## §4.2 Arzéla – Ascoli Theorem

$\{f_n\}$ sequence of functions on $[a,b]$.

$\nexists$ $f_{n_k} \rightrightarrows f_\infty$.

- uniformly bounded

  $\exists M > 0$ s.t. $|f_n(x)| \le M$ $\forall n \in \mathbb{N}$, $x \in [a,b]$

  ↑ indep. of $x, n$.

  e.g. $\left\{ \sin\left( \dfrac{n^{\lfloor n \rfloor} \cos(\lceil n^2 \rceil)}{n e^n + e^{n^e} + \sum a^n \cos(b^n \pi x)} \right) \right\}$

- equicontinuity

  $\{f_n\}$ is equicontinuous on $[a,b]$

  $\overset{\text{def}}{\iff}$ $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

  $|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$ $\forall n \in \mathbb{N}$.

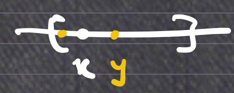
  e.g. $\{\sin(\frac{x}{n})\}$

$\forall \varepsilon > 0$ $\exists \delta = \varepsilon$, then $|x-y| < \delta$

$$\Rightarrow \Big|\underbrace{\sin\left(\tfrac{x}{n}\right)}_{g_n(x) = \sin \frac{x}{n}} - \sin\left(\tfrac{y}{n}\right)\Big| = \underbrace{\Big|\tfrac{1}{n}\cos\tfrac{c}{n}\Big|}_{g_n'(c)} |x-y|$$

$\le \frac{1}{n}|x-y| < \frac{\delta}{n} \le \delta = \varepsilon$

non-e.c. $\{\underbrace{nx}_{f_n(x)}\}_{n=1}^{\infty}$




Take $\varepsilon_0 = 1$, $\forall \delta > 0$, $\exists x, y$ s.t. $|x-y| = \frac{\delta}{2}$

$\exists n_0 \ge \frac{2}{\delta}$, $|f_n(x) - f_n(y)| = n|x-y| = n\frac{\delta}{2} \ge 1 = \varepsilon_0$

- if $|f_n'(x)| \le M$ $\forall n \in [a,b], x \in [a,b]$

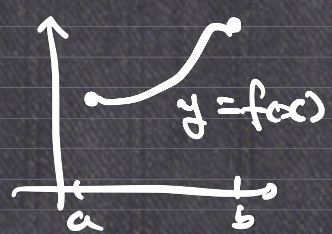
  $\Rightarrow \{f_n\}$ is equicontinuous on $[a,b]$.

Arzela-Ascoli

if $\{f_n\}$ is ① uniformly bounded on $[a,b]$.
and ② equicontinuous

then $\exists \{f_{n_k}\} \rightrightarrows f_\infty$ on $[a,b]$.

- $L(f) := \int_a^b \sqrt{1+f'(x)^2}\,dx$




- inf $L(f)$ exists.
  $f \in C^1, f(a), f(b)$

$\exists f_n \in C^1[a,b]$ s.t. $L(f_n) \to \inf_{f \in C^1} L(f)$.

e.g. Given $\{f_n\} \in C[a,b]$.

$$|f_n(x)| \le M \quad \forall x \in [a,b],\ n \in \mathbb{N}.$$

$$F_n(x) := \int_a^x f_n(t)\,dt$$

want: $\exists\ F_{n_k} \rightrightarrows F_\infty$ on $[a,b]$.

① $\{F_n\}$ is uniformly bounded on $[a,b]$

Proof:
$$|F_n(x)| = \left|\int_a^x f_n(t)\,dt\right|$$
$$\le \int_a^x |f_n(t)|\,dt$$
$$\le \int_a^x M\,dt = M(x-a)$$
$$\le \underline{M(b-a)}$$

② $\{F_n\}$ is equicontinuous on $[a,b]$.

Proof: $F_n'(x) = f_n(x)$

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Ex. 4.28

Given $f_n : [a,b] \to \mathbb{R}$, $C^\infty$

$\forall k \in \mathbb{N} \cup \{0\}$, $\exists C_k > 0$ s.t.

$$\| f_n^{(k)} \|_{[a,b]} \le C_k$$

then show $\exists \{f_{n_j}\}$ s.t. $f_{n_j}^{(k)} \rightrightarrows f_\infty^{(k)}$ on $[a,b]$

$\forall k \in \mathbb{N} \cup \{0\}$

**Proof:**

$\|f_n\| \le C_0$, $\|f_n'\| \le C_1$

$\Rightarrow \exists f_{n_j} \rightrightarrows f_\infty$. $\quad f_{1,n} \rightrightarrows f_\infty$.

$f_{1,n}$

$f_n$ : $\boxed{f_1}$ $f_2$ $f_3$ $f_4$ $f_5$ $f_6$ $\boxed{f_7}$, $f_8$ $f_9$ $f_{10}$

$\|f_{1,n}'\| \le C_1$, $\|f_{1,n}''\| \le C_2 \Rightarrow \exists f_{1,n_j}' \rightrightarrows f_\infty$

$\boxed{f_1}$ $\cancel{f_2}$ $f_3$ $\cancel{f_4}$ $f_5$ $\cancel{f_6}$ $\boxed{f_7}$ $\cancel{f_8}$ $f_9$ $\cancel{f_{10}}$ $f_{11}$

$f_{2,n}$

$\{f_{1,n}\} \supset \{f_{2,n}\} \supset \{f_{3,n}\} \supset \cdots$

construct $\boxed{\{f_{n,n}\}_{n=1}^\infty}$ = subsequence of each $\{f_{k,n}\}_{n=1}^\infty$ (except the first k-1 terms)

□