(X, d) metric

$x_n \to y \overset{\text{def}}{\iff} \forall \varepsilon > 0, \exists N > 0$ s.t.

$\underset{X}{\uparrow} \quad \underset{X}{\uparrow} \qquad n \geq N \Rightarrow d(x_n, y) < \varepsilon$

Cauchy sequence:

$\{x_n\} \in X$ is Cauchy

$\overset{\text{def}}{\iff} \forall \varepsilon > 0, \exists N > 0$ s.t.

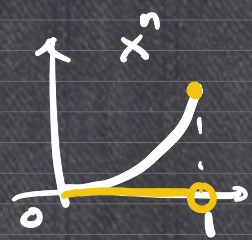
$m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon.$

e.g. $X = C[0,1]$

$$\|f\|_p := \left(\int_0^1 |f(x)|^p\right)^{\frac{1}{p}}$$

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

$$\boxed{f_n(x) = x^n}$$



$$\|f_n - 0\|_p = \left(\int_0^1 |x^n|^p \, dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 x^{pn} \, dx\right)^{\frac{1}{p}} = \left(\left[\frac{x^{pn+1}}{pn+1}\right]_{x=0}^{x=1}\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{pn+1}\right)^{p} \to 0 \text{ as } n \to \infty.$$

$$|f_n(y)-g(y)| \le \|f_n - g\|_\infty := \sup_{x\in[0,1]} |f_n(x) - g(x)|$$
$$\forall y \in [0,1] \qquad \uparrow$$

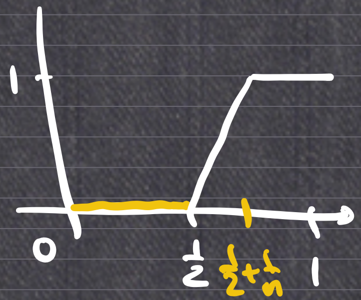
$$\|f_n - g\|_\infty \to 0 \implies f_n(y) \to g(y) \quad \forall y \in [0,1]$$

(X,d) is a <u>complete</u> metric space
$\overset{\text{def}}{\iff}$ $\forall$ Cauchy sequence in (X,d) converges to a limit in X.

$(V, \|\ \|)$ is said to be a <u>Banach</u> space
$\overset{\text{def}}{\iff}$ $\forall$ Cauchy sequence in $(V, \|\ \|)$ converges to a limit in V.

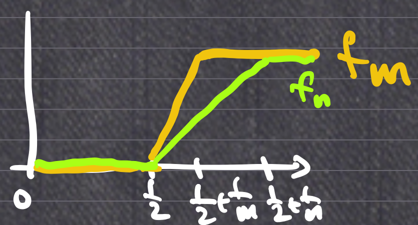
<u>e.g.</u> $X = C[0,1]$, $\|f\|_1 = \int_0^1 |f(x)|\, dx$

$f_n(x) =$



$\forall \varepsilon > 0, \ \exists N > 2/\varepsilon$

$$\|f_m - f_n\|_1 = \int_0^1 |f_m(x) - f_n(x)|\,dx \qquad \underline{m > n \geq N}$$

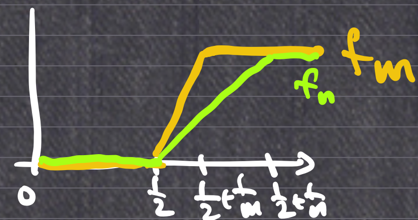


$$= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right)$$

$$\leq \frac{1}{2N} < \varepsilon.$$

$\{f_n\}$ is Cauchy wrt $\| \ \|_1$.

<u>WTS</u>: $f_n \not\to \ \forall g \in C[0,1]$



Assume $f_n \to g \in (C[0,1], \| \ \|_1)$

$$\underset{\text{as } n\to\infty}{0 \leftarrow} \|f_n - g\|_1 = \int_0^1 |f_n(x) - g(x)|\,dx$$

$$= \int_0^{\frac{1}{2}} |f_n(x) - g(x)|\,dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \cdots$$

$$+ \int_{\frac{1}{2}+\frac{1}{n}}^1 \ \cdots \ dx$$

$$= \int_0^{1/2} |g(x)|\,dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - g(x)|\,dx$$

$$+ \int_{\frac{1}{2}+\frac{1}{n}}^1 |1 - g(x)|\,dx$$

$$\Rightarrow \lim_{n\to\infty} \underline{\int_0^{1/2} |g(x)|\,dx} = 0 \Rightarrow \int_0^{1/2} |g(x)|\,dx = 0$$

$$\Rightarrow \underline{g(x) = 0 \text{ on } [0, \tfrac{1}{2}]}$$

$$\lim_{n\to\infty} \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - g(x)| dx = 0$$

$$\lim_{n\to\infty} \int_{\frac{1}{2}+\frac{1}{n}}^{1} |1-g(x)| dx = 0$$

$$\Rightarrow \int_{\lim_{n\to\infty}(\frac{1}{2}+\frac{1}{n})}^{1} |1-g(x)| dx \Rightarrow \int_{\frac{1}{2}}^{1} |1-g(x)| dx$$

$$\Rightarrow \underline{g(x) = 1 \text{ on } [\tfrac{1}{2}, 1]}$$

$g \notin C[0,1]$

Conclusion: $(C[0,1], \|\cdot\|_1)$ is not a Banach space.

$C[0,1]$, $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$

$$\|f_n - g\|_\infty = \sup_{x\in[0,1]} |f_n(x) - g(x)|$$

$f_n \to g$ in $\| \ \|_\infty$

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $n \geq N$

$$\Longrightarrow \sup_{x\in[0,1]} |f_n(x) - g(x)| < \varepsilon.$$

$\Rightarrow \quad \forall x \in [0,1], \quad |f_n(x) - g(x)| < \varepsilon.$

$\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \geq N, \forall x \in [0,1]$
$|f_n(x) - g(x)| < \varepsilon.$

Take $\{f_n\}$ Cauchy seq in $(C[0,1], \| \ \|_\infty)$.

$$|f_n(x) - f_m(x)| \leq \sup_{[0,1]} |f_n - f_m| < \varepsilon/2$$
$\forall x \in [0,1]$

$\forall x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$.

$\Rightarrow \quad f_n(x) \to g(x).$

$$\|f_n - g\|_\infty = \sup_{[0,1]} |f_n - g|$$

$$|f_n(x) - g(x)| \leq \underbrace{|f_n(x) - f_N(x)|}_{< \varepsilon/2} + \underbrace{|f_N(x) - g(x)|}_{< \varepsilon/2}$$