Heine–Borel

$K \subset \mathbb{R}$, TFAE:

① K is closed and bounded

② K is sequentially compact
($\forall$ seq. $\{x_n\}$ in K, $\exists \{x_{n_k}\} \to L \in K$.)

③ K is compact
(every open cover of K has a finite subcover)

① $\Longleftrightarrow$ ② : Bolzano–Weierstrass. & K is closed.

② $\Longrightarrow$ ③ : Given K is sequentially compact.

Need: any open cover $\{U_\alpha\}_{\alpha\in A}$ of K has a finite subcover.

$\downarrow$

Step 1: .. countable ..

$$U_\alpha = \coprod_{i=1}^{\infty} (c_{\alpha,i}, d_{\alpha,i})$$

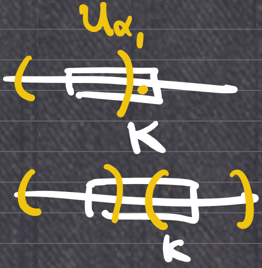
$$= \coprod_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (p_{\alpha,i,j}, q_{\alpha,i,j})$$

$$K \subset \bigcup_{\alpha\in A} U_\alpha = \underbrace{\bigcup_{\alpha\in A} \coprod_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (p_{\alpha,i,j}, q_{\alpha,i,j})}_{= \bigcup_{k=1}^{\infty} (p_k, q_k)}$$

$\subset \bigcup_{n=1}^{\infty} \mathcal{U}_{\alpha_k}$.

<u>Step 2</u>: Assume opposite

K cannot be covered by finitely many $\mathcal{U}_{\alpha_n}$'s.



$\exists x_1 \in K$, $x_1 \notin \mathcal{U}_{\alpha_1}$

$\exists x_2 \in K$, $x_2 \notin \mathcal{U}_{\alpha_1} \cup \mathcal{U}_{\alpha_2}$

$\exists x_3 \in K$, $x_3 \notin \mathcal{U}_{\alpha_1} \cup \mathcal{U}_{\alpha_2} \cup \mathcal{U}_{\alpha_3}$

$\exists x_4 \in K$ $x_4 \notin \mathcal{U}_{\alpha_1} \cup \mathcal{U}_{\alpha_2} \cup \mathcal{U}_{\alpha_3} \cup \mathcal{U}_{\alpha_4}$

$\vdots$

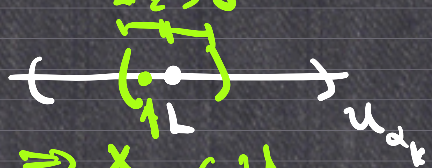
$x_{n_j} \in K$ $\quad x_{n_j} \notin \mathcal{U}_{\alpha_1} \cup \cdots \cup \mathcal{U}_{\alpha_{n_j}}$

seq. compact. $\downarrow$

$L \in K$

$L \in K \subset \bigcup_{k=1}^{\infty} \mathcal{U}_{\alpha_k}$

$\Rightarrow \exists k$ s.t. $L \in \mathcal{U}_{\alpha_k}$

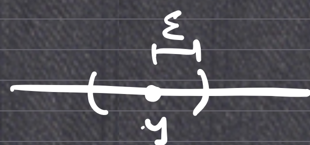


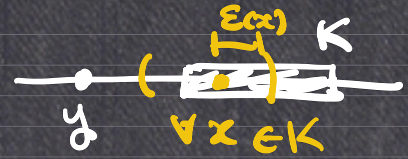
$j \gg 1 \Rightarrow x_{n_j} \in \mathcal{U}_{\alpha_k}$

③ $\Rightarrow$ ①: Given K is compact.

- Prove K is closed.

<u>Proof</u>: $\forall y \in \mathbb{R} \setminus K$





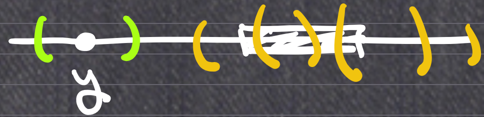
$\forall x \in K$, choose $\varepsilon(x) > 0$ s.t.

$y \notin (x - \varepsilon(x), x + \varepsilon(x))$

$$K = \bigcup_{x \in K} \{x\} \subset \bigcup_{x \in K} (x - \varepsilon(x), x + \varepsilon(x))$$

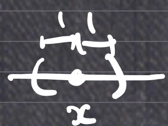
K compact $\Longrightarrow \exists x_1, \dots, x_n \in K$ s.t.

$$K \subset \bigcup_{i=1}^{n} (x_i - \varepsilon(x_i), x_i + \varepsilon(x_i))$$



- Prove K is bounded.

<u>Proof</u>: $\forall x \in K$, consider $(x-1, x+1)$



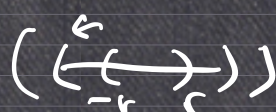
$$K = \bigcup_{x \in K} \{x\} \subset \bigcup_{x \in K} (x-1, x+1)$$

compact $\Downarrow$

$\exists x_1, \dots, x_N \in K$ s.t.

$$K \subset \bigcup_{i=1}^{N} (x_i - 1, x_i + 1)$$

$$K \subset \bigcup_{r > 0} (-r, r) = \mathbb{R}$$



Extreme Value Theorem.

$f : [a,b] \to \mathbb{R}$ continuous.

$\exists\, x_0, x_1 \in [a,b]$ s.t.

$$f(x_0) = \inf_{[a,b]} f \in \mathbb{R}$$

$$f(x_1) = \sup_{[a,b]} f \in \mathbb{R}.$$

Goal: $f([a,b])$ is closed and bounded.

Proof ①: Show $f([a,b])$ is sequentially compact.

Take sequence $\{y_n\} \in f([a,b])$

$$y_n = f(x_n) \quad \exists\, x_n \in \underbrace{[a,b]}.$$

sequentially compact

$$y_{n_j} = \underbrace{f(x_{n_j})}_{\downarrow \atop f(L)} \qquad \exists\, x_{n_j} \to L \in [a,b].$$

Proof ②: Show $f([a,b])$ is compact.

Take any open cover $\{U_\alpha\}_{\alpha \in A}$ of $f([a,b])$

$$f([a,b]) \subset \bigcup_{\alpha \in A} U_\alpha.$$

f continuous $\Leftrightarrow f^{-1}(\text{open})$ is open.

$$\Rightarrow \quad [a,b] \subset f^{-1}\Big(\bigcup_{\alpha \in A} U_\alpha\Big)$$

$$= \bigcup_{\alpha \in A} f^{-1}(U_\alpha).$$

open.

$[a,b]$ is compact

$\Rightarrow \exists \alpha_1, \dots, \alpha_n$ s.t. $[a,b] \subset \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$

$\Rightarrow f([a,b]) \subset \bigcup_{i=1}^{n} U_{\alpha_i}$

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§2: Metric Spaces.

$$\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t. } 0<\overset{d(x,a)}{|x-a|}<\delta \Longrightarrow \underset{D(f(x),L)}{|f(x)-L|}<\varepsilon.$$

Let $X \neq \emptyset$.

$d: X \times X \to [0,\infty)$ is a metric (distance function)

if d satisfies:

① $d(x,y) \geq 0 \quad \forall x,y \in X$, and equality holds $\Leftrightarrow x=y$.

② $d(x,y) = d(y,x) \quad \forall x,y \in X$

③ $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X$.

then call (X,d) is called a metric space.

$(\mathbb{R}, |x-y|)$ ✓

$\vec{x} = (x_1, \dots, x_n)$

$\mathbb{R}^n$, $d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}.$$

- $X \neq \phi$

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

discrete.

① ✓

② ✓

③ x, y, z

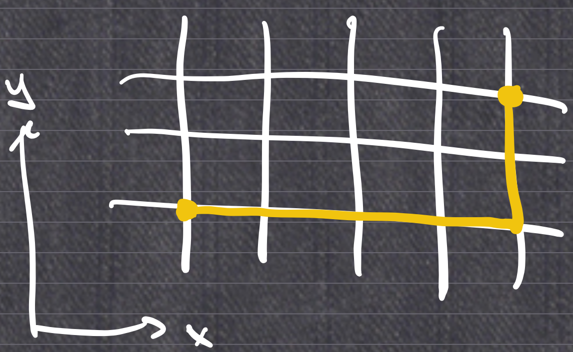
$x = y$, , $x = y = z$

$\neq$

z . . . . .

- $\mathbb{R}^2$:

$$d(\vec{a}, \vec{b}) = |x_1 - x_2| + |y_1 - y_2|$$



$\vec{a} = (x_1, y_1)$

$\vec{b} = (x_2, y_2)$