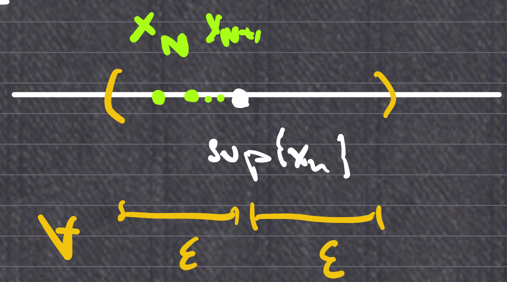
Thm:

$\{x_n\}$ monotone increasing, and bound above,

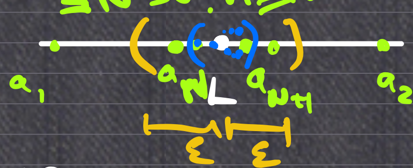
then $\lim\limits_{n\to\infty} x_n = \underbrace{\sup}\{x_n\}$.

Proof:



$\{a_n\} \to L$

$\exists N > 0$. $n \geq N$



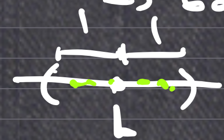
$\forall \varepsilon > 0$, $\exists N > 0$

s.t. $n > N \Rightarrow |a_n - L| < \varepsilon$.

Thm Bolzano - Weierstrass.

Any bounded sequence $\{x_n\}$ must have a converging subsequence.

converges $\Rightarrow$ bd.



x_n

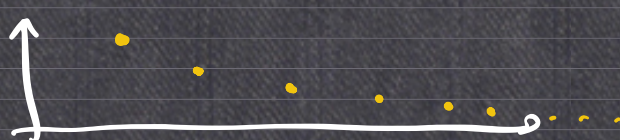
$\{1, -1, 1, -1, \ldots\}$ $n \geq N$

$(-1)^n$

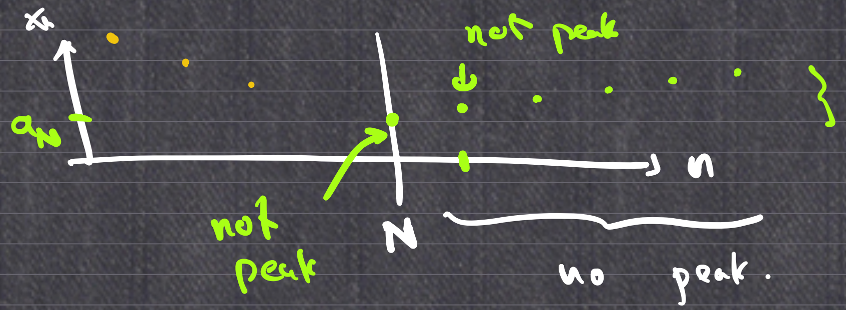
$\boxed{\text{Peak}}$

x_n is called a peak if $x_n \geq x_m$ $\forall m \geq n$.

Proof: Case ① $\exists$ infinitely peaks

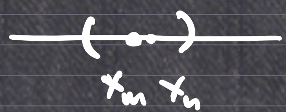


Case ② only finitely many peaks.

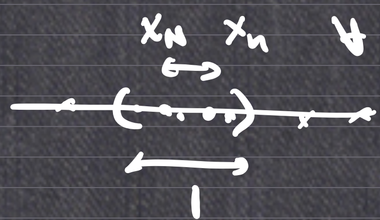


Thm Cauchy seq. must converge.

Cauchy seq. $\left(\forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon \right)$



Outline: ① $\forall n > N$ $\quad |x_n - x_N| < 1$



Cauchy $\Rightarrow$ bounded.

② BW $\Rightarrow$ $\{x_n\}$ has a converging subseq. $\{x_{n_k}\} \to L \in \mathbb{R}$

$$
\begin{aligned}
|x_n - L| &= |(x_n - x_{n_k}) + (x_{n_k} - L)| \\
&\leq |x_n - x_{n_k}| + |x_{n_k} - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}
$$

§1.2 limsup liminf.

$\{x_n\}$ bounded seq.

$\underbrace{\mathrm{LIM}\{x_n\}}_{\text{limit set of } \{x_n\}} := \{$ set of limits of converging subseq. of $\{x_n\}\}$

$\to -1$

$\{(-1)^n\} = (-1, 1, -1, 1, -1, 1, -1, \dots\}$

$\to 1$

$\mathrm{LIM}\{(-1)^n\} = \{-1, 1\}.$

$\limsup\limits_{n\to\infty} x_n = \sup \mathrm{LIM}\{x_n\}$ $\qquad$ $\limsup (-1)^n = 1$

$\liminf\limits_{n\to\infty} x_n = \inf \mathrm{LIM}\{x_n\}.$ $\qquad$ $\liminf (-1)^n = -1$

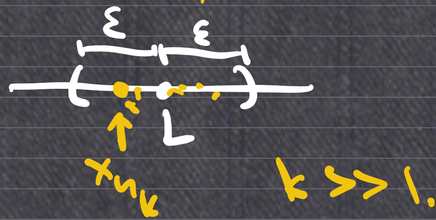
Def: L is a limit point of $\{x_n\}$ $\overset{\text{def}}{\iff}$ $\exists \{x_{n_k}\} \to L$.

Prop 1.24: $\{x_n\} \subset \mathbb{R}$. $L \in \mathbb{R}$

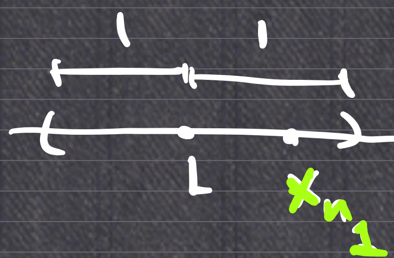
① $L \in \mathrm{LIM}\{x_n\}$

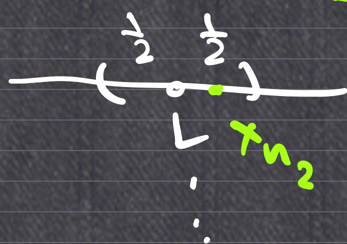
$\iff$ ② $\forall \varepsilon > 0$, $\exists$ infinitely n's s.t. $x_n \in (L-\varepsilon, L+\varepsilon)$

① ⇒ ② : $\exists\, x_{n_k} \to L$



② ⇒ ① :

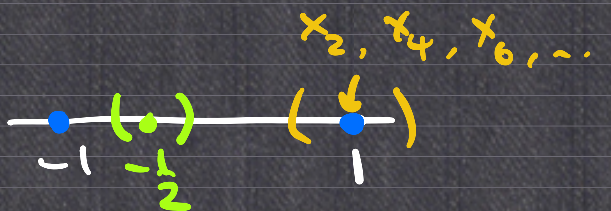




$n_2 > n_1$

$\exists\, x_{n_k}$ s.t. $x_{n_k} \in \left(L - \frac{1}{k}, L + \frac{1}{k}\right)$

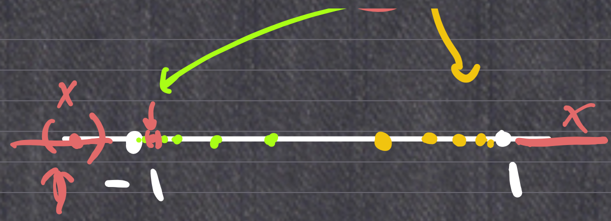
$\mathrm{LIM}\{(-1)^n\} = \{-1, 1\}$



e.g.

$$\{x_n\} = \left\{ \begin{array}{ll} 1-\frac{1}{2}, & -1+\frac{1}{2}, \\ 1-\frac{1}{3}, & -1+\frac{1}{3} \\ 1-\frac{1}{4}, & -1+\frac{1}{4} \\ \vdots & \end{array} \right\} \qquad \to -1$$

$1, -1 \in \mathrm{LIM}.$


x
-1
1
x