Claim: $\mathcal{L}^*([a,b]) = b-a$.

$$\mathcal{L}^*(E) = \inf\left\{ \sum_i |b_i - a_i| \;\middle|\; E \subset \bigcup_{i=1}^{\infty} (a_i, b_i),\ a_i \le b_i \right\}$$

$(\le)$ $[a,b] \subset \underline{(a-\varepsilon, b+\varepsilon)}$

$\forall \varepsilon > 0$, $\mathcal{L}^*([a,b]) \le (b+\varepsilon) - (a-\varepsilon) = b-a+2\varepsilon$

let $\varepsilon \to 0^+$ $\Longrightarrow$ $\mathcal{L}^*([a,b]) \le b-a$.

$(\ge)$ Take any arbitrary $\bigcup_{i=1}^{\infty} (a_i, b_i) \supset [a,b]$

$\overset{\text{Heine-Borel}}{\Longrightarrow}$ $\exists\, (a_{i_1}, b_{i_1}), \ldots, (a_{i_N}, b_{i_N})$

s.t. $\bigcup_{j=1}^{N} (a_{i_j}, b_{i_j}) \supset [a,b]$

$\underset{\text{on } N}{\overset{\text{induction}}{\Longrightarrow}}$ $\sum_{j=1}^{N} (b_{i_j} - a_{i_j}) \ge b-a$

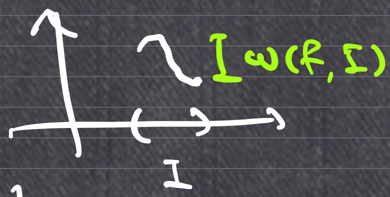
$\Rightarrow$ $\sum_{i=1}^{\infty} (b_i - a_i) \ge \sum_{j=1}^{N} (b_{i_j} - a_{i_j}) \ge b-a$.

$\mathcal{L}^*([a,b]) = \inf\{ \cdots \} \ge b-a$.

$(\Longrightarrow)$ Suppose $f: [a,b] \to [m, M]$ Riem. integrable

Want: $\mathcal{L}^*(D_f) = 0$.

$$\omega(f, \underset{\text{open interval}}{I}) := \sup\{|f(x)-f(y)| : x, y \in I\}$$



$$\omega(f, x_0) := \inf\{\omega(f, I) \mid I \ni x_0\}$$

$$D_f = \{x_0 \mid \omega(f, x_0) > 0\}.$$

$\subset$: $x_0 \in D_f \Rightarrow \exists \varepsilon_0 > 0, \forall \delta > 0$
$\exists x, y \in (x_0 - \delta, x_0 + \delta)$
s.t. $|f(x) - f(y)| \geq \varepsilon_0$

$\Rightarrow$ $I := (x_0 - \delta, x_0 + \delta)$

$\Rightarrow \omega(f, I) \geq \varepsilon_0$

$\Rightarrow \omega(f, x_0) = \inf\{\underbrace{\omega(f, I)}_{\geq \varepsilon_0} \mid I \ni x_0\}$
$\geq \varepsilon_0$

$$D_f = \{x_0 \mid \omega(f, x_0) > 0\} = \bigcup_{k \in \mathbb{N}} \{x_0 \mid \omega(f, x_0) \geq \tfrac{1}{k}\}$$

Fix $k \in \mathbb{N}$, consider $\Omega_k := \{x_0 \mid \omega(f, x_0) \geq \frac{1}{k}\}$

want $\mathcal{L}^*(\Omega_k) = 0$.

$f : [a, b] \to [m, M]$ Riem. Intg.

$\forall \varepsilon > 0$, $\exists P = \{x_i\}$ s.t. $U(f, P) - L(f, P) < ?$

$$\Omega_k \setminus \{x_0, x_1, \dots, x_n\} \subset \bigcup_{\Omega_k \cap (x_{i-1}, x_i) \neq \phi} (x_{i-1}, x_i)$$

$$U(f,P) - L(f,P)$$

$$= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

$$M_i := \sup_{[x_{i-1}, x_i]} f \qquad \geq \sum_{\Omega_k \cap (x_{i-1}, x_i) \neq \phi} (M_i - m_i)(x_i - x_{i-1}) \geq \frac{1}{2k} \sum_{\Omega_k \cap (x_{i-1}, x_i) \neq \phi} (x_i - x_{i-1})$$

$\exists\, x, y$ s.t. $|f(x) - f(y)| > \frac{1}{2k}$.

I

$\omega(f, I) \geq \omega(f, y) \geq \frac{1}{k}$.

$> \frac{1}{2k}$

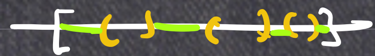
$y \in \Omega_k \leftarrow \omega(f, y) \geq \frac{1}{k}$.

$(\Leftarrow)$ Given $\mathcal{L}^*(D_f) = 0$.

$\forall \varepsilon > 0$, $\exists \bigcup_{i=1}^{\infty} (a_i, b_i) \supset D_f$ s.t.

$$\sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon.$$

$K := [a,b] \setminus \bigcup_{i=1}^{\infty} (a_i, b_i)$



$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$.

$$\mathcal{U}(f,P) - L(f,P) = \sum_i (M_i - m_i)(x_i - x_{i-1})$$

$$= \sum_{K\cap[x_{i-1},x_i]\neq\phi} (M_i - m_i)(x_i - x_{i-1}) + \sum_{K\cap[x_{i-1},x_i]=\phi} (M_i - m_i)(x_i - x_{i-1})$$