$$\liminf_{n\to\infty} x_n = \inf \operatorname{LIM}\{x_n\}$$

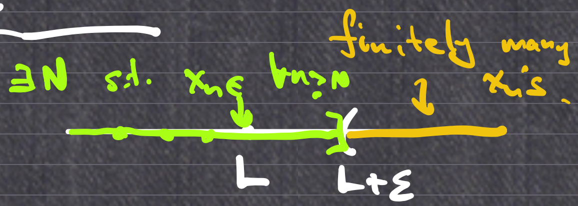
$$\limsup_{n\to} x_n = \sup \operatorname{LIM}\{x_n\}.$$

$$M_n := \sup\{x_n, x_{n+1}, x_{n+2}, \dots\} \longrightarrow \{x_n\} \text{ bounded}$$

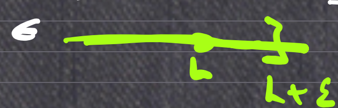
$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, \dots\}.$$

Claim: $\lim_{n\to\infty} M_n = \limsup_{n\to\infty} x_n =: L$

Proof: $\forall \varepsilon > 0$,



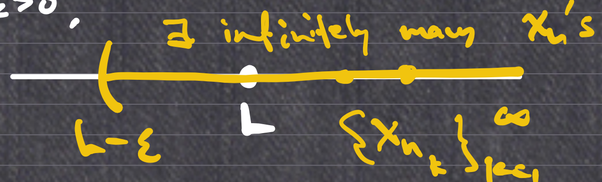
$\Rightarrow n \geq N \Rightarrow M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\} \leq L + \varepsilon$



$\Downarrow$

$\lim_{n\to\infty} M_n \leq L + \varepsilon.$ $\xRightarrow{\text{let } \varepsilon \to 0^+}$ $\boxed{\lim_{n\to\infty} M_n \leq L}$ ✓

$\geq$? $\forall \varepsilon > 0$,



$$M_n = \sup\underbrace{\{x_n, x_{n+1}, x_{n+2}, \dots\}}_{\ni x_{n_k} \ \exists k} \geq x_{n_k} > L - \varepsilon$$

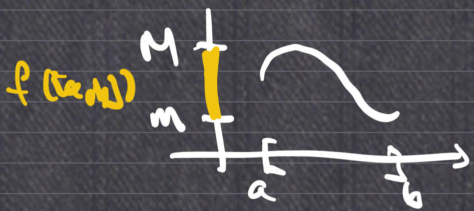
$\Downarrow$

$\lim_{n\to\infty} M_n \geq L - \varepsilon$ $\xrightarrow{\text{let } \varepsilon \to 0^+}$ $\boxed{\lim_{n\to\infty} M_n \geq L}$ ✓

EVT . IVT

EVT: $f: [a,b] \to \mathbb{R}$ cts — max. min must exists.

$f([a,b])$ is closed and bounded



IVT: $f: [a,b] \to \mathbb{R}$ cts

$$\boxed{f(a) < 0 < f(b) \Rightarrow \exists c \in (a,b) \text{ s.t. } f(c) = 0.}$$

$f(\text{interval}) = \text{interval}$.

$f(\text{connected set}) = \text{connected set}$.

§1.3

$\emptyset \neq S$ is countable

$\overset{\text{def}}{\iff} \exists f: S \to \mathbb{N}$ injective.

- $S = \{$ FF, Ivan, Yan Min, Henry Cheug $\}$

$f \downarrow$

$\{380, 1, 2, 3\} \subset \mathbb{N}$.

- $S = \{2, 4, 6, 8, 10, \dots\}$

$$\begin{array}{c|ccccccc} S = & \{2, & 4, & 6, & 8, & 10, & \dots\} \\ f \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{N} & 1 & 2 & 3 & 4 & 5 & \dots \end{array}$$

$f(2) = 1$
$f(4) = 2$
$f(6) = 3$

$$f(2k) = k \quad \forall k \in \mathbb{N}.$$

- $\mathbb{N} \times \mathbb{N} = \{(m, n) : \ m, n \in \mathbb{N}\}.$

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \dots \\ (2,1) & (2,2) & (2,3) & \dots \\ (3,1) & (3,2) & (3,3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

- $S \subset \underset{\text{countable}}{T} \implies S$ countable.

$\exists f : T \to \mathbb{N}$ injective.

$f|_S : S \to \mathbb{N}, \quad f|_S(x) = f(x)$, $x \in S$

$f|_S(x) = f|_S(y) \quad \exists x, y \in S$

$\implies f(x) = f(y)$

$\implies x = y. \qquad$ (f is injective)

- $S \xrightarrow[\text{injective}]{\exists g} \underset{\text{countable}}{T} \implies S$ countable.

$$\begin{tikzcd} S \arrow[r, "g", "\text{inj.}"'] \arrow[rr, bend right, "f\circ g"'] & T \arrow[r, "f", "\text{inj.}"'] & \mathbb{N} \end{tikzcd}$$

$f(g(x))$

\circ

$\{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$

- $\mathbb{Q}$ is countable

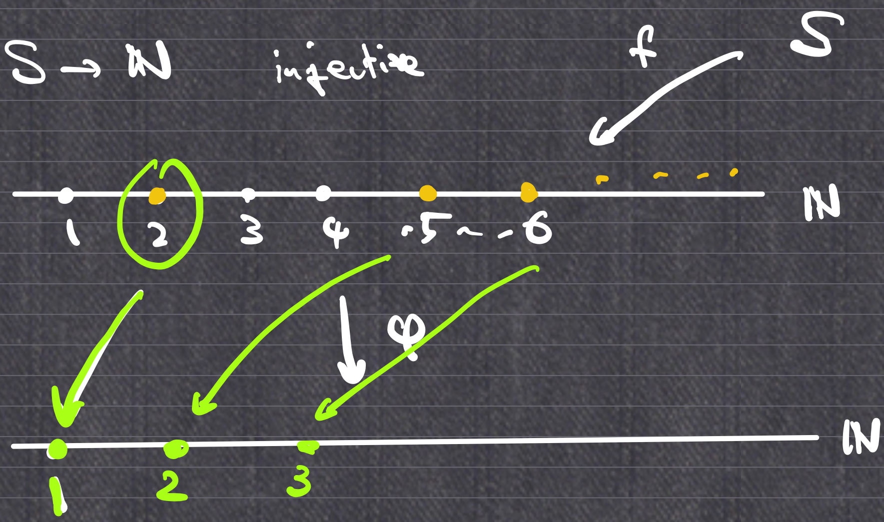
$f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$

$\frac{m}{n}$ in simplest form, $\boxed{n > 0}$ $\gcd(m,n) = 1$

$$f\left(\frac{m}{n}\right) = \begin{cases} (m,n) & \text{if } m \neq 0 \\ (0,0) & \text{if } m = 0. \end{cases}$$

S infinite set, countable.

$f : S \to \mathbb{N}$ injective



$g := \varphi \circ f : S \to \mathbb{N}$ bijection.

$g^{-1} : \mathbb{N} \to S$

<u>e.g.</u> $(0,1)$ is uncountable.

<u>Proof:</u> Assume $\underbrace{(0,1)}_{\text{infinite set}}$ is countable

$f: \mathbb{N} \to (0,1)$ bijection

$$f(1) = 0.a_{11}a_{12}a_{13}a_{14}\cdots$$
$$f(2) = 0.a_{21}a_{22}a_{23}a_{24}\cdots$$
$$f(3) = 0.a_{31}a_{32}a_{33}a_{34}\cdots$$
$$\vdots$$

$$x := 0.b_1 b_2 b_3 b_4 \cdots \neq f(n) \quad \forall n.$$
$$b_1 \neq a_{11},\ b_2 \neq a_{22},\ b_3 \neq a_{33},\ b_4 \neq a_{44}$$

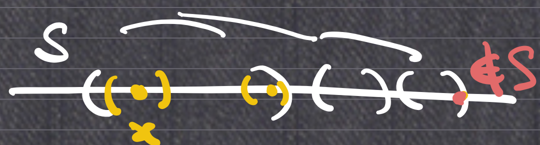
- S, T countable $\Rightarrow$ $S \cup T$ is countable.

$$\mathbb{Q} \cup \underbrace{(\mathbb{R} \setminus \mathbb{Q})}_{\Downarrow\ \text{uncountable}} = \mathbb{R} \supset (0,1)$$

open, closed.

↑

- $S \subset \mathbb{R}$ is <u>open</u> if $\forall x \in S$, $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset S$



- $S \subset \mathbb{R}$ is <u>closed</u> $\overset{\text{def}}{\iff}$ <u>$\mathbb{R}\setminus S$ is open</u>.

$S = $ ——[——]——

$\mathbb{R}\setminus S = $ ——)—— (——

$T = $ ——[——)—— not closed.

$\mathbb{R}\setminus T = $ ——)——[—— not open

- $\emptyset$, $\mathbb{R}$ both open and closed

$\mathbb{R}\setminus\emptyset = \mathbb{R}$ open $\Downarrow$ $\emptyset$ is closed $\quad$ $\mathbb{R}\setminus\mathbb{R} = \emptyset$ open $\Downarrow$ $\mathbb{R}$ is closed

✓ open: $\forall x \in \emptyset$, $\exists \varepsilon > 0$, s.t. $(x-\varepsilon, x+\varepsilon) \subset \emptyset$

(~~not open~~: $\exists x \in \emptyset$, $\forall \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \not\subset \emptyset$).