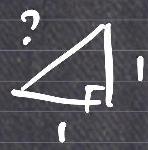
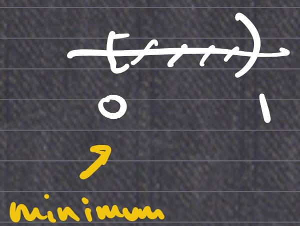
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$



supremum and infimum.

generalization of:

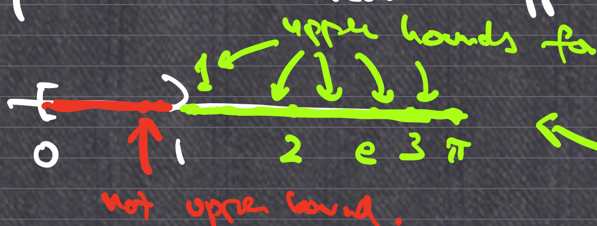
maximum and minimum.



y = maximum of S

$\overset{\text{def}}{\iff} \begin{cases} y \geq x & \forall x \in S \\ y \in S. \end{cases}$

supremum = least upper bound



$\boxed{\sup [0,1) = 1}$

y = "The" supremum of S

$\overset{\text{def}}{\iff} \begin{cases} y \text{ is an upper bound for } S. \\ \underbrace{\text{if } x \text{ is an upper bound for } S\text{, then } y \leq x}_{\text{if } y > x\text{, then } x \text{ is not an upper bound for } S.}. \end{cases}$

$y =: \underline{\sup S}$

infimum := greatest lower bound.

Claim: supremum of a set $S \neq \emptyset$ is unique.

Proof: let L and M be supremums of S.

$L \geq M$ — L is a sup $\Rightarrow$ is an upper bound; M is sup $\Rightarrow$ least upper bound.

$L \leq M$ left as an exercise.

$\therefore \boxed{L = M}$

COMPLETENESS AXIOM:

if $\emptyset \neq S \subset \mathbb{R}$ is $\underbrace{\text{bounded from above}}_{\exists y \in \mathbb{R} \text{ s.t. } y \geq x \ \forall x \in S}$.

then $\sup S$ exists $\in \mathbb{R}$

S $\quad y$

Dedekind cut

Cauchy completion.

$S := \mathbb{Q} \cap (-\sqrt{2}, \sqrt{2})$

$\hat{\mathbb{Q}}$ $\quad \sup S = \sqrt{2} \notin \mathbb{Q}$

$-\sqrt{2}$ $\quad \sqrt{2}$

Prop 1.5 $\emptyset \neq S \subset \mathbb{R}$ bounded above.

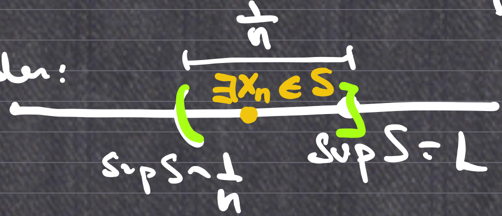
$L = \sup S$

$\iff \begin{cases} ① & L \text{ is an upper bound for } S. \\ ② & \exists \{x_n\} \in \underline{\underline{S}} \text{ s.t. } x_n \to L. \end{cases}$

($\Rightarrow$) Given $L = \sup S$

$\Rightarrow$ ① is true (by definition of $\sup S$)

$\forall n \in \mathbb{N}$, consider:



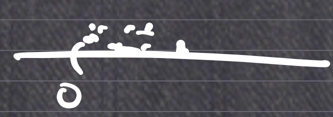
$\Rightarrow \sup S - \frac{1}{n} < x_n \leq \sup S \quad \forall n \in \mathbb{N}$

$$\begin{array}{ccc} \downarrow & \Downarrow & \downarrow \\ \sup S & \underline{\sup S} & \sup S \end{array}$$

($\Leftarrow$) Exercise.

Ex: $S = \left\{ \frac{1}{n} + \frac{1}{2^m} : m, n \in \mathbb{N} \right\}$.

WTS: $\inf S = 0$



① $0 < \frac{1}{n} + \frac{1}{2^m} \quad \forall m, n \in \mathbb{N}$

$\Rightarrow$ 0 is a lower bound for S.

② $\exists \left\{ \frac{1}{n} + \frac{1}{2^n} \right\} \to 0$

$$\underbrace{\qquad}_{\in S}$$

$\therefore\ 0 = \inf S.$

<u>Ex</u>: $\emptyset \neq S \subset (0, \infty)$

$$T := \left\{ \frac{1}{x} : x \in S \right\}.$$

<u>Claim</u>: $\sup S = \dfrac{1}{\inf T}$

<u>Proof</u>: ① $\forall x \in S,$ $\quad \underset{\in S}{x} = \dfrac{1}{\underset{\in T}{\frac{1}{x}}} \leq \underbrace{\dfrac{1}{\inf T}}_{\substack{\Downarrow \\ \text{an upper bound for } S.}}$ $\quad \dfrac{1}{x} \geq \inf T$

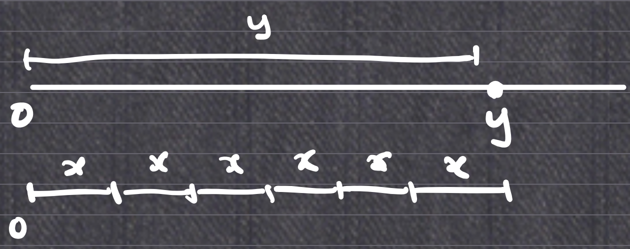
WTS:

② $\exists \underset{\in S}{?} \to \dfrac{1}{\inf T}$

$\exists \underbrace{\{y_n\}}\in T$ s.t. $y_n \to \inf T.$

$\underbrace{\dfrac{1}{y_n} \in S} \longrightarrow \dfrac{1}{\inf T} \longrightarrow \sup S = \dfrac{1}{\inf T}$

<u>Theorem</u>: (Archimede's Principle)



Given any $\underline{x, y > 0}$, ($\exists\, n \in \mathbb{N}$ s.t. $nx > y$.) *

<u>Proof</u>: $S := \{ nx : n \in \mathbb{N} \} \neq \emptyset$

Assume (*) is false:

$\forall n \in \mathbb{N}$ s.t. $nx \leq y$.

y is an upper bound for S.

$\Rightarrow \sup S$ exists in $\mathbb{R}$ (completeness)

$= \underbrace{(m+1)x}_{\in S}$

$\exists\, z \in S$

$z = mx$, $\exists\, m \in \mathbb{N}$.

$\sup - \frac{x}{2} < mx < \sup$

$\sup + \frac{x}{2} < mx + x < \sup + 1$

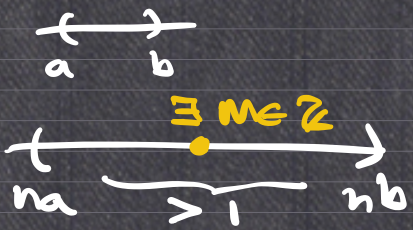
$\boxed{\begin{array}{l} \text{(Exercise)} \\ \text{Prove if } (a,b) \subset \mathbb{R} \text{ with } b-a>1 \\ \text{then } \exists\, n \in \mathbb{Z} \text{ s.t. } a<n<b. \end{array}}$

<u>Prop</u>: (Density of $\mathbb{Q}$)

$\forall \underbrace{a<b}_{\in\mathbb{R}}$, then $\underbrace{(a,b) \cap \mathbb{Q} \neq \emptyset}$.

$\exists\, r \in \mathbb{Q}$ s.t. $a < r < b$.

<u>Proof</u>:



AP $\{ \exists\, n \in \mathbb{N}$

$\{ \exists\, n \in \mathbb{N}$ s.t. $n(b-a) > 1$

$$na < m < nb$$

$$\Rightarrow a < \frac{m}{n} < b$$

$$\in \mathbb{Q}.$$