MATH1023 Homework, Part 4

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Exercise 1.3.2. Prove that if x_n is bounded for sufficiently large n, i.e. $|x_n| \leq B$ for $n \geq N$, then x_n is still bounded.
Solution: Consider N such that $n \geq N$ implies $|x_n| \leq B$. Let $B' = \max\{B, |x_1|, |x_2|, ..., |x_{N-1}|\}$. For $1 \leq n < N$, we have $|x_n| \leq B'$ by the definition of maximum. For $n \geq N$, we have $|x_n| \leq B'$ since $|x_n| \leq B \leq B'$. Hence, x_n is bounded by B'.

Exercise 1.3.5 (2). Show the convergence of sequences:
1. $x_n = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + ... + \frac{1}{n^3}$;
2. $x_n = \frac{1}{1^{2.4}} + \frac{1}{2^{2.4}} + \frac{1}{3^{2.4}} + ... + \frac{1}{n^{2.4}}$;
3. $x_n = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{(2n-1)(2n+1)}$
4. $x_n = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + ... + \frac{1}{n(n+1)(n+2)}$;
5. $\frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$.

Solution: It is obvious that all the given sequences are increasing, so the problem reduces to showing that all these sequences are bounded above.
1. Since $\frac{1}{n^3} \leq \frac{1}{n^2}$, we have
$$x_n = \sum_{k=1}^{n} \frac{1}{k^3} \leq \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n} < 2,$$
as has been shown in Example 1.3.1.
2. Since $\frac{1}{n^{2.4}} \leq \frac{1}{n^2}$, the boundedness follows from Example 1.3.1 as above.
3. Let us write an upper bound:
$$x_n = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + ... + \frac{1}{(2n-1)(2n+1)} = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} =$$
$$= \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{2}{(2k-1)(2k+1)} = \frac{1}{2} \cdot \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) = \frac{1}{2} \cdot \left(1 - \frac{1}{2n+1} \right) \leq \frac{1}{2}.$$
4. Noting that $\frac{1}{n(n+1)(n+2)} \leq \frac{1}{n(n+1)}$, we write
$$x_n = \sum_{k=1}^{n} \frac{1}{n(n+1)(n+2)} \leq \sum_{k=1}^{n} \frac{1}{n(n+1)} = \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1} < 1.$$
5. As we know from the convergence of $\frac{n^2}{n!}$ to zero, $n! > n^2$ for sufficiently large n, say, starting at $n = N$. Now, we write the upper bound for x_n (for $n > N$):
$$x_n = \sum_{k=1}^{n} \frac{1}{k!} = \sum_{k=1}^{N-1} \frac{1}{k!} + \sum_{k=N}^{n} \frac{1}{k!} \leq x_{N-1} + \sum_{k=N}^{n} \frac{1}{k^2} \leq x_{N-1} + \sum_{k=1}^{n} \frac{1}{k^2} \leq x_{N-1} + 2,$$
making use of Example 1.3.1. Now, since x_n is bounded for $n > N$, it is also bounded for all n (see Exercise 1.3.2).

Exercise 1.3.6. Suppose a sequence x_n satisfies $x_{n+1} = \sqrt{2 + x_n}$.
1. Prove that if $-2 < x_1 < 2$, then x_n is increasing and converges to 2.
2. Prove that if $x_1 > 2$, then x_n is decreasing and converges to 2.

Solution:

1. We will prove by induction that x_n is increasing: $x_{n+1} > x_n$.
   - Base: $n = 1$. If $x_1 < 0$, then we have $x_1 < 0 < \sqrt{2 + x_1} = x_2$. Otherwise, we solve the characteristic inequality:

$$\begin{aligned} x_1 < \sqrt{2 + x_1} &\Longleftarrow \\ x_1^2 < 2 + x_1 &\Longleftarrow \\ x_1^2 - x_1 - 2 < 0 &\Longleftarrow \\ (x_1 + 1)(x_1 - 2) < 0 &. \end{aligned}$$

   The last inequality holds for all admissible x_1, hence so does the inequality $x_1 < x_2$.
   - Step: $n \to n + 1$. As in Example 1.3.2, we have

$$x_{n+1} = \sqrt{2 + x_n} > \sqrt{2 + x_{n-1}} = x_n.$$

   Now, if $x < 2$, we have $\sqrt{2 + x} < \sqrt{2 + 2} = \sqrt{4} = 2$, meaning that $x_n < 2$ for all n (a trivial proof by induction). Thus, x_n is both increasing and bounded above. Hence x_n has a limit, say l. Taking the limit of both sides of

$$x_{n+1}^2 = 2 + x_n$$

   and applying the aruthmetic property, we have $l^2 = 2 + l$ and $l = 2$.
2. We prove $x_{n+1} < x_n$ analogically by induction:
   - Base: $n = 1$. We have

$$\begin{aligned} \sqrt{2 + x_1} < x_1 &\Longleftarrow \\ 2 + x_1 < x_1^2 &\Longleftarrow \\ x_1^2 - x_1 - 2 > 0 &\Longleftarrow \\ (x_1 + 1)(x_1 - 2) > 0 &. \end{aligned}$$

   The last inequality holds for all $x_1 > 2$, and hence so does $x_2 \le x_1$.
   - Step: $n \to n + 1$. By analogy with Example 1.3.2.

   If $x > 2$, then $\sqrt{2 + x} > \sqrt{2 + 2} = \sqrt{4} = 2$, and so $x_n > 2$ for all n. Being decreasing and bounded below, x_n has a limit l. Similarly to the previous case, the recursive relation $x_{n+1} = \sqrt{2 + x_n}$ leads to l being equal to 2.

Exercise 1.3.12. Explain the *continued fraction expansion*

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}.$$

What if 2 on the right side is changed to some other positive number?

Solution: As in Example 1.3.2, this "infinite fraction" can be thought of as the limit of a recursive sequence, in this case with the property

$$x_{n+1} = 1 + \frac{1}{1 + x_n}.$$

The initial term, x_1, we will set to an arbitrary positive number. Our task is to prove that the resulting sequence converges to $\sqrt{2}$.

First, we see that, since $0 < 1 + \frac{1}{1+x} < 2$ for all $x > 0$, we have $0 < x_n < 2$ for all n (a trivial proof by induction). That is, the sequence x_n is bounded both above and below.

Consider two subsequences of x_n: $y_n = x_{2n-1}$ and $z_n = x_{2n}$. If $y_{n+1} \leq y_n$, then we have

$$y_{n+2} = 1 + \frac{1}{2 + \frac{1}{1+y_{n+1}}} \leq 1 + \frac{1}{2 + \frac{1}{1+y_n}} = y_{n+1}.$$

Similarly, if $y_{n+1} \geq y_n$, then $y_{n+2} \geq y_{n+1}$. The same applies to z_n. In other words, both y_n and z_n are monotonous. Since they are also bounded, they both have limits, l_1 and l_2. Both of these numbers have to satisfy the equation

$$x = 1 + \frac{1}{2 + \frac{1}{1+x}},$$

by the logic of taking the limit of both sides of the recursive property of y_n and z_n. Finally, we see with trivial algebra that the only positive root of this equation is $\sqrt{2}$. Hence, $l_1 = l_2 = \sqrt{2}$, which means that the original sequence x_n, as a union of y_n and z_n, converges to $\sqrt{2}$.

Exercise 1.3.17. Extend Example 1.3.3 to a proof of $\lim\limits_{n\to\infty} n^p a^n = 0$ for $|a| < 1$. *Solution:* We first tackle the case when $0 \leq a < 1$. Denote $n^p a^n$ by x_n. Consider the d'Alambertian quotients:

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^p a^{n+1}}{n^p a^n} = a\left(1 + \frac{1}{n}\right)^p. \tag{1}$$

For sufficiently large n (say, from $n = N$), the last expression in Formula 1 will be less than 1, since $a < 1$ and $\left(1 + \frac{1}{n}\right)^p$ converges to 1. Since all x_n are positive, this means that $x_{n+1} < x_n$ starting from $x = N$. In other words, x_n is decreasing for sufficiently large n. Since it is also bounded below by 0, we see that x_n has a limit l. We have

$$l = \lim_{n\to\infty} n^p a^n = a \cdot \lim_{n\to\infty}\left(\left(\frac{n}{n-1}\right)^p (n-1)^p a^{n-1}\right) = a \cdot \lim_{n\to\infty}\left(1 + \frac{1}{n-1}\right)^p \cdot \lim_{n\to\infty} (n-1)^p a^{n-1} =$$
$$= a \cdot 1 \cdot l = al,$$

from which it follows that $l = 0$, since $a \neq 1$.

If $-1 < a \leq 0$, then we see that $-n^p |a|^n \leq n^p a^n \leq n^p a^n$, and the limit follows from the sandwich rule.

Exercise 1.3.19 (2). Find the limit of

$$\left(1 - \frac{1}{n}\right)^n.$$

Solution: We write

$$\left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \frac{1}{\left(1 + \frac{1}{n-1}\right)^n} = \left(\frac{1}{\left(1 + \frac{1}{n-1}\right)^{n-1}}\right)^{\frac{n}{n-1}} \underset{n\to\infty}{\to} \left(\frac{1}{e}\right)^1 = \frac{1}{e},$$

by the arithmetic rule, seeing that $\left(1 + \frac{1}{n-1}\right)^{n-1} \to e$.

Exercise 1.3.22. If x_n is a Cauchy sequence, is $|x_n|$ also a Cauchy sequence? What about the converse?
Solution: It is true. Let x_n be a Cauchy sequence. For proving $|x_n|$ to be Cauchy, consider an arbitrary $\varepsilon > 0$. Then, there is N such that $n, m > N$ implies $|x_n - x_m| < \varepsilon$. However, we have the triangle inequality

$$||x_n| - |x_m|| \le |x_n - x_m|,$$

and thus we have

$$||x_n| - |x_m|| \le |x_n - x_m| < \varepsilon$$

for $n, m > N$. Hence, $|x_n|$ is Cauchy.

The converse fails, as can easily be seen from the example of $x_n = (-1)^n$.

Exercise 1.3.23 (1,3). Use the Cauchy criterion to determine the convergence or divergence of

1. $$x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}};$$

3. $$x_n = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}.$$

Solution:
1. With $\varepsilon = \frac{1}{\sqrt{2}}$, for any N consider $n = N$ and $m = N$. We have

$$x_m - x_n = x_{2N} - x_N = \frac{1}{\sqrt{N+1}} + \frac{1}{\sqrt{N+2}} + \cdots + \frac{1}{\sqrt{2N}} \ge N \cdot \frac{1}{\sqrt{2N}} = \sqrt{\frac{N}{2}} \ge \varepsilon,$$

   and shus the Cuchy criterion fails.
3. For all $n > 1$, we have $\frac{n-1}{n} \ge \frac{1}{2}$. Now, take $\varepsilon = \frac{1}{2}$ and for all N take $n = N$ and $m = N + 1$. We write

$$x_m - x_n = x_{N+1} - x_N = \frac{N}{N+1} \ge \frac{1}{2} = \varepsilon,$$

   and thus the Cauchy criterion fails.

Exercise 1.4.2. Prove that $\lim\limits_{n\to\infty} x_n = +\infty$ if and only if $x_n > 0$ for sufficiently large n and $\lim\limits_{n\to\infty} \frac{1}{x_n} = 0$.
Solution:
$\Longrightarrow$: Since $x_n > B$ for sufficiently large n, this also applies for $B = 0$, meaning that $x_n > 0$ for sufficiently large n. Then, for all $\varepsilon > 0$, we have $x_n > \frac{1}{\varepsilon}$ for sufficiently large n, and thus $\frac{1}{x_n} < \varepsilon$. Therefore, $\frac{1}{x_n}$ converges to 0.
$\Longleftarrow$: Let B be a real number. If $B \le 0$, we have $x_n > 0 \ge B$ for sufficiently large n. If $B > 0$, then there is N such that $n > N$ implies $\frac{1}{x_n} < \frac{1}{B}$, or $x_n > B$. Hence, x_n diverges to $+\infty$.

Exercise 1.4.3 (1,3). Rigorously prove divergence to infinity. Determine $\pm\infty$ if possible:

1. $$x_n = \frac{n^2 - n + 1}{n + 1};$$

3. $$x_n = \frac{a^n}{n}, \quad |a| > 1.$$

Solution:
1. We have

$$\frac{n^2 - n + 1}{n + 1} = n - \frac{2n - 1}{n + 1} > n - 2,$$

which is greater than any B chosen in advance, for sufficiently large n. Hence, the sequence diverges to $+\infty$.

3. We first tackle the case where $a > 0$. Consider $y_n = \frac{1}{x_n} = n\left(\frac{1}{a}\right)^n$. Since $\left|\frac{1}{a}\right| < 1$, we see that $y_n \underset{n\to\infty}{\to} 0$ as per Exercise 1.3.17. Moreover, y_n is obviously positive for all n. Thus, by Exercise 1.4.2 we have that x_n diverges to $+\infty$.

   If $a < 0$, then the subsequences of odd and even terms, x_{2n-1} and x_{2n}, diverge to $-\infty$ and $+\infty$ respectively, which is seen by applying the logic of the previous case. Hence, the sequence x_n diverges to ∞, but the sign cannot be determined.

Exercise 1.4.6 (2). Prove the extended arithmetic rule $l + (+\infty) = +\infty$.
Solution: Let $x_n \underset{n\to\infty}{\to} l \in \mathbb{R}$ and $y_n \underset{n\to\infty}{\to} +\infty$. We are tasked with proving that $(x_n + y_n) \underset{n\to\infty}{\to} +\infty$. Let $B \in \mathbb{R}$ be arbitrary. For sufficiently large n, we have $x_n > l - 1$ and $y_n > B - (l - 1)$. Hence,

$$x_n + y_n > l - 1 + B - (l - 1) = B,$$

q.e.d.

Exercise 1.4.7. Construct sequences x_n and y_n, such that both diverge to infinity, but $x_n + y_n$ can have any of the following behaviors:
1. $\lim\limits_{n\to\infty} (x_n + y_n) = \infty$;
2. $\lim\limits_{n\to\infty} (x_n + y_n) = 2$;
3. $x_n + y_n$ is bounded but does not converge.

Solution:
1. Take $x_n = y_n = (-1)^n n \to \infty$. We have $x_n + y_n = 2 \cdot (-1)^n n \to \infty$.
2. Take $x_n = n \to +\infty, \quad y_n = 2 - n \to -\infty$. Their sum equals 2 for all n and thus converges to 2.
3. Take $x_n = n \to +\infty, \quad y_n = -n + (-1)^n < -n + 1 \to -\infty$. Then, $x_n + y_n = (-1)^n$, which is bounded but does not converge.

Exercise 1.4.10. Prove the extended orger rule: If $\lim\limits_{n\to\infty} x_n = l \in \mathbb{R}$ and $\lim\limits_{n\to\infty} y_n = +\infty$, then $x_n < y_n$ for sufficiently large n.
Solution: For sufficiently large n (say, for $n > N_1$), we have $|x_n - l| < 1$ and thus $x_n < l + 1$. Also, for sufficiently large n (say, for $n > N_2$) we have $y_n > l + 1$. Then, for $n > \max(N_1, N_2)$

$$x_n < l + 1 < y_n,$$

q.e.d.

Exercise 1.4.12. Prove that $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = l$ and $|l| > 1$, then x_n diverges to infinity.
Solution: We easily see that

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = |l| > 1.$$

Hence, by the order rule, we have $\frac{|x_{n+1}|}{|x_n|} > a > 1$ for sufficiently large n, where $1 < a < |l|$. Say that this holds for $n > N$. Then, for such n, we can write

$$|x_n| = |x_N| \cdot \frac{|x_{N+1}|}{|x_N|} \cdot \ldots \cdot \frac{|x_n|}{|x_{n-1}|} > |x_N| \cdot a^{n-N} \to +\infty.$$

Therefore, we see that $|x_n|$ diverges to $+\infty$, and thus x_n diverges to ∞.

Exercise 1.4.13 (2, 4). Explain the infinities. Determine the sign if possible.

2. $x_n = \frac{n!}{a^n + b^n}, \quad a + b \neq 0;$
4. $x_n = \frac{1}{\sqrt[n]{n} - \sqrt[n]{2n}}.$

<u>*Solution:*</u>

2. Consider $y_n = \frac{1}{x_n} = \frac{a^n}{n!} + \frac{b^n}{n!}$. We see that $y_n \underset{n\to\infty}{\to} 0$, meaning that x_n diverges to ∞ by the extended arithmetic rule. Now, assume without loss of generality that $|a| \geq |b|$. If $a > 0$, then $a^n + b^n > 0$ for sufficiently large n, and $x_n \underset{n\to\infty}{\to} +\infty$. If $a < 0$, then for odd n we have $a^n + b^n < 0$, and thus the sign cannot be determined.
4. Since

$$y_n = \frac{1}{x_n} = \sqrt[n]{n} - \sqrt[n]{2n} \underset{n\to\infty}{\to} 1 - 1 = 0,$$

we have $x_n \underset{n\to\infty}{\to} \infty$ by the extended order rule. Further, we see that $y_n < 0$ for all n, meaning that $x_n < 0$ and thus $x_n \underset{n\to\infty}{\to} -\infty$.