$f : \mathbb{R}^n \to \mathbb{R}$ is k-times diff^{ble} at $\vec{a}$

$\overset{\text{def}}{\Longleftrightarrow}$ ① k-th order partial derivatives exist at $\vec{a}$

② $(k-1)$-th partial derivatives are differentiable at $\vec{a}$

e.g. $f_{x_1 \cdots x_{k-1}}(\vec{x}) = f_{x_1 \cdots x_{k-1}}(\vec{a})$

$$+ \sum_j \frac{\partial}{\partial x_j} f_{x_1 \cdots x_{k-1}}(\vec{a}) \cdot (x_j - a_j)$$

$$+ o(|\vec{x} - \vec{a}|) \text{ as } \vec{x} \to \vec{a}.$$

$C^1 \Rightarrow$ differentiable

$(k-1)$-th partials $C^1 \Rightarrow$ ② ✓

C^k : k-th order partials are continuous

$C^k \Longrightarrow$ k-order differentiable.

$f(x,y) = \log(1+x^2+y^2)$ is $C^\infty = \bigcap_{k=1}^{\infty} C^k$.

$$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{2y}{1+x^2+y^2}$$

$$\frac{\partial^k f}{\partial \cdots} = \frac{?}{(1+x^2+y^2)^?}$$

If $f(x,y)$ is C^2, then $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

Proof: $(a,b) \in \mathbb{R}^2$ fixed

$$\Delta(h,k) = f(a+h, b+k) - f(a+h, b) - (f(a, b+k) - f(a,b))$$

$\forall h, k \neq 0.$

$g(x,y) = f(a+x, y) - f(a,y)$

$$= g(h, b+k) - g(h, b)$$

$$= \frac{\partial g}{\partial y}(h, c_1) \cdot k$$

$c_1 \in (b, b+k)$

$c_2 \in (a, a+h)$

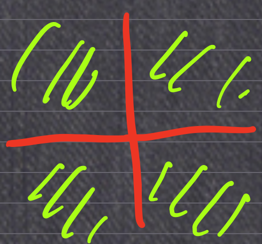
$$= \left(\frac{\partial f}{\partial y}(a+h, c_1) - \frac{\partial f}{\partial y}(a, c_1)\right) k$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Big|_{(c_2, c_1)} \cdot hk$$

$$\Delta(h,k) = f(a+h, b+k) - f(a+h, b) - (f(a, b+k) - f(a,b))$$

$$= \cdots = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\Big|_{(c_3, c_4)} hk.$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}_{\text{cts}}\Big|_{(c_2, c_1)} = \underline{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}\Big|_{(c_3, c_4)}$$



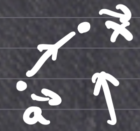
$\downarrow \quad (h,k) \to (0,0)$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a,b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a,b)$$

Given $f(\vec{x})$ is twice diff' at $\vec{a}$

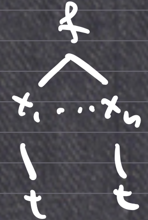
$\vec{x} = (x_1, \dots, x_n)$

$$g(t) = f(\underbrace{(1-t)\vec{a} + t\vec{x}})$$



$\underline{(1-t)\vec{a} + t\vec{x}}$

$$g'(t) = \sum_j \frac{\partial f}{\partial x_j}((1-t)\vec{a}+t\vec{x}) \cdot \underbrace{\frac{dx_j}{dt}}_{(x_j - a_j)}$$



$$\frac{d}{dt}\left(\underline{(1-t)a_j + t x_j}\right) = x_j - a_j$$

$$g'(0) = \sum_j \frac{\partial f}{\partial x_j}(\vec{a}) \cdot (x_j - a_j)$$

$$g''(0) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \, (x_i - a_i)(x_j - a_j)$$

$$g(t) = g(0) + g'(0)\,t + \frac{g''(0)}{2!} t^2 + ?$$

$$\text{error term} = f(\vec{x}) - \Big(f(\vec{a}) + \sum_i \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i - a_i)(x_j - a_j)\Big)$$

$$= g(1) - \Big(g(0) + g'(0)\cdot 1 + \frac{g''(0)}{2!} \cdot 1^2\Big)$$

let $E(t) = g(t) - \Big(g(0) + g'(0)\,t + \frac{g''(0)}{2!} \cdot t^2\Big)$

$= o(|x-a|^2)$?

$$\frac{E(1) - \cancel{E(0)}}{1^2 - 0^2} = \frac{E'(c)}{2c} = \frac{1}{2c}\big(g'(c) - g'(0) - g''(0)\,c\big)$$

$$E(1) = \frac{1}{2c}\left(g'(c) - g'(0)\right) - \frac{1}{2}g''(0)$$

$$= \frac{1}{2c}\sum_i \left(\boxed{\frac{\partial f}{\partial x_i}((1-c)\vec{a}+c\vec{x})} - \frac{\partial f}{\partial x_i}(\vec{a}) \right)(x_i - a_i)$$

$$- \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \cdot (x_i - a_i)(x_j - a_j)$$

$$= \frac{1}{2\cancel{c}}\sum_i \Bigg(\cancel{\frac{\partial f}{\partial x_i}(\vec{a})} + \cancel{\sum_j \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}(\vec{a})\,\big((1-c)a_j + cx_j - a_j\big)}$$

$\cancel{c}(x_j - a_j)$

$$+ o\left(\left|(1-c)\vec{a}+c\vec{x} - \vec{a}\right|\right) = o(|\vec{x}-\vec{a}|)$$

$$- \cancel{\frac{\partial f}{\partial x_i}(\vec{a})} \Bigg) \cdot (x_i - a_i)$$

$$\cancel{- \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) (x_i - a_i)(x_j - a_j)}$$

$f(0,0)$ $\quad\quad$ $\nabla f(0,0) = \vec{0}$

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

$$f(x,y) = f(0,0) + \cancel{f_x(0,0)} \cdot x + \cancel{f_y(0,0)} \cdot y$$

$$+ \frac{1}{2!}\Big(f_{xx}(0,0)\,x^2 + 2 f_{xy}(0,0)\,xy + f_{yy}(0,0)\,y^2 \Big)$$

$Q(x,y)$

$$+ o(|(x,y)|^2)$$

$$\overset{\shortparallel}{[x\ y]\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix}}$$

$$\text{If } \begin{cases} (2 f_{xy}(0,0))^2 - 4 f_{xx}(0,0) f_{yy}(0,0) < 0 \\ f_{xx}(0,0) > 0 \end{cases}$$

$$\Longrightarrow \quad \text{then } Q(x,y) \geq 0 \quad (= \text{ holds iff } (x,y) = (0,0))$$