Given $|f_n(x)| \le M \quad \forall n \in \mathbb{N}, \quad x \in [a,b]$

$\forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon) > 0$ s.t.

$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \quad \forall n \in \mathbb{N}.$

Goal: $f_{n_j} \rightrightarrows f_\infty$ on $[a,b]$.

Proof: Step ① $\exists f_{n_k}$ s.t. $f_{n_k}(r) \longrightarrow L(r) \quad \forall r \in [a,b] \cap \mathbb{Q}$

$\mathbb{Q} \cap [a,b] = \{r_1, r_2, r_3, r_4, \dots\}$

$\overset{BW}{\Rightarrow} \exists f_{1,n}$ s.t. $f_{1,n}(r_1) \longrightarrow L_1 \quad (\because |f_n(r_1)| \le M)$

$|f_{1,n}(r_2)| \le M \overset{BW}{\Rightarrow} f_{2,n}(r_2) \to L_2$

$\vdots$

$\exists \{f_n\} \supset \{f_{1,n}\} \supset \{f_{2,n}\} \supset \{f_{3,n}\} \supset \cdots$

s.t. $f_{j,n}(r_j) \longrightarrow L_j$

$(f_{j,n}(r_\ell) \longrightarrow L_\ell$ if $\ell \le j$
as $n \to \infty$

then

$f_{n,n}(r) \longrightarrow L(r) \quad \forall r \in \mathbb{Q} \cap [a,b]$

by standard diagonalization.

Step ②: $f_{n,n} \rightrightarrows f_\infty$ on $[a,b]$.

want: $\forall \varepsilon > 0, \exists N > 0$ s.t. $m, n \ge N \Rightarrow \|f_{m,m} - f_{n,n}\|$

$< \varepsilon.$

Assume not:

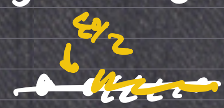
$\exists \varepsilon_0 > 0,\ \forall N > 0$ s.t. $\exists\, m, n \geq N$

$\sup \| f_{m,m} - f_{n,n} \| \geq \varepsilon_0.$

$\vdots$

$\exists\ m_1 < m_2 < m_3 < \cdots$ s.t. $\| f_{m_j, m_j} - f_{n_j, n_j} \| \geq \varepsilon_0$

$n_1 < n_2 < n_3 < \cdots$



$\forall j: \underline{\sup_{x \in [a,b]} | f_{m_j, m_j}(x) - f_{n_j, n_j}(x) |} \geq \varepsilon_0 > \varepsilon_0/2$

$\exists\ x_j \in [a,b]$ s.t.

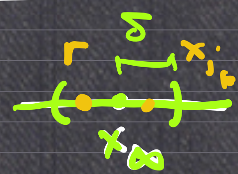
$\underbrace{| f_{m_j, m_j}(x_j) - f_{n_j, n_j}(x_j) | > \varepsilon_0/2.}$

BW

$\exists x_{j_k}$

$\downarrow$

$x_\infty \in [a,b]$



$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0$ s.t.

$\underline{|x - y| < \delta \Rightarrow | f_n(x) - f_n(y) | < \varepsilon \quad \forall n \in \mathbb{N}.}$

$\frac{\varepsilon_0}{2} < | f_{m_{j_k}, m_{j_k}}(x_{j_k}) - f_{n_{j_k}, n_{j_k}}(x_{j_k}) |$

$\leq | f_{m_{j_k}, m_{j_k}}(x_{j_k}) - f_{m_{j_k}, m_{j_k}}(r) |$

$+ | f_{m_{j_k}, m_{j_k}}(r) - f_{n_{j_k}, n_{j_k}}(r) |$

$$+\left| f_{n_{j_k}}(r) - f_{n_{j_k'} n_{j_k}}(x_{j_k}) \right|$$