Midterm date:

22 March (Saturday) 2pm-5pm

Open:

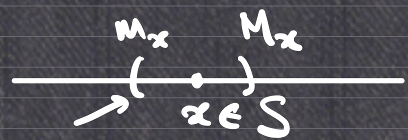
$\forall x \in \mathcal{U},\ \exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset \mathcal{U}$

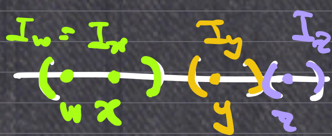
closed: E is closed $\overset{\text{def}}{\Longleftrightarrow}$ $\mathbb{R} \setminus E$ is open.

<u>Prop 1.42</u>: Any non-empty open set $S \subset \mathbb{R}$ is a disjoint union of countably many open intervals

(i.e. $S = \bigsqcup_{j=1}^{\infty} (a_j, b_j)$)

<u>Proof</u>:





I_x = largest open interval $\ni x$, $\subset S$

$$S = \bigcup_{x \in S} \underbrace{\{x\}}_{\subset I_x} \subset \boxed{\bigcup_{x \in S} \underbrace{I_x}_{\subset S} \subset S}$$

$$\Downarrow$$

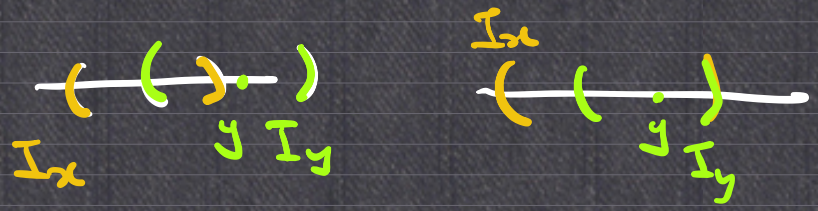
$$= \qquad\qquad =$$

$$S = \bigcup_{x \in S} I_x$$

Given $x, y \in S$, $x \neq y$

$I_x = I_y$ or $I_x \cap I_y = \emptyset$

Otherwise



$$S = \bigsqcup_{x \in T} I_x \neq \emptyset$$

← open interval

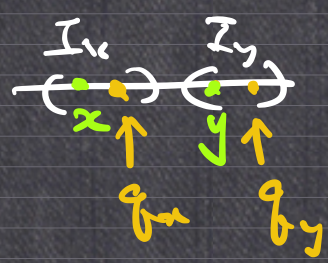
$T \subset S$

$\quad\ \cup$

$\quad\ x$

$\quad\ \cap$

$\quad\ I_x \ni q_x \in \mathbb{Q}$



$$\varphi : T \to \mathbb{Q} \quad \leftarrow \text{countable}$$

injective

$$x \mapsto q_x$$

- any union of open sets is open.

$$\boxed{\bigcup_{\alpha \in A} U_\alpha}$$

$$\forall x \in \underbrace{\bigcup_{\alpha \in A} U_\alpha}_{T} \implies x \in \underbrace{U_\alpha}_{\text{open}} \quad \exists \alpha \in A$$

$$\implies \exists \varepsilon > 0 \text{ s.t. } (x-\varepsilon, x+\varepsilon) \subset U_\alpha \subset T$$

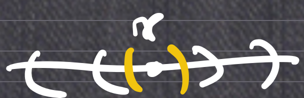
- <u>finite</u> intersection of open sets is open.

$$\underbrace{U_1 , U_2}_{\text{open}}$$

$$\forall x \in U_1 \cap U_2, \implies \begin{array}{l} x \in U_1 \\ x \in U_2 \end{array}$$

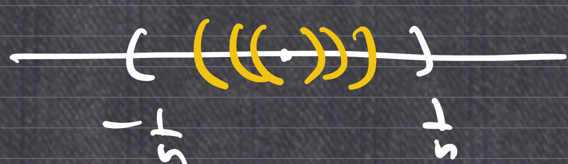
$\exists \varepsilon_1 > 0$ s.t. $(x-\varepsilon_1, x+\varepsilon_1) \subset U_1$

$\exists \varepsilon_2 > 0$ s.t. $(x-\varepsilon_2, x+\varepsilon_2) \subset U_2$



$\varepsilon := \frac{1}{2}\min(\varepsilon_1, \varepsilon_2)$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$$



$$= \{0\}$$

Prop 1.48:

$S \subset \mathbb{R}$

① S is closed ($\mathbb{R}\setminus S$ is open)

$\Longleftrightarrow$ ② any convergent seq. $\{x_n\}$ in S has limit in S.

$\boxed{\text{Given } x_n \to L}$

$$a \le x_n \le b \quad \forall n \;\Downarrow\; a \le L \le b$$

$$x_n \in [a,b] \quad \forall n \;\Downarrow\; L \in [a,b]$$

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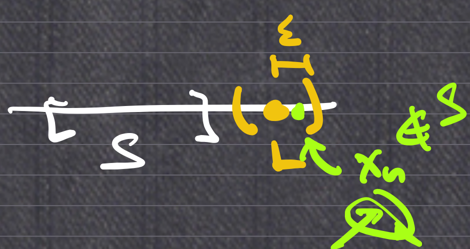
① $\Rightarrow$ ② Given S is closed.

$\Longleftrightarrow$ $\mathbb{R}\setminus S$ is open.

Assume ② is not true.

$\exists$ conv. seq $\{x_n\}$ in S $\to$ $L \notin S$. $\Rightarrow$ $\underline{L \in \mathbb{R}\setminus S}$ (open)

$\exists \varepsilon > 0$ s.t. $(L-\varepsilon, L+\varepsilon) \subset \mathbb{R}\setminus S$

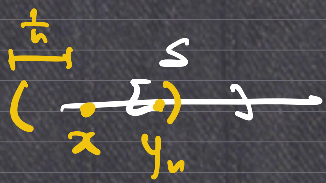


$n >> N$.

② $\Rightarrow$ ① Consider $x \in \mathbb{R} \setminus S$

WANT: $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subset \mathbb{R}\setminus S$.

If not: $\forall \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \not\subset \mathbb{R}\setminus S$



$\Rightarrow \exists\, y \in (x-\varepsilon, x+\varepsilon)$ but $\underline{y \notin \mathbb{R}\setminus S}$.

$\iff y \in S$.

$\forall n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$, $\exists\, y_n \in (x - \frac{1}{n}, x + \frac{1}{n})$

$x \in S$

e.g.: $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$.

$$\left.\begin{array}{l} x_n = \frac{1}{n} \in S \quad \forall n \\ \quad\downarrow \\ \quad 0 \notin S \end{array}\right\} \quad S \text{ is not closed.}$$

e.g.: $\{x_n\} \subset S$ (closed) bounded sequence.

BW $\Rightarrow \exists \{x_{n_k}\} \to \underbrace{L \in S}$

$\uparrow$ S closed.

Definition:

$S \subset \mathbb{R}$ is <u>sequentially</u> <u>compact</u>

$\overset{\text{def}}{\iff}$ any seq. $\{x_n\} \in S$ has a convergent subseq $\{x_{n_k}\} \longrightarrow L \in S$.

$\boxed{\text{On } \mathbb{R}}$

S closed and bounded $\Longleftrightarrow$ S is sequentially compact

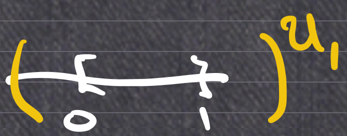
$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

S is compact

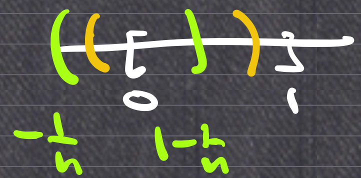
<u>Definition</u> $S \subset \mathbb{R}$ is <u>compact</u>

$\overset{\text{def}}{\Longleftrightarrow}$ any open cover of S has a finite subcover.

$$S \subset \bigcup_{\alpha \in A} \underbrace{U_\alpha}_{\text{open sets}} \Rightarrow \text{say } \{U_\alpha\} \text{ is an open cover of } S.$$



$\{U_1\}$



$$[0,1] \subset \left(\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1-\frac{1}{n}\right)\right) \cup \left(\frac{1}{2}, \frac{3}{2}\right)$$

$\bigcup_{n=1} \left\{\left(-\frac{1}{n}, 1-\frac{1}{n}\right)\right\} \cup \left\{\left(\frac{1}{2}, \frac{3}{2}\right)\right\}$

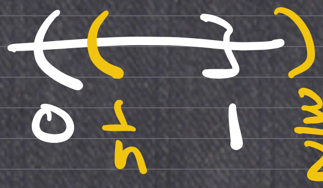
open cover of $[0,1]$.

$n=2$ $\qquad$ $n=3$

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{2}{3}\right) \cup \underline{\left(\frac{1}{2}, \frac{3}{2}\right)} \supset [0,1]$$

$\left\{\left(-\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{3}{2}\right)\right\}$

is a finite subcover of $[0,1]$.



$$\subset \underbrace{\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, \frac{3}{2}\right)}$$

$$[0,1] \subset \left(0, \frac{3}{2}\right)$$

$\left\{ \left(\frac{1}{n}, \frac{3}{2}\right) : n \in \mathbb{N} \right\}$ is open cover of $[0,1]$.

Ex: Prove $[0,1]$ is compact.

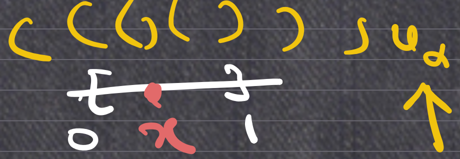
Proof: Assume not.

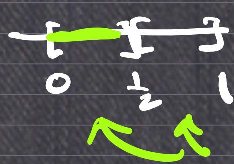
compact: any open cover of $[0,1]$ has finite subcover

not compact: $\exists$ open cover of $[0,1]$ has no finite subcover.

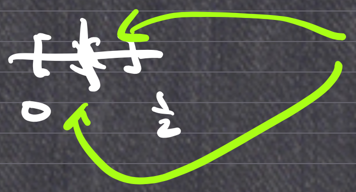
$\therefore \exists \underset{\text{open}}{\{U_\alpha\}_{\alpha \in A}}$ s.t. $[0,1] \subset \bigcup_{\alpha \in A} U_\alpha$

but $\{U_\alpha\}$ has no finite subcover.

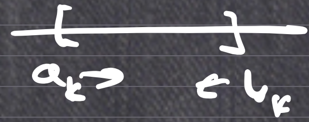




at least one cannot be covered by



finitely many $\mathcal{U}_\alpha$'s.



$\exists\ \underset{[0,1]}{\underset{\shortparallel}{I_1}} \supset I_2 \supset I_3 \supset \cdots$ s.t.

$\forall k$ I_k cannot be covered by finitely many $\mathcal{U}_\alpha$'s.

$\bigcap_{k \in \mathbb{N}}^{\infty} \underset{[a_k,\, b_k]}{\underset{\shortparallel}{I_k}} = \{x\}$

$x \in [0,1] \subset \bigcup_{\alpha \in A} \mathcal{U}_\alpha$

$\Downarrow$

$\exists \alpha \in A$ s.t.

$x \in \mathcal{U}_\alpha$.