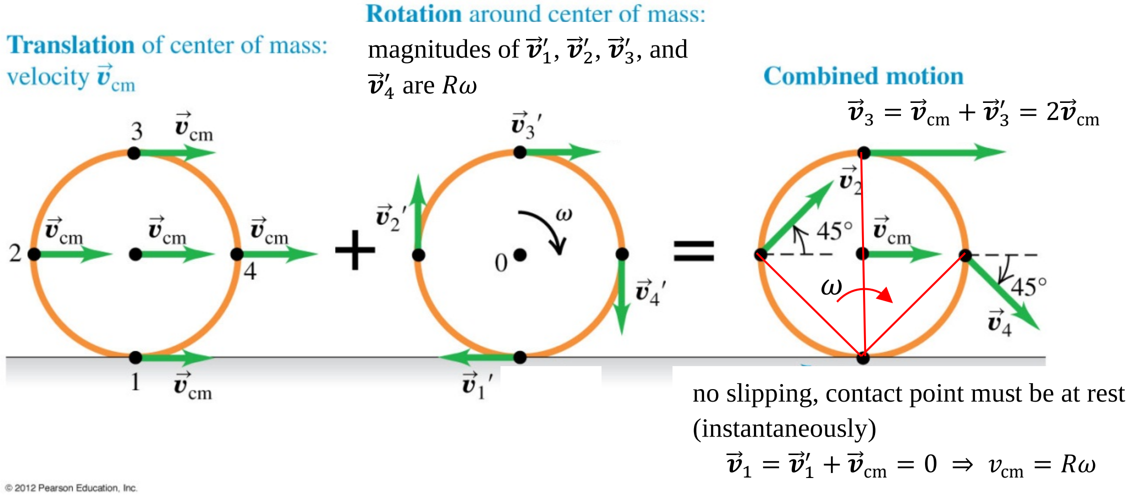
Dynamics of Rigid Bodies III

Intended Learning Outcomes – after this lecture you will learn:
1. the ideal case of rolling without slipping.
2. rolling friction in realistic cases.
3. work and power in rotation motion.
Textbook Reference: 10.3, 10.4

Rolling without slipping



no slipping, contact point must be at rest (instantaneously)

$$\vec{v}_1 = \vec{v}'_1 + \vec{v}_{cm} = 0 \Rightarrow v_{cm} = R\omega$$

Two possible ways to look at this problem:

translation of CM (as if no rotation) + rotation about CM (as if no translation)

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$$

same

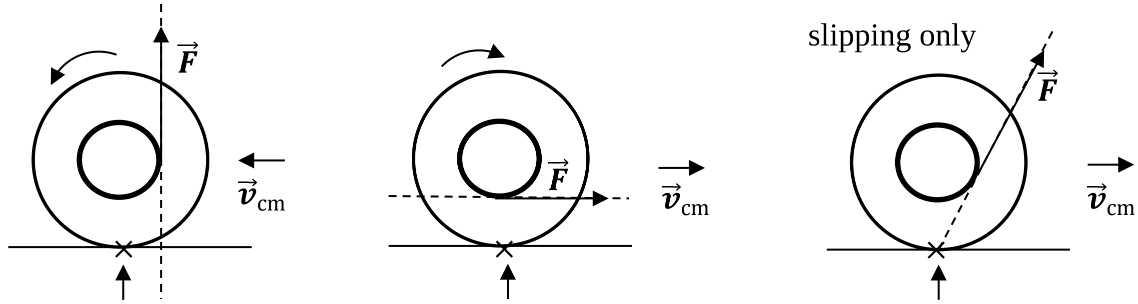
a single rotation about an **instantaneous axis of rotation** (as if no translation)

$$K = \frac{1}{2}(I + MR^2)\omega^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$$

parallel axis theorem

$v_{cm} = R\omega$

Demonstration: a rolling spool to show that the contact point with the floor is an instantaneous axis of rotation



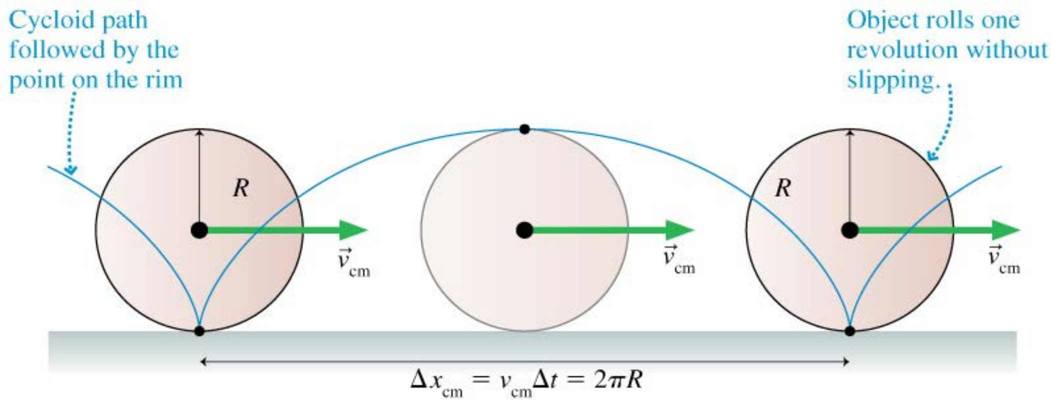
Line of $\vec{F}$ to the right of rotation axis, counterclockwise	Line of $\vec{F}$ above rotation axis, clockwise	Line of $\vec{F}$ through rotation axis, no rotation

Mathematical Digression any point on the rim of a rolling circle without slipping traces out a *cycloid* which has parametric form,

$$x = R(\omega t - \sin \omega t)$$
$$y = R(1 - \cos \omega t)$$

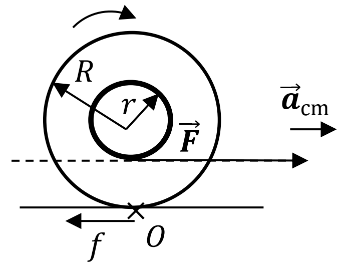
Or in Cartesian form,

$$x = R \cos^{-1}\left(1 - \frac{y}{R}\right) - \sqrt{y(2R - y)}$$



Example

A spool of mass M is being pulled by a string with tension F as shown without slipping. Find the acceleration of the CM of the spool, a_{cm}.



Condition for no slipping: $v_{\text{cm}} = R\omega \;\; \Rightarrow a_{\text{cm}} = \alpha R$

Method 1: as a single rotation about instantaneous axis of rotation O

Moment of inertia about O: $I_O = I_{\text{cm}} + MR^2$

$$\tau = F(R-r) = I_O\left(\frac{a_{\text{cm}}}{R}\right) \;\; \Rightarrow \;\; a_{\text{cm}} = \frac{FR(R-r)}{I_{\text{cm}} + MR^2}$$

Method 2: as a translation plus rotation about CM

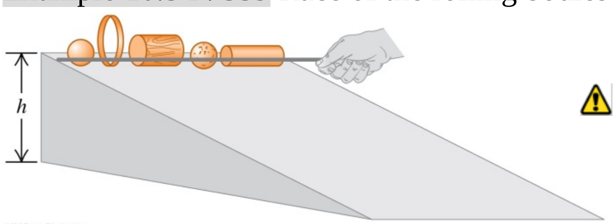
Translation: $F - f = Ma_{\text{cm}}$

Rotation about CM: $fR - Fr = I_{\text{cm}}\left(\frac{a_{\text{cm}}}{R}\right)$

$$\Rightarrow \;\; a_{\text{cm}} = \frac{FR(R-r)}{I_{\text{cm}} + MR^2}, \quad f = \left(\frac{I_{\text{cm}} + MRr}{I_{\text{cm}} + MR^2}\right)F$$

Demonstration: [Ring and disk]

Example 10.5 P. 339 Race of the rolling bodies



⚠ Rolling without slipping, friction does no work

What determines which body rolls down the incline fastest?

Suppose a rigid body's moment of inertia about its symmetry axis is $I = cMR^2$

$$\underbrace{0}_{\text{initial KE}} + \underbrace{Mgh}_{\text{initial PE}} = \underbrace{\tfrac{1}{2}Mv_{\text{cm}}^2}_{\text{translation KE of CM}} + \underbrace{\tfrac{1}{2}cMR^2\left(\frac{v_{\text{cm}}}{R}\right)^2}_{\text{rotation KE about a fixed axis through CM}} + \underbrace{0}_{\text{final PE}}$$

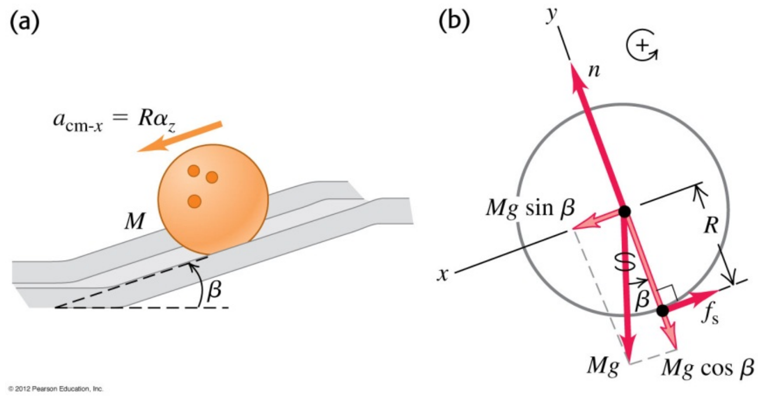
$$\Rightarrow \quad v_{cm} = \sqrt{\frac{2gh}{1+c}}$$ ⚠ depends on c only, independent of M and R

Rigid body with smaller c rolls faster : solid sphere ($c = \frac{2}{5}$) > solid cylinder ($c = \frac{1}{2}$) > thin walled hollow sphere ($c = \frac{2}{3}$) > thin walled hollow cylinder ($c = 1$)

Role of friction: Example 10.7 P. 341

⚠ Rolling without slipping is not possible without friction.

Consider a rigid sphere going freely down an inclined plane. If no friction, no torque about the center and the sphere slides down the plane without rolling.



Assume rolling without slipping, friction must be (static / dynamics) and must point (upward / downward) along the plane. $v_{\text{cm}} = R\omega \quad \Rightarrow \quad a_{\text{cm}} = R\alpha$

Translation of CM: $Mg \sin\beta - f = Ma_{\text{cm}}$

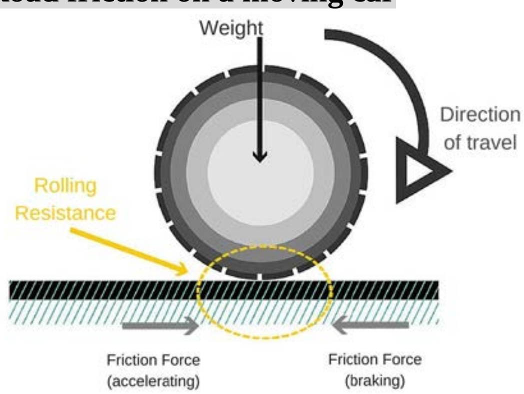
Rotation of sphere about its center: $fR = I_{\text{cm}}\alpha = \left(\frac{2}{5}MR^2\right)(a_{\text{cm}}/R)$

Get $a_{\text{cm}} = \frac{5}{7} g \sin\beta$ and $f = \frac{2}{7} Mg \sin\beta$

⚠ If the sphere is rolling uphill with no slipping, the friction will point (upward / downward) along the plane because its effect is to *decelerate* the rotation.

Demonstration: [Rolling vs sliding] Rolling is slower than sliding because part of the PE is converted into rotation KE

Road friction on a moving car

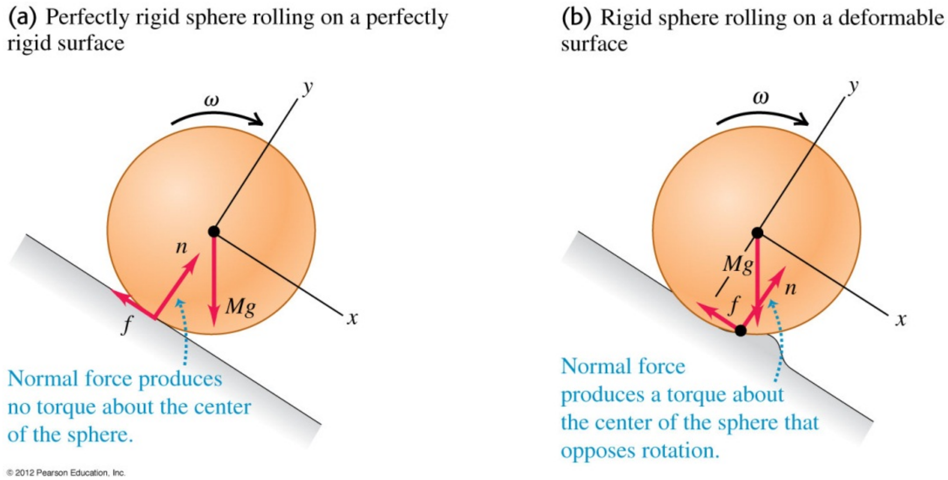


Check to make sure in the accelerating (or decelerating) case, the road friction is in a direction such that it

a) accelerates (or decelerates) the CM of the car, as required by $\vec{\boldsymbol{F}}_{\text{ex}} = M\vec{\boldsymbol{a}}_{\text{CM}}$;
b) produces a torque to oppose the torque by the engine (or the brake) that increases (or decreases) ω

Puzzle: For rolling without slipping, friction does no work. Therefore a vehicle will go on forever if there is no air resistance, just like a magnetic levitated train. Too good to be true! ☹

In reality energy is lost because the floor and/or the rolling body are deformed, e.g. vehicle tyre.



Energy is lost because:

1. due to deformation, normal reaction produces a torque opposing the rotation.
2. sliding of the deformed surfaces causes energy lost.

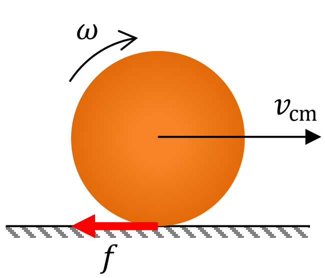
These two effects give rise to **rolling friction**.

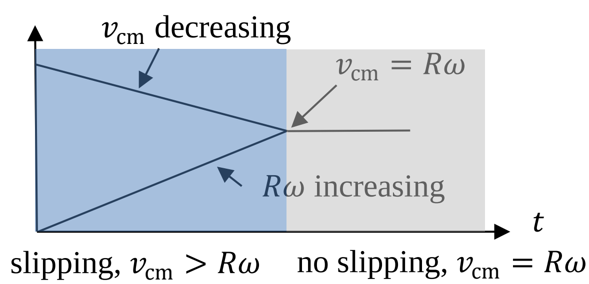
Consequence: trains, with metal wheels on metal tracks, are more fuel efficient than vehicles with rubber tires.

Example a bowling ball (from Feynman Lectures https://www.feynmanlectures.caltech.edu/info/solutions/bowling_ball_rolling_sol_2.pdf)

A uniform bowling ball of radius R and mass M is tossed with speed v_0 without rolling on an alley with coefficient of kinetic friction μ. How far does the ball go before it starts rolling without slipping?

Physical picture:





Initially, friction is kinetic and backwards $f = \mu M g$, then $a_{\text{cm}} = -f/M = -\mu g$, and $\alpha = fR/\frac{2}{5}MR^2 = 5\mu g/2R$

$$v_{\text{cm}}(t) = v_0 - \mu g t, \qquad \omega(t) = \frac{5\mu g}{2R} t$$

⚠ At this moment the contact point is NOT an instantaneous axis of rotation

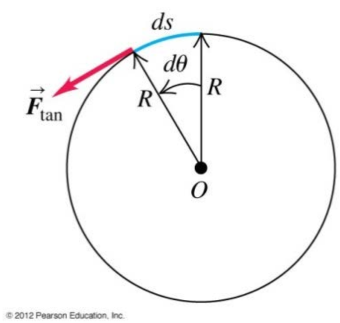
When rolling without slipping, $v_{\text{cm}}(t_1) = R\omega(t_1) \Rightarrow t_1 = \frac{2v_0}{7\mu g}$

Distant travelled is $v_0 t_1 - \frac{1}{2}\mu g t_1^2 = \frac{12v_0^2}{49\mu g}$

Afterwards (i.e., for $t > t_1$) it rolls without slipping at constant speed $v_{\text{cm}}(t) = v_0 - \mu g t_1 = \frac{5}{7}v_0$ with no friction.

Work and power in rotational motion

A particle or rigid body, being pushed by an external force, is undergoing circular motion about a fixed axis (such as a merry-go-round).



⚠ only the tangential component F_{tan} does work – no displacement along the radial and z directions.

Work done after going through angle $d\theta$

$$dW = F_{\text{tan}}(Rd\theta) = \tau d\theta$$

$$\Rightarrow \quad \boxed{W = \int \tau\, d\theta} \qquad \text{c.f. in translation,} \quad W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$$

By changing variable

$$\tau d\theta = (I\alpha)d\theta = I\frac{d\omega}{dt}d\theta = I(d\omega)\omega$$

$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega d\omega = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2$$

This is the **work-energy theorem** for rotational motion.

How about power?

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

c.f. $P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$ for translational motion.

Question: You apply equal torques to two different cylinders, one of which has a moment of inertial twice as large as the other. Each cylinder is initially at rest and is free to rotate about its fixed symmetry axis. After one complete rotation, the cylinder with larger moment of inertia will have (larger / smaller / the same) kinetic energy as the other one.

Answer: see inverted text on P. 344 in textbook