# MATH1023 Homework, Part 2

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**Exercise 1.1.31 (11).** Find the limit $(n \to \infty)$ of $(an^2 + b)^{\frac{c}{n+d}}$, where $a > 0$.
*Solution:* In the following inequalities the $\le$ sign will mean "less than or equal to, for sufficiantly large $n$". We will also assume for now that $c \ge 0$. Trivially, we have

$$1 \le (an^2 + b)^{\frac{c}{n+d}} \le ((a+1)n^2)^{\frac{c}{n+d}} \le ((a+1)n^2)^{\frac{2c}{n}} = (a+1)^{\frac{2c}{n}} \cdot \left(\sqrt[n]{n}\right)^{4c} \underset{n\to\infty}{\to} 1,$$

since $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, and thus the limit in question is equal to 1. When $c < 0$, the similar result follows from the arithmetic rule by considering $b_n = 1/a_n$, where $a_n$ is the original sequence.

**Exercise 1.1.32 (5).** Find the limit $(n \to \infty)$ of $(n - \cos(n))^{\frac{1}{n+\sin(n)}}$, where $a > 0$.
*Solution:*

$$1 \le (n - \cos(n))^{\frac{1}{n+\sin(n)}} \le (2n)^{\frac{2}{n}} = 2^{\frac{2}{n}} \cdot \left(\sqrt[n]{n}\right)^2 \underset{n\to\infty}{\to} 1,$$

thus the original limit is 1.

**Exercise 1.1.34 (6).** Find the limit $(n \to \infty)$ of $\sqrt[n]{4^{2n-1} + (-1)^n 5^n}$.
*Solution:* We will rewrite $4^{2n-1}$ as $\frac{1}{4}16^n$. It is obvious that starting at some $n$, we have

$$\frac{1}{8}16^n \le (-1)^n 5^n \le \frac{1}{8}16^n.$$

Consequently, we can write

$$16 \underset{n\to\infty}{\leftarrow} 16 \cdot \sqrt[n]{\frac{1}{8}} = \sqrt[n]{\frac{1}{8}16^n} \le \sqrt[n]{\frac{1}{4}16^n + (-1)^n 5^n} \le \sqrt[n]{\frac{3}{8}16^n} = 16 \cdot \sqrt[n]{\frac{3}{8}} \underset{n\to\infty}{\to} 16,$$

which means that 16 is the answer by the sandwich rule.

**Exercise 1.1.35 (8).** Find the limit $(n \to \infty)$ of $(a^n - b^n)^{\frac{n}{n^2-1}}$, where $a > b > 0$.
*Solution:* We will first establish that $a_n := (a^n - b^n)^{\frac{1}{n}} \to a$. Indeed,

$$a \ge (a^n - b^n)^{\frac{1}{n}} = a \cdot \left(1 - \left(\frac{b}{a}\right)^n\right)^{\frac{1}{n}} \ge a \cdot \left(1 - \left(\frac{b}{a}\right)\right)^{\frac{1}{n}} \underset{n\to\infty}{\to} a,$$

which leads to $a_n \to a$ by the sandwith rule. Now, since $\frac{n}{n^2-1} > \frac{1}{n}$ and $a^n - b^n > 1$ for sufficiently large $n$, we write

$$a \underset{n\to\infty}{\leftarrow} a \cdot a^{\frac{1}{n^2-1}} = a^{\frac{n^2}{n^2-1}} > (a^n - b^n)^{\frac{n}{n^2-1}} > (a^n - b^n)^{\frac{1}{n}} \underset{n\to\infty}{\to} a,$$

and conclude that the limit in question is $a$ by the sandwich rule.

**Exercise 1.1.38.** Show that $\lim\limits_{n\to\infty} n^2 a^n = 0$ for $|a| < 1$ in two ways (Example 1.1.12, and the fact that $\lim\limits_{n\to\infty} na^n = 0$ for $|a| < 1$).
*Solution:*
**(first method)** We represent $|a|$ as $\frac{1}{1+b}$ where $b > 0$ (since $|a| < 1$). Now,

$$\left|n^2a^n\right| = \frac{n^2}{(1+b)^n} = \frac{n^2}{1+nb+\frac{n(n-1)}{2}b^2+\frac{n(n-1)(n-2)}{6}b^3+\ldots} \leq \frac{6n^2}{n(n-1)(n-2)b^3} = \frac{6}{b^3}\cdot\frac{n}{n-1}\cdot\frac{1}{n-2}.$$

By recognizing that $\frac{n}{n-1} \to 1$ and $\frac{1}{n-2} \to 0$, we see that $\left|n^2a^n\right|$ tends to 0, meaning that $n^2a^n$ tends to 0 as well, by Exercise 1.1.13.

**(second method)** We have that $\lim\limits_{n\to\infty} na^n = 0$ for $|a| < 1$. Considering that $\sqrt[n]{n} \to 1$, we see that

$$\left|\sqrt[n]{na^n}\right| = |a|\sqrt[n]{n} \to |a|,$$

and thus, for a sufficiently small $\varepsilon > 0$ and sufficiently large $n$, we have

$$\left|\sqrt[n]{na^n}\right| < |a| + \varepsilon < 1.$$

Now, we return to the sequence in question and consider its absolute value:

$$\left|n^2a^n\right| = \left|n\cdot\left(\sqrt[n]{na^n}\right)^n\right| = n\cdot\left(\left|\sqrt[n]{na^n}\right|\right)^n \leq n\cdot(|a|+\varepsilon)^n \underset{n\to\infty}{\to} 0,$$

since $|a| + \varepsilon < 1$.

**<u>Exercise 1.1.41.</u>** Show that $\lim\limits_{n\to\infty}\frac{a^n}{\sqrt{n!}} = 0$ for $a = 4$.
*<u>Solution:</u>* For sufficiently large $n$, we write

$$0 \leq \frac{4^n}{\sqrt{n!}} = \sqrt{\frac{16^n}{n!}} = \sqrt{\frac{16^{16}}{16!}\cdot\frac{16}{17}\cdot\frac{16}{18}\cdot\ldots\cdot\frac{16}{n}} \leq \sqrt{\frac{16^{16}}{16!}\cdot\frac{16}{n}} = A\cdot\frac{1}{\sqrt{n}} \underset{n\to\infty}{\to} 0,$$

and the result follows from the sandwich rule.

**<u>Exercise 1.1.43 (7).</u>** Find the limit $(n\to\infty)$ of $(n3^n)/(1+2^n)^2$.
*<u>Solution:</u>* First we will find the limit of $(n3^n)/(2^n)^2$. It follows simply from Example 1.1.13 by the representation

$$\frac{n3^n}{(2^n)^2} = n\cdot\left(\frac{3}{4}\right)^n,$$

that the limit is equal to 0. Now, we consider the ratio

$$\frac{\frac{n3^n}{(2^n)^2}}{\frac{n3^n}{(1+2^n)^2}} = \left(1+\frac{1}{2^n}\right)^2 \underset{n\to\infty}{\to} 1,$$

and thus the limit of $(n3^n)/(1+2^n)^2$ is also 0, by the arithmetic rule of division.

**<u>Exercise 1.1.45.</u>** Show that $\lim\limits_{n\to\infty}\frac{n^pa^n}{\sqrt{n!}} = 0$ for any $a$.
*<u>Solution:</u>* We pick $b$ such that $|b| > |a|$. Now, by Example 1.1.15, we have

$$\frac{n^pa^n}{b^n} = n^p\cdot\left(\frac{a}{b}\right)^n \underset{n\to\infty}{\to} 0.$$

We also have $\frac{b^n}{\sqrt{n!}} \underset{n\to\infty}{\to} 0$ by Exercise 1.1.41. Now, multiplying the two sequences, we see that the limit in question is also 0.

**<u>Exercise 1.1.51 (6).</u>** Find the limit $(n\to\infty)$ of $\sqrt[n-2]{n2^{3n}+\frac{3^{2n-1}}{n^2}}$.
*<u>Solution:</u>* For simplicity we will first take the $n$-th root instead of the $(n-2)$-th root. We have

$$\sqrt[n]{n2^{3n}+\frac{3^{2n-1}}{n^2}} = \sqrt[n]{\frac{9^n}{3n^2}}\cdot\sqrt[n]{1+\frac{3n^3 8^n}{9^n}} = 9\cdot\frac{1}{\sqrt[n]{3}}\cdot\left(\frac{1}{\sqrt[n]{n}}\right)^2\cdot\sqrt[n]{1+3n^3\left(\frac{8}{9}\right)^n}\underset{n\to\infty}{\to} 9\cdot 1\cdot 1\cdot 1 = 9.$$

Now, we express $\frac{1}{n-2}$ as $\frac{1}{n}+\frac{2}{n(n-2)}$, and see that

$$\sqrt[n-2]{n2^{3n}+\frac{3^{2n-1}}{n^2}} = \left(\sqrt[n]{n2^{3n}+\frac{3^{2n-1}}{n^2}}\right)\cdot\left(\sqrt[n]{n2^{3n}+\frac{3^{2n-1}}{n^2}}\right)^{\frac{2}{n-2}}\underset{n\to\infty}{\to} 9\cdot 1 = 9,$$

due to the arithmetic rule $\lim\limits_{n\to\infty} x_n y_n = \left(\lim\limits_{n\to\infty} x_n\right)\left(\lim\limits_{n\to\infty} y_n\right)$. In the above derivation, to conclude that the right part of the product approaches 1, we used the fact that if $x_n\to\xi>0$, then $x_n^{\frac{2}{n-2}}\to 1$. This is due to the sandwich rule: let $0<\alpha<\xi<\beta$. Starting from some point, we have $\alpha<x_n<\beta$. Now,

$$1\underset{n\to\infty}{\leftarrow}\alpha^{\frac{2}{n-2}}<x_n^{\frac{2}{n-2}}<\beta^{\frac{2}{n-2}}\underset{n\to\infty}{\to}1.$$

Now we can be sure that the answer is indeed 9.

**Exercise 1.1.52 (11).** Find the limit ($n\to\infty$) of $((n+1)a^n+b^n)^{\frac{n}{n^2-1}}$.
*Solution:* For simplicity and out of concern for the well-definition of real exponentiation, we will assume that $a>b>0$. Consider the sequence $((n+1)a^n)^{\frac{n}{n^2-1}}$. By considering its quotient with the original sequence and writing inequalities, we see that

$$1\leq\frac{((n+1)a^n+b^n)^{\frac{n}{n^2-1}}}{((n+1)a^n)^{\frac{n}{n^2-1}}} = \left(1+\frac{1}{n+1}\left(\frac{b}{a}\right)^n\right)^{\frac{n}{n^2-1}}\leq\left(1+\left(\frac{b}{a}\right)^n\right)^{\frac{n}{n^2-1}}\leq 1+\left(\frac{b}{a}\right)^n\to 1,$$

and the quotient approaches 1 by the sandwich rule. By the arithmetic rule, this means that we may instead find the limit of $((n+1)a^n)^{\frac{n}{n^2-1}}$. Indeed,

$$((n+1)a^n)^{\frac{n}{n^2-1}} = \left(\sqrt[n+1]{n+1}\right)^{\frac{n}{n-1}}\cdot a^{\frac{n^2}{n^2-1}} = \sqrt[n+1]{n+1}\cdot\left(\sqrt[n+1]{n+1}\right)^{\frac{1}{n-1}}\cdot a\cdot a^{\frac{1}{n^2-1}}\underset{n\to\infty}{\to}1\cdot 1\cdot a\cdot 1 = a.$$

If $b<a\leq 0$, then the given expression is not defined for odd $n$. If, now, $b\leq 0<a$ and $a<|b|$, then the expression is again not defined for odd $n$. If, however, $b\leq 0<a$, and $a\geq|b|$, then the expression becomes well-defined since $(n+1)a^n>|b^n|$. In this case we bound the expression (for sufficiently large $n$) as follows:

$$\left(\frac{n+1}{2}a^n+|b|^n\right)^{\frac{n}{n^2-1}}\leq((n+1)a^n+b^n)^{\frac{n}{n^2-1}}\leq((n+1)a^n+|b|^n)^{\frac{n}{n^2-1}}.$$

As already established, both the leftmost and the rightmost sequence approach $a$, and thus the original sequence also approaches $a$ by the sandwich rule.

**Exercise 1.1.58.** Prove that if $\lim\limits_{n\to\infty}\left|\frac{x_n}{x_{n-1}}\right| = l<1$, then $x_n$ converges to 0.
*Solution:* By the order rule, for sufficiently large $n$ (say, from $N$) we have

$$\left|\frac{x_n}{x_{n-1}}\right|<l+\varepsilon<1,$$

where $\varepsilon$ is a small positive number. Now, for $n>N$ we express $|x_n|$ as

$$|x_n| = |x_N|\cdot\left|\frac{x_{N+1}}{x_N}\right|\cdot\left|\frac{x_{N+2}}{x_{N+1}}\right|\cdot\ldots\cdot\left|\frac{x_n}{x_{n-1}}\right|<|x_N|\cdot(l+\varepsilon)^{n-N}\underset{n\to\infty}{\to}0,$$

since $l + \varepsilon < 1$. Therefore, $x_n$ converges to 0.

**Exercise 1.1.59 (7).** Find $a$ such that the sequence $x_n = \frac{a^{n^2}}{\sqrt{n!}}$ converges to 0.
*Solution:* Consider the d'Alambertian ratios:

$$\left|\frac{x_n}{x_{n-1}}\right| = \left|\frac{a^{2n-1}}{\sqrt{n}}\right|.$$

These ratios approach 0 when $|a| \leq 1$, and in that case the original sequence $x_n$ will approach 0 as well (by Exercise 1.1.58). If, on the other hand, $|a| > 1$, then the ratios diverge to infinity:

$$\left|\frac{\sqrt{n}}{a^{2n-1}}\right| \underset{n\to\infty}{\to} 0 \Longrightarrow \left|\frac{a^{2n-1}}{\sqrt{n}}\right| \underset{n\to\infty}{\to} \infty.$$

In the above implication, the premise can be obtained from Example 1.1.13. Therefore, since the ratios diverge to infinity, the original sequence diverges as well. We demonstrate this by picking $M > 1$ and stating that $\left|\frac{x_n}{x_{n-1}}\right| > M$ for sufficiently large $n$ (say, $n > N$). Now, for $n > N$ we have

$$|x_n| = |x_N| \cdot \left|\frac{x_{N+1}}{x_N}\right| \cdot \left|\frac{x_{N+2}}{x_{N+1}}\right| \cdot \ldots \cdot \left|\frac{x_n}{x_{n-1}}\right| \geq |x_N| \cdot M^{n-N} \underset{n\to\infty}{\to} \infty,$$

and thus $x_n \to \infty$ by the order rule, meaning that $x_n$ cannot converge to a finite number.

**Exercise 1.1.61 (2, 6).** Explain the convergence/divergence of sequences.

1. $x_n = n^{\frac{(-1)^n}{n}}$
   *Solution:* Consider the odd subsequence:

   $$x_{2n-1} = (2n-1)^{-\frac{1}{2n-1}} = \frac{1}{\sqrt[2n-1]{2n-1}} \underset{n\to\infty}{\to} 1,$$

   by Example 1.1.9. For the even subsequence, the situation is analogous: $x_{2n} \underset{n\to\infty}{\to} 1$. As a result, the entire sequence also converges to 1, as a union of two subsequences converging to 1.

2. $x_n = \sqrt{n}\left(\sqrt{n + (-1)^n} - \sqrt{n}\right)$
   *Solution:* First consider the even subsequence: $x_k = \sqrt{k}\left(\sqrt{k+1} - \sqrt{k}\right)$ with $k$ even. We multiply and divide by the conjugate:

   $$\sqrt{k}\left(\sqrt{k+1} - \sqrt{k}\right) = \frac{\sqrt{k}\left(\sqrt{k+1} - \sqrt{k}\right)\left(\sqrt{k+1} + \sqrt{k}\right)}{\sqrt{k+1} + \sqrt{k}} = \frac{\sqrt{k}}{\sqrt{k+1} + \sqrt{k}} \underset{k\to\infty}{\to} \frac{1}{2}.$$

   By analogy, we see that the odd subsequence, $x_k = \sqrt{k}\left(\sqrt{k-1} - \sqrt{k}\right)$ with $k$ odd, approaches $-\frac{1}{2}$. As a result, the main sequence diverges, since two of its subsequences converge to different values.

**Exercise 1.1.63.** Extend Exercise 1.1.27, by dropping the requirement that all $x_n \geq 1$: If $\lim\limits_{n\to\infty} x_n = 1$ and $y_n$ is bounded, then $\lim\limits_{n\to\infty} x_n^{y_n} = 1$.
*Solution:* Since $y_n$ is bounded, there are $p, q \in \mathbb{R}$ such that $p \leq y_n \leq q$ for all $n$. Consider the subsequence $x_{k_n}$ of those elements that are not greater than 1. Then, we have

$$x_{k_n}^q \leq x_{k_n}^{y_{k_n}} \leq x_{k_n}^p.$$

Both the leftmost and the rightmost sequences converge to 1 by Example 1.1.23, and thus the middle sequence converges to 1 by the sandwich rule. Now, an analogous argument (with reversed inequalities) proves that the subsequence $x_{m_n}$ of all elements $\geq 1$ also has $x_{m_n}^{y_{m_n}}$ converge to 1. As a result, $x_n^{y_n}$ converges to 1 as a union of two subsequences converging to 1.