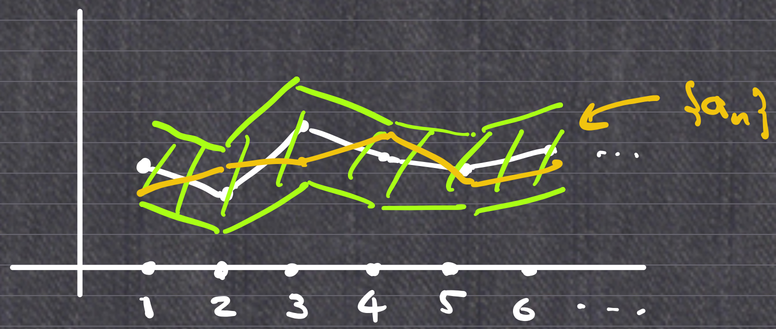
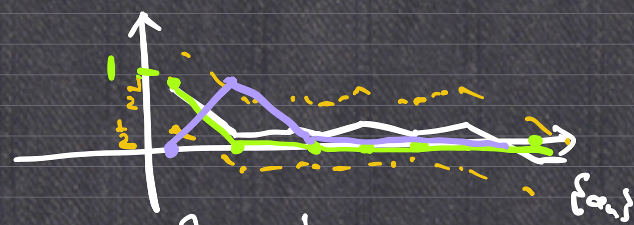


$$B_\varepsilon(\{a_n\}) = \left\{ \{b_n\} \in \ell^\infty(\mathbb{R}) \;\middle|\; \sup_{n\in\mathbb{N}} |a_n - b_n| < \varepsilon \right\}$$

$$e_i = \{0, 0, \dots, \underset{i}{1}, 0, 0 \dots\}$$

$$S = \{e_1, e_2, e_3, \dots\}$$

$B_{1/4}(\{a_n\}) \ni$



at most one of e_i's.

① K complete and totally bounded

② K sequentially compact

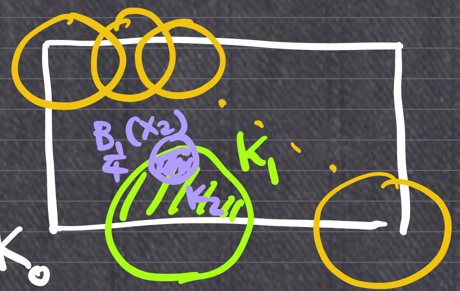
③ K compact

① ⇒ ③ ⇒ ② ⇒ ①.

① ⇒ ③ : Given K is complete, totally bounded.

Take an open cover $\bigcup_{\alpha\in A} U_\alpha \supset K$.

Assume no finite subcover for K.



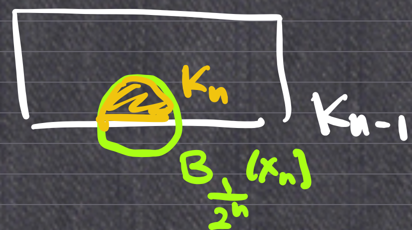
$$K \subset B_{\frac12}(x_1)\cup\cdots\cup B_{\frac12}(x_n)$$

$$K \cap B_{\frac12}(x_i)$$

$\exists\, B_{\frac12}(x_1) \Leftarrow K_1 = K_0 \cap B_{\frac12}(x_1)$ has no finite subcover.

$\exists\, B_{\frac14}(x_2)$ $\quad K_2 = K_1 \cap B_{\frac{1}{2^2}}(x_2)$ has no finite subcover.

$\vdots$



$K_n = K_{n-1} \cap B_{\frac{1}{2^n}}(x_n)$. has no finite subcover.

$$K_n \neq \emptyset \Rightarrow \exists\, y_n \in K_n \subset B_{\frac{1}{2^n}}(x_n)$$

$$y_{n+1} \in K_{n+1} \subset K_n \subset B_{\frac{1}{2^n}}(x_n)$$

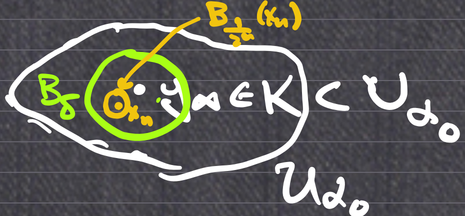
$d(y_n, y_{n+1}) \le \frac{1}{2^{n-1}} \Rightarrow \{y_n\}$ is Cauchy.

$d(x_n, y_n) \le \frac{1}{2^n}$

K complete

$y_n \to y_\infty \in K$

x_n



③ $\Rightarrow$ ② : Given K is compact

<u>want</u>: $\{x_n\} \subset K$. $\rightsquigarrow$? $\exists \{x_{n_j}\} \to x_\infty \in K$

$$\overline{S_n := \{x_n, x_{n+1}, x_{n+2}, \dots\}}$$

$$S := \bigcap_{n=1}^{\infty} S_n$$

<u>want</u>: $S \cap K \neq \emptyset$.

Assume $S \cap K = \emptyset$, $\left(\bigcap_{n=1}^{\infty} S_n\right) \cap K = \emptyset$.

$$\Leftrightarrow \quad K \subset X - \bigcap_{n=1}^{\infty} S_n$$
$$= \bigcup_{n=1}^{\infty} (X - S_n)$$

K compact

$\exists S_{n_1}, \dots, S_{n_k}$ s.t. $\quad n_1 < \dots < n_k$

$$K \subset (X - S_{n_1}) \cup \dots \cup (X - S_{n_k}) = X - (S_{n_1} \cap \dots \cap S_{n_k})$$

$$\subset X - \underbrace{\{x_{n_k}, x_{n_{k+1}}, \dots\}}_{\text{in } K}$$

$\therefore \underbrace{S \cap K}_{\ni\, y} \neq \emptyset$,

$$y \in S = \bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} \overline{\{x_n, x_{n+1}, \dots\}}$$

$y \in \overline{S_1 = \{x_1, x_2, \dots\}}$

$B_1(y) \ni x_{m_1}$ $\quad m_1 \geq 1.$

$y \in \overline{S_{m_1+1} = \{x_{m_1+1}, x_{m_1+2} \dots\}}$

$1/2$

$B_{\frac{1}{2}}(y) \ni x_{m_2}$ $\quad m_2 > m_1 \geq 1.$

$\forall j . \exists x_{m_j} \in B_{\frac{1}{j}}(y)$, $\quad m_j > m_{j-1} > \dots > m_2 > m_1$

$d(x_{m_j}, y) < \frac{1}{j} \to 0.$

② ⇒ ① : sequentially compact ⇒ complete, totally bounded.

__want:__ $\forall \varepsilon > 0, \exists B_\varepsilon(x_1) \dots, B_\varepsilon(x_n)$

$\bigcup_{i=1}^{N} B_\varepsilon(x_i) \supset K$

Assume not: $\exists \varepsilon > 0$, $\quad K \not\subset$ any $\bigcup_{i=1}^{N} B_\varepsilon(x_i)$

Pick $x_1 \in K$, $\quad B_\varepsilon(x_1) \not\supset K$ $\quad$ B K

$\Rightarrow \exists x_2 \in K$

but, $x_2 \notin B_\varepsilon(x_1)$

$B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \not\supset K$

$\Rightarrow \exists x_3 \in K, \quad x_3 \notin B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$

$\vdots$

$\exists x_n \in K$, s.t. $d(x_n, x_m) \geq \varepsilon. \quad \forall m \neq n$