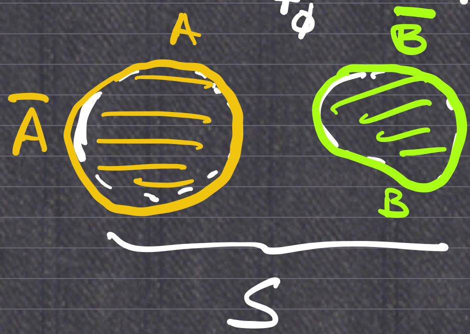
## §2.4 $(X, d)$ metric space

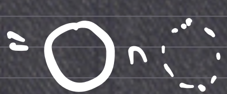
$S$ is said to **disconnected**

$\overset{def}{\iff}$ $S = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$ where $\begin{cases} \bar{A} \cap B = \emptyset \\ A \cap \bar{B} = \emptyset \end{cases}$

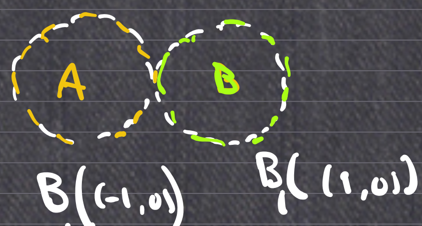



connected $\overset{def}{\iff}$ not disconnected

$\bar{A} \cap B$



$= \emptyset$




**Prop. 2.67** A subset $S \subset \mathbb{R}$ is connected ①

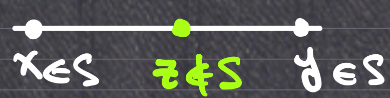
$\iff$ $\forall x, y \in S,\ x < y,$
$\Rightarrow [x, y] \subset S.$ ②

$(\Rightarrow)$ Suppose $S$ is connected

Assume ② is false

$\exists\ x, y \in S,\ x < y$ s.t. $[x, y] \not\subset S$

$\Rightarrow \exists\ z \in (x, y)$
$z \notin S.$



Take $A=(-\infty,z)\cap S$, $B=(z,+\infty)\cap S$

$S=A\cup B$. $\bar{A}\cap B = \overline{(-\infty,z)\cap S}\cap B$

$\subset \overline{(-\infty,z)}\cap\bar{S}\cap B$

$= \underbrace{(-\infty,z]\cap\bar{S}\cap(z,+\infty)}_{=\phi}\cap S$

$=\phi$

$\overline{X\cap Y}\subset\bar{X}\cap\bar{Y}$

$A\cap\bar{B}=\phi$ similarly.

☐

$(\Leftarrow)$ Given $\forall x,y\in S$ where $x<y \Rightarrow [x,y]\subset S$.

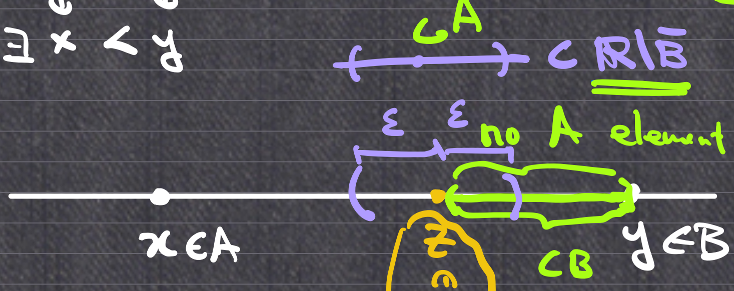
Assume $S$ is disconnected.

$S = A\cup B$ s.t. $\bar{A}\cap B=\phi$, $\bar{B}\cap A=\phi$.

$\exists\, x<y$, $x\in A$, $y\in B$

$z\notin\bar{B}$

$\Leftrightarrow z\in\mathbb{R}\setminus\bar{B}$ (open)




let $z=\sup(A\cap[x,y])$

$\uparrow$

$a_n\in A\cap[x,y]$

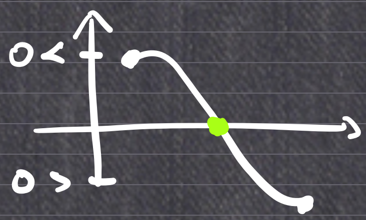
$\Rightarrow z\in\bar{A}$

$z\in\bar{A} \Rightarrow z\notin B \Rightarrow \boxed{z\in A}$

$z\in S=A\cup B$

☐

Ex: $S$ connected $\Rightarrow \bar{S}$ connected.




$f: X \to Y$

"IVT" on metric spaces:

$$\left\{\begin{array}{l} S \text{ connected} \\ f \text{ is continuous} \end{array}\right. \Rightarrow \underline{f(S) \text{ is connected.}}$$

**Proof:** Assume $f(S)$ is disconnected.

$$\Rightarrow \quad f(S) = \underset{Y}{\underset{\cap}{A}} \cup \underset{Y}{\underset{\cap}{B}}, \qquad \bar{A} \cap B = \emptyset, \quad A \cap \bar{B} = \emptyset.$$

$$\Rightarrow \quad S \subset f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$\Rightarrow \quad S = \underbrace{(f^{-1}(A) \cap S)} \cup (f^{-1}(B) \cap S)$$

$$\overline{f^{-1}(A) \cap S} \cap (f^{-1}(B) \cap S)$$

$$\subset \overline{f^{-1}(A)} \cap \bar{S} \cap f^{-1}(B) \cap S$$

$$\subset \underbrace{f^{-1}(\bar{A}) \cap f^{-1}(B)} \cap S$$

$$\subset f^{-1}(\underbrace{\bar{A} \cap B}_{\emptyset}) \cap S = \emptyset.$$

$f$ continuous $\Rightarrow \overline{f^{-1}(A)} \subset f^{-1}(\bar{A})$

HW

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$S$ is Path-connected $\overset{\text{def}}{\iff} \forall x, y \in S.\ \exists \gamma : [0,1] \to S$ continuous s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

**Good news:** path-connected $\Longrightarrow$ connected.

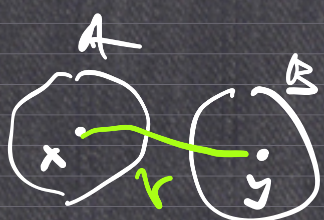
**Bad news:** connected $\not\Longrightarrow$ path-connected.

Given: $S$ is path-connected

**Want:** $S$ is connected.

Assume not (i.e. $S$ is disconnected)

$$S = \underset{\neq \phi}{A} \cup \underset{\neq \phi}{B}, \quad \bar{A} \cap B = \phi, \quad A \cap \bar{B} = \phi.$$




$\exists\, x \in A$

$\exists\, y \in B$

$\exists\, \gamma : [0,1] \to S$ continuous.

$\gamma(0) = x, \quad \gamma(1) = y.$

$$\underbrace{\gamma^{-1}(S)}_{[0,1]} = \gamma^{-1}(A \cup B) = \gamma^{-1}(A) \cup \gamma^{-1}(B)$$

$$\overline{\gamma^{-1}(A)} \cap \gamma^{-1}(B) \subset \gamma^{-1}(\bar{A}) \cap \gamma^{-1}(B) = \gamma^{-1}(\bar{A} \cap B) = \gamma^{-1}(\phi) = \phi$$

$\Gamma = \{y = \sin\frac{1}{x}\}$

$x \in (0,1]$

$S = \Gamma \cup (\{0\} \times [-1,1])$