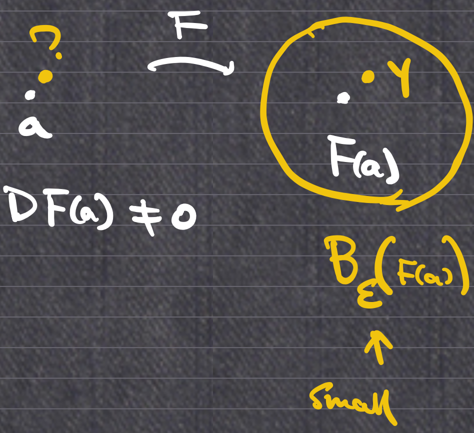
Step ①

Want: $\forall y \sim F(a)$

show $\exists x \sim a$

s.t. $F(x) = y$.



$\det DF(a) \neq 0$

$$T_y : \overline{B_\delta(a)} \longrightarrow \mathbb{R}^n$$

$$T_y(x) := x + DF(a)^{-1}(y - F(x))$$

☆ If $T_y(x) = x$ $\exists x$,

then $\cancel{x+}\ DF(a)^{-1}(y - F(x)) = \cancel{x}\ 0$

$\Rightarrow y = F(x)$.

Next: prove T_y is a contraction:

$$\|T_y(x_1) - T_y(x_2)\|_2$$

$$= \| x_1 + DF(a)^{-1}(\cancel{y} - F(x_1)) - x_2 - DF(a)^{-1}(\cancel{y} - F(x_2))\|$$

$$= \| (x_1 - x_2) - DF(a)^{-1}(F(x_1) - F(x_2))\|$$

$$= \| DF(a)^{-1}\big((DF(a)x_1 - F(x_1)) - (DF(a)x_2 - F(x_2))\big)\|$$

$$\leq \|DF(a)^{-1}\| \, \|G(x_1) - G(x_2)\|$$

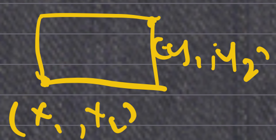
$\vdots$

$$G(x) := DF(a)\,x - F(x)$$

$$\le \|DF(a)^{-1}\| \cdot C \sup_{B_\delta(a)} \|DG\| \; \| x_1 - x_2 \|$$

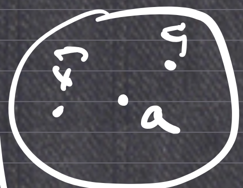
$$|f(x_1,x_2) - f(y_1,y_2)|$$

$$= |\underbrace{f(x_1,x_2) - f(x_1,y_2)} + f(x_1,y_2) - f(y_1,y_2)|$$



$$= \left| \frac{\partial f}{\partial y}\Big|_? (x_2-y_2) + \frac{\partial f}{\partial x}\Big|_? (x_1-y_1)\right|$$

$$\le \sup_{B_\varepsilon(?)} \|Df\|_2 \; \|(x_1,x_2)-(y_1,y_2)\|$$



$$= C\|DF(a)^{-1}\| \underbrace{\sup_{z\in \overline{B_\delta(a)}} \|DF(a)-DF(z)\|}\cdot\|x_1-x_2\|$$

F is $C^1 \Rightarrow DF(x)$ is C^0

$\Rightarrow \exists \delta$ s.t. $\displaystyle\sup_{z\in \overline{B_\delta(a)}} \|DF(a)-DF(z)\| \le \frac{1}{2C\|DF(a)^{-1}\|}$

$\exists \delta > 0$ s.t.

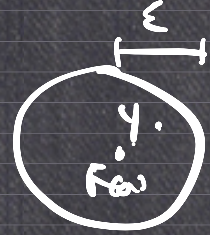
$T_y : \overline{B_\delta(a)} \longrightarrow \mathbb{R}^n$

is contraction:

$$\|T_y(x_1) - T_y(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|.$$

<u>**Next**</u>: choose smaller δ', ε s.t.

$T_y : \overline{B_{\delta'}(a)} \to \overline{B_{\delta'}(a)}$ when



$$\|T_y(x) - a\|$$

$$= \|T_y(x) - T_y(a)\| + \|T_y(a) - a\|$$

$$\le \tfrac{1}{2}\|x-a\| + \|\underbrace{\cancel{a} + DF(a)^{-1}(y - F(a))}_{T_y(a)} \cancel{-a}\|$$

$$\le \tfrac{1}{2}\|x-a\| + \|DF(a)^{-1}\| \underbrace{\|y - F(a)\|}_{<\varepsilon}.$$

$$\le \tfrac{1}{2}\delta' + \underbrace{\varepsilon\|DF(a)^{-1}\|}_{<\frac{1}{2}\delta'} < \delta'$$

Step ④ : F^{-1} is differentiable at $\underbrace{F(x_0)}_{y_0}$

$$F^{-1}(y) - F^{-1}(y_0) - DF(x_0)^{-1}(y - y_0)$$

as $y \to y_0$
$\Leftrightarrow x \to x_0$
$y = F(x)$

$$= F^{-1}(F(x)) - F^{-1}(F(x_0)) - DF(x_0)^{-1}\big(\underbrace{F(x) - F(x_0)}\big)$$

$$= F^{-1}(F(x)) - F^{-1}(F(x_0)) - DF(x_0)^{-1}\big(DF(x_0)(x - x_0) + E(x)\big)$$

$$= \cancel{F^{-1}(f(x)) - F^{-1}(f(x_0))}$$

x, x_0

$$\cancel{\sim (x-x_0)} + \tilde{E}(x)$$

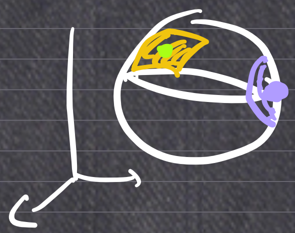
$$\in o(|x-x_0|) = o(|y-y_0|)$$

$E(x) \in o(|x-x_0|)$ as $x \to x_0$

$|x_1-x_2|$

$\le ?\ |y_1-y_2|$

$f(x,y,z)=0 \longrightarrow$



If $\frac{\partial f}{\partial x}(p) \neq 0 \Rightarrow$ near p, $x = g(y,z)$.

$\vdots$

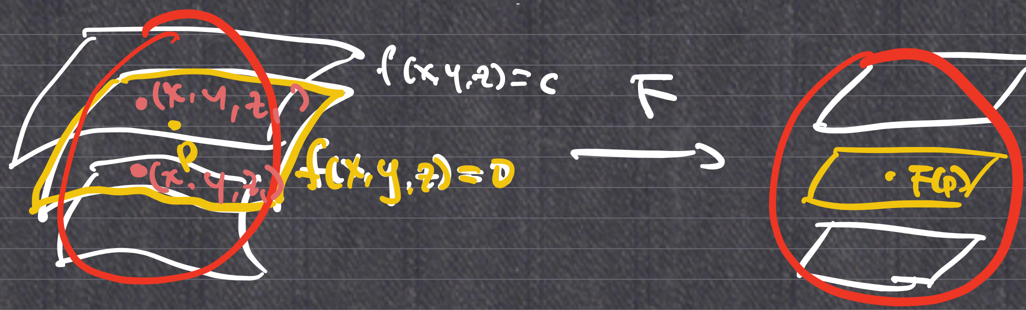
$\frac{\partial f}{\partial z}(p) \neq 0 \Rightarrow$ near p, $z = h(x,y)$.

Given : $f(x,y,z)$ is C^1, consider $f(x,y,z)=0$

$\exists p$ s.t. $\frac{\partial f}{\partial z}(p) \neq 0$

Consider $F(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^3$

$$F(x,y,z) = \begin{bmatrix} x \\ y \\ f(x,y,z) \end{bmatrix}$$



$$DF(x,y,z)_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}_p = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & \underset{\neq 0}{*} \end{bmatrix}}_{\text{invertible}}$$