$x \in S'$

$x \in \partial S$

$$\partial S \cup S = S' \cup S =: \bar{S}.$$

Claim: $\bar{S}$ is closed. (X, d)

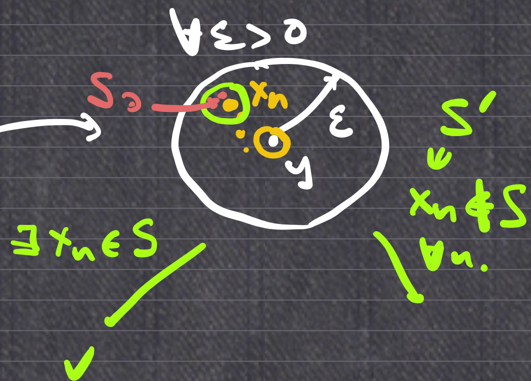
Proof: Let $\{x_n\} \to y \in X$

$\cap$
$\bar{S}$

need: $y \in \bar{S}$.

Case ①: $y \in S \subset \bar{S}$

Case ②: $y \notin S$ $\longrightarrow$



$\partial S \subset S \iff$ $\quad$ $S' \subset S \iff$

Prop 2.34 $\bar{S} = S \iff S$ is closed.

$(\Longrightarrow)$ trivial

$(\Leftarrow)$ Assume S is closed.

Goal: $\partial S \subset S$ $\quad \forall \varepsilon > 0$

$\forall x \in \partial S,$

$\forall n \in \mathbb{N}, \ \exists\, x_n \in S \cap B_{\frac{1}{n}}(x), \quad d(x_n, x) < \frac{1}{n}$

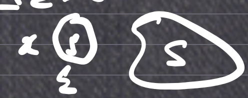
$\downarrow$ $\quad \uparrow$ closed $\quad\quad \downarrow$

$x \in S$ $\quad\quad\quad x$

Alternately: $\forall x \in \partial S$, assume $x \notin S$

$\Rightarrow x \in \underbrace{X \setminus S}_{\text{open}}.$

$\exists \varepsilon > 0$



$$\overline{S} = \bigcap_{\substack{S \subset E \\ \uparrow \\ \text{closed}}} E$$

$$\overline{B_\varepsilon(x)} \ \cancel{\supseteq} \ \{y \in X \mid d(x,y) \leq \varepsilon\}$$

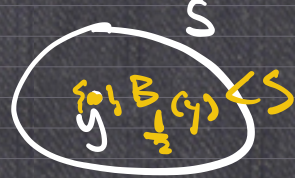
$$B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$$

(X, d) discrete metric

$$B_1(x) = \{y \in X : d(x,y) < 1\} = \{x\}.$$

$$C_1(x) = \{y \in X : d(x,y) \le 1\} = X.$$

$$\overline{B_1(x)} = B_1(x).$$



<u>dense</u> $S \subset (X, d)$ is dense $\overset{\text{def}}{\iff} \overline{S} = X.$

- $\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}$ is dense in $\mathbb{R}$.

  $\overline{\mathbb{R}\setminus\mathbb{Q}} = \mathbb{R}$, $\mathbb{R}\setminus\mathbb{Q}$ " — — .

$$f(x+y) = f(x) f(y) \quad \forall x, y \in \mathbb{R}.$$

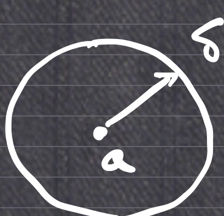
$$S^\circ := S \setminus \partial S.$$

$$f : (X, d_X) \to (Y, d_Y)$$

$\lim_{x \to a} f(x) = p$

$\overset{\text{def}}{\iff} \forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < d_X(x, a) < \delta$ $\quad x \in B^X_\delta(a) \setminus \{a\}.$

$\Rightarrow d_Y(f(x), p) < \varepsilon.$



$S \subset X$

$a \in S'$

$$\lim_{\substack{x \to a \\ x \in S}} f(x) = p$$

$\overset{\text{def}}{\iff} \forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < d_X(x, a) < \delta, x \in S$

$\Rightarrow d_Y(f(x), p) < \varepsilon.$

$f : \mathbb{R} \to \mathbb{R}$

$$\lim_{\substack{x \to a \\ x \in (a, a+1)}} f(x) = L \iff \lim_{x \to a^+} f(x) = L$$

$\mathbb{1}_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$, $\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

$\lim_{\substack{x \to a \\ x \in \mathbb{R}}} \mathbb{1}_{\mathbb{Q}}(x)$ does not exist

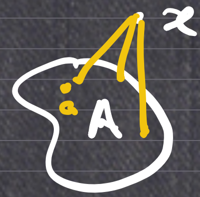
$$\lim_{\substack{x \to a \\ x \in \mathbb{Q}}} \underbrace{\mathbb{1}_{\mathbb{Q}}(x)}_{1} = 1 \quad \Big| \quad \lim_{\substack{x \to a \\ x \in \mathbb{R} \setminus \mathbb{Q}}} \mathbb{1}_{\mathbb{Q}}(x) = 0.$$

$f : (X, d_X) \to (Y, d_Y)$ is continuous

at $x = a$

$\overset{\text{def}}{\iff} \lim_{x \to a} f(x) = f(a)$

e.g. 2.44: $A \subset (X, d)$



$$d_A : X \to \mathbb{R}$$

$$d_A(x) := \inf\{d(x,a) : a \in A\}$$

Claim: d_A is continuous on X.

Proof: Want: $|d_A(x) - d_A(y)| \le d(x,y)$.

$$d(x,a) \le d(x,y) + d(y,a) \quad \forall a \in A.$$

$$\overset{\inf_{a\in A}}{\Longrightarrow} \quad d_A(x) \le \inf\{d(x,y) + d(y,a) : a \in A\}.$$

$$= d(x,y) + \underbrace{\inf\{d(y,a) : a \in A\}}_{d_A(y)}$$

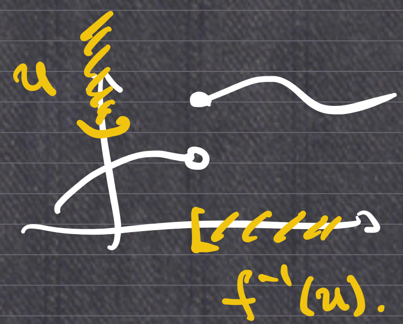
$$\Rightarrow \quad d_A(x) - d_A(y) \le d(x,y).$$

Let $f : (X, d_X) \to (Y, d_Y)$, TFAE:

① f is continuous on X

② $f^{-1}(U)$ is open in X for any open set $U \subset Y$.

$$f^{-1}(\mathcal{U}) := \{x \in X : f(x) \in \mathcal{U}\}$$



e.g. $\quad f(x,y) = x^2 + 4xy + y^3 - x^3y^2 \quad$ in $\mathbb{R}^2$

$$\Sigma := \{(x,y) \in \mathbb{R}^2 : f(x,y) > 0\}.$$

$$\parallel$$

$$f^{-1}\big((0,\infty)\big) = \{(x,y) \in \mathbb{R} : f(x,y) \in (0,\infty)\}$$

e.g. $\quad M^{n \times n}$ = set of all $n \times n$ real matrices.

$$\cong \mathbb{R}^{n^2}$$

Σ := set of all invertible matrices.

$$\det : M^{n \times n} \to \mathbb{R}$$

$\Sigma = \det^{-1}(\mathbb{R}\setminus\{0\}) = \{A \in M^{n\times n} : \det(A) \neq 0\}$

is open in $M^{2\times 2}$.

Let $f : (X, d_X) \to (Y, d_Y)$, TFAE:

① f is continuous on X

② $f^{-1}(U)$ is open in X for any open set $U \subset Y$.

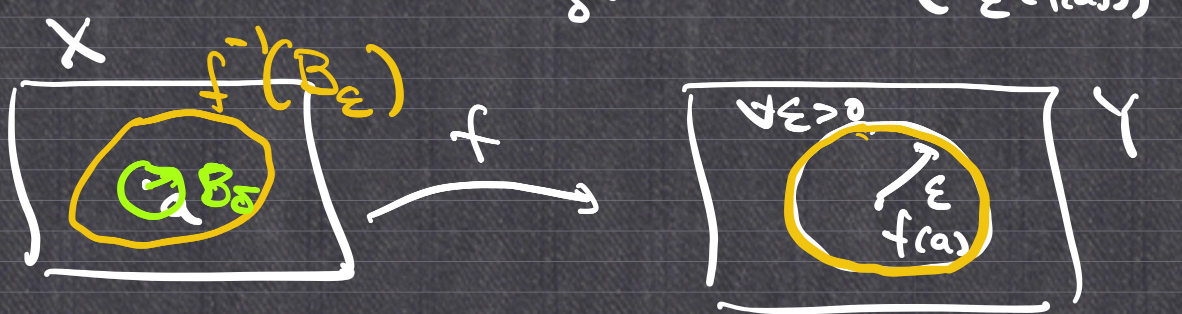
$\Rightarrow$ $\forall a \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $\underbrace{d_X(x,a) < \delta}_{x \in B_\delta(a)} \Rightarrow \underbrace{d_Y(f(x), f(a)) < \varepsilon}_{f(x) \in B_\varepsilon(f(a))}$

$\Updownarrow$

$x \in f^{-1}(B_\varepsilon(f(a)))$

$$\underbrace{\qquad\qquad\qquad\qquad}_{B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a)))}$$



① $\Rightarrow$ ② :

Take $\mathcal{U} \subset Y$ open.