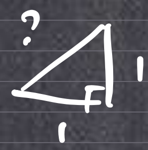
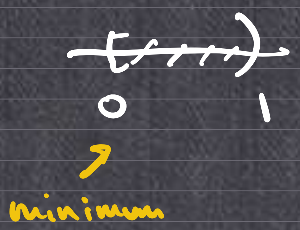
$\mathbb{N} < \mathbb{Z} < \mathbb{Q} < \underline{\mathbb{R}}$



supremum and infimum.

generalization of:

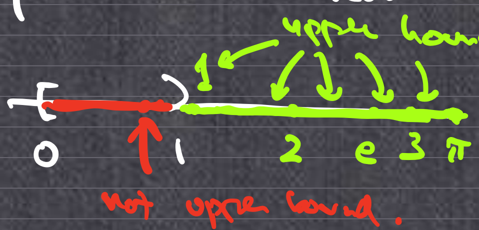
maximum and minimum.



y = maximum of S

$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} y \geq x & \forall x \in S \\ y \in S. \end{cases}$

supremum = least upper bound



$\boxed{\sup\, [0,1) = 1}$

y = "The" supremum of S

$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} y \text{ is an upper bound for } S. \\ \text{if } x \text{ is an upper bound for } S, \text{ then } y \leq x. \end{cases}$

if $y > x$, then x is not an upper bound for S.

$y =: \underline{\sup S}$

infimum := greatest lower bound.

Claim: Supremum of a set $S \neq \emptyset$ is unique.

Proof: Let L and M be supremums of S.

$$L \geq M$$

L is a sup $\Rightarrow$ is an upper bound

M is sup $\Rightarrow$ least upper bound

$$L \leq M$$

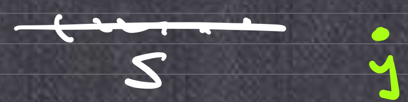
left as an exercise.

$\therefore \boxed{L = M}$

COMPLETENESS AXIOM:

if $\emptyset \neq S \subset \mathbb{R}$ is $\underbrace{\text{bounded from above}}_{\exists y \in \mathbb{R} \text{ s.t. } y \geq x \ \forall x \in S}$.

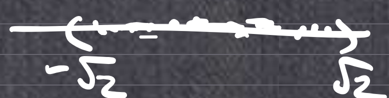
then $\sup S$ exists $\in \mathbb{R}$



Dedekind cut

Cauchy completion.

$$S := \mathbb{Q} \cap (-\sqrt{2}, \sqrt{2})$$



$\hat{\mathbb{Q}}$

$$\sup S = \sqrt{2} \notin \mathbb{Q}$$

Prop 1.5 $\emptyset \neq S \subset \mathbb{R}$ bounded above.

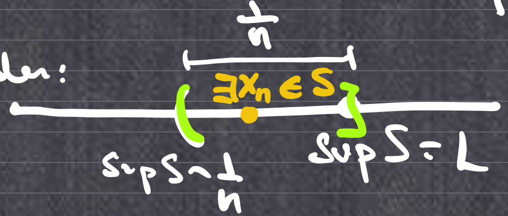
$L = \sup S$

$\Leftrightarrow \begin{cases} ① \ L \text{ is an upper bound for } S. \\ ② \ \exists \{x_n\} \in \underline{\underline{S}} \text{ s.t. } x_n \to L. \end{cases}$

$(\Rightarrow)$ Given $L = \sup S$

$\Rightarrow$ ① is true (by definition of $\sup S$)

$\forall n \in \mathbb{N}$, consider:



$$\Rightarrow \sup S - \frac{1}{n} < x_n \leq \sup S \quad \forall n \in \mathbb{N}$$

$$\begin{array}{ccccc} \downarrow & & \Downarrow & & \downarrow \\ \sup S & & \underline{\sup S} & & \sup S \end{array}$$

$(\Leftarrow)$ Exercise.

Ex: $S = \left\{ \frac{1}{n} + \frac{1}{2^m} : m, n \in \mathbb{N} \right\}$.

WTS: $\inf S = 0$

① $0 < \frac{1}{n} + \frac{1}{2^m} \quad \forall m, n \in \mathbb{N}$

$\Rightarrow 0$ is a lower bound for S.

② $\exists \left\{ \frac{1}{n} + \frac{1}{2^n} \right\} \to 0$

$\underbrace{}_{\in S}$

$\therefore\ 0 = \inf S.$

<u>Ex</u>: $\emptyset \neq S \subset (0, \infty)$

$$T := \left\{ \frac{1}{x} : x \in S \right\}.$$

<u>Claim</u>: $\sup S = \dfrac{1}{\inf T}$

<u>Proof</u>: ① $\forall x \in S$, $\quad \underset{\in S}{x} = \dfrac{1}{\underbrace{\frac{1}{x}}_{\in T}} \le \underbrace{\dfrac{1}{\inf T}}_{\substack{\Downarrow \\ \text{an upper bound for } S.}}$, $\quad \dfrac{1}{x} \geq \inf T$

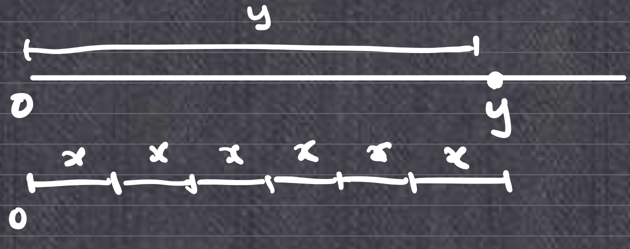
WTS:

② $\exists \underset{\in S}{?} \to \dfrac{1}{\inf T}$

$\exists \underline{\{y_n\} \in T}$ s.t. $y_n \to \inf T$.

$\underline{\dfrac{1}{y_n} \in S} \longrightarrow \dfrac{1}{\inf T} \longrightarrow \sup S = \dfrac{1}{\inf T}$

<u>Theorem</u>: (Archimede's Principle)



Given any $x, y > 0$, ($\exists n \in \mathbb{N}$ s.t. $nx > y$.)*

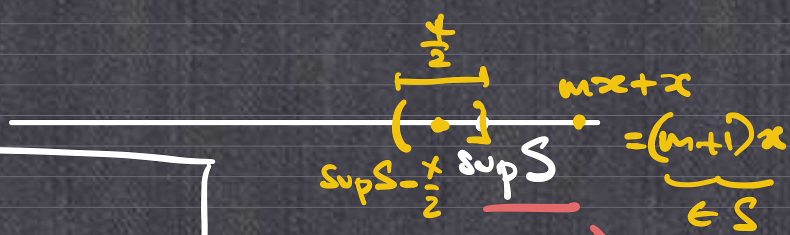
Proof: $S := \{nx : n \in \mathbb{N}\} \neq \emptyset$

Assume (*) is false:

$\forall n \in \mathbb{N}$ s.t. $nx \le y$.

y is an upper bound for S.

$\Rightarrow \sup S$ exists in $\mathbb{R}$ — completeness



$\exists z \in S$, $z = mx$, $\exists m \in \mathbb{N}$.

$\sup - \frac{x}{2} < mx < \sup$

$\sup + \frac{x}{2} < \underbrace{mx + x} < \sup + 1$

$\boxed{\begin{array}{l}\text{(Exercise)} \\ \text{Prove if } (a,b) \subset \mathbb{R} \text{ with } b - a > 1 \\ \text{then } \exists n \in \mathbb{Z} \text{ s.t. } a < n < b.\end{array}}$

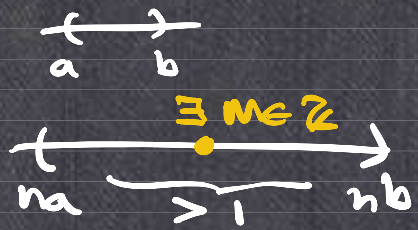
Prop: (Density of $\mathbb{Q}$)

$\forall \underbrace{a < b}_{\in \mathbb{R}}$, then $\underbrace{(a,b) \cap \mathbb{Q} \neq \emptyset}$.

$\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

Proof:

AP $\{\exists n \in \mathbb{N}$



$\exists n \in \mathbb{N}$ s.t. $n(b-a) > 1$

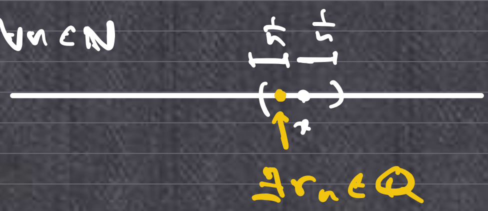
$$na < m < nb$$

$$\Rightarrow a < \frac{m}{n} < b$$

$$\frac{m}{n} \in \mathbb{Q}.$$

Claim: $\forall x \in \mathbb{R}$, $\exists \{r_n\} \in \mathbb{Q}$ s.t. $r_n \to x$.

$\forall n \in \mathbb{N}$

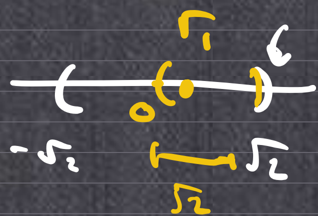


$$\begin{array}{ccccc} x - \frac{1}{n} & < & r_n & < & x + \frac{1}{n} \\ \downarrow & & \Downarrow & & \downarrow \\ x & & x & & x \end{array}$$

$S = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

① $\sqrt{2}$ is an upper bound for S

② $\exists \{r_n\} \in \mathbb{Q} \cap (-\sqrt{2}, \sqrt{2}) \to \sqrt{2}$



$\Rightarrow \sqrt{2} = \sup S.$

- $\emptyset \neq S \subset T \subset \mathbb{R}$

$$\underbrace{\sup S}_{\text{least ub for } S} \leq \underbrace{\sup T}_{\text{upper bd for } S}$$

$$\underbrace{\inf S}_{\text{greatest lower b.}} \geq \underbrace{\inf T}_{\text{a lower bound for } S}$$

of S

- $\sup \emptyset = -\infty$

  $\inf \emptyset = +\infty$

Any real $x \in \mathbb{R}$ is an upper b. for $\emptyset$.

Proof by contradiction :

Assume $\exists x \in \mathbb{R}$ which is not an upper bound for $\emptyset$.

$\Rightarrow$ not (x is an upper bound for $\emptyset$)

$\Rightarrow$ not ($\forall y \in \emptyset, \quad x \geq y$)

$\Rightarrow$ $\exists y \in \emptyset$, s.t. $x < y$.

---

x^y

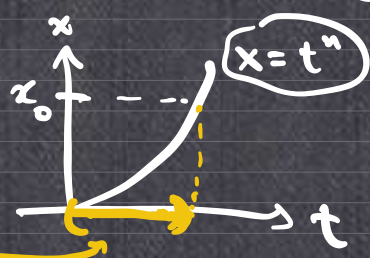
$x^3 := x \cdot x \cdot x$ $\qquad\qquad x^m \cdot x^n = x^{m+n}$.

$x^{\frac{1}{n}} = \sqrt[n]{x}$

Prop: $\forall x_0 > 1$, $n \in \mathbb{N}$, $\exists !\, t > 0$ s.t. $t^n = x_0$

Proof: $S = \{t > 0 : t^n < x_0\}$

Claim: $(\sup S)^n = x$.



Case ①: $\underline{x_0 > 1}$

$\phi \neq S \ni \{1\}$.

$\underline{\text{Claim}}$: $1+x_0$ is an upper bound for S.

$\underline{\text{Pf}}$: By contradiction. $\exists t \in S$ s.t.

$t \geq 1+x_0$

$$\Rightarrow x_0 > \underbrace{t^n}_{t \text{ in } S} \geq (1+x_0)^n > 1+x_0$$

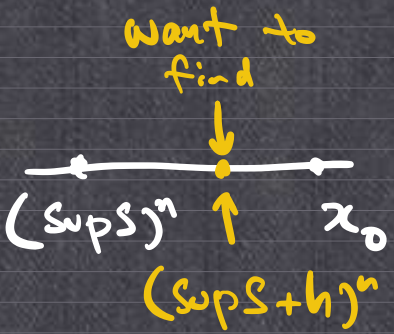
$\phi \neq S$, bound above $\Rightarrow \sup S$

$\underline{\text{Claim}}$. $(\sup S)^n = x_0$

$\underline{\text{Proof}}$: $\exists \{t_k\} \in S \longrightarrow \sup S$.

$$t_k^n < x_0$$
$$\downarrow k \to \infty$$
$$(\sup S)^n \leq x_0$$

$\underline{\text{Next}}$: Assume $(\sup S)^n < x_0$. Try to derive contradict^n



$\exists h > 0$ small

Need. $(\sup S+h)^n - (\sup S)^n$

$\forall h \in (0,1)$ $\qquad < x_0 - (\sup S)^n$

$$(\sup S + h)^n - (\sup S)^n$$

$$= \underbrace{(\sup S + h - \sup S)}_{h}\left(\sum_{j=0}^{n-1} \underbrace{(\sup S+h)^j}_{\le \sup S + 1} (\sup S)^{n-j-1}\right)$$

$$\le h \underbrace{\sum_{j=0}^{n-1} (\sup S + 1)^j (\sup S)^{n-j-1}}_{\overset{\shortparallel}{C}} = Ch$$

Choose $h = \min\left\{\frac{1}{2}, \frac{x_0 - (\sup S)^n}{2C}\right\}$.

$$\Rightarrow (\sup S + h)^n - (\sup S)^n$$

$$\overset{h<1}{\Rightarrow} \le Ch \le C \cdot \frac{x_0 - (\sup S)^n}{2C} < x_0 - (\sup S)^n$$

$$\Rightarrow \sup S + h \in S$$

uniqueness:

If $y_1^n = y_2^n = x_0$

$$\Rightarrow 0 = y_1^n - y_2^n = (y_1 - y_2)(\underbrace{+ + + \cdots}_{+})$$

Case ②: $x_0 = 1$ trivial.

Case ③: $x_0 \in (0,1)$ $\rightsquigarrow$ $\frac{1}{x_0} > 1 \Rightarrow \exists t > 0$ s.t. $t^n = \frac{1}{x_0} \Rightarrow (\frac{1}{t})^n = x_0$

$\underline{x>0}$: $x^{\frac{1}{n}}$ ✓ $\longrightarrow$ $\underbrace{x^{\frac{m}{n}} := \left(x^{\frac{1}{n}}\right)^m}$

check: $\frac{m}{n} = \frac{p}{q} \Rightarrow \left(x^{\frac{1}{n}}\right)^m = \left(x^{\frac{1}{q}}\right)^p$

$p, q, m, n \in \mathbb{N}$

$\boxed{x>1}$, $\boxed{y>0}$

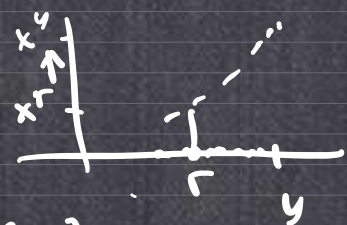
$x^y = ?$

$x^{\sqrt{2}}$

$S(x,\sqrt{2}) = \{x^1, x^{1.4}, x^{1.41}, \ldots\}$

$S(x,y) := \{x^r \mid 0<r<y,\ r\in\mathbb{Q}\}$.

$x^y := \sup S(x,y)$



$\underline{\text{Check}}$ $r \in \mathbb{Q}_+$, $x^r = \sup S(x,r)$

$= \sup\{x^s \mid 0<s<r,\ s\in\mathbb{Q}\}$

$\forall s \in (0,r)\cap\mathbb{Q}$, $x^s < \underbrace{x^r}_{\substack{\text{upper} \\ \text{bd for } S(x,r)}}$ $\Rightarrow$ $\sup(x,r) \leq x^r$.

$s_j \in \mathbb{Q} \xrightarrow{\nearrow} r$ $\qquad s_j - r \to 0$

$\underset{\substack{\uparrow \\ S(x,r)}}{x^{s_j}} \xrightarrow{?} x^r$

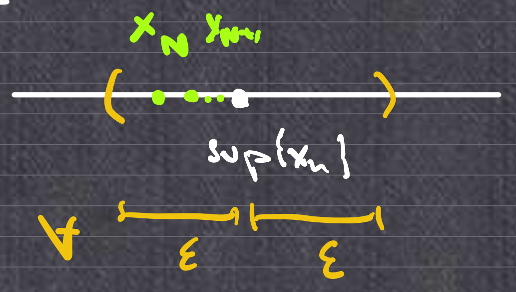
$|x^{s_j} - x^r| = |x^r(x^{s_j - r} - 1)| = |x^r|\,\underbrace{|x^{s_j-r} - 1|}_{\downarrow \atop 0}$

Thm:

$\{x_n\}$ monotone increasing, and bound above,

then $\lim\limits_{n\to\infty} x_n = \sup\{x_n\}$.

Proof:



$\{a_n\} \to L$

$\exists N > 0,\ n \geq N$

$\forall \varepsilon > 0,\ \exists N > 0$ s.t. $n > N \Rightarrow |a_n - L| < \varepsilon$.

Thm Bolzano-Weierstrass.

Any bounded sequence $\{x_n\}$ must have a converging subsequence.

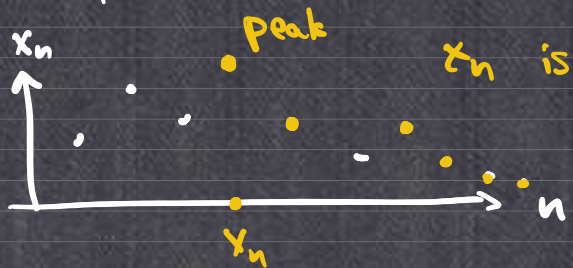
converges $\Rightarrow$ bd.

x_n

$\{1, -1, 1, -1, \ldots\}$ $n \geq N$

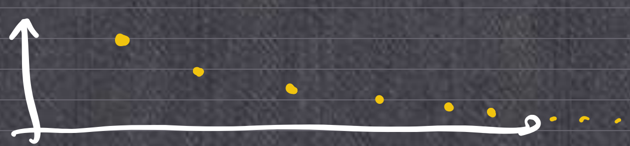
$(-1)^n$

$\boxed{\text{Peak}}$

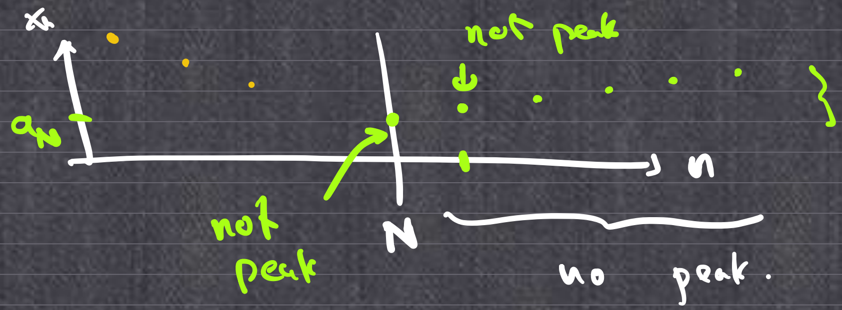


x_n is called a peak if $x_n \geq x_m$ $\forall m \geq n$.

Proof: Case ① $\exists$ infinitely peaks

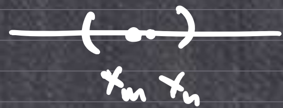


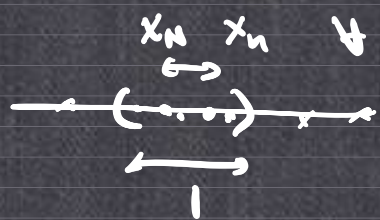
Case ② only finitely many peaks.



Thm Cauchy seq. must converge.

Cauchy seq. $\left(\forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon \right)$



Outline: ① 

$\forall n > N$ $|x_n - x_N| < 1$

Cauchy $\Rightarrow$ bounded.

② BW $\Rightarrow$ $\{x_n\}$ has a converging subseq. $\{x_{n_k}\} \to L \in \mathbb{R}$

$$|x_n - L| = |(x_n - x_{n_k}) + (x_{n_k} - L)|$$
$$\leq |x_n - x_{n_k}| + |x_{n_k} - L|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

§1.2 limsup liminf.

$\{x_n\}$ bounded seq.

$\underline{LIM\{x_n\}}$:= { set of limits of converging subseq. of $\{x_n\}$ }

limit set of $\{x_n\}$

$\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, -1, \dots\}$ → -1 ; → 1

$LIM\{(-1)^n\} = \{-1, 1\}$.

$\limsup_{n\to\infty} x_n = \sup LIM\{x_n\}$

$\liminf_{n\to\infty} x_n = \inf LIM\{x_n\}$.

$\limsup (-1)^n = 1$

$\liminf (-1)^n = -1$

Def: L is a limit point of $\{x_n\}$ $\overset{def}{\iff}$ $\exists \{x_{n_k}\} \to L$.

Prop 1.24: $\{x_n\} \subset \mathbb{R}$. $L \in \mathbb{R}$

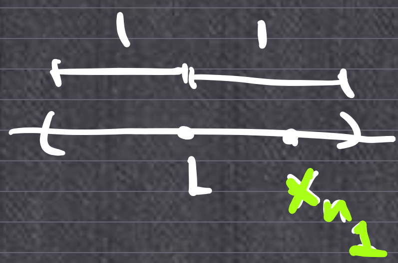
① $L \in LIM\{x_n\}$

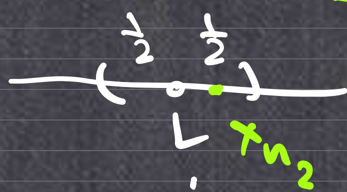
$\iff$ ② $\forall \varepsilon > 0$, $\exists$ infinitely n's s.t. $x_n \in (L-\varepsilon, L+\varepsilon)$

① $\Rightarrow$ ② : $\exists\, x_{n_k} \to L$

$k \gg 1.$

② $\Rightarrow$ ① :





$n_2 > n_1$

$\vdots$

$\exists\, x_{n_k}$ s.t. $x_{n_k} \in \left(L - \frac{1}{k}, L + \frac{1}{k}\right)$

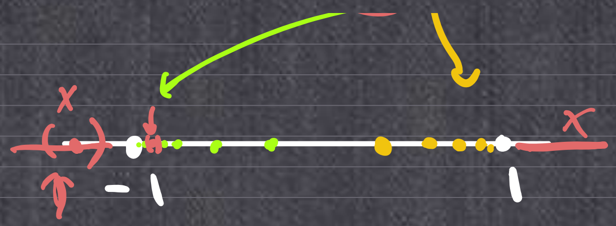
$\text{LIM}\{(-1)^n\} = \{-1, 1\}$

e.g.

$$\{x_n\} = \left\{ 1-\tfrac{1}{2},\ -1+\tfrac{1}{2},\ 1-\tfrac{1}{3},\ -1+\tfrac{1}{3},\ 1-\tfrac{1}{4},\ -1+\tfrac{1}{4},\ \ldots \right\}$$

$\to -1$

$1, -1 \in \text{LIM}.$


x
x
-1
1

$$\limsup_{n\to\infty} x_n = \sup \mathrm{LIM}\{x_n\}$$

$$\liminf_{n\to\infty} x_n = \inf \mathrm{LIM}\{x_n\}.$$

Ex. 1.27

$$x_n = \begin{cases} 1/p_1 & \text{if } n = p_1^{\alpha_1} \\ \dfrac{p_{m_1}}{p_{m_k}} & \text{if otherwise} \end{cases}$$

$p_{m_1}^{\alpha_1} \cdots p_{m_k}^{\alpha_k}$

$p_{m_1} < p_{m_2} < \cdots < p_{m_k}$

$k \geq 2.$

$0 < x_n < 1$

$\mathrm{LIM}\{x_n\} \subset [0,1]$

$\limsup \leq 1$

$\liminf \geq 0.$

list of primes:

$\{p_1, p_2, p_3, p_4, \ldots\}$

$\{x_{p_1}, x_{p_2}, x_{p_3}, \ldots\}$

$= \{\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \ldots\}$

$\liminf = 0. \leftarrow \to 0$

$x_6 = x_{2\cdot 3} = \frac{2}{3}, \quad x_{12} = x_{2^2\cdot 3} = \frac{2}{3}$

$x_{p_n p_{n+1}} = \frac{p_n}{p_{n+1}} \to ?$

$$\boxed{\lim_{n\to\infty} \frac{p_n}{n \log n} = 1}$$

Prime Number Theorem.

$$x_{p_n p_{n+1}} = \frac{p_n}{p_{n+1}} = \underbrace{\frac{p_n}{n\log n}}_{\downarrow\, 1} \cdot \underbrace{\frac{(n+1)\log(n+1)}{p_{n+1}}}_{\downarrow\, 1} \cdot \underbrace{\frac{n\log n}{(n+1)\log(n+1)}}_{\downarrow\, 1}$$

$$\downarrow$$

$$\boxed{\limsup x_n = 1}$$

$$x_{p_{2n} p_{3n}} = \frac{p_{2n}}{p_{3n}} \sim \frac{2n\log(2n)}{3n\log(3n)} \to \frac{2}{3}$$

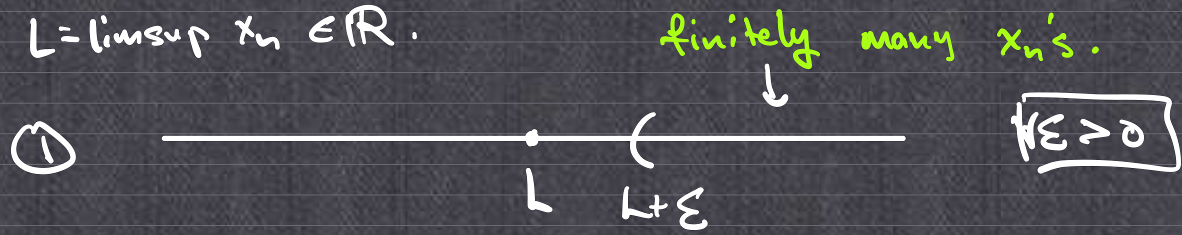
$$\left|\frac{p}{n\log n} - 1\right| < ?$$

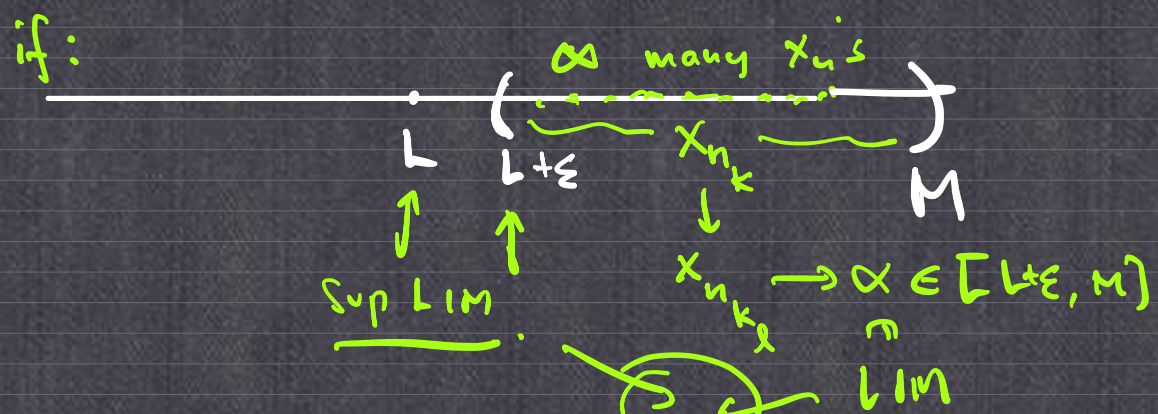
$x_n = n \to +\infty$

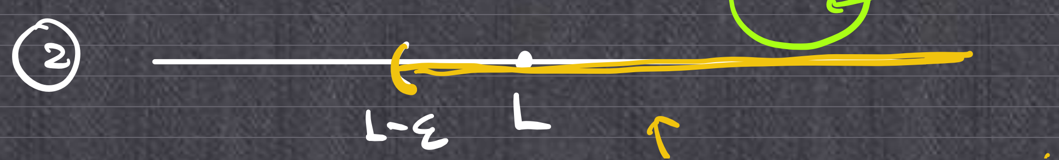
$LM\{x_n\} = \{+\infty\}$

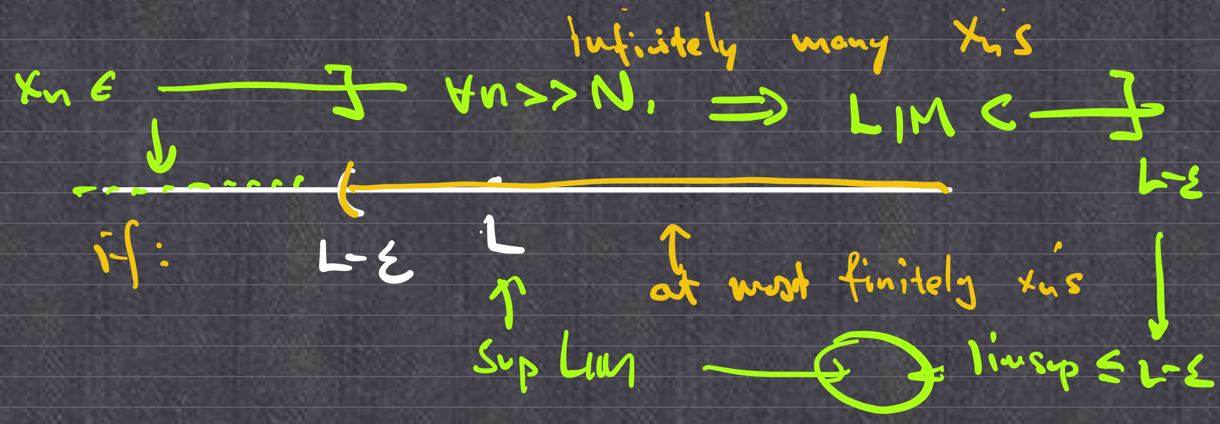
$\limsup x_n = +\infty$.

$L = \limsup x_n \in \mathbb{R}$.

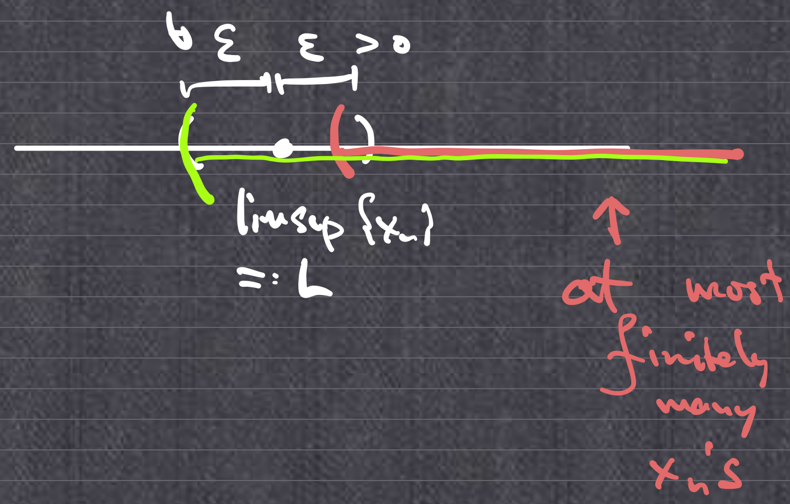








$$\boxed{\limsup x_n \in \text{LIM}}$$



Example 1.30

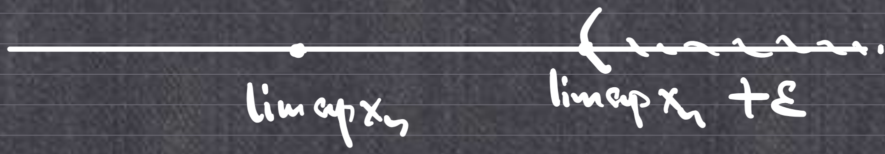
$\{x_n\}$

$$\liminf x_n \leq \liminf \frac{x_1+\dots+x_n}{n} \leq \underbrace{\limsup \frac{x_1+\dots+x_n}{n} \leq \limsup x_n}$$

WTP:

$\forall \varepsilon > 0$

$$\limsup \frac{x_1+\dots+x_n}{n} \leq \limsup x_n + \varepsilon \quad \xRightarrow{\text{let } \varepsilon \to 0^+} \quad \dots \leq \dots$$



at most finitely many x_n.

$\exists N > 0$ s.t. $\Leftarrow$

$n > N \Rightarrow \quad x_n \le \limsup x_k + \varepsilon$

$$\frac{x_1 + \cdots + x_n}{n} = \frac{x_1 + \cdots + x_N}{n} + \frac{x_{N+1} + \cdots + x_n}{n}$$

$$\frac{x_1 + \cdots + x_n}{n} \le \frac{x_1 + \cdots + x_N}{n} + \frac{(n-N)(\limsup + \varepsilon)}{n}$$

$$\limsup \frac{x_1 + \cdots + x_n}{n} \le \limsup \left(\underbrace{\frac{x_1 + \cdots + x_N}{n} + \frac{(n-N)}{n}(\limsup x_k + \varepsilon)}_{\text{conv.}} \right)$$

$\overset{\text{lim.}}{=}$

$$= 0 + 1 \cdot (\limsup x_k + \varepsilon)$$

$$= \limsup x_k + \varepsilon$$

$$\liminf_{n\to\infty} x_n = \inf \mathrm{LIM}\{x_n\}$$

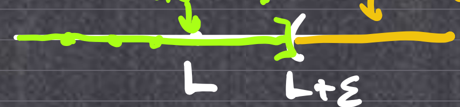
$$\limsup_{n\to} x_n = \sup \mathrm{LIM}\{x_n\}.$$

$$M_n := \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$$ → $\{x_n\}$ bounded

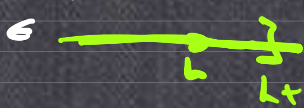
$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, \dots\}.$$

Claim: $\lim\limits_{n\to\infty} M_n = \limsup\limits_{n\to\infty} x_n =: L$

Proof: $\forall \varepsilon > 0$,



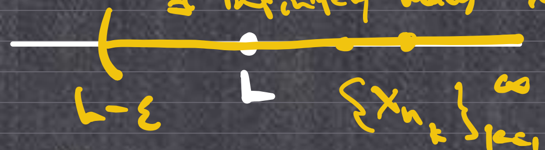
$\Rightarrow n \geq N \Rightarrow M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\} \leq L+\varepsilon$



$\Downarrow$

$\lim\limits_{n\to\infty} M_n \leq L+\varepsilon.$ $\xRightarrow{\text{let } \varepsilon\to 0^+}$ $\boxed{\lim\limits_{n\to\infty} M_n \leq L}$ ✓

$\geq$? $\forall \varepsilon > 0$,



$$M_n = \sup\underbrace{\{x_n, x_{n+1}, x_{n+2}, \dots\}}_{\ni\, x_{n_k} \ \exists k.} \geq x_{n_k} > L-\varepsilon$$

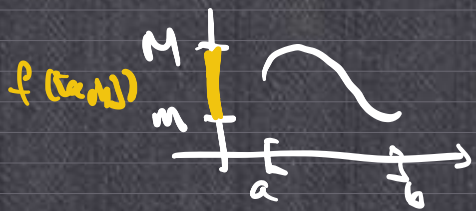
$\Downarrow$

$\lim\limits_{n\to\infty} M_n \geq L-\varepsilon$ $\xRightarrow{\text{let } \varepsilon\to 0^+}$ $\boxed{\lim\limits_{n\to\infty} M_n \geq L}$ ✓

EVT . IVT

EVT: $f: [a,b] \to \mathbb{R}$ cts   max . min must exists.

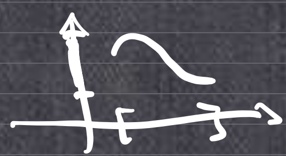
$f([a,b])$ is closed and bounded



IVT: $f: [a,b] \to \mathbb{R}$ cts

$$\boxed{f(a) < 0 < f(b) \Rightarrow \exists c \in (a,b) \text{ s.t. } f(c) = 0.}$$

f(interval) = interval.



f(connected set) = connected set.

§1.3

$\emptyset \neq S$ is countable

$\overset{\text{def}}{\iff}$ $\exists f: S \to \mathbb{N}$ injective.

- $S = \{$ FF, Ivan, Yan Min, Henry Cheng $\}$

$$\begin{array}{ccccccccc} S = \{ & \text{FF}, & \text{Ivan}, & \text{Yan Min}, & \text{Henry Cheng} & \} & & \\ f \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\ \{ & 380, & 1, & 2, & 3 & \} & \subset & \mathbb{N}. \end{array}$$

- $S = \{2, 4, 6, 8, 10, \dots\}$

$$\begin{array}{cccccc} 2 & 4 & 6 & 8 & 10 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 2 & 3 & 4 & 5 & \dots \end{array}$$

$f \downarrow$ $\mathbb{N}$

$f(2) = 1$
$f(4) = 2$
$f(6) = 3$

$$f(2k) = k \quad \forall k \in \mathbb{N}.$$

- $\mathbb{N} \times \mathbb{N} = \{(m,n) : \ m, n \in \mathbb{N}\}.$

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \dots \\ (2,1) & (2,2) & (2,3) & \dots \\ (3,1) & (3,2) & (3,3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

- $S \subset T \implies S$ countable.
  T countable

$\exists f : T \to \mathbb{N}$ injective.

$f|_S : S \to \mathbb{N}$, $\quad f|_S(x) = f(x)$
$x \in S$

$f|_S(x) = f|_S(y) \qquad \exists x, y \in S$

$\implies f(x) = f(y)$

$\implies x = y. \qquad$ (f is injective)

- $S \xrightarrow[\text{injective}]{\exists g} T \implies S$ countable.
  T countable

$$\begin{tikzcd} S \arrow[r, "g", "\text{inj.}"'] \arrow[rr, bend right, "f \circ g"'] & T \arrow[r, "f", "\text{inj.}"'] & \mathbb{N} \end{tikzcd}$$

$f(g(x))$

\circ

- $\mathbb{Q}$ is countable $\{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$

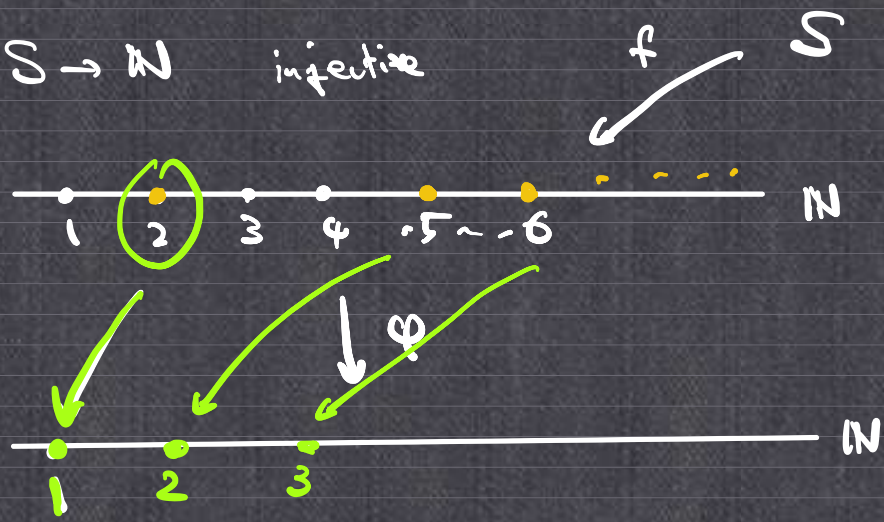
$f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$

$\frac{m}{n}$ in simplest form, $\boxed{n > 0}$

$$f\left(\frac{m}{n}\right) = \begin{cases} (m, n) & \text{if } m \neq 0 \quad \gcd(m,n) = 1 \\ (0, 0) & \text{if } m = 0. \end{cases}$$

S infinite set, countable.

$f : S \to \mathbb{N}$ injective



$g := \varphi \circ f : S \to \mathbb{N}$ bijection.

$g^{-1} : \mathbb{N} \to S$

e.g. $(0,1)$ is uncountable.

Proof: Assume $\underbrace{(0,1)}_{\text{infinite set}}$ is countable

$f: \mathbb{N} \to (0,1)$ bijection

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ f(2) &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ f(3) &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ &\vdots \end{aligned}$$

$$x := 0.\underset{\neq a_{11}}{b_1}\,\underset{\neq a_{22}}{b_2}\,\underset{\neq a_{33}}{b_3}\,\underset{\neq a_{44}}{b_4}\cdots \neq f(n) \quad \forall n.$$

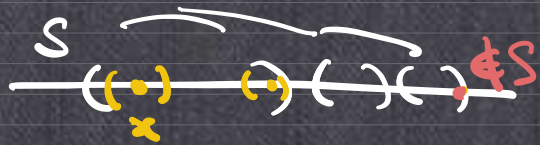
- S, T countable $\Rightarrow S \cup T$ is countable.

$$\mathbb{Q} \cup \underbrace{(\mathbb{R}\setminus\mathbb{Q})}_{\Downarrow \text{ uncountable}} = \mathbb{R} \supset (0,1)$$

open, closed.

↑

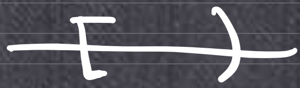
- $S \subset \mathbb{R}$ is open if $\forall x \in S, \exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset S$



- $S \subset \mathbb{R}$ is <u>closed</u> $\overset{def}{\iff}$ <u>$\mathbb{R} \setminus S$ is open</u>.

$S =$ 

$\mathbb{R} \setminus S =$ 

$T =$  not closed.

$\mathbb{R} \setminus T =$  not open

- $\emptyset$, $\mathbb{R}$ both open and closed

<u>open</u> $\Downarrow$ $\mathbb{R} \setminus \emptyset$ is <u>closed</u>   <u>open</u> $\Downarrow$ $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is <u>closed</u>

$\mathbb{R}$ ✓ open: $\forall x \in \emptyset$, $\exists \varepsilon > 0$, s.t. $(x-\varepsilon, x+\varepsilon) \subset \emptyset$

(~~not open~~: $\exists x \in \emptyset$, $\forall \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \not\subset \emptyset$).

Midterm date:

22 March 2pm - 5pm
(Saturday)

$\mathcal{U}$

Open :

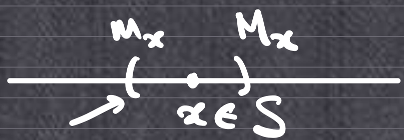
$\forall x \in \mathcal{U}, \ \exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset \mathcal{U}$

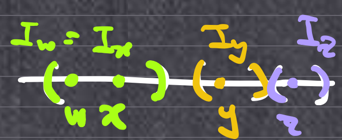
closed : E is closed $\overset{\text{def}}{\Longleftrightarrow}$ $\mathbb{R} \setminus E$ is open.

Prop 1.42 : Any non-empty open set $S \subset \mathbb{R}$ is a disjoint union of countably many open intervals

(i.e. $S = \coprod_{j=1}^{\infty} (a_j, b_j)$)

Proof :





I_x = largest open interval $\ni x$ $\subset S$

$$S = \bigcup_{x \in S} \underbrace{\{x\}}_{\subset I_x} \subset \boxed{\bigcup_{x \in S} \underbrace{I_x}_{\subset S} \subset S}$$

$\Downarrow$

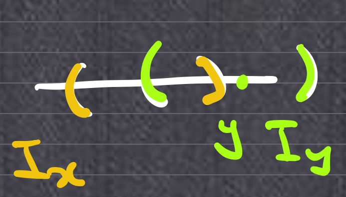
$=$ $=$

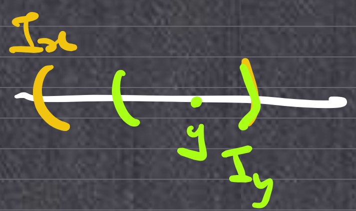
$$S = \bigcup_{x\in S} I_x$$

Given $x, y \in S$, $x \neq y$

$I_x = I_y$ or $I_x \cap I_y = \emptyset$

Otherwise



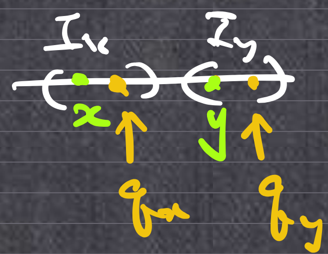


$$S = \bigsqcup_{x\in T} I_x^{\neq\emptyset}$$ ← open interval

$T \subset S$

$T \ni x$

$x \in I_x \ni q_x \in \mathbb{Q}$



$\varphi : T \longrightarrow \mathbb{Q}$ ← countable

$x \longmapsto q_x$ injective

- any union of open sets is open.

$$\boxed{\bigcup_{\alpha\in A} U_\alpha}$$

$$\forall x \in \underbrace{\bigcup_{\alpha\in A} U_\alpha}_{T} \Rightarrow x \in \underbrace{U_\alpha}_{\text{open}} \quad \exists \alpha\in A$$

$$\Rightarrow \exists \varepsilon>0 \text{ s.t. } (x-\varepsilon, x+\varepsilon) \subset U_\alpha \subset T$$

- <u>finite</u> intersection of open sets is open.

$$\underbrace{U_1 , U_2}_{\text{open}}$$

$$\forall x\in U_1\cap U_2 , \Rightarrow x\in U_1, \ x\in U_2$$

$\exists\, \varepsilon_1>0$ s.t. $(x-\varepsilon_1, x+\varepsilon_1)\subset U_1$

$\exists\, \varepsilon_2>0$ s.t. $(x-\varepsilon_2, x+\varepsilon_2)\subset U_2$

$\varepsilon := \frac{1}{2}\min(\varepsilon_1,\varepsilon_2)$

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right) = \{0\}$$

Prop 1.48:

$S \subset \mathbb{R}$

① S is closed ($\mathbb{R}\setminus S$ is open)

$\Longleftrightarrow$ ② any convergent seq. $\{x_n\}$ in S has limit in S.

$\boxed{\text{Given } x_n \to L}$

$a \le x_n \le b \quad \forall n$	$x_n \in [a,b] \quad \forall n$
$\Downarrow$	$\Downarrow$
$a \le L \le b$	$L \in [a,b]$

---

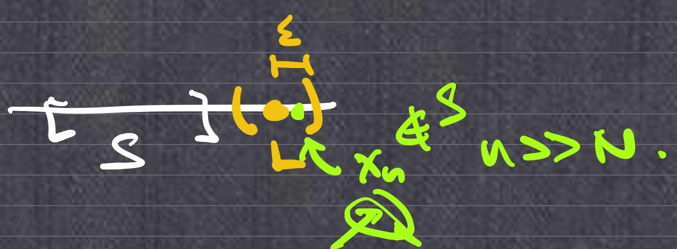
① $\Rightarrow$ ② Given S is closed.

$\Longleftrightarrow$ $\mathbb{R}\setminus S$ is open.

Assume ② is not true.

$\exists$ conv. seq $\{x_n\}$ in S $\to$ $L \notin S$. $\Rightarrow$ $L \in \mathbb{R}\setminus S$ (open)

$\exists\, \varepsilon > 0$ s.t. $(L-\varepsilon, L+\varepsilon) \subset \mathbb{R}\setminus S$

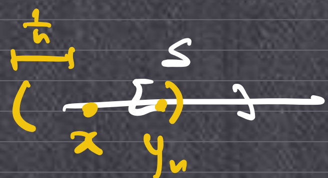


② $\Rightarrow$ ① Consider $x \in \mathbb{R} \setminus S$

WANT: $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset \mathbb{R}\setminus S$.

If not: $\forall \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \not\subset \mathbb{R}\setminus S$

$\Rightarrow \exists y \in (x-\varepsilon, x+\varepsilon)$ but $y \notin \mathbb{R}\setminus S$ $\iff y \in S$.



$\forall n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$, $\exists y_n \in (x-\frac{1}{n}, x+\frac{1}{n})$

$y_n \to x$, $y_n \in S$

$x \in S$

e.g. $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

$x_n = \frac{1}{n} \in S \;\; \forall n$

$\downarrow$

$0 \notin S$

$\Big\}$ S is not closed.

e.g. $\{x_n\} \subset S$ (closed) bounded sequence.

BW $\Rightarrow$ $\exists \{x_{n_k}\} \to \underbrace{L \in S}_{S \text{ closed.}}$

Definition:

$S \subset \mathbb{R}$ is sequentially compact

$\overset{\text{def}}{\iff}$ any seq. $\{x_n\} \in S$ has a convergent subseq $\{x_{n_k}\} \to L \in S$.

$\boxed{\text{On } \mathbb{R}}$

S closed and bounded $\Longleftrightarrow$ S is sequentially compact

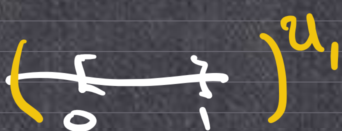
$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

S is compact

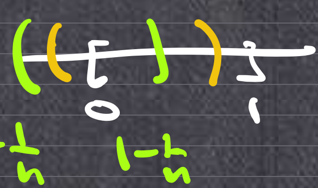
<u>Definition</u> $\quad S \subset \mathbb{R}$ is <u>compact</u>

$\overset{\text{def}}{\Longleftrightarrow}$ any open cover of S has a finite subcover.

$S \subset \bigcup_{\alpha \in A} \underbrace{U_\alpha}_{\text{open sets}} \Rightarrow$ say $\{U_\alpha\}$ is an open cover of S.



$\{u_1\}$



$[0,1] \subset \left(\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1-\frac{1}{n}\right)\right) \cup \left(\frac{1}{2}, \frac{3}{2}\right)$

$\bigcup_{n=1}^{\infty} \{(-\frac{1}{n}, 1-\frac{1}{n})\} \cup \{(\frac{1}{2}, \frac{3}{2})\}$.

open cover of $[0,1]$.

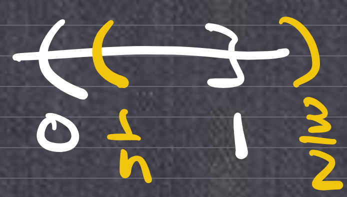
$\overset{n=2}{\swarrow} \qquad \overset{n=3}{\swarrow}$

$\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{2}{3}\right)$

$\cup \underline{\left(\frac{1}{2}, \frac{3}{2}\right)}$

$\supset [0,1]$

$\{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{3}, \frac{2}{3}), (\frac{1}{2}, \frac{3}{2})\}$ is a finite subcover of $[0,1]$.



$$[0,1] \subset \underbrace{\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, \frac{3}{2}\right)}_{}$$

$$[0,1] \subset \left(0, \frac{3}{2}\right)$$

$\left\{ \left(\frac{1}{n}, \frac{3}{2}\right) : n \in \mathbb{N} \right\}$ is open cover of $[0,1]$.

Ex: Prove $[0,1]$ is compact.

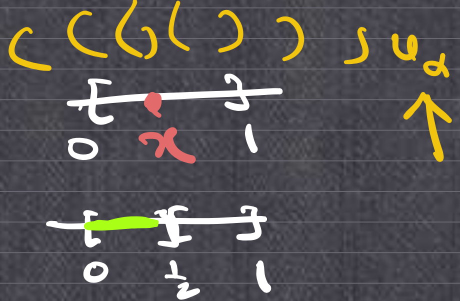
Proof: Assume not.

compact: any open cover of $[0,1]$ has finite subcover

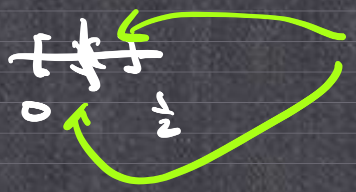
not compact: $\exists$ open cover of $[0,1]$ has no finite subcover.

$\therefore \exists \{\underset{\text{open}}{U_\alpha}\}_{\alpha \in A}$ s.t. $[0,1] \subset \bigcup_{\alpha \in A} U_\alpha$

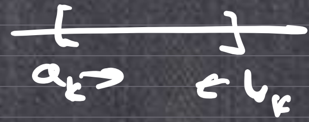
but $\{U_\alpha\}$ has no finite subcover.



at least one cannot be covered by



finitely many U_α's.



$\exists \; I_1 \supset I_2 \supset I_3 \supset \cdots$ s.t.

$I_1 = [0,1]$

$\forall k$ I_k cannot be covered by finitely many U_α's.

$$\bigcap_{k\in\mathbb{N}}^{\infty} I_k = \{x\}, \qquad I_k = [a_k, b_k]$$

$$x \in [0,1] \subset \bigcup_{\alpha \in A} U_\alpha$$

$$\Downarrow$$

$\exists \alpha \in A$ s.t.

$x \in U_\alpha$.

Heine–Borel

$K \subset \mathbb{R}$, TFAE:

① K is closed and bounded

② K is sequentially compact
($\forall$ seq. $\{x_n\}$ in K, $\exists \{x_{n_k}\} \to L \in K$.)

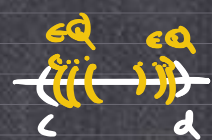
③ K is compact
(every open cover of K has a finite subcover)

① $\iff$ ② : Bolzano – Weierstrass. & K is closed.

② $\Rightarrow$ ③ : Given K is sequentially compact.

Need: any open cover $\{U_\alpha\}_{\alpha \in A}$ of K has a finite subcover.
$\downarrow$
countable

Step 1: $\therefore$ . .



$$U_\alpha = \coprod_{i=1}^{\infty} (c_{\alpha,i}, d_{\alpha,i})$$

$$= \coprod_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (p_{\alpha,i,j}, q_{\alpha,i,j})$$

$$K \subset \bigcup_{\alpha \in A} U_\alpha = \underbrace{\bigcup_{\alpha \in A} \coprod_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (p_{\alpha,i,j}, q_{\alpha,i,j})}_{= \bigcup_{k=1}^{\infty} (p_k, q_k)}$$

$\subset \bigcup_{n=1}^{\infty} \mathcal{U}_{\alpha_k}.$

Step 2: Assume opposite

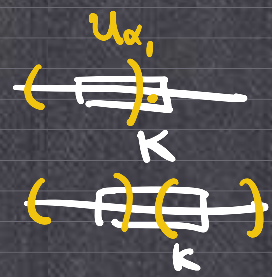
K cannot be covered by finitely many $\mathcal{U}_{\alpha_n}$'s.

$\exists x_1 \in K, \quad x_1 \notin \mathcal{U}_{\alpha_1}$

$\exists x_2 \in K, \quad x_2 \notin \mathcal{U}_{\alpha_1} \cup \mathcal{U}_{\alpha_2}$

$\exists x_3 \in K, \quad x_3 \notin \mathcal{U}_{\alpha_1} \cup \mathcal{U}_{\alpha_2} \cup \mathcal{U}_{\alpha_3}$

$\exists x_4 \in K \quad x_4 \notin \mathcal{U}_{\alpha_1} \cup \mathcal{U}_{\alpha_2} \cup \mathcal{U}_{\alpha_3} \cup \mathcal{U}_{\alpha_4}$



$\vdots$

$x_{n_j} \in K \quad x_{n_j} \notin \mathcal{U}_{\alpha_1} \cup \cdots \cup \mathcal{U}_{\alpha_{n_j}}$

seq. compact.

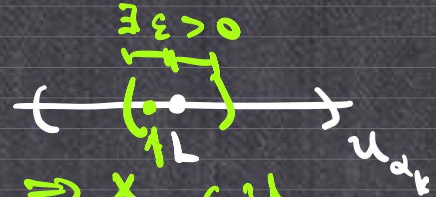
$\downarrow$

$L \in K$

$L \in K \subset \bigcup_{k=1}^{\infty} \mathcal{U}_{\alpha_k}$

$\Rightarrow \exists k$ s.t.

$L \in \mathcal{U}_{\alpha_k}$

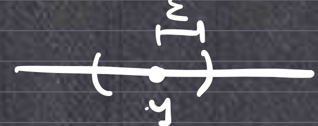


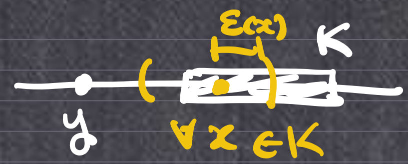
$j \gg 1 \Rightarrow x_{n_j} \in \mathcal{U}_{\alpha_k}$

③ $\Rightarrow$ ①: Given K is compact.

- Prove K is closed.

Proof: $\forall y \in \mathbb{R} \setminus K$





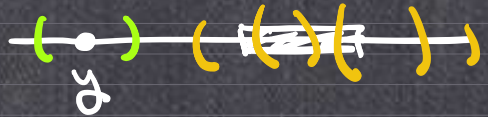
∀x ∈ K, choose $\varepsilon(x) > 0$ s.t.

$y \notin (x-\varepsilon(x),\ x+\varepsilon(x))$

$$K = \bigcup_{x\in K}\{x\} \subset \bigcup_{x\in K}(x-\varepsilon(x),\ x+\varepsilon(x))$$

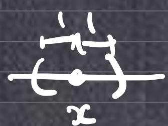
K compact $\Longrightarrow \exists\, x_1, \ldots, x_n \in K$ s.t.

$$K \subset \bigcup_{i=1}^{n}(x_i-\varepsilon(x_i),\ x_i+\varepsilon(x_i))$$



- Prove K is bounded.

<u>Proof</u>: $\forall x \in K$, consider $(x-1,\ x+1)$



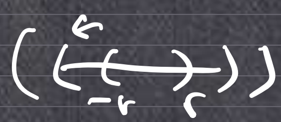
$$K = \bigcup_{x\in K}\{x\} \subset \bigcup_{x\in K}(x-1,\ x+1)$$

compact $\Downarrow$

$\exists\, x_1, \ldots, x_N \in K$ s.t.

$$K \subset \bigcup_{i=1}^{N}(x_i-1,\ x_i+1)$$

$$K \subset \bigcup_{r>0}(-r,\ r) = \mathbb{R}$$



Extreme Value Theorem.

$f: [a,b] \to \mathbb{R}$ continuous.

$\exists\, x_0, x_1 \in [a,b]$ s.t.

$$f(x_0) = \inf_{[a,b]} f \in \mathbb{R}$$

$$f(x_1) = \sup_{[a,b]} f \in \mathbb{R}.$$

Goal: $f([a,b])$ is closed and bounded.

Proof ①: Show $f([a,b])$ is sequentially compact.

Take sequence $\{y_n\} \in f([a,b])$

$y_n = f(x_n)$ $\quad \exists x_n \in [a,b]$.

sequentially compact

$y_{n_j} = f(\underbrace{x_{n_j}}_{\downarrow})$ $\qquad \exists x_{n_j} \to L \in [a,b]$.

$f(L)$

Proof ②: Show $f([a,b])$ is compact.

Take any open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of $f([a,b])$

$$f([a,b]) \subset \bigcup_{\alpha \in A} \mathcal{U}_\alpha.$$

f continuous $\Leftrightarrow f^{-1}(\text{open})$ is open.

$$\Rightarrow [a,b] \subset f^{-1}\Big(\bigcup_{\alpha \in A} \mathcal{U}_\alpha\Big)$$

$$= \bigcup_{\alpha \in A} f^{-1}(\mathcal{U}_\alpha).$$

open.

$[a,b]$ is compact

$\Rightarrow \exists\, \alpha_1, \dots, \alpha_n$ s.t. $[a,b] \subset \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$

$\Rightarrow f([a,b]) \subset \bigcup_{i=1}^{n} U_{\alpha_i}$

---

§2: Metric Spaces.

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } 0 < \overbrace{|x-a|}^{d(x,a)} < \delta \Longrightarrow \underbrace{|f(x) - L|}_{D(f(x),L)} < \varepsilon.$$

let $X \neq \phi$.

$d : X \times X \to [0, \infty)$ is a metric (distance function)

if d satisfies:

① $d(x,y) \geq 0 \quad \forall x, y \in X$,
and equality holds $\Leftrightarrow x = y$.

② $d(x,y) = d(y,x) \quad \forall x, y \in X$

③ $d(x,z) \leq d(x,y) + d(y,z)$
$\forall x, y, z \in X$.

then call (X,d) is called a metric space.

$(\mathbb{R}, |x-y|)$ ✓ $\qquad \vec{x} = (x_1, \dots, x_n)$

$$\mathbb{R}^n, \quad d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}.$$

- $X \neq \phi$

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

discrete.

① ✓ $\qquad$ ③ $\quad x, y, z$

② ✓

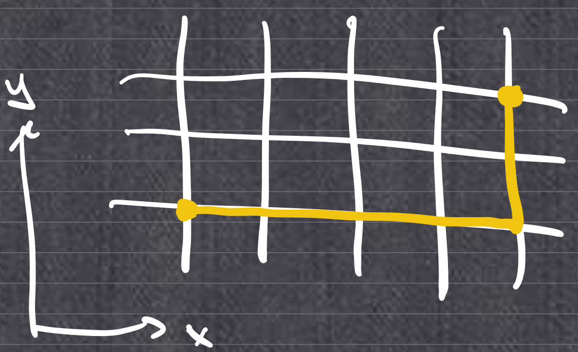
$x = y, \quad , x = y = z$

$\neq$

$z \quad \dots$

- $\mathbb{R}^2$:

$$d(\vec{a}, \vec{b}) = |x_1 - x_2| + |y_1 - y_2|$$



$\vec{a} = (x_1, y_1)$

$\vec{b} = (x_2, y_2)$

(X, d) metric

$$\underset{X}{\underset{\uparrow}{x_n}} \to \underset{X}{\underset{\uparrow}{y}} \overset{\text{def}}{\iff} \forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } n \geq N \Rightarrow d(x_n, y) < \varepsilon$$

Cauchy sequence:

$\{x_n\} \in X$ is Cauchy

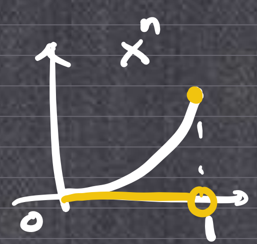
$$\overset{\text{def}}{\iff} \forall \varepsilon > 0,\ \exists N > 0 \text{ s.t. } m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon.$$

e.g. $X = C[0,1]$

$$\|f\|_p := \left(\int_0^1 |f(x)|^p\right)^{\frac{1}{p}}$$

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

$$\boxed{f_n(x) = x^n}$$



$$\|f_n - 0\|_p = \left(\int_0^1 |x^n|^p\, dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 x^{pn}\, dx\right)^{\frac{1}{p}} = \left(\left[\frac{x^{pn+1}}{pn+1}\right]_{x=0}^{x=1}\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{pn+1}\right)^{p} \to 0 \quad \text{as } n \to \infty.$$

$|f_n(y)-g(y)| \le \|f_n-g\|_\infty := \sup\limits_{x\in[0,1]} |f_n(x)-g(x)|$

$\forall y \in [0,1]$

$$\|f_n-g\|_\infty \to 0 \implies f_n(y)\to g(y) \quad \forall y\in[0,1]$$

(X,d) is a <u>complete</u> metric space

$\overset{def}{\iff}$ $\forall$ Cauchy sequence in (X,d) converges to a limit in X.

$(V, \|\ \|)$ is said to be a <u>Banach</u> space

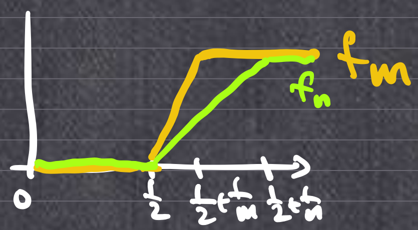
$\overset{def}{\iff}$ $\forall$ Cauchy sequence in $(V,\|\ \|)$ converges to a limit in V.

<u>eg.</u> $X = C[0,1]$, $\|f\|_1 = \int_0^1 |f(x)|\,dx$

$f_n(x) =$

$\forall \varepsilon > 0, \ \exists N > 2/\varepsilon$

$$\|f_m - f_n\|_1 = \int_0^1 |f_m(x) - f_n(x)|\,dx \qquad \underline{m > n \geq N}$$



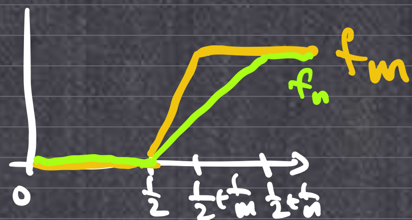
$$= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right)$$

$$\leq \frac{1}{2N} < \varepsilon.$$

$\{f_n\}$ is Cauchy wrt $\|\ \|_1$.

$\underline{\text{WTS}}$: $f_n \not\to g \ \forall g \in C[0,1]$

Assume $f_n \to g \in (C[0,1], \|\ \|_1)$



$$0 \underset{\substack{\text{as} \\ n \to \infty}}{\longleftarrow} \|f_n - g\|_1 = \int_0^1 |f_n(x) - g(x)|\,dx$$

$$= \int_0^{\frac{1}{2}} |f_n(x) - g(x)|\,dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \cdots$$

$$+ \int_{\frac{1}{2}+\frac{1}{n}}^{1} \cdots \,dx$$

$$= \int_0^{1/2} |g(x)|\,dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - g(x)|\,dx$$

$$+ \int_{\frac{1}{2}+\frac{1}{n}}^{1} |1 - g(x)|\,dx$$

$$\Rightarrow \lim_{n \to \infty} \underline{\int_0^{1/2} |g(x)|\,dx} = 0 \Rightarrow \int_0^{1/2} |g(x)|\,dx = 0$$

$$\Rightarrow \underline{g(x) = 0 \text{ on } [0, \tfrac{1}{2}]}$$

$$\lim_{n\to\infty} \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - g(x)|\,dx = 0$$

$$\lim_{n\to\infty} \int_{\frac{1}{2}+\frac{1}{n}}^{1} |1-g(x)|\,dx = 0$$

$$\Rightarrow \int_{\lim_{n\to\infty}(\frac{1}{2}+\frac{1}{n})}^{1} |1-g(x)|\,dx \Rightarrow \int_{\frac{1}{2}}^{1} |1-g(x)|\,dx$$

$$\Rightarrow \underline{g(x) = 1 \text{ on } [\tfrac{1}{2}, 1]}$$

$g \notin C[0,1]$

Conclusion: $(C[0,1], \|\cdot\|_1)$ is not a Banach space.

$$C[0,1], \quad \|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

$$\|f_n - g\|_\infty = \sup_{x\in[0,1]} |f_n(x) - g(x)|$$

$f_n \to g$ in $\| \ \|_\infty$

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $n \geq N$

$$\Longrightarrow \sup_{x\in[0,1]} |f_n(x) - g(x)| < \varepsilon.$$

$\Rightarrow \quad \forall x \in [0,1], \quad |f_n(x) - g(x)| < \varepsilon.$

$\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \geq N, \forall x \in [0,1]$
$|f_n(x) - g(x)| < \varepsilon.$

Take $\{f_n\}$ Cauchy seq in $(C[0,1], \|\ \|_\infty)$.

$$|f_n(x) - f_m(x)| \leq \sup_{[0,1]} |f_n - f_m| < \varepsilon/2$$
$\forall x \in [0,1]$

$\forall x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$.

$$\Rightarrow \quad f_n(x) \to g(x).$$

$$\|f_n - g\|_\infty = \sup_{[0,1]} |f_n - g|$$

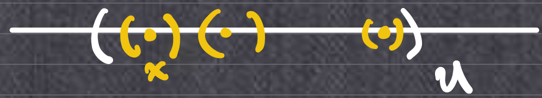
$$|f_n(x) - g(x)| \leq \underbrace{|f_n(x) - f_N(x)|}_{< \varepsilon/2} + \underbrace{|f_N(x) - g(x)|}_{< \varepsilon/2}$$

§2.2

$U \subset \mathbb{R}$ is open

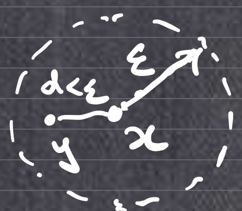
$\overset{\text{def}}{\iff}$ $\forall x \in U,\ \exists \varepsilon > 0$ s.t.

$$\boxed{(x-\varepsilon,\ x+\varepsilon)} \subset U.$$



(X, d) metric space

$$B_\varepsilon(x) = \{\, y \in X : d(x,y) < \varepsilon \,\}$$



$U \subset X$ is <u>open</u> in X

$\overset{\text{def}}{\iff}$ $\forall x \in U,\ \exists \varepsilon > 0$ s.t.

$$B_\varepsilon(x) \subset U.$$

E is closed in X

$\overset{\text{def}}{\iff}$ $\underbrace{X \setminus E}_{E^c}$ is open.

<u>e.g.</u> (X, d) metric

$U := B_r(a)$

$\forall x \in B_r(a)$

take $\varepsilon := \frac{1}{2}(r - d(x,a)) < r - d(x,a)$

<u>Claim</u>: $B_\varepsilon(x) \subset B_r(a)$

<u>Proof</u>: $\forall y \in B_\varepsilon(x), \Rightarrow d(x,y) < \varepsilon$.

$$\Rightarrow d(y,a) \le \underbrace{d(y,x)}_{<\varepsilon} + \underbrace{d(x,a)}_{d(x,a)}$$

$$< r - \cancel{d(x,a)} + \cancel{d(x,a)}$$

$\Rightarrow y \in B_r(a)$.

$\mathbb{R}^2$, $d_1((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$

$B_1((0,0)) =$ 

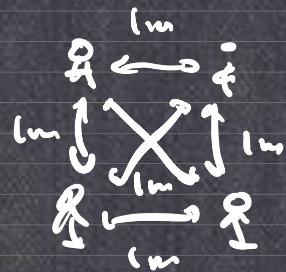
$\mathbb{R}^2$, $d_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|, |y_1-y_2|\}$

$B_1((0,0)) =$ 

$\mathbb{R}^2$, $d_0(\vec{x},\vec{y}) = \begin{cases} 1 & \text{if } \vec{x} \neq \vec{y} \\ 0 & \text{if } \vec{x} = \vec{y} \end{cases}$

$B_2((0,0)) = \mathbb{R}^2$

$B_{\frac{1}{2}}((0,0)) = \{(0,0)\}$



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e.g. $X = C[0,1]$

$\|f\|_1 = \int_0^1 |f(x)|\,dx \rightsquigarrow B_1^{d_1}(0)$

$\|f\|_\infty = \max\limits_{[0,1]} |f(x)| \rightsquigarrow B_1^{d_\infty}(0)$

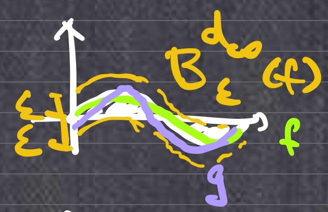
$B_1^{d_1}(0) = \{ f \in C[0,1] \mid \int_0^1 |f(x)|\,dx < 1 \}$

$B_1^{d_\infty}(0) = \{ f \in C[0,1] \mid \sup\limits_{[0,1]} |f(x)| < 1 \}$

① $B_1^{d_1}(0)$ is open in (X, d_∞) ?

② $B_1^{d_\infty}(0)$ is open in (X, d_1) ?

① $f \in B_1^{d_1}(0) \longrightarrow$



$\int_0^1 |f(x)|\,dx < 1$

want $\underline{B_\varepsilon^{d_\infty}(f)} \subset \underline{B_1^{d_1}(0)}$

$\forall g \in B_\varepsilon^{d_\infty}(f), \quad d_\infty(f,g) < \varepsilon$

$\Rightarrow \sup_{[0,1]} |f(x)-g(x)| < \varepsilon.$

$\Rightarrow |f(x)-g(x)| < \varepsilon \quad \forall x \in [0,1].$

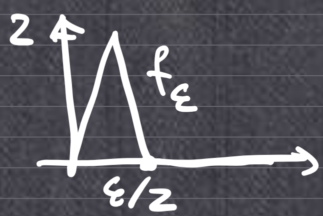
$$\int_0^1 |g(x)|\,dx \le \int_0^1 \underbrace{|g(x)-f(x)|}_{<\varepsilon} + |f(x)|\,dx$$

$$< \varepsilon + \underbrace{\int_0^1 |f(x)|\,dx} < 1$$

Take $\varepsilon = \underbrace{1-\int_0^1 |f(x)|\,dx}_{>0}.$

② $B_1^{d_\infty}(0)$ ~~open?~~ in (X, d_1).

$0 \in B_1^{d_\infty}(0) = \mathcal{U}$



$\|f_\varepsilon\|_{d_1} = \frac{\varepsilon}{2} < \varepsilon$

$\Rightarrow f_\varepsilon \in B_\varepsilon^{d_1}(0).$

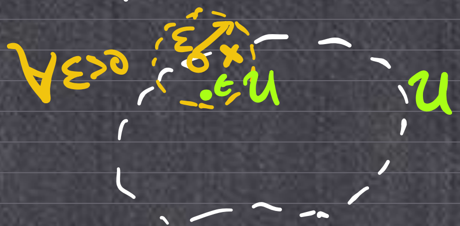
$$f_\varepsilon \notin B_1^{d_\infty}(0)$$

$\forall \varepsilon > 0$, $\exists f_\varepsilon \in B_\varepsilon^{d_1}(0)$ but $f_\varepsilon \notin B_1^{d_\infty}(0)$

$$\Rightarrow \quad B_\varepsilon^{d_1}(0) \not\subset B_1^{d_\infty}(0).$$

Def 2.29

limit point



$\Rightarrow$ x is a limit point of $\mathcal{U}$.

$\mathcal{U}' :=$ the set of all limit points of $\mathcal{U}$.

boundary point : $\forall \varepsilon > 0$

$\Rightarrow$ x is a boundary point of $\mathcal{U}$

$\partial \mathcal{U} :=$ the set of boundary points of $\mathcal{U}$.

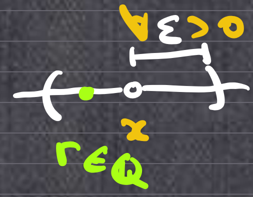
e.g. $\quad \mathbb{Q} \subset (\mathbb{R}, |x-y|)$

$\forall x \in \mathbb{R}$, $\quad B_\varepsilon(x) \cap \mathbb{Q} \neq \phi$

$\qquad\qquad B_\varepsilon(x) \cap \mathbb{Q}^c \neq \phi$

$\Rightarrow \quad x \in \partial\mathbb{Q}.$

$\mathbb{R} \subset \partial\mathbb{Q} \subset \mathbb{R} \quad \Rightarrow \quad \partial\mathbb{Q} = \mathbb{R}.$



$\forall x \in \mathbb{R}, \quad \forall \varepsilon > 0,$

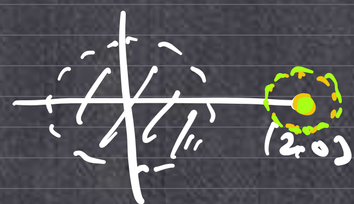
$\mathbb{Q} \cap \big((x-\varepsilon, x+\varepsilon) \setminus \{x\}\big) \neq \emptyset$

$\Rightarrow \quad x \in \mathbb{Q}' \qquad \Big\} \Rightarrow \mathbb{Q}' = \mathbb{R}.$

<u>eg. 2.30</u>

$\mathbb{R}^2, \; \|\vec{x} - \vec{y}\|_2$

$S := B_1(0) \cup \{(2,0)\}$



$(2,0) \notin S'.$

$(2,0) \in \partial S.$

<u>Prop. 2.32</u> : $\quad S' \cup S = \partial S \cup S$

"$\subset$" : $\quad \forall x \in S' \cup S$

Case ① : if $x \in S$, $\quad x \in \partial S \cup S.$

Case ② : if $x \notin S$, then $x \in S'.$

$\Rightarrow \quad \forall \varepsilon > 0, \; \big(B_\varepsilon(x) \setminus \{x\}\big) \cap S \neq \emptyset.$

$\Rightarrow \quad x \in \partial S.$

$$\boxed{\overline{S} := \partial S \cup S}$$

closure of S.

$x \in S'$

$x \in \partial S$

$$\partial S \cup S = S' \cup S =: \bar{S}.$$

Claim: $\bar{S}$ is closed. (X, d)

Proof: Let $\{x_n\} \to y \in X$

$\{x_n\} \subset \bar{S}$ ← **need:** $y \in \bar{S}$.

Case ①: $y \in S \subset \bar{S}$

$\forall \varepsilon > 0$

Case ②: $y \notin S$ ⟶

S' $\Downarrow$ $x_n \notin S$ $\forall n$.

$\exists x_n \in S$ ✓

$\partial S \subset S \iff S' \subset S \iff$

Prop 2.34 $\bar{S} = S \iff S$ is closed.

$(\Longrightarrow)$ trivial

($\Leftarrow$) Assume S is closed.

Goal: $\partial S \subset S$ $\quad \forall \varepsilon > 0$

$\forall x \in \partial S,$

$\forall n \in \mathbb{N}, \ \exists x_n \in S \cap B_{\frac{1}{n}}(x), \quad d(x_n, x) < \frac{1}{n}$

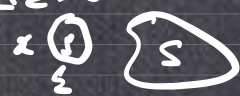
$\downarrow$ $\quad \uparrow$ closed $\quad \downarrow$

$x \in S$ $\quad\quad x$

Alternately: $\forall x \in \partial S$, assume $x \notin S$

$\Rightarrow x \in \underbrace{X \setminus S}_{\text{open}}.$

$\exists \varepsilon > 0$



$$\overline{S} = \bigcap_{\substack{S \subset E \\ \uparrow \\ \text{closed}}} E$$

$\overline{B_\varepsilon(x)}$

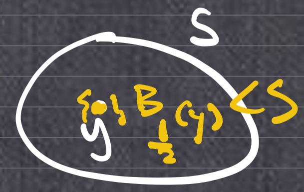
$\cancel{\supsetneq} \{y \in X \mid d(x,y) \leq \varepsilon\}$ $\quad\Big|\quad$ $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$

(X, d) discrete metric

$B_1(x) = \{y \in X : d(x,y) < 1\} = \{x\}.$

$$C_1(x) = \{y \in X : d(x,y) \le 1\} = X.$$

$$\overline{B_1(x)} = B_1(x).$$



<u>**dense**</u> $S \subset (X, d)$ is dense $\overset{\text{def}}{\iff} \overline{S} = X.$

- $\overline{\mathbb{Q}} = \mathbb{R}$, $\quad \mathbb{Q}$ is dense in $\mathbb{R}$.
- $\overline{\mathbb{R}\setminus\mathbb{Q}} = \mathbb{R}$, $\quad \mathbb{R}\setminus\mathbb{Q}$ " — — .

$$f(x+y) = f(x)\,f(y) \quad \forall x, y \in \mathbb{R}.$$

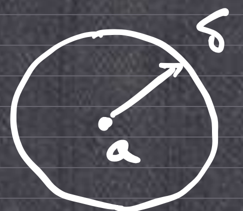
$$S^\circ := S \setminus \partial S.$$

$$f : (X, d_X) \to (Y, d_Y)$$

$\lim_{x \to a} f(x) = p$

$\overset{\text{def}}{\iff} \forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < d_X(x, a) < \delta \quad$ ($x \in B^X_\delta(a) \setminus \{a\}$.)

$\Rightarrow d_Y(f(x), p) < \varepsilon.$



$S \subset X$

$a \in S'$

$$\lim_{\substack{x \to a \\ x \in S}} f(x) = p$$

$$\overset{\text{def}}{\Longleftrightarrow} \forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. } 0 < d_X(x,a) < \delta, \ x \in S$$
$$\Rightarrow d_Y(f(x), p) < \varepsilon.$$

$f : \mathbb{R} \to \mathbb{R}$

$$\lim_{\substack{x \to a \\ x \in (a, a+1)}} f(x) = L \iff \lim_{x \to a^+} f(x) = L$$

$$\mathbb{1}_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}, \quad \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$\lim_{\substack{x \to a \\ x \in \mathbb{R}}} \mathbb{1}_{\mathbb{Q}}(x)$ does not exist

$$\lim_{\substack{x \to a \\ x \in \mathbb{Q}}} \underbrace{\mathbb{1}_{\mathbb{Q}}(x)}_{1} = 1 \quad \Big| \quad \lim_{\substack{x \to a \\ x \in \mathbb{R} \setminus \mathbb{Q}}} \mathbb{1}_{\mathbb{Q}}(x) = 0.$$

$f : (X, d_X) \to (Y, d_Y)$ is continuous at $x = a$

$$\overset{\text{def}}{\Longleftrightarrow} \lim_{x \to a} f(x) = f(a)$$

e.g. 2.44 : $A \subset (X, d)$

$$d_A : X \to \mathbb{R}$$

$$d_A(x) := \inf \{ d(x,a) : a \in A \}$$

Claim: d_A is continuous on X.

Proof: Want: $|d_A(x) - d_A(y)| \leq d(x,y)$.

$$d(x,a) \leq d(x,y) + d(y,a) \quad \forall a \in A.$$

$\inf_{a \in A} \Rightarrow$ $$d_A(x) \leq \inf \{ d(x,y) + d(y,a) : a \in A \}.$$

$$= d(x,y) + \underbrace{\inf \{ d(y,a) : a \in A \}}_{d_A(y)}$$

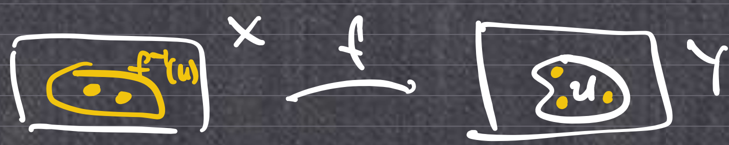
$$\Rightarrow \quad d_A(x) - d_A(y) \leq d(x,y).$$

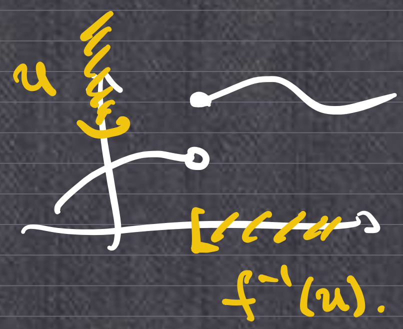
Let $f : (X, d_X) \to (Y, d_Y)$, TFAE:

① f is continuous on X

② $f^{-1}(U)$ is open in X for any open set $U \subset Y$.

$f^{-1}(\mathcal{U}) := \{x \in X : f(x) \in \mathcal{U}\}$





e.g. $f(x,y) = x^2 + 4xy + y^3 - x^3y^2$ in $\mathbb{R}^2$

$$\Sigma := \{(x,y) \in \mathbb{R}^2 : f(x,y) > 0\}.$$

$$\parallel$$

$$f^{-1}\big((0,\infty)\big) = \{(x,y) \in \mathbb{R} : f(x,y) \in (0,\infty)\}$$

e.g. $M^{n\times n}$ = set of all $n\times n$ real matrices.

$\cong \mathbb{R}^{n^2}$

Σ := set of all invertible matrices.

$\det : M^{n\times n} \to \mathbb{R}$

$\Sigma = \det^{-1}(\mathbb{R}\setminus\{0\}) = \{A \in M^{n\times n} : \det(A) \neq 0\}$

is open in $M^{2\times 2}$.

Let $f : (X, d_X) \to (Y, d_Y)$, TFAE:

① f is continuous on X

② $f^{-1}(U)$ is open in X for any open set $U \subset Y$.

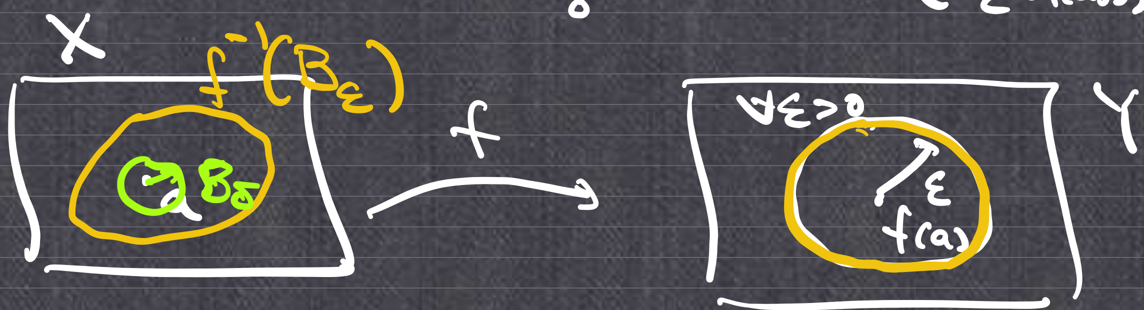
$\Rightarrow$ $\forall a \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $\underbrace{d_X(x,a) < \delta}_{x \in B_\delta(a)} \Rightarrow \underbrace{d_Y(f(x), f(a)) < \varepsilon}_{f(x) \in B_\varepsilon(f(a))}$

$$\Updownarrow$$

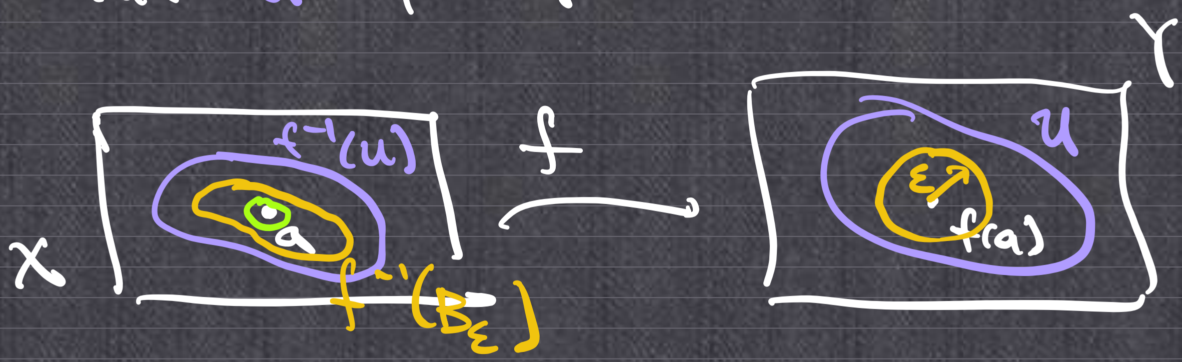
$$x \in f^{-1}(B_\varepsilon(f(a)))$$

$$B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a)))$$



① $\Rightarrow$ ② :

Take $\mathcal{U} \subset Y$ open .



$f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \\ & \gcd(m,n) = 1, \ m > 0, \ n > 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Ex: f is continuous at $x \in [0,1] \setminus \mathbb{Q}$

but is not continuous at $x \in \mathbb{Q} \cap [0,1]$.

$f(r) > 0 \quad \forall r \in \mathbb{Q} \cap [0,1]$.

$s_n \notin \mathbb{Q} \qquad \underbrace{f(s_n)}_{=0} \to 0$

$s \notin \mathbb{Q}$

<u>Question:</u> ? $\exists f : \mathbb{R} \to \mathbb{R}$

s.t. f is continuous at $a \in \mathbb{Q}$

but not continuous at $a \notin \mathbb{Q}$.

$D_f = \{a \in \mathbb{R} : f \text{ is not continuous at } a\}$.

$$\left. \begin{array}{l} \parallel \\ \bigcup_{i=1}^{\infty} E_i \leftarrow \text{closed.} \end{array} \right\} \quad \text{HW2.}$$

If $\exists f$ is continuous at $a \in \mathbb{Q}$

discontinuous at $a \notin \mathbb{Q}$.

$\Longrightarrow \quad D_f = \mathbb{R} \setminus \mathbb{Q}.$

$$\mathbb{R} = \underbrace{D_f}_{\mathbb{R}\setminus\mathbb{Q}} \cup \mathbb{Q} = \bigcup_{i=1}^{\infty} E_i \cup \{r_1, r_2, \dots\}$$

$$= \bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} \{r_j\}$$

$\bar{E}_i^{\circ} = \phi$: if not, $\exists x \in E_i^{\circ}$

$\Rightarrow \exists \varepsilon > 0$ s.t.

$(x-\varepsilon, x+\varepsilon) \subset E_i \subset \mathbb{R}\setminus\mathbb{Q}$

Baire (c) $\Longrightarrow \quad \overline{\mathbb{R} - \bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} \{r_j\}} = \mathbb{R}$

$$\left(\bigcup_{i=1}^{\infty} E_i \cup \bigcup_{j=1}^{\infty} \{r_j\} \right)^{\circ} = \phi$$

$\mathbb{R}^{\circ} = \phi$

Given: (X, d) complete

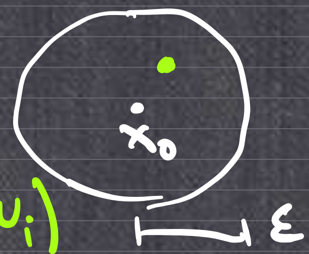
$\mathcal{U}_1, \mathcal{U}_2, \dots$ open in X s.t. $\overline{\mathcal{U}_i} = X$ $\forall i$

Need: $\overline{\bigcap_{i=1}^{\infty} \mathcal{U}_i} = X$.

↓

$\forall x_0 \in X$, $\forall \varepsilon > 0$.

show $\exists\, y \in B_\varepsilon(x_0) \cap (\bigcap_i U_i)$

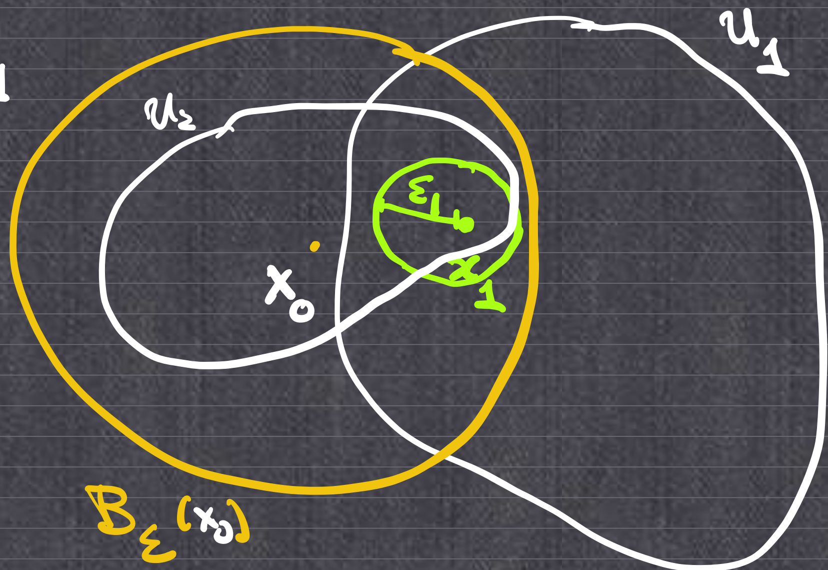


$\exists\, \overline{B_{\varepsilon_1}(x_1)} \subset B_\varepsilon(x_0) \cap U_1$

$\exists\, \overline{B_{\varepsilon_2}(x_2)} \subset B_{\varepsilon_1}(x_1) \cap U_2$

$\exists\, \overline{B_{\varepsilon_3}(x_3)} \subset B_{\varepsilon_2}(x_2) \cap U_3$

$\vdots$



Require $\varepsilon_n < \frac{1}{2^n}$.

$d(x_n, x_{n+1}) \leq \frac{c}{2^n}$. $\Longrightarrow$ $\{x_n\}$ Cauchy.

$X \ni y$

$\forall n \in \mathbb{N}$, $x_n, x_{n+1}, x_{n+2}, \dots \in B_{\varepsilon_n}(x_n)$

$\Downarrow$

$y \in \overline{B_{\varepsilon_n}(x_n)}$

$$y \in B_{\varepsilon_{n-1}}(x_n) \leftarrow \subset B_{\varepsilon_{n-1}}(x_{n-1}) \cap \mathcal{U}_{n-1}$$

$$\subset \cdots \subset B_{\varepsilon_0}(x_0) \qquad \subset \mathcal{U}_{n-1}$$

$\therefore$ $y \in \mathcal{U}_{n-1}$ $\forall n$ and $y \in B_\varepsilon(x_0)$

$$\Rightarrow \boxed{y \in \bigcap_{i=1}^{\infty} \mathcal{U}_i \cap B_\varepsilon(x_0)}$$

■

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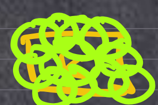
§ 2.3 : Compact set.



$\forall \varepsilon > 0$, $\exists B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)$ s.t. $K \subset B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_n)$

(X, d) metric space, $K \subset X$

TFAE :

① K is <u>complete</u> and <u>totally bounded</u>

② K is sequentially compact

$\forall \{x_n\} \subset K$, $\exists \{x_{n_j}\} \to y \in K$.

③ K is <u>compact</u>

every open cover of K has a finite subcover.

<u>Ex</u>: $X = \ell_\infty(\mathbb{R}) = \{ \{x_n\}_{n=1}^\infty : \{x_n\} \text{ bounded} \}$.

$$\| \{x_n\}_{n=1}^\infty \|_\infty = \sup_{n \in \mathbb{N}} |x_n|$$

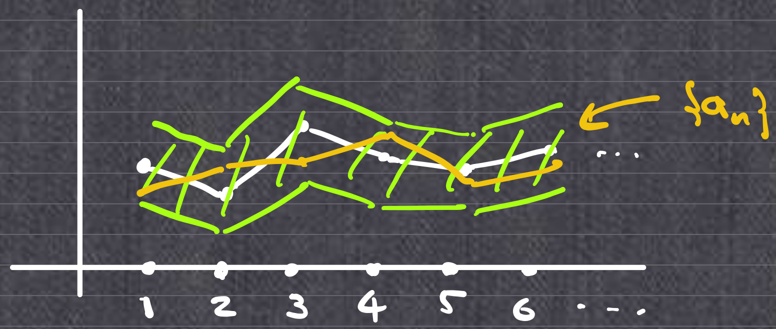
$$e_i = \{0, 0, \ldots, \underset{i}{1}, 0, 0 \ldots \}$$

$\{e_i\}_{i=1}^{\infty}$ is bounded.

$$d(e_i, e_j) = \|e_i - e_j\|_\infty = 1 \qquad i \neq j$$

any $\{e_{i_n}\}$ is not Cauchy

$\Rightarrow$ $\{e_{i_k}\}$ does not converge.

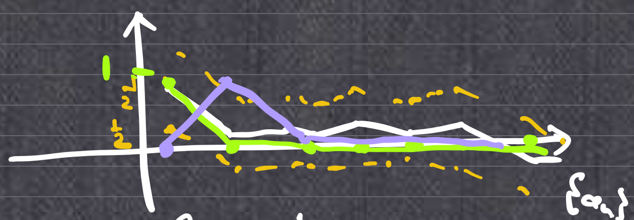


$$B_\varepsilon(\{a_n\}) = \{\{b_n\} \in \ell^\infty(\mathbb{R}) \mid \sup_{n\in\mathbb{N}} |a_n - b_n| < \varepsilon\}$$

$$e_i = \{0, 0, \dots, \underset{i}{1}, 0, 0 \dots\}$$

$$S = \{e_1, e_2, e_3, \dots\}$$

$B_{1/4}(\{a_n\}) \ni$



at most one of e_i's.

① K complete and totally bounded

② K sequentially compact

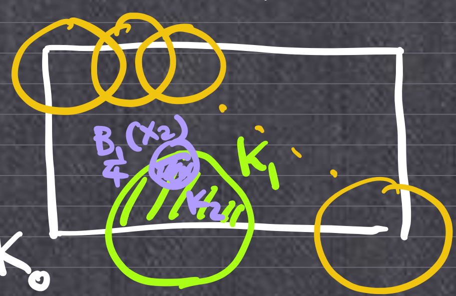
③ K compact

① ⇒ ③ ⇒ ② ⇒ ①.

① ⇒ ③ : Given K is complete, totally bounded.

Take an open cover $\bigcup_{\alpha \in A} U_\alpha \supset K$.

Assume no finite subcover for K.



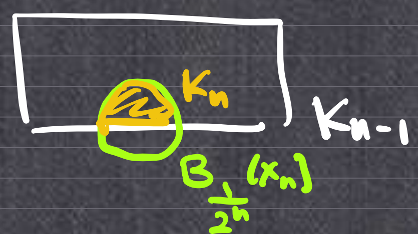
$$K \subset B_{\frac{1}{2}}(x_1) \cup \cdots \cup B_{\frac{1}{2}}(x_n)$$

$$K \cap B_{\frac{1}{2}}(x_i)$$

$\exists\, B_{\frac{1}{2}}(x_1) \leftarrow K_1 = K_0 \cap B_{\frac{1}{2}}(x_1)$ has no finite subcover.

$\exists\, B_{\frac{1}{4}}(x_2)$ $\quad K_2 = K_1 \cap B_{\frac{1}{2^2}}(x_2)$ has no finite subcover.

$\vdots$



$K_n = K_{n-1} \cap B_{\frac{1}{2^n}}(x_n)$, has no finite subcover.

$$K_n \neq \emptyset \Rightarrow \exists\, y_n \in K_n \subset B_{\frac{1}{2^n}}(x_n)$$

$$y_{n+1} \in K_{n+1} \subset K_n \subset B_{\frac{1}{2^n}}(x_n)$$

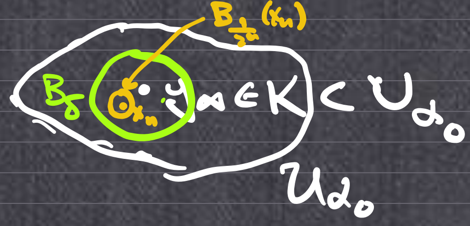
$$d(y_n, y_{n+1}) \leq \frac{1}{2^{n-1}} \Rightarrow \{y_n\} \text{ is Cauchy.}$$

$$d(x_n, y_n) \leq \frac{1}{2^n}$$

K complete

$$y_n \to y_\infty \in K$$

x_n



③ $\Rightarrow$ ② : Given K is compact

<u>want</u>: $\{x_n\} \subset K$. $\rightsquigarrow$? $\exists \{x_{n_j}\} \to x_\infty \in K$

$$S_n := \overline{\{x_n, x_{n+1}, x_{n+2}, \dots\}}$$

$$S := \bigcap_{n=1}^{\infty} S_n$$

<u>want</u>: $S \cap K \neq \emptyset$.

Assume $S \cap K = \emptyset$, $\left(\bigcap_{n=1}^{\infty} S_n\right) \cap K = \emptyset$.

$$\iff K \subset X - \bigcap_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} (X - S_n)$$

K compact

$\exists S_{n_1}, \dots, S_{n_k}$ s.t. $\quad n_1 < \dots < n_k$

$$K \subset (X - S_{n_1}) \cup \dots \cup (X - S_{n_k}) = X - (S_{n_1} \cap \dots \cap S_{n_k})$$

$$\subset X - \underbrace{\{x_{n_k}, x_{n_{k+1}}, \dots\}}_{\text{in } K}$$

$\therefore \underbrace{S \cap K}_{\ni y} \neq \emptyset$,

$$y \in S = \bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} \overline{\{x_n, x_{n+1}, \dots\}}$$

$y \in S_1 = \overline{\{x_1, x_2, \dots\}}$

$B_1(y) \ni x_{m_1}$ $\qquad m_1 \geq 1.$

$y \in S_{m_1+1} = \overline{\{x_{m_1+1}, x_{m_1+2}, \dots\}}$

$B_{\frac{1}{2}}(y) \ni x_{m_2}$ $\qquad m_2 > m_1 \geq 1.$

$\forall j. \exists x_{m_j} \in B_{\frac{1}{j}}(y)$, $\qquad m_j > m_{j-1} > \dots > m_2 > m_1$

$d(x_{m_j}, y) < \frac{1}{j} \to 0.$

② ⇒ ① : sequentially compact ⇒ complete.

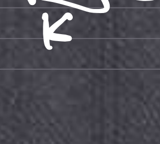
totally bounded.

want: $\forall \varepsilon > 0, \exists B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)$

$\bigcup_{i=1}^{N} B_\varepsilon(x_i) \supset K$

Assume not: $\exists \varepsilon > 0$, $\quad K \not\subset$ any $\bigcup_{i=1}^{N} B_\varepsilon(x_i)$

Pick $x_1 \in K$, $\quad B_\varepsilon(x_1) \not\supset K$



$\Rightarrow \exists x_2 \in K$

but. $x_2 \notin B_\varepsilon(x_1)$

$B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \not\supset K$

$\Rightarrow \exists x_3 \in K, \quad x_3 \notin B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$

$\vdots$

$\exists x_n \in K$, s.t. $d(x_n, x_m) \geq \varepsilon$. $\forall m \neq n$

§2.4 (X, d) metric space

S is said to **disconnected**

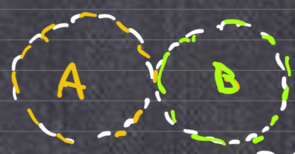
$\overset{\text{def}}{\Longleftrightarrow}$ $S = \underset{\neq \phi}{A} \cup B_{\neq \phi}$ where $\begin{cases} \bar{A} \cap B = \phi \\ A \cap \bar{B} = \phi \end{cases}$

connected $\overset{\text{def}}{\Longleftrightarrow}$ not disconnected

$\bar{A} \cap B$

$=$ 

$= \phi$



Prop. 2.67 A subset $S \subset \mathbb{R}$ is connected ①

$\Longleftrightarrow$ $\forall x, y \in S,\ x < y,$
$\Rightarrow [x, y] \subset S.$ ②

$(\Rightarrow)$ Suppose S is connected

Assume ② is false

$\exists\, x, y \in S,\ x < y$ s.t. $[x, y] \not\subset S$

$\Rightarrow \exists\, z \in (x, y)$
$z \notin S.$

Take $A = (-\infty, z) \cap S$, $B = (z, +\infty) \cap S$

$S = A \cup B$. $\bar{A} \cap B = \overline{(-\infty, z) \cap S} \cap B$

$\overline{X \cap Y} \subset \bar{X} \cap \bar{Y}$

$$\subset \overline{(-\infty, z)} \cap \bar{S} \cap B$$

$$= \underbrace{(-\infty, z]}\cap \bar{S} \cap \underbrace{(z, +\infty)} \cap S$$

$$= \phi \qquad = \phi$$

$A \cap \bar{B} = \phi$ similarly.

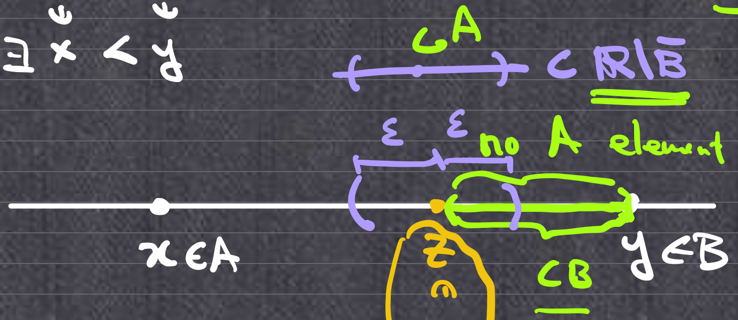
☒

($\Leftarrow$) Given $\forall x, y \in S$ where $x < y \Rightarrow [x, y] \subset S$.

Assume S is disconnected.

$$S = A \cup B \quad \text{s.t. } \bar{A} \cap B = \phi, \ \bar{B} \cap A = \phi.$$

$\exists\, x < y$, $x \in A$, $y \in B$



$z \notin \bar{B}$

$\Leftrightarrow z \in \underbrace{\mathbb{R} \setminus \bar{B}}_{\text{open}}$

let $z = \sup(A \cap [x, y])$

$\uparrow$

$a_n \in A \cap [x, y]$

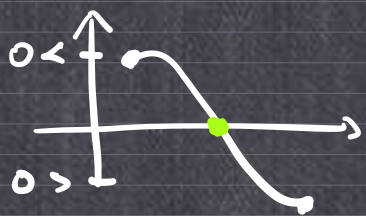
$\Rightarrow z \in \bar{A}$

$z \in \bar{A} \Rightarrow z \notin B \Rightarrow \boxed{z \in A}$

$z \in S = A \cup B$

☐

Ex: S connected $\Rightarrow \bar{S}$ connected.



$f: X \to Y$

"IVT" on metric spaces:

$$\left\{\begin{array}{l} S \text{ connected} \\ f \text{ is continuous} \end{array}\right. \Rightarrow \underline{f(S) \text{ is connected.}}$$

Proof: Assume $f(S)$ is disconnected.

$$\Rightarrow \quad f(S) = \underset{\substack{\cap \\ Y}}{A} \cup \underset{\substack{\cap \\ Y}}{B}, \qquad \begin{array}{l} \bar{A} \cap B = \phi \\ A \cap \bar{B} = \phi. \end{array}$$

$$\Rightarrow \quad S \subset f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$\Rightarrow \quad S = \underbrace{(f^{-1}(A) \cap S)} \cup (f^{-1}(B) \cap S)$$

$$\overline{f^{-1}(A) \cap S} \cap (f^{-1}(B) \cap S)$$

$$\subset \overline{f^{-1}(A)} \cap \bar{S} \cap f^{-1}(B) \cap S$$

$$\subset \underbrace{f^{-1}(\bar{A}) \cap f^{-1}(B)} \cap S$$

$$\subset f^{-1}(\underbrace{\bar{A} \cap B}_{\phi}) \cap S = \phi.$$

f continuous $\Rightarrow$ $\overline{f^{-1}(A)} \subset f^{-1}(\bar{A})$ HW

S is Path-connected $\overset{\text{def}}{\Longleftrightarrow}$ $\forall x, y \in S.\ \exists \gamma : [0,1] \to S$ continuous s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

Good news: path-connected $\Longrightarrow$ connected.

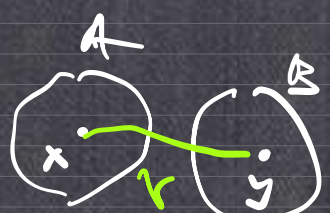
Bad news: connected $\not\Longrightarrow$ path-connected.

Given: S is path-connected

Want: S is connected.

↳ Assume not (i.e. S is disconnected)

$S = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $\bar{A} \cap B = \emptyset$, $A \cap \bar{B} = \emptyset$.



$\exists x \in A$, $\exists y \in B$

$\exists \gamma : [0,1] \to S$ continuous.

$\gamma(0) = x$, $\gamma(1) = y$.

$$\underbrace{\gamma^{-1}(S)}_{[0,1]} = \gamma^{-1}(A \cup B) = \gamma^{-1}(A) \cup \gamma^{-1}(B)$$

$$\overline{\gamma^{-1}(A)} \cap \gamma^{-1}(B) \subset \gamma^{-1}(\bar{A}) \cap \gamma^{-1}(B) = \gamma^{-1}(\bar{A} \cap B) = \gamma^{-1}(\emptyset) = \emptyset$$

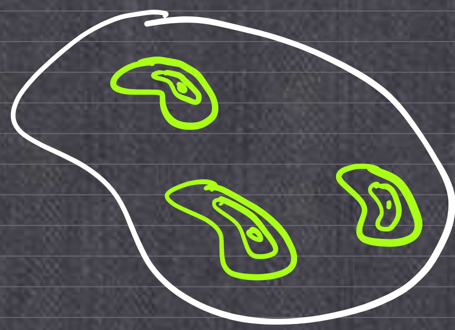
$\Gamma = \{ y = \sin \frac{1}{x} \}$

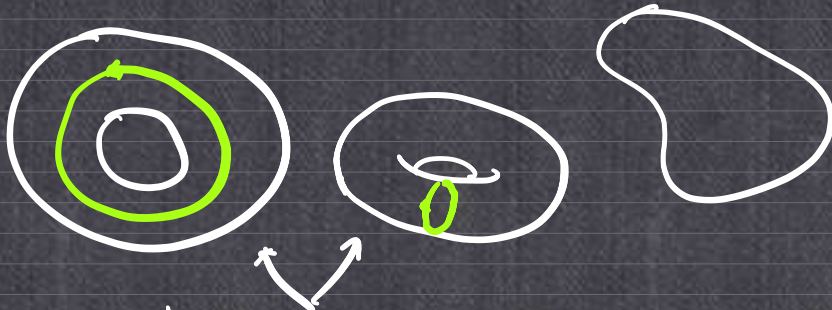
$x \in (0,1]$

$S = \Gamma \cup (\{0\} \times [-1,1])$

S is Simply-connected

$\overset{\text{def}}{\iff}$ $\forall$ loop in S
can contract to
a point without
leaving S.





not simply-connected

Fundamental Theorem of Algebra

$$p(z) = \sum_{n=0}^{k} a_n z^n \quad , \; a_n \in \mathbb{C} \qquad (a_0 \neq 0)$$

must a complex root $\alpha \in \mathbb{C}$.

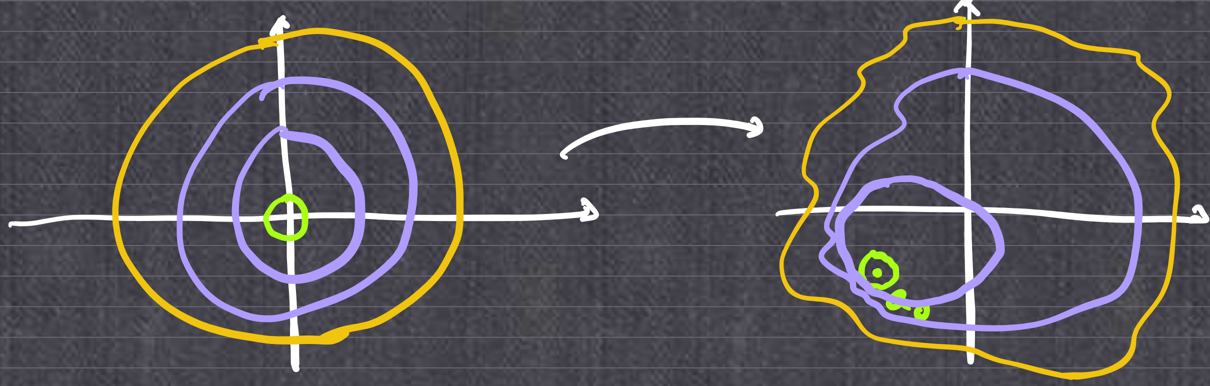
$p : \mathbb{C} \to \mathbb{C}$

$f(z) = z^n$

$$p(z) = a_n z^n + \cdots + a_1 z + a_0$$

$|z| \sim$ small, $p(z) \approx a_0 \neq 0$

$|z|$ large, $p(z) \approx a_n z^n$



$\underline{\text{Ch. 3}}$ $\boxed{F : \mathbb{R}^m \to \mathbb{R}^n}$

$f : \mathbb{R}^n \to \mathbb{R}$.
$(x_1, \ldots, x_n)$

$e_i = (0, \ldots, \underset{i}{1}, \ldots, 0)$

$$\frac{\partial f}{\partial x_i}\Big|_{\underbrace{(a_1, \ldots, a_n)}_{\vec{a}}} := \lim_{t \to 0} \frac{f(a_1, \ldots, a_i + t, \ldots a_n) - f(a_1, \ldots, a_n)}{t}$$

$$= \lim_{t \to 0} \frac{f(\vec{a} + t\vec{e_i}) - f(\vec{a})}{t}$$

$$(D_{\vec{u}} f)(a_1, \ldots, a_n) = \lim_{t \to 0} \frac{f(a_1 + tu_1, \ldots, a_n + tu_n) - f(a_1, \ldots, a_n)}{t}$$

$\vec{u} = (u_1, \ldots, u_n)$ $\quad = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(0+t,0)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^2\cdot 0}{t^4+0^2} - 0}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

$\vec{u} = (u_1, u_2)$

$$D_{\vec{u}} f(0,0) = \lim_{t\to 0} \frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \left(\frac{(tu_1)^2(tu_2)}{(tu_1)^4 + (tu_2)^2} - 0 \right) \Big/ t$$

$$= \lim_{t\to 0} \frac{t\, u_1 u_2}{t^2 u_1^4 + u_2^2} = 0.$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$r(t) = (t, t^2) \longrightarrow 0$ as $t \to 0$.

$$\lim_{t\to 0} f(r(t)) = \lim_{t\to 0} \frac{t^2(t^2)}{t^4+(t^2)^2} = \frac{1}{2}.$$

$$\lim_{x\to 0,\, y=0} f(x,y) = \lim_{x\to 0,\, y=0} 0 = 0 \quad \times$$

$f: \mathbb{R} \to \mathbb{R}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\Longleftrightarrow \quad f(x) = f(a) + f'(a)(x-a) + \underbrace{o((x-a))}_{= h(x)}$$

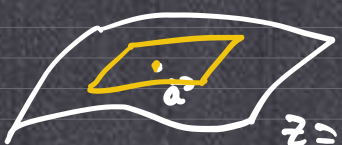
s.t. $\lim_{x \to a} \frac{h(x)}{x-a} = 0.$

$F: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$

$\overset{\text{def}}{\Longleftrightarrow}$ ① $\frac{\partial F}{\partial x_i}(\vec{a})$ exists $\forall i = 1, 2, \ldots, n$

② $F(\vec{x}) = \boxed{F(\vec{a}) + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\vec{a})(x_i - a_i)}$

$+ \underbrace{o(|\vec{x} - \vec{a}|)}_{= h(\vec{x})}$ s.t. $\lim_{\vec{x} \to \vec{a}} \frac{h(\vec{x})}{|\vec{x} - \vec{a}|} = 0$



$z = F(x,y)$

$F(\vec{a}) + \nabla F(\vec{a}) \cdot (\vec{x} - \vec{a})$

e.g. $f(x,y) = |xy|$ differentiable at $(0,0)$?

$\frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \left(f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)\cdot(y-0)\right)}{\|(x,y)-(0,0)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{|xy| - 0}{\sqrt{x^2+y^2}} = \lim_{\substack{r\to 0^+\\ \theta \text{ anything}}} \frac{r^2|\sin\theta\cos\theta|}{r}$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \lim_{\substack{r\to 0^+\\ \theta \text{ anything}}} r|\sin\theta\cos\theta|$$

$$\underset{\downarrow\\0}{0} \le \underset{\Downarrow\\0}{\underline{r|\sin\theta\cos\theta|}} \le \underset{\downarrow\\0}{r}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{y} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r\sin\theta} = \lim_{r\to 0} r\frac{\cos^2\theta}{\sin\theta}$$

$$\frac{|xy|}{\sqrt{x^2+y^2}} \le \underbrace{|x|\cdot 1}_{\downarrow\ 0}$$

$$\frac{|y|}{\sqrt{x^2+y^2}} \le \underbrace{\frac{|y|}{\sqrt{y^2}}}_{=1}$$

$f(x,y)$ is differentiable at (a,b)

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \text{①} & \frac{\partial f}{\partial x}(a,b) \text{ and } \frac{\partial f}{\partial y}(a,b) \text{ exist} \\ \text{②} & f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) \end{cases}$

$$+\, o\big(|(x,y)-(a,b)|\big) \quad \text{as } (x,y)\to(a,b)$$

$\overset{\text{HW3}}{\Longleftrightarrow}$ $\exists\, c_1, c_2 \in \mathbb{R}$ s.t.

$$f(x,y) = f(a,b) + c_1(x-a) + c_2(y-b) + o\big(|(x,y)-(a,b)|\big)$$

as $(x,y)\to(a,b)$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - c_1(x-a) - c_2(y-b)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

$F(x,y) = \big(u(x,y),\, v(x,y)\big) \;:\; \mathbb{R}^2 \to \mathbb{R}^2$

$u(x,y) = \; \cdot \;\; - \;\; -$

$v(x,y) = \; - \;\; - \; \cdot$

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} + \frac{\partial(u,v)}{\partial(x,y)}(a,b) \begin{bmatrix} x-a \\ y-b \end{bmatrix} + o\big(|(x,y)-(a,b)|\big)$$

C^1 : partial derivatives exist and are continuous

$\Downarrow$

differentiable

$$f(x,y) = \log(1+x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{2y}{1+x^2+y^2} \qquad \text{continuous.}$$

$\Rightarrow$ f is C^1 on $\mathbb{R}^2$.

$\Rightarrow$ f is differentiable on $\mathbb{R}^2$

__Proof__ ($C^1 \Rightarrow$ differentiable)

$$f(x,y) - f(a,b) = f(x,y) - f(a,y) + \underline{f(a,y) - f(a,b)}$$

$$\begin{array}{ccc} (a,y) & \longrightarrow & (x,y) \\ \uparrow & & \\ (a,b) & & \end{array}$$

$$= \frac{\partial f}{\partial x}(c_1,y)\,(x-a) + \frac{\partial f}{\partial y}(a,c_2)\cdot(y-b)$$

$\exists c_1 \in (a,x) \cup (x,a)$ $\quad \exists c_2 \in (b,y) \cup (y,b)$

$$\boxed{\lim_{(x,y)\to(a,b)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(a,b)}$$

$\Leftrightarrow$ $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(a,b) + o(1)$ as $(x,y)\to(a,b)$

$$= \left(\frac{\partial f}{\partial x}(a,b) + o(1)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b) + o(1)\right)(y-b)$$

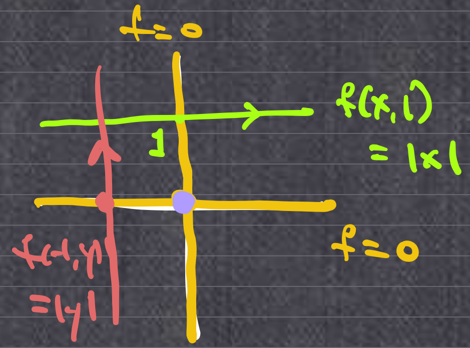
$$= \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) + o(\underbrace{|x-a|}_{\le \sqrt{(x-a)^2+(y-b)^2}}) + o(|y-b|).$$

$$= o(|(x,y)-(a,b)|).$$

differentiable $\not\Longrightarrow C^1$.

$f(x,y) = |xy|$.

$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$



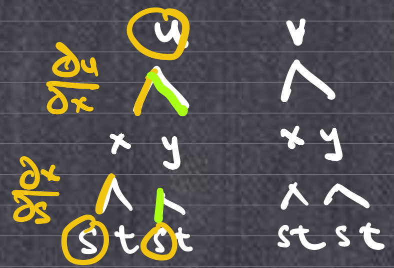
$G : \mathbb{R}^m \to \mathbb{R}^n$

$F : \mathbb{R}^n \to \mathbb{R}^k$

$F \circ G$

$(u,v) = F(x,y)$

$(x,y) = G(s,t)$



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

$$D(F \circ G)(a,b) = DF(G(a,b)) \cdot DG(a,b)$$

$$\frac{\partial(u,v)}{\partial(s,t)}(a,b) = \frac{\partial(u,v)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(s,t)}$$

$$\begin{array}{c} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \\ \begin{matrix} \uparrow \quad & \quad \uparrow \\ \frac{\partial}{\partial s} \quad & \quad \frac{\partial}{\partial t} \end{matrix} \end{array} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$F\circ G(\vec{x}) \qquad \vec{y} = G(\vec{x})$$

$$F(\vec{y}).$$

$$G(\vec{x}) = G(\vec{a}) + DG(\vec{a})(\vec{x}-\vec{a}) + o(|\vec{x}-\vec{a}|) \quad \text{as } \vec{x}\to\vec{a}$$

$$F(\vec{y}) = F(G(a)) + DF(G(a))(\vec{y}-G(a)) + o(|\vec{y}-G(a)|) \quad \text{as } \vec{y}\to G(a)$$

$$\Rightarrow F(G(x)) = F(G(a)) + DF(G(a))(G(x)-G(a)) + o(|G(x)-G(a)|)$$

$$= F(G(a)) + DF(G(a))\cdot\left(DG(a)\cdot(x-a) + o(|x-a|)\right) + o(|G(x)-G(a)|)$$

$$= F(G(a)) + DF(G(a))\cdot DG(a)\,(x-a) + DF(G(a))\cdot o(|x-a|) + o(|G(x)-G(a)|)$$

$$D_{\vec{u}} f(a,b) = \lim_{t\to 0}\frac{f(a+tu_1, b+tu_2) - f(a,b)}{t}$$

$$\vec{u} = (u_1, u_2) \qquad = \lim_{t\to 0}\frac{F(t)-F(0)}{t}$$

$$F(t) := f(\underbrace{a+tu_1}_{x}, \underbrace{b+tu_2}_{y}) \qquad = F'(0).$$

$F = f(\cdot,\cdot)$

$$\begin{array}{cc} x & y \\ | & | \\ t & t \end{array}$$

$$F' = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2$$

$$F'(0) = \frac{\partial f}{\partial x}(a,b)\, u_1 + \frac{\partial f}{\partial y}(a,b)\, u_2$$

$$= \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right) \cdot (u_1, u_2)$$

§ 3.2:

$f: \mathbb{R}^n \to \mathbb{R}$ is k-times diff^{ble} at $\vec{a}$

$\overset{\text{def}}{\iff}$ ① k-th order partial derivatives exist at $\vec{a}$

② $(k-1)$-th partial derivatives are differentiable at $\vec{a}$

e.g. $f_{x_1 \cdots x_{k-1}}(\vec{x}) = f_{x_1 \cdots x_{k-1}}(\vec{a})$

$+ \sum_j \frac{\partial}{\partial x_j} f_{x_1 \cdots x_{k-1}}(\vec{a}) \cdot (x_j - a_j)$

$+ o(|\vec{x} - \vec{a}|)$ as $\vec{x} \to \vec{a}$.

$C^1 \Rightarrow$ differentiable

$(k-1)$-th partials C^1 $\Rightarrow$ ② ✓

C^k : k-th order partials are continuous

$C^k \Rightarrow$ k-order differentiable.

$f(x,y) = \log(1+x^2+y^2)$ is $C^\infty = \bigcap_{k=1}^{\infty} C^k$.

$\frac{\partial f}{\partial x} = \frac{2x}{1+x^2+y^2}$. $\frac{\partial f}{\partial y} = \frac{2y}{1+x^2+y^2}$

$\frac{\partial^k f}{\partial \cdots} = \frac{?}{(1+x^2+y^2)^?}$

If $f(x,y)$ is C^2, then $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

<u>Proof</u>: $(a,b) \in \mathbb{R}^2$ fixed

$$\Delta(h,k) = f(a+h, b+k) - f(a+h, b) - \left(f(a, b+k) - f(a,b)\right)$$

$\forall h,k \neq 0.$

$g(x,y) = f(a+x, y) - f(a,y)$

$$= g(h, b+k) - g(h, b)$$

$$= \frac{\partial g}{\partial y}(h, c_1) \cdot k$$

$c_1 \in (b, b+k)$

$c_2 \in (a, a+h)$

$$= \left(\frac{\partial f}{\partial y}(a+h, c_1) - \frac{\partial f}{\partial y}(a, c_1)\right) k$$

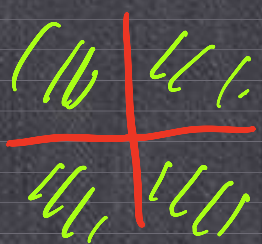
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Big|_{(c_2, c_1)} \cdot hk$$

$$\Delta(h,k) = f(a+h, b+k) - f(a+h, b) - \left(f(a, b+k) - f(a,b)\right)$$

$$= \cdots = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\Big|_{(c_3, c_4)} hk.$$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}_{\text{cts}}\Big|_{(c_2, c_1)} = \underline{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}\Big|_{(c_3, c_4)}$$

$\downarrow \quad \downarrow \; (h,k) \to (0,0) \quad \downarrow$



$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a,b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a,b)$$

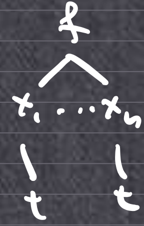
Given $f(\vec{x})$ is twice diff' at $\vec{a}$

$\overset{\shortparallel}{(x_1, \dots, x_n)}$

$$g(t) = f(\underbrace{(1-t)\vec{a} + t\vec{x}})$$

$\underbrace{(1-t)\vec{a} + t\vec{x}}$

$$g'(t) = \sum_j \frac{\partial f}{\partial x_j}((1-t)\vec{a} + t\vec{x}) \cdot \underbrace{\frac{dx_j}{dt}}_{(x_j - a_j)}$$



$$\frac{d}{dt}\underbrace{\begin{pmatrix}(1-t)a_j \\ +tx_j\end{pmatrix}} = x_j - a_j$$

$$g'(0) = \sum_j \frac{\partial f}{\partial x_j}(\vec{a}) \cdot (x_j - a_j)$$

$$g''(0) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})\,(x_i - a_i)(x_j - a_j)$$

$$g(t) = g(0) + g'(0)\,t + \frac{g''(0)}{2!}t^2 + ?$$

error term $= f(\vec{x}) - \Big(f(\vec{a}) + \sum_i \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i)$

$\qquad + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})\,(x_i - a_i)(x_j - a_j)\Big)$

$$= g(1) - \left(g(0) + g'(0)\cdot 1 + \frac{g''(0)}{2!}\cdot 1^2\right)$$

let $E(t) = g(t) - \left(g(0) + g'(0)\,t + \frac{g''(0)}{2!}\cdot t^2\right)$

$= o(|x-a|^2)$?

$$\frac{E(1) - \cancel{E(0)}}{1^2 - 0^2} = \frac{E'(c)}{2c} = \frac{1}{2c}\left(g'(c) - g'(0) - g''(0)\,c\right)$$

$$E(1) = \frac{1}{2c}\left(\underline{g'(c)} - \underline{g'(0)}\right) - \underline{\frac{1}{2}g''(0)}$$

$$= \frac{1}{2c}\sum_i \left(\boxed{\underline{\frac{\partial f}{\partial x_i}\left((1-c)\vec{a} + c\vec{x}\right)}} - \underline{\frac{\partial f}{\partial x_i}(\vec{a})} \right)(x_i - a_i)$$

$$\underline{- \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \cdot (x_i - a_i)(x_j - a_j)}$$

$$= \frac{1}{2\cancel{c}}\sum_i \Bigg(\cancel{\frac{\partial f}{\partial x_i}(\vec{a})} + \cancel{\sum_j \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}(\vec{a})} \underbrace{\left(\cancel{(1}-c)a_j + cx_j - \cancel{a_j}\right)}_{\cancel{c}(x_j - a_j)}$$

$$+ \, o\left(\underbrace{\left|\cancel{(1}-c)\vec{a} + c\vec{x} - \cancel{\vec{a}}\right|}_{= o(|\vec{x} - \vec{a}|)}\right)$$

$$- \cancel{\frac{\partial f}{\partial x_i}(\vec{a})} \Bigg) \cdot (x_i - a_i)$$

$$\cancel{- \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \cdot (x_i - a_i)(x_j - a_j)}$$

$f(0,0)$ $\quad \nabla f(0,0) = \vec{0}$

$$\underline{\frac{\partial f}{\partial x}(0,0)} = \underline{\frac{\partial f}{\partial y}(0,0) = 0}.$$

$$f(x,y) = f(0,0) + \cancel{f_x(0,0)} \cdot x + \cancel{f_y(0,0)} \cdot y$$

$$+ \frac{1}{2!}\left(f_{xx}(0,0)\,x^2 + 2f_{xy}(0,0)\,xy + f_{yy}(0,0)\,y^2\right)$$

$Q(x,y)$

$$+\, o\left(|(x,y)|^2 \right)$$

$$\parallel$$

$$[x\ y]\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{If } \begin{cases} \left(2 f_{xy}(0,0)\right)^2 - 4 f_{xx}(0,0) f_{yy}(0,0) < 0 \\ f_{xx}(0,0) > 0 \end{cases}$$

$$\Longrightarrow \quad \text{then } Q(x,y) \geq 0 \quad (= \text{ holds iff } (x,y) = (0,0))$$

If $\quad F \circ G = id \quad$ and $\quad G \circ F = id$

then $\quad \underbrace{D(F \circ G)}_{DF \cdot DG} = D(id) = I$

$\underbrace{D(G \circ F)}_{DG \cdot DF} = D(id) = I$

$$\frac{\partial(x_1, \dots, x_n)}{\partial(x_1, \dots, x_n)} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$\Rightarrow \quad DG = (DF)^{-1}$

$\therefore \quad F$ bijective $\Longrightarrow$ DF is invertible

and $D(F^{-1}) = (DF)^{-1}$.

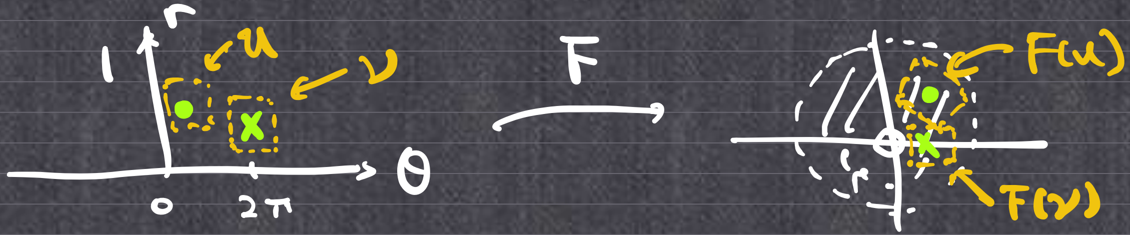
$$F : (0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$

$$(r, \theta) \mapsto (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$$

$$DF = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{array}{c} \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} \\ \uparrow \qquad \uparrow \\ \frac{\partial}{\partial r} \qquad \frac{\partial}{\partial \theta} \end{array} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det DF = r > 0$

but F is not injective.



e.g. 3.25

$$(*)\ \begin{cases} x - y^2 = a \\ x^2 + y + y^3 = b \end{cases}$$

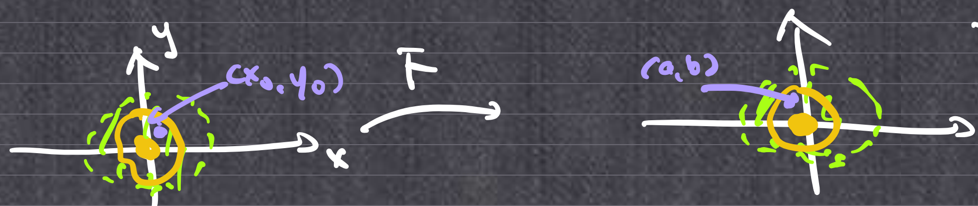
Claim: when (a,b) small then $(*)$ has a solution.

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = (x - y^2,\ x^2 + y + y^3)$

$F(0,0) = (0,0)$, $DF = \begin{bmatrix} 1 & -2y \\ 2x & 1+3y^2 \end{bmatrix}$

$\Rightarrow DF(0,0) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{invertible}}$



e.g. 3.26

$F: (0,\infty) \times (0,\pi) \times (0, 2\pi) \to \mathbb{R}^3$

$(\rho,\ \phi,\ \theta) \longmapsto (\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho\cos\phi)$

$$\det \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi \neq 0$$

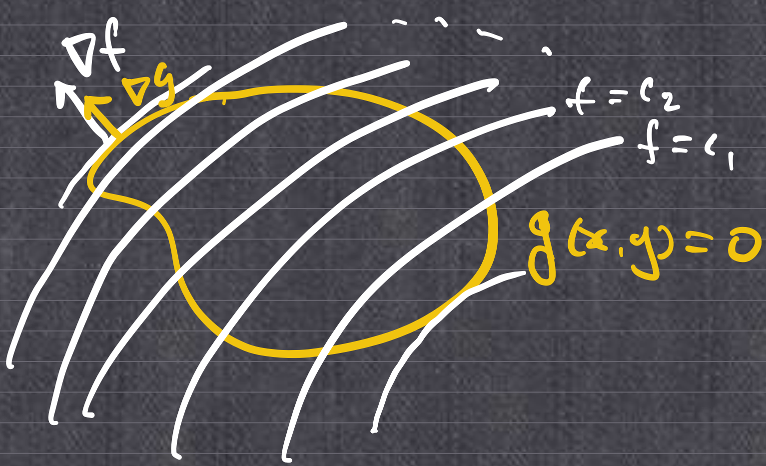
$\Rightarrow F^{-1}$ locally exists and $\underbrace{C^\infty}_{(\because\, F \text{ is } C^\infty)}$

$\underline{3.3.4}$ Lagrange's multiplier.

optimize: $f(x,y)$

subject to: $g(x,y) = 0$

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0. \end{cases}$ $\Longleftrightarrow$ $\{\nabla f, \nabla g\}$ are linearly dependent.



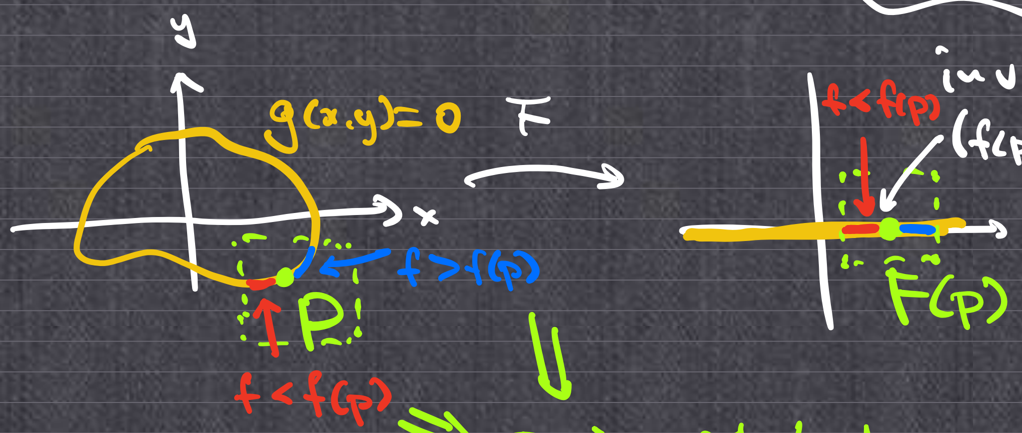
Claim: $\{\nabla f(p), \nabla g(p)\}$ are linearly indep.

$\Rightarrow$ p is not max/min for f under constraint $\{g=0\}$.

Proof:

$$F(x,y) = (f(x,y), g(x,y))$$

$$DF(p) \begin{bmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \\ \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y}(p) \end{bmatrix} = \underbrace{\begin{bmatrix} - \nabla f(p) - \\ - \nabla g(p) - \end{bmatrix}}_{\text{invertible}}$$



$\Rightarrow$ p is not local max/min for f under $\{g=0\}$.

Banach Contraction Mapping

(X, d) complete metric space.

$f: X \to X$. given $\exists\, \alpha \in (0,1)$

s.t. $d(f(x), f(y)) \leq \alpha\, d(x, y)$.

$\forall x, y \in X$.

$\Longrightarrow$ then $\exists!\ x_0 \in X$ s.t. $f(x_0) = x_0$.

$f \circ f$

Proof: Let $x_1 \in X$ any point.

$x_n := f(x_{n-1}) \quad \forall n \geq 2$

$$\underline{d(x_{n+1}, x_n)} = d(f(x_n), f(x_{n-1}))$$
$$\leq \alpha\, d(x_n, x_{n-1})$$
$$\leq \alpha^2\, d(x_{n-1}, x_{n-2})$$
$$\leq \cdots \leq \alpha^{n-1}\, \underline{d(x_2, x_1)}$$

$\alpha \in (0,1)$ assume $\neq 0$

$$\underset{m \geq n \geq N}{d(x_m, x_n)} \leq \cdots \leq O(\alpha^N)_{\to 0}$$

$\{x_n\}$ is Cauchy $\overset{\text{complete}}{\Longrightarrow}$ $\boxed{x_n \to x_0}$
$\exists\, x_0 \in X$

$$\begin{array}{ccc} x_n & = & \underbrace{f(x_{n-1})} \\ \downarrow & & \downarrow \ f \text{ cts} \\ x_0 & = & f(x_0). \end{array}$$

$$(*) \begin{cases} y'(x) = F(y(x)) \\ y(0) = 1 \end{cases} \qquad \begin{cases} y'(x) = \underline{\sin(y^2(x)+1)} \\ y(0) = 1 \end{cases}$$

$F(y) = \sin(y^2+1)$

$$\Longleftrightarrow \quad y(x) = 1 + \int_0^x F(y(t))\,dt$$

chosen later

$$\Phi : C[0,s] \longrightarrow C[0,s]$$

$$\Phi(f) := 1 + \int_0^x F(f(t))\,dt$$

$$\|\Phi(f) - \Phi(g)\|_\infty$$

$$= \left\| \int_0^x F(f(t))\,dt - \int_0^x F(g(t))\,dt \right\|_\infty$$

$$= \sup_{x \in [0,s]} \left| \int_0^x F(f(t)) - F(g(t)) \, dt \right|$$

$$\leq \sup_{x \in [0,s]} \int_0^x | F(f(t)) - F(g(t)) | \, dt$$

$$\leq \sup_{x \in [0,s]} \int_0^x L \underbrace{|f(t) - g(t)|} dt$$

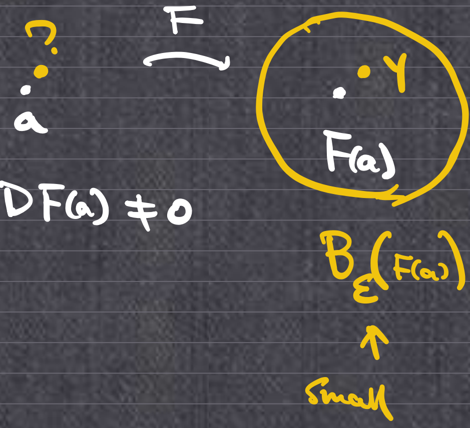
Assume $|F(y_1) - F(y_2)| \leq L |y_1 - y_2|$

$$\leq L \sup_{x \in [0,s]} \int_0^x \|f - g\|_\infty \, dt$$

$$= \sup_{[0,s]} L \|f - g\|_\infty \cdot x \leq Ls \|f - g\|_\infty$$

Step ①

Want: $\forall y \sim F(a)$
show $\exists x \sim a$
st. $F(x) = y$.



$\det DF(a) \neq 0$

$$T_y : \overline{B_\delta(a)} \longrightarrow \mathbb{R}^n$$

$$T_y(x) := x + DF(a)^{-1}\left(y - F(x)\right)$$

★ If $T_y(x) = x$ $\exists x$,

then $\cancel{x+}\ \underbrace{DF(a)^{-1}}\ \underbrace{(y - F(x))} = \cancel{x}\ 0$

$\Rightarrow y = F(x)$.

Next: prove T_y is a contraction.

$$\|T_y(x_1) - T_y(x_2)\|_2$$

$$= \| x_1 + DF(a)^{-1}(\cancel{y} - F(x_1)) - x_2 - DF(a)^{-1}(\cancel{y} - F(x_2)) \|$$

$$= \| (x_1 - x_2) - DF(a)^{-1}\left(F(x_1) - F(x_2)\right) \|$$

$$= \| DF(a)^{-1}\left((DF(a)x_1 - F(x_1)) - (DF(a)x_2 - F(x_2))\right) \|$$

$$\leq \|DF(a)^{-1}\| \, \| G(x_1) - G(x_2) \|$$

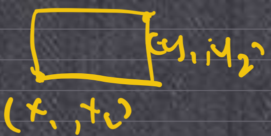
$\vdots$

$$G(x) := DF(a)\, x - F(x)$$

$$\le \|DF(a)^{-1}\| \cdot C \sup_{B_\delta(a)} \|DG\| \; \| x_1 - x_2 \|$$

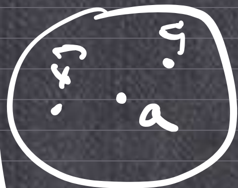
$$|f(x_1, x_2) - f(y_1, y_2)|$$

$$= |\underbrace{f(x_1, x_2) - f(x_1, y_2)} + f(x_1, y_2) - f(y_1, y_2)|$$



$$= \left| \frac{\partial f}{\partial y}\Big|_? (x_2 - y_2) + \frac{\partial f}{\partial x}\Big|_? (x_1 - y_1) \right|$$

$$\le \sup_{B_\varepsilon(?)} \| Df \|_2 \; \|(x_1, x_2) - (y_1, y_2)\|$$



$$= C \|DF(a)^{-1}\| \underbrace{\sup_{z \in \overline{B_\delta(a)}} \| DF(a) - DF(z) \|} \cdot \|x_1 - x_2\|$$

F is C^1 $\Rightarrow$ $DF(x)$ is C^0

$\Rightarrow \exists \delta$ s.t. $\displaystyle \sup_{z \in \overline{B_\delta(a)}} \| DF(a) - DF(z) \| \le \frac{1}{2C\|DF(a)^{-1}\|}$

$\exists \delta > 0$ s.t.

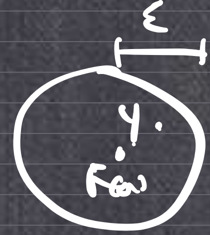
$T_y : \overline{B_\delta(a)} \longrightarrow \mathbb{R}^n$

is contraction:

$$\| T_y(x_1) - T_y(x_2) \| \le \frac{1}{2} \|x_1 - x_2\|.$$

Next: choose smaller δ', ε s.t.

$T_y : \overline{B_{\delta'}(a)} \to \overline{B_{\delta'}(a)}$ when



$$\|T_y(x) - a\|$$

$$= \|T_y(x) - T_y(a)\| + \|T_y(a) - a\|$$

$$\le \frac{1}{2}\|x-a\| + \|\underbrace{\cancel{a} + DF(a)^{-1}(y - F(a))}_{T_y(a)} \cancel{-a}\|$$

$$\le \frac{1}{2}\|x-a\| + \|DF(a)^{-1}\| \underbrace{\|y - F(a)\|}_{<\varepsilon}.$$

$$\le \frac{1}{2}\delta' + \underbrace{\varepsilon\|DF(a)^{-1}\|}_{<\frac{1}{2}\delta'} < \delta'$$

<u>Step ④</u>: F^{-1} is differentiable at $\underbrace{F(x_0)}_{y_0}$

$$F^{-1}(y) - F^{-1}(y_0) - DF(x_0)^{-1}(y - y_0)$$

$$= F^{-1}(F(x)) - F^{-1}(F(x_0)) - DF(x_0)^{-1}(\underbrace{F(x) - F(x_0)})$$

as $y \to y_0$
$\Leftrightarrow x \to x_0$
$y = F(x)$

$$= F^{-1}(F(x)) - F^{-1}(F(x_0)) - DF(x_0)^{-1}\left(DF(x_0)(x - x_0) + E(x)\right)$$

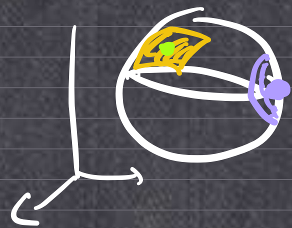
$$= \cancel{F^{-1}(f(x)) - F^{-1}(f(x_0))}$$

$$\cancel{\sim (x - x_0)} + \underbrace{\tilde{E}(x)}_{\in o(|x-x_0|)} = o(|y-y_0|)$$

$E(x) \in o(|x-x_0|)$ as $x \to x_0$

$|x_1 - x_2| \overset{?}{\leq} |y_1 - y_2|$

$f(x,y,z) = 0 \longrightarrow$



If $\frac{\partial f}{\partial x}(p) \neq 0 \Rightarrow$ near p, $x = g(y,z)$.

$\vdots$

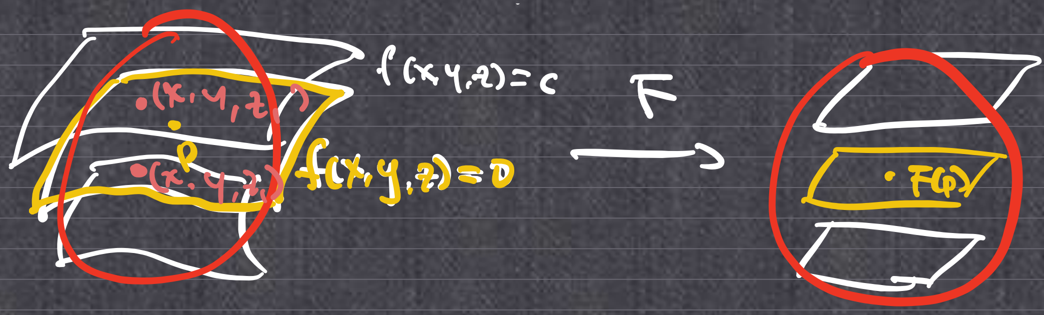
$\frac{\partial f}{\partial z}(p) \neq 0 \Rightarrow$ near p, $z = h(x,y)$.

Given: $f(x,y,z)$ is C^1, consider $f(x,y,z) = 0$

$\exists p$ s.t. $\frac{\partial f}{\partial z}(p) \neq 0$

Consider $F(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^3$

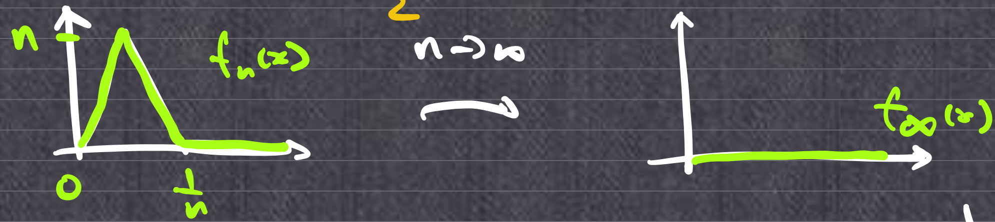
$$F(x,y,z) = \begin{bmatrix} x \\ y \\ f(x,y,z) \end{bmatrix}$$



$$DF(x,y,z)_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}_p = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & \underset{\uparrow}{\overset{}{*}}_{\neq 0} \end{bmatrix}}_{\text{invertible}}$$

§4 : Uniform convergence.

$$\frac{1}{2} = \lim_{n\to\infty} \underbrace{\int_0^1 f_n(x)\,dx}_{=\frac{1}{2}} \neq \int_0^1 \underbrace{\lim_{n\to\infty} f_n(x)}_{=0}\,dx = 0$$



if $x>0$, $f_n(x)=0 \quad \forall n > \frac{1}{x}$.

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

if $x=0$, $f_n(0) = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

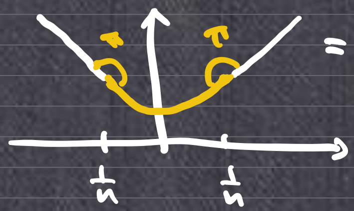
- $$\lim_{x\to a}\lim_{n\to\infty} f_n(x) \overset{?}{=} \lim_{n\to\infty}\lim_{x\to a} f_n(x)$$

Let $f_n(x) = x^n$

$$\lim_{x\to 1^-}\lim_{n\to\infty} f_n(x) = \lim_{x\to 1^-} \underbrace{\overset{=0}{\lim_{n\to\infty} x^n}}_{x<1} = 0$$

$$\lim_{n\to\infty}\lim_{x\to 1^-} f_n(x) = \lim_{n\to\infty} \underbrace{\lim_{x\to 1^-} x^n}_{=1} = 1$$

- $$\lim_{n\to\infty} \frac{d}{dx} f_n(x) \neq \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$$

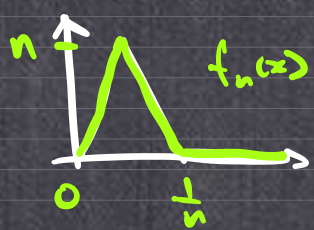


$= |x|$ if $x \gtrsim 0$

- $f_n : D \subset \mathbb{R} \to \mathbb{R}$ is said to converge uniformly to g on $E \subset D$

$\overset{\text{def}}{\iff}$ $\boxed{\lim_{n\to\infty} \|f_n - g\|_E = 0}$

$$\boxed{\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - g(x)| = 0}$$

- 

$E = [0, 1]$

if $\|f_n - f\|_E \xrightarrow{\text{as } n\to\infty} 0$

then $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$.

Proof: $|f_n(x) - f(x)| \le \|f_n - f\|_E$ $\forall x \in E$

$$\|f_n - 0\|_{[0,1]} = \sup_{x\in[0,1]} |f_n(x)| = \sup_{x\in[0,1]} n$$

$$= n \to \infty \quad \text{as } n \to \infty.$$

$\therefore$ f_n does not converges uniformly to 0 on $[0,1]$

However. $\sup_{x\in[\frac{1}{2},1]} \underbrace{|f_n(x) - \cancel{f}(x)|}_{=0} = 0$ if $n > 2$ (with $f = 0$)

f_n <u>converges uniformly</u> to 0 on $[\frac{1}{2}, 1]$

$\boxed{n \geq N}$

$f_n \rightrightarrows f$ on E

uniform convergence on E.

$$\lim_{n\to\infty} \underbrace{\sup_{x\in E} |f_n(x) - f(x)|} = 0$$

$\Leftrightarrow \forall \varepsilon > 0, \ \exists \underbrace{N}_{\text{indep. of } x}^{=N(\varepsilon)} > 0$ s.t. $n \geq N$

$\Rightarrow \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$

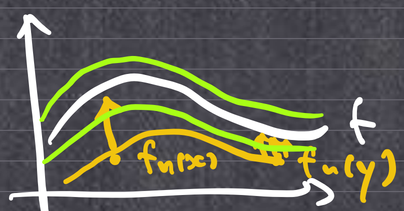
$\Rightarrow \forall x \in E, \ |f_n(x) - f(x)| < \varepsilon$

$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in E$ (pointwise convergence)

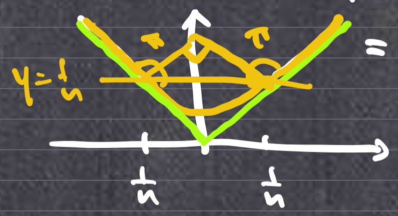
$N = N(\varepsilon, x)$

$\Longleftrightarrow \forall \varepsilon > 0, \ \forall x \in E, \ \exists N > 0$ s.t. $n \geq N$

$|f_n(x) - f(x)| < \varepsilon.$



- $f_n(x) \longrightarrow |x|$ as $n \to \infty$



$= |x|$ if $x \gtrsim 0$

$$\sup_{x\in\mathbb{R}} \underbrace{\left|f_n(x) - f(x)\right|}_{< \frac{1}{n}} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

$$f_n \rightrightarrows f = |x| \quad \text{on } \mathbb{R}.$$

$L(x) := \sum_{n=1}^{\infty} g_n(x)$ converges uniformly on E

$\iff \lim_{k\to\infty} \underbrace{\sum_{n=1}^{k} g_n(x)}_{\text{seq. of functions}}$ converges uniformly $L(x)$ on E.

Weierstrass M-test:

If $\exists$ sequence of real number $\{M_n\}$ s.t.

$$\begin{cases} ① & |g_n(x)| \le M_n \quad \forall n \in \mathbb{N},\ \forall x \in E \\ ② & \sum_{n=1}^{\infty} M_n \text{ converges (as a series of real number)} \end{cases}$$

Then: $\sum_{n=1}^{\infty} g_n$ converges uniformly on E.

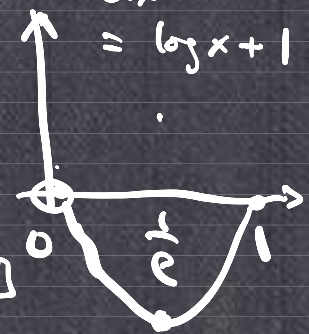
Ex: $$\sum_{k=1}^{\infty} \underbrace{\frac{\sin(k^{|x|}x^2)}{2^k}}_{g_k(x)}$$

- $|g_k(x)| \le \underbrace{\frac{1}{2^k}}_{M_k} \quad \forall x \in \mathbb{R}$
- $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$

$\xRightarrow{\text{Weierstrass M-test}}$ $\sum_{k=1}^{\infty} \frac{\sin(k^{|x|}x^2)}{2^k}$ converges uniformly on $\mathbb{R}$

- $\sum_{k=1}^{\infty} (x \log x)^k \qquad x \in [0,1].$

$\frac{d}{dx} x\log x = \log x + 1$



- $|(x\log x)^k| \le \left|\left(\frac{1}{e}\log\frac{1}{e}\right)^k\right| = \left(\frac{1}{e}\right)^k \quad \forall x \in [0,1]$
- $\sum \left(\frac{1}{e}\right)^k$ converges

$\Rightarrow \quad \sum_{k=1}^{\infty} (x\log x)^k$ converges uniformly on $[0,1]$.

Proof of Weierstrass's M-test

$$\begin{cases} |g_n(x)| \le M_n & \forall x \in E,\ n \in \mathbb{N}. \\ \sum M_n < \infty \end{cases}$$

- $\sum_{n=1}^{\infty} g_n(x)$ is defined $\forall x \in E$.

Proof: $\left| \sum_{n=1}^{m} g_n(x) - \sum_{n=1}^{k} g_n(x) \right| \qquad m > k$

$$= \left| \sum_{n=k+1}^{m} g_n(x) \right| \le \sum_{n=k+1}^{m} \underbrace{|g_n(x)|}_{\le M_n} \le \sum_{n=k+1}^{m} M_n$$

$$= \left| \underline{\sum_{n=1}^{m} M_n} - \sum_{n=1}^{k} M_n \right|$$

- uniformly converge:

$$\sup_{x \in E} \left| \sum_{n=1}^{m} g_n(x) - \sum_{n=1}^{k} g_n(x) \right|$$

$$\le \sup_{x \in E} \cdot - \cdots \le \sup_{x \in E} \left| \sum_{n=1}^{m} M_n - \sum_{n=1}^{k} M_n \right|$$

$$= \left| \sum_{n=1}^{m} M_n - \sum_{n=1}^{k} M_n \right|$$

$f_n \rightrightarrows f$ on $[a,b]$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \overset{= f}{\underline{\lim_{n\to\infty} f_n(x)}}\,dx$$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$\leq \int_a^b \underbrace{|f_n(x) - f(x)|}_{\leq \|f_n - f\|_{[a,b]}}\,dx \leq \int_a^b \|f_n - f\|_{[a,b]}\,dx = \underbrace{\|f_n - f\|_{[a,b]}(b-a)}_{\downarrow\ 0}$$

- $$\int_0^1 \frac{1}{x^x}\,dx = \int_0^1 x^{-x}\,dx = \int_0^1 e^{\log x^{-x}}\,dx$$

$$= \int_0^1 e^{-x\log x}\,dx \qquad e^y = \sum \frac{y^n}{n!}$$

$$= \int_0^1 \sum_{n=0}^{\infty} \underbrace{\frac{(-x\log x)^n}{n!}}\,dx$$

$$\left| \frac{(-x\log x)^n}{n!} \right| \leq \frac{\left(\frac{1}{e}\right)^n}{n!} \leq \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \underbrace{\int_0^1 \frac{(-x\log x)^n}{n!}\,dx} \qquad \sum \frac{1}{n!} \text{ converges.}$$

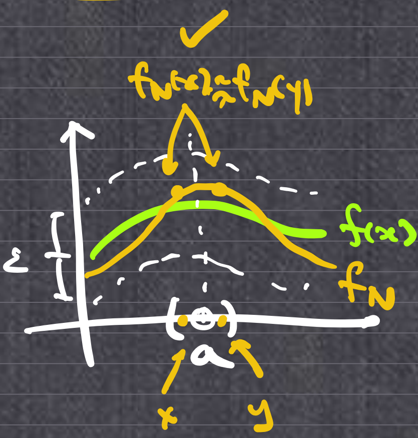
$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^n}$$

Given: $\begin{cases} f_n \rightrightarrows f \text{ on } D, & a \in D' \\ \lim\limits_{x\to a} f_n(x) \text{ exists } \forall n. \end{cases}$

Want: $\underbrace{\lim\limits_{x\to a} \lim\limits_{n\to\infty} f_n(x)}_{f(x)} = \underbrace{\lim\limits_{n\to\infty} \lim\limits_{x\to a} f_n(x)}_{\checkmark}$

Claim. $\lim\limits_{x\to a} f(x)$ exists



$$\boxed{\begin{aligned} &|f(x) - f(y)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| \\ &\quad + |f_N(y) - f(y)| \end{aligned}}$$

$\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t. $n \ge N$

$\Rightarrow \quad \|f_n - f\|_D < \varepsilon$

$\Rightarrow \forall x \in D, \quad |f_n(x) - f(x)| < \varepsilon$
$\forall n \ge N.$

$\lim\limits_{x\to a} |f_n(x) - f(x)| \le \varepsilon$

$\Rightarrow \left| \underbrace{\lim\limits_{x\to a} f_n(x)}_{\downarrow} - \underbrace{\lim\limits_{x\to a} f(x)}_{\downarrow} \right| \le \varepsilon \quad \forall n \ge N$

$\Rightarrow \left| \underbrace{\lim\limits_{x\to a} f_n(x)}_{\text{seq. of number}} - L \right| \le \varepsilon. \quad \forall n \ge N.$

$\lim\limits_{n\to\infty} \left(\lim\limits_{x\to a} f_n(x) \right) = \lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \lim\limits_{n\to\infty} f_n(x)$

Cor: if further f_n is continuous at $x=a$.

then
$$\lim_{x\to a}\underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)} = \lim_{n\to\infty}\underbrace{\lim_{x\to a} f_n(x)}_{f_n(a)}$$

$$\underbrace{\lim_{x\to a} f(x)}_{} \qquad \underbrace{\lim_{n\to\infty} f_n(a)}_{} = f(a)$$

e.g. $\quad f_n(x) = x^n \quad$ on $[0,1]$.

$$\zeta(x) = \sum_{n=1}^{\infty}\frac{1}{n^x} \quad \text{converges when} \quad x>1$$

$\forall \varepsilon>0$, $\quad \zeta(x) = \sum_{n=1}^{\infty}\frac{1}{n^x}$ converges uniformly on $(1+\varepsilon,\infty)$.

$$\left\{\begin{array}{l} \left|\frac{1}{n^x}\right| = \frac{1}{n^x} \le \frac{1}{n^{1+\varepsilon}} \quad \text{on } x\in(1+\varepsilon,\infty) \\ \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} \text{ converges} \end{array}\right.$$

$\xRightarrow{\text{Weierstrass M-test}}$ $\quad \zeta(x) = \sum_{n=1}^{\infty}\underbrace{\frac{1}{n^x}}_{\text{continuous}}$ converges uniformly on $(1+\varepsilon,\infty)$.

$\zeta(x)$ is continuous on $(1+\varepsilon, \infty)$ $\forall \varepsilon > 0$.

uniform con. on $(1, \infty)$ $\not\Downarrow$

$\Downarrow$

$\zeta(x)$ is continuous on $\bigcup_{\varepsilon>0} (1+\varepsilon, \infty) = (1, \infty)$.

$\sum_{n=0}^{\infty} a_n x^n$ <u>converge</u> on $(-R, R)$

$\Rightarrow$ converges at $x = -R+\varepsilon$ and $x = R-\varepsilon$.

$\overset{?}{\Longrightarrow}$ uniform converge on $[-R+2\varepsilon, R-2\varepsilon]$.

$$\begin{cases} \cdot\ f_n \to f & \text{on } (a,b) \\ \cdot\ f_n' \rightrightarrows g & \text{on } (a,b) \end{cases} \qquad x_0 \in (a,b)$$

$$\Longrightarrow \underbrace{\lim_{n\to\infty} \frac{d}{dx}\Big|_{x=x_0} f_n(x)}_{=\lim_{n\to\infty} f_n'(x_0) = g(x_0)} = \frac{d}{dx}\Big|_{x=x_0} \lim_{n\to\infty} f_n(x) \quad = \frac{d}{dx}\Big|_{x=x_0} f(x)$$

<u>Proof</u>:

$$\text{LHS} = \lim_{n\to\infty} \lim_{x\to x_0} \frac{f_n(x) - f_n(x_0)}{x - x_0}$$

$$\text{RHS} = \lim_{x\to x_0} \frac{\lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \lim_{n \to \infty} \underbrace{\frac{f_n(x) - f_n(x_0)}{x - x_0}}_{g_n(x)}$$

Need: $g_n \rightrightarrows ?$ on $(a,b) \setminus \{x_0\}$.

$$|g_n(x) - g_m(x)| = \left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right|$$

$$= \frac{\left| \overset{g(x)}{(f_n(x) - f_m(x))} - \overset{g(x_0)}{(f_n(x_0) - f_m(x_0))} \right|}{|x - x_0|}$$

$$= \frac{|g'(c) \cancel{(x - x_0)}|}{\cancel{|x - x_0|}} = |f_n'(c) - f_m'(c)|$$

$$\leq \| f_n' - f_m' \|_{(a,b) \setminus \{x_0\}}.$$

e.g.

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \qquad \boxed{x > 1}$$

$$\sum_{n=1}^{\infty} \frac{d}{dx} \frac{1}{n^x}$$

$$= \sum_{n=1}^{\infty} - \frac{1}{(n^x)^2} \frac{d}{dx} n^x$$

$$= \sum_{n=1}^{\infty} \left(- \frac{1}{n^{2x}} \cdot n^x \log n \right)$$

$$= - \sum_{n=1}^{\infty} \frac{1}{n^x} \log n \qquad x \in (1+\varepsilon, \infty)$$

$$\left|\frac{\log n}{n^x}\right| \le \frac{\log n}{n^{1+\varepsilon}} \le \frac{n^{\varepsilon/2}}{n^{1+\varepsilon}} = \frac{1}{n^{1+\varepsilon/2}}$$

$\log n \le n^{\varepsilon/2}$ if $n >> 1$

$\forall x \in (1+\varepsilon, \infty)$

$\sum \frac{1}{n^{1+\varepsilon/2}}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{\log n}{n^x}$ converges uniformly on $(1+\varepsilon, \infty)$

$$\therefore \quad \frac{d}{dx}\sum_{n=1}^{\infty}\frac{1}{n^x} = \sum_{n=1}^{\infty}\frac{d}{dx}\frac{1}{n^x} \quad \text{on } (1+\varepsilon, \infty)$$

$\forall \varepsilon > 0.$

$\Rightarrow$ " = " on $\bigcup_{\varepsilon>0}(1+\varepsilon, \infty)$

$= (1, \infty).$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \quad \text{on} \quad (-R, R)$$

ex: $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ $\forall \varepsilon > 0$

$$\sum_{n=1}^{\infty}\frac{d}{dx}a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} \le 1 \quad \text{on } x \in [-R+\varepsilon, R-\varepsilon]$$

$$\Rightarrow \limsup_{n\to\infty} \underbrace{\sqrt[n]{|n a_n x^{n-1}|}}_{n^{\frac{1}{n}} \to 1} = \limsup_{n\to\infty} n^{\frac{1}{n}} \sqrt[n]{|a_n x^n|} \cdot |x|^{-\frac{1}{n}}$$

§4.2 Arzéla – Ascoli Theorem

$\{f_n\}$ sequence of functions on $[a,b]$.

$\nexists \quad f_{n_k} \rightrightarrows f_\infty$.

- uniformly bounded

  $\exists M > 0$ s.t. $\quad |f_n(x)| \le M \quad \forall n \in \mathbb{N}$, $x \in [a,b]$

  (M) indep. of x, n.

  e.g. $\left\{ \sin\left(\dfrac{n^{\lfloor n \rfloor} \cos(\lceil n^2 \rceil)}{n e^n + e^{n^e} + \sum a^n \cos(b^n \pi x)} \right) \right\}$

- equicontinuity

  $\{f_n\}$ is equicontinuous on $[a,b]$

  $\overset{\text{def}}{\Longleftrightarrow} \quad \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

  $|x-y| < \delta \Rightarrow \quad |f_n(x) - f_n(y)| < \varepsilon$
  $\forall n \in \mathbb{N}$.

  e.g. $\{\sin(\frac{x}{n})\}$

$\forall \varepsilon > 0 \quad \exists \delta = \varepsilon$, then $|x-y| < \delta$

$$\Rightarrow \left| \underbrace{\sin\left(\frac{x}{n}\right)}_{g_n(x) = \sin\frac{x}{n}} - \sin\left(\frac{y}{n}\right) \right| = \left| \underbrace{\frac{1}{n}\cos\frac{c}{n}}_{g_n'(c)} \right| |x-y|$$

$\le \frac{1}{n}|x-y| < \frac{\delta}{n} \le \delta = \epsilon$

non-e.q. $\{\underbrace{nx}_{f_n(x)}\}_{n=1}^{\infty}$

Take $\epsilon_0 = 1$, $\forall \delta > 0$. $\exists x, y$ s.t. $|x-y| = \frac{\delta}{2}$

$\exists n_0 \ge \frac{2}{\delta}$, $|f_n(x) - f_n(y)| = n|x-y| = n\frac{\delta}{2} \ge 1 = \epsilon_0$

- if $|f_n'(x)| \le M$ $\forall n \in [a,b], x \in [a,b]$

$\Rightarrow \{f_n\}$ is equicontinuous on $[a,b]$.

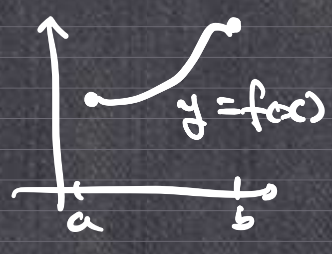
Arzela-Ascoli

if $\{f_n\}$ is ① uniformly bounded on $[a,b]$.

and ② equicontinuous

then $\exists \{f_{n_k}\} \rightrightarrows f_\infty$ on $[a,b]$.

- $L(f) := \int_a^b \sqrt{1+f'(x)^2}\, dx$



- $\inf\limits_{f \in C^1, f(a), f(b)} L(f)$ exists.

$\exists f_n \in C^1[a,b]$ s.t. $L(f_n) \to \inf\limits_{f \in C^1} L(f)$.

e.g. Given $\{f_n\} \in C[a,b]$.

$$|f_n(x)| \le M \quad \forall x \in [a,b],\ n \in \mathbb{N}.$$

$$F_n(x) := \int_a^x f_n(t)\,dt$$

want: $\exists\ F_{n_k} \rightrightarrows F_\infty$ on $[a,b]$.

① $\{F_n\}$ is uniformly bounded on $[a,b]$

Proof:
$$|F_n(x)| = \left|\int_a^x f_n(t)\,dt\right|$$
$$\le \int_a^x |f_n(t)|\,dt$$
$$\le \int_a^x M\,dt = M(x-a)$$
$$\le \underline{M(b-a)}$$

② $\{F_n\}$ is equicontinuous on $[a,b]$.

Proof: $F_n'(x) = f_n(x)$

Ex. 4.28

Given $f_n : [a,b] \to \mathbb{R}$, C^∞

$\forall k \in \mathbb{N} \cup \{0\}$, $\exists C_k > 0$ s.t.

$$\| f_n^{(k)} \|_{[a,b]} \le C_k$$

then show $\exists \{f_{n_j}\}$ s.t. $f_{n_j}^{(k)} \rightrightarrows f_\infty^{(k)}$ on $[a,b]$

$\forall\, k \in \mathbb{N} \cup \{0\}$

Proof:

$\|f_n\| \le C_0$, $\|f_n'\| \le C_1$

$\Rightarrow \quad \exists\, f_{n_j} \rightrightarrows f_\infty$. $\qquad f_{1,n} \rightrightarrows f_\infty$.

$f_{1,n}$

f_n : $\boxed{f_1}$ f_2 f_3 f_4 f_5 f_6 $\boxed{f_7}$, f_8 f_9 f_{10}

$\|f'_{1,n}\| \le C_1$, $\quad \|f''_{1,n}\| \le C_2 \Rightarrow \exists\, f'_{1,n_j} \rightrightarrows f_\infty$

$\boxed{f_1}$ $\cancel{f_2}$ f_3 $\cancel{f_4}$ f_5 $\cancel{f_6}$ $\boxed{f_7}$ $\cancel{f_8}$ f_9 $\cancel{f_{10}}$ f_{11}

$f_{2,n}$

$$\{f_{1,n}\} \supset \{f_{2,n}\} \supset \{f_{3,n}\} \supset \cdots$$

$\square$

construct $\boxed{\{f_{n,n}\}_{n=1}^{\infty}}$ = subsequence of each $\{f_{k,n}\}_{n=1}^{\infty}$ (except the first $k-1$ terms)

Given $|f_n(x)| \le M \quad \forall n \in \mathbb{N}, \quad x \in [a,b]$

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \quad \forall n \in \mathbb{N}.$

Goal: $f_{n_j} \rightrightarrows f_\infty$. on $[a,b]$.

Proof: Step ① $\exists f_{n_k}$ s.t. $f_{n_k}(r) \longrightarrow L(r) \quad \forall r \in [a,b] \cap \mathbb{Q}$

$\mathbb{Q} \cap [a,b] = \{r_1, r_2, r_3, r_4, \dots\}$

$\overset{BW}{\Rightarrow} \exists f_{1,n}$ s.t. $f_{1,n}(r_1) \longrightarrow L_1 \quad (\because |f_n(r_1)| \le M)$

$|f_{1,n}(r_2)| \le M \overset{BW}{\Rightarrow} f_{2,n}(r_2) \to L_2$

$\vdots$

$\exists \{f_n\} \supset \{f_{1,n}\} \supset \{f_{2,n}\} \supset \{f_{3,n}\} \supset \cdots$

s.t. $f_{j,n}(r_j) \longrightarrow L_j$

$(f_{j,n}(r_\ell) \longrightarrow L_\ell$ if $\ell \le j$
as $n \to \infty$

then

$f_{n,n}(r) \longrightarrow L(r) \quad \forall r \in \mathbb{Q} \cap [a,b]$

by standard diagonalization.

Step ②: $f_{n,n} \rightrightarrows f_\infty$ on $[a,b]$.

want: $\forall \varepsilon > 0, \exists N > 0$ s.t. $m, n \ge N \Rightarrow \|f_{m,m} - f_{n,n}\|$

$< \varepsilon.$

Assume not:

$$\exists \varepsilon_0 > 0, \ \forall N > 0 \text{ s.t. } \exists m, n \geq N$$

$$\text{s.t. } \|f_{m,m} - f_{n,n}\| \geq \varepsilon_0.$$

$\vdots$

$$\exists \ m_1 < m_2 < m_3 < \cdots \quad \text{s.t. } \|f_{m_j,m_j} - f_{n_j,n_j}\| \geq \varepsilon_0$$
$$n_1 < n_2 < n_3 < \cdots$$

$\varepsilon_0/2$

$0 \quad \varepsilon_0$

$$\forall j: \underbrace{\sup_{x \in [a,b]} |f_{m_j,m_j}(x) - f_{n_j,n_j}(x)|} \geq \varepsilon_0 > \varepsilon_0/2$$

$$\exists \ x_j \in [a,b] \text{ s.t.}$$

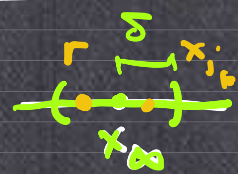
$$\underbrace{|f_{m_j,m_j}(x_j) - f_{n_j,n_j}(x_j)| > \varepsilon_0/2.}$$

BW

$\exists x_{j_k}$

$\downarrow$

$x_\infty \in [a,b]$



$\forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon) > 0$ s.t.

$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \quad \forall n \in \mathbb{N}.$

$$\frac{\varepsilon_0}{2} < |f_{m_{j_k},m_{j_k}}(x_{j_k}) - f_{n_{j_k},n_{j_k}}(x_{j_k})|$$

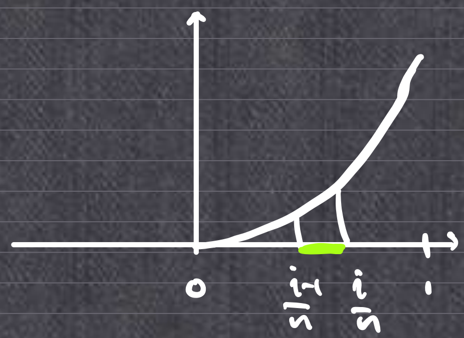
$$\leq |f_{m_{j_k},m_{j_k}}(x_{j_k}) - f_{m_{j_k},m_{j_k}}(r)|$$
$$+ |f_{m_{j_k},m_{j_k}}(r) - f_{n_{j_k},n_{j_k}}(r)|$$

$$+ \left| f_{n_{j_k}}(r) - f_{n_{j_{k}'} n_{j_k}}(x_{j_k}) \right|$$

e.g. $f(x) = x^2$ on $[0,1]$

Let $P_n : 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < \frac{n}{n} = 1$ $\quad x_i = \frac{i}{n}, \; 0 \le i \le n$

$$[\;\; 0 \;\; \tfrac{1}{n} \;\; \tfrac{2}{n} \;\; \tfrac{3}{n} \cdots 1 \;\;]$$



$$U(f, P_n) = \sum_{i=1}^{n} \underbrace{\left(\frac{i}{n}\right)^2}_{f(x_i)} \frac{1}{n}$$

$$= \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2)$$

$$= \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) \to \frac{1}{3}$$

as $n \to \infty$

$$L(f, P_n) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}(0^2 + 1^2 + \cdots + (n-1)^2)$$

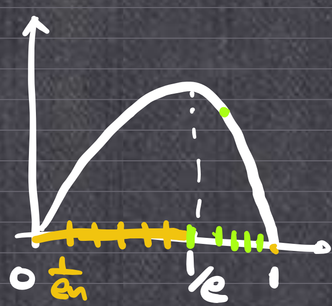
$$= \frac{1}{n^3} \cdot \frac{1}{6}(n-1)n(2n-1) \to \frac{1}{3}.$$

$\exists \{P_n\}$ s.t. $\lim_{n\to\infty} L(f, P_n) = \frac{1}{3} = \lim_{n\to\infty} U(f, P_n)$

$\therefore$ x^2 is Riemann integrable

and $\int_0^1 f(x)\,dx = \frac{1}{3}$

$$f(x) = \begin{cases} -x\log x & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$



$$\left(1-\frac{1}{e}\right)\frac{1}{n}$$

(ii) $\sup_P L(f,P) = \inf_P U(f,P)$

$\Downarrow$

(i) $\exists \{P_k\}$ s.t. $\lim_{k\to\infty} U(f,P_k) = \lim_{k\to\infty} L(f,P_k)$.

$\exists \{P_k\}$ s.t. $\lim_{k\to\infty} L(f,P_k) = \sup_P L(f,P)$

$\exists \{Q_k\}$ s.t. $\lim_{k\to\infty} U(f,Q_k) = \inf_P U(f,P)$

$P \subseteq Q \implies$ $U(f,P) \geq U(f,Q)$

$L(f,P) \leq L(f,Q)$

e.g. $\{x_1,\dots,x_n\}$ $\{x_1,\dots,y,\dots,x_n\}$

$R_k := P_k \cup Q_k$

$\Rightarrow$ $L(f,P_k) \leq L(f,R_k) \leq U(f,R_k) \leq U(f,Q_k)$

$\downarrow$ $\sup_P L(f,P)$

$\downarrow$ $\inf_P U(f,P)$

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

P = any partition of $[0,1]$.

$$U(\chi_{\mathbb{Q}}, P) = \sum \underbrace{\sup}_{1} \cdot \Delta x_i = 1$$

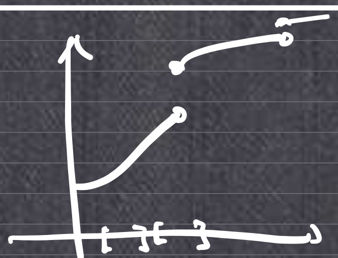
$$L(\chi_{\mathbb{Q}}, P) = \sum \underbrace{\inf}_{0} \cdot \Delta x_i = 0.$$

- Continuous on $[a,b]$ $\Longrightarrow$ Riemann integrable on $[a,b]$.

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right) (x_i - x_{i-1})$$

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t. $|x-y| < \delta$
$\Rightarrow |f(x) - f(y)| < \varepsilon/?$

$$\le \sum_{i=1}^{n} \varepsilon/? \, (x_i - x_{i-1}) = \varepsilon/? \, (b-a)$$



Ex: Given $f:[a,b]\to[m,M]$ Riem. integrable

$\varphi:[m,M]\to\mathbb{R}$ continuous ($\Longrightarrow$ uniformly (cts))

then $\varphi\circ f:[a,b]\to\mathbb{R}$ Riem. integrable.

Proof: $\forall\varepsilon>0,\ \exists\delta>0$ s.t. $y_1,y_2\in[m,M]$ and $|y_1-y_2|<\delta$

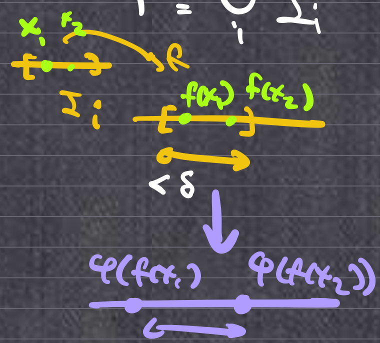
$$\Longrightarrow |\varphi(y_1)-\varphi(y_2)|<\boxed{\varepsilon}$$

$\exists P$ of $[a,b]$ s.t. $U(f,P)-L(f,P)<\cancel{?}\ \delta^2$

$$\Longrightarrow \underbrace{\sum_i\Big(\sup_{I_i}f-\inf_{I_i}f\Big)}\,|I_i|<\cancel{?}\ \delta^2$$

if small

$$P=\bigcup_i I_i$$



$$U(\varphi\circ f,P)-L(\varphi\circ f,P)$$

$$=\sum_i\Big(\sup_{I_i}\varphi\circ f-\inf_{I_i}\varphi\circ f\Big)|I_i|$$

$$|\varphi(f(x_1))-\varphi(f(x_2))|$$

$$=\sum_{\substack{\sup_{I_i}f-\inf_{I_i}f<\delta}}\underbrace{\Big(\sup_{I_i}\varphi\circ f-\inf_{I_i}\varphi\circ f\Big)}_{\Longrightarrow\ <\varepsilon}|I_i| \quad ✓ \quad —①$$

$$+\sum_{\boxed{\sup_{I_i}f-\inf_{I_i}f\geq\delta}}\Big(\sup_{I_i}\varphi\circ f-\inf_{I_i}\varphi\circ f\Big)|I_i| \quad —②$$

Recall $\displaystyle\sum_j\Big(\sup_{I_j}f-\inf_{I_j}f\Big)|I_j|<\cancel{?}\ \delta^2$

$$\sum\delta|I_i|\leq\sum_{\sup_{I_i}f-\inf_{I_i}f\geq\delta}\underbrace{\Big(\sup_{I_i}f-\inf_{I_i}f\Big)}|I_i|<\cancel{?}\ \delta^2$$

$$\Longrightarrow |I_i|\leq\delta$$

$$② \leq \sum_{\sup_{I_i} f - \inf_{I_i} f \geq \delta} \left(\sup_{I_i} \varphi\circ f - \inf_{I_i} \varphi\circ f\right)|I_i|$$

$$\leq M \sum_{\cdots \geq \delta} |I_i| < M\delta.$$

$E \subset \mathbb{R}$

Lebesgue outer measure $\qquad \emptyset \subset \emptyset\cup\emptyset\cup\emptyset\cup\cdots$

$$\mathcal{L}^*(E) = \inf\left\{ \sum_{i=1}^{\infty}(b_i-a_i) \;\middle|\; E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\} \qquad \boxed{a_i \leq b_i}$$

• $\underbrace{\boxed{\mathcal{L}^*([a,b]) = b-a}}_{3043}$

$\mathcal{L}^*(E) = 0$

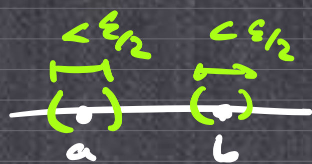
inf=0 $\quad \varepsilon > 0$

$\iff \forall \varepsilon > 0, \ \exists \bigcup_{i=1}^{\infty}(a_i, b_i) \supset E$

s.t. $\sum_i (b_i - a_i) < \varepsilon$.

$\mathcal{L}^*(\{a,b\}) = 0$

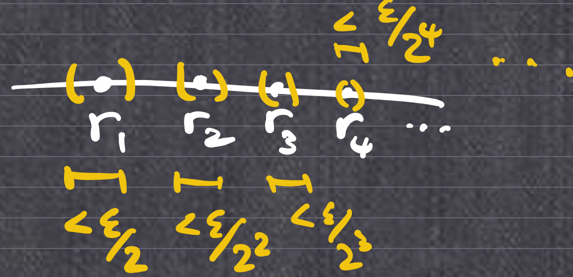
$a \neq b$



proof: $\forall \varepsilon > 0$, pick $\{a,b\} \subset (a-\frac{\varepsilon}{5}, a+\frac{\varepsilon}{5})$

$\underbrace{\cup (b-\frac{\varepsilon}{5}, b+\frac{\varepsilon}{5})}_{\text{total length } < \varepsilon}.$

$$\mathcal{L}^*\left(\bigcup_{i=1}^{\infty}\{r_i\}\right) = 0$$



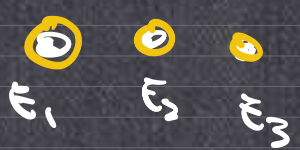
any countable set has zero Lebesgue measure.

- Given $\{E_i\}_{i=1}^{\infty}$ s.t. $\underline{\mathcal{L}^*(E_i) = 0}$ $\forall i$

$$\Rightarrow \mathcal{L}^*\left(\bigcup_{i=1}^{\infty} E_i\right) = 0.$$

$\forall \varepsilon > 0, j \in \mathbb{N}, \exists \bigcup_i (a_i^j, b_i^j) \supset E_j$

s.t. $\sum_i (b_i^j - a_i^j) < \varepsilon/2^j$



then $\bigcup_j E_j \subset \bigcup_j \bigcup_i (a_i^j, b_i^j)$

and $\sum_j \sum_i (b_i^j - a_i^j)$

$< \sum_j \varepsilon/2^j = \varepsilon.$

Lebesgue Theorem :

$f : [a,b] \to [m,M]$ is Riemann integrable

$$\Longleftrightarrow \mathcal{L}^*(D_f) = 0.$$

$$\overset{\shortparallel}{\{x_0 \in [a,b] \mid f \text{ is } \underline{\text{NOT}} \text{ continuous at } x_0\}}.$$

- $f : [a,b] \to \mathbb{R}$ continuous

  <u>Proof</u>: $D_f = \emptyset \Rightarrow \mathcal{L}^*(D_f) = 0$.

- $f : [a,b] \to \mathbb{R}$ is monotone

  <u>Proof</u>: D_f is countable

  $\Rightarrow \mathcal{L}^*(D_f) = 0$.

- $f, g : [a,b] \to [m, M]$ Riem. integrable

  $\Rightarrow f+g$ is Riem. integrable

  <u>Proof</u>: If f, g are cts at x_0

  $\Rightarrow f+g$ is cts at x_0

  Contrapositive: $f+g$ is not cts at x_0

  $\Rightarrow f$ or g is not cts at x_0

  $\therefore x_0 \in D_{f+g} \Rightarrow x_0 \in D_f \cup D_g$

  $\Rightarrow D_{f+g} \subset D_f \cup D_g$

- $f : [a,b] \to [m, M]$ Riem. integ.

  $\varphi : [m, M] \to \mathbb{R}$ continuous

  $\Rightarrow \varphi \circ f$ is Riem. integ.

Proof: If f is continuous at x_0

$\Rightarrow$ $\varphi \circ f$ is continuous at x_0

If $\varphi \circ f$ is not cts at x_0

then f is not cts at x_0

$D_{\varphi \circ f} \subset D_f$

<u>Claim:</u> $\mathcal{L}^*([a,b]) = b-a$.

$$\mathcal{L}^*(E) = \inf\left\{ \sum_i |b_i - a_i| \;\middle|\; E \subset \bigcup_{i=1}^{\infty} (a_i, b_i),\ a_i \le b_i \right\}$$

($\le$) $\quad [a,b] \subset \underline{(a-\varepsilon, b+\varepsilon)}$

$\forall \varepsilon > 0$, $\quad \mathcal{L}^*([a,b]) \le (b+\varepsilon) - (a-\varepsilon) = b-a+2\varepsilon$

let $\varepsilon \to 0^+ \implies \mathcal{L}^*([a,b]) \le b-a$.

($\ge$) Take any arbitrary $\bigcup_{i=1}^{\infty} (a_i, b_i) \supset [a,b]$

$$\overset{\text{Heine-Borel}}{\implies} \exists\, (a_{i_1}, b_{i_1}), \dots, (a_{i_N}, b_{i_N})$$

s.t. $\bigcup_{j=1}^{N} (a_{i_j}, b_{i_j}) \supset [a,b]$

$$\xRightarrow[\text{on } N]{\text{induction}} \sum_{j=1}^{N} (b_{i_j} - a_{i_j}) \ge b-a$$

$$\Rightarrow \sum_{i=1}^{\infty} (b_i - a_i) \ge \sum_{j=1}^{N} (b_{i_j} - a_{i_j}) \ge b-a.$$

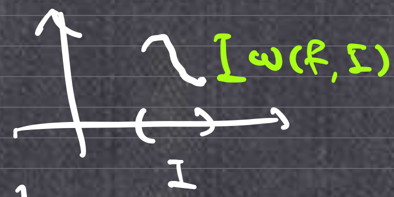
$$\mathcal{L}^*([a,b]) = \inf\{ \cdots \} \ge b-a.$$

($\Longrightarrow$) Suppose $f: [a,b] \to [m,M]$ Riem. integrable

<u>Want</u>: $\mathcal{L}^*(D_f) = 0$.

$$\omega(f, I) := \sup\{|f(x) - f(y)| : x, y \in I\}$$

I " open interval



$\omega(f, x_0)$

$:= \inf\{\omega(f, I) \mid I \ni x_0\}$

$D_f = \{x_0 \mid \omega(f, x_0) > 0\}$.

$\subset$: $x_0 \in D_f \Rightarrow \exists \varepsilon_0 > 0, \forall \delta > 0$

$\exists x, y \in (x_0 - \delta, x_0 + \delta)$

s.t. $|f(x) - f(y)| \geq \varepsilon_0$

$\Rightarrow$ $I := (x_0 - \delta, x_0 + \delta)$

$\Rightarrow \omega(f, I) \geq \varepsilon_0$

$\Rightarrow \omega(f, x_0) = \inf\{\underbrace{\omega(f, I)}_{\geq \varepsilon_0} \mid I \ni x_0\}$

$\geq \varepsilon_0$

$$D_f = \{x_0 \mid \omega(f, x_0) > 0\} = \bigcup_{k \in \mathbb{N}} \{x_0 \mid \omega(f, x_0) \geq \tfrac{1}{k}\}$$

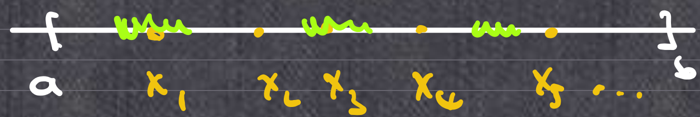
Fix $k \in \mathbb{N}$, consider $\Omega_k := \{x_0 \mid \omega(f, x_0) \geq \frac{1}{k}\}$

want $\mathcal{L}^*(\Omega_k) = 0$.

$f : [a, b] \to [m, M]$ Riem. Intg.

$\forall \varepsilon > 0$, $\exists P = \{x_i\}$ s.t. $U(f, P) - L(f, P) < ?$

$$\Omega_k \setminus \{x_0, x_1, \dots, x_n\} \subset \bigcup_{\Omega_k \cap (x_{i-1}, x_i) \neq \phi} (x_{i-1}, x_i)$$



$$U(f,P) - L(f,P)$$

$$= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

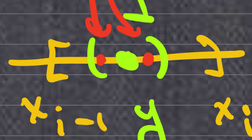
$$M_i := \sup_{[x_{i-1}, x_i]} f \qquad \geq \sum_{\Omega_k \cap (x_{i-1}, x_i) \neq \phi} (M_i - m_i)(x_i - x_{i-1}) \geq \frac{1}{2k} \sum_{\Omega_k \cap (x_{i-1}, x_i) \neq \phi} (x_i - x_{i-1})$$

$\exists\, x, y$ s.t. $|f(x) - f(y)| > \frac{1}{2k}$.

I

$\omega(f, I) \geq \omega(f, y) \geq \frac{1}{k}$.

$\geq \frac{1}{2k}$



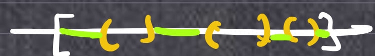
$y \in \Omega_k \leftarrow \omega(f, y) \geq \frac{1}{k}$.

$(\Leftarrow)$ Given $\mathcal{L}^*(D_f) = 0$.

$\forall \varepsilon > 0$, $\exists \bigcup_{i=1}^{\infty} (a_i, b_i) \supset D_f$ s.t.

$$\sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon.$$

$K := [a,b] \setminus \bigcup_{i=1}^{\infty} (a_i, b_i)$



$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$.

$$\mathcal{U}(f,P) - L(f,P) = \sum_i (M_i - m_i)(x_i - x_{i-1})$$

$$= \sum_{K \cap [x_{i-1}, x_i] \neq \emptyset} (M_i - m_i)(x_i - x_{i-1}) + \sum_{K \cap [x_{i-1}, x_i] = \emptyset} (M_i - m_i)(x_i - x_{i-1})$$