Dynamics of Rigid Bodies II

Intended Learning Outcomes – after this lecture you will learn:
1. vector product
2. torque, and the Newton's second law in rotational dynamics
3. how to deal with a rigid body rotating about a moving axis, e.g. a yo-yo

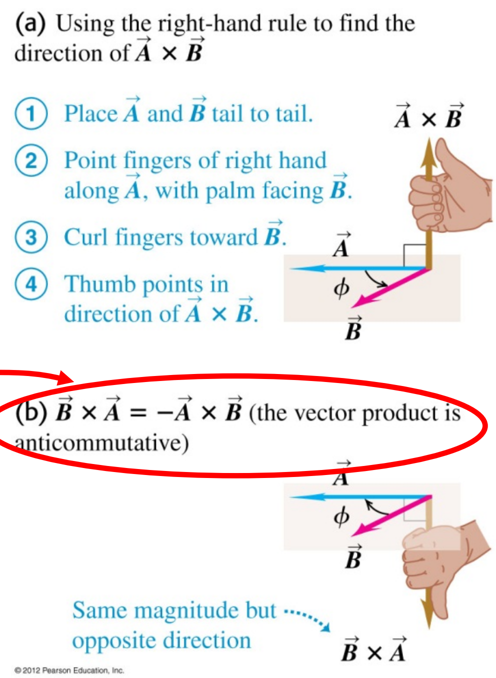
Textbook Reference: Ch 10.1, 10.2, 10.3

Vector (cross) product:

$\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$

Magnitude: $C = AB \sin\phi$

direction determined by *Right Hand Rule*

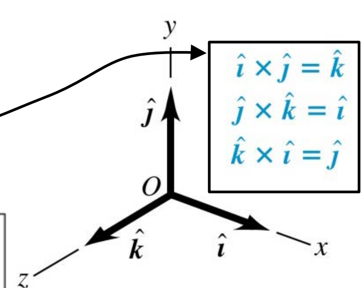


Important!

Special cases:

(i) if $\vec{\boldsymbol{A}} \parallel \vec{\boldsymbol{B}}$, $|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = 0$, in particular, $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

(ii) if $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$, $|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = AB$, in particular,



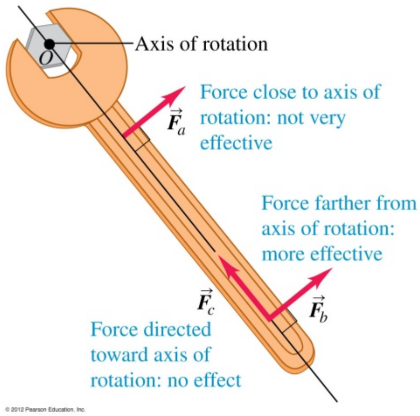
In analytical form (no need to memorize this)

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

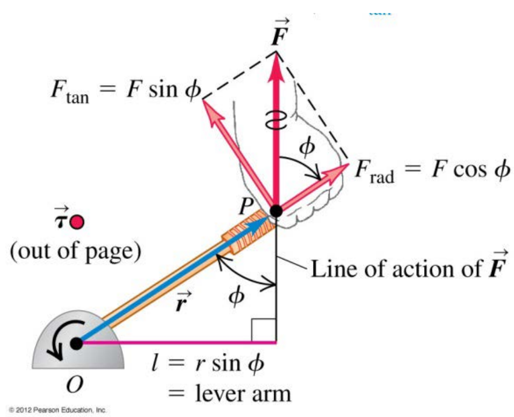
don't worry if you have not learnt determinants in high school

Torque

Besides magnitude and direction, the **line of action** of a force is important because it produces rotation effect.



$\vec{\boldsymbol{F}}_a$ and $\vec{\boldsymbol{F}}_b$ have the same magnitudes and directions, but different line of action: they produce different physical effects – which force would you apply if you were to tighten/loosen the screw?



Define **torque** about a point O as a vector

$$\boxed{\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}}$$

⚠ $\vec{\boldsymbol{\tau}}$ is $\perp$ to both $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$

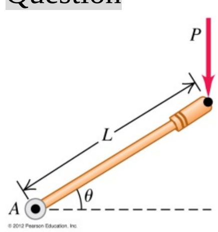
Magnitude: $\tau = r\underbrace{(F \sin \phi)}_{\text{component of } \vec{\boldsymbol{F}} \perp \text{ to } \vec{\boldsymbol{r}}} = \underbrace{(r \sin \phi)}_{\perp \text{ distance from } O \text{ to line of actions of } \vec{\boldsymbol{F}}}F$

Direction gives the sense of rotation about O through the right-hand-rule.

Notation: ⊙ out of the plane ⊗ into the plane

SI unit for torque: N·m (just like work done)

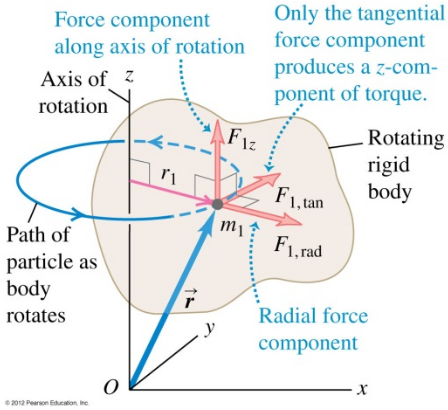
Question



A force P is applied to one end of a lever of length L. The magnitude of the torque of this force about point A is ($PL \sin \theta$ / $PL \cos \theta$ / $PL \tan \theta$)

Answer: see inverted text on P. 333 of textbook

Suppose a rigid body is rotating about a fixed axis which we arbitrarily call the *z* axis. m_1 is a small part of the total mass.



$F_{1,\text{rad}}$, $F_{1,\text{tan}}$, and $F_{1,z}$ are the 3 components of the total force acting on m_1

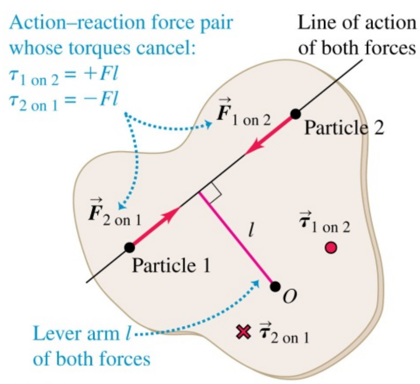
Only $F_{1,\text{tan}}$ produces the desired rotation, $F_{1,\text{rad}}$ and $F_{1,z}$ produce some other effects which are irrelevant to the rotation about the *z* axis.

$$F_{1,\text{tan}} = m_1 a_{1,\text{tan}} = m_1(r_1\alpha_z)$$

$$\underbrace{F_{1,\text{tan}} r_1}_{\text{torque on } m_1 \text{ about } z,\ \tau_{1z}} = m_1 r_1^2 \alpha_z$$

Sum over all mass in the body, since they all have the same α_z

$$\sum \tau_{iz} = \left(\sum m_r r_i^2\right)\alpha_z = I\alpha_z$$



⚠ Need to consider torque due to external forces only. Internal forces (action and reaction pairs) produce equal and opposite torques which have no net rotational effect.

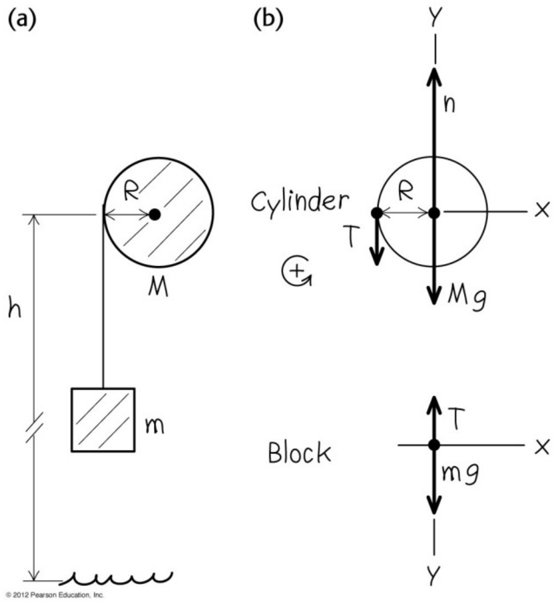
Conclusion: for rigid body rotation about a fixed axis,

$$\boxed{\sum \tau_{\text{ext}} = I\alpha}$$

c.f. Newton's second law $\sum \vec{\boldsymbol{F}}_{\text{ext}} = M\vec{\boldsymbol{a}}$

Example 10.3 P. 335 An unwinding cable

Pulley rotates about a fixed axis. What is the acceleration a of the block?



For the cylinder

$$\underbrace{TR}_{\text{torque due to } T} = \underbrace{\left(\frac{1}{2}MR^2\right)}_{\text{moment of inertia of cylinder}} \underbrace{\left(\frac{a}{R}\right)}_{\text{angular acceleration}}$$

i.e.

$$T = \frac{1}{2}Ma$$

For the block

$$mg - T = ma$$

Therefore

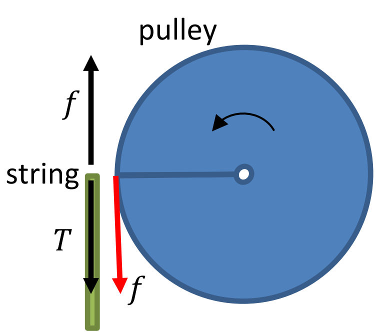
$$a = \frac{g}{1 + M/2m}$$

Suppose the block is initially at rest at height h. At the moment it hits the floor:

$$v^2 = 0 + 2\left(\frac{g}{1 + M/2m}\right)h \quad \Longrightarrow \quad v = \sqrt{\frac{2gh}{1 + M/2m}}$$

c.f. lecture 8 in which we get the same result using energy conservation.

How does the tension pull the pulley? Through the friction f between the string and pulley

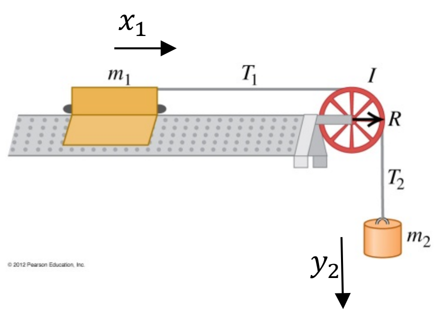


A simplified argument:

For a light string, $f = T$, i.e., friction pulling the pulley is the same (magnitude and direction) as the tension in the string. The reaction pair of f pulls the pulley. In particular, if there is no friction, m is free falling and T is zero too.

Digression: if you are interested, you can read https://doi.org/10.1119/1.5016040 for a detail analysis of this problem.

Question



Mass m_1 slides on a frictionless track. The pulley has moment of inertia I about its rotation axis, and the string does not slip nor stretch. When the hanging mass m_2 is released, arrange the forces T_1, T_2, and $m_2 g$ in increasing order of magnitude.

Answer: see inverted text on P. 336 of textbook

Equation of motion of masses and pulley:

$$m_2 g - T_2 = m_2 \ddot{y}_2, \qquad T_1 = m_1 \ddot{x}_1, \qquad (T_2 - T_1)R = I\ddot{\theta}$$

Constraints: $\Delta y_2 = \Delta x_1 = R\Delta\theta \;\Rightarrow\; \ddot{y}_2 = \ddot{x}_1 = R\ddot{\theta}$, we get

$$a = \frac{1}{m_1 + m_2 + I/R^2} m_2 g, \qquad T_1 = \frac{m_1}{m_1 + m_2 + I/R^2} m_2 g, \;\; T_2 = \frac{m_1 + I/R^2}{m_1 + m_2 + I/R^2} m_2 g$$

Demonstration: [falling faster than g] – same angular acceleration (same rod), the far end of the rod has linear acceleration larger than g.

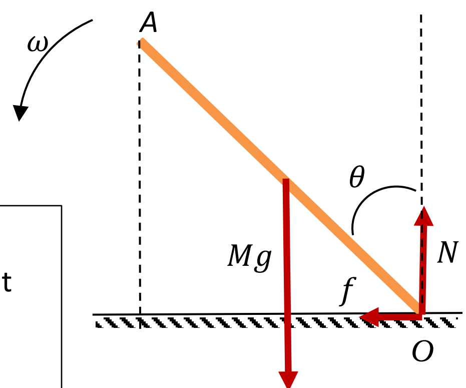
Falling faster than g

A uniform rod of mass M and length L. Suppose friction f is large enough so that there is no slipping at the point O. Want to find downward acceleration of point A.

Tangential acceleration of A is $L\ddot{\theta}$

Radial acceleration of A is $L\dot{\theta}^2$

$$\ddot{x}_A = -\left(L\ddot{\theta}\right)\cos\theta + \left(L\dot{\theta}^2\right)\sin\theta$$
$$\ddot{y}_A = -\left(L\ddot{\theta}\right)\sin\theta - \left(L\dot{\theta}^2\right)\cos\theta$$



Alternatively, you can directly differentiate the coordinates of A, $x_A = -L\sin\theta$ and $y_A = L\cos\theta$ to get

$$\dot{x}_A = -L\cos\theta\,\dot{\theta}, \quad \ddot{x}_A = -L\cos\theta\,\ddot{\theta} + L\sin\theta\,\dot{\theta}^2$$
$$\dot{y}_A = -L\sin\theta\,\dot{\theta}, \quad \ddot{y}_A = -L\sin\theta\,\ddot{\theta} - L\cos\theta\,\dot{\theta}^2$$

From torque equation at O

$$\frac{MgL}{2}\sin\theta = \left(\frac{1}{3}ML^2\right)\ddot{\theta} \Rightarrow \quad \ddot{\theta} = \frac{3g}{2L}\sin\theta$$

From conservation of energy

$$\frac{MgL}{2}\cos\theta_0 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\dot{\theta}^2 + \frac{MgL}{2}\cos\theta \;\;\Rightarrow\;\; \dot{\theta}^2 = \frac{3g}{L}(\cos\theta_0 - \cos\theta)$$

Alternatively, you can get $\dot{\theta}^2$ by integrating $\ddot{\theta}$ from the torque equation:

$$\ddot{\theta}\,d\theta = \dot{\theta}\,d\dot{\theta} = \frac{3g}{2L}\sin\theta\,d\theta \;\;\Rightarrow\;\; \dot{\theta}^2 = \frac{3g}{L}(\cos\theta_0 - \cos\theta)$$

Therefore, vertical acceleration of point A (taking upwards as positive)

$$\ddot{y}_A = -L\sin\theta\,\ddot{\theta} - L\cos\theta\,\dot{\theta}^2 = -\frac{3}{2}g(\sin^2\theta - 2\cos^2\theta + 2\cos\theta_0\cos\theta)$$

Note that when θ is large enough, $|\ddot{y}_A| \geq g$, i.e., point A falls faster than g.

To find f and N at the contact point, write down equations of motion of the CM,

$$f = -M\ddot{x}_{\text{CM}} = -M\left(-\frac{L}{2}\cos\theta\,\ddot{\theta} + \frac{L}{2}\sin\theta\,\dot{\theta}^2\right)$$

$$N - Mg = M\ddot{y}_{\text{CM}} = M\frac{\ddot{y}_A}{2}$$

We get

$$f = \frac{1}{4}Mg\sin\theta\,(9\cos\theta - 6\cos\theta_i)$$

$$N = \frac{1}{4}Mg(\sin^2\theta + 10\cos^2\theta - 6\cos\theta\cos\theta_i)$$

⚠ Note that the torque equation $\sum\tau_{\text{ext}} = I\ddot{\theta}$ holds at *O* and the CM, but not at *A* (check them). *A* is understandable because it is moving and therefore not a rotation axis. But the CM is also moving. Why does the torque equation hold? This implies that there is something special about the CM.

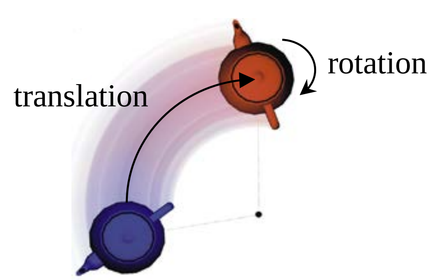
We know how to deal with:

1. translation of a point particle (or CM of a rigid body): $\sum\vec{\boldsymbol{F}}_{\text{ext}} = m\vec{\boldsymbol{a}}$
2. rotation of a rigid body about a *fixed* axis: $\sum\tau_{\text{ext}} = I\alpha$

In general, a rigid body is rotating about a *moving* axis, i.e., has both types of motion simultaneously.

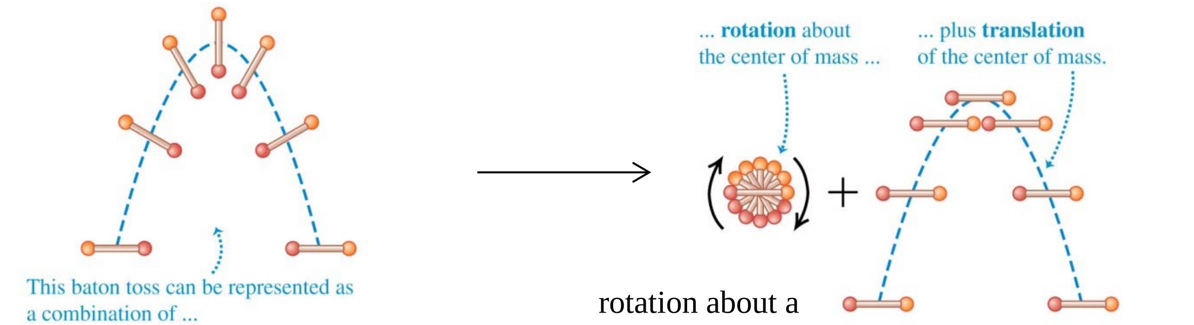
Theorem: (stated without proof here)
Every possible motion of a rigid body can be represented as

i. a translational motion of the CM (as if no rotation), plus
ii. a rotation about an axis through its CM (as if no translation)



⚠ This does not only apply to the position and orientation of the rigid body. It applies to the dynamics, e.g. torque and energy, **even when the CM is accelerating**.

e.g. tossing a baton

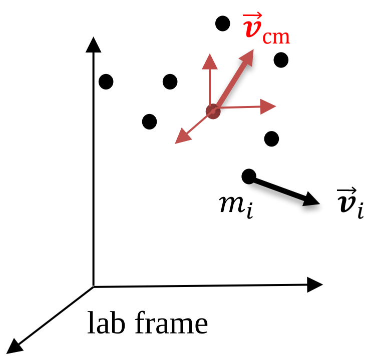


rotation about a moving axis → rotation about a fixed axis through CM + translation of CM (considered as a particle)

KE measured in lab frame (both translation and rotation)	=	$\frac{1}{2}I\omega^2$ as if no translation	+	$\frac{1}{2}Mv_{\text{cm}}^2$ as if no rotation

Energy consideration Second part of the **König Theorem**

First consider a system consisting of particles, CM moving with velocity $\vec{\boldsymbol{v}}_{\text{cm}}$ relative to the lab



$\vec{\boldsymbol{v}}_i'$ velocity of m_i relative to CM, its velocity relative to the lab is $\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{v}}_i'$

KE of m_i in lab frame:

$$K_i = \frac{1}{2}m_i(\vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{v}}_i')\cdot(\vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{v}}_i')$$
$$= \frac{1}{2}m_i(v_{\text{cm}}^2 + 2\vec{\boldsymbol{v}}_{\text{cm}}\cdot\vec{\boldsymbol{v}}_i' + v_i'^2)$$

Total KE of the rigid body in lab frame:

$$K = \sum K_i$$
$$= \frac{1}{2}\underbrace{\left(\sum m_i\right)}_{M} v_{\text{cm}}^2 + \vec{\boldsymbol{v}}_{\text{cm}}\cdot\underbrace{\left(\sum m_i\vec{\boldsymbol{v}}_i'\right)}_{\text{Velocity of CM relative to CM}} + \underbrace{\sum\left(\frac{1}{2}m_i v_i'^2\right)}_{\text{KE in CM frame}}$$

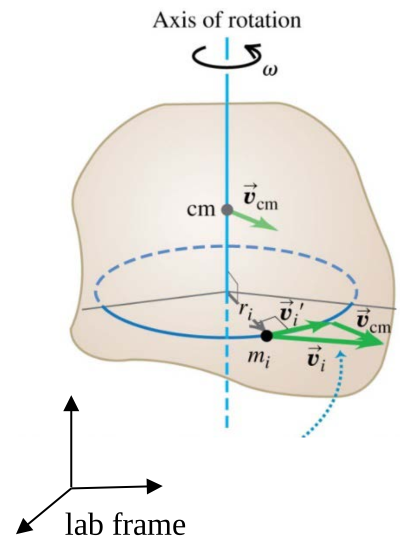
$$\boxed{K = \frac{1}{2}Mv_{\text{cm}}^2 + \sum\left(\frac{1}{2}m_i v_i'^2\right)}$$

In words, the **KE in the lab frame is the KE of the CM plus the KE in the CM frame**

⚠ The CM frame can be accelerating, but cannot be rotating in the lab frame

Corollary:

1. The total energy in the CM frame is the minimal energy observed in any *inertial* frame.
2. A completely inelastic collision means the KE lost (in lab frame) is a maximum.

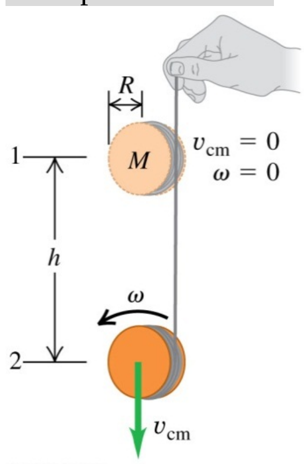


As a special case, if the system is a rigid body rotating about an axis with fixed orientation, then

$$\sum \tfrac{1}{2} m_i v_i'^2 = \sum \tfrac{1}{2} m_i r_i^2 \omega^2 = \tfrac{1}{2} I \omega^2$$

$$\boxed{K = \underbrace{\tfrac{1}{2} M v_{cm}^2}_{\text{as if no rotation}} + \underbrace{\tfrac{1}{2} I \omega^2}_{\text{as if no translation}}}$$

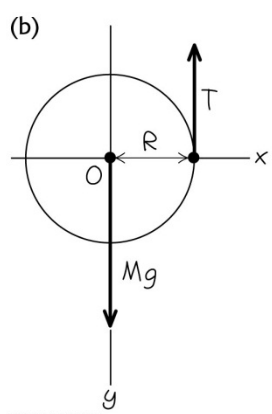
Example 10.6 P. 340 A yo-yo



To find v_{cm} at point 2, need energy conservation

$$\underset{\text{initial KE}}{0} + M\underset{\text{initial PE}}{gh} = \underbrace{\tfrac{1}{2} M v_{cm}^2}_{\text{translation KE of CM}} + \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2} M R^2\right)\left(\frac{v_{cm}}{R}\right)^2}_{\text{rotation KE about a fixed axis through its CM}} + \underset{\text{final PE}}{0}$$

$$\Rightarrow \qquad v_{cm} = \sqrt{\tfrac{4}{3} g h} \qquad \textit{c.f.} \text{ for free falling } v_{cm} = \sqrt{2gh}$$



To find the downward acceleration of the yo-yo, need dynamic equations

Translation of CM: $\quad Mg - T = M a_{cm}$

Rotation of cylinder about its axis: $\quad TR = I_{cm} \alpha = \left(\tfrac{1}{2} M R^2\right)(a_{cm}/R)$

Get

$$a_{cm} = \tfrac{2}{3} g$$
$$T = \tfrac{1}{3} M g$$

Challenge Exercise "Antigravity" fly wheel

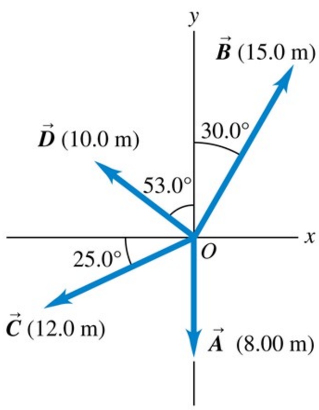
A Maxwell's wheel is usually used to demonstrate the conversion between kinetic and potential energies. If you place the whole apparatus on a scale, you will find that it weighs *less* by a certain amount while the wheel is ascending and descending. See https://www.youtube.com/watch?v=pwx12kwgOAM .
Calculate this amount. Does it depend on:

1. The mass of the wheel?
2. The mass of the whole apparatus?
3. Whether the wheel is ascending or descending?

Clicker Question:

Q1.14

Consider the vectors shown. What is the cross product $\vec{A} \times \vec{C}$?



A. $(96.0\ \mathrm{m}^2)\sin 25.0°\hat{k}$

B. $(96.0\ \mathrm{m}^2)\cos 25.0°\hat{k}$

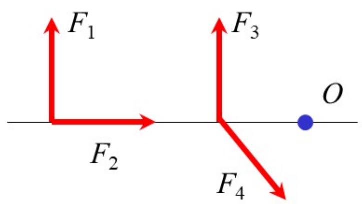
C. $-(96.0\ \mathrm{m}^2)\sin 25.0°\hat{k}$

D. $-(96.0\ \mathrm{m}^2)\cos 25.0°\hat{k}$

E. none of these

Q10.2

Which of the four forces shown here produces a torque about O that is directed *out of* the plane of the drawing?



A. F_1

B. F_2

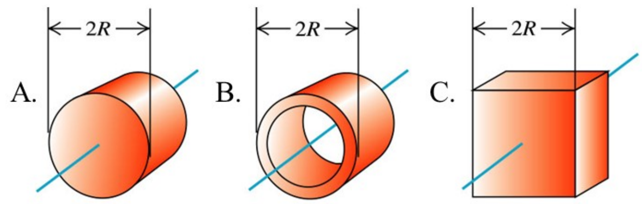
C. F_3

D. F_4

E. more than one of these

Q-RT9.2

Objects A, B, and C all have the same mass, all have the same outer dimension, and are all uniform. Each object is rotating about an axis through its center (shown in blue). All three objects have the same rotational kinetic energy.

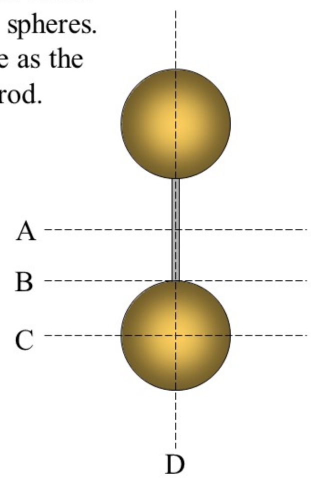


Rank these objects in order of their *angular speed* of rotation, from fastest to slowest.

Q-RT9.3

Two identical uniform solid spheres are attached by a solid uniform thin rod. The rod lies on a line connecting the centers of mass of the two spheres. Axes A, B, C, and D are in the same plane as the centers of mass of the spheres and of the rod.

For the combined object of two spheres plus rod, **rank** the object's *moments of inertia* about the four axes, from largest to smallest.



Ans: Q1.14) D, Q10.2) D, Q-RT9.2) ACB, Q-RT9.3) CBAD