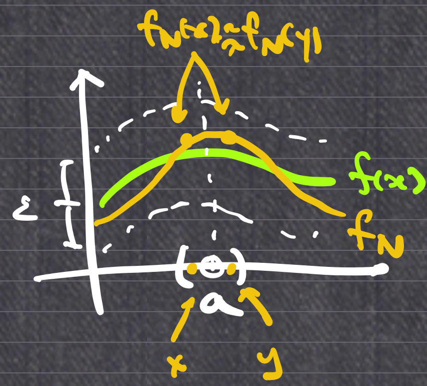
Given: $\begin{cases} f_n \rightrightarrows f \text{ on } D, & a \in D' \\ \lim_{x\to a} f_n(x) \text{ exists } \forall n. \end{cases}$

Want: $$\lim_{x\to a} \underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)} = \underbrace{\lim_{n\to\infty} \lim_{x\to a} f_n(x)}_{\checkmark}$$

Claim: $\lim_{x\to a} f(x)$ exists



$$\boxed{\begin{aligned} &|f(x) - f(y)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| \\ &\quad + |f_N(y) - f(y)| \end{aligned}}$$

$\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t. $n \ge N$

$\Rightarrow \quad \|f_n - f\|_D < \varepsilon$

$\Rightarrow \forall x \in D, \quad |f_n(x) - f(x)| < \varepsilon \quad \forall n \ge N.$

$$\lim_{x\to a} |f_n(x) - f(x)| \le \varepsilon$$

$$\Rightarrow \Big| \underbrace{\lim_{x\to a} f_n(x)}_{L_n} - \underbrace{\lim_{x\to a} f(x)}_{L} \Big| \le \varepsilon \quad \forall n \ge N$$

$$\Rightarrow \Big| \underbrace{\lim_{x\to a} f_n(x)}_{\text{seq. of number}} - L \Big| \le \varepsilon. \quad \forall n \ge N.$$

$$\lim_{n\to\infty} \left(\lim_{x\to a} f_n(x) \right) = \lim_{x\to a} f(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x)$$

Cor: if further f_n is continuous at $x=a$.

then

$$\underbrace{\lim_{x\to a}\underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)}}_{\lim_{x\to a} f(x)} = \underbrace{\lim_{n\to\infty}\underbrace{\lim_{x\to a} f_n(x)}_{f_n(a)}}_{= f(a)}$$

e.g. $f_n(x) = x^n$ on $[0,1]$.

$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ converges when $x > 1$

$\forall \varepsilon > 0$, $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $(1+\varepsilon, \infty)$.

$$\begin{cases} \left|\frac{1}{n^x}\right| = \frac{1}{n^x} \le \frac{1}{n^{1+\varepsilon}} & \text{on } x \in (1+\varepsilon, \infty) \\ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \text{ converges} & \end{cases}$$

$\xRightarrow{\text{Weierstrass M-test}}$ $\zeta(x) = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^x}}_{\text{continuous}}$ converges uniformly on $(1+\varepsilon, \infty)$.

$\zeta(x)$ is continuous on $(1+\varepsilon, \infty)$ $\forall \varepsilon > 0$.

$\Downarrow$

$\zeta(x)$ is continuous on $\bigcup_{\varepsilon > 0} (1+\varepsilon, \infty) = (1, \infty)$.

$\cancel{\Downarrow}$

uniform conv. on $(1,\infty)$

$\sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$

$\Rightarrow$ converges at $x = -R+\varepsilon$ and $x = R-\varepsilon$.

$\overset{?}{\Longrightarrow}$ uniform convergence on $[-R+2\varepsilon, R-2\varepsilon]$.

$$\begin{cases} \cdot\ f_n \to f & \text{on } (a,b) \\ \cdot\ f_n' \rightrightarrows g & \text{on } (a,b) \end{cases} \qquad x_0 \in (a,b)$$

$$\Longrightarrow \underbrace{\lim_{n\to\infty} \frac{d}{dx}\Big|_{x=x_0} f_n(x)}_{= \lim_{n\to\infty} f_n'(x_0) = g(x_0)} = \frac{d}{dx}\Big|_{x=x_0} \lim_{n\to\infty} f_n(x) \quad = \frac{d}{dx}\Big|_{x=x_0} f(x)$$

Proof: $$\text{LHS} = \lim_{n\to\infty} \lim_{x\to x_0} \frac{f_n(x) - f_n(x_0)}{x - x_0}$$

$$\text{RHS} = \lim_{x\to x_0} \frac{\lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \lim_{n \to \infty} \underbrace{\frac{f_n(x) - f_n(x_0)}{x - x_0}}_{g_n(x)}$$

Need: $g_n \rightrightarrows$? on $(a,b) \setminus \{x_0\}$.

$$|g_n(x) - g_m(x)| = \left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right|$$

$$= \frac{|\overset{g(x)}{(f_n(x) - f_m(x))} - \overset{g(x_0)}{(f_n(x_0) - f_m(x_0))}|}{|x - x_0|}$$

$$= \frac{|g'(c) \cancel{(x - x_0)}|}{\cancel{|x - x_0|}} = |f_n'(c) - f_m'(c)|$$

$$\leq \| f_n' - f_m' \|_{(a,b) \setminus \{x_0\}}.$$

e.g.

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \qquad \boxed{x > 1}$$

$$\sum_{n=1}^{\infty} \frac{d}{dx} \frac{1}{n^x}$$

$$= \sum_{n=1}^{\infty} - \frac{1}{(n^x)^2} \frac{d}{dx} n^x$$

$$= \sum_{n=1}^{\infty} \left(- \frac{1}{n^{2x}} \cdot n^x \log n \right)$$

$$= - \sum_{n=1}^{\infty} \frac{1}{n^x} \log n \qquad x \in (1+\varepsilon, \infty)$$

$$\left\{\begin{aligned} \left|\frac{\log n}{n^x}\right| &\le \frac{\log n}{n^{1+\varepsilon}} \\ &\le \frac{n^{\varepsilon/2}}{n^{1+\varepsilon}} = \frac{1}{n^{1+\varepsilon/2}} \\ \sum \frac{1}{n^{1+\varepsilon/2}} &\text{ converges} \end{aligned}\right.$$

$\log n \le n^{\varepsilon/2}$ if $n >> 1$

$\forall x \in (1+\varepsilon, \infty)$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^x}$ converges uniformly. on $(1+\varepsilon, \infty)$

$\therefore \displaystyle\frac{d}{dx}\sum_{n=1}^{\infty}\frac{1}{n^x} = \sum_{n=1}^{\infty}\frac{d}{dx}\frac{1}{n^x}$ on $(1+\varepsilon, \infty)$ $\forall \varepsilon > 0$.

$\Rightarrow$ " $=$ " on $\displaystyle\bigcup_{\varepsilon>0}(1+\varepsilon, \infty) = (1, \infty)$.

$f(x) = \displaystyle\sum_{n=1}^{\infty} a_n x^n$ on $(-R, R)$

ex: $\displaystyle\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ $\forall \varepsilon > 0$

$$\sum_{n=1}^{\infty}\frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} \le 1 \quad \text{on } x \in [-R+\varepsilon, R-\varepsilon]$$

$$\Rightarrow \limsup_{n\to\infty} \underbrace{\sqrt[n]{|n a_n x^{n-1}|}}_{n^{\frac{1}{n}} \to 1} = \limsup_{n\to\infty} n^{\frac{1}{n}} \sqrt[n]{|a_n x^n|} \cdot |x|^{-\frac{1}{n}}$$