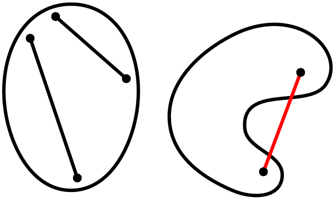
The Structure of Convexity

Roman Maksimovich

Belgrade, Serbia

November 29, 2023

Introduction



- Convexity is an extremely popular concept.
- It is rarely studied as an abstract idea.
- 'Topology-like' proof methodology is pleasant.

Convex space

Definition

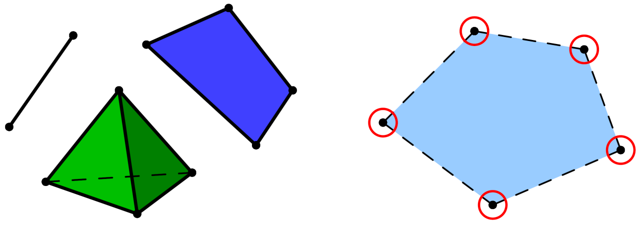
$(X, \mathcal{C})$ is a *convex space* if:

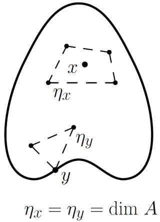
- $\varnothing$, X lie in $\mathcal{C}$;
- For every $\mathcal{A} \subset \mathcal{C}$ we have $\bigcap \mathcal{A} \in \mathcal{C}$;
- For every *net* $\mathcal{N} \subset \mathcal{C}$ we have $\bigcup \mathcal{N} \in \mathcal{C}$.

Definition

Convex hull $\langle A \rangle$ — the smallest convex set containing A.

Polytopes, freedom, dimension

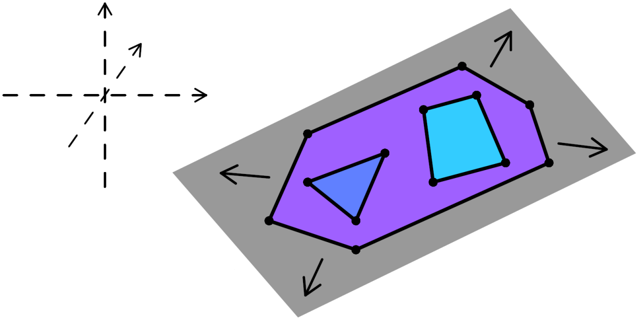




Hyperplanes

Definition

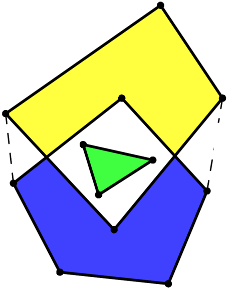
Hyperplane — union of a **maximal** net of polytopes of the same dimension.

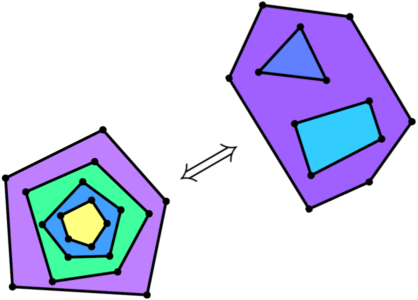


The Polytope Union Lemma

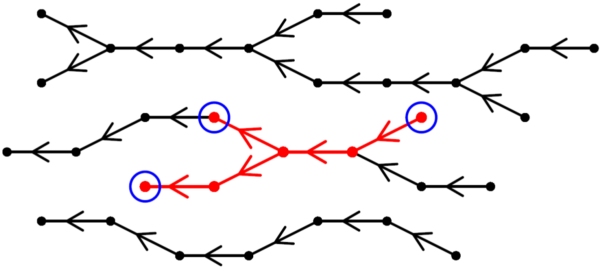
Lemma

Let P, Q, L be polytopes of equal dimension, $L \subset P \cap Q$. Then the dimension of $\langle P \cup Q \rangle$ is m.





Order convexity



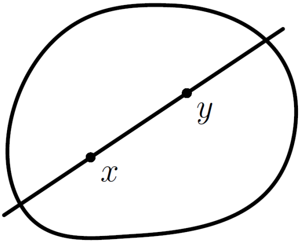
Theorem

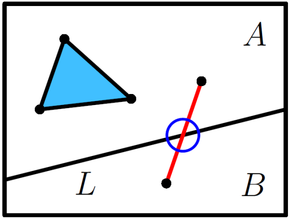
Every ordered convex space is free, i.e. contains only free polytopes.

n-Affinity

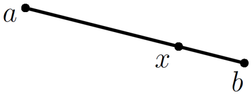
Definition

1-Affine convex space $\iff$ each segment's convexity is induced by a linear order.

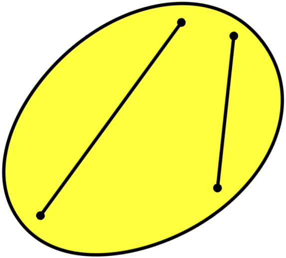




Metric convexity



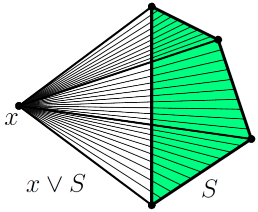
$$d(a, b) = d(a, x) + d(x, b)$$

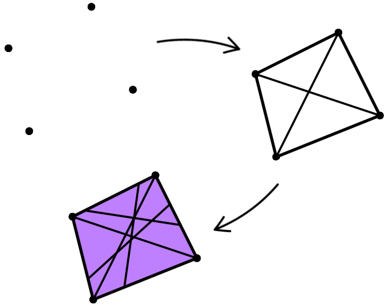


Definition

Convex $\iff$ contains the segment connecting every pair of points.

Join, Finite-segmentiality





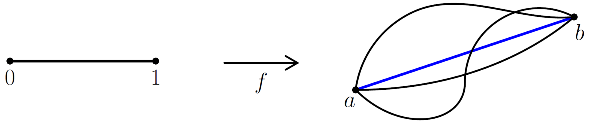
Theorem

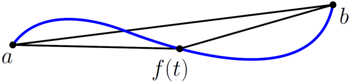
2-Affine + TPUL + Finite-segmential $\implies$ *Free.*

Uniquely Geodesic Metric Spaces

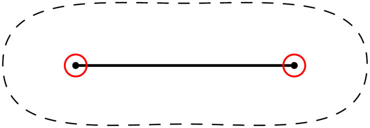
Definition

UGS: There is a unique path f such that $|f| = d(a, b)$.

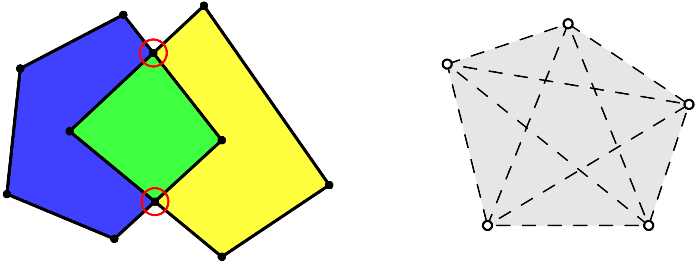




$$d(a,b) = d(a, f(t)) + d(f(t), b)$$



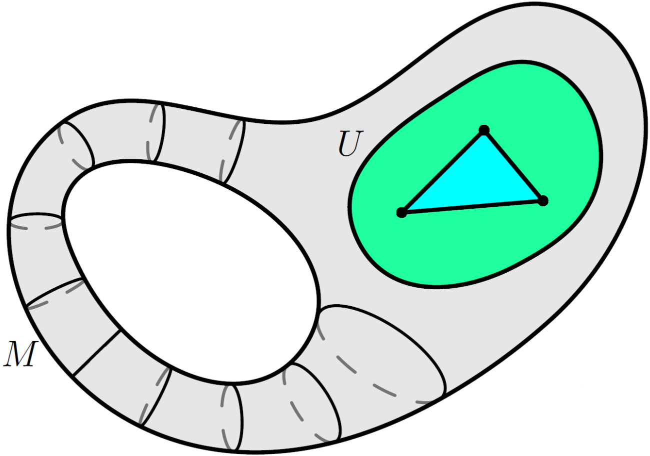
The Polytope Intersection Lemma



Definition

Free + Finite-dimensional + TPUL + TPIL $\implies$ Topology

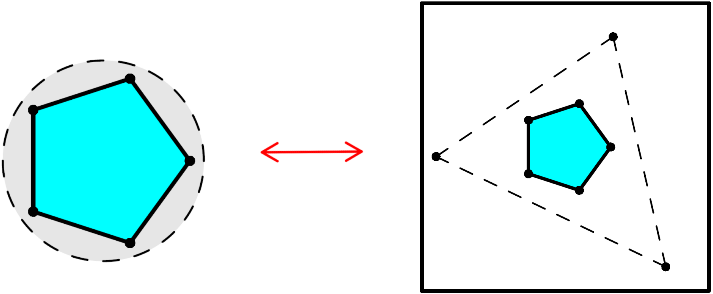
Local convexity



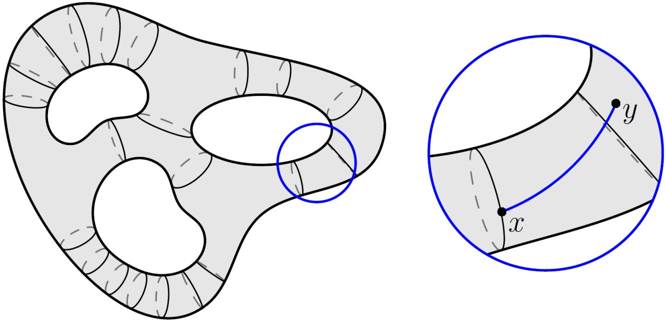
Local isomorphism

Lemma

$\mathbb{R}^2$ is not isomorphic to B^2, but they are locally isomorphic.



Riemannian manifolds



Lemma

All Riemannian manifolds are locally uniquely geodesic.

Summary of results

- **Internal theory:** Finite nature of convexity, technical statements, hyperplane properties, TPUL and its connection to hyperplanes.
- **Inducing structures:** Freedom of order convexities, Linear and 1-affine space properties, *n*-affinity, sufficient conditions for join-commutativity and freedom, attributes of UGS.
- **Induced structures:** Polytope interior, convex topology, Riemannian convexity.

Thank you for your attention!