Ex: Given $f:[a,b]\to[m,M]$ Riem. integrable

$\varphi:[m,M]\to\mathbb{R}$ continuous $\Longleftrightarrow$ uniformly (cts)

then $\varphi\circ f:[a,b]\to\mathbb{R}$ Riem. integrable.

Proof: $\forall\varepsilon>0,\ \exists\delta>0$ s.t. $y_1,y_2\in[m,M]$ and $|y_1-y_2|<\delta$

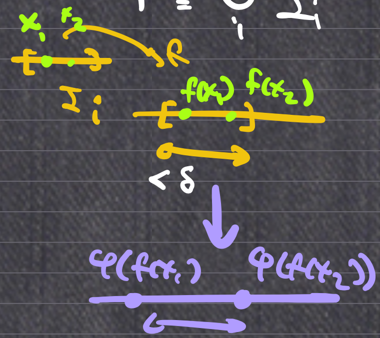
$\Longrightarrow |\varphi(y_1)-\varphi(y_2)|<\boxed{\varepsilon}$

$\exists P$ of $[a,b]$ s.t. $U(f,P)-L(f,P) < \cancel{?}\ \delta^2$

$$\Rightarrow \sum_i \underbrace{\left(\sup_{I_i} f - \inf_{I_i} f\right)}_{} |I_i| < \cancel{?}\ \delta^2$$

← if small

$P=\bigcup_i I_i$



$$U(\varphi\circ f,P)-L(\varphi\circ f,P)$$

$$=\sum_i\left(\sup_{I_i}\varphi\circ f - \inf_{I_i}\varphi\circ f\right)|I_i|$$

$$|\varphi(f(x_1))-\varphi(f(x_2))|$$

$$=\sum_{\underbrace{\sup_{I_i} f-\inf_{I_i} f<\delta}_{}} \underbrace{\left(\sup_{I_i}\varphi\circ f - \inf_{I_i}\varphi\circ f\right)|I_i|}_{\Rightarrow\ <\varepsilon} \quad ✓ \quad — ①$$

$$+\sum_{\boxed{\sup_{I_i} f-\inf_{I_i} f\ge\delta}} \left(\sup_{I_i}\varphi\circ f - \inf_{I_i}\varphi\circ f\right)|I_i| \quad — ②$$

Recall $\sum_j\left(\sup_{I_j} f - \inf_{I_j} f\right)|I_j| < \cancel{?}\ \delta^2$

$$\ge\delta|I_i|\le \sum_{\sup_{I_i} f-\inf_{I_i} f\ge\delta} \underbrace{\left(\sup_{I_i} f - \inf_{I_i} f\right)}_{}|I_i| < \cancel{?}\ \delta^2$$

$\Rightarrow |I_i|\le\delta$

$$② \le \sum_{\sup_{I_i} f - \inf_{I_i} f \ge \delta} \left(\sup_{I_i} \varphi\circ f - \inf_{I_i} \varphi\circ f\right)|I_i|$$

$$\le M \sum_{\cdots \ge \delta} |I_i| < M\delta .$$

$E \subset \mathbb{R}$

Lebesgue outer measure $\qquad \emptyset \subset \emptyset \cup \emptyset \cup \emptyset \cup \cdots$

$$\mathcal{L}^*(E) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \;\middle|\; E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\} \qquad \boxed{a_i \le b_i}$$

- $\boxed{\underline{\mathcal{L}^*([a,b]) = b - a}}$

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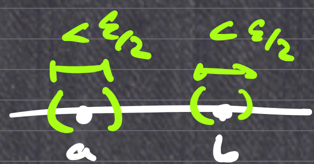
$\inf = 0 \qquad \varepsilon > 0$

$$\mathcal{L}^*(E) = 0$$

$$\Longleftrightarrow \quad \forall \varepsilon > 0, \ \exists \bigcup_{i=1}^{\infty} (a_i, b_i) \supset E$$

$$\text{s.t.} \quad \sum_i (b_i - a_i) < \varepsilon .$$

$\mathcal{L}^*(\{a,b\}) = 0$

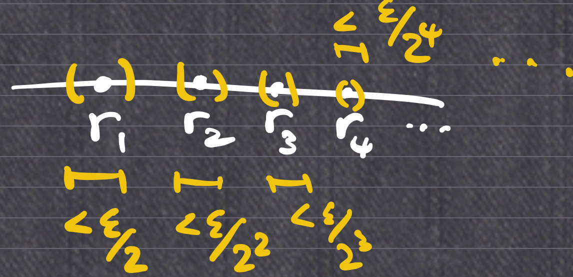


$a \ne b$

proof: $\forall \varepsilon > 0$, pick $\{a,b\} \subset (a - \frac{\varepsilon}{5}, a + \frac{\varepsilon}{5}) \cup (b - \frac{\varepsilon}{5}, b + \frac{\varepsilon}{5})$.

$\underbrace{\qquad\qquad}_{\text{total length } < \varepsilon}$.

$$\mathcal{L}^*\left(\bigcup_{i=1}^{\infty}\{r_i\}\right) = 0$$



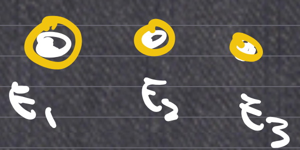
any countable set has zero Lebesgue measure.

- Given $\{E_i\}_{i=1}^{\infty}$ s.t. $\underline{\mathcal{L}^*(E_i) = 0}\ \forall i$

$$\Rightarrow \mathcal{L}^*\left(\bigcup_{i=1}^{\infty} E_i\right) = 0.$$

$\forall \varepsilon > 0, j \in \mathbb{N}, \exists \bigcup_i (a_i^j, b_i^j) \supset E_j$

s.t. $\sum_i (b_i^j - a_i^j) < \varepsilon/2^j$



then $\bigcup_j E_j \subset \bigcup_j \bigcup_i (a_i^j, b_i^j)$

and $\sum_j \sum_i (b_i^j - a_i^j)$

$< \sum_j \varepsilon/2^j = \varepsilon.$

Lebesgue Theorem :

$f : [a,b] \to [m,M]$ is Riemann integrable

$$\Longleftrightarrow \mathcal{L}^*(D_f) = 0.$$

$D_f = \{x_0 \in [a,b] \mid f$ is <u>NOT</u> continuous at $x_0\}$.

- $f : [a,b] \to \mathbb{R}$ continuous

<u>Proof</u>: $D_f = \emptyset \Rightarrow \mathcal{L}^*(D_f) = 0$.

- $f : [a,b] \to \mathbb{R}$ is monotone

<u>Proof</u>: D_f is countable

$\Rightarrow \mathcal{L}^*(D_f) = 0$.

- $f, g : [a,b] \to [m,M]$ Riem. integrable

$\Rightarrow f+g$ is Riem. integrable

<u>Proof</u>: If f, g are cts at x_0

$\Rightarrow f+g$ is cts at x_0

Contrapositive: $f+g$ is not cts at x_0

$\Rightarrow f$ or g is not cts at x_0

$\therefore x_0 \in D_{f+g} \Rightarrow x_0 \in D_f \cup D_g$

$\Rightarrow D_{f+g} \subset D_f \cup D_g$

- $f : [a,b] \to [m,M]$ Riem. integ.

$\varphi : [m,M] \to \mathbb{R}$ continuous

$\Rightarrow \varphi \circ f$ is Riem. integ.

Proof: If f is continuous at x_0

$\Rightarrow$ $\varphi \circ f$ is continuous at x_0

If $\varphi \circ f$ is not cts at x_0

then f is not cts at x_0

$$D_{\varphi \circ f} \subset D_f$$