(y, a) : a \in A \}}_{d_A(y)} \\ \Rightarrow d_A(x) - d_A(y) & \leq d(x, y). \end{aligned}$$

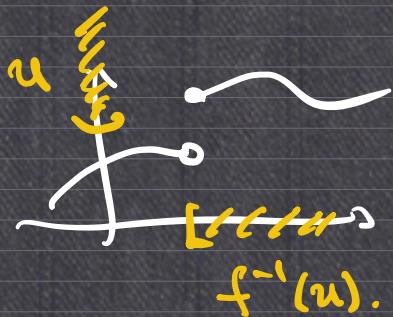
Let $f: (X, d_X) \rightarrow (Y, d_Y)$, TFAE:

① f is continuous on X

② $f^{-1}(U)$ is open in X

for any open set $U \subset Y$.

$$f^{-1}(U) := \{x \in X : f(x) \in U\}$$



e.g. $f(x,y) = x^2 + 4xy + y^3 - x^3y^2$ in \mathbb{R}^2

$$\Sigma := \{(x,y) \in \mathbb{R}^2 : f(x,y) > c\}.$$

"

$$f^{-1}((0, \infty)) = \{(x,y) \in \mathbb{R}^2 : f(x,y) \in (0, \infty)\}$$

e.g. $M^{n \times n} = \text{set of all } n \times n$
real matrices.

$$\approx \mathbb{R}^{n^2}$$

$\Sigma := \text{set of all invertible matrices.}$

$$\det : M^{n \times n} \rightarrow \mathbb{R}$$

$\Sigma = \det^{-1}(\mathbb{R} \setminus \{0\}) = \{A \in M^{n \times n} : \det(A) \neq 0\}$
 is open in $M^{n \times n}$.

Let $f: (X, d_X) \rightarrow (Y, d_Y)$, TFAE:

① f is continuous in X

② $f^{-1}(U)$ is open in X

for any open set $U \subset Y$.

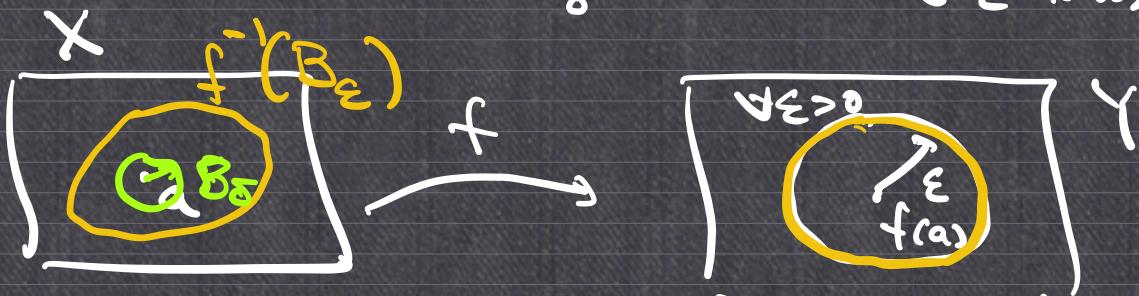
$\Rightarrow \forall a \in X, \forall \varepsilon > 0, \exists \delta > 0$

$$\text{s.t. } \underbrace{d_X(x, a) < \delta}_{x \in B_\delta(a)} \Rightarrow \underbrace{d_f(f(x), f(a)) < \varepsilon}_{f(x) \in B_\varepsilon(f(a))}$$

↑

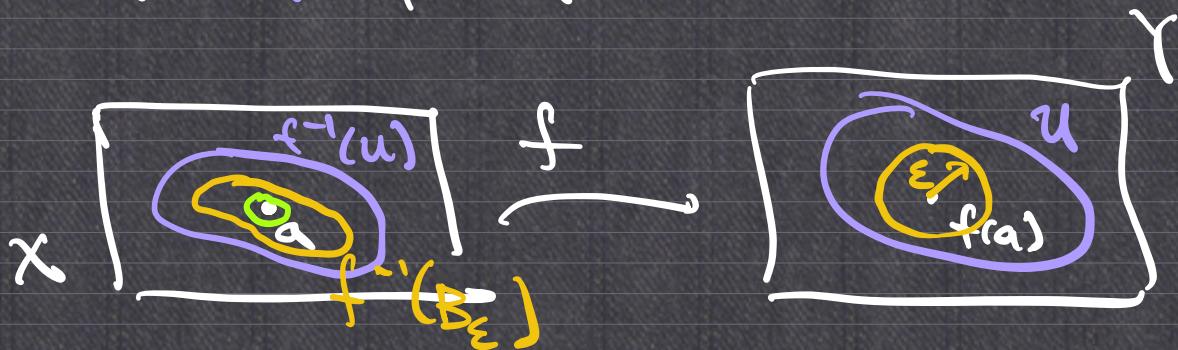
$$x \in f^{-1}(B_\varepsilon(f(a)))$$

$$B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a)))$$



$\textcircled{1} \Rightarrow \textcircled{2}$:

Take $u \subset Y$ open .



$$f : [0, 1] \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \\ 1 & \text{if } x=0 \end{cases}$$

$\text{qcd}(m, n) = 1$

Ex: f is continuous at $x \in [0, 1] \setminus \mathbb{Q}$

but is not continuous at $x \in \mathbb{Q} \cap [0, 1]$.

$$\overrightarrow{f(r)} = 0 \quad \forall r \in \mathbb{Q} \cap [0, 1].$$

$s_n \not\in \mathbb{Q}$ $\underbrace{f(s_n)}_{= 0} \rightarrow 0$



Question: ? $\exists f : \mathbb{R} \rightarrow \mathbb{R}$

s.t. f is continuous at $a \in \mathbb{Q}$

but not continuous at $a \notin \mathbb{Q}$.

$$D_f = \{a \in \mathbb{R} : f \text{ is not continuous at } a\}.$$

$$\bigcup_{i=1}^{\infty} E_i \leftarrow \text{closed.} \quad \left\{ \text{HW 2.} \right.$$

If f is continuous at $a \in \mathbb{Q}$
discontinuous at $a \notin \mathbb{Q}$.

$$\Rightarrow D_f = \mathbb{R} \setminus \mathbb{Q}.$$

$$\begin{aligned}\overline{\mathbb{R}} &= \overline{D_f} \cup \mathbb{Q} = \overline{\underbrace{\bigcup_{i=1}^{\infty} E_i}_{\mathbb{R} \setminus \mathbb{Q}}} \cup \{r_1, r_2, \dots\} \\ &= \overline{\bigcup_{i=1}^{\infty} E_i} \cup \overline{\bigcup_{j=1}^{\infty} \{r_j\}}\end{aligned}$$

$E_i^o = \emptyset$: if not, $\exists x \in E_i^o$
 $\Rightarrow \exists \varepsilon > 0$ s.t.
 $(x - \varepsilon, x + \varepsilon) \subset E_i \subset \mathbb{R} \setminus \mathbb{Q}$

Baire (\mathbb{C}) $\Rightarrow \overline{\mathbb{R} \setminus \bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} \{r_j\}} = \mathbb{R}$

$$\left(\overline{\bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} \{r_j\}} \right)^o = \emptyset$$



$$\mathbb{R}^o = \emptyset$$

Given: (X, d) complete

U_1, U_2, \dots open in X s.t. $\overline{\bigcap_{i=1}^{\infty} U_i} = X$

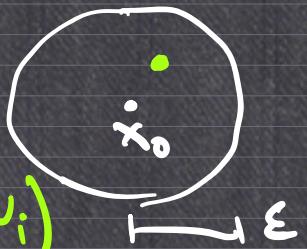
Need:

$$\bigcap_{i=1}^{\infty} \overline{U_i} = X.$$



$\forall x_0 \in X, \forall \varepsilon > 0$.

Show $\exists y \in \overline{B_\varepsilon(x_0)} \cap U_i$

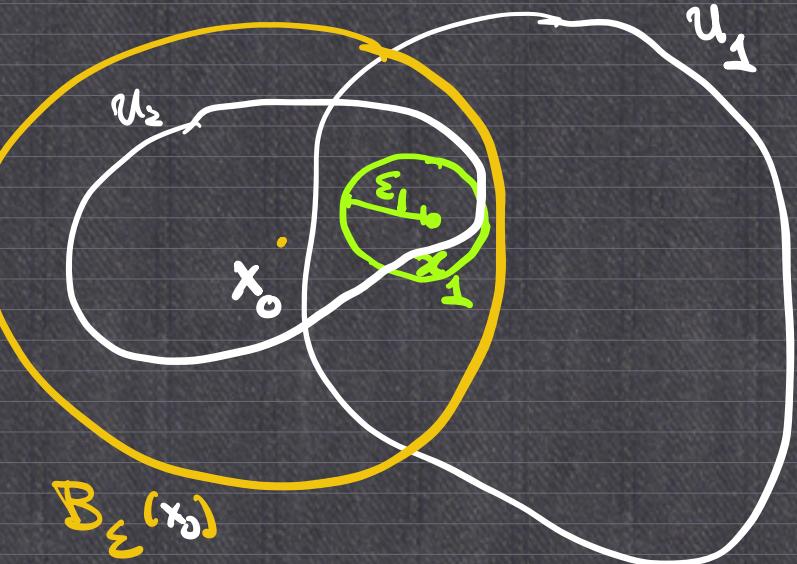


$\exists \overline{B_{\varepsilon_1}(x_1)} \subset B_\varepsilon(x_0) \cap U_1$

$\exists \overline{B_{\varepsilon_2}(x_2)} \subset B_{\varepsilon_1}(x_1) \cap U_2$

$\exists \overline{B_{\varepsilon_3}(x_3)} \subset B_{\varepsilon_2}(x_2) \cap U_3$

\vdots



Require $\varepsilon_n < \frac{1}{2^n}$.



$d(x_n, x_{n+1}) \leq \frac{c}{2^n} \Rightarrow \{x_n\}$
converges.

$X \ni y$

$\forall n \in \mathbb{N}, x_n, x_{n+1}, x_{n+2}, \dots \in \overline{B_{\varepsilon_n}(x_n)}$

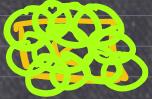


$y \in \overline{\overline{B_{\varepsilon_n}(x_n)}}$

$$\begin{aligned}
 y \in B_{\varepsilon_{n-1}}(x_n) &\subset B_{\varepsilon_{n-1}}(x_{n-1}) \cap U_{n-1} \\
 &\subset \dots \subset B_{\varepsilon_0}(x_0) \subset U_{n-1} \\
 \therefore y \in U_{n-1} \text{ for } n \text{ and } y \in B_{\varepsilon_0}(x_0) \\
 \Rightarrow \underbrace{y \in \bigcap_{i=1}^{\infty} U_i \cap B_{\varepsilon_0}(x_0)}_{\text{def}}
 \end{aligned}$$

∴

§ 2.3 : Compact set.



$$\begin{cases}
 \forall \varepsilon > 0, \exists B_\varepsilon(x_1), \dots, B_\varepsilon(x_n) \\
 \text{s.t. } K \subset B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_n)
 \end{cases}$$

(X, d) metric space, $K \subset X$

TFAE :

- ① K is complete and totally bounded
- ② K is sequentially compact
 $\forall \{x_n\} \subset K, \exists \{x_{n_j}\} \rightarrow y \in K.$
- ③ K is compact
every open cover of K has a finite subcover.

E_x: $X = l_\infty(\mathbb{R}) = \left\{ \{x_n\}_{n=1}^\infty : \{x_n\} \text{ bounded} \right\}.$

$$\| \{x_n\}_{n=1}^\infty \|_\infty = \sup_{n \in \mathbb{N}} |x_n|$$

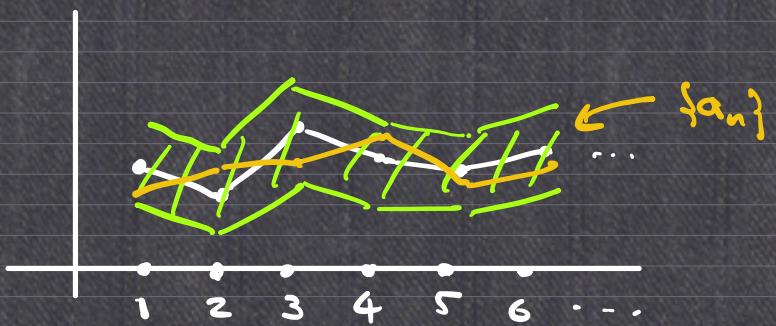
$$e_i = \{0, 0, \dots, \underset{i}{1}, 0, 0, \dots\}$$

$\{e_i\}_{i=1}^{\infty}$ is bounded.

$$d(e_i, e_j) = \|e_i - e_j\|_{\infty} = 1$$

any $\{e_{i_n}\}$ is not Cauchy

$\Rightarrow \{e_{i_n}\}$ does not converge.



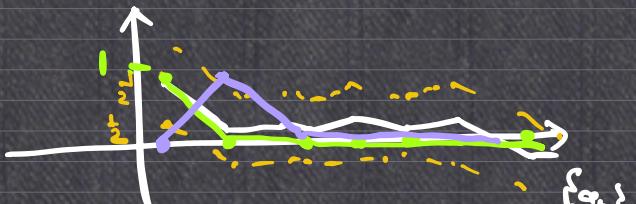
$$B_\varepsilon(\{a_n\}) = \left\{ \{b_n\} \subset l^\infty(\mathbb{R}) \mid \sup_{n \in \mathbb{N}} |a_n - b_n| < \varepsilon \right\}$$

$$e_i = \underbrace{\{0, 0, \dots, 1, 0, 0, \dots\}}_i$$

$$S = \{e_1, e_2, e_3, \dots\}$$

$$B_{1/4}(\{a_n\}) \ni$$

at most one of e_i 's.



① K complete and totally bounded

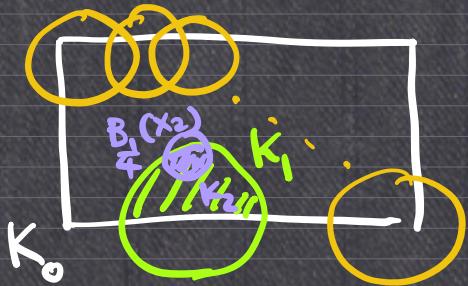
② K sequentially compact

③ K compact

① \Rightarrow ③ \Rightarrow ② \Rightarrow ① .

① \Rightarrow ③ : Given K is complete, totally bounded.

Take an open cover $\bigcup_{\alpha \in A} U_\alpha \supset K$.
 Assume no finite subcover for K .



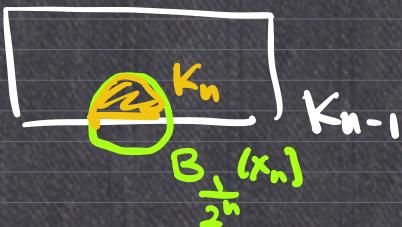
$$K \subset B_{\frac{1}{2}}(x_1) \cup \dots \cup B_{\frac{1}{2}}(x_n)$$

$$K \cap B_{\frac{1}{2}}(x_i)$$

$\exists B_{\frac{1}{2}}(x_1) \subset F_1 = K_0 \cap B_{\frac{1}{2}}(x_1)$ has no finite subcover.

$\exists B_{\frac{1}{4}}(x_2) \quad K_2 = K_1 \cap B_{\frac{1}{4}}(x_2)$ has no finite subcover.

⋮



$$K_n = K_{n-1} \cap B_{\frac{1}{2^n}}(x_n).$$

has no finite subcover.

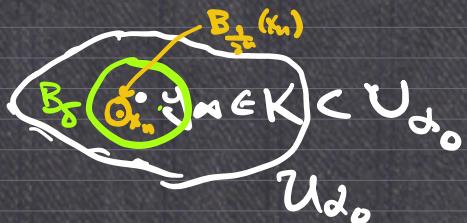
$K_n \neq \emptyset \Rightarrow \exists y_n \in K_n \subset B_{\frac{1}{2^n}}(x_n)$

$y_{n+1} \in K_{n+1} \subset K_n \subset B_{\frac{1}{2^n}}(x_n)$

$d(y_n, y_{n+1}) \leq \frac{1}{2^{n+1}} \Rightarrow \{y_n\}$ is

Cauchy.

$$d(x_n, y_n) \leq \frac{1}{2^n}$$



$y_n \rightarrow y_\infty \in K$

$y_n \rightarrow y_\infty \in K$

$\textcircled{3} \Rightarrow \textcircled{2}$: Given K is compact

want: $\{x_n\} \subset K \rightsquigarrow \exists \{x_{n_j}\} \rightarrow x_0 \in K$

$$S_n := \overline{\{x_n, x_{n+1}, x_{n+2}, \dots\}}$$

$$S := \bigcap_{n=1}^{\infty} S_n$$

want: $S \cap K \neq \emptyset$.

Assume $S \cap K = \emptyset$, $\left(\bigcap_{n=1}^{\infty} S_n \right) \cap K = \emptyset$.

$$\Leftrightarrow K \subset X - \bigcap_{n=1}^{\infty} S_n \\ = \bigcup_{n=1}^{\infty} (X - S_n)$$

K compact

$\exists S_{u_1}, \dots, S_{u_K}$ s.t. $u_1 < \dots < u_K$

$$K \subset (X - S_{u_1}) \cup \dots \cup (X - S_{u_K}) = X - (S_{u_1} \cup \dots \cup S_{u_K})$$

$$\subset X - \underbrace{\{x_{u_K}, x_{u_{K+1}}, \dots\}}_{\text{in } K}$$



$\therefore \underbrace{S \cap K}_{\neq \emptyset} \neq \emptyset$,

$$y \in S = \bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} \overline{\{x_n, x_{n+1}, \dots\}}$$

$$y \in S_1 = \overline{\{x_1, x_2, \dots\}}$$

 $B_1(y) \ni x_{m_1} \quad m_1 \geq 1.$

$$\text{② } y \in S_{m_1+1} = \overline{\{x_{m_1+1}, x_{m_1+2}, \dots\}}$$

$$B_{1/2}(y) \ni x_{m_2} \quad m_2 > m_1 \geq 1.$$

$$\forall j. \exists x_{m_j} \in B_{\frac{1}{j}}(y), \quad m_j > m_{j-1} > \dots > m_2 > m_1$$

$$d(x_{m_j}, y) < \frac{1}{j} \rightarrow 0.$$

② \Rightarrow ① : sequentially compact \Rightarrow complete.
totally bounded.

Want: $\forall \varepsilon > 0, \exists B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)$

$$\bigcup_{i=1}^n B_\varepsilon(x_i) \supset K$$

Assume not: $\nexists \varepsilon > 0, K \not\subset \text{any } \bigcup_{i=1}^n B_\varepsilon(x_i)$

Pick $x_1 \in K, B_\varepsilon(x_1) \not\supset K \Rightarrow \exists x_2 \in K$



but. $x_2 \notin B_\varepsilon(x_i)$

$B_\varepsilon(x_i) \cup B_\varepsilon(x_2) \not\supset K$

$\Rightarrow \exists x_3 \in K, x_3 \notin B_\varepsilon(x_i) \cup B_\varepsilon(x_2)$

⋮

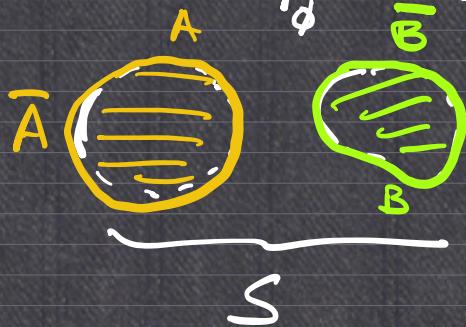
$\exists x_n \in K, \text{ s.t. } d(x_n, x_m) \geq \varepsilon. \quad \forall m < n$

§ 2.4

(X, d) metric space

S is said to **disconnected**

$\Leftrightarrow S = A \cup B$, where $\begin{cases} \overline{A} \cap B = \emptyset \\ A \cap \overline{B} = \emptyset \end{cases}$



connected

\Leftrightarrow not disconnected

$$\begin{aligned} \overline{A} \cap B &= \text{O} \cap \text{O}^c = \emptyset \\ &= \emptyset \end{aligned}$$

Diagram showing two overlapping circles, A and B, with dashed lines indicating their boundaries. The intersection of their closures is empty, meaning they are connected.

Prop. 2.67 A subset $S \subset \mathbb{R}$ is connected \Leftrightarrow ①

$\Leftrightarrow \forall x, y \in S, x < y, \left\{ \begin{array}{l} \exists z \in S, x < z < y \\ [x, y] \subset S \end{array} \right\}$ ②

(\Rightarrow) Suppose S is connected

Assume ② is false

$\exists x, y \in S, x < y$ s.t. $[x, y] \not\subset S$

$$x \in S \quad z \notin S \quad y \in S$$

$\Rightarrow \exists z \in (x, y) \quad z \notin S$

Take $A = (-\infty, z) \cap S$, $B = (z, +\infty) \cap S$

$$S = A \cup B, \quad \bar{A} \cap B = \overline{(-\infty, z) \cap S} \cap B$$

$$\overline{x \wedge y} < \bar{x} \wedge \bar{y}$$

$$A \cap \bar{B} = \emptyset \quad \text{similarly.} \quad = \emptyset \quad = \emptyset$$

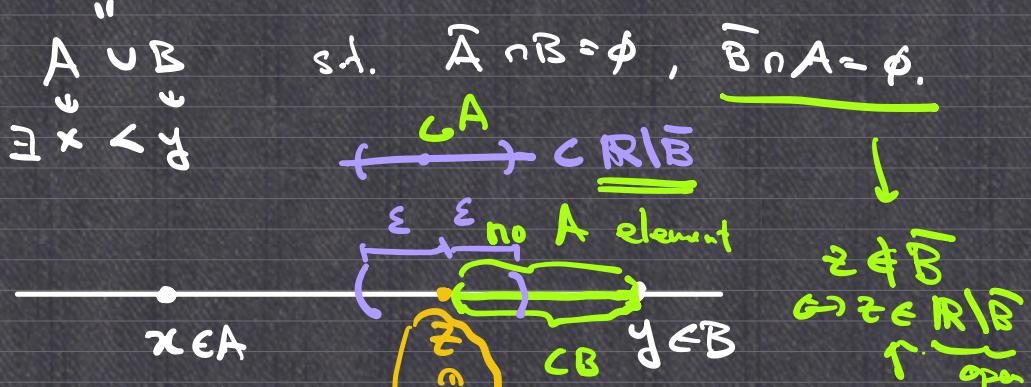
$$< \overline{(-\infty, z)} \cap \bar{S} \cap B$$

$$= \underbrace{(-\infty, z]}_{\text{closed}} \cap \bar{S} \cap \underbrace{(z, +\infty) \cap S}_{\text{open}}$$

□

(\Leftarrow) Given $\forall x, y \in S$ where $x < y \Rightarrow [x, y] \subset S$.

Assume S is disconnected.



let $z = \sup(A \cap [x, y])$

\uparrow

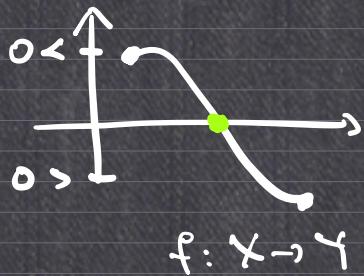
$a_n \in A \cap [x, y]$

$\Rightarrow z \in \bar{A} \quad z \in S = A \cup B$

$\Rightarrow z \not\in B \Rightarrow z \in A$

□

Ex: S connected $\Rightarrow \bar{S}$ connected.



"INT" on metric spaces:

$\left\{ \begin{array}{l} S \text{ connected} \\ f \text{ is continuous} \end{array} \right. \Rightarrow f(S) \text{ is connected.}$

Proof: Assume $f(S)$ is disconnected.

$$\Rightarrow f(S) = \overline{A} \cup \overline{B}, \quad \overline{A} \cap \overline{B} = \emptyset$$

$$A \cap B = \emptyset.$$

$$\Rightarrow S \subset f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

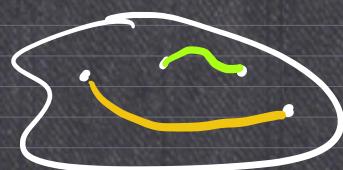
$$\Rightarrow S = \underbrace{(f^{-1}(A) \cap S)}_{\substack{\text{f continuous} \\ \Rightarrow \overline{f^{-1}(A)} \subset f^{-1}(\overline{A})}} \cup \underbrace{(f^{-1}(B) \cap S)}_{\substack{\text{f continuous} \\ \Rightarrow \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})}}$$

$$\begin{aligned} & \overline{f^{-1}(A) \cap S} \cap (f^{-1}(B) \cap S) \\ & \subset \overline{f^{-1}(A)} \cap \overline{S} \cap f^{-1}(B) \cap S \\ & \subset \underbrace{f^{-1}(\overline{A})}_{\substack{\text{f continuous} \\ \Rightarrow \overline{f^{-1}(A)} \subset f^{-1}(\overline{A})}} \cap \underbrace{f^{-1}(B)}_{\substack{\text{f continuous} \\ \Rightarrow \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})}} \cap S \\ & \subset f^{-1}(\overline{A} \cap B) \cap S = \emptyset. \end{aligned}$$

Hw

S is Path-connected.

$\Leftrightarrow \forall x, y \in S. \exists \gamma: [0, 1] \rightarrow S$ continuous s.t. $\gamma(0) = x$ and $\gamma(1) = y$.



Good news: path-connected \Rightarrow connected.

Bad news: connected $\not\Rightarrow$ path-connected.

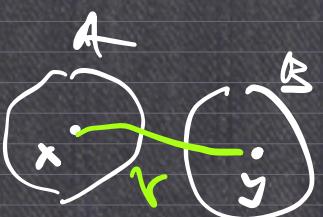
Given: S is path-connected

Want: S is connected.

\hookrightarrow Assume not (i.e. S is disconnected)

$$S = A \cup B, \quad \overline{A} \cap B = \emptyset$$

$$\neq \neq \quad A \cap \overline{B} = \emptyset.$$



$$\begin{cases} x \in A \\ y \in B \end{cases}$$

$$\exists \gamma: [0,1] \rightarrow S$$

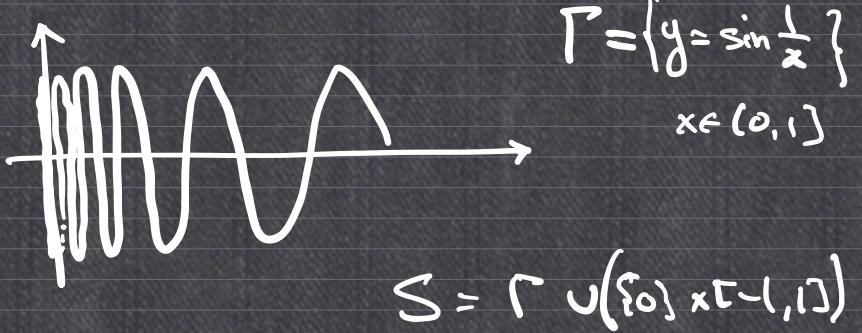
continuous.

$$\gamma(0) = x, \quad \gamma(1) = y.$$

$$\underbrace{\gamma^{-1}(S)}_{[0,1]} = \gamma^{-1}(A \cup B)$$

$$= \gamma^{-1}(A) \cup \gamma^{-1}(B)$$

$$\begin{aligned} \overline{\gamma^{-1}(A)} \cap \gamma^{-1}(B) &< \gamma^{-1}(\overline{A}) \cap \gamma^{-1}(B) \\ &= \gamma^{-1}(\overline{A} \cap B) = \gamma(\emptyset) \\ &= \emptyset \end{aligned}$$



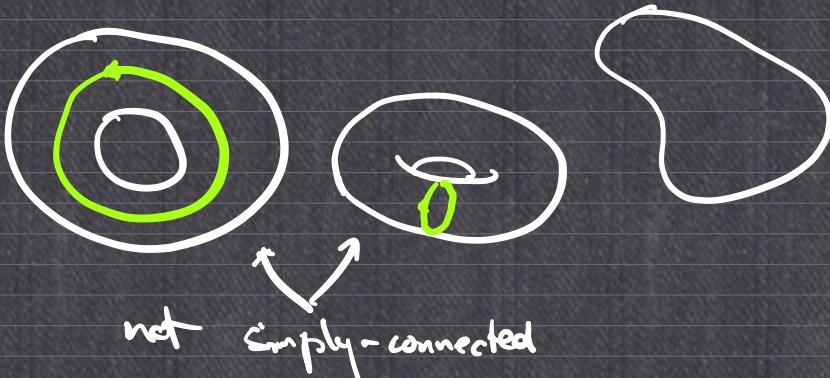
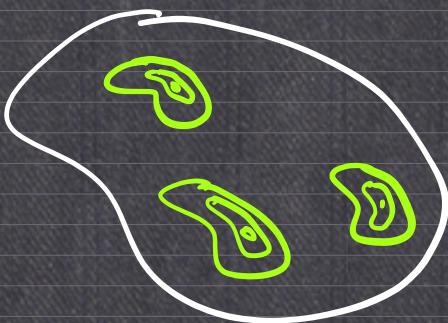
S is Simply-connected

def

A loop in S

can contract to

a point without
leaving S .



Fundamental Theorem of Algebra

$$p(z) = \sum_{n=0}^k a_n z^n, \quad a_n \in \mathbb{C} \quad (a_0 \neq 0)$$

ment a complex root $\alpha \in \mathbb{C}$.

$$p : \mathbb{C} \rightarrow \mathbb{C}$$

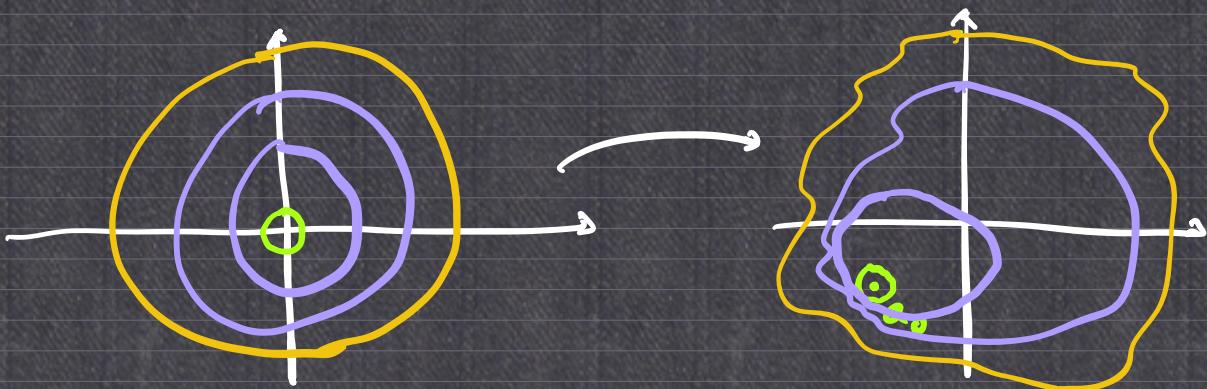


$$f(z) = z^n$$

$$p(z) = a_n z^n + \dots + a_1 z + a_0$$

$|z| \sim \text{small}$, $p(z) \approx a_0$

$|z| \text{ large}$, $p(z) \approx a_n z^n$



Ch. 3

$$F: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R} . \\ (x_1, \dots, x_n)$$

$$e_i = (0, \dots, \underset{i}{1}, \dots, 0)$$

$$\frac{\partial f}{\partial x_i} \Big|_{\underbrace{(a_1, \dots, a_n)}_{\vec{a}}} := \lim_{t \rightarrow 0} \frac{f(a_1, \dots, e_i + t, \dots, a_n) - f(a_1, \dots, a_n)}{t} \\ = \lim_{t \rightarrow 0} \frac{f(\vec{a} + t \vec{e}_i) - f(\vec{a})}{t}$$

$$(D_{\vec{u}} f)(a_1, \dots, a_n) = \lim_{t \rightarrow 0} \frac{f(a_1 + tu_1, \dots, a_n + tu_n) - f(a_1, \dots, a_n)}{t}$$

$$\vec{u} = (u_1, \dots, u_n) \Rightarrow \lim_{t \rightarrow 0} \frac{f(\vec{a} + t \vec{u}) - f(\vec{a})}{t}$$

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{t \rightarrow 0} \frac{f(0+t,0) - f(0,0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{t^2 \cdot 0}{t^4+0^2} - 0}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0. \end{aligned}$$

$$\vec{u} = (u_1, u_2)$$

$$\begin{aligned} D_{\vec{u}} f(0,0) &= \lim_{t \rightarrow 0} \frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} \\ &= \lim_{t \rightarrow 0} \left(\frac{(tu_1)^2 (tu_2)}{(tu_1)^4 + (tu_2)^2} - 0 \right) / t \\ &= \lim_{t \rightarrow 0} \frac{t u_1 u_2}{t^2 u_1^4 + u_2^2} = 0. \end{aligned}$$

$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.

$$\gamma(t) = (t, t^2) \rightarrow 0 \text{ as } t \rightarrow 0.$$

$$\lim_{t \rightarrow 0} f(\gamma(t)) = \lim_{t \rightarrow 0} \frac{t^2(t^2)}{t^4 + (t^2)^2} = \frac{1}{2}.$$

$$\lim_{x \rightarrow 0, y=0} f(x,y) = \lim_{x \rightarrow 0, y=0} 0 = 0 \quad \times$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$\Leftrightarrow f(x) = f(a) + f'(a)(x-a) + \underbrace{o((x-a))}_{= h(x)}$$

s.t. $\lim_{x \rightarrow a} \frac{h(x)}{x-a} = 0$.

$F: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at \vec{a}

\Leftrightarrow

① $\frac{\partial F}{\partial x_i}(\vec{a})$ exists $\forall i = 1, 2, \dots, n$

② $F(\vec{x}) = F(\vec{a}) + \sum_{i=1}^n \frac{\partial F}{\partial x_i}(\vec{a})(x_i - a_i) + o(|\vec{x} - \vec{a}|)$



$z = F(x, y)$

$+ o(|\vec{x} - \vec{a}|)$

$= h(\vec{x})$ s.t. $\lim_{\vec{x} \rightarrow \vec{a}} \frac{h(\vec{x})}{|\vec{x} - \vec{a}|} = 0$

$F(\vec{a}) + \nabla F(\vec{a}) \cdot (\vec{x} - \vec{a})$

e.g. $f(x, y) = |xy|$ differentiable at $(0, 0)$?

$$\frac{\partial f}{\partial x}(0, 0) = 0, \quad \frac{\partial f}{\partial y}(0, 0) = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - (f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0))}{|(x,y) - (0,0)|}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{\sqrt{x^2+y^2}} = \lim_{\substack{r \rightarrow 0^+ \\ \theta \text{ anything}}} \frac{r^2 |\sin \theta \cos \theta|}{r}$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

$$= \lim_{\substack{r \rightarrow 0^+ \\ \theta \text{ anything}}} r |\sin \theta \cos \theta|$$

$$0 \leq \underbrace{r |\sin \theta \cos \theta|}_{0} \leq r$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{y} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta}{r \sin \theta} = \lim_{r \rightarrow 0} r \frac{\cos^2 \theta}{\sin \theta}$$

$$\frac{|xy|}{\sqrt{x^2+y^2}} \leq \underbrace{(x^2+y^2)^{\frac{1}{2}}}_0$$

$$\frac{|y|}{\sqrt{x^2+y^2}} \leq \frac{|y|}{\sqrt{y^2}} = 1$$

$f(x,y)$ is differentiable at (a,b)

def $\begin{cases} \textcircled{1} & \frac{\partial f}{\partial x}(a,b) \text{ and } \frac{\partial f}{\partial y}(a,b) \text{ exist} \\ \textcircled{2} & f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b) \cdot (x-a) + \frac{\partial f}{\partial y}(a,b) \cdot (y-b) \end{cases}$

$$+ o((x,y)-(a,b)) \text{ as } (x,y) \rightarrow (a,b)$$

$\Leftrightarrow \exists c_1, c_2 \in \mathbb{R} \text{ s.t.}$

$$f(x,y) = f(a,b) + c_1(x-a) + c_2(y-b) + o((x,y)-(a,b)) \text{ as } (x,y) \rightarrow (a,b)$$

$\hookrightarrow \lim_{(x,y) \rightarrow (a,b)} \frac{f(x,y) - f(a,b) - c_1(x-a) - c_2(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$

$$F(x,y) = (u(x,y), v(x,y)) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$u(x,y) = \cdot \quad - \quad -$$

$$v(x,y) = - \quad - \quad -$$

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} + \frac{\partial(u,v)}{\partial(x,y)}(a,b) \begin{bmatrix} x-a \\ y-b \end{bmatrix} + o((x,y)-(a,b))$$

C^1 : partial derivatives exist and are continuous.



differentiable

$$f(x,y) = \log(1+x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2} \quad \frac{\partial f}{\partial y} = \frac{2y}{1+x^2+y^2} \quad \text{continuous.}$$

$\Rightarrow f$ is C^1 on \mathbb{R}^2 .

$\Rightarrow f$ is differentiable on \mathbb{R}^2

Proof ($C^1 \Rightarrow$ differentiable)

$$f(x,y) - f(a,b) = f(x,y) - f(a,y) + f(a,y) - f(a,b)$$

$$\begin{array}{lcl} \stackrel{(a,y)}{\downarrow} \rightarrow (x,y) & = & \frac{\partial f}{\partial x}(c_1, y) \cdot (x-a) + \frac{\partial f}{\partial y}(a, c_2) \cdot (y-b) \\ (a,b) & & \exists c_1 \in (a,x) \quad \exists c_2 \in (b,y) \end{array}$$

$$\boxed{\lim_{\substack{(x,y) \\ \rightarrow (a,b)}} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(a,b)} = \left(\frac{\partial f}{\partial x}(a,b) + o(1) \right) (x-a) + \left(\frac{\partial f}{\partial y}(a,b) + o(1) \right) (y-b)$$

$$\Leftrightarrow \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(a,b) + o(1)$$

as $(x,y) \rightarrow (a,b)$

$$= \frac{\partial f}{\partial x}(a,b) \cdot (x-a)$$

$$+ \frac{\partial f}{\partial y}(a,b) \cdot (y-b)$$

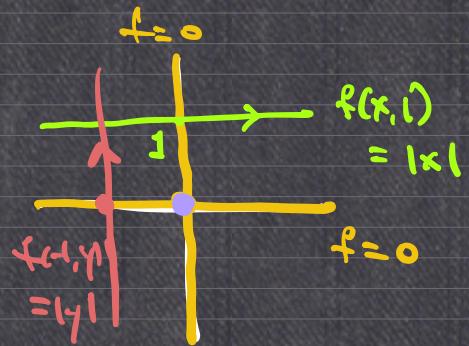
$$+ o(\underbrace{|x-a|}_{\pi}) + o(\underbrace{|y-b|}_{\pi}) \leq \sqrt{(x-a)^2 + (y-b)^2}$$

$$= o(|(x,y) - (a,b)|).$$

differentiable $\not\Rightarrow C^1$.

$$f(x,y) = |xy|.$$

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$



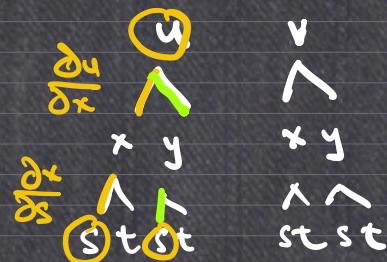
$$G: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

$$F: \mathbb{R}^n \rightarrow \mathbb{R}^k$$

$$F \circ G$$

$$(u,v) = F(x,y)$$

$$(x,y) = G(s,t)$$



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}.$$

$$D(F \circ G)(a,b) = DF(G(a,b)) \cdot D(G(a,b))$$

$$\frac{\partial(u,v)}{\partial(s,t)}(a,b) = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(s,t)}$$

$$\begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$F \circ g(\vec{x})$$

$$\vec{y} = G(\vec{x})$$

$$F(\vec{y}).$$

$$G(\vec{x}) = G(\vec{a}) + Dg(\vec{a})(\vec{x} - \vec{a}) + o(|\vec{x} - \vec{a}|)$$

$$F(\vec{y}) = F(G(a)) + DF(G(a)) \xrightarrow{\text{as } \vec{x} \rightarrow \vec{a}} (\vec{y} - G(a)) \\ + o(|\vec{y} - G(a)|) \quad \text{as } \vec{y} \rightarrow G(a)$$

$$\Rightarrow F(G(x)) = F(G(a)) + DF(G(a)) (G(x) - G(a)) \\ + o(|G(x) - G(a)|)$$

$$= F(G(a)) + DF(G(a)) \cdot (Dg(a) \cdot (x-a) + \\ o(|x-a|)) \\ + o(|G(x) - G(a)|)$$

$$= F(G(a)) + DF(G(a)) \cdot Dg(a) (x-a) \\ + DF(G(a)) \cdot o(|x-a|) + o(|G(x) - G(a)|)$$

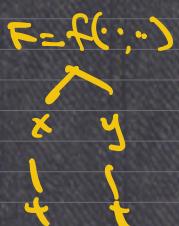
$$D_{\vec{u}} f(a, b) = \lim_{t \rightarrow 0} \frac{f(a+tu_1, b+tu_2) - f(a, b)}{t}$$

$$\vec{u} = (u_1, u_2)$$

$$= \lim_{t \rightarrow 0} \frac{F(t) - F(0)}{t}$$

$$F(t) := f(\underbrace{a+tu_1}_x, \underbrace{b+tu_2}_y)$$

$$= F'(0).$$

$$F = f(\cdot, \cdot)$$


$$F^I = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

$$\begin{aligned} F^I(0) &= \frac{\partial f}{\partial x}(a, b) u_1 + \frac{\partial f}{\partial y}(a, b) u_2 \\ &= \left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right) \cdot (u_1, u_2) \end{aligned}$$

§ 3.2 :

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ is k -times diff^b at \vec{a}

$\overset{\text{def}}{\iff}$ ① k -th order partial derivatives exist at \vec{a}

② $(k-1)$ -th partial derivatives are differentiable
at \vec{a}

e.g. $f_{x_1 \dots x_{k-1}}(\vec{x}) = f_{x_1 \dots x_{k-1}}(\vec{a})$

$$+ \sum_j \frac{\partial}{\partial x_j} f_{x_1 \dots x_{k-1}}(\vec{a}) \cdot (x_j - a_j)$$

$$+ o(|\vec{x} - \vec{a}|) \text{ as } \vec{x} \rightarrow \vec{a}.$$

$C^1 \Rightarrow$ differentiable

$(k-1)$ -th partials $C^1 \Rightarrow$ ② ✓

C^k : k -th order partials are continuous

$C^k \Rightarrow k$ -order differentiable.

$f(x,y) = \log(1+x^2+y^2)$ is $C^\infty = \bigcap_{k=1}^{\infty} C^k$.

$$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{1+x^2+y^2}$$

$$\frac{\partial^k f}{\partial \dots} = \frac{?}{(1+x^2+y^2)^{?}}$$

If $f(x,y)$ is C^2 , then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Proof:

$$\Delta(h,k) = \begin{aligned} & f(a+h, b+k) - f(a+h, b) \\ & - (f(a, b+k) - f(a, b)) \end{aligned}$$

$g(x,y)$

$$= f(a+x, y) - f(a, y)$$

$$= g(h, b+k) - g(h, b)$$

$$= \frac{\partial g}{\partial y}(h, c_1) \cdot k$$

$$\begin{aligned} c_1 &\in (b, b+k) & = \left(\frac{\partial f}{\partial y}(a+h, c_1) - \frac{\partial f}{\partial y}(a, c_1) \right) k \\ c_2 &\in (a, a+h) & = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \Big|_{(c_2, c_1)} \cdot h k \end{aligned}$$

$$\Delta(h,k) = f(a+h, b+k) - f(a+h, b)$$

$$- (f(a, b+k) - f(a, b))$$

$$= \dots = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \Big|_{(c_3, c_4)} h k .$$

\Rightarrow

$$\underbrace{\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \Big|_{(c_2, c_1)}}_{\text{cts}} = \underbrace{\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \Big|_{(c_3, c_4)}}_{\text{cts}}$$

$\begin{array}{|c|c|} \hline 1/0 & 0/- \\ \hline 0/1 & 0/0 \\ \hline \end{array}$

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) (a, b) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) (a, b)$$

$\downarrow \quad \downarrow \quad (h, k) \rightarrow (0, 0) \quad \downarrow$

↓

Given $f(\vec{x})$ is twice diff at \vec{a}
 (x_1, \dots, x_n)

$$g(t) = f\left(\underbrace{(1-t)\vec{a} + t\vec{x}}_{(1-t)\vec{a} + t\vec{x}}\right)$$

$$(1-t)\vec{a} + t\vec{x}$$

$$g'(t) = \sum_j \frac{\partial f}{\partial x_j}((1-t)\vec{a} + t\vec{x}) \cdot \frac{dx_j}{dt} \quad \begin{matrix} f \\ \downarrow \\ (x_j - a_j) \end{matrix} \quad \begin{matrix} \vec{a} \\ \downarrow \\ t \end{matrix} \quad \begin{matrix} \vec{x} \\ \downarrow \\ t \end{matrix} \quad \begin{matrix} \frac{\partial}{\partial t} \\ \underbrace{(1-t)a_j + t x_j} \end{matrix}$$

$$g'(0) = \sum_j \frac{\partial f}{\partial x_j}(\vec{a}) \cdot (x_j - a_j) \quad = x_j - a_j$$

$$g''(0) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) (x_i - a_i)(x_j - a_j)$$

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2!}t^2 + \textcircled{?}$$

$$\begin{aligned} \text{error term} &= f(\vec{x}) - \left(f(\vec{a}) + \sum_i \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \right. \\ &\quad \left. + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i - a_i)(x_j - a_j) \right) \end{aligned}$$

$$= g(1) - (g(0) + g'(0) \cdot 1 + \frac{g''(0)}{2!} \cdot 1^2)$$

$$\text{let } E(t) = g(t) - (g(0) + g'(0)t + \frac{g''(0)}{2!}t^2)$$

$$\frac{E(1) - E(0)}{1^2 - 0^2} = \frac{E'(c)}{2c} = \frac{1}{2c} (g'(c) - g'(0) - g''(0)c)$$

$$E(\alpha) = \frac{1}{2c} \left(g'(c) - g'(\alpha) \right) - \frac{1}{2} g''(\alpha)$$

$$= \frac{1}{2c} \sum_i \left(\underbrace{\frac{\partial f}{\partial x_i}((1-c)\vec{a} + c\vec{x})}_{\text{blue box}} - \underbrace{\frac{\partial f}{\partial x_i}(\vec{a})}_{\text{green line}} \right) (x_i - a_i)$$

$$- \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \cdot (x_i - a_i) (x_j - a_j)$$

$$= \cancel{\frac{1}{2c} \sum_i} \left(\cancel{\frac{\partial f}{\partial x_i}(\vec{a})} + \sum_j \cancel{\frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}(\vec{a})} \underbrace{((1-c)a_j + c x_j - a_j)}_{\cancel{c(x_j - a_j)}} \right.$$

$$+ \cancel{o(|(1-c)\vec{a} + c\vec{x} - \vec{a}|)} \underbrace{= o(|\vec{x} - \vec{a}|)}_{\text{green line}}$$

$$\left. - \cancel{\frac{\partial^2 f}{\partial x_i}(\vec{a})} \right) \cdot (x_i - a_i)$$

$$- \frac{1}{2} \sum_{i,j} \cancel{\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})} \cdot (x_i - a_i) (x_j - a_j)$$

$$f(0,0)$$

$$\nabla f(0,0) = \vec{0}$$

$$\underbrace{\frac{\partial f}{\partial x}(0,0)}_{\text{green line}} = \underbrace{\frac{\partial f}{\partial y}(0,0)}_{\text{green line}} = 0.$$

$$f(x,y) = f(0,0) + f_x(0,0) \cdot x + f_y(0,0) \cdot y$$

$$+ \frac{1}{2!} \left(f_{xx}(0,0) x^2 + 2 f_{xy}(0,0) xy + f_{yy}(0,0) y^2 \right)$$

$Q(x,y)$

$$+ O(|(x,y)|^2)$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix}$$

If $\left\{ \begin{array}{l} \left(2f_{xy}(0,0) \right)^2 - 4f_{xx}(0,0)f_{yy}(0,0) < 0 \\ f_{xx}(0,0) > 0 \end{array} \right.$

\Rightarrow then $Q(x,y) \geq 0$ ($=$ holds if $(x,y) = (0,0)$)

If $F \circ G = \text{id}$ and $G \circ F = \text{id}$

then $\underbrace{D(F \circ G)}_{DF \cdot DG} = D(\text{id}) = I$

$$\underbrace{D(G \circ F)}_{DG \cdot DF} = D(\text{id}) = I \quad \frac{\partial(x_1, \dots, x_n)}{\partial(x_1, \dots, x_n)} = \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ 0 & 0 & \ddots & \\ \vdots & & & 1 \end{bmatrix}$$

$$\Rightarrow DG = (DF)^{-1}$$

$\therefore F$ bijective $\Rightarrow DF$ is invertible
and $D(F^{-1}) = (DF)^{-1}$.

$$F: (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$

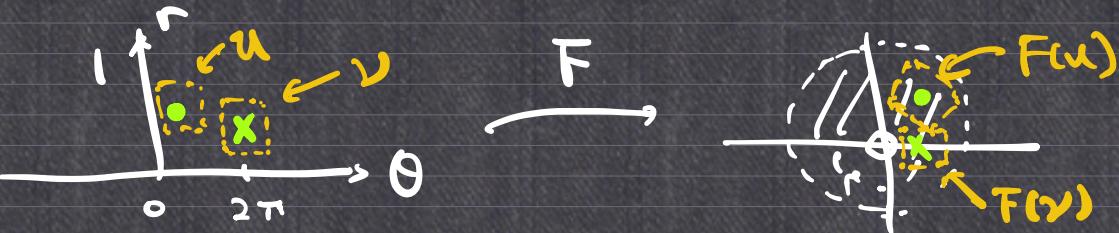
$$(r, \theta) \mapsto (\underbrace{r \cos \theta}_x, \underbrace{r \sin \theta}_y)$$

$$DF = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

$$\frac{\partial}{\partial r} \quad \frac{\partial}{\partial \theta}$$

$$\det DF = r > 0$$

but F is not injective.



e.g. 3.25

$$(*) \begin{cases} x - y^2 = a \\ x^2 + y + y^3 = b \end{cases}$$

Claim: when
 (a, b) small
then (*)
has a solution.

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$F(x, y) = (x - y^2, x^2 + y + y^3)$$

$$F(0, 0) = (0, 0), \quad DF = \begin{bmatrix} 1 & -2y \\ 2x & 1+3y^2 \end{bmatrix} \Rightarrow DF(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



e.g. 3.26

$$F: (0, \infty) \times (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{R}^3$$

$$(r, \phi, \theta) \mapsto (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$$

$$\det \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi \neq 0$$

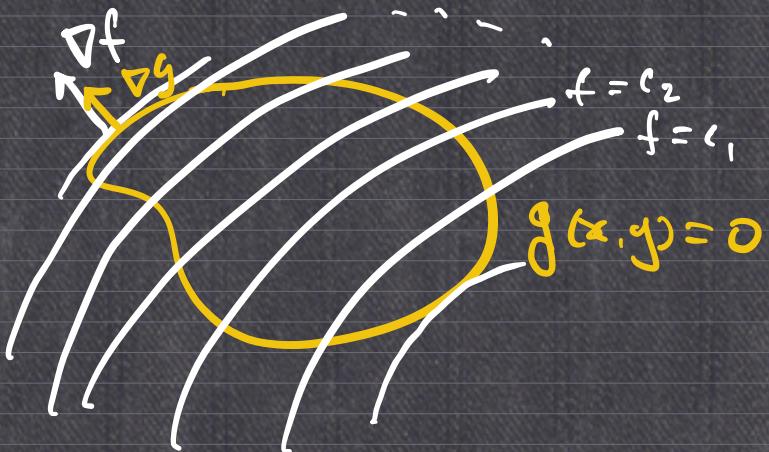
$\Rightarrow F^{-1}$ locally exists and C^∞
 $(\because F \text{ is } C^\infty)$

3.3.4 Lagrange's multiplier.

optimize: $f(x, y)$

subject to: $g(x, y) = 0$

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y) = 0 \end{cases} \iff \begin{cases} \nabla f, \nabla g \text{ are linearly dependent.} \end{cases}$

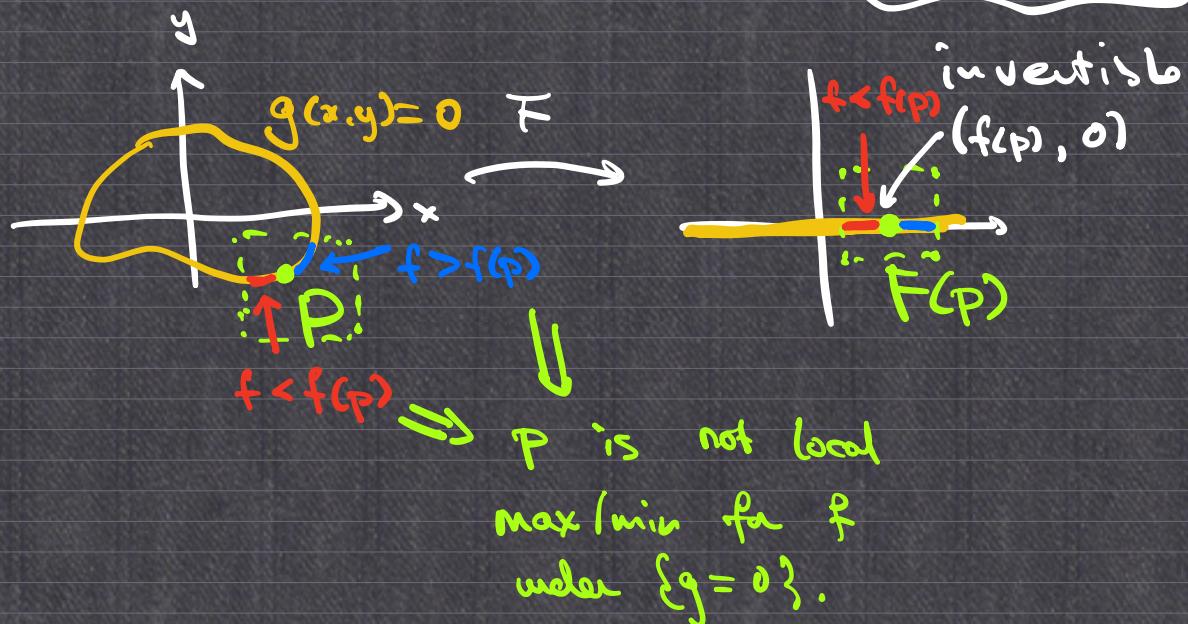


Claim: $\{\nabla f(p), \nabla g(p)\}$ are linearly indep.
 $\Rightarrow p$ is not max/min for f
under constraint $\{g=0\}$.

Proof:

$$F(x, y) = (f(x, y), g(x, y))$$

$$DF(p) \begin{bmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \\ \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y}(p) \end{bmatrix} = \begin{bmatrix} -\nabla f(p) - \\ -\nabla g(p) - \end{bmatrix}$$



Banach Contraction Mapping

(X, d) complete metric space.

$f: X \rightarrow X$. given $\exists \alpha \in \underline{(0, 1)}$

s.t. $d(f(x), f(y)) \leq \alpha d(x, y)$.

$\forall x, y \in X$.

\Rightarrow then $\exists! x_0 \in X$ s.t. $f(x_0) = x_0$.



$f \circ f$

Proof: Let $x_1 \in X$ any point.

$$x_n := f(x_{n-1}) \quad \forall n \geq 2$$

$$\underline{d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))}$$

$$\leq \alpha d(x_n, x_{n-1})$$

$$\leq \alpha^2 d(x_{n-1}, x_{n-2})$$

$$\leq \dots \leq \underbrace{\alpha^{n-1} d(x_2, x_1)}$$

$$\alpha \in (0, 1)$$

assume $\neq 0$

$$\begin{aligned} d(x_m, x_n) &\leq \dots \leq O(\alpha^N)_{\epsilon, \delta} \\ m \geq n \geq N \end{aligned}$$

$\{x_n\}$ is Cauchy $\xrightarrow{\text{complete}} \boxed{x_n \rightarrow x_0}$

$$\exists x_0 \in X$$

$$x_n = \underbrace{f(x_{n-1})}_{\substack{\downarrow \\ f \text{ cts}}} \\ x_0 = f(x_0).$$

$$\left. \begin{array}{l} y'(x) = F(y(x)) \\ y(0) = 1 \end{array} \right| \quad \left. \begin{array}{l} y'(x) = \sin(y^2(x+1)) \\ y(0) = 1 \end{array} \right|$$

$$\Leftrightarrow y(x) = 1 + \int_0^x F(y(t)) dt \quad F(y) = \sin(y^2+1)$$

$$\Phi : C[0, s] \xrightarrow{\text{chosen later}} C[0, s]$$

$$\Phi(f) := 1 + \int_0^x F(f(\omega)) dt$$

$$\|\Phi(f) - \Phi(g)\|_\infty$$

$$= \left\| \int_0^x F(f(\omega)) dt - \int_0^x F(g(\omega)) dt \right\|_\infty$$

$$= \sup_{x \in [0, \zeta]} \left| \int_0^x F(f(\tau)) - F(g(\tau)) \, d\tau \right|$$

$$\leq \sup_{x \in [0, \zeta]} \int_0^x |F(f(\tau)) - F(g(\tau))| \, d\tau$$

$$\leq \sup_{x \in [0, \zeta]} \int_0^x L \underbrace{|f(\tau) - g(\tau)|}_{(F(y_1) - F(y_2))} \, d\tau$$

Assume
 $|F(y_1) - F(y_2)|$

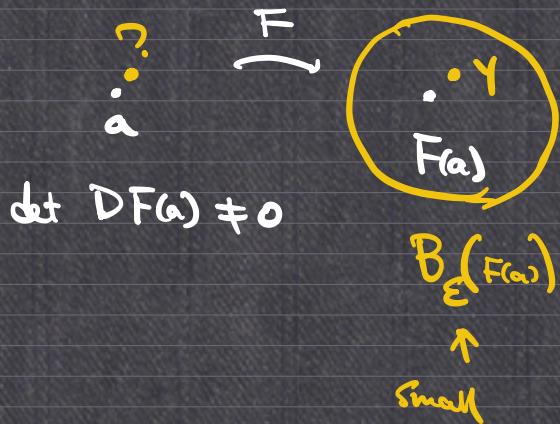
$$\leq L |y_1 - y_2|$$

$$\leq L \sup_{x \in [0, \zeta]} \int_0^x \|f - g\|_\infty \, d\tau$$

$$= \sup_{[0, \zeta]} L \|f - g\|_\infty \cdot \zeta \leq L \|f - g\|_\infty$$

Step ①

Want: $\forall y \sim F(a)$



$\det DF(a) \neq 0$

show $\exists x \sim a$

s.t. $F(x) = y$.

$$T_y : \overline{B_\epsilon(a)} \longrightarrow \mathbb{R}^n$$

$$T_y(x) := x + DF(a)^{-1}(y - F(x))$$

* If $T_y(x) = x \quad \exists x,$

$$\text{then } \cancel{x} + \underbrace{DF(a)^{-1}(y - F(x))}_{= 0} = \cancel{x} \Rightarrow y \sim F(x).$$

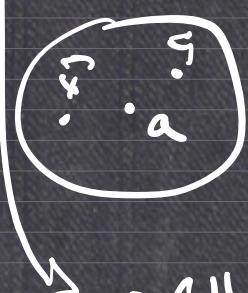
Next: prove T_y is a contraction.

$$\begin{aligned}
& \|T_y(x_1) - T_y(x_2)\|_2 \\
&= \|x_1 + DF(a)^{-1}(y - F(x_1)) - x_2 - DF(a)^{-1}(y - F(x_2))\| \\
&= \|(x_1 - x_2) - DF(a)^{-1}(F(x_1) - F(x_2))\| \\
&= \|DF(a)^{-1}((DF(a)x_1 - F(x_1)) - (DF(a)x_2 - F(x_2)))\| \\
&\leq \|DF(a)^{-1}\| \|G(x_1) - G(x_2)\|
\end{aligned}$$

:

$$G(x) := DF(a)X - F(x)$$

$$\begin{aligned}
 &\leq \|Df(a)^{-1}\| \cdot C \sup_{B_\delta(a)} \|Df\| \|x_1 - x_2\| \\
 &\quad |f(x_1, x_2) - f(y_1, y_2)| \\
 &= \underbrace{|f(x_1, x_2) - f(x_1, y_2) + f(x_1, y_2) - f(y_1, y_2)|}_{\text{Diagram: } \begin{array}{c} \square \\ \text{---} \\ (x_1, x_2) \end{array} \quad \begin{array}{c} \square \\ \text{---} \\ (y_1, y_2) \end{array}} = \left| \frac{\partial f}{\partial y}|_{x_1} (x_2 - y_2) + \frac{\partial f}{\partial x}|_{y_1} (x_1 - y_1) \right| \\
 &\leq \sup_{B_\varepsilon(\cdot)} \|Df\|_2 \| (x_1, x_2) - (y_1, y_2) \|
 \end{aligned}$$



$$= C \left\| Df(a)^{-1} \right\| \underbrace{\sup_{z \in B_\delta(a)} \|Df(a) - Df(z)\|} \cdot \|x_1 - x_2\|$$

$\overbrace{\phantom{\sup_{z \in B_\delta(a)} \|Df(a) - Df(z)\|}}$

F is $C^1 \Rightarrow Df(x)$ is C^0

$$\Rightarrow \exists \delta \text{ s.t. } \sup_{z \in B_\delta(a)} \|Df(a) - Df(z)\|$$

$$\begin{aligned}
 &\exists \delta > 0 \text{ s.t. } \leq \frac{1}{2C\|Df(a)\|} \\
 &\bar{T}_y: \overline{B_\delta(a)} \rightarrow \mathbb{R}^n
 \end{aligned}$$

is contraction.

$$\|\bar{T}_y(x_1) - \bar{T}_y(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|.$$

Next: choose smaller δ', ε s.t.

$$T_y : \overline{B_g(a)} \rightarrow \overline{B_g(\omega)} \quad \text{when } \varepsilon$$



$$\|T_y(x) - a\|$$

$$\begin{aligned} &= \|T_y(x) - T_y(a)\| + \|T_y(a) - a\| \\ &\leq \underbrace{\frac{1}{2}\|x-a\| + \|\alpha + DF(a)^{-1}(y-F(a))\|}_{\|T_y(a)\|} \end{aligned}$$

$$\begin{aligned} &\leq \underbrace{\frac{1}{2}\|x-a\| + \|DF(a)^{-1}\| \underbrace{\|y-F(a)\|}_{<\varepsilon}}_{<\varepsilon} \\ &\leq \frac{1}{2}\delta' + \underbrace{\varepsilon\|DF(a)^{-1}\|}_{<\frac{1}{2}\delta'} < \delta' \end{aligned}$$

Step ④ : F^{-1} is differentiable at $\underbrace{F(x_0)}_y$

$$F^{-1}(y) - F^{-1}(y_0) = DF(x_0)^{-1}(y-y_0)$$

$$\begin{aligned} &= F^{-1}(F(x)) - F^{-1}(F(x_0)) \\ &\quad - DF(x_0)^{-1} \underbrace{(F(x) - F(x_0))}_{\text{as } y = F(x)} \end{aligned}$$

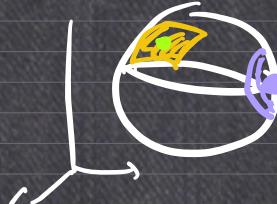
$$= F^{-1}(F(x)) - F^{-1}(F(x_0)) - DF(x_0)^{-1} \left(DF(x_0)(x-x_0) + E(x) \right)$$

$$= \cancel{F^{-1}(f(x))} - \cancel{F^{-1}(f(x_0))} + \tilde{E}(x)$$

$$\sim (x - x_0) + \underbrace{\tilde{E}(x)}_{\in O(|x - x_0|)} = o(|y - y_0|)$$

$E(x) \in o(|x - x_0|)$
as $x \rightarrow x_0$

$$|x_1 - x_2| \leq ? |y_1 - y_2|$$

$$f(x, y, z) = 0 \rightarrow$$


$$\frac{\partial f}{\partial x}(P) \neq 0 \Rightarrow \text{near } P, \quad x = g(y, z).$$

;

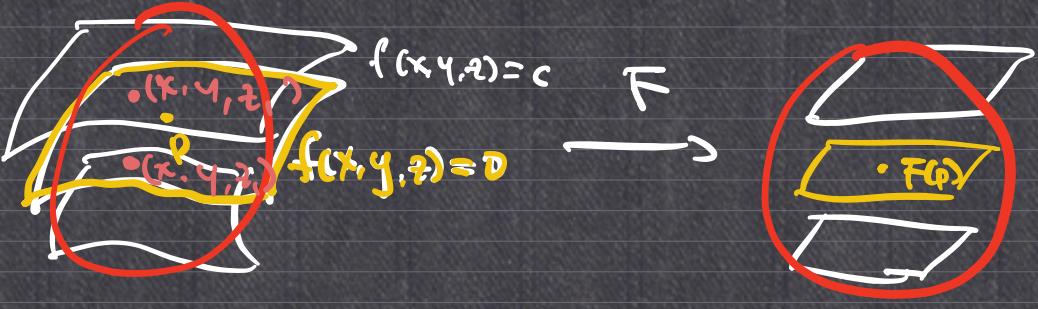
$$\frac{\partial f}{\partial z}(P) \neq 0 \Rightarrow \text{near } P, \quad z = h(x, y).$$

Given : $f(x, y, z) \in C^1$, consider $f(x, y, z) = 0$
 $\exists P$ s.t. $\frac{\partial f}{\partial z}(P) \neq 0$

Consider $F(x, y, z) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$F(x, y, z) = \begin{bmatrix} x \\ y \\ f(x, y, z) \end{bmatrix}$$





$$DF(x,y,z)_{\mathbf{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}_{\mathbf{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix}$$

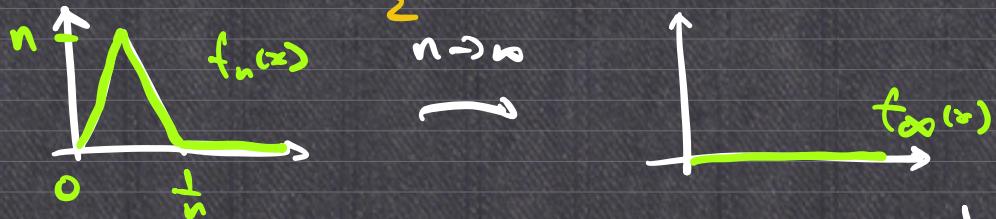
$\overbrace{\quad\quad\quad}^{f^0 \uparrow}$

invertible

§ 4 : Uniform convergence.

$$\frac{1}{2} = \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx \neq \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx = 0$$

$\frac{1}{2}$
 $= \int_0^1 f_\infty(x) dx$



$$\text{if } x > 0, f_n(x) = 0 \quad \forall n > \frac{1}{x}.$$

$$\Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 0$$

$$\text{if } x = 0, f_n(0) = 0 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \lim_{n \rightarrow \infty} f_n(0) = 0$$

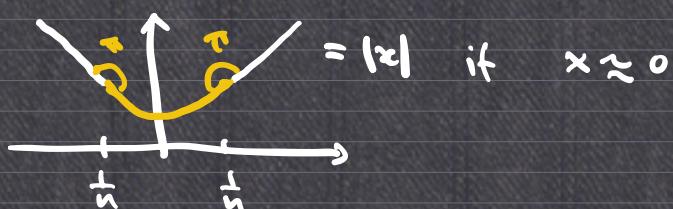
- $\lim_{x \rightarrow a} \lim_{n \rightarrow \infty} f_n(x) \stackrel{?}{=} \lim_{n \rightarrow \infty} \lim_{x \rightarrow a} f_n(x)$

Let $f_n(x) = x^n$

$$\lim_{x \rightarrow 1^-} \lim_{n \rightarrow \infty} f_n(x) = \lim_{x \rightarrow 1^-} \lim_{n \rightarrow \infty} x^n = 0$$

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow 1^-} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow 1^-} x^n = 1$$

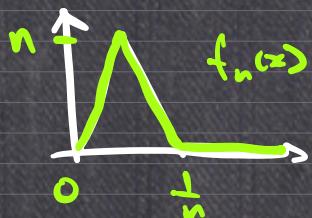
- $\lim_{n \rightarrow \infty} \frac{d}{dx} f_n(x) \neq \frac{d}{dx} \left(\lim_{n \rightarrow \infty} f_n(x) \right)$



- $f_n : D \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to converge
uniformly to g on $E \subset D$

def $\boxed{\lim_{n \rightarrow \infty} \|f_n - g\|_E = 0}$

$\boxed{\lim_{n \rightarrow \infty} \sup_{x \in E} |f_n(x) - g(x)| = 0}$



$$E = [0, 1]$$

If $\|f_n - f\|_E \rightarrow 0$ as $n \rightarrow \infty$

Part: $|f_n(x) - f(x)| \leq \|f_n - f\|_{\mathcal{E}}$
 $\forall x \in A$

$$\|f_n - 0\|_{[0,1]} = \sup_{x \in [0,1]} |f_n(x)| = \sup_{x \in [0,1]} n$$

$= n \rightarrow \infty \text{ as } n \rightarrow \infty.$

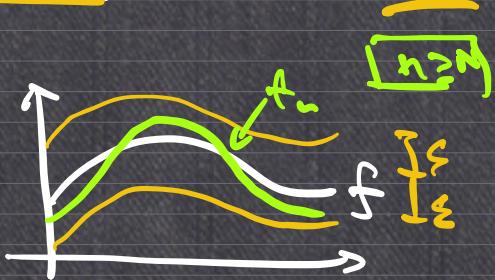
$\therefore f_n$ does not converge uniformly to 0 on \mathbb{R}

However

$$\sup_{x \in [\frac{1}{2}, 1]} |f_n(x) - \overline{f(x)}| = 0 \text{ if } n > 2$$

f_n converges uniformly to 0 on $[0, 1]$

$f_n \xrightarrow{\text{def}} f$ on E
uniformly converges
on E .



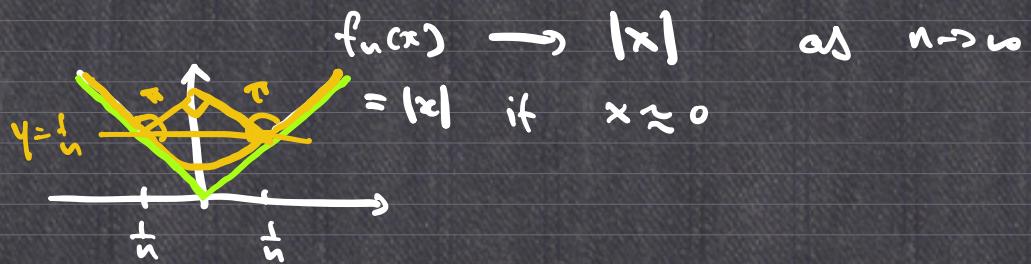
$$\limsup_{n \rightarrow \infty} \underbrace{\sup_{x \in E} |f_n(x) - f(x)|}_{\leq N(\varepsilon)} = 0$$

$\Leftrightarrow \forall \varepsilon > 0, \exists \underbrace{N > 0}_{\text{indep. } \forall x} \text{ s.t. } n \geq N \Rightarrow \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$
 $\Rightarrow \forall x \in E, |f_n(x) - f(x)| < \varepsilon$

$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in E \quad (\text{pointwise convergence})$

$\Leftrightarrow \forall \varepsilon > 0, \forall x \in E, \exists N > 0 \text{ s.t. } n \geq N$
 $|f_n(x) - f(x)| < \varepsilon.$





$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

$\underbrace{\qquad\qquad\qquad}_{< \frac{1}{n}}$

$$f_n \xrightarrow{\text{def}} f = |x| \quad \text{on } \mathbb{R}.$$

$L(x) := \sum_{n=1}^{\infty} g_n(x)$ converges uniformly on E

$\Leftrightarrow \lim_{k \rightarrow \infty} \underbrace{\sum_{n=1}^k g_n(x)}_{\text{seq. of functions}} \text{ converges uniformly } L(x) \text{ on } E.$

Weierstrass M-test :

If \exists sequence of real numbers $\{M_n\}$ s.t.

- ① $|g_n(x)| \leq M_n \quad \forall n \in \mathbb{N}, \forall x \in E$
- ② $\sum_{n=1}^{\infty} M_n$ converges (as a series of real numbers)

then : $\sum_{n=1}^{\infty} g_n$ converges uniformly on E .

$$\text{Ex: } \sum_{k=1}^{\infty} \frac{\sin(k|x| x^2)}{2^k}$$

$\underbrace{\phantom{\sum_{k=1}^{\infty}}}_{g_k(x)}$

$$\left. \begin{array}{l} \cdot |g_k(x)| \leq \frac{1}{2^k} \quad \forall x \in \mathbb{R} \\ \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty. \end{array} \right\} \begin{array}{l} \text{Weierstrass} \\ \text{M-test} \end{array} \quad \sum_{k=1}^{\infty} \frac{\sin(k|x| x^2)}{2^k}$$

Converges uniformly on \mathbb{R}

$$\cdot \sum_{k=1}^{\infty} (x \log x)^k \quad x \in (0, 1].$$

$$\left\{ \begin{array}{l} \cdot |(x \log x)^k| \\ \leq \left| \left(\frac{1}{e} \log \frac{1}{e}\right)^k \right| = \left(\frac{1}{e}\right)^k \quad \forall x \in (0, 1] \\ \cdot \sum \left(\frac{1}{e}\right)^k \text{ converges} \end{array} \right.$$

$$\Rightarrow \sum_{k=1}^{\infty} (x \log x)^k \text{ converges uniformly on } (0, 1].$$

Proof of Weierstrass's M-test

$$\left\{ \begin{array}{l} |g_n(x)| \leq M_n \quad \forall x \in E, n \in \mathbb{N}, \\ \sum M_n < \infty \end{array} \right.$$

- $\sum_{n=1}^{\infty} g_n(x)$ is defined $\forall x \in E$.

Proof:

$$\begin{aligned} & \left| \sum_{n=1}^m g_n(x) - \sum_{n=1}^k g_n(x) \right| \quad m > k \\ &= \left| \sum_{n=k+1}^m g_n(x) \right| \leq \sum_{n=k+1}^m \underbrace{|g_n(x)|}_{\leq M_n} \leq \sum_{n=k+1}^m M_n \\ &= \left| \underbrace{\sum_{n=1}^m M_n}_{\sum_{n=1}^k M_n} - \sum_{n=1}^k M_n \right| \end{aligned}$$

- uniformly converge:

$$\begin{aligned} & \sup_{x \in E} \left| \sum_{n=1}^m g_n(x) - \sum_{n=1}^k g_n(x) \right| \\ &\leq \sup_{x \in E} \dots \leq \sup_{x \in E} \left| \sum_{n=1}^m M_n - \sum_{n=1}^k M_n \right| \\ &= \left| \sum_{n=1}^m M_n - \sum_{n=1}^k M_n \right| \end{aligned}$$

(*)

$$\begin{aligned} f_n &\xrightarrow{*} f \text{ on } [a, b] \\ \Rightarrow \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx &= \int_a^b \underbrace{\lim_{n \rightarrow \infty} f_n(x)} dx = f \end{aligned}$$

$$\begin{aligned}
& \left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| \\
& \leq \int_a^b \underbrace{|f_n(x) - f(x)|}_{\leq \|f_n - f\|_{[a,b]}} dx \leq \int_a^b \|f_n - f\|_{[a,b]} dx \\
& \quad \downarrow \\
& \quad \boxed{0}
\end{aligned}$$

$$\cdot \int_0^1 \frac{1}{x^x} dx = \int_0^1 x^{-x} dx = \int_0^1 e^{\log x^{-x}} dx$$

$$\begin{aligned}
& = \int_0^1 e^{-x \log x} dx \\
& = \int_0^1 \sum_{n=0}^{\infty} \frac{(-x \log x)^n}{n!} dx
\end{aligned}$$

$$e^y = \sum \frac{y^n}{n!}$$

$$\left\{ \underbrace{\frac{(-x \log x)^n}{n!}}_{\text{term}} \right\} \leq \frac{\left(\frac{1}{e}\right)^n}{n!} \leq \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \int_0^1 \underbrace{\frac{(-x \log x)^n}{n!}}_{\text{term}} dx \quad \sum \frac{1}{n!} \text{ converges.}$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{n+1}}_{\text{term}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^n}$$

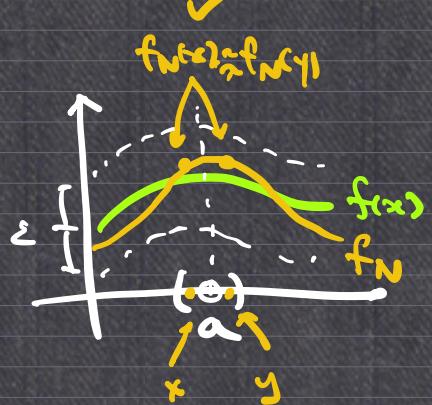
Given : $\begin{cases} f_n \rightarrow f \text{ on } D, & a \in D' \\ \lim_{x \rightarrow a} f_n(x) \text{ exists } \forall n. \end{cases}$

Want : $\lim_{x \rightarrow a} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow a} f_n(x)$

$f(x)$ ✓

Claim : $\lim_{x \rightarrow a} f(x)$ exists

$$\begin{aligned} & |f(x) - f(y)| \\ & \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| \\ & \quad + |f_N(y) - f(y)| \end{aligned}$$



$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N$

$$\Rightarrow \|f_n - f\|_D < \varepsilon$$

$$\Rightarrow \forall x \in D, |f_n(x) - f(x)| < \varepsilon$$

$\forall n \geq N.$

$$\lim_{x \rightarrow a} |f_n(x) - f(x)| \leq \varepsilon$$

$$\Rightarrow |\lim_{x \rightarrow a} f_n(x) - \lim_{x \rightarrow a} f(x)| \leq \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \left| \underbrace{\lim_{x \rightarrow a} f(x)}_{\text{seq. of numbers.}} - L \right| \leq \varepsilon. \quad \forall n \geq N.$$

$$\lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow a} f_n(x) \right) = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \lim_{n \rightarrow \infty} f_n(x)$$

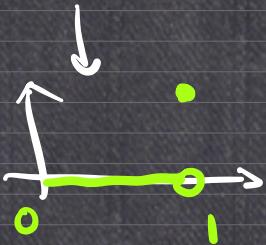
Cor: if further f_n is continuous at $x=a$.

then

$$\lim_{x \rightarrow a} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow a} f_n(x)$$

$$\underbrace{f(x)}_{\lim_{x \rightarrow a} f(x)} = \underbrace{\lim_{n \rightarrow \infty} f_n(a)}_{= f(a)}$$

e.g. $f_n(x) = x^n$ on $[0, 1]$.



$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

converges when $x > 1$

$\forall \varepsilon > 0$, $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly
on $(1+\varepsilon, \infty)$.

$$\left\{ \begin{array}{l} \left| \frac{1}{n^x} \right| = \frac{1}{n^x} \leq \frac{1}{n^{1+\varepsilon}} \\ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \text{ converges} \end{array} \right.$$

as $x \in (1+\varepsilon, \infty)$

Weierstrass
 α -test

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

converges uniformly
continuous on $(1+\varepsilon, \infty)$.

$\zeta(x)$ is continuous
on $(1+\epsilon, \infty)$ $\forall \epsilon > 0$.


uniform conv. on $(1, \infty)$

$\zeta(x)$ is continuous on $\bigcup_{\epsilon > 0} (1+\epsilon, \infty) = (1, \infty)$.

$\sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$

\Rightarrow converges at $x = -R + \epsilon$
and $x = R - \epsilon$.

$\stackrel{?}{\Rightarrow}$ uniform convergence on $[-R+2\epsilon, R-2\epsilon]$.

$$\left\{ \begin{array}{l} \cdot f_n \rightarrow f \text{ on } (a, b) \quad x_0 \in (a, b) \\ \cdot f'_n \rightarrow g \text{ on } (a, b) \\ \Rightarrow \lim_{n \rightarrow \infty} f'_n(x_0) = g(x_0) \\ \qquad \qquad \qquad = \frac{d}{dx} \Big|_{x=x_0} f(x) \end{array} \right.$$

Proof: LHS = $\lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} \frac{f_n(x) - f_n(x_0)}{x - x_0}$

RHS = $\lim_{x \rightarrow x_0} \frac{\lim_{n \rightarrow \infty} f_n(x) - \lim_{n \rightarrow \infty} f_n(x_0)}{x - x_0}$

$$= \lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} \underbrace{\frac{f_n(x) - f_n(x_0)}{|x - x_0|}}_{g_n(x)}$$

Need: $g_n \xrightarrow{x \rightarrow x_0}$? on $(a, b) \setminus \{x_0\}$.

$$\begin{aligned} |g_n(x) - g_m(x)| &= \left| \frac{f_n(x) - f_n(x_0)}{|x - x_0|} - \frac{f_m(x) - f_m(x_0)}{|x - x_0|} \right| \\ &= \frac{|(f_n(x) - f_n(x_0)) - (f_m(x) - f_m(x_0))|}{|x - x_0|} \\ &= \frac{|g'(c) \cancel{(x - x_0)}|}{|x - x_0|} = |f_n'(c) - f_m'(c)| \\ &\leq \|f_n' - f_m'\|_{(a, b) \setminus \{x_0\}}. \end{aligned}$$

e.g.

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad |x > 1|$$

$$\sum_{n=1}^{\infty} \frac{d}{dx} \frac{1}{n^x}$$

$$= \sum_{n=1}^{\infty} -\frac{1}{(n^x)^2} \frac{d}{dx} n^x$$

$$= \sum_{n=1}^{\infty} \left(-\frac{1}{n^{2x}} \cdot n^x \log n \right)$$

$$= - \sum_{n=1}^{\infty} \frac{1}{n^x} \log n$$

$x \in (1 + \varepsilon, \infty)$

$$\left\{ \begin{array}{l} \left| \frac{\log n}{n^x} \right| \leq \frac{\log n}{n^{1+\varepsilon}} \\ \quad \quad \quad \leq \frac{n^{\varepsilon/2}}{n^{1+\varepsilon}} = \frac{1}{n^{1-\varepsilon/2}} \end{array} \right.$$

$\log n \leq n^{\varepsilon/2}$
if $n >> 1$

$\forall x \in (1+\varepsilon, \infty)$

$\sum \frac{1}{n^{1+\varepsilon/2}}$ converges

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\log n}{n^x} \text{ converges uniformly on } (1+\varepsilon, \infty)$$

$$\therefore \frac{d}{dx} \sum_{n=1}^{\infty} \frac{1}{n^x} = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{1}{n^x} \text{ on } (1+\varepsilon, \infty)$$

$\forall \varepsilon > 0$

$$\Rightarrow \downarrow \quad = \quad \uparrow \quad \text{on } \bigcup_{\varepsilon > 0} (1+\varepsilon, \infty)$$

$= (1, \infty)$.

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \quad \text{on } (-R, R)$$

Ex: $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on

$$\left(\sum_{n=1}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} \right) \quad \text{for } \varepsilon > 0$$

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n x^n|} \leq 1 \quad \text{on } x \in [-R+\epsilon, R-\epsilon]$$

$$\Rightarrow \limsup_{n \rightarrow \infty} \underbrace{\sqrt[n]{|a_n x^{kn}|}}_{n^{\frac{1}{kn}}} = \limsup_{n \rightarrow \infty} n^{\frac{1}{kn}} \sqrt[n]{|a_n x^n|} \cdot |x|^{\frac{-1}{n}}$$

§ 4.2 Arzéla - Ascoli Theorem

$\{f_n\}$ sequence of functions on $[a, b]$.

$$\exists f_n \rightharpoonup f_\infty.$$

- uniformly bounded

$\exists M > 0$ s.t. $|f_n(x)| \leq M \quad \forall n \in \mathbb{N} \quad x \in [a, b]$
 t. indep. of x, n .

$$\text{e.g. } \left\{ \sin \left(\frac{n^{\ln n} \cos(Tn^2)}{n^e + e^n + \sum a^n \cos(b^n x)} \right) \right\}$$

- equicontinuity

$\{f_n\}$ is equicontinuous on $[a, b]$

$\underset{\text{def}}{\Leftrightarrow} \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \quad \forall n \in \mathbb{N}.$$

$$\text{e.g. } \left\{ \sin \left(\frac{x}{n} \right) \right\}$$

$\forall \varepsilon > 0 \quad \exists \delta = \varepsilon$, then $|x - y| < \delta$

$$\Rightarrow \left| \underbrace{\sin \left(\frac{x}{n} \right) - \sin \left(\frac{y}{n} \right)}_{g_n(x) = \sin \frac{x}{n}} \right| = \underbrace{\left| \frac{1}{n} \cos \frac{c}{n} \right|}_{g'_n(c)} |x - y|$$

$$\leq \frac{1}{n} |x-y| < \frac{\delta}{n} \leq \delta = \epsilon$$

non-R.g. $\{f_n(x)\}_{n=1}^{\infty}$

$$f_n(x)$$



$\exists \epsilon_0 = 1, \forall \delta > 0 . \exists x, y \text{ s.t. } |x-y| = \frac{\delta}{2} \geq 1 = \epsilon_0$

$$\exists n_0 \geq \frac{2}{\delta}, |f_{n_0}(x) - f_{n_0}(y)| = n_0 |x-y| = n_0 \frac{\delta}{2} \geq 1 = \epsilon_0$$

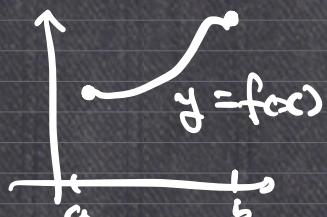
\bullet if $|f_n'(x)| \leq M \quad \forall n \in [a,b], x \in [a,b]$
 $\Rightarrow \{f_n\}$ is equicontinuous on $[a,b]$.

Arzela-Ascoli

if $\{f_n\}$ is
 ① uniformly bounded on $[a,b]$.
 and ② equicontinuous.

then $\exists \{f_{n_k}\} \rightarrow f_\infty$ on $[a,b]$.

$$\bullet L(f) := \int_a^b \sqrt{1+f'(x)^2} dx$$



$\bullet \inf L(f)$ exists.
 $f \in C^1, f(a), f(b)$

$\exists f_n \in C^1[a,b] \text{ s.t. } L(f_n) \rightarrow \inf_{f \in C^1} L(f).$

e.g. Given $\{f_n\} \subset C[a,b]$.

$$|f_n(x)| \leq M \quad \forall x \in [a,b], n \in \mathbb{N}.$$

$$F_n(x) := \int_a^x f_n(t) dt$$

want: $\exists F_{n_k} \rightrightarrows F_\infty$ on $[a,b]$.

① $\{F_n\}$ is uniformly bounded on $[a,b]$

$$\text{Proof: } |F_n(x)| = \left| \int_a^x f_n(t) dt \right|$$

$$\leq \int_a^x |f_n(t)| dt$$

$$\leq \int_a^x M dt = M(x-a)$$

$$\leq \underline{M(b-a)}$$

② $\{F_n\}$ is equicontinuous on $[a,b]$.

$$\text{Proof: } F'_n(x) = f_n(x)$$

E x. 4.28

Given $f_n: [a,b] \rightarrow \mathbb{R}, C^\infty$

$\forall k \in \mathbb{N} \cup \{0\}$, $\exists C_k > 0$ s.t.

$$\|f_n^{(k)}\|_{[a,b]} \leq C_k$$

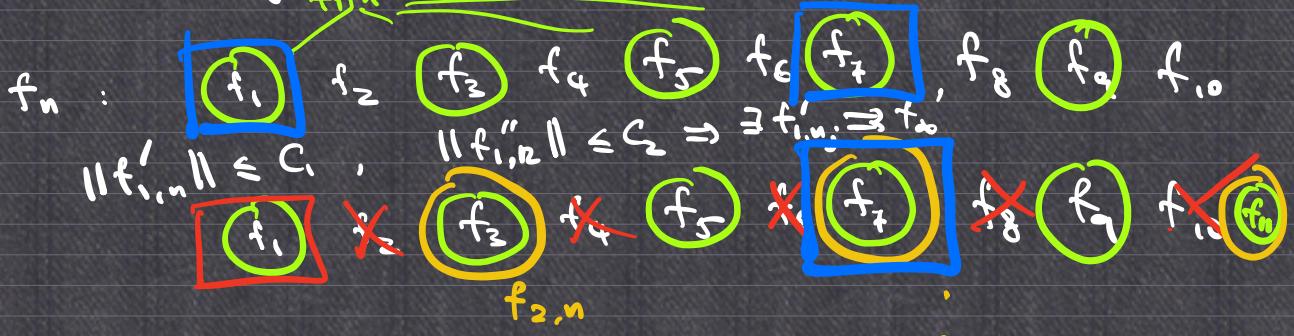
then show $\exists \{f_{n_j}\}$ s.t. $f_{n_j}^{(k)} \rightrightarrows f_\infty^{(k)}$ on $[a,b]$

$\forall k \in \mathbb{N} \cup \{\infty\}$

Proof:

$$\|f_n\| \leq C_0, \|f'_n\| \leq C_1$$

$$\Rightarrow \exists f_{n_j} \xrightarrow{j} f_\infty.$$



$$\|f'_{1,n}\| \leq C_1, \|f''_{1,n}\| \leq C_2 \Rightarrow \exists f'_{1,n} \xrightarrow{n} f_\infty$$

$$\{f_{1,n}\} > \{f_{2,n}\} > \{f_{3,n}\} > \dots$$

□

construct

$\{f_{n,n}\}_{n=1}^\infty$

= subsequence of
each $\{f_{k,n}\}_{n=1}^\infty$
(except the first $k-1$
terms)

Given $|f_{n,\infty}| \leq M \quad \forall n \in \mathbb{N}, \quad x \in [a,b]$

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \quad \forall n \in \mathbb{N}.$

Goal: $f_{n,j} \xrightarrow{\text{Bw}} f_\infty$ on $[a,b]$.

Proof: Step ① $\exists f_{n_k}$ s.t. $f_{n_k}(r) \rightarrow L(r) \quad \forall r \in [a,b]$ � Q

$$\mathbb{Q} \cap [a,b] = \{r_1, r_2, r_3, r_4, \dots\}$$

$\stackrel{\text{Bw}}{\Rightarrow} \exists f_{1,n} \text{ s.t. } f_{1,n}(r_1) \rightarrow L_1 \quad (\because |f_n(r_1)| \leq M)$

$|f_{1,n}(r_2)| \leq M \stackrel{\text{Bw}}{\Rightarrow} f_{2,n}(r_2) \rightarrow L_2$
 \vdots

$\exists \{f_n\} > \{f_{1,n}\} > \{f_{2,n}\} > \{f_{3,n}\} > \dots$

s.t. $f_{j,n}(r_j) \rightarrow L_j$

$\left(f_{j,n}(r_\ell) \rightarrow L_\ell \right) \text{ if } \ell \leq j$
 $\text{as } n \rightarrow \infty$

then

$f_{n,n}(r) \rightarrow L(r) \quad \forall r \in \mathbb{Q} \cap [a,b]$

by standard diagonalization.

Step ② : $f_{n,n} \xrightarrow{\text{Bw}} f_\infty$ on $[a,b]$.

want: $\forall \varepsilon > 0, \exists N > 0$ s.t. $m, n \geq N \Rightarrow \|f_{m,m} - f_{n,n}\|$

$\leq \xi.$

Assume $\neg A$:

$\exists n \in \omega . \forall m \in \omega . \exists A \subseteq N . \phi_A$

$$\text{but } \|f_{m,m} - f_{n,m}\| \geq \varepsilon_0.$$

10

$$\exists m_1 < m_2 < m_3 < \dots \text{ s.t. } \|f_{m_j, m_j} - f_{n_j, n_j}\| \geq \varepsilon_0$$

$\forall \varepsilon_0$

$$A_j : \sup_{x \in [a, b]} |f_{m_j, m_j}(x) - f_{n_j, n_j}(x)| \leq \varepsilon_0 < \frac{\varepsilon_0}{3}$$

$$\exists x_j \in [a, b] \text{ s.t.}$$

$$\sum_{j \in k} x_j \cdot B_{ij} \geq f_{m_j, m_j}(x_j) - f_{n_j, n_j}(x_j) > \varepsilon_0/2.$$

$$x_\infty \in \{a, b\}$$

$$|f_{m_j, m_j}(x_j) - f_{n_j, n_j}(x_j)| > \varepsilon_0/2.$$

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 \text{ s.t.}$$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

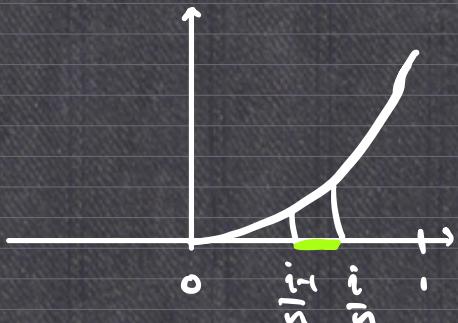
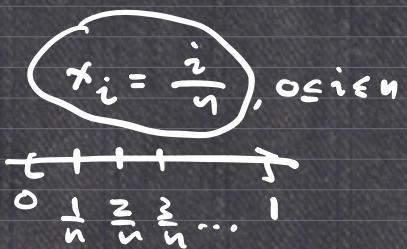
$$\frac{\epsilon_0}{2} < |f_{w_{j_k}, w_{j_k}}(x_{j_k}) - f_{u_{j_k}, u_{j_k}}(x_{j_k})|$$

$$\leq \left| f_{m_{jk}, m_{jk}}(x_{jk}) - f_{m_{jk}, m_{jk}}(r) \right| + \left| f_{m_{jk}, m_{jk}}(r) - f_{n_{jk}, n_{jk}}(r) \right|$$

$$+ |f_{n_{jk}}(r) - f_{n_{jk} n_{jk}}(x_{jk})|$$

e.g. $f(x) = x^2$ on $[0, 1]$

Let $P_n : 0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < \frac{n}{n} = 1$



$$U(f, P_n) = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n^3} (1^2 + 2^2 + \dots + n^2)$$

$$= \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) \rightarrow \frac{1}{3}$$

as $n \rightarrow \infty$

$$L(f, P_n) = \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} (0^2 + 1^2 + \dots + (n-1)^2)$$

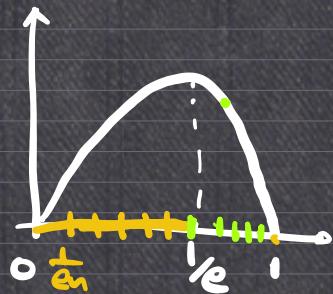
$$= \frac{1}{n^3} \cdot \frac{1}{6} (n-1)n(2n-1) \rightarrow \frac{1}{3}$$

$$\exists \{P_n\} \text{ s.t. } \lim_{n \rightarrow \infty} L(f, P_n) = \frac{1}{3} = \lim_{n \rightarrow \infty} U(f, P_n)$$

$\therefore x^2$ is Riemann integrable

$$\text{and } \int_0^1 f(x) dx = \frac{1}{3}$$

$$f(x) = \begin{cases} -x \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$



$$(i) \sup_P L(f, P) = \inf_P U(f, P)$$

$$\underbrace{(1-\frac{1}{n})}_{\approx}^{\approx n}$$

\downarrow

$$\text{i.e. } \exists \{P_k\} \text{ s.t. } \lim_{k \rightarrow \infty} U(f, P_k) = \lim_{k \rightarrow \infty} L(f, P_k)$$

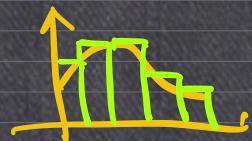
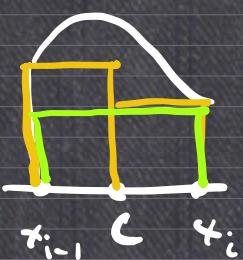
$$\exists \{P_k\} \text{ s.t. } \lim_{k \rightarrow \infty} L(f, P_k) = \sup_P L(f, P)$$

$$\exists \{Q_k\} \text{ s.t. } \lim_{k \rightarrow \infty} U(f, Q_k) = \inf_P U(f, P)$$

$$P \subseteq Q \implies U(f, P) \geq U(f, Q)$$

$$L(f, P) \leq L(f, Q)$$

e.g. $\{x_1, \dots, x_n\}$ $\{x_1, \dots, y_1, \dots, x_n\}$



$$R_k := P_k \cup Q_k$$

$$\Rightarrow L(f, P_k) \leq L(f, R_k) \leq U(f, R_k) \leq U(f, Q_k)$$

$$\downarrow$$

$$\sup_P L(f, P)$$

$$\downarrow$$

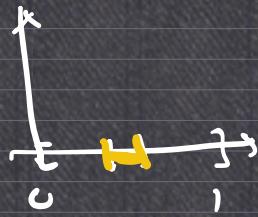
$$\inf_P U(f, P)$$

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

P = any partition of $[0, 1]$.

$$U(\chi_{\mathbb{Q}}, P) = \sum_{i=1}^n \sup_{[x_{i-1}, x_i]} \Delta x_i = 1$$

$$L(\chi_{\mathbb{Q}}, P) = \sum_{i=1}^n \inf_{[x_{i-1}, x_i]} \Delta x_i = 0.$$

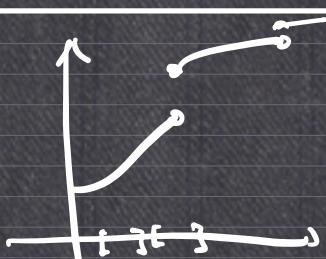


• Continuous \Rightarrow Riemann integrable on $[a, b]$

$$U(f, P) - L(f, P) = \sum_{i=1}^n (\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f) (x_i - x_{i-1})$$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{n}$

$$\leq \sum_{i=1}^n \frac{\varepsilon}{n} (x_i - x_{i-1}) = \frac{\varepsilon}{n} (b - a)$$



Ex: Given $f: [a, b] \rightarrow [m, M]$ Riem. integrable
 $\varphi: [m, M] \rightarrow \mathbb{R}$ continuous (\Rightarrow uniformly continuous)

then $\varphi \circ f: [a, b] \rightarrow \mathbb{R}$ Riem. integrable.

Proof: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $y_1, y_2 \in [m, M]$ and $|y_1 - y_2| < \delta \Rightarrow |\varphi(y_1) - \varphi(y_2)| < \frac{\varepsilon}{\sup f - \inf f}$

$\exists P$ of $[a, b]$ s.t. $U(f, P) - L(f, P) < \frac{\varepsilon}{\sup f - \inf f}$

$$\Rightarrow \sum_i (\sup_{I_i} f - \inf_{I_i} f) |I_i| < \frac{\varepsilon}{\sup f - \inf f}$$

$$\begin{aligned} U(\varphi \circ f, P) - L(\varphi \circ f, P) &\Rightarrow \sum_i (\sup_{I_i} \varphi \circ f - \inf_{I_i} \varphi \circ f) |I_i| \\ &= \sum_i (\sup_{I_i} \varphi(f(x_2)) - \inf_{I_i} \varphi(f(x_1))) |I_i| \end{aligned}$$

$$= \sum_{\substack{\sup f - \inf f < \delta \\ I_i}} (\sup_{I_i} \varphi \circ f - \inf_{I_i} \varphi \circ f) |I_i| \quad \text{--- (1)}$$

$$+ \sum_{\substack{\sup f - \inf f \geq \delta \\ I_i}} (\sup_{I_i} \varphi \circ f - \inf_{I_i} \varphi \circ f) |I_i| \quad \text{--- (2)}$$

Recall

$$\sum_j (\sup_{I_j} f - \inf_{I_j} f) |I_j| < \frac{\varepsilon}{\sup f - \inf f} \delta^2$$

$$\begin{aligned} 2\delta |I_i| &\leq \sum_{\substack{\sup f - \inf f \geq \delta \\ I_i}} (\sup_{I_i} f - \inf_{I_i} f) |I_i| < \frac{\varepsilon}{\sup f - \inf f} \delta^2 \\ &\Rightarrow |I_i| \leq \delta \end{aligned}$$

$$\begin{aligned}
 ② &\leq \sum_{\substack{\text{Sup } f - \inf f \geq \delta \\ I_i \\ I_i}} (\sup_{I_i} \varphi \circ f - \inf_{I_i} \varphi \circ f) |I_i| \\
 &\leq M \cdot \sum_{\substack{\text{Sup } f - \inf f \geq \delta \\ I_i}} |I_i| < M\delta.
 \end{aligned}$$

$E \subset \mathbb{R}$

Lebesgue outer measure

$\phi < \phi \cup \phi \cup \phi \cup \dots$

$$\begin{aligned}
 L^*(E) &= \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \mid E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\} \\
 &\cdot \boxed{L^*([a, b]) = b - a} \quad \text{3043}
 \end{aligned}$$



$$L^*(E) = 0$$

$$\inf = 0 \quad \varepsilon > 0$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \bigcup_{i=1}^{\infty} (a_i, b_i) \supset E \quad \text{s.t.} \quad \sum_i (b_i - a_i) < \varepsilon.$$

$$L^*([a, b]) = 0$$

$a \neq b$



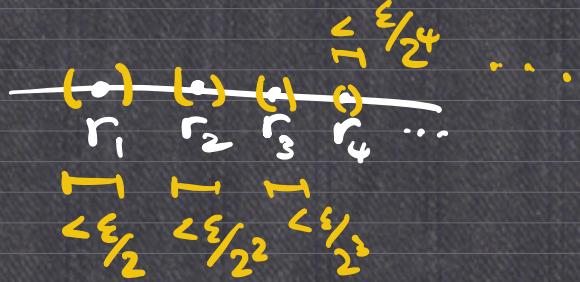
Proof: $\forall \varepsilon > 0$, pick $\{a, b\} \subset (a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2})$

$$\underbrace{(b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})}_{\text{total length}}.$$

$$\leq \varepsilon.$$

$$\mathcal{L}^*(\bigcup_{i=1}^{\infty} \{r_i\}) = 0$$

any countable set has zero Lebesgue measure.



- Given $\{E_i\}_{i=1}^{\infty}$ s.t. $\underline{\mathcal{L}}^*(E_i) = 0 \quad \forall i$
- $$\Rightarrow \underline{\mathcal{L}}^*(\bigcup_{i=1}^{\infty} E_i) = 0.$$

$$\forall \epsilon > 0, \exists j \in \mathbb{N}, \exists \bigcup_i (a_i^j, b_i^j) \supset E_j$$

$$\text{s.t. } \sum_i (b_i^j - a_i^j) < \epsilon / \sum a_i^j$$

$\epsilon_1, \epsilon_2, \epsilon_3$

$$\text{then } \bigcup_j E_j \subset \bigcup_j \bigcup_i (a_i^j, b_i^j)$$

$$\text{and } \sum_j \sum_i (b_i^j - a_i^j) \\ < \sum_j \epsilon_2^j = \epsilon.$$

Lebesgue Theorem :

$f: [a, b] \rightarrow [m, M]$ is Riemann integrable

$$\Leftrightarrow \mathcal{L}^*(D_f) = 0.$$

ii

$\{x_0 \in [a, b] \mid f \text{ is Not continuous at } x_0\}.$

- $f: [a,b] \rightarrow \mathbb{R}$ continuous
Proof: $D_f = \emptyset \Rightarrow \mathcal{L}^*(D_f) = 0$.
 - $f: [a,b] \rightarrow \mathbb{R}$ is monotone
Proof: D_f is countable
 $\Rightarrow \mathcal{L}^*(D_f) = 0$. 
 - $f, g: [a,b] \rightarrow [\underline{m}, M]$ Riem. integrable
 $\Rightarrow f+g$ is Riem. integrable
- Proof: If f, g are cts at x_0
 $\Rightarrow f+g$ is cts at x_0
-
- Contrapositive: $f+g$ is not cts at x_0
 $\Rightarrow f$ or g is not cts at x_0
- $$\therefore x_0 \in D_{f+g} \Rightarrow x_0 \in D_f \cup D_g$$
- $$\Rightarrow D_{f+g} \subset D_f \cup D_g$$
- $f: [a,b] \rightarrow [\underline{m}, M]$ Riem. integ.
 $\varphi: [\underline{m}, M] \rightarrow \mathbb{R}$ continuous
 $\Rightarrow \varphi \circ f$ is Riem. integ.

Proof: If f is continuous at x_0

$\Rightarrow \varphi \circ f$ is continuous at x_0

If $\varphi \circ f$ is not cts at x_0

then f is not cts at x_0

$$D_{\varphi \circ f} \subset D_f$$

Claim: $\mathcal{L}^*([a, b]) = b - a$.

$$\mathcal{L}^*(E) = \inf \left\{ \sum_i (b_i - a_i) \mid E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \quad a_i < b_i \right\}$$

$$(\leq) \quad [a, b] \subset \underline{(a-\varepsilon, b+\varepsilon)}$$

$$\forall \varepsilon > 0, \quad \mathcal{L}^*([a, b]) \leq (b + \varepsilon) - (a - \varepsilon) = b - a + 2\varepsilon$$

$$\text{let } \varepsilon \rightarrow 0^+ \Rightarrow \mathcal{L}^*([a, b]) \leq b - a.$$

$$(\geq) \quad \text{Take any arbitrary } \bigcup_{i=1}^n (a_i, b_i) \supset [a, b]$$

Heine-Borel

$$\Rightarrow \exists (a_{i_1}, b_{i_1}), \dots, (a_{i_N}, b_{i_N})$$

$$\text{s.t. } \bigcup_{j=1}^N (a_{i_j}, b_{i_j}) \supset [a, b]$$

$$\begin{matrix} \text{induction} \\ \text{on } N \end{matrix} \Rightarrow \sum_{j=1}^N (b_{i_j} - a_{i_j}) \geq b - a$$

$$\Rightarrow \sum_{i=1}^{\infty} (b_i - a_i) \geq \sum_{j=1}^N (b_{i_j} - a_{i_j}) \geq b - a.$$

$$\mathcal{L}^*([a, b]) = \inf \{ \dots \} \geq b - a.$$

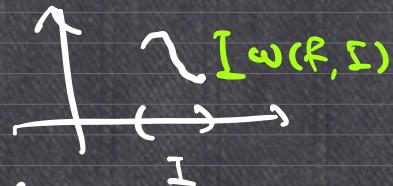
\Rightarrow Suppose $f: [a, b] \rightarrow [m, M]$ Riem. integrable

$$\text{Want: } \mathcal{L}^*(D_f) = 0.$$

$$\omega(f, I) := \sup_{\substack{\text{open interval} \\ I}} \{ |f(x) - f(y)| : x, y \in I \}$$

$$\omega(f, x_0)$$

$$:= \inf \{ \omega(f, I) \mid I \ni x_0 \}$$



$$D_f = \{x_0 \mid \omega(f, x_0) > 0\}.$$

$$\hookrightarrow : x_0 \in D_f \Rightarrow \exists \varepsilon_0 > 0, \forall \delta > 0$$

$$\exists x, y \in (x_0 - \delta, x_0 + \delta) \\ \text{s.t. } |f(x) - f(y)| \geq \varepsilon_0$$

$$\Rightarrow I := (x_0 - \delta, x_0 + \delta)$$

$$\Rightarrow \omega(f, I) \geq \varepsilon_0$$

$$\Rightarrow \omega(f, x_0) \leq \inf \{ \overbrace{\omega(f, I)}^{\geq \varepsilon_0} \mid I \ni x_0 \} \leq \varepsilon_0 \leq \varepsilon.$$

$$D_f = \{x_0 \mid \omega(f, x_0) > 0\} = \bigcup_{k \in \mathbb{N}} \{x_0 \mid \omega(f, x_0) \geq \frac{1}{k}\}$$

Fix $k \in \mathbb{N}$, consider $\Omega_k := \{x_0 \mid \omega(f, x_0) \geq \frac{1}{k}\}$
 want $\mathcal{L}^*(\Omega_k) = 0$.

$f: [a, b] \rightarrow [m, M]$ Riem. Integ.

$\forall \varepsilon > 0, \exists P = \{x_i\}$ s.t. $U(f, P) - L(f, P) < \varepsilon$?

$$\Omega_k \setminus \{x_0, x_1, \dots, x_n\} \subset \bigcup_{\Omega_k \cap (x_{i-1}, x_i) \neq \emptyset} (x_{i-1}, x_i)$$



$$U(f, P) - L(f, P)$$

$$= \sum_{i=1}^n (M_i - m_i) (x_i - x_{i-1})$$

$$M_i = \sup_{[x_{i-1}, x_i]} f \geq \sum_{\Omega_k \cap (x_{i-1}, x_i) \neq \emptyset} (M_i - m_i) (x_i - x_{i-1}) \geq \frac{1}{2k} \sum_{\Omega_k \cap (x_{i-1}, x_i) \neq \emptyset} (x_i - x_{i-1})$$

$\exists \epsilon > 0$ s.t. $|f(x_1) - f(x_2)| > \frac{1}{2k}$.

$$\text{If } x_{i-1} < y < x_i \in \Omega_k \text{ then } \omega(f, I) \geq \omega(f, \gamma) \geq \frac{1}{k}.$$

$$\Rightarrow \omega(f, \gamma) \geq \frac{1}{2k}.$$

(\Leftarrow) Given $\omega(\mathcal{D}_f) = 0$.

$\forall \epsilon > 0$, $\exists \bigcup_{i=1}^{\infty} (a_i, b_i) > \mathcal{D}_f$ s.t.

$$\sum_{i=1}^{\infty} (b_i - a_i) < \epsilon.$$

$$K := [a, b] \setminus \bigcup_{i=1}^{\infty} (a_i, b_i)$$

$$P = \{a = x_0 < x_1 < \dots < x_{n-1} < x_n = b\}.$$

$$U(f, P) - L(f, P) = \sum_i (M_i - m_i) (x_i - x_{i-1})$$

$$= \sum_{K \cap [x_{i-1}, x_i] \neq \emptyset} (M_i - m_i) (x_i - x_{i-1}) + \sum_{K \cap [x_{i-1}, x_i] = \emptyset} (M_i - m_i) (x_i - x_{i-1})$$



$$\begin{aligned} & |f(x) - f(\eta)| \\ & \xrightarrow{x \in K} \\ & x \quad \eta \end{aligned}$$