MATH1023 Homework, Part 3

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Due date: Fri, Sep 27

PLEASE READ

When proving $x_n \to_{n\to\infty} l$ using the formal definition of limit, it is sufficient to show that for every $\varepsilon > 0$ *there exists* N, such that $|x_n - l| < \varepsilon$ for $n > N$. It is *not required* to provide *the exact value* for N. For example, proving that $\frac{1}{\sqrt{n}} \to 0$ by saying that $\frac{1}{n} < \delta^2$ for sufficiently large n for all $\delta > 0$ (with a formal definition of "sufficiently large") is sound mathematical logic based on the convergence of $\frac{1}{n}$ to 0, and I will appeal if you count this as a mistake. Providing exact values of N is redundant and it runs against the whole intuition of limit — that it doesn't matter how many first terms to exclude, as long as the desired behavior is achieved starting *from some point.*

We will now give a rigorous definition for "sufficiently large", in order to simplify our proofs.

Definition 1. Let $P(n)$ be a predicate on the set of natural numbers $\mathbb{N}$, that is, $P(n)$ is either tue or false for all n. We say that $P(n)$ holds *for sufficiently large n,* if there exists $N \in \mathbb{N}$ such that $n > N$ implies the truth of $P(n)$.

Note 1. We will assume throughout the proofs that for all $a \in \mathbb{R}$, $a < n$ for sufficiently large n, and for all $a > 0$, $\frac{1}{n} < a$ for sufficiently large n. These two statements are trivial and they follow from the formal convergence of $\frac{1}{n}$ to 0.

Note 2. The definition of limit can be re-formulated as follows:

$$\lim_{n\to\infty} x_n = l \iff \text{for all } \varepsilon > 0,\ |x_n - l| < \varepsilon \text{ for sufficiently large } n.$$

In fact, this definition is perfectly identical to the traditional one, but it is more convenient in proofs.

Lemma 1. *If predicates $P(n)$ and $Q(n)$ both hold for sufficiently large n, then their conjunction also holds for sufficiently large n.*

Proof: Let N_P be such that $n > N_P$ implies $P(n)$, and N_Q be such that $n > N_Q$ implies $Q(n)$. Take $N = \max\{N_P, N_Q\}$. Now, for $n > N$ we have both $n > N_P$ and $n > N_Q$, and thus the conjunction $P(n) \wedge Q(n)$ holds. ■

Exercise 1.2.1 (1). Explain $\frac{1}{2}n^2 < n^2 + (-1)^n n - 5 < 2n^2$ for sufficiently large n.
Solution: The left inequality is equivalent to

$$(-1)^{n+1} n + 5 < \frac{1}{2}n^2.$$

For sufficiently large n, we have $5 < n$ and $4/n < 1$ (since $4/n \to 0$). Thus,

$$(-1)^{n+1} n + 5 \le \left|(-1)^{n+1} n + 5\right| \le n + 5 < 2n = \frac{2n^2}{n} = \frac{4}{n} \cdot \frac{1}{2}n^2 < \frac{1}{2}n^2.$$

It immediately follows that

$$(-1)^n n - 5 \le \left|(-1)^n n - 5\right| = \left|(-1)^{n+1} n + 5\right| < \frac{1}{2}n^2 < n^2,$$

$$n^2 + (-1)^n n - 5 < \frac{1}{2}n^2 < 2n^2,$$

proving the right inequality as well.

Exercise 1.2.2 (3). Rigorously find the limit of

$$\frac{2n^2 - 3n + 2}{3n^2 - 4n + 1}.$$

Solution: We claim that the limit is $2/3$. To prove this, consider an arbitrarily small $\varepsilon > 0$. For sufficiently large n, we have

$$\left|\frac{2n^2 - 3n + 2}{3n^2 - 4n + 1} - \frac{2}{3}\right| = \frac{\frac{4}{3}n - \frac{1}{3}}{3n^2 - 4n + 1} < \frac{2n}{2n^2} = \frac{1}{n},$$

which is less than ε for sufficiently large n, since $\frac{1}{n} \underset{n\to\infty}{\to} 0$. Hence, by Lemma 1 the given sequence indeed tends to $\frac{2}{3}$.

Exercise 1.2.2 (6). Rigorously find the limit of

$$\frac{\sqrt{n} + a}{n + b}.$$

Solution: We claim that the limit is 0. Consider an arbitrary $\delta > 0$. For sufficiently large n, we have $0 < \sqrt{n} + a < 2\sqrt{n}$, $n + b > \frac{1}{2}n$, and $1/n < (\delta/4)^2$. Taking N to be the maximum of the respective thresholds ($N = \max\{N_1, N_2, N_3\}$), we can see that all three properties hold for $n > N$. Now, for such n we have

$$\left|\frac{\sqrt{n} + a}{n + b} - 0\right| = \frac{\sqrt{n} + a}{n + b} < \frac{2\sqrt{n}}{\frac{1}{2}n} = 4\frac{1}{\sqrt{n}} < \delta,$$

and we are done.

Exercise 1.2.4 (3). Rigorously find the limit of

$$\sqrt{n + a} - \sqrt{n + b}$$

Solution: Let $\delta > 0$ be arbitrary. Without loss of generality, assume that $a > b$. Now, for sufficiently large n we have

$$\left|\sqrt{n + a} - \sqrt{n + b} - 0\right| = \frac{\left(\sqrt{n + a} - \sqrt{n + b}\right)\left(\sqrt{n + a} + \sqrt{n + b}\right)}{\sqrt{n + a} + \sqrt{n + b}} = \frac{a - b}{\sqrt{n + a} + \sqrt{n + b}} < \frac{2(a - b)}{\sqrt{n}} < \delta,$$

because

$$\frac{1}{n} < \left(\frac{\delta}{2(a - b)}\right)^2$$

for sufficiently large n. Therefore, $\sqrt{n + a} - \sqrt{n + b} \underset{n\to\infty}{\to} 0$.

If we have $a < b$, then we write

$$\sqrt{n + a} - \sqrt{n + b} = -\left(\sqrt{n + b} - \sqrt{n + a}\right) \underset{n\to\infty}{\to} -0 = 0,$$

reducing the problem to a case we already considered.

Exercise 1.2.7. Rigorously prove $\lim\limits_{n\to\infty} \frac{n^{5.4}}{n!} = 0$. Then prove $\lim\limits_{n\to\infty} \frac{n^p}{n!} = 0$.
Solution: We will first prove that $\frac{n^m}{n!} \underset{n\to\infty}{\to} 0$, where m is a positive integer. Let $\delta > 0$. We write

$$\begin{aligned}\frac{n^m}{n!} &= \frac{n}{n} \cdot \frac{n}{n-1} \cdot \ldots \cdot \frac{n}{n-(m-1)} \cdot \frac{1}{(n-m)!} \\ &= \left(1 + \frac{1}{n-1}\right)\left(1 + \frac{1}{n-2}\right)\ldots\left(1 + \frac{1}{n-(m-1)}\right) \cdot \frac{1}{(n-m)!} \\ &\leq m\left(1 + \frac{1}{n-m}\right) \cdot \frac{1}{(n-m)!}.\end{aligned}$$

For sufficiently large n, $\frac{1}{(n-m)!}$ will be less than $\delta/(2m)$ (since $\frac{1}{(n-m)!} \leq \frac{1}{n-m}$) and $\frac{1}{n-m}$ will be less than 1. Consequently, we have

$$m\left(1 + \frac{1}{n-m}\right) \cdot \frac{1}{(n-m)!} < m \cdot (1+1) \cdot \frac{\delta}{2m} = \delta$$

for sufficiently large n, which proves that $\frac{n^m}{n!}$ approaches 0.

We will now return to the solution. Let $\delta > 0$ be freely chosen. Write

$$\frac{n^{5.4}}{n!} \leq \frac{n^6}{n!} < \delta,$$

where the last inequality holds for sufficiently large n since $\frac{n^6}{n!} \to 0$. This proves that $\frac{n^{5.4}}{n!}$ also tends to 0.

As for the general case $p \in \mathbb{R}$, we employ a similar tactic. There is a positive integer m such that $p < m$. Similarly, we have

$$\frac{n^p}{n!} < \frac{n^m}{n!} < \delta$$

for sufficiently large n, where $\delta > 0$ is arbitrarily chosen in advance. Hence, $\frac{n^p}{n!} \to 0$.

Exercise 1.2.9. Rigorously prove $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$.
Solution: Let $\varepsilon > 0$ be arbitrary. We simply write

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \ldots \cdot \frac{n-1}{n} \cdot \frac{n}{n} \leq \frac{1}{n} < \varepsilon$$

for sufficiently large n, since $1/n \to 0$. Hence the given limit is also 0.

Exercise 1.2.10. Prove that if $\lim\limits_{n\to\infty} x_n = l$, then $\lim\limits_{n\to\infty} |x_n| = |l|$.
Solution: Consider an $\varepsilon > 0$. For sufficiently large n, we have $|x_n - l| < \varepsilon$. Additionally, we have the triangle inequalities for x_n and l:

$$\begin{aligned}|x_n| \leq |l| + |x_n - l| &\Longrightarrow |x_n| - |l| \leq |x_n - l|, \\ |l| \leq |x_n| + |l - x_n| &\Longrightarrow |l| - |x_n| \leq |x_n - l|.\end{aligned}$$

Combining these two inequalities, we see that $||x_n| - |l|| \leq |x_n - |l||$. Finally, we write

$$||x_n| - |l|| \leq |x_n - l| < \varepsilon$$

for sufficiently large n, proving that $|x_n| \underset{n\to\infty}{\to} |l|$.

Exercise 1.2.13. Suppose $\lim_{n\to\infty} x_n = 0$.
1. If $x_n \geq 0$, prove that $\lim_{n\to\infty} \sqrt{x_n} = 0$.
2. If $x_n \geq 0$ and $p > 0$, prove thath $\lim_{n\to\infty} x_n^p = 0$.
3. Prove that $\lim_{n\to\infty} \sqrt[3]{x_n} = 0$.

Solution:
1. Let $\delta > 0$. Starting at some index, we have $x_n < \delta^2$, by the definition of convergence. Then, starting at the same index, we have
$$\sqrt{x_n} < \sqrt{\delta^2} = \delta,$$
and we are done.
2. For sufficiently large n, we have $x_n < \delta^{\frac{1}{p}}$ and $x_n^p < \delta$, where $\delta > 0$ is freely chosen in advance. Therefore, $x_n^p \to 0$.
3. For any $\delta > 0$, we have $|x_n| < \delta^3$ starting at some index $n = N$. Thus, we have $\left|\sqrt[3]{x_n}\right| = \sqrt[3]{|x_n|} < \delta$ starting from the same index N.

Exercise 1.2.16. For $a > 0$, by taking $x_n = y_n - l$ in Example 1.2.13 and Exercise 1.2.15, and using Exercise 1.2.11, rigorously argue that
$$\lim_{n\to\infty} y_n = l \Longrightarrow \lim_{n\to\infty} a^{y_n} = a^l.$$

Solution: With Example 1.2.13 and Exercise 1.2.15 at hand (Exercise 1.2.15 follows trivially from Example 1.2.13 by cosidering $b = 1/a$ and arguing that $\lim_{n\to\infty} a^{x_n} = \lim_{n\to\infty} \left(\frac{1}{b}\right)^{x_n} = 1/\left(\lim_{n\to\infty} b^{x_n}\right) = 1$), we can conclude that for $x_n \to 0$ and $a > 0$ we have $a^{x_n} \to 1$. Now assume that $y_n \to l$. Defining x_n as $y_n - l$, we see that $x_n \to 0$ and conclude that $a^{x_n} \to 0$. Finally, by using Exercise 1.2.11, we write
$$a^{y_n} = a^{x_n + l} = a^{x_n} \cdot a^l \to 1 \cdot a^l = a^l,$$
and we are done.

Exercise 1.2.19. Prove that if a sequence x_n converges to l, then any subsequence x_{n_k} also converges to l.
Solution: Consider an arbitrary $\varepsilon > 0$. Then, since $x_n \to l$, there exists $N \in \mathbb{N}$ such that $|x_n - l| < \varepsilon$ for all $n > N$. Take $K = N$. Now, for all $k > K$, we have $n_k \geq k > N$, and thus
$$\left|x_{n_k} - l\right| < \varepsilon,$$
q.e.d.

Exercise 1.2.21. Suppose two sequences x_n and y_n satisfy $x_n = y_{n+K}$ for a constant integer K and sufficiently large n. Prove that $\lim_{n\to\infty}$ exists iff $\lim_{n\to\infty} y_n$ exists and the two limit values are equal.
Solution: Suppose that for all $n > M$, we have $x_n = y_{n+K}$. Let $l \in \mathbb{R}$ be arbitrary. Assume, for now, that $x_n \underset{n\to\infty}{\to} l$. We will prove that y_n approaches l as well. Consider some $\varepsilon > 0$. Then, there is an integer N such that $|x_n - l| < \varepsilon$ for $n > N$. By taking $N' = \max\{N + K, M + K\}$, we have
$$|y_n - l| = |x_{n-K} - l| < \varepsilon,$$
for $n > N'$, since $n - K > N' - K \geq N$. This proves that y_n approaches l. Conversely, an analogical argument proves that if y_n converges to l, then x_n also converges to l. Therefore, we have
$$x_n \underset{n\to\infty}{\to} l \Longleftrightarrow y_n \underset{n\to\infty}{\to} l,$$

which proves the desired statement. Indeed, if $\lim_{n\to\infty} x_n$ exists and is equal to l, then $\lim_{n\to\infty} y_n$ anso exists and is equal to l, and vice versa.

Exercise 1.2.22 (1). Prove that the following is equivalent to the definition of $\lim_{n\to\infty} x_n = l$:

$$\text{For any } 1 > \varepsilon > 0, \text{ there is } N, \text{ such that } n > N \text{ implies } |x_n - l| < \varepsilon. \tag{1}$$

Solution: It is clear that the definition of limit implies Condition 1, because the latter is just a restriction of the former. If something holds for all $\varepsilon > 0$, it holds for al $1 > \varepsilon > 0$ as well. Now, let us prove that if the above condition holds, then $x_n \underset{n\to\infty}{\to} l$. Let $\varepsilon > 0$ be arbitrary, and define $\delta = \min\{\varepsilon, \frac{1}{2}\}$. Now, by the condition we assumed, there is a number N such that $|x_n - l| < \delta$ for $n > N$. Finally, for $n > N$ we have

$$|x_n - l| < \min\left\{\varepsilon, \frac{1}{2}\right\} \leq \varepsilon,$$

and we are done.

Exercise 1.2.23. Which are equivalent to the definition of $\lim_{n\to\infty} x_n = l$?

1. For $\varepsilon = 0.001$, we have $N = 1000$, such that $n > N$ implies $|x_n - l| < \varepsilon$ – NO.
2. For any $0.001 > \varepsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \varepsilon$ – YES.
3. For any $\varepsilon > 0.001$, there is N, such that $n > N$ implies $|x_n - l| < \varepsilon$ – NO.
4. For any $\varepsilon > 0$, there is a natural number N, such that $n > N$ implies $|x_n - l| < \frac{1}{2}\varepsilon$ – YES.
5. For any $\varepsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < 2\varepsilon^2$ – YES.
6. For any $\varepsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \varepsilon + 1$ – NO.
7. For any $\varepsilon > 0$, we have $N = 1000$, such that $n > N$ implies $|x_n - l| < \varepsilon$ – NO.
8. For any $\varepsilon > 0$, there are infinitely many n, such that $|x_n - l| < \varepsilon$ – NO.
9. For infinitely many $\varepsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \varepsilon$ – NO.
10. For any $\varepsilon > 0$, there is N, such that $n > N$ implies $l - 2\varepsilon < x_n < l + \varepsilon$ – YES.