Calculus

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Contents

Chapter 1

Sequence

1.1 Intuition of Limit

A *sequence* is an infinite list

$$x_1, x_2, \dots, x_n, \dots .$$

We call x_n the n-th term of the sequence, and call n the *index* of the term. In this course, we always assume that all the terms are real numbers. Here are some examples

$$\begin{aligned}
x_n = n\colon &\quad 1,\ 2,\ 3,\ \dots,\ n,\ \dots;\\
y_n = 2\colon &\quad 2,\ 2,\ 2,\ \dots,\ 2,\ \dots;\\
z_n = \tfrac{1}{n}\colon &\quad 1,\ \tfrac{1}{2},\ \dots,\ \tfrac{1}{n},\ \dots;\\
u_n = (-1)^n\colon &\quad 1,\ -1,\ 1,\ \dots,\ (-1)^n,\ \dots;\\
v_n = \sin n\colon &\quad \sin 1,\ \sin 2,\ \sin 3,\ \dots,\ \sin n,\ \dots .
\end{aligned}$$

Note that the index does not have to start from 1. For example, the sequence u_n actually starts from $n = 0$ (or any even integer). Moreover, a sequence does not have to be given by a formula. For example, the decimal expansions of π give a sequence that has no formula

$$w_n\colon 3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ 3.141592,\ \dots .$$

If n is the number of digits after the decimal point, then the sequence w_n starts at $n = 0$.

The limit of a sequence is the trend of the terms as n gets larger. We find that x_n becomes arbitrarily large, y_n remains constant, and z_n gets arbitrarily small. This means that x_n diverges, y_n approaches 2, and z_n approaches 0. Moreover, u_n and v_n jump around and do not approach anything. Finally, w_n approximates π up to the n-th decimal place, and therefore approaches π.

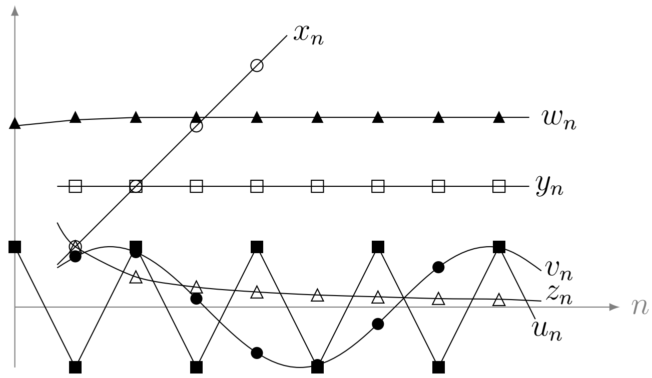


Figure 1.1.1: Sequence and trend.

Definition 1.1.1 (Intuitive). If x_n approaches a finite number l when n gets larger and larger, then the sequence x_n *converges* to the *limit* l, and we write

$$\lim_{n\to\infty} x_n = l.$$

A sequence *diverges* if it does not approach a specific finite number when n gets larger.

The sequences y_n, z_n, w_n converge respectively to 2, 0 and π. The sequences x_n, u_n, v_n diverge.

We may express the definition as the following

$$n \to \infty \implies x_n \to l.$$

The intuition is that x_n is better and better approximation of l. For example, we measure a length l by a meter ruler and get x_1, then by a centimeter ruler and get x_2, and then by a millimeter ruler and get x_3, and so on. Then $x_1, x_2, x_3, \dots$ is a sequence that approximates l within a meter, a centimeter, a millimeter, etc. The squence x_n converges to l.

We use approximations everywhere, in everyday life. The limit is similar to "I have about 100 dollars in my pocket", or "China has about 1.4 billion people".

Since the limit describes the behavior when n gets very large, we have the following property.

Proposition 1.1.2. *If y_n is obtained from x_n by adding, deleting, or changing finitely many terms, then* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

The equality in the proposition means that x_n converges if and only if y_n converges. Moreover, the two limits have equal value when both converge.

Example 1.1.1. The sequence (starting from $n = 1$)

$$\tfrac{1}{\sqrt{n+2}}\colon \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{6}}, \dots$$

is obtained by deleting the first two terms from the sequence

$$\tfrac{1}{\sqrt{n}}\colon \tfrac{1}{\sqrt{1}}, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{4}}, \dots.$$

By $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and Proposition 1.1.2, we get $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+2}} = 0$.

In general, we have $\lim_{n\to\infty} x_{n+k} = \lim_{n\to\infty} x_n$ for any integer k.

The example assumes $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, which is intuitively obvious. Although mathematics is inspired by intuition, a critical feature of mathematics is rigorous logic. This means that we need to be clear what basic facts are assumed in any argument. For the moment, we always assume that we already know

$$\lim_{n\to\infty} c = c; \quad \lim_{n\to\infty} \frac{1}{n^p} = 0 \text{ for } p > 0.$$

After we rigorously establish the two limits in Examples 1.2.2 and 1.2.3, the conclusions based on the two limits become rigorous.

1.1.1 Arithmetic Rule

If I have about 100 dollars in my left pocket, and about 200 dollars in my right pocket, then I have about $100 + 200 = 300$ dollars in total.

If I measure a screen, and find that length is 40cm long and the height is 30cm (both are approximations of real length and height), then the screen area is (approximately) $40\text{cm} \times 30\text{cm} = 1200\text{cm}^2$.

If $x \approx k$ and $y \approx l$, then $x + y \approx k + l$, and $xy \approx kl$, and $\frac{x}{y} \approx \frac{k}{l}$.

Proposition 1.1.3 (Arithmetic Rule). *Suppose* $\lim_{n\to\infty} x_n = k$ *and* $\lim_{n\to\infty} y_n = l$. *Then*

$$\lim_{n\to\infty} (x_n + y_n) = k + l, \quad \lim_{n\to\infty} cx_n = ck, \quad \lim_{n\to\infty} x_n y_n = kl, \quad \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{k}{l},$$

where c is a constant and $l \neq 0$ in the last equality.

Based on our intuition, the arithmetic rule must be correct. Although we will establish the rule by rigorously going through the logic, the correctness of the rule is not due to the proof. If your rigorous mathematical argument shows that the arithmetic rule is wrong, then your mathematics is wrong. You should modify your mathematics to fit the intuition.

We also remark that the equality

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

in the proposition is of different nature from the equality in Proposition 1.1.2. The convergence of the limits on two sides are *not* equivalent: If the two limits on the right converge, then the limit on the left also converges and the two sides are equal. However, for $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$, the limit $\lim_{n\to\infty}(x_n + y_n) = 0$ on the left converges, but both limits on the right diverge.

Exercise 1.1.1. Explain that $\lim_{n\to\infty} x_n = l$ if and only if $\lim_{n\to\infty}(x_n - l) = 0$.

Exercise 1.1.2. Explain that, if $\lim_{n\to\infty} x_n = k$ and $\lim_{n\to\infty} y_n = l$, then $\lim_{n\to\infty}(x_n - y_n) = k - l$.

Exercise 1.1.3. Explain that, if $\lim_{n\to\infty} x_n = l$ and p is a positive integer, then $\lim_{n\to\infty} x_n^p = l^p$. Moreover, if $l \neq 0$, then for any integer p, we have $\lim_{n\to\infty} x_n^p = l^p$.

Exercise 1.1.4. Suppose x_n and y_n converge. Explain that $\lim_{n\to\infty} x_n y_n = 0$ implies either $\lim_{n\to\infty} x_n = 0$ or $\lim_{n\to\infty} y_n = 0$. Moreover, explain that the conclusion fails if x_n and y_n are not assumed to converge.

Example 1.1.2. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{2n^2+n}{n^2-n-1} &= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{1-\frac{1}{n}-\frac{1}{n^2}} = \frac{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)} \\
&= \frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n}} \\
&= \frac{2+0}{1-0-0\cdot 0} = 2.
\end{aligned}$$

The limits $\lim_{n\to\infty} c = c$ and $\lim_{n\to\infty}\frac{1}{n} = 0$ are used in the fourth equality. The arithmetic rule is used in the second and third equalities. In fact, the logic goes backwards: Since the fourth term makes sense (the involved limits converge), by the arithmetic rule, we know the third term makes sense and has the same value. Once we know the third term makes sense, then we also know the second term makes sense and has the same value.

Exercise 1.1.5. Find the limits.

1. $\frac{n+2}{n-3}$.
2. $\frac{n+2}{n^2-3}$.
3. $\frac{n^2+4n-2}{2n^3-n+3}$.
4. $\frac{(n+1)(n+2)}{2n^2-1}$.
5. $\frac{2n^2-1}{(n+1)(n+2)}$.
6. $\frac{(n^2+1)(n+2)}{(n+1)(n^2+2)}$.
7. $\frac{(2-n)^3}{2n^3+3n-1}$.
8. $\frac{(n^2+3)^3}{(n^3-2)^2}$.

Exercise 1.1.6. Find the limits.

1. $\frac{\sqrt{n}+2}{\sqrt{n}-3}$.
2. $\frac{\sqrt{n}+2}{n-3}$.
3. $\frac{2\sqrt{n}-3n+2}{3\sqrt{n}-4n+1}$.
4. $\frac{\sqrt[3]{n}+4\sqrt{n}-2}{2\sqrt[3]{n}-n+3}$.
5. $\frac{2n-1}{(\sqrt{n}+1)(\sqrt{n}+2)}$.
6. $\frac{(\sqrt{n}+1)(n+2)}{(n+1)(\sqrt{n}+2)}$.
7. $\frac{(2-\sqrt[3]{n})^3}{2\sqrt[3]{n}+3n-1}$.
8. $\frac{(\sqrt[3]{n}+3)^3}{(\sqrt{n}-2)^2}$.

Exercise 1.1.7. Find the limits.

1. $\frac{n+a}{n+b}$.
2. $\frac{\sqrt{n}+a}{n+b}$.
3. $\frac{n+a}{n^2+bn+c}$.
4. $\frac{\sqrt{n}+a}{n+b\sqrt{n}+c}$.
5. $\frac{cn+d}{(\sqrt{n}+a)(\sqrt{n}+b)}$.
6. $\frac{an^3+b}{(c\sqrt{n}+d)^6}$.
7. $\frac{(a\sqrt[3]{n}+b)^2}{(c\sqrt{n}+d)^3}$.
8. $\frac{(a\sqrt{n}+b)^2}{(c\sqrt[3]{n}+d)^3}$.

Exercise 1.1.8. Show that

$$\lim_{n\to\infty}\frac{a_pn^p+a_{p-1}n^{p-1}+\cdots+a_1n+a_0}{b_qn^q+b_{q-1}n^{q-1}+\cdots+b_1n+b_0}=\begin{cases}0, & \text{if } p<q,\\ \frac{a_p}{b_q}, & \text{if } p=q \text{ and } b_q\neq 0.\end{cases}$$

Exercise 1.1.9. Find the limits.

1. $\frac{10^{10}n}{n^2-10}$.
2. $\frac{5^5(2n+1)^2-10^{10}}{10n^2-5}$.
3. $\frac{5^5(2\sqrt{n}+1)^2-10^{10}}{10n-5}$.

Exercise 1.1.10. Find the limits.

1. $\frac{n}{n+1}-\frac{n}{n-1}$.
2. $\frac{n^2}{n+1}-\frac{n^2}{n-1}$.
3. $\frac{n}{\sqrt{n}+1}-\frac{n}{\sqrt{n}-1}$.
4. $\frac{n+a}{n+b}-\frac{n+c}{n+d}$.
5. $\frac{n^2+a}{n+b}-\frac{n^2+c}{n+d}$.
6. $\frac{n+a}{\sqrt{n}+b}-\frac{n+c}{\sqrt{n}+d}$.
7. $\frac{n^3+a}{n^2+b}-\frac{n^3+c}{n^2+d}$.
8. $\frac{\sqrt{n}+a}{\sqrt[3]{n}+b}-\frac{\sqrt{n}+c}{\sqrt[3]{n}+d}$.

Exercise 1.1.11. Find the limits.

1. $\frac{n^2+a_1n+a_0}{n+b}-\frac{n^2+c_1n+c_0}{n+d}$.
2. $\frac{n^2+a_1n+a_0}{n^2+b_1n+b_0}-\frac{n^2+c_1n+c_0}{n^2+d_1n+d_0}$.
3. $\left(\frac{n+a}{n+b}\right)^2-\left(\frac{n+c}{n+d}\right)^2$.
4. $\left(\frac{n^2+a}{n+b}\right)^2-\left(\frac{n^2+c}{n+d}\right)^2$.

Exercise 1.1.12. Find the limits, $p,q>0$.

1. $\frac{n^p+a}{n^q+b}$.
2. $\frac{an^p+bn^q+c}{an^q+bn^p+c}$.
3. $\frac{n^p+a}{n^q+b}-\frac{n^p+c}{n^q+d}$.
4. $\frac{n^{2p}+a_1n^p+a_2}{n^{2q}+b_1n^q+b_2}$.

1.1.2 Sandwich Rule

Trump is taller than Biden, Biden is taller than Harris. If both Trump and Harris are about 6 feet tall, then Biden is also about 6 feet tall.

If $x\le y\le z$, and $x\approx l$, and $z\approx l$, then $y\approx l$.

Proposition 1.1.4 (Sandwich Rule). *Suppose $x_n\le y_n\le z_n$ for sufficiently large n. If $\lim_{n\to\infty}x_n=\lim_{n\to\infty}z_n=l$, then $\lim_{n\to\infty}y_n=l$.*

We remark that something holds for sufficiently large n is the same as something fails for only finitely many n.

Example 1.1.3. By $2n-3=n+(n-3)>n$ for sufficiently large n (in fact, $n>3$ is enough), we have

$$0<\frac{1}{\sqrt{2n-3}}<\frac{1}{\sqrt{n}}.$$

Then by $\lim_{n\to\infty}0=\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ and the sandwich rule, we get $\lim_{n\to\infty}\frac{1}{\sqrt{2n-3}}=0$.

On the other hand, for sufficiently large n, we have $n+1<2n$ and $n-1>\frac{n}{2}$. Therefore

$$0<\frac{\sqrt{n+1}}{n-1}<\frac{\sqrt{2n}}{\frac{n}{2}}=\frac{2\sqrt{2}}{\sqrt{n}}.$$

By $\lim_{n\to\infty}\frac{2\sqrt{2}}{\sqrt{n}}=2\sqrt{2}\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ (arithmetic rule used) and the sandwich rule, we get $\lim_{n\to\infty}\frac{\sqrt{n+1}}{n-1}=0$.

Example 1.1.4. By $-1\le\sin n\le 1$, we have

$$-\frac{1}{n}\le\frac{\sin n}{n}\le\frac{1}{n},\quad \frac{1}{\sqrt{n+1}}\le\frac{1}{\sqrt{n+\sin n}}\le\frac{1}{\sqrt{n-1}}.$$

By $\lim_{n\to\infty}\frac{1}{n}=0$, $-\lim_{n\to\infty}\frac{1}{n}=0$, and the sandwich rule, we get $\lim_{n\to\infty}\frac{\sin n}{n}=0$. Moreover, by $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}}=\lim_{n\to\infty}\frac{1}{\sqrt{n-1}}=0$ (see argument in Example 1.1.1) and the sandwich rule, we get $\lim_{n\to\infty}\frac{1}{\sqrt{n+\sin n}}=0$.

Exercise 1.1.13. Explain that $\lim_{n\to\infty}|x_n|=0$ implies $\lim_{n\to\infty}x_n=0$.

Exercise 1.1.14. Find the limits, $a,c>0$.

1. $\frac{1}{\sqrt{3n-4}}$.
2. $\frac{\sqrt{2n+3}}{4n-1}$.
3. $\frac{1}{\sqrt{an+b}}$.
4. $\frac{\sqrt{an+b}}{cn+d}$.

Exercise 1.1.15. Find the limits.

1. $\frac{\cos n}{n}$.
2. $\frac{(-1)^n}{n}$.
3. $\frac{\sin\sqrt{n}}{n}$.
4. $\frac{\cos n}{\sqrt{n-2}}$.
5. $\frac{1}{n+(-1)^n}$.
6. $\frac{\cos n}{n+(-1)^n}$.
7. $\frac{\cos n}{\sqrt{n+(-1)^n 2}}$.
8. $\frac{\cos n}{\sqrt{n+\sin\sqrt{n}}}$.
9. $\frac{(-1)^n}{\sqrt{n+(-1)^n}}$.
10. $\frac{2+(-1)^n 3}{\sqrt[3]{n^2-2\cos n}}$.
11. $\frac{\sin n+(-1)^n\cos n}{\sqrt{n}+(-1)^n}$.
12. $\frac{|\sin n+\cos n|}{n}$.
13. $\frac{3\sqrt{n}+2}{2n+(-1)^n 3}$.
14. $\frac{\sqrt{n}\sin n+\cos n}{n-1}$.
15. $\frac{n+\sin\sqrt{n}}{n+\cos 2n}$.
16. $\frac{\sqrt{n}+\sin n}{\sqrt{n}-\cos n}$.
17. $\frac{(-1)^n(n+1)}{n^2+(-1)^{n+1}}$.
18. $\frac{(-1)^n(n+10)^2-10^{10}}{10(-1)^n n^2-5}$.

Exercise 1.1.16. Find the limits.

1. $\frac{\sqrt{n}+a}{n+(-1)^n b}$.
2. $\frac{1}{\sqrt[3]{n^2+an+b}}$.
3. $\frac{\sqrt{n+c}+d}{\sqrt[3]{n^2+an+b}}$.
4. $\frac{n+(-1)^n a}{n+(-1)^n b}$.
5. $\frac{(-1)^n(an+b)}{n^2+c(-1)^{n+1}n+d}$.
6. $\frac{(-1)^n(an+b)^2+c}{(-1)^n n^2+d}$.
7. $\frac{\cos\sqrt{n}+a}{n+b\sin n}$.
8. $\frac{\cos\sqrt{n}+a}{\sqrt{n+b\sin n}}$.
9. $\frac{an+b\sin n}{cn+d\sin n}$.

Exercise 1.1.17. Find the limits, $p>0$.

1. $\frac{\sin\sqrt{n}}{n^p}$.
2. $\frac{\sin(n+1)}{n^p+(-1)^n}$.
3. $\frac{a\sin n+b}{n^p+c}$.
4. $\frac{a\cos(\sin n)}{n^p-b\sin n}$.

Example 1.1.5. For $a>0$, the sequence $\sqrt{n+a}-\sqrt{n}$ satisfies

$$0<\sqrt{n+a}-\sqrt{n}=\frac{(\sqrt{n+a}-\sqrt{n})(\sqrt{n+a}+\sqrt{n})}{\sqrt{n+a}+\sqrt{n}}=\frac{a}{\sqrt{n+a}+\sqrt{n}}<\frac{a}{\sqrt{n}}.$$

By $\lim_{n\to\infty}\frac{a}{\sqrt{n}}=0$ and the sandwich rule, we get $\lim_{n\to\infty}(\sqrt{n+a}-\sqrt{n})=0$. Similar argument shows the same limit for $a<0$.

Example 1.1.6. The sequence $\sqrt{\frac{n+2}{n}}$ satisfies

$$1<\sqrt{\frac{n+2}{n}}<\frac{n+2}{n}=1+2\frac{1}{n}.$$

By $\lim_{n\to\infty}\left(1+2\frac{1}{n}\right)=1+2\cdot 0=1$ and the sandwich rule, we get $\lim_{n\to\infty}\sqrt{\frac{n+2}{n}}=1$.

Exercise 1.1.18. Show that $\lim_{n\to\infty}(\sqrt{n+a}-\sqrt{n})=0$ for $a<0$.

Exercise 1.1.19. Use the idea of Example 1.1.5 to estimate $\sqrt{\frac{n+2}{n}}-1$ and then find $\lim_{n\to\infty}\sqrt{\frac{n+2}{n}}$.

Exercise 1.1.20. Show that $\lim_{n\to\infty}\sqrt{\frac{n+a}{n+b}}=1$. You may need separate argument for $a>b$ and $a<b$.

Exercise 1.1.21. Find the limits.

1. $\sqrt{n+a}-\sqrt{n+b}$.
2. $\frac{\sqrt{n+a}}{\sqrt{n+c}+\sqrt{n+d}}$.
3. $\frac{\sqrt{n+a}\sqrt{n+b}}{\sqrt{n+c}\sqrt{n+d}}$.
4. $\frac{\sqrt{n+a}+b}{\sqrt{n+c}+d}$.
5. $\sqrt{n}(\sqrt{n+a}-\sqrt{n+b})$.
6. $\sqrt{n+c}(\sqrt{n+a}-\sqrt{n+b})$.

7. $\sqrt{n^2+an+b}-\sqrt{n^2+an}$.
8. $\sqrt{n^2+an+b}-\sqrt{n^2+cn+d}$.
9. $\sqrt{n+a}\sqrt{n+b}-\sqrt{n+c}\sqrt{n+d}$.
10. $\sqrt{\frac{n}{n^2+n+1}}$.
11. $\sqrt{\frac{n+a}{n^2+bn+c}}$.
12. $\frac{n}{\sqrt{n^2+n+1}}$.
13. $\frac{n+a}{\sqrt{n^2+bn+c}}$.
14. $\sqrt{\frac{n^2+an+b}{n^2+cn+d}}$.

Exercise 1.1.22. Find the limits.

1. $\sqrt{n+a\sin n}-\sqrt{n+b\cos n}$.
2. $\sqrt{\frac{n+a\sin n}{n+b\cos n}}$.
3. $\sqrt{\frac{n+(-1)^n a}{n+(-1)^n b}}$.
4. $\frac{\sqrt{n+a}+\sin n}{\sqrt{n+c}+(-1)^n}$.
5. $\sqrt{n+(-1)^n}(\sqrt{n+a}-\sqrt{n+b})$.
6. $\sqrt{n^2+an+\sin n}-\sqrt{n^2+bn+\cos n}$.
7. $\sqrt{\frac{n^2+an+\sin n}{n^2+bn+\cos n}}$.
8. $\frac{\sqrt{n^2+an+b}}{n+(-1)^n c}$.

Exercise 1.1.23. Find the limits.

1. $\sqrt[3]{n+a}-\sqrt[3]{n+b}$.
2. $\sqrt[3]{\frac{n+a}{n+b}}$.
3. $\sqrt[3]{n^2}(\sqrt[3]{n+a}-\sqrt[3]{n+b})$.
4. $\sqrt[3]{n}(\sqrt[3]{\sqrt{n}+a}-\sqrt[3]{\sqrt{n}+b})$.

Exercise 1.1.24. Find the limits.

1. $\left(\frac{n-2}{n+1}\right)^5$.
2. $\left(\frac{n-2}{n+1}\right)^{5.4}$.
3. $\left(\frac{n-2}{n+1}\right)^{-\sqrt{2}}$.
4. $\left(\frac{n+a}{n+b}\right)^p$.

Exercise 1.1.25. Find the limits.

1. $\left(\frac{\sqrt{n}+a\sin n}{\sqrt{n}+b\cos 2n}\right)^p$.
2. $\left(\frac{n^2+an+b}{n^2+(-1)^n c}\right)^p$.
3. $\left(\frac{n+a}{n^2+bn+c}\right)^p$.

Example 1.1.7. Exercise 1.1.3 says that, for integr p, we have

$$\lim_{n\to\infty} x_n = l \implies \lim_{n\to\infty} x_n^p = l^p.$$

We expect this to be true for general p, as long as l^p makes sense. See Example 1.1.23.

We argue for the following special case: If $x_n \ge 1$ and $\lim_{n\to\infty} x_n = 1$, then $\lim_{n\to\infty} x_n^{3.4} = 1^{3.4} = 1$.

By $3 < 3.4 < 4$ and $x_n \ge 1$, we have

$$x_n^3 \le x_n^{3.4} \le x_n^4.$$

By the arithmetic rule, we already know $\lim_{n\to\infty} x_n^3 = 1$ and $\lim_{n\to\infty} x_n^4 = 1$. Then by the sandwich rule, we get $\lim_{n\to\infty} x_n^{3.4} = 1$.

Exercise 1.1.26. Suppose $\lim_{n\to\infty} x_n = 1$.

1. If $x_n \ge 1$, show that $\lim_{n\to\infty} x_n^{-3.4} = 1$.
2. If $x_n \le 1$, also show that $\lim_{n\to\infty} x_n^{-3.4} = 1$.
3. If $x_n \ge 1$, show that $\lim_{n\to\infty} x_n^p = 1$ for general p.

Exercise 1.1.27. Suppose $\lim_{n\to\infty} x_n = 1$, and $x_n \ge 1$, and y_n is bounded. Prove that $\lim_{n\to\infty} x_n^{y_n} = 1$.

Exercise 1.1.28. By taking $y_n = \frac{x_n}{l}$, for $l > 0$, explain that the statement

$$\lim_{n\to\infty} x_n = l \implies \lim_{n\to\infty} x_n^p = l^p$$

is the same as

$$\lim_{n\to\infty} y_n = 1 \implies \lim_{n\to\infty} y_n^p = 1.$$

Therefore its remains to consider the case $l = 1$ and x_n may sometimes > 1 and sometimes < 1.

1.1.3 Basic Limits

We establish some basic limits. Our argument is based on the limit rules we learned so far, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 1.1.8. We show that

$$\lim_{n\to\infty} \sqrt[n]{a} = 1, \text{ for } a > 0.$$

First assume $a \ge 1$. Then $x_n = \sqrt[n]{a} - 1 \ge 0$, and

$$a = (1 + x_n)^n = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots + x_n^n > nx_n.$$

This implies

$$0 \le x_n < \frac{a}{n}.$$

By the sandwich rule and $\lim_{n\to\infty} \frac{a}{n} = 0$, we get $\lim_{n\to\infty} x_n = 0$. Then by the arithmetic rule, this further implies

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} (x_n + 1) = 0 + 1 = 1.$$

For the case $0 < a \le 1$, let $b = \frac{1}{a} \ge 1$. Then by the arithmetic rule,

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{b}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{b}} = \frac{1}{1} = 1.$$

Exercise 1.1.29. Use induction to prove $(1+x)^n > 1+nx$ for $x>0$. Then for $a>1$, prove $\sqrt[n]{a} < 1+\frac{a}{n}$.

Example 1.1.9. Example 1.1.8 can be extended to

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Let $x_n = \sqrt[n]{n} - 1$. Then we have $x_n > 0$ for sufficiently large n (in fact, $n \ge 2$ is enough), and

$$n = (1+x_n)^n = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots + x_n^n > \frac{n(n-1)}{2}x_n^2.$$

This implies

$$0 \le x_n < \frac{\sqrt{2}}{\sqrt{n-1}}.$$

By $\lim_{n\to\infty} \frac{\sqrt{2}}{\sqrt{n-1}} = 0$ (see Example 1.1.1 or 1.1.3) and the sandwich rule, we get $\lim_{n\to\infty} x_n = 0$. This further implies

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} x_n + 1 = 1.$$

Example 1.1.10. The following "n-th root type" limits can be compared with the limits in Examples 1.1.8 and 1.1.9

$$\begin{aligned} &1 < \sqrt[n]{n+1} < \sqrt[n]{2n} = \sqrt[n]{2}\,\sqrt[n]{n}, \\ &1 < n^{\frac{1}{n+1}} < \sqrt[n]{n}, \\ &1 < (n^2-n)^{\frac{n}{n^2-1}} < (n^2)^{\frac{n}{n^2/2}} = (\sqrt[n]{n})^4. \end{aligned}$$

By Examples 1.1.8, 1.1.9 and the arithmetic rule, the sequences on the right converge to 1. Then by the sandwich rule, we get

$$\lim_{n\to\infty} \sqrt[n]{n+1} = \lim_{n\to\infty} n^{\frac{1}{n+1}} = \lim_{n\to\infty} (n^2-n)^{\frac{n}{n^2-1}} = 1.$$

Example 1.1.11. We have

$$3 = \sqrt[n]{3^n} < \sqrt[n]{2^n+3^n} < \sqrt[n]{3^n+3^n} = 3\sqrt[n]{2}.$$

By Example 1.1.8, we have $\lim_{n\to\infty} 3\sqrt[n]{2} = 3$. Then by the sandwich rule, we get $\lim_{n\to\infty} \sqrt[n]{2^n+3^n} = 3$.

For another example, we have

$$3^n > 3^n - 2^n = 3^{n-1} + 2\cdot 3^{n-1} - 2\cdot 2^{n-1} > 3^{n-1}.$$

Taking the n-th root, we get

$$3 > \sqrt[n]{3^n-2^n} > 3\frac{1}{\sqrt[n]{3}}.$$

By $\lim_{n\to\infty} 3\frac{1}{\sqrt[n]{3}} = 3$ and the sandwich rule, we get $\lim_{n\to\infty} \sqrt[n]{3^n-2^n} = 3$.

Exercise 1.1.30. If $0 < a \le x_n \le b$ for some constants a, b and sufficiently large n, prove that $\lim_{n\to\infty} \sqrt[n]{x_n} = 1$.

Exercise 1.1.31. Find the limits, $a > 0$.

1. $n^{\frac{1}{2n}}$.
2. $n^{\frac{2}{n}}$.
3. $n^{\frac{c}{n}}$.
4. $(n+1)^{\frac{c}{n}}$.
5. $(an+b)^{\frac{c}{n}}$.
6. $(an^2+b)^{\frac{c}{n}}$.
7. $(\sqrt{n}+1)^{\frac{c}{n}}$.
8. $(n-2)^{\frac{1}{n+3}}$.
9. $(an+b)^{\frac{c}{n+d}}$.
10. $(an+b)^{\frac{cn}{n^2+dn+e}}$.
11. $(an^2+b)^{\frac{c}{n+d}}$.
12. $(an^2+b)^{\frac{cn+d}{n^2+en+f}}$.

Exercise 1.1.32. Find the limits, $a > 0$.

1. $\sqrt[n]{n+\sin n}$.
2. $\sqrt[n]{an+b\sin n}$.
3. $\sqrt[n]{n+(-1)^n \sin n}$.
4. $\sqrt[n]{an+(-1)^n b\sin n}$.
5. $(n-\cos n)^{\frac{1}{n+\sin n}}$.
6. $(an+b\sin n)^{\frac{n}{n^2+c\cos n}}$.

Exercise 1.1.33. Find the limits, $p, q > 0$.

1. $\sqrt[n]{n^p+\sin n}$.
2. $\sqrt[n]{n^p+n^q}$.
3. $\sqrt[n+2]{n^p+n^q}$.
4. $\sqrt[n-2]{n^p+n^q}$.

Exercise 1.1.34. Find the limits.

1. $\sqrt[n]{5^n-4^n}$.
2. $\sqrt[n]{5^n-3\cdot 4^n}$.
3. $\sqrt[n]{5^n-3\cdot 4^n+2^n}$.
4. $\sqrt[n]{5^n-3\cdot 4^n-2^n}$.
5. $\sqrt[n]{4^{2n-1}-5^n}$.
6. $\sqrt[n]{4^{2n-1}+(-1)^n 5^n}$.
7. $(5^n-4^n)^{\frac{1}{n+1}}$.
8. $(5^n-4^n)^{\frac{1}{n-2}}$.
9. $(5^n-4^n)^{\frac{n+1}{n^2+1}}$.

Exercise 1.1.35. Find the limits, $a > b > 0$.

1. $\sqrt[n]{a^n+b^n}$.
2. $\sqrt[n]{a^n-b^n}$.
3. $\sqrt[n]{a^n+(-1)^n b^n}$.
4. $\sqrt[n]{a^n b^{2n+1}}$.
5. $\sqrt[n+2]{a^n+b^n}$.
6. $\sqrt[n-2]{a^n-b^n}$.
7. $(a^n+b^n)^{\frac{n}{n^2-1}}$.
8. $(a^n-b^n)^{\frac{n}{n^2-1}}$.
9. $(a^n-(-1)^n b^n)^{\frac{n}{n^2-1}}$.

Exercise 1.1.36. For $a, b, c > 0$, find $\lim_{n\to\infty} \sqrt[n]{a^n+b^n+c^n}$.

Exercise 1.1.37. For $a \ge 1$, prove $\lim_{n\to\infty} \sqrt[n]{a} = 1$ by using

$$a-1 = (\sqrt[n]{a}-1)\left((\sqrt[n]{a})^{n-1}+(\sqrt[n]{a})^{n-2}+\cdots+\sqrt[n]{a}+1\right).$$

Example 1.1.12. We show that

$$\lim_{n\to\infty} a^n = 0, \text{ for } |a| < 1.$$

First assume $0 < a < 1$ and write $a = \frac{1}{1+b}$. Then $b > 0$ and

$$0 < a^n = \frac{1}{(1+b)^n} = \frac{1}{1 + nb + \frac{n(n-1)}{2}b^2 + \cdots + b^n} < \frac{1}{nb}.$$

By $\lim_{n\to\infty} \frac{1}{nb} = 0$ and the sandwich rule, we get $\lim_{n\to\infty} a^n = 0$.

If $-1 < a < 0$, then $0 < |a| < 1$ and $\lim_{n\to\infty} |a^n| = \lim_{n\to\infty} |a|^n = 0$. By Exercise 1.1.13, we get $\lim_{n\to\infty} a^n = 0$.

Example 1.1.13. Example 1.1.12 can be extended to

$$\lim_{n\to\infty} na^n = 0, \text{ for } |a| < 1.$$

This follows from

$$0 < na^n = \frac{n}{(1+b)^n} = \frac{n}{1 + nb + \frac{n(n-1)}{2}b^2 + \cdots + b^n} < \frac{n}{\frac{n(n-1)}{2}b^2} = \frac{2}{(n-1)b^2},$$

the limit $\lim_{n\to\infty} \frac{2}{(n-1)b^2} = 0$, and the sandwich rule.

Exercises 1.1.38 and 1.1.39 show how to argue

$$\lim_{n\to\infty} n^p a^n = 0, \text{ for } |a| < 1.$$

Exercise 1.1.38. Show that $\lim_{n\to\infty} n^2 a^n = 0$ for $|a| < 1$ in two ways. The first is by using the ideas from Examples 1.1.12 and 1.1.13. The second is by using $\lim_{n\to\infty} na^n = 0$ for $|a| < 1$.

Exercise 1.1.39. Show that $\lim_{n\to\infty} n^{5.4} a^n = 0$ for $|a| < 1$. What about $\lim_{n\to\infty} n^{-5.4} a^n$? What about $\lim_{n\to\infty} n^p a^n$?

Example 1.1.14. We show that

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0, \text{ for any } a,$$

for the special case $a = 4$. For $n > 4$, we have

$$0 < \frac{4^n}{n!} = \frac{4\cdot 4\cdot 4\cdot 4}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{4}{5}\cdot\frac{4}{6}\cdots\frac{4}{n-1}\cdot\frac{4}{n} \le \frac{4\cdot 4\cdot 4\cdot 4}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{4}{n} = \frac{4^5}{4!}\frac{1}{n}.$$

We note that $\le$ is obtained by dropping terms that are obviously < 1. By $\lim_{n\to\infty} \frac{4^5}{4!}\frac{1}{n} = 0$ and the sandwich rule, we get $\lim_{n\to\infty} \frac{4^n}{n!} = 0$.

Exercise 1.1.40 suggests how to show the limit in general.

Exercise 1.1.40. Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for $a = 5.4$ and $a = -5.4$, by considering $n > 6$.

Exercise 1.1.41. Show that $\lim_{n\to\infty} \frac{a^n}{\sqrt{n!}} = 0$ for $a = 4$. Of course the limit holds for all a.

Exercise 1.1.42. Show that $\lim_{n\to\infty} \frac{n!a^n}{(2n)!} = 0$ for any a. In fact, we have $\lim_{n\to\infty} \frac{(n!)^p a^n}{(2n)!} = 0$ for $p < 2$.

Exercise 1.1.43. Find the limits.

1. $\frac{n+1}{2^n}$.
2. $\frac{n^2}{2^n}$.
3. $n^{99}0.99^n$.
4. $\frac{(n^2+1)^{1001}}{1.001^{n-2}}$.
5. $\frac{n+2^n}{3^n}$.
6. $\frac{n2^n+(-3)^n}{4^n}$.
7. $\frac{n3^n}{(1+2^n)^2}$.
8. $\frac{5^n-n6^{n+1}}{3^{2n-1}-2^{3n+1}}$.

Exercise 1.1.44. Find the limits.

1. $\frac{n^2+3n+5^n}{n!}$.
2. $\frac{n^2+3n+5^n}{n!-n^2+2^n}$.
3. $\frac{n^2+n3^n+5!}{n!}$.
4. $\frac{n^2 3^{n+5}+5\cdot(n-1)!}{(n+1)!}$.
5. $\frac{n^2+n!+(n-1)!}{3^n-n!+(n-1)!}$.
6. $\frac{2^n n!+3^n(n-1)!}{4^n(2n-1)!-5^n n!}$.

Example 1.1.15. We show that

$$\lim_{n\to\infty} \frac{n^p a^n}{n!} = 0, \text{ for any } a.$$

We pick any b satisfying $|b| > |a|$ (say $b = 2|a|$, for example). Then by $c = \frac{a}{b}$ satisfying $|c| < 1$ and Example 1.1.13, we have

$$\lim_{n\to\infty} \frac{n^p a^n}{b^n} = \lim_{n\to\infty} n^p c^n = 0.$$

By Example 1.1.14, we also have

$$\lim_{n\to\infty} \frac{b^n}{n!} = 0.$$

By multiplying the two limits together, we get the limit at the beginning.

Exercise 1.1.45. Show that $\lim_{n\to\infty} \frac{n^p a^n}{\sqrt{n!}} = 0$ for any a.

Exercise 1.1.46. Show that $\lim_{n\to\infty} \frac{n^p n! a^n}{(2n)!} = 0$ for any a.

Exercise 1.1.47. Prove $\sqrt[n]{n!} > \sqrt{\frac{n}{2}}$. Then use this to prove $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$.

Exercise 1.1.48. Prove $\frac{n!}{n^n} < \frac{1}{n}$ and $\frac{(n!)^2}{(2n)!} < \frac{1}{n+1}$ for $n > 1$. Then use this to prove $\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} = 0$. What about $\lim_{n\to\infty} \frac{(n!)^k}{(kn)!}$?

1.1.4 Order Rule

China has about 1.4 billion people, and United States has about 0.35 billion people. Without knowing the exact populations of the two countries, we definitely know China has more people than United States.

The converse is also (almost) true. In pictures, we see Trump is taller than Biden. We expect that their official heights, accurate up to inch, Trump's height (6 feet 3 inch) would be higher than Biden's (6 feet 0 inch).

Suppose $x \approx k$ and $y \approx l$. Then $x < y$ and $k < l$ should be related. In the exact statement, we need to be careful about the difference between $\le$ and $<$.

Proposition 1.1.5 (Order Rule). *Suppose* $\lim_{n\to\infty} x_n = k$ *and* $\lim_{n\to\infty} y_n = l$.

1. *If $x_n \le y_n$ for sufficiently large n, then $k \le l$.*
2. *If $k < l$, then $x_n < y_n$ for sufficiently large n.*

By taking $y_n = l$, we get the following special cases for a converging sequence x_n.

1. If $x_n \le l$ for sufficiently large n, then $\lim_{n\to\infty} x_n \le l$.
2. If $\lim_{n\to\infty} x_n < l$, then $x_n < l$ for sufficiently large n.

Similar statements with reversed inequalities also hold.

In the two statements in the order rule, $\le$ and $<$ cannot be exchanged. For example, we have $x_n = \frac{1}{n^2} < y_n = \frac{1}{n}$, but $\lim_{n\to\infty} x_n = 0 \not< \lim_{n\to\infty} y_n = 0$. The example also satisfies $\lim_{n\to\infty} x_n = 0 \ge \lim_{n\to\infty} y_n = 0$ but $x_n \not\ge y_n$, even for sufficiently large n.

Example 1.1.16. By $\lim_{n\to\infty} \frac{2n^2+n}{n^2-n-1} = 2$ and the order rule, we know $1 < \frac{2n^2+n}{n^2-n-1} < 3$ for sufficiently large n. This implies $1 < \sqrt[n]{\frac{2n^2+n}{n^2-n-1}} < \sqrt[n]{3}$ for sufficiently large n. By $\lim_{n\to\infty} \sqrt[n]{3} = 1$ and the sandwich rule, we get

$$\lim_{n\to\infty} \sqrt[n]{\frac{2n^2+n}{n^2-n-1}} = 1.$$

Example 1.1.17. We showed $\lim_{n\to\infty} \sqrt[n]{3^n - 2^n} = 3$ in Example 1.1.11. Here we use a different method, with the help of the order rule.

By $\sqrt[n]{3^n - 2^n} = 3\sqrt[n]{1 - \left(\frac{2}{3}\right)^n}$, we only need to find $\lim_{n\to\infty} \sqrt[n]{1 - \left(\frac{2}{3}\right)^n}$. By Example 1.1.12, we have $\lim_{n\to\infty} \left(1 - \left(\frac{2}{3}\right)^n\right) = 1$. Then by the order rule, we get (for sufficiently large n)

$$\frac{1}{2} < 1 - \left(\frac{2}{3}\right)^n < 2.$$

This implies that

$$\frac{1}{\sqrt[n]{2}} < \sqrt[n]{1-\left(\frac{2}{3}\right)^n} < \sqrt[n]{2}.$$

Then by Example 1.1.8 and the sandwich rule, we get

$$\lim_{n\to\infty} \sqrt[n]{1-\left(\frac{2}{3}\right)^n} = 1, \quad \lim_{n\to\infty} \sqrt[n]{3^n-2^n} = 3\lim_{n\to\infty} \sqrt[n]{1-\left(\frac{2}{3}\right)^n} = 3.$$

Exercise 1.1.49. Prove that $\lim_{n\to\infty} x_n = l > 0$ implies $\lim_{n\to\infty} \sqrt[n]{x_n} = 1$. Moreover, find a sequence satisfying $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \sqrt[n]{x_n} = 1$. Can we have x_n converging to 0 and $\sqrt[n]{x_n}$ converging to 0.32? Can we have x_n converging to 0 and $\sqrt[n]{x_n}$ converging to 3.2?

Exercise 1.1.50. Find the limits.

1. $\sqrt[n]{5^n - n4^n}$.
2. $\sqrt[n+2]{5^n - n4^n}$.
3. $\sqrt[n-2]{5^n - n4^n}$.
4. $\sqrt[n]{5^n - (-1)^n n4^n}$.
5. $(5^n - n4^n)^{\frac{n-1}{n^2+1}}$.
6. $(5^n - (-1)^n n4^n)^{\frac{n+(-1)^n}{n^2+1}}$.

Exercise 1.1.51. Find the limits.

1. $\sqrt[n]{\frac{1}{n}5^n - n4^n}$.
2. $\sqrt[n]{\frac{1}{n}5^n - (-1)^n n4^n}$.
3. $\sqrt[n+2]{\frac{1}{n}5^n - n4^n \sin n}$.
4. $(n^2 4^{2n-1} - 5^n)^{\frac{n-1}{n^2+1}}$.
5. $\sqrt[n]{n2^n + 4^{n+1} + \frac{3^{n-1}}{n}}$.
6. $\sqrt[n-2]{n2^{3n} + \frac{3^{2n-1}}{n^2}}$.
7. $\sqrt[n-2]{2^{3n} + \frac{n-1}{n^2+1}3^{2n-1}}$.
8. $\left(n2^{3n} + \frac{3^{2n-1}}{n^2}\right)^{\frac{n-1}{n^2}}$.
9. $\left(n2^{3n} + \frac{3^{2n-1}}{n^2}\right)^{\frac{1}{n^2}}$.

Exercise 1.1.52. Find the limits, $a > b$.

1. $\sqrt[n]{a^{n+1} + b^n}$.
2. $\sqrt[n]{a^{n+1} + (-1)^n b^n}$.
3. $\sqrt[n-2]{a^{n+1} + (-1)^n b^n}$.
4. $\sqrt[n]{4a^n - 5b^n}$.
5. $\sqrt[n]{4a^n + 5b^{2n+1}}$.
6. $\sqrt[n]{a + b^n}$.
7. $\sqrt[n]{an + b^n}$.
8. $\sqrt[n-2]{na^n + (n^2+1)b^n}$.
9. $\sqrt[n+2]{\frac{1}{n}a^n + nb^{n+1}}$.
10. $(a^n + b^n)^{\frac{n+1}{n^2+1}}$.
11. $((n+1)a^n + b^n)^{\frac{n}{n^2-1}}$.
12. $(a^n + b^n)^{\frac{1}{n^2-1}}$.
13. $(a^n + (-1)^n b^n)^{\frac{(-1)^n}{n^2-1}}$.

Exercise 1.1.53. Find the limits, $a, b, c > 0$.

1. $\sqrt[n]{n^2 a^n + nb^n + 2c^n}$.
2. $\sqrt[n]{a^n(b^n + 1) + nc^n}$.
3. $\sqrt[n]{(n + \sin n)a^n + b^n + n^2 c^n}$.
4. $\sqrt[n]{a^n(n + b^n(1 + nc^n))}$.

Exercise 1.1.54. Suppose a polynomial $p(n) = a_p n^p + a_{n-1}n^{p-1} + \cdots + a_1 n + a_0$ has the leading coefficient $a_p > 0$. Prove that $p(n) > 0$ for sufficiently large n.

Exercise 1.1.55. Suppose $a, b, c > 0$, and p, q, r are polynomials with positive leading coefficients. Find the limit of $\sqrt[n]{p(n)a^n + q(n)b^n + r(n)c^n}$.

Exercise 1.1.56. Find the limits, $a, b, p, q > 0$.

1. $\sqrt[n]{an^p + b\sin n}$.
2. $\sqrt[n]{an^p + bn^q}$.
3. $\sqrt[n+2]{an^p + bn^q}$.
4. $\sqrt[n-2]{an^p + bn^q}$.
5. $\sqrt[n^2]{an^p + bn^q}$.
6. $(an^p + bn^q)^{\frac{1}{n^2-1}}$.

Example 1.1.18. The sequence $x_n = \frac{(n!)^2 3^n}{(2n)!}$ satisfies

$$\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = \lim_{n\to\infty} \frac{3n^2}{2n(2n-1)} = \frac{3}{4} = 0.75.$$

By the order rule, we have $\frac{x_n}{x_{n-1}} < 0.8$ for sufficiently large n, say for $n > N$ (in fact, $N = 8$ is enough). Then for $n > N$, we have

$$0 < x_n = \frac{x_n}{x_{n-1}} \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_{N+1}}{x_N} x_N < 0.8^{n-N} x_N = C \cdot 0.8^n, \quad C = 0.8^{-N} x_N.$$

By Example 1.1.12, we have $\lim_{n\to\infty} 0.8^n = 0$. Since C is a constant, by the sandwich rule, we get $\lim_{n\to\infty} x_n = 0$.

Exercises 1.1.57 and 1.1.58 summarise the idea of the example.

Exercise 1.1.57. Prove that if $\left|\frac{x_n}{x_{n-1}}\right| \le c$ for a constant $c < 1$, then x_n converges to 0.

Exercise 1.1.58. Prove that if $\lim_{n\to\infty} \left|\frac{x_n}{x_{n-1}}\right| = l < 1$, then x_n converges to 0.

Exercise 1.1.59. Find a such that the sequence converges to 0, $p, q > 0$.

1. $\frac{(2n)!}{(n!)^2} a^n$.
2. $\frac{(n!)^2}{(3n)!} a^n$.
3. $\frac{(n!)^3}{(3n)!} a^n$.
4. $\frac{\sqrt{(2n)!}}{n!} a^n$.
5. $\sqrt{n!} a^{n^2}$.
6. $\frac{a^{n^2}}{n!}$.
7. $\frac{a^{n^2}}{\sqrt{n!}}$.
8. $(n!)^p a^n$.
9. $\frac{a^n}{(n!)^p}$.
10. $\frac{n^p}{\sqrt{n!}} a^n$.
11. $\frac{n^p n!}{\sqrt{(2n)!}} a^n$.
12. $\frac{n^q}{(n!)^p} a^n$.
13. $\frac{(n!)^p}{((2n)!)^q} a^n$.
14. $\frac{n^5 (n!)^p}{((2n)!)^q} a^n$.
15. $\frac{(3n)!}{n!(2n)!} a^n$.

1.1.5 Subsequence

A *subsequence* is obtained by choosing infinitely many terms from a sequence. We denote a subsequence by

$$x_{n_k} : x_{n_1},\ x_{n_2},\ \ldots,\ x_{n_k},\ \ldots,$$

where the indices satisfy

$$n_1 < n_2 < \cdots < n_k < \cdots .$$

The following are some examples

$$\begin{aligned}
x_{2k}&\colon x_2,\ x_4,\ x_6,\ x_8,\ \dots, x_{2k},\ \dots,\\
x_{2k-1}&\colon x_1,\ x_3,\ x_5,\ x_7,\ \dots, x_{2k-1},\ \dots,\\
x_{2^k}&\colon x_2,\ x_4,\ x_8,\ x_{16},\ \dots, x_{2^k},\ \dots,\\
x_{k!}&\colon x_1,\ x_2,\ x_6,\ x_{24},\ \dots, x_{k!},\ \dots .
\end{aligned}$$

If x_n starts at $n = 1$, then $n_1 \geq 1$, which further implies $n_k \geq k$ for all k.

The trend of the subsequence must follow the trend of the whole sequence.

Proposition 1.1.6. *If a sequence converges to l, then any subsequence converges to l. Conversely, if a sequence is the union of finitely many subsequences that all converge to the same limit l, then the whole sequence converges to l.*

Example 1.1.19. Since $\frac{1}{n^2}$ is a subsequence of $\frac{1}{n}$ (by taking x_{n^2}), $\lim_{n\to\infty} \frac{1}{n} = 0$ implies $\lim_{n\to\infty} \frac{1}{n^2} = 0$. By the same reason, we know $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ implies $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 1.1.20. The sequence $\frac{n+(-1)^n 3}{n-(-1)^n 2}$ is the union of the odd subsequence $\frac{(2k-1)-3}{(2k-1)+2} = \frac{2k-4}{2k+1}$ and the even subsequence $\frac{2k+3}{2k-2}$. Both subsequences converge to 1, either by direct calculation, or by regarding them also as subsequences of $\frac{n-4}{n+1}$ and $\frac{n+3}{n-2}$, both converging to 1. Then we conclude $\lim_{n\to\infty} \frac{n+(-1)^n 3}{n-(-1)^n 2} = 1$.

Example 1.1.21. The sequence $(-1)^n$ has one subsequence $(-1)^{2k} = 1$ converging to 1 and another subsequence $(-1)^{2k-1} = -1$ converging to -1. Since the two limits are different, by Proposition 1.1.6, the sequence $(-1)^n$ diverges.

Example 1.1.22. The sequence $\sin na$ converges to 0 when a is an integer multiple of π. Now assume $0 < a < \pi$. For any natural number k, the interval $[k\pi, (k+1)\pi]$ of length π contains the following interval of length a (both intervals have the same middle point $(k+\frac{1}{2})\pi$ and respective "radii" $\frac{\pi}{2}$ and $\frac{a}{2}$)

$$[a_k, b_k] = \left[k\pi + \frac{\pi - a}{2}, (k+1)\pi - \frac{\pi - a}{2}\right].$$

For even k, we have $\sin x \geq \sin\left(\frac{\pi-a}{2}\right) = \cos\frac{a}{2} > 0$ on $[a_k, b_k]$. For odd k, we have $\sin x \leq -\cos\frac{a}{2} < 0$ on $[a_k, b_k]$.

The arithmetic sequence $a, 2a, 3a, \dots$ has increment a, which is the length of $[a_k, b_k]$. Therefore for each k, we have $n_k a \in [a_k, b_k]$ for some natural number n_k. Then $\sin n_{2k} a$ is a subsequence of $\sin na$ satisfying $\sin n_{2k} \geq \cos\frac{a}{2} > 0$, and

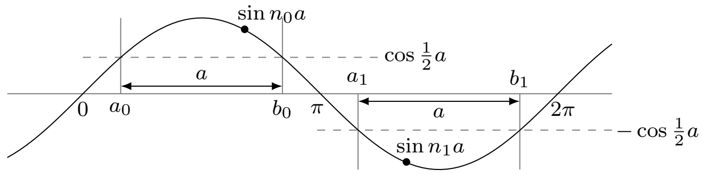


Figure 1.1.2: $\sin n_k a$.

$\sin n_{2k+1}a$ is a subsequence satisfying $\sin n_{2k+1} \le -\cos\frac{a}{2}$. By the order rule, the two subsequences cannot converge to the same limit. Therefore the sequence $\sin na$ diverges.

Now for general a that is not an integer multiple of π, we have $a = 2N\pi \pm b$ for an integer N and b satisfying $0 < b < \pi$. Then we have $\sin na = \pm\sin nb$. Since we already know that $\sin nb$ diverges, we also know $\sin na$ diverges.

We conclude that $\sin na$ converges if and only if a is an integer multiple of π.

Exercise 1.1.60. Find the limit.

1. $\sqrt{n!+1} - \sqrt{n!-1}$.
2. $(n!)^{\frac{1}{n!}}$.
3. $((n+1)!)^{\frac{1}{n!}}$.
4. $((n+(-1)^n)!)^{\frac{1}{n!}}$.
5. $(n!)^{\frac{1}{(n+1)!}}$.
6. $(2^{n^2-1} + 3^{n^2})^{\frac{1}{n^2}}$.

Exercise 1.1.61. Explain convergence or divergence.

1. $2^{(-1)^n}$.
2. $n^{\frac{(-1)^n}{n}}$.
3. $n^{(-1)^n}$.
4. $\frac{(-1)^n n+3}{n-(-1)^n 2}$.
5. $\frac{(-1)^n n^2}{n^3-1}$.
6. $\sqrt{n}\left(\sqrt{n+(-1)^n} - \sqrt{n}\right)$.
7. $(2^{(-1)^n n} + 3^n)^{\frac{1}{n}}$.
8. $(2^n + 3^{(-1)^n n})^{\frac{1}{n}}$.
9. $\tan\frac{n\pi}{3}$.
10. $(-1)^n \sin\frac{n\pi}{3}$.
11. $\sin\frac{n\pi}{2}\cos\frac{n\pi}{3}$.
12. $\frac{n\sin\frac{n\pi}{3}}{n\cos\frac{n\pi}{2}+2}$.
13. $\frac{n-\sin\frac{n\pi}{3}}{n+2\cos\frac{n\pi}{2}}$.

Exercise 1.1.62. Find all a such that the sequence $\cos na$ converges.

Example 1.1.23. In Example 1.1.7 and Exercise 1.1.26, we used the arithmetic rule and the sandwich rule to prove that, if $\lim_{n\to\infty} x_n = 1$, then with the additional assumption of all $x_n \ge 1$ or all $x_n \le 1$, we get $\lim_{n\to\infty} x_n^p = 1$.

Without the additional assumption, we may divide x_n into two subsequences x'_k and x''_k (short for x_{m_k} and x_{n_k}), respectively consisting of all those $x_n \ge 1$ and all those $x_n \le 1$.

By Proposition 1.1.6, the assumption $\lim_{n\to\infty} x_n = 1$ implies that $\lim_{k\to\infty} x'_k = 1$ and $\lim_{k\to\infty} x''_k = 1$. By the earlier example and exercise, we get $\lim_{k\to\infty} {x'_k}^p = 1$ and $\lim_{k\to\infty} {x''_k}^p = 1$. Since the sequence x_n^p is the union of two subsequences ${x'_k}^p$ and ${x''_k}^p$, by Proposition 1.1.6 again, we get $\lim_{n\to\infty} x_n^p = 1$.

Exercise 1.1.63. Extend Exercise 1.1.27, by dropping the requirement that all $x_n \geq 1$: If $\lim_{n\to\infty} x_n = 1$ and y_n is bounded, then $\lim_{n\to\infty} x_n^{y_n} = 1$.

Exercise 1.1.64. Use Example 1.1.23 and Exercise 1.1.28 to prove that

$$\lim_{n\to\infty} x_n = l > 0 \implies \lim_{n\to\infty} x_n^p = l^p.$$

In fact, for $l < 0$, as long as l^p makes sense (this means $p = \frac{m}{n}$, with m integer and n odd integer), the implication is also true.

1.2 Rigorous Definition of Limit

The statement $\lim_{n\to\infty} \frac{1}{n} = 0$ means that $\frac{1}{n}$ gets smaller and smaller as n gets larger and larger. To make the statement rigorous, we need to be more specific about smaller and larger.

Is 1000 large? The answer depends on the context. A village of 1000 people is large, and a settlement of 1000 people is hardly a city. Similarly, a rope of diameter less than one millimeter is considered thin. But the hair is considered thin only if the diameter is less than 0.05 millimeter.

Large or small makes sense only when compared with some reference quantity. We say n is in the thousands if $n > 1000$ and in the millions if $n > 10000000$. The reference quantities 1000 and 1000000 give a sense of the scale of largeness. In this spirit, the statement $\lim_{n\to\infty} \frac{1}{n} = 0$ means the following list of infinitely many implications, where the numbers on the right of the inequalities are the reference quantities

$$\begin{aligned}
n > 1 &\implies \left|\frac{1}{n} - 0\right| < 1,\\
n > 10 &\implies \left|\frac{1}{n} - 0\right| < 0.1,\\
n > 100 &\implies \left|\frac{1}{n} - 0\right| < 0.01,\\
&\ \vdots\\
n > 1000000 &\implies \left|\frac{1}{n} - 0\right| < 0.000001,\\
&\ \vdots
\end{aligned}$$

For another example, $\lim_{n\to\infty} \frac{2^n}{n!} = 0$ means the following list of infinitely many implications

$$n > 10 \implies \left|\frac{2^n}{n!} - 0\right| < 0.0003,$$
$$n > 20 \implies \left|\frac{2^n}{n!} - 0\right| < 0.0000000000005,$$
$$\vdots$$

The general shape of the implications is

$$n > N \implies |x_n - l| < \epsilon.$$

Note that the relation between N (reference quantity that measures the largeness of n) and ϵ (reference quantity that measures the smallness of $|x_n - l|$) may be different for different limits.

The problem with infinitely many implications is that our language is finite. In practice, we cannot verify all the implications one by one. Even if we have verified the truth of the first one million implications, there is no guarantee that the one million and the first implication is true. To mathematically establish the truth of all implications, we have to formulate one finite statement that includes the consideration for *all* N and *all* ϵ.

1.2.1 Rigorous Definition and Proof

Definition 1.2.1 (Rigorous). A sequence x_n converges to a finite number l, and denoted $\lim_{n\to\infty} x_n = l$, if for any $\epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \epsilon$.

In case N is a natural number (which can always be arranged if needed), the definition means that, for any given horizontal ϵ-band around l (shaded area), we can find N, such that all the terms $x_{N+1}, x_{N+2}, x_{N+3}, \dots$ after N lie in the band in Figure 1.2.1.

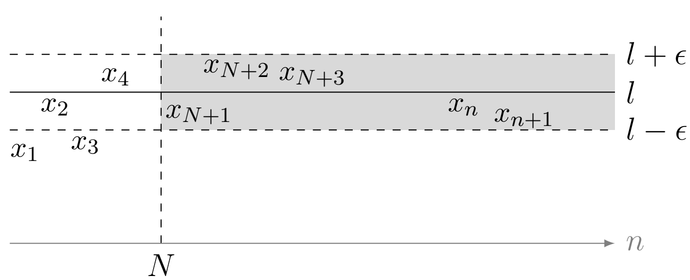


Figure 1.2.1: $n > N$ implies $|x_n - l| < \epsilon$.

Example 1.2.1. For any $\epsilon > 0$, choose $N = \frac{1}{\epsilon}$. Then

$$n > N \implies \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \epsilon.$$

This verifies the rigorous definition of $\lim_{n\to\infty} \frac{1}{n} = 0$.

By applying the rigorous definition to $\epsilon = 0.1, 0.01, \dots$ (and using the formula $N = \frac{1}{\epsilon}$), we recover the infinitely many implications we wish to achieve.

Example 1.2.2. For the constant sequence $x_n = c$, we rigorously prove

$$\lim_{n\to\infty} c = c.$$

For any $\epsilon > 0$, choose $N = 0$. Then

$$n > 0 \implies |x_n - c| = |c - c| = 0 < \epsilon.$$

In fact, the right side is always true, regardless of the left side.

Example 1.2.3. We rigorously prove

$$\lim_{n\to\infty} \frac{1}{n^p} = 0, \text{ for } p > 0.$$

For any $\epsilon > 0$, choose $N = \frac{1}{\epsilon^{\frac{1}{p}}}$. Then

$$n > N \implies \left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \frac{1}{N^p} = \epsilon.$$

We need to be more specific on the logical foundation for the argument. The foundation is the theory of real numbers. We assume the basic knowledge of real numbers, which are the four arithmetic operations $x+y, x-y, xy, \frac{x}{y}$, the exponential operation x^y (for $x > 0$), the order[1] $x < y$, and the properties for these operations. For example, we assume that we already know $x > y > 0$ implies $\frac{1}{x} < \frac{1}{y}$ and $x^p > y^p$ for $p > 0$. These properties are used in the examples above.

More important than the knowledge assumed is the knowledge not assumed. The knowledge not assumed cannot be used until after it is established. In particular, we do not assume any knowledge about the logarithm. The logarithm and its properties will be rigorously established in Example **??** as the inverse of the exponential.

Example 1.2.4. We try to rigorously prove $\lim_{n\to\infty} \frac{1}{n^2-2} = 0$. This means we need to achieve the following

$$\left|\frac{1}{n^2-2} - 0\right| = \frac{1}{n^2-2} < \epsilon.$$

[1] This is the same as $y > x$. Moreover, $x \le y$ means $x < y$ or $x = y$.

The inequality is equivalent to $n > \sqrt{2 + \frac{1}{\epsilon}}$. Therefore we get

$$n > N = \sqrt{2 + \frac{1}{\epsilon}} \implies \left| \frac{1}{n^2 - 2} - 0 \right| = \frac{1}{n^2 - 2} < \frac{1}{N^2 - 2} = \epsilon.$$

We remark our choice of N actually gives $n > N \iff |x_n - l| < \epsilon$. In fact, the choices of N in Examples 1.2.1 and 1.2.3 also give the $\iff$ relation. It is not always the case that we can calculate the exact N for $\iff$ implication. For the definition of limit, it is enough to find N that gives the $\implies$ implication.

We may use $n^2 - 2 > n$ for sufficiently large n. Specifically, we know $n^2 > 2n$ for $n > 2$. Then for $n > 2$, we have $n^2 - 2 > 2n - 2 > n$ and

$$\frac{1}{n^2 - 2} < \frac{1}{n}.$$

It is enough to achieve $\frac{1}{n} < \epsilon$. Then we get the following

$$n > \max\left\{2, \frac{1}{\epsilon}\right\} \implies \frac{1}{n^2 - 2} < \frac{1}{n} < \epsilon.$$

The condition $n > \max\{2, \frac{1}{\epsilon}\}$ means $n > 2$ and $n > \frac{1}{\epsilon}$. The first $<$ is due to $n > 2$, and the second $<$ is due to $n > \frac{1}{\epsilon}$.

Alternatively, we may split half of n^2 to cover 2. Specifically, we have $\frac{1}{2}n^2 > 2$ for $n > 2$. Then we get

$$\frac{1}{n^2 - 2} = \frac{1}{\frac{1}{2}n^2 + (\frac{1}{2}n^2 - 2)} < \frac{1}{\frac{1}{2}n^2} = \frac{2}{n^2}.$$

It is sufficient to achieve $\frac{2}{n^2} < \epsilon$. Then we get the following

$$n > \max\left\{2, \sqrt{\frac{2}{\epsilon}}\right\} \implies \frac{1}{n^2 - 2} < \frac{2}{n^2} < \epsilon.$$

In both alternative methods, the formulae for $N(\epsilon)$ are simpler than the exact choice $\sqrt{2 + \frac{1}{\epsilon}}$. The price you pay is that the estimations are valid only for sufficiently large n. Therefore N is the maximum of a large number and a simple formula for ϵ.

Example 1.2.5. Consider the limit in Example 1.1.2. For $n > 1$, we have

$$\left| \frac{2n^2 + n}{n^2 - n - 1} - 2 \right| = \frac{3n + 2}{n^2 - n - 1} < \frac{4n}{\frac{1}{2}n^2} = \frac{8}{n}.$$

Here we use $n > 2$ implies $3n + 2 < 3n + n = 4n$. We also use $\frac{1}{4}n^2 > n$ for $n > 4$ and $\frac{1}{4}n^2 > 1$ for $n > 2$ to get

$$n^2 - n - 1 = \tfrac{1}{2}n^2 + (\tfrac{1}{4}n^2 - n) + (\tfrac{1}{4}n^2 - 1) > \tfrac{1}{2}n^2 \text{ for } n > 4.$$

Of course, $n^2 - n - 1 > \frac{1}{2}n^2$ precisely for $n > 2$. But the inequality for sufficiently large n is enough for us.

The discussion above is the analysis of the problem, which you may write on your scratch paper. The formal rigorous argument you are supposed to present is the following: For any $\epsilon > 0$, we have

$$n > \max\left\{4, \frac{8}{\epsilon}\right\} \implies \frac{3n+2}{n^2-n-1} < \frac{4n}{\frac{1}{2}n^2} = \frac{8}{n} < \epsilon.$$

Exercise 1.2.1. Explain the following are between $\frac{1}{2}n^2$ and $2n^2$, for sufficiently large n.

1. $n^2 + (-1)^n n - 5$.
2. $n^2 - n\sin n + 2\sqrt{n} - 4$.
3. $n^2 + 6n^{1.5} - 10n^{1.2} + 1$.

Exercise 1.2.2. Rigorously prove the limits, $p > 0$.

1. $\frac{n+a}{n+b}$.
2. $\frac{n^2-1}{n^2+1}$.
3. $\frac{2n^2-3n+2}{3n^2-4n+1}$.
4. $\frac{n+(-1)^n}{n-3}$.
5. $\frac{\sqrt{n}+2}{\sqrt{n}-3}$.
6. $\frac{\sqrt{n}+a}{n+b}$.
7. $\frac{2\sqrt{n}-3n+2}{3\sqrt{n}-4n+1}$.
8. $\frac{2}{\sqrt{n}+\sin\sqrt{n}}$.
9. $\frac{(-1)^n\sqrt{n}}{n-\sqrt{n}\sin n+3}$.
10. $\frac{(n+1)(n+2)}{2n^2-1}$.
11. $\frac{an+\cos n}{n(n+(-1)^n)}$.
12. $\frac{(a\sqrt[3]{n}+b)^2}{(c\sqrt{n}+d)^3}$.
13. $\frac{n^2+a}{n+b} - \frac{n^2+c}{n+d}$.
14. $\frac{a}{\sqrt{n^p+b}}$.
15. $\frac{n^p+a}{n^p+b}$.
16. $\frac{a\sin n+b}{n^p+c}$.

Example 1.2.6. To rigorously prove the limit in Example 1.1.5, we estimate the difference between the sequence and the expected limit

$$|\sqrt{n+a} - \sqrt{n} - 0| = \frac{(n+a)-n}{\sqrt{n+a}+\sqrt{n}} < \frac{|a|}{\sqrt{n}}.$$

This shows that for any $\epsilon > 0$, it is sufficient to have $\frac{|a|}{\sqrt{n}} < \epsilon$, or $n > \frac{a^2}{\epsilon^2}$. In other words, we should choose $N = \frac{a^2}{\epsilon^2}$.

The analysis leads to the formal argument: For any $\epsilon > 0$, choose $N = \frac{a^2}{\epsilon^2}$. Then

$$n > N \implies |\sqrt{n+a} - \sqrt{n} - 0| = \frac{|a|}{\sqrt{n+a}+\sqrt{n}} < \frac{|a|}{\sqrt{n}} < \frac{|a|}{\sqrt{N}} = \epsilon.$$

Example 1.2.7. In Example 1.1.6, we argued that $\lim_{n\to\infty}\sqrt{\frac{n+2}{n}} = 1$. To make the argument rigorous, we use the estimation in Example 1.2.6

$$\left|\sqrt{\frac{n+2}{n}} - 1\right| = \frac{\sqrt{n+2}-\sqrt{n}}{\sqrt{n}} = \frac{(n+2)-n}{\sqrt{n}(\sqrt{n+2}+\sqrt{n})} < \frac{2}{n}.$$

The estimation suggests that it is sufficient to have $\frac{2}{n} < \epsilon$. Then we get the following rigorous argument for the limit

$$n > \frac{2}{\epsilon} \implies \left|\sqrt{\frac{n+2}{n}} - 1\right| = \frac{\sqrt{n+2}-\sqrt{n}}{\sqrt{n}} = \frac{2}{\sqrt{n}(\sqrt{n+2}+\sqrt{n})} < \frac{2}{n} < \epsilon.$$

Exercise 1.2.3. Use the estimation in Example 1.1.6 to rigorously prove $\lim_{n\to\infty}\sqrt{\frac{n+2}{n}}=1$.

Exercise 1.2.4. Rigorously prove the limits.

1. $\frac{1}{\sqrt{an+b}}$, $a>0$.
2. $\sqrt{n+a}-\sqrt{n+b}$.
3. $\sqrt{n}(\sqrt{n+a}-\sqrt{n+b})$.
4. $\sqrt{n^2+n+1}-\sqrt{n^2-n-1}$.
5. $\sqrt{\frac{n+a}{n+b}}$.
6. $\sqrt[3]{n+1}-\sqrt[3]{n}$.

1.2.2 Proof of Basic Limits

We rigorously prove important basic limits.

Example 1.2.8. The estimation in Example 1.1.8 tells us that $|\sqrt[n]{a}-1|=\sqrt[n]{a}-1<\frac{a}{n}$ for $a>1$ (this can also be obtained from Exercise 1.1.29). This suggests that for any $\epsilon>0$, we may choose $N=\frac{a}{\epsilon}$. Then

$$n>\frac{a}{\epsilon}\implies|\sqrt[n]{a}-1|<\frac{a}{n}<\epsilon.$$

This rigorously proves that $\lim_{n\to\infty}\sqrt[n]{a}=1$ in case $a\ge 1$.

Similarly, we may use the estimation in Example 1.1.9 to get a rigorous proof of $\lim_{n\to\infty}\sqrt[n]{n}=1$.

Exercise 1.2.5. Rigorously prove $\lim_{n\to\infty}\sqrt[n]{n}=1$.

Example 1.2.9. We try to rigorously prove $\lim_{n\to\infty}n^2a^n=0$ for $|a|<1$.

Using the idea of Example 1.1.12, we write $|a|=\frac{1}{1+b}$. Then $|a|<1$ implies $b>0$, and for $n>2$, we have

$$\begin{aligned}|n^2a^n-0|=n^2|a|^n&=\frac{n^2}{(1+b)^n}\\&=\frac{n^2}{1+nb+\frac{n(n-1)}{2!}b^2+\frac{n(n-1)(n-2)}{3!}b^3+\cdots+b^n}\\&<\frac{n^2}{\frac{n(n-1)(n-2)}{3!}b^3}=\frac{3!n}{(n-1)(n-2)b^3}<\frac{6n}{\frac{n}{2}\frac{n}{2}b^3}=\frac{24}{nb^3}.\end{aligned}$$

Since $\frac{24}{nb^3}<\epsilon$ is the same as $n>\frac{24}{b^3\epsilon}$, we have

$$n>\max\left\{2,\frac{24}{b^3\epsilon}\right\}\implies|n^2a^n-0|<\frac{24}{nb^3}<\epsilon.$$

It is clear from the proof that we generally have

$$\lim_{n\to\infty}n^pa^n=0,\ \text{for any } p \text{ and } |a|<1.$$

Example 1.2.10. We rigorously prove $\lim_{n\to\infty}\frac{a^n}{n!}=0$ in Example 1.1.14.

Choose a natural number M satisfying $|a|<M$. Then for $n>M$, we have

$$\left|\frac{a^n}{n!}\right|<\frac{M^n}{n!}=\frac{M\cdot M\cdots M}{1\cdot 2\cdots M}\cdot\frac{M}{M+1}\cdot\frac{M}{M+2}\cdots\frac{M}{n}\le\frac{M^M}{M!}\cdot\frac{M}{n}=\frac{M^{M+1}}{M!}\cdot\frac{1}{n}.$$

Therefore for any $\epsilon>0$, we have

$$n>\max\left\{M,\frac{M^{M+1}}{M!\epsilon}\right\}\implies\left|\frac{a^n}{n!}-0\right|<\frac{M^{M+1}}{M!}\cdot\frac{1}{n}<\frac{M^{M+1}}{M!}\cdot\frac{1}{\frac{M^{M+1}}{M!\epsilon}}=\epsilon.$$

Exercise 1.2.6. For $|a|<1$, rigorously prove $\lim_{n\to\infty}n^{2.5}a^n=0$. Then prove $\lim_{n\to\infty}n^pa^n=0$.

Exercise 1.2.7. Rigorously prove $\lim_{n\to\infty}\frac{n^{5.4}}{n!}=0$. Then prove $\lim_{n\to\infty}\frac{n^p}{n!}=0$.

Exercise 1.2.8. Rigorously prove $\lim_{n\to\infty}\frac{n^{5.4}3^n}{n!}=0$. Then prove $\lim_{n\to\infty}\frac{n^pa^n}{n!}=0$.

Exercise 1.2.9. Rigorously prove $\lim_{n\to\infty}\frac{n!}{n^n}=0$.

Example 1.2.11. Exercise 1.1.64 says $\lim_{n\to\infty}x_n=l>0$ implies $\lim_{n\to\infty}x_n^p=l^p$. This is obtained after many steps, using the arithmetic rule, sandwich rule, and subsequences.

Here we rigorously prove the result for the special case $p=\frac{1}{2}$. First we clarify the problem. The limit $\lim_{n\to\infty}x_n=l$ means the implication

$$\text{For any } \epsilon>0, \text{ there is } N, \text{ such that } n>N\implies|x_n-l|<\epsilon.$$

The limit $\lim_{n\to\infty}\sqrt{x_n}=\sqrt{l}$ means the implication

$$\text{For any } \epsilon>0, \text{ there is } N, \text{ such that } n>N\implies|\sqrt{x_n}-\sqrt{l}|<\epsilon.$$

We need to argue that the first implication implies the second implication.

We have

$$|\sqrt{x_n}-\sqrt{l}|=\frac{|(\sqrt{x_n}-\sqrt{l})(\sqrt{x_n}+\sqrt{l})|}{\sqrt{x_n}+\sqrt{l}}=\frac{|x_n-l|}{\sqrt{x_n}+\sqrt{l}}\le\frac{|x_n-l|}{\sqrt{l}}.$$

Therefore for any given $\epsilon>0$, the second implication will hold as long as $\frac{|x_n-l|}{\sqrt{l}}<\epsilon$, or $|x_n-l|<\sqrt{l}\epsilon$. The inequality $|x_n-l|<\sqrt{l}\epsilon$ can be achieved from the first implication, *provided* we apply the first implication to $\sqrt{l}\epsilon$ in place of ϵ.

The analysis leads to the following formal proof. Let $\epsilon>0$. By applying the definition of $\lim_{n\to\infty}x_n=l$ to $\sqrt{l}\epsilon>0$, there is N, such that

$$n>N\implies|x_n-l|<\sqrt{l}\epsilon.$$

Then

$$\begin{aligned} n > N &\implies |x_n - l| < \sqrt{l}\epsilon \\ &\implies |\sqrt{x_n} - \sqrt{l}| = \frac{|(\sqrt{x_n} - \sqrt{l})(\sqrt{x_n} + \sqrt{l})|}{\sqrt{x_n} + \sqrt{l}} = \frac{|x_n - l|}{\sqrt{x_n} + \sqrt{l}} \le \frac{|x_n - l|}{\sqrt{l}} < \epsilon. \end{aligned}$$

In the argument, we take advantage of the fact that the definition of limit can be applied to *any* positive number, $\sqrt{l}\epsilon$ for example, instead of the given positive number ϵ.

Exercise 1.2.10. Prove that if $\lim_{n\to\infty} x_n = l$, then $\lim_{n\to\infty} |x_n| = |l|$.

Exercise 1.2.11. Prove that if $\lim_{n\to\infty} x_n = l$, then $\lim_{n\to\infty} cx_n = cl$.

Exercise 1.2.12. Prove that $\lim_{n\to\infty} |x_n - l| = 0$ if and only if $\lim_{n\to\infty} x_n = l$.

Exercise 1.2.13. Suppose $\lim_{n\to\infty} x_n = 0$.

1. If $x_n \ge 0$, prove that $\lim_{n\to\infty} \sqrt{x_n} = 0$.
2. If $x_n \ge 0$ and $p > 0$, prove that $\lim_{n\to\infty} x_n^p = 0$.
3. Prove that $\lim_{n\to\infty} \sqrt[3]{x_n} = 0$. This allows x_n to be negative.

This extends the result in Exercise 1.1.64 to $l = 0$.

Exercise 1.2.14. Suppose $x_n \ge 0$ for sufficiently large n and $\lim_{n\to\infty} x_n = 0$. Suppose $y_n \ge p$ for sufficiently large n and some constant $p > 0$. Prove that $\lim_{n\to\infty} x_n^{y_n} = 0$.

Example 1.2.12. Continuing Example 1.2.11, we rigorously prove the $p = -1$ case of Exercise 1.1.64

$$\lim_{n\to\infty} y_n = l \ne 0 \implies \lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{l}.$$

We note that l can be a negative number.

Suppose $|y_n - l| < \epsilon'$. Then we estimated the error for the limit we wish to prove

$$\left|\frac{1}{y_n} - \frac{1}{l}\right| = \frac{|y_n - l|}{|y_n||l|} < \frac{\epsilon'}{|y_n||l|}.$$

To have the right side $\le \epsilon$, we need positive lower bound for $|y_n|$. If $|y_n| > r > 0$, then $\frac{\epsilon'}{|y_n||l|} < \frac{\epsilon'}{r|l|}$, and we may choose ϵ' to satisfy $\frac{\epsilon'}{r|l|} \le \epsilon$.

Intuitively, if y is close to $l \ne 0$, then we should have $|y| > \frac{|l|}{2}$. This is true as along as y is within $\frac{|l|}{2}$ of l

$$|y - l| < \frac{|l|}{2} \implies |y| = |(y - l) + l| > |l| - |y - l| > \frac{|l|}{2}.$$

Therefore we should require $\epsilon' \le \frac{|l|}{2}$. Then we may choose $r = \frac{|l|}{2}$ and $\epsilon' \le r|l| = \frac{l^2}{2}\epsilon$.

Now we can write down the formal proof. For any $\epsilon > 0$, we have $\epsilon' = \min\{\frac{|l|}{2}, \frac{l^2}{2}\epsilon\} > 0$. For this $\epsilon' > 0$, there is N, such that $n > N$ implies $|y_n - l| < \epsilon'$. Then $n > N$ further implies

$$|y_n| > |(y_n - l) + l| > |l| - |y_n - l| > |l| - \epsilon' \ge \frac{|l|}{2},$$

and

$$\left|\frac{1}{y_n} - \frac{1}{l}\right| = \frac{|y_n - l|}{|y_n||l|} < \frac{\epsilon'}{\frac{|l|}{2}|l|} \le \epsilon.$$

Example 1.2.13. In Examples 1.1.8 and 1.2.8, we get $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$, for $a > 0$. We expect the more general property

$$\lim_{n\to\infty} x_n = 0 \implies \lim_{n\to\infty} a^{x_n} = 1.$$

We rigorously prove the general property for $a > 1$. The idea is to compare a^{x_n} with $a^{\frac{1}{n}}$. Specifically, for any natural number k, by $\lim_{n\to\infty} x_n = 0$, there is N, such that (applying the definition of limit to $\epsilon = \frac{1}{k} > 0$)

$$n > N \implies |x_n| < \frac{1}{k}.$$

Then we get $-\frac{1}{k} < x_n < \frac{1}{k}$. By $a > 1$, this implies $a^{-\frac{1}{k}} < a^{x_n} < a^{\frac{1}{k}}$. Since the left and right are close to 1, we are in the sandwich situation, and the middle term a^{x_n} should also be close to 1.

The analysis leads to the following proof. We already know

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1, \quad \lim_{n\to\infty} a^{-\frac{1}{n}} = \frac{1}{\lim_{n\to\infty} a^{\frac{1}{n}}} = 1.$$

This implies that, for any $\epsilon > 0$, there is a natural number k, such that

$$|a^{\frac{1}{k}} - 1| < \epsilon, \quad |a^{-\frac{1}{k}} - 1| < \epsilon.$$

Then by $\lim_{n\to\infty} x_n = 0$ and applying the definition of limit to $\frac{1}{k} > 0$, there is N, such that

$$n > N \implies |x_n| < \frac{1}{k} \iff -\frac{1}{k} < x_n < \frac{1}{k}.$$

By $a > 1$ and $|a^{\frac{1}{k}} - 1| < \epsilon$, this implies

$$a^{x_n} - 1 < a^{\frac{1}{k}} - 1 < \epsilon.$$

By $a > 1$ and $|a^{-\frac{1}{k}} - 1| < \epsilon$, this also implies

$$a^{x_n} - 1 > a^{-\frac{1}{k}} - 1 > -\epsilon.$$

Combining all, we get

$$n > N \implies -\frac{1}{k} < x_n < \frac{1}{k} \implies -\epsilon < a^{x_n} - 1 < \epsilon \iff |a^{x_n} - 1| < \epsilon.$$

Exercise 1.2.15. Use Examples 1.2.12 and 1.2.13 to rigorously argue that $\lim_{n\to\infty} x_n = 0$ implies $\lim_{n\to\infty} a^{x_n} = 1$ for $0 < a < 1$.

Exercise 1.2.16. For $a > 0$, by taking $x_n = y_n - l$ in Example 1.2.13 and Exercise 1.2.15, and using Exercise 1.2.11, rigorously argue that

$$\lim_{n\to\infty} y_n = l \implies \lim_{n\to\infty} a^{y_n} = a^l.$$

1.2.3 Proof of Limit Rules

Example 1.2.14. We prove the arithmetic rule in Proposition 1.1.3

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

Specifically, we consider three statements:

1. $\lim_{n\to\infty} x_n = k$: For any $\epsilon_1 > 0$, there is N_1, such that $n > N_1 \implies |x_n - k| < \epsilon_1$.
2. $\lim_{n\to\infty} y_n = l$: For any $\epsilon_2 > 0$, there is N_2, such that $n > N_2 \implies |y_n - l| < \epsilon_2$.
3. $\lim_{n\to\infty}(x_n + y_n) = k + l$: For any $\epsilon > 0$, there is N, such that $n > N \implies |(x_n + y_n) - (k + l)| < \epsilon$.

We need to argue that the first and second statements together imply the third statement. In Example 1.2.11, we see that ϵ and N in the three statements need not be the same.

The structure of the argument is

$$\begin{aligned} n > N = \max\{N_1, N_2\} &\iff n > N_1 \text{ and } n > N_2 \\ &\implies |x_n - k| < \epsilon_1 \text{ and } |y_n - l| < \epsilon_2 \\ &\overset{?}{\implies} |(x_n + y_n) - (k + l)| < \epsilon. \end{aligned}$$

Only $\overset{?}{\implies}$ needs to be explained. We have the following implication

$$|x_n - l| < \epsilon_1 \text{ and } |y_n - k| < \epsilon_2 \implies |(x_n + y_n) - (l + k)| \le |x_n - l| + |y_n - k| < \epsilon_1 + \epsilon_2.$$

To get $|(x_n + y_n) - (l + k)| < \epsilon$ on the right, it is sufficient to have $\epsilon_1 + \epsilon_2 \le \epsilon$. For example, we may take $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$.

The analysis above leads to the following formal proof. For any $\epsilon > 0$, apply the definition of $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} y_n = k$ to $\frac{\epsilon}{2} > 0$. We find N_1 and N_2, such that

$$\begin{aligned} n > N_1 &\implies |x_n - l| < \tfrac{\epsilon}{2}, \\ n > N_2 &\implies |y_n - k| < \tfrac{\epsilon}{2}. \end{aligned}$$

Then

$$\begin{aligned} n > \max\{N_1, N_2\} &\implies |x_n - l| < \tfrac{\epsilon}{2} \text{ and } |y_n - k| < \tfrac{\epsilon}{2} \\ &\implies |(x_n + y_n) - (l + k)| \le |x_n - l| + |y_n - k| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Exercise 1.2.17. Prove $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$.

The proof of limit properties is to show some known implications imply a new implication. The results of the implications are the estimations of errors $|x_n - l|$. Examples 1.2.11 and 1.2.14 show that the key is to analyse how the known error estimations imply the error estimation we want.

Example 1.2.15. We prove the arithmetic rule in Proposition 1.1.3

$$\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n.$$

The key is that small errors $|x-k| < \epsilon_1$ and $|y-l| < \epsilon_2$ imply small error $|xy-kl| < \epsilon$. The products are the areas of rectangles, and the error $|xy - kl|$ is the area of the shaded part, which consists of two thin rectangles of width ϵ_1, ϵ_2. Intuitively, if the two rectangles are very thin, then the area of the shaded part is very small. We simply need to quantify the intuition.

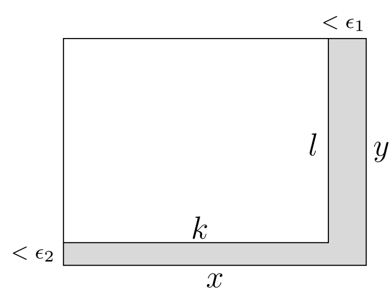


Figure 1.2.2: The error in product.

Suppose $|x - k| < \epsilon_1$ and $|y - l| < \epsilon_2$. Guided by Figure 1.2.2, we have the estimation of $|xy - kl|$ in terms of $|x - k|$ and $|y - l|$

$$\begin{aligned} |xy - kl| &= |(x-k)y + k(y-l)| \le |x-k||y| + |k||y-l| \\ &\le |x-k|(|y-l| + |l|) + |k||y-l| < \epsilon_1(\epsilon_2 + |l|) + |k|\epsilon_2. \end{aligned}$$

For the right side to be $< \epsilon$, it is sufficient to arrange ϵ_1, ϵ_2 to satisfy

$$\epsilon_1(1 + |l|) \le \tfrac{\epsilon}{2}, \quad \epsilon_2 \le 1, \quad |k|\epsilon_2 \le \tfrac{\epsilon}{2}.$$

This leads to the following formal proof. For any $\epsilon > 0$, we take

$$\epsilon_1 = \frac{\epsilon}{2(1 + |l|)}, \quad \epsilon_2 = \min\left\{1, \frac{\epsilon}{2|k|}\right\}.$$

By $\lim_{n\to\infty} x_n = k$ and $\lim_{n\to\infty} y_n = l$, there is N, such that[2]

$$\begin{aligned} n > N &\implies |x_n - k| < \epsilon_1 \text{ and } |y_n - l| < \epsilon_2 \\ &\implies |x_n y_n - kl| = |(x_n - k)y_n + k(y_n - l)| \le \cdots < \epsilon_1(\epsilon_2 + |l|) + |k|\epsilon_2 \le \epsilon. \end{aligned}$$

Here $\cdots$ is filled by our estimation of $|xy - kl|$ above.

Exercise 1.2.18. Use Examples 1.2.12 and 1.2.15 to rigorously argue the following arithmetic rule in Proposition 1.1.3: If $\lim_{n\to\infty} x_n = k$ and $\lim_{n\to\infty} y_n = l \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{k}{l}$.

Example 1.2.16. The error estimation for the proof of the sandwich rule in Proposition 1.1.4 is based on the following intuition: If $x < y < z$, and both x, z are within ϵ of l, then y is also within ϵ of l.

$$x \le y \le z,\ |x - l| < \epsilon,\ |z - l| < \epsilon \implies |y - l| < \epsilon.$$

For example, if both Trump and Harris are 6 feet, give or take 1 inch, then Biden is also 6 feet, give or take 1 inch.

Now the formal proof. Suppose $x_n \le y_n \le z_n$ for $n > N_0$ (i.e., for sufficiently large n), and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$. For any $\epsilon > 0$, there is N_1, such that

$$n > N_1 \implies |x_n - l| < \epsilon \text{ and } |z_n - l| < \epsilon.$$

Then

$$\begin{aligned} n > N = \max\{N_0, N_1\} &\implies x_n \le y_n \le z_n, \text{ and } |x_n - l| < \epsilon, \text{ and } |z_n - l| < \epsilon \\ &\implies l - \epsilon < x_n \le y_n \le z_n < l + \epsilon \\ &\iff |y_n - l| < \epsilon. \end{aligned}$$

Example 1.2.17. Consider the second part of the order rule in Proposition 1.1.5. The intuition for the proof is the following. Trump's official height of 6 feet 3 inch, and Biden's 6 feet 0 inch. If the official heights are accurate within 1 inch, then Trump's real height is at least 6 feet 2 inch and Biden's is at most 6 feet 1 inch. Therefore Trump is really taller than Biden.

Mathematically, we need the accuracy to be within half of the difference between the two. Therefore for $k < l$, we take $\epsilon = \frac{l-k}{2} > 0$. Then the key argument is the following

$$|x - k| < \epsilon \text{ and } |y - l| < \epsilon \implies y - x > l - k - 2\epsilon = 0.$$

We leave the reader to write the formal proof.

The following is a proof of the first part of the order rule. By $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} y_n = k$, for any $\epsilon > 0$, there is N, such that

$$n > N \implies |x_n - k| < \epsilon \text{ and } |y_n - l| < \epsilon.$$

[2] We may always combine $n > N_1$ and $n > N_2$, by simply write $n > N = \max\{N_1, N_2\}$.

Picking any $n > N$, we get

$$k - \epsilon < x_n \le y_n < l + \epsilon.$$

Therefore we have $k - l < 2\epsilon$ for any $\epsilon > 0$. The property is the same as $k - l \le 0$.

It is a very common technique to first show $x \le \epsilon$ for any $\epsilon > 0$, and then conclude $x \le 0$.

Finally, we remark that the second part of the order rule implies the first part. In other words, the proof above can be replaced by a more conceptual argument.

The second part is equivalent to its contrapositive

$$\text{Not "}x_n < y_n \text{ for large } n\text{"} \implies \text{Not "}k < l\text{"}.$$

When you think of the left side hard enough, you realize the statement is the same as the following

$$\text{For any } N\text{, there is } n > N \text{ satisfying } x_n \ge y_n \implies k \ge l.$$

Then we have

$$\begin{aligned} &\text{There is } N\text{, such that } x_n \ge y_n \text{ for all } n > N \\ \implies &\text{For any } N\text{, there is } n > N \text{ satisfying } x_n \ge y_n \\ \implies &k \ge l. \end{aligned}$$

The first $\implies$ actually means something happens for all large n implies you can find arbitrarily large n for something to happen[3]. Now if you switch (x, k) with (y, l), then you get the first part of the order rule.

Example 1.2.18. We prove a special case of the converse direction of Proposition 1.1.6: Suppose x_n is the union of two subsequences x_{m_k} and x_{n_k}. If $\lim_{k\to\infty} x_{m_k} = l$ and $\lim_{k\to\infty} x_{n_k} = l$, then $\lim_{n\to\infty} x_n = l$.

For any $\epsilon > 0$, by $\lim_{k\to\infty} x_{m_k} = l$ and $\lim_{k\to\infty} x_{n_k} = l$, there is a natural number K, such that

$$k > K \implies |x_{m_k} - l| < \epsilon \text{ and } |x_{n_k} - l| < \epsilon.$$

We claim that

$$n > \max\{m_K, n_K\} \implies |x_n - l| < \epsilon.$$

For $n > m_K, n_K$, since x_n is the union of two subsequences x_{m_k} and x_{n_k}, we know $n = m_k$ or $n = n_k$. Suppose $n = m_k$. By $n = m_k > m_K$, and $m_1 < m_2 < m_3 < \dots$, we know $k > K$. This implies $|x_n - l| = |x_{m_k} - l| < \epsilon$. The proof for the case $n = n_k$ is the same.

The difficulty of this example is actually to write down a mathematical proof. An important part is to choose the right notation. If we simply denote the subspaces by x'_n and x''_n as in Example 1.1.23, then it is impossible to write a proper proof.

[3] Isn't this obvious?

Exercise 1.2.19. Prove the first part of Proposition 1.1.6: If a sequence x_n converges to l, then any subsequence x_{n_k} also converges to l.

As commented before the proposition, you may assume $n_k \geq k$ for all k.

Exercise 1.2.20. Prove the converse part of Proposition 1.1.6 for the case the sequence is the union of three subsequences. Can you write the proof for the general case the sequence is the union of finitely many subsequence?

Exercise 1.2.21. Suppose two sequences x_n and y_n satisfy $x_n = y_{n+k}$ for a constant integer k and sufficiently large n. Prove that $\lim_{n\to\infty} x_n$ converges if and only if $\lim_{n\to\infty} y_n$ converges. Moreover, the two limit values are the same.

This rigorously proves Proposition 1.1.2.

After rigorously proving basic limits and the basic properties in Propositions 1.1.2, 1.1.3, 1.1.4, 1.1.5, 1.1.6, all the limits we discussed so far are rigorously confirmed.

Finally, we have been quite liberal in choosing ϵ and other ingredients in the definition of limit. The following expands on this observation.

Example 1.2.19. We claim that the definition of limit is equivalent to the following

$$\text{For any } \epsilon > 0, \text{ there is } N, \text{ such that } n > N \implies |x_n - l| \leq \epsilon.$$

In other words, we want to show that the implication above is equivalent to the implication below

$$\text{For any } \epsilon > 0, \text{ there is } N, \text{ such that } n > N \implies |x_n - l| < \epsilon.$$

Suppose we have the first implication. Then for any $\epsilon > 0$, we apply the implication to $\epsilon' = \frac{\epsilon}{2} > 0$. Then for any $\epsilon > 0$, there is N, such that

$$n > N \implies |x_n - l| \leq \epsilon' = \frac{\epsilon}{2} < \epsilon.$$

Therefore we get the second implication.

The converse is easier and is left to the reader.

Exercise 1.2.22. Prove the following are equivalent to the definition of $\lim_{n\to\infty} x_n = l$.

1. For any $1 > \epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \epsilon$.
2. For any $\epsilon > 0$, there is a natural number N, such that $n > N$ implies $|x_n - l| < \epsilon$.
3. For any $\epsilon > 0$, there is N, such that $n \geq N$ implies $|x_n - l| \leq \epsilon$.
4. For any $\epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \epsilon^2$.
5. For any $\epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| \leq 2\epsilon$.

6. For any natural number k, there is N, such that $n > N$ implies $|x_n - l| < \frac{1}{k}$.

Exercise 1.2.23. Which are equivalent to the definition of $\lim_{n\to\infty} x_n = l$?

1. For $\epsilon = 0.001$, we have $N = 1000$, such that $n > N$ implies $|x_n - l| < \epsilon$.
2. For any $0.001 \geq \epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \epsilon$.
3. For any $\epsilon > 0.001$, there is N, such that $n > N$ implies $|x_n - l| < \epsilon$.
4. For any $\epsilon > 0$, there is a natural number N, such that $n > N$ implies $|x_n - l| \leq \frac{1}{2}\epsilon$.
5. For any $\epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < 2\epsilon^2$.
6. For any $\epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \epsilon + 1$.
7. For any $\epsilon > 0$, we have $N = 1000$, such that $n > N$ implies $|x_n - l| < \epsilon$.
8. For any $\epsilon > 0$, there are infinitely many n, such that $|x_n - l| < \epsilon$.
9. For infinitely many $\epsilon > 0$, there is N, such that $n > N$ implies $|x_n - l| < \epsilon$.
10. For any $\epsilon > 0$, there is N, such that $n > N$ implies $l - 2\epsilon < x_n \leq l + \epsilon$.

1.3 Convergence Criterion

move to after subsequence

If I have about 100 dollars, give or take 10 dollars then my real money is between 90 (lower bound) and 110 (upper bound) dollars.

Theorem 1.3.1. *If x_n converges, then $|x_n| \leq B$ for a constant B and all n.*

Any convergent sequence is bounded. The number B is a *bound* for the sequence.

If $x_n \leq B$ for all n, then we say x_n is *bounded above*, and B is an *upper bound*. If $x_n \geq B$ for all n, then we say x_n is *bounded below*, and B is a *lower bound*. A sequence is bounded if and only if it is bounded above and bounded below.

The sequences n, $\frac{n^2+(-1)^n}{n+1}$ diverge because they are not bounded. On the other hand, the sequences $(-1)^n, \sin na$ (a is not integer multiple of π) are bounded but diverge. Therefore the converse of Theorem 1.3.1 is not true in general.

Exercise 1.3.1. If x_n and y_n are bounded, prove that $x_n + y_n$, $x_n - y_n$, $x_n y_n$ are bounded. What if x_n and y_n are bounded above or bounded below?

Exercise 1.3.2. Prove that if x_n is bounded for sufficiently large n, i.e., $|x_n| \leq B$ for $n \geq N$, then x_n is still bounded.

Exercise 1.3.3. Suppose x_n is the union of two subsequences x'_n and x''_n. Prove that x_n is bounded if and only if both x'_n and x''_n are bounded. The statement can be extended to finitely many subsequences.

Exercise 1.3.4. Suppose $\lim_{n\to\infty} x_n = 0$ and y_n is bounded. Prove that $\lim_{n\to\infty} x_n y_n = 0$.

1.3.1 Monotone Sequence

The converse of Theorem 1.3.1 holds under some additional assumption. A sequence x_n is *increasing* if

$$x_1 \le x_2 \le x_3 \le \cdots \le x_n \le x_{n+1} \le \cdots .$$

It is *strictly increasing* if

$$x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} < \cdots .$$

We can similarly define *decreasing* and *strictly decreasing* sequences. A sequence is *monotone* if it is either increasing or decreasing.

The sequences $\frac{1}{n}$, $\frac{1}{2^n}$, $\sqrt[n]{2}$ are (strictly) decreasing. The sequences $-\frac{1}{n}$, n are increasing.

Theorem 1.3.2. *A monotone sequence converges if and only if it is bounded.*

An increasing sequence x_n is always bounded below by its first term x_1. Therefore the sequence is bounded if and only if it is bounded above. Similarly, a decreasing sequence is bounded if and only if it is bounded below.

The world record for 100 meter dash is a decreasing sequence bounded below by 0. The proposition reflects the intuition that there is a limit on how fast a human can run. We note that the proposition does not tell us the exact value of the limit, just like we do not know the exact limit of the human ability.

Example 1.3.1. Consider the sequence

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

The sequence is clearly increasing. Moreover, the following shows the sequence is bounded above

$$\begin{aligned} x_n &\le 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

Therefore the sequence converges.

The limit of the sequence is the sum of the *infinite series*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots.$$

We will see that the sum is actually $\frac{\pi^2}{6}$.

Exercise 1.3.5. Show the convergence of sequences.

1. $x_n=\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}$.
2. $x_n=\frac{1}{1^{2.4}}+\frac{1}{2^{2.4}}+\frac{1}{3^{2.4}}+\cdots+\frac{1}{n^{2.4}}$.
3. $x_n=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}$.
4. $x_n=\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)}$.
5. $x_n=\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$.

Example 1.3.2. The number $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ is the limit of the sequence x_n inductively given by

$$x_1=\sqrt{2},\quad x_{n+1}=\sqrt{2+x_n}.$$

After trying the first couple of terms, we expect the sequence to be increasing. This can be verified by induction. We have $x_2=\sqrt{2+\sqrt{2}}>x_1=\sqrt{2}$. Moreover, if we assume $x_n>x_{n-1}$, then

$$x_{n+1}=\sqrt{2+x_n}>\sqrt{2+x_{n-1}}=x_n.$$

This proves inductively that x_n is indeed increasing.

Next we show that x_n is bounded above. For an increasing sequence, we expect its limit to be the upper bound. Therefore we find the hypothetical limit l first. Taking the limit on both sides of the equality $x_{n+1}^2=2+x_n$ and applying the arithmetic rule, we get $l^2=2+l$. The solution is $l=2$ or -1. Since $x_n>0$, by the order rule, we must have $l\geq 0$. Therefore $l=2$.

The hypothetical limit value suggests that $x_n<2$ for all n. Again we verify this by induction. We already have $x_1=\sqrt{2}<2$. If we assume $x_n<2$, then

$$x_{n+1}=\sqrt{2+x_n}<\sqrt{2+2}=2.$$

This proves inductively that $x_n<2$ for all n.

We conclude that x_n is increasing and bounded above. By Theorem 1.3.2, the sequence converges, and the hypothetical limit value 2 is the real limit value.

Figure 1.3.1 suggests that our conclusion actually depends only on the general shape of the graph of the function, and has little to do with the exact formula $\sqrt{2+x}$.

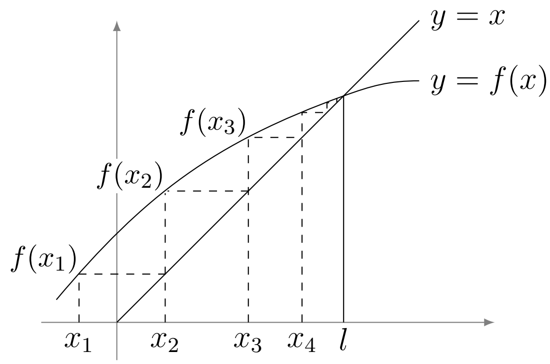


Figure 1.3.1: Limit of inductively defined sequence.

Exercise 1.3.6. Suppose a sequence x_n satisfies $x_{n+1} = \sqrt{2 + x_n}$.

1. Prove that if $-2 < x_1 < 2$, then x_n is increasing and converges to 2.

2. Prove that if $x_1 > 2$, then x_n is decreasing and converges to 2.

Exercise 1.3.7. Suppose a sequence x_n satisfies $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{x_n})$, and $x_1 > 0$. Prove that starting from x_2, the sequence is decreasing, bounded, and converges to 1. What if $x_1 < 0$?

Exercise 1.3.8. Suppose a sequence x_n satisfies $x_{n+1} = \frac{1}{2}(x_n^2 + x_n)$. Prove the following.

1. If $x_1 > 1$, then the sequence is increasing and diverges.

2. If $0 < x_1 < 1$, then the sequence is decreasing and converges to 0.

3. If $-1 < x_1 < 0$, then the sequence is increasing and converges to 0.

4. If $-2 < x_1 < -1$, then the sequence is decreasing for $n \geq 2$ and converges to 0.

5. If $x_1 < -2$, then the sequence is increasing for $n \geq 2$ and diverges.

Exercise 1.3.9. For the three functions $f(x)$ in Figure 1.3.2, study the convergence of the sequence x_n defined by $x_{n+1} = f(x_n)$. Your answer depends on the initial value x_1.

Exercise 1.3.10. Determine the convergence of inductively defined sequences. Your answer may depend on the initial value x_1.

1. $x_{n+1} = x_n^2$.
2. $x_{n+1} = \frac{x_n^2+1}{2}$.
3. $x_{n+1} = 2x_n^2 - 1$.
4. $x_{n+1} = \frac{1}{x_n}$.
5. $x_{n+1} = 1 + \frac{1}{x_n}$.
6. $x_{n+1} = 2 - \frac{1}{x_n}$.

Exercise 1.3.11. Determine the convergence of inductively defined sequences, $a > 0$. In some cases, the sequence may not be defined after certain number of terms.

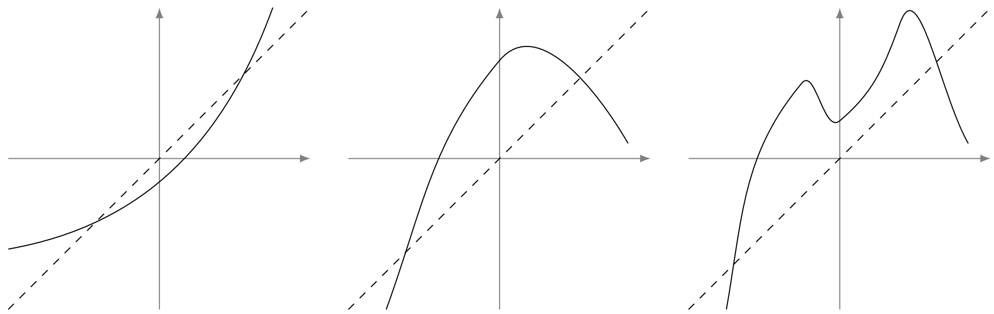


Figure 1.3.2: Three functions

1. $x_{n+1} = \sqrt{a + x_n}$.
2. $x_{n+1} = \sqrt{x_n - a}$.
3. $x_{n+1} = \sqrt{a - x_n}$.
4. $x_{n+1} = \sqrt[3]{a + x_n}$.
5. $x_{n+1} = \sqrt[3]{x_n - a}$.
6. $x_{n+1} = \sqrt[3]{a - x_n}$.

Exercise 1.3.12. Explain the *continued fraction expansion*

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}.$$

What if 2 on the right side is changed to some other positive number?

Exercise 1.3.13. Study the limit of the sequences $\sin(\sin(\sin \cdots a)))$ and $\cos(\cos(\cos \cdots a)))$, where the trigonometric functions are applied n times.

Exercise 1.3.14. For any $a, b > 0$, define a sequence by

$$x_1 = a, \quad x_2 = b, \quad x_n = \frac{x_{n-1} + x_{n-2}}{2}.$$

Prove that the sequence converges.

Exercise 1.3.15. The arithmetic and the geometric means of $a, b > 0$ are $\frac{a+b}{2}$ and $\sqrt{ab}$. By repeating the process, we get two sequences defined by

$$x_1 = a, \quad y_1 = b, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_n y_n}.$$

Prove that $x_n \geq x_{n+1} \geq y_{n+1} \geq y_n$ for $n \geq 2$, and the two sequences converge to the same limit.

Exercise 1.3.16. The Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

is defined by $x_0 = x_1 = 1$ and $x_{n+1} = x_n + x_{n-1}$. Consider the sequence $y_n = \frac{x_{n+1}}{x_n}$.

1. Find the relation between y_{n+1} and y_n.
2. Assuming y_n converges, find the limit l.
3. Use the relation between y_{n+2} and y_n to prove that l is the upper bound of y_{2k} and the lower bound of y_{2k+1}.
4. Prove that the subsequence y_{2k} is increasing and the subsequence y_{2k+1} is decreasing.
5. Prove that the sequence y_n converges to l.

Exercise 1.3.17. To find $\sqrt{a}$ for $a > 0$, we start with a guess $x_1 > 0$ of the value of $\sqrt{a}$. Noting that x_1 and $\frac{a}{x_1}$ are on the two sides of $\sqrt{a}$, it is reasonable to choose the average $x_2 = \frac{1}{2}(x_1 + \frac{a}{x_1})$ as the next guess. This leads to the inductive formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

as a way of numerically computing better and better approximate values of $\sqrt{a}$.

1. Prove that $\lim_{n\to\infty} x_n = \sqrt{a}$.
2. We may also use weighted average $x_{n+1} = \frac{1}{3}(x_n + 2\frac{a}{x_n})$ as the next guess. Do we still have $\lim_{n\to\infty} x_n = \sqrt{a}$ for the weighted average?
3. Compare the two methods for specific values of a and b (say $a = 4$, $b = 1$). Which way is faster?
4. Can you come up with a similar scheme for numerically computing $\sqrt[3]{a}$? What choice of the weight gives you the fastest method?

1.3.2 Application of Monotone Sequence

Example 1.3.3. We give another argument for $\lim_{n\to\infty} a^n = 0$ in Example 1.1.12.

For $0 < a < 1$ the sequence a^n is decreasing and satisfies $0 < a^n < 1$. Therefore a^n converges to a limit l. By the remark in Example 1.1.1, we also have $\lim_{n\to\infty} a^{n-1} = l$. Then by the arithmetic rule, we have

$$l = \lim_{n\to\infty} a^n = \lim_{n\to\infty} a \cdot a^{n-1} = a \lim_{n\to\infty} a^{n-1} = al.$$

Since $a \neq 1$, we get $l = 0$.

For the case $-1 < a < 0$, we may consider the even and odd subsequences of a^n and apply Proposition 1.1.6. Another way is to apply the sandwich rule to $-|a|^n \leq a^n \leq |a|^n$.

Example 1.3.4. We consider the sequence $x_n = \frac{(n!)^2}{(2n)!}a^n$, where $a > 0$. In Example 1.1.18, we showed that, for $a = 3$, the sequence converges to 0.

We have

$$\lim_{n\to\infty}\frac{x_n}{x_{n-1}}=\lim_{n\to\infty}\frac{an^2}{2n(2n-1)}=\frac{a}{4}.$$

If $0<a<4$, then by the order rule, we have $\frac{x_n}{x_{n-1}}<1$ for sufficiently large n. Since x_n is always positive, this implies $x_{n-1}>x_n$, i.e., the sequence is decreasing for sufficiently large n. Since 0 is the lower bound, the sequence converges.

Let $\lim_{n\to\infty}x_n=l$. Then we also have $\lim_{n\to\infty}x_{n-1}=l$. If $l\neq 0$, then

$$\lim_{n\to\infty}\frac{x_n}{x_{n-1}}=\frac{\lim_{n\to\infty}x_n}{\lim_{n\to\infty}x_{n-1}}=\frac{l}{l}\neq\frac{a}{4}.$$

The contradiction shows that $l=0$.

If $a>4$, then x_n is increasing for $n>N$. If x_n converges, then by the order rule, the limit $l\geq x_N>0$. Then we still get $\lim_{n\to\infty}\frac{x_n}{x_{n-1}}=\frac{l}{l}\neq\frac{a}{4}$, a contradiction. Therefore the sequence diverges for $a>4$.

Exercise 1.3.18. Extend Example 1.3.3 to a proof of $\lim_{n\to\infty}n^pa^n=0$ for $|a|<1$.

Exercise 1.3.19. Study the limit of $x_n=\frac{(n!)^2}{(2n)!}a^n$, for $a<0$.

Example 1.3.5 (Definition of e). For the sequence $\left(1+\frac{1}{n}\right)^n$, we compare two consecutive terms by their binomial expansions

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n&=1+n\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots3\cdot2\cdot1}{n!}\frac{1}{n^n}\\&=1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots\\&\quad+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right),\\\left(1+\frac{1}{n+1}\right)^{n+1}&=1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots\\&\quad+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)\\&\quad+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right).\end{aligned}$$

A close examination shows that the sequence is increasing. Moreover, by the calculation in Example 1.3.1, the first expansion implies

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n&<1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\\&<1+1+\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{(n-1)n}<3.\end{aligned}$$

By Theorem 1.3.2, the sequence converges. We denote the limit by e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845904\cdots.$$

Exercise 1.3.20. Find the limits.

1. $(1+\frac{1}{n+1})^n$.
2. $(1+\frac{1}{n})^{n+1}$.
3. $(1-\frac{1}{n})^n$.
4. $(1+\frac{1}{2n})^n$.
5. $(1+\frac{2}{n})^n$.
6. $(1+\frac{a}{n})^n$.
7. $(1+\frac{1}{n}-\frac{1}{n^2})^n$.
8. $(1+\frac{a}{n}+\frac{b}{n^2})^n$.
9. $(\frac{n+2}{n-2})^n$.
10. $(\frac{n+a}{n+b})^{n+c}$.
11. $(1+\frac{n}{n^2-1})^{n+1}$.
12. $(1+\frac{n+a}{n^2+b})^{n+c}$.
13. $(1+\frac{1}{n})^{\frac{n^2}{n+1}}$.
14. $(1+\frac{a}{n})^{\frac{n^2}{n+b}}$.
15. $(\frac{n+a}{n+b})^{\frac{n^2+c}{n+d}}$.
16. $(1+\frac{(-1)^n}{n^2-1})^n$.
17. $(1+\frac{n}{n^2+(-1)^n})^{n+1}$.
18. $(\frac{n}{n+(-1)^n})^{(-1)^n n}$.
19. $(\frac{n-1}{n})^{\frac{n^2}{n+(-1)^n}}$.
20. $(\frac{n-1}{n+2})^{\frac{n^2+(-1)^n n}{n+1}}$.

Exercise 1.3.21. Let $x_n = \left(1+\frac{1}{n}\right)^{n+1}$.

1. Use induction to prove $(1+x)^n \ge 1+nx$ for $x > -1$ and any natural number n.
2. Use the first part to prove $\frac{x_{n-1}}{x_n} > 1$. This shows that x_n is decreasing.
3. Prove that $\lim_{n\to\infty} x_n = e$.
4. Prove that $\left(1-\frac{1}{n}\right)^n$ is increasing and converges to e^{-1}.

Exercise 1.3.22. Prove that for $n > k$, we have

$$\left(1+\frac{1}{n}\right)^n \ge 1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k}{n}\right).$$

Then use Proposition 1.1.5 to show that

$$e \ge 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!} \ge \left(1+\frac{1}{k}\right)^k.$$

Finally, prove

$$\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right) = e.$$

1.3.3 Cauchy Criterion

Theorem 1.3.2 gives a special case that we know the convergence of a sequence without knowing the actual limit value. Since the definition of limit explicitly uses the limit value, it cannot be directly used to derive the convergence of the sequence. The following provides the general criterion for convergence, without referring to the actual limit value.

Theorem 1.3.3 (Cauchy Criterion). *A sequence x_n converges if and only if for any $\epsilon > 0$, there is N, such that*

$$m, n > N \implies |x_m - x_n| < \epsilon.$$

We call a sequence satisfying the property in the theorem a *Cauchy sequence.* A sequence converges if and only if it is a Cauchy sequence.

The necessity is easy to see. If $\lim_{n\to\infty} x_n = l$, then for big m, n, both x_m and x_n are very close to l (say within $\frac{\epsilon}{2}$). This implies that x_m and x_n are very close (within $\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$).

The proof of sufficiency is much more difficult and relies on the following deep result that touches the essential difference between the real and rational numbers.

Theorem 1.3.4 (Bolzano-Weierstrass). *Any bounded sequence has a convergent subsequence.*

Using the theorem, the converse may be proved by the following steps.

1. A Cauchy sequence is bounded.
2. By Bolzano-Weierstrass Theorem, the sequence has a convergent subsequence.
3. If a Cauchy sequence has a subsequence converging to l, then the whole sequence converges to l.

Example 1.3.6. In Example 1.1.21, we argued that the sequence $(-1)^n$ diverges because two subsequences converge to different limits. Alternatively, we may apply the Cauchy criterion. For $\epsilon = 1$ and any N, we pick any $n > N$ and pick $m = n+1$. Then $m, n > N$ and $|x_m - x_n| = |(-1)^{n+1} - (-1)^n| = 2 > \epsilon$. This means that the Cauchy criterion fails, and therefore the sequence diverges.

Example 1.3.7. In Example 1.3.1, we argued the convergence of

$$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

by increasing and bounded property. Alternatively, for $m > n$, we have

$$\begin{aligned}
|x_m - x_n| &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2} \\
&\le \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(m-1)m} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
&= \frac{1}{n} - \frac{1}{m}.
\end{aligned}$$

Then for any $\epsilon > 0$, we have

$$m > n > N = \frac{1}{\epsilon} \implies |x_m - x_n| < \frac{1}{n} < \frac{1}{N} = \epsilon.$$

Therefore x_n is a Cauchy sequence and converges.

We note that the same argument can be used to show the convergence of

$$x_n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots + (-1)^n \frac{1}{n^2}.$$

The method in Example 1.3.1 cannot be used here because the sequence is not monotone.

Example 1.3.8 (Harmonic Series). The following is the partial sum of the *harmonic series*

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For any n, we have

$$x_{2n} - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

For $\epsilon = \frac{1}{2}$ and any N, we choose a natural number $n > N$ and also choose $m = 2n > N$. Then

$$|x_m - x_n| = x_{2n} - x_n \geq \epsilon.$$

Therefore the sequence fails the Cauchy criterion and diverges.

Exercise 1.3.23. If x_n is a Cauchy sequence, is $|x_n|$ also a Cauchy sequence? What about the converse?

Exercise 1.3.24. Use Cauchy criterion to determine the convergence.

1. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.
2. $1 - \frac{1}{2^3} + \frac{1}{3^3} - \cdots + \frac{(-1)^{n+1}}{n^3}$.
3. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}$.
4. $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}$.
5. $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$.
6. $1 + \frac{2}{1!} + \frac{3}{2!} + \cdots + \frac{n}{(n-1)!}$.

Example 1.3.9. Theorem 1.3.4 says a bounded sequence has converging subsequences. How about the limit values of such subsequences?

It is easy to see that the limits of the converging subsequences in $(-1)^n$ are ± 1. In fact, given finitely many numbers, it is easy to construct sequences with the given numbers as all the limit values of converging subsequences.

For an extreme example, let us list all finite decimal expressions in $(0, 1)$ as a sequence

$$x_n\colon \quad 0.1,\ 0.2,\ \ldots,\ 0.9,\ 0.01,\ 0.02,\ \ldots, 0.99,\ 0.001,\ 0.002,\ \ldots,\ 0.999,\ \ldots .$$

The number $0.318309\cdots$ is the limit of the following subsequence

$$0.3,\ 0.31,\ 0.318,\ 0.3183,\ 0.31830,\ 0.318309,\ \dots .$$

It is easy to see that any number in $[0,1]$ is the limit of a convergent subsequence of x_n.

Exercise 1.3.25. Construct a sequence such that the limits of convergent subsequences are exactly $\frac{1}{n}$, $n \in \mathbb{N}$ and 0.

Exercise 1.3.26. Construct a sequence such that any number is the limit of some convergent subsequence.

Exercise 1.3.27. Use any suitable method or theorem to determine convergence.

1. $\frac{1}{2},\ \frac{2}{1},\ \frac{2}{3},\ \frac{3}{2},\ \dots,\ \frac{n}{n+1},\ \frac{n+1}{n},\ \dots$.
2. $\frac{1}{2},\ -\frac{2}{1},\ \frac{2}{3},\ -\frac{3}{2},\ \dots,\ \frac{n}{n+1},\ -\frac{n+1}{n},\ \dots$.
3. $1+\frac{1}{2^2}+\frac{2}{3^2}+\cdots+\frac{n-1}{n^2}$.
4. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(2n-1)2n}$.
5. $\frac{1}{1\cdot 2}-\frac{1}{3\cdot 4}+\cdots+(-1)^{n+1}\frac{1}{(2n-1)2n}$.
6. $\frac{2}{1\cdot 3}+\frac{3}{2\cdot 4}+\cdots+\frac{n}{(n-1)(n+1)}$.
7. $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$.
8. $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$.
9. $\sqrt{1+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{3}+\cdots+\sqrt{\frac{1}{n}}}}}$.
10. $\sqrt{1+\sqrt{2+\sqrt{3+\cdots+\sqrt{n}}}}$.

1.4 Infinity

1.4.1 Divergence to Infinity

A sequence may diverge for various reasons. The sequence n diverges because it can become arbitrarily large. The bounded sequence $(-1)^n$ diverges because it has two subsequences with different limits.

Definition 1.4.1. A sequence *diverges to infinity*, denoted $\lim_{n\to\infty} x_n = \infty$, if for any B, there is N, such that $n > N$ implies $|x_n| > B$.

In the definition, the infinity means that the absolute value (or the magnitude) of the sequence can become arbitrarily large. If we further take into account of the signs, then we get the following definitions.

Definition 1.4.2. A sequence *diverges to* $+\infty$, denoted $\lim_{n\to\infty} x_n = +\infty$, if for any B, there is N, such that $n > N$ implies $x_n > B$. A sequence *diverges to* $-\infty$, denoted $\lim_{n\to\infty} x_n = -\infty$, if for any B, there is N, such that $n > N$ implies $x_n < B$.

Figure 1.4.1 shows the meaning of $\lim_{n\to\infty} x_n = +\infty$. We also note that, in the definition of $\lim_{n\to\infty} x_n = +\infty$, we may additionally assume $B > 0$ (or $B > 100$) without loss of generality.

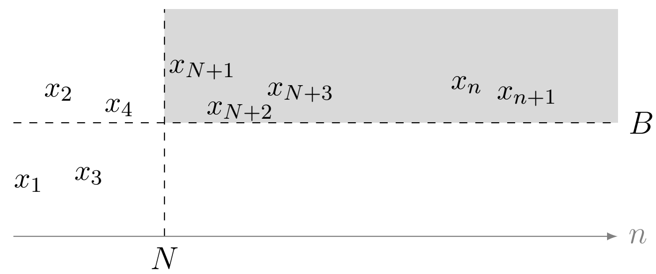


Figure 1.4.1: $n > N$ implies $x_n > B$.

Example 1.4.1. If $p > 0$, then for any $B > 0$, we have

$$n > B^{\frac{1}{p}} \implies n^p > B.$$

This proves

$$\lim_{n\to\infty} n^p = +\infty, \text{ for } p > 0.$$

Example 1.4.2. Example 1.1.12 may be extended to show that $\lim_{n\to\infty} a^n = \infty$ for $|a| > 1$. Specifically, let $|a| = 1 + b$. Then $|a| > 1$ implies $b > 0$, and we have

$$|a^n| = (1+b)^n = 1 + nb + \frac{n(n-1)}{2}b^2 + \cdots + b^n > nb.$$

For any B, we then have

$$n > \frac{B}{b} \implies |a^n| > nb > B.$$

This proves $\lim_{n\to\infty} a^n = \infty$ for $|a| > 1$. If we consider the sign, this also proves $\lim_{n\to\infty} a^n = +\infty$ for $a > 1$.

Example 1.4.3. Suppose $x_n \neq 0$. We prove that $\lim_{n\to\infty} x_n = 0$ implies $\lim_{n\to\infty} \frac{1}{x_n} = \infty$. Actually the converse is also true and the proof is left to the reader.

If $\lim_{n\to\infty} x_n = 0$, then for any $B > 0$, we apply the definition of the limit to $\frac{1}{B} > 0$ to get N, such that

$$n > N \implies |x_n| < \frac{1}{B} \implies \left|\frac{1}{x_n}\right| > B.$$

This proves $\lim_{n\to\infty} \frac{1}{x_n} = \infty$.

Applying what we just proved to the limit in Example 1.1.12, we get another proof of $\lim_{n\to\infty} a^n = \infty$ for $|a| > 1$.

Exercise 1.4.1. Let $x_n \neq 0$. Prove that $\lim_{n\to\infty} \frac{1}{x_n} = \infty$ implies $\lim_{n\to\infty} x_n = 0$.

Exercise 1.4.2. Prove that $\lim_{n\to\infty} x_n = +\infty$ if and only if $x_n > 0$ for sufficiently large n and $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

Exercise 1.4.3. Rigorously prove divergence to infinity. Determine $\pm\infty$ if possible.

1. $\frac{n^2-n+1}{n+1}$.
2. $\frac{n}{\sqrt{n}+1}$.
3. $\frac{a^n}{n}$, $|a| > 1$.
4. $n^p a^n$, $|a| > 1$.
5. $\frac{n!}{4^n}$.
6. $n!a^n$.

We know a bounded increasing sequence x_n converges. If the sequence is not bounded, then it is not bounded above. This means that, for any B, there is a term $x_N > B$. Then by the increasing property, we get

$$n > N \implies x_n \geq x_N > B.$$

This means x_n diverges to $+\infty$. We get an enhancement of Theorem 1.3.2.

Theorem 1.4.3. *Suppose x_n is an increasing sequence.*

1. *If x_n is bounded, then x_n converges.*
2. *If x_n is not bounded, then x_n diverges to $+\infty$.*

Since there are only two possibilities for an increasing sequence, the two statements are actually if and only if. For example, the divergence of the harmonic series in Example 1.3.8 shows that the series diverges to $+\infty$.

There is also similar theorem for decreasing sequences.

Exercise 1.4.4. For $a > 1$, use Theorem 1.4.3 to prove $\lim_{n\to\infty} a^n = +\infty$.

Exercise 1.4.5. Prove $\lim_{n\to\infty} \frac{a^n}{n^2} = +\infty$ for $a > 1$.

1.4.2 Arithmetic Rule for Infinity

A sequence x_n is an *infinitesimal* if $\lim_{n\to\infty} x_n = 0$. Example 1.4.3 and Exercise 1.4.1 show that a sequence is an infinetesimal if and only if its reciprocal is an infinity.

Many properties of the finite limit can be extended to infinity. For example, we have $\frac{l}{0} = \infty$ for $l \neq 0$. This means that, if $\lim_{n\to\infty} x_n = l \neq 0$ and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$. For example,

$$\lim_{n\to\infty} \frac{2n^2+1}{n-1} = \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{\frac{1}{n}-\frac{1}{n^2}} = \frac{\lim_{n\to\infty}\left(2+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)} = \frac{2}{0} = \infty.$$

Note that l in $\frac{l}{0}$ can represent any sequence converging to l, and is not necessarily a constant.

The following are more extensions of the arithmetic rules to infinity. The rules are symbolically denoted by "arithmetic equalities", and the exact meaning of the rules are also given.

- $\frac{l}{\infty} = 0$: If $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} y_n = \infty$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$.
- $(+\infty) + (+\infty) = +\infty$: If $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = +\infty$, then $\lim_{n\to\infty}(x_n + y_n) = +\infty$.
- $(-\infty) + l = -\infty$: If $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} y_n = l$, then $\lim_{n\to\infty}(x_n + y_n) = -\infty$.
- $(+\infty) \cdot l = -\infty$ for $l < 0$: If $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = l < 0$, then $\lim_{n\to\infty} x_n y_n = -\infty$.
- $\frac{l}{0^+} = +\infty$ for $l > 0$: If $\lim_{n\to\infty} x_n = l > 0$, $\lim_{n\to\infty} y_n = 0$ and $y_n > 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = +\infty$.

On the other hand, we should always be cautious not to overextend the arithmetic equalities. For example, the following "arithmetic rules" are actually wrong

$$\infty + \infty = \infty, \quad \frac{+\infty}{-\infty} = -1, \quad 0 \cdot \infty = 0, \quad 0 \cdot \infty = \infty.$$

A counterexample for the first equality is $x_n = n$ and $y_n = -n$, for which we have $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} y_n = \infty$ and $\lim_{n\to\infty}(x_n + y_n) = 0$. In general, one needs to use common sense to decide whether certain extended arithmetic rule makes sense.

Example 1.4.4. By Example 1.4.1 and the extended arithmetic rule, we have

$$\lim_{n\to\infty}(n^3 - 3n + 1) = \lim_{n\to\infty} n^3 \left(1 - \frac{3}{n^2} + \frac{1}{n^3}\right) = (+\infty) \cdot 1 = +\infty.$$

In general, any non-constant polynomial of n diverges to ∞, and for rational functions, we have

$$\lim_{n\to\infty} \frac{a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0}{b_q n^q + b_{q-1} n^{q-1} + \cdots + b_1 n + b_0} = \begin{cases} +\infty, & \text{if } p > q,\ a_p b_q > 0, \\ -\infty, & \text{if } p > q,\ a_p b_q < 0, \\ \frac{a_p}{b_q}, & \text{if } p = q,\ b_q \neq 0, \\ 0, & \text{if } p < q,\ b_q \neq 0. \end{cases}$$

Exercise 1.4.6. Prove the extended arithmetic rules

$$\frac{l}{0} = \infty, \quad l + (+\infty) = +\infty, \quad (+\infty) \cdot (-\infty) = -\infty, \quad \frac{l}{0^-} = -\infty \text{ for } l > 0.$$

Exercise 1.4.7. Construct sequences x_n and y_n, such that both diverge to infinity, but $x_n + y_n$ can have any of the following behaviors.

1. $\lim_{n\to\infty}(x_n + y_n) = \infty$.

2. $\lim_{n\to\infty}(x_n + y_n) = 2$.

3. $x_n + y_n$ is bounded but does not converge.

4. $x_n + y_n$ is not bounded and does not diverge to infinity.

The exercise shows that $\infty + \infty$ has no definite meaning.

Exercise 1.4.8. Prove that if $p > 0$, then $\lim_{n\to\infty} x_n = +\infty$ implies $\lim_{n\to\infty} x_n^p = +\infty$. What about the case $p < 0$?

Exercise 1.4.9. Prove the extended sandwich rule: If $x_n \le y_n$ for sufficiently large n, then $\lim_{n\to\infty} x_n = +\infty$ implies $\lim_{n\to\infty} y_n = +\infty$.

Exercise 1.4.10. Prove the extended order rule: If $\lim_{n\to\infty} x_n = l$ is finite and $\lim_{n\to\infty} y_n = +\infty$, then $x_n < y_n$ for sufficiently large n.

Exercise 1.4.11. Suppose $\lim_{n\to\infty} x_n = l > 1$. Prove that $\lim_{n\to\infty} x_n^n = +\infty$.

Exercise 1.4.12. Prove that if $\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = l$ and $|l| > 1$, then x_n diverges to infinity.

Exercise 1.4.13. Explain the infinities. Determine the sign of infinity if possible.

1. $\frac{n+\sin 2n}{\sqrt{n}-\cos n}$.
2. $\frac{n!}{a^n+b^n}$, $a+b \ne 0$.
3. $\frac{1}{\sqrt[n]{n}-1}$.
4. $\frac{1}{\sqrt[n]{n}-\sqrt[n]{2n}}$.
5. $n(\sqrt{n+2}-\sqrt{n})$.
6. $\frac{(-1)^n n^2}{n-1}$.
7. $\frac{3^n-2^n}{n}$.
8. $\frac{3^n-2^n}{n^3+n^2}$.
9. $\left(1+\frac{1}{n}\right)^{n^2}$.

1.5 Series

A *series* is an infinite sum

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \cdots .$$

We have seen the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (Examples 1.3.1 and 1.3.7), $\sum_{n=0}^{\infty} \frac{1}{n!}$ (Exercise 1.3.22), $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ (Example 1.3.7), $\sum_{n=1}^{\infty} \frac{1}{n}$ (Example 1.3.8) in earlier sections.

1.5.1 Sum of Series

The *partial sum* of a series $\sum_{n=1}^{\infty} x_n$ is

$$s_n = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n.$$

If the partial sum converges to l, then the series *converges* and has *sum* (or *value*) l

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = l.$$

If the partial sum diverges, then the series *diverges*.

The arithmetic rule (Proposition 1.1.3) of the sequence limit implies

$$\sum (x_n + y_n) = \sum x_n + \sum y_n, \quad \sum a x_n = a \sum x_n.$$

However, there is no formula for $\sum x_n y_n$ or $\sum \frac{x_n}{y_n}$.

Example 1.5.1. The decimal expansion

$$\pi = 3.1415926\cdots = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \cdots$$

is a series that converges to π.

Example 1.5.2. The partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is calculated in Example 1.3.1

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Exercise 1.5.1. Suppose the partial sum $s_n = \frac{n}{2n+1}$. Find the series $\sum x_n$ and its sum.

Exercise 1.5.2. Compute the partial sum and the sum of series.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$.
2. $\sum_{n=0}^{\infty} \frac{1}{(a+nd)(a+(n+1)d)}$.
3. $\sum_{n=2}^{\infty} \frac{1}{n(n+1)(n+2)}$.

Exercise 1.5.3. Suppose $x_n > 0$. Find the sum of the series $\sum_{n=1}^{\infty} \frac{x_n}{(1+x_1)(1+x_2)\cdots(1+x_n)}$. Then find $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

Exercise 1.5.4. The *Fibonacci sequence* $1, 1, 2, 3, 5, \ldots$ is defined recursively by $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$. Prove the following

$$\frac{1}{a_{n-1}a_{n+1}} = \frac{1}{a_{n-1}a_n} - \frac{1}{a_n a_{n+1}}, \quad \sum_{n=2}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = 1, \quad \sum_{n=2}^{\infty} \frac{a_n}{a_{n-1}a_{n+1}} = 2.$$

Example 1.5.3. The *geometric series* is

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots .$$

The partial sum $s_n = 1 + x + x^2 + \cdots + x^n$ satisfies

$$(1-x)s_n = (1 + x + x^2 + \cdots + x^n) - (x + x^2 + x^3 + \cdots + x^{n+1}) = 1 - x^{n+1}.$$

Therefore

$$s_n = \frac{1 - x^{n+1}}{1 - x}, \quad \sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \frac{1}{1-x}, & \text{if } |x| < 1, \\ \text{diverges}, & \text{if } |x| \geq 1. \end{cases}$$

We note that the formula s_n does not work for $x = 1$, and the sum of the series needs to be separated argued for $x = 1$.

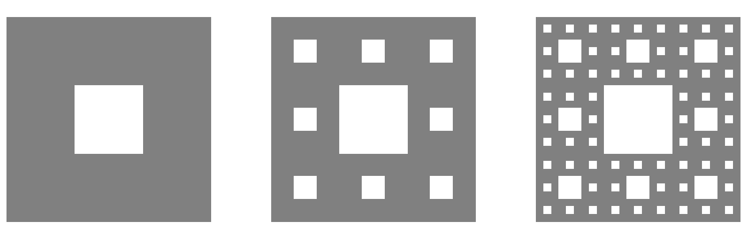
Exercise 1.5.5. Find the partial sum of $\sum_{n=1}^{\infty} nx^n$, by multiplying $1 - x$. Then find the sum.

Exercise 1.5.6. Decimal expressions of rational numbers have repeating patterns. For example, we have

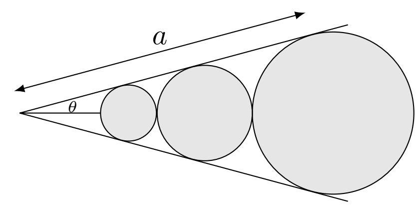
$$\begin{aligned} 1.2\overline{34} = 1.2343434\cdots &= 1.2 + \frac{34}{1000} + \frac{34}{100000} + \frac{34}{10000000} + \cdots \\ &= 1.2 + \frac{34}{1000} \sum_{n=0}^{\infty} \frac{1}{100^n} = 1.2 + \frac{34}{1000} \frac{1}{1 - \frac{1}{100}} = \frac{611}{495}. \end{aligned}$$

1. Find rational expressions for $1.\overline{23}$, $1.\overline{230}$, $1.\overline{023}$.
2. Final the decimal based series representing the rational numbers $\frac{5}{12}$, $\frac{43}{35}$.

Exercise 1.5.7. The *Sierpinski carpet* is obtained from the unit square by successively deleting "one third squares". Find the area of the carpet.



Exercise 1.5.8. Show that the ratio of the radii of the two consecutive disks is $\frac{1-\sin\theta}{1+\sin\theta}$. Then find the total area of infinitely many disks.



Exercise 1.5.9. Two lines L and L' form an angle θ at P. A boy starts on L at distance a from P and walk to L' along shortest path. After reaching L', he walks back to L along shortest path. Then he walks to L' again along shortest path, and keeps walking back and forth. What is the total length of his trip?

Example 1.5.4. The conclusion of Exercise 1.3.22 is

$$\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots=e.$$

In general, we have

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x}{2!}+\cdots+\frac{x^n}{n!}+\cdots=e^x.$$

The argument requires the Taylor expansion of the function e^x, and the remainder formula.

1.5.2 Convergence of Series

It is rare to find the sum of series. We will concentrate on the convergence of series.

First, if finitely many terms in a series are modified, added or dropped, then the new partial sum s'_n and the original partial sum s_n are related by $s'_n = s_{n+n_0}+C$ for some constants n_0 and C. This implies that the convergence is not affected, although the value of the sum may be affected.

The following is a simple necessary condition for the convergence of series. The converse is not true, as shown given by the harmonic series $\sum\frac{1}{n}$ in Example 1.3.8.

Theorem 1.5.1. *If $\sum x_n$ converges, then $\lim_{n\to\infty}x_n=0$.*

The theorem is a consequence of

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}(s_n-s_{n-1})=\lim_{n\to\infty}s_n-\lim_{n\to\infty}s_{n-1}=l-l=0.$$

By the theorem, the series $\sum 1$, $\sum n$, $\sum(-1)^n$, $\sum\frac{n}{n+1}$ diverge. By Example 1.1.22, the series $\sum\sin na$ converges if and only if a is an integer multiple of π.

If $x_n \geq 0$ for sufficiently large n, then the partial sum sequence s_n is increasing for big n. Then by Theorem 1.3.2, we get the following.

Theorem 1.5.2. *If $x_n \geq 0$, then $\sum x_n$ converges if and only if the partial sums are bounded.*

Example 1.5.5. Consider the series $\sum \frac{1}{n^p}$. For $p > 1$, we have have estimation similar to Example 1.3.8

$$s_{2n} - s_n = \frac{1}{(n+1)^p} + \frac{1}{(n+2)^p} + \cdots + \frac{1}{(2n)^p} < n\frac{1}{n^p} = \frac{1}{n^{p-1}}.$$

For $n = 2^{k-1}$, this becomes

$$s_{2^k} - s_{2^{k-1}} < \frac{1}{2^{(p-1)(k-1)}} = r^{k-1}, \text{ where } 0 < r = \frac{1}{2^{p-1}} < 1.$$

Adding the inequalities together, we get (the equality is from Example 1.5.3)

$$s_{2^k} - s_1 < 1 + r + r^2 + \cdots + r^{k-1} = \frac{1 - r^{n+1}}{1 - r} < \frac{1}{1 - r}.$$

Therefore the partial sums are bounded, and the series $\sum \frac{1}{n^p}$ converges for $p > 1$.

For $p \leq 1$, the partial sum is bigger than the partial sum of the harmonic series

$$s_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Since the harmonic series diverges to $+\infty$, we know $\sum \frac{1}{n^p}$ diverges to $+\infty$ for $p \leq 1$.

The following is a simple criterion for convergence.

Proposition 1.5.3 (Leibniz Test). *If x_n is decreasing for sufficiently large n, and $\lim_{n\to\infty} x_n = 0$, then $\sum(-1)^n x_n$ converges.*

For $x_n > 0$, we call the series

$$\sum_{n=0}^{\infty}(-1)^n x_n = x_0 - x_1 + x_2 - x_3 + \cdots$$

an *alternating series*. If x_n is decreasing (for all n), then the odd partial sum

$$\begin{aligned} s_{2n+1} &= (x_0 - x_1) + (x_2 - x_3) + (x_4 - x_5) + \cdots + (x_{2n} - x_{2n+1}) \\ &= x_0 - (x_1 - x_2) - (x_3 - x_4) - \cdots - (x_{2n-1} - x_{2n}) - x_{2n+1} \end{aligned}$$

is increasing and has upper bound x_0. Therefore $\lim s_{2n+1}$ converges. By $s_{2n} = s_{2n+1} - x_{2n+1}$ and $\lim x_{2n+1} = 0$, we have $\lim s_{2n} = \lim s_{2n+1}$. Therefore the whole partial sum sequence converges.

Exercise 1.5.10. Suppose x_n is decreasing and $\lim_{n\to\infty} x_n = 0$. Show that the even partial sum s_{2n} of an alternating series is decreasing and has lower bound 0.

Example 1.5.6. By the Leibniz test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \cdots + \frac{(-1)^{n+1}}{n^p} + \cdots$$

converges for $p > 0$.

Note that by taking the absolute value, the series in Example 1.5.6 becomes the series in Example 1.5.5. For $p > 1$, both series converge. For $0 < p \le 1$, the alternating series converges, but the absolute value series diverges.

1.5.3 Comparison Test

If we apply the Cauchy criterion (Theorem 1.3.3) to the partial sum, we know a series $\sum x_n$ converges if and only if for any $\epsilon > 0$, there is N, such that (since $|s_m - s_n|$ is symmetric in m and n, we may always assume $n > m$)

$$n > m > N \implies |s_m - s_n| = |x_{m+1} + x_{m+2} + \cdots + x_n| < \epsilon.$$

We may further modify the criterion by taking $m + 1$ to be m.

Theorem 1.5.4 (Cauchy Criterion). *A series $\sum x_n$ converges if and only if for any $\epsilon > 0$, there is N, such that*

$$n \ge m > N \implies |x_m + x_{m+1} + \cdots + x_n| < \epsilon.$$

By taking $m = n$, Theorem 1.5.1 is a special case of the Cauchy criterion. A special and very useful case of the Cauchy criterion is the following.

Theorem 1.5.5 (Comparison Test). *Suppose $|x_n| \le y_n$ for sufficiently large n. If $\sum y_n$ converges, then $\sum x_n$ also converges.*

In the theorem, $|x_n| \le y_n$ implies

$$|x_m + x_{m+1} + \cdots + x_n| \le |x_m| + |x_{m+1}| + \cdots + |x_n| \le y_m + y_{m+1} + \cdots + y_n.$$

We also note that $y_i \ge 0$. Therefore the Cauchy criterion for the convergence of $\sum y_n$ implies the Cauchy criterion for the convergence of $\sum x_n$.

For the special case $y_n = |x_n|$, the comparison test says that the convergence of $\sum |x_n|$ implies the convergence of $\sum x_n$. Therefore there are three possibilities to any series

1. *Absolute Convergence*: $\sum |x_n|$ converges, which implies $\sum x_n$ converges.
2. *Conditional Convergence*: $\sum x_n$ converges, and $\sum |x_n|$ diverges.
3. *Divergence*: $\sum x_n$ diverges, which implies $\sum |x_n|$ diverges.

The series $\sum \frac{(-1)^n}{n^p}$ absolutely converges for $p > 1$, and conditionally converges for $0 < p \leq 1$.

Example 1.5.7. For $\sum \frac{n+\sin n}{n^3+n+2}$, we observe

$$\lim_{n\to\infty} \frac{\frac{n+\sin n}{n^3+n+2}}{\frac{1}{n^2}} = 1 < 2.$$

By the order rule, we get

$$0 < \frac{n+\sin n}{n^3+n+2} < \frac{2}{n^2}$$

for sufficiently large n. Then by the convergence of $\sum \frac{2}{n^2} = 2\sum \frac{1}{n^2}$ and the comparison test, we know $\sum \frac{n+\sin n}{n^3+n+2}$ converges.

Similarly, by the comparison

$$\lim_{n\to\infty} \frac{\frac{2^n+n^2}{\sqrt{5^{n-1}-n^4 3^n}}}{\left(\frac{2}{\sqrt{5}}\right)^n} = \sqrt{5},$$

and the convergence of $\sum \left(\frac{2}{\sqrt{5}}\right)^n$, the series $\sum \frac{2^n+n^2}{\sqrt{5^{n-1}-n^4 3^n}}$ converges.

Exercise 1.5.11. Prove that if $a_n > 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges. Moreover, prove that the converse is not true.

Exercise 1.5.12. Prove that if $\sum a_n^2$, then $\sum \frac{a_n}{n}$ converges.

Exercise 1.5.13. Prove that if $\sum a_n^2$ and $\sum b_n^2$ converge, then $\sum a_n b_n$ and $\sum (a_n + b_n)^2$ converge.

Exercise 1.5.14. Determine the convergence, $b, d, p, q > 0$.

1. $\sum \frac{\sqrt{4n^5+5n^4}}{3n^2-2n^3}$.
2. $\sum \frac{3n^2-2n^3}{\sqrt{4n^5+5n^4}}$.
3. $\sum \frac{3n^2+(-1)^n 2n^3}{4n^5+5n^4}$.
4. $\sum \frac{1}{(a+nb)^p}$.
5. $\sum \frac{(c+nd)^q}{(a+nb)^p}$.
6. $\sum \frac{1}{(a+nb)^p(c+nd)^q}$.

Example 1.5.8. For $p > 1$, by $\frac{|\sin na|}{n^p} \leq \frac{1}{n^p}$, the convergence of $\sum \frac{1}{n^p}$, and the comparison test, we know $\sum \frac{\sin na}{n^p}$ absolutely converges. The series also converges if a is a multiple of π, because all the terms are 0.

It remains to consider the case $0 < p \le 1$ and a is not a multiple of π. We apply the idea of Example 1.1.22. Up to adding integer multiples of π, we may assume $0 < a < \pi$. Then for any natural number k, there is a natural number n_k satisfying

$$n_k a \in [a_k, b_k] = \left[k\pi + \frac{\pi - a}{2}, (k+1)\pi - \frac{\pi - a}{2}\right].$$

Then $|\sin n_k a| \ge \cos\frac{a}{2}$ and $n_k^p \le n_k < (k+1)\pi$. For $n \ge n_K$, we have $n_k \le n$ for all $1 \le k \le K$, and

$$\sum_{i=1}^{n} \frac{|\sin na|}{n^p} \ge \sum_{k=1}^{K} \frac{|\sin n_k a|}{n_k} \ge \left(\cos\frac{a}{2}\right)\sum_{k=1}^{K}\frac{1}{n_k^p} \ge \left(\frac{1}{\pi}\cos\frac{a}{2}\right)\sum_{k=1}^{K}\frac{1}{k+1}.$$

By the divergence of the harmonic series, and $\cos\frac{a}{2} > 0$, the right side can be arbitrarily large. Therefore $\sum \frac{|\sin na|}{n^p}$ diverges.

We will show that $\sum \frac{\sin na}{n^p}$ converges (using the method that is essentially the same as the integration by parts). Then $\sum \frac{\sin na}{n^p}$ conditionally converges for $0 < p \le 1$ and a not an integer multiple of π.

Exercise 1.5.15. Discuss the absolute convergence of $\sum \frac{\cos na}{n^p}$.

We already know that $\sum r^n$ converges when $0 < r < 1$. Applying the comparison test to $y_n = r^n$, the comparison $|x_n| \le r^n$ is the same as $\sqrt[n]{|x_n|} \le r$.

Theorem 1.5.6 (Root Test). *If $\sqrt[n]{|x_n|} \le r$ for some $r < 1$ and sufficiently large n, then $\sum x_n$ absolutely converges.*

Example 1.5.9. To determine the convergence of $\sum(n^5 + 2n + 3)x^n$, we note that $\lim_{n\to\infty} \sqrt[n]{|(n^5 + 2n + 3)x^n|} = |x|$. If $|x| < 1$, then we can pick r satisfying $|x| < r < 1$. By $\lim_{n\to\infty} \sqrt[n]{|(n^5 + 2n + 3)x^n|} < r$ and the order rule, we get $\sqrt[n]{|(n^5 + 2n + 3)x^n|} < r$ for sufficiently large n. Then by the root test, we conclude that $\sum(n^5 + 2n + 3)x^n$ converges for $|x| < 1$.

If $|x| \ge 1$, then the term $(n^5 + 2n + 3)x^n$ of the series does not converge to 0. By Theorem 1.5.1, the series diverges for $|x| \ge 1$.

The example suggests that, in practice, it is often more convenient to use the limit version of the root test.

If $\lim_{n\to\infty} \sqrt[n]{|x_n|} < 1$, then we fix r satisfying $\lim_{n\to\infty} \sqrt[n]{|x_n|} < r < 1$. By the order rule, we have $\sqrt[n]{|x_n|} < r$ for sufficiently large n. Then by the root test, we know $\sum x_n$ converges.

If $\lim_{n\to\infty} \sqrt[n]{|x_n|} > 1$, then we have $\sqrt[n]{|x_n|} > 1$ for sufficiently large n. This implies $|x_n| > 1$. By Theorem 1.5.1, we know $\sum x_n$ diverges.

Example 1.5.10. Consider the series $\sum n^p r^n$. For the convergence, it is necessary to have $\lim_{n\to\infty} n^p r^n = 0$. This means $|r| < 1$ or $|r| = 1$ and $p < 0$.

We have $\lim_{n\to\infty} \sqrt[n]{n^p r^n} = \lim_{n\to\infty} r\sqrt[n]{n^p} = r$. Here we get $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$ by Example 1.1.10. Therefore $\sum n^p r^n$ absolutely converges for $|r| < 1$.

It remains to consider $r = \pm 1$. The series is $\sum \frac{1}{n^{-p}}$ or $\sum \frac{(-1)^n}{n^{-p}}$. By Examples 1.5.5 and 1.5.6, we know $\sum \frac{1}{n^{-p}}$ converges if and only if $p < -1$, and $\sum \frac{(-1)^n}{n^{-p}}$ converges if and only if $p < 0$.

Exercise 1.5.16. The decimal representations of positive real numbers are actually the sum of series. For example,

$$\pi = 3.1415926\cdots = 3 + 0.1 + 0.04 + 0.001 + 0.0005 + 0.00009 + 0.000002 + 0.0000006 + \cdots.$$

Explain why the expression always converges.

Exercise 1.5.17. Determine the convergence.

1. $\sum \frac{1}{5^n - 1}$.
2. $\sum \frac{3^{n+1}}{5^{n-1} - n^2 2^n}$.
3. $\sum \frac{5^{n-1} - n^2 2^n}{3^{n+1}}$.

Exercise 1.5.18. Determine the convergence.

1. $\sum x^{n^2}$.
2. $\sum n x^{n^2}$.
3. $\sum n^2 x^{n^2}$.

Exercise 1.5.19. Determine the convergence, $a, b > 0$.

1. $\sum (a^n + b^n)^p$.
2. $\sum n^p x^n$.
3. $\sum \frac{n^2}{na^n + b^n}$.
4. $\sum \frac{1}{\sqrt[n]{a^n + b^n}}$.
5. $\sum \frac{n^p}{\left(a + \frac{b}{n}\right)^n}$.
6. $\sum n^3 \left(\frac{a + (-1)^n}{b + (-1)^n}\right)^n$.

Next we turn to another way of comparing series. If we know two sequences have the same initial term, and the change of one sequence is always smaller than the change of another sequence, then the first sequence is smaller than the second sequence.

Specifically, we measure the change by the quotient. Suppose $x_n, y_n > 0$, and $\frac{x_{n+1}}{x_n} \le \frac{y_{n+1}}{y_n}$ for $n \ge N$. Then for $C = \frac{x_N}{y_N}$, we have

$$x_n = x_N \frac{x_{N+1}}{x_N} \frac{x_{N+2}}{x_{N+1}} \cdots \frac{x_n}{x_{n-1}} \le C y_N \frac{y_{N+1}}{y_N} \frac{y_{N+2}}{y_{N+1}} \cdots \frac{y_n}{y_{n-1}} = C y_n.$$

By the comparison test, if $\sum y_n$ converges, then $\sum x_n$ converges.

Theorem 1.5.7 (Ratio Test). *Suppose* $\left|\frac{x_{n+1}}{x_n}\right| \le \frac{y_{n+1}}{y_n}$ *for sufficiently large* n. *If* $\sum y_n$ *converges, then* $\sum x_n$ *converges.*

We note that the assumption implies that the terms y_n have the same sign for sufficiently large n. By changing all y_n to $-y_n$ if necessary, we may assume that $y_n > 0$ for sufficiently large n.

Applying the ratio test to $y_n = r^n$, we get the first of the following.

1. If $|\frac{x_{n+1}}{x_n}| \le r$ for some $r < 1$ and sufficiently large n, then $\sum x_n$ absolutely converges.

2. If $|\frac{x_{n+1}}{x_n}| \ge 1$ for sufficiently large n, then $\sum x_n$ diverges.

In the second statement, we have $|x_{n+1}| \ge |x_n|$. This implies x_n does not converge to 0. Therefore the series diverges.

Similar to the root test, we also have the limit version of the ratio test.

1. If $\lim_{n\to\infty} |\frac{x_{n+1}}{x_n}| < 1$, then $\sum x_n$ absolutely converges.

2. If $\lim_{n\to\infty} |\frac{x_{n+1}}{x_n}| > 1$, then $\sum x_n$ diverges.

Example 1.5.11. The series $\sum \frac{(n!)^2}{(2n)!}x^n$ satisfies

$$\lim_{n\to\infty}\left|\frac{x_n}{x_{n-1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n!)^2}{(2n)!}x^n}{\frac{((n-1)!)^2}{(2n-2)!}x^{n-1}}\right| = \lim_{n\to\infty}\frac{n^2}{2n(2n-1)}|x| = \frac{|x|}{4}.$$

By the limit version of the ratio test, the series absolutely converges for $|x| < 4$, and diverges for $|x| > 4$.

For $|x| = 4$, we actually have

$$\left|\frac{x_n}{x_{n-1}}\right| = \frac{n^2}{2n(2n-1)}4 > 1.$$

Again by the ratio test, the series diverges.

The comparison idea is already used in the similar Examples 1.1.18 and 1.3.4 (also see Exercises 1.1.57 and 1.1.58).

Exercise 1.5.20. Determine convergence.

1. $\frac{4}{2} + \frac{4\cdot 7}{2\cdot 6} + \frac{4\cdot 7\cdot 10}{2\cdot 6\cdot 10} + \cdots$.

2. $\frac{2}{4} + \frac{2\cdot 6}{4\cdot 7} + \frac{2\cdot 6\cdot 10}{4\cdot 7\cdot 10} + \cdots$.

3. $\frac{2}{4} + \frac{2\cdot 5}{4\cdot 7} + \frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10} + \cdots$.

4. $\frac{2}{4\cdot 7} + \frac{2\cdot 5}{4\cdot 7\cdot 10} + \frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10\cdot 13} + \cdots$.

Exercise 1.5.21. Determine convergence.

1. $\sum \frac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)}$.

2. $\sum \frac{a(a+1^p)\cdots(a+n^p)}{b(b+1^p)\cdots(b+n^p)}$.

3. $\sum \frac{a^p(a+c)^p\cdots(a+nc)^p}{b^q(b+d)^q\cdots(b+nd)^q}\frac{1}{n^r}$.

4. $\sum \frac{(a+c)(a+2c)^2\cdots(a+nc)^n}{(b+d)(b+2d)^2\cdots(b+nd)^n}$.

5. $\sum \frac{(a_1+b_1 1+c_1 1^2)\cdots(a_1+b_1 n+c_1 n^2)}{(a_2+b_2 1+c_2 1^2)\cdots(a_2+b_2 n+c_2 n^2)}$.
6. $\sum \frac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)}\frac{c(c+1)\cdots(c+n)}{d(d+1)\cdots(d+n)}$.

Exercise 1.5.22. Determine convergence. There might be come special values of x for which you cannot yet make conclusion.

1. $\sum \frac{(n!)^2}{(2n)!}x^n$.
2. $\sum \frac{(3n)!}{(n!)^3}x^n$.
3. $\sum \frac{n!(2n)!}{(3n)!}x^n$.
4. $\sum \frac{n^n}{n!}x^n$.
5. $\sum \frac{n!}{n^n}x^n$.
6. $\sum \frac{n!}{(n+1)^n}x^n$.
7. $\sum \frac{n^{n+1}}{(n+1)!}x^n$.
8. $\sum \frac{(2n)!}{n^{2n}}x^n$.

1.6 Multivariable Limit

A multivariable is a vector in the Euclidean space. We restrict our discussion to 2-variables, which are vectors $\vec{x} = (x, y)$ in $\mathbb{R}^2$.

A sequence in $\mathbb{R}^2$ is $\vec{x}_n = (x_n, y_n)$, which can be regarded as two sequences x_n and y_n. If the sequence converges, then the limit should also be a vector $\vec{l} = (k, l)$. We may copy the definition of the limit of number sequence, and define that $\vec{x}_n$ converges to $\vec{l}$ if, for any $\epsilon > 0$, there is N, such that

$$n > N \implies |\vec{x}_n - \vec{l}| < \epsilon.$$

We should understand the distance $|\vec{x} - \vec{l}|$ as the length of the vector $\vec{x} - \vec{l}$, that generalizes the absolute value of number $x - l$.

So the first thing we need to clarify in defining the vector sequence limit is the meaning of the length of vectors. The most commonly used length is the *Euclidean length*

$$\|(x, y)\|_2 = \sqrt{x^2 + y^2}.$$

However, this are other useful lengths such as

$$\|(x, y)\|_1 = |x| + |y|, \quad \|(x, y)\|_\infty = \max\{|x|, |y|\}.$$

In everyday life, we measure the distance in various ways, depending on our purpose. The usual distance between two places is the shortest route between the two, not the straight line distance (unless you fly) given by $\|\cdot\|_2$ above. In a grid-like city, such distance is exactly given by $\|\cdot\|_1$, sometimes called the Taxicab length. If you want to move between the two places in fastest way, then you should use the time to drive as the distance. If you want to move in cheapest way, then you should use lowest public transportation fare as the distance.

Definition 1.6.1. A *norm* (i.e., *length*) on a vector space is a function $\|\vec{x}\|$ satisfying

1. *Positivity*: $\|\vec{x}\| \ge 0$, and $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0} = (0, 0, \dots, 0)$.
2. *Scalar Property*: $\|c\vec{x}\| = |c|\|\vec{x}\|$.

3. *Triangle Inequality*: $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$.

If $p \ge 1$, then
$$\|(x,y)\|_p = (|x|^p + |y|^p)^{\frac{1}{p}}$$
is a norm[4], called the L^p-*norm*. The Euclidean norm is the L^2-norm, and the Taxicab norm is the L^1-norm.

Exercise 1.6.1. Explain $\lim_{n\to\infty} \|(x,y)\|_n = \|(x,y)\|_\infty$.

Exercise 1.6.2. Show that a norm on $\mathbb{R}^1$ is $\|x\| = a|x|$ for a constant $a > 0$. This is why we did not discuss norm in the previous sections.

Exercise 1.6.3. Verify that $\|(x,y)\|_1, \|(x,y)\|_2, \|(x,y)\|_\infty$ are norms.

Exercise 1.6.4. Suppose $\|\vec{x}\|$ and $\|\vec{x}\|'$ are norms. Verify that $\|\vec{x}\| + \|\vec{x}\|'$ is also a norm. Therefore L^p-norms are not the only norms.

Definition 1.6.2. A sequence $\vec{x}_n$ converges to $\vec{l}$ with respect to a norm $\|?\|$, and denoted $\lim_{n\to\infty} \vec{x}_n = \vec{l}$, if for any $\epsilon > 0$, there is N, such that $n > N$ implies $\|\vec{x}_n - \vec{l}\| < \epsilon$.

Example 1.6.1. The limit $\lim_{n\to\infty}(x_n, y_n) = (k,l)$ with respect to the Euclidean norm means that, for any $\epsilon > 0$, there is N, such that
$$n > N \implies \|(x_n,y_n) - (k,l)\|_2 = \sqrt{(x_n-k)^2 + (y_n-l)^2} < \epsilon.$$
By
$$|x_n - k| < \sqrt{(x_n-k)^2 + (y_n-l)^2}, \quad |y_n - l| < \sqrt{(x_n-k)^2 + (y_n-l)^2},$$
we get
$$n > N \implies |x_n - k| < \epsilon \text{ and } |y_n - l| < \epsilon.$$
Therefore we proved
$$\lim_{n\to\infty}(x_n,y_n) = (k,l) \implies \lim_{n\to\infty} x_n = k \text{ and } \lim_{n\to\infty} y_n = l.$$

Conversely, suppose $\lim_{n\to\infty} x_n = k$ and $\lim_{n\to\infty} y_n = l$. Then for any $\epsilon > 0$, there is N, such that
$$\begin{aligned} n > N &\implies |x_n - k| < \epsilon \text{ and } |y_n - l| < \epsilon \\ &\implies \|(x_n,y_n) - (k,l)\|_2 \le \sqrt{\epsilon^2 + \epsilon^2} = \sqrt{2}\epsilon. \end{aligned}$$
By Example 1.2.19 and the subsequent Exercise 1.2.22, this proves $\lim_{n\to\infty}(x_n,y_n) = (k,l)$.

[4]A proof uses the convexity of the function x^p.

Exercise 1.6.5. Prove that $\lim_{n\to\infty}(x_n, y_n) = (k, l)$ with respect to the Taxicab norm $\|?\|_1$ if and only if $\lim_{n\to\infty} x_n = k$ and $\lim_{n\to\infty} y_n = l$.

Exercise 1.6.6. Prove that $\lim_{n\to\infty}(x_n, y_n) = (k, l)$ with respect to the norm $\|?\|_\infty$ if and only if $\lim_{n\to\infty} x_n = k$ and $\lim_{n\to\infty} y_n = l$.

It is annoying that the limit is defined with respect to the choice of norm. However, Example 1.6.1 and Exercises 1.6.5 and 1.6.6 suggest that the different choice does not make any difference. In fact, we have

$$\|(x,y)\|_\infty = \max\{|x|,|y|\} \le \|(x,y)\|_2 = \sqrt{x^2+y^2} \le \sqrt{2}\max\{|x|,|y|\} = \|(x,y)\|_\infty.$$

Then

$$\|\vec{x}_n - \vec{l}\|_2 < \epsilon \implies \|\vec{x}_n - \vec{l}\|_\infty < \epsilon,$$

and

$$\|\vec{x}_n - \vec{l}\|_\infty < \epsilon \implies \|\vec{x}_n - \vec{l}\|_2 < \sqrt{2}\epsilon.$$

By Example 1.2.19 and the subsequent Exercise 1.2.22, we conclude that $\vec{x}_n$ converges to $\vec{l}$ in L^2-norm if and only if $\vec{x}_n$ converges to $\vec{l}$ in L^∞-norm.

Exercise 1.6.7. Find the relation between the L^p-norm and the L^∞-norm.

Exercise 1.6.8. Extend the relation between the L^2-norm and L^∞-norm on $\mathbb{R}^n$.

Two norms $\|\vec{x}\|$ and $\|\vec{x}\|'$ are *equivalent* if there are $c, c' > 0$, such that

$$c\|\vec{x}\|' \le \|\vec{x}\| \le c'\|\vec{x}\|'.$$

It is easy to see that the convergence with respect to $\|\vec{x}\|$ is the same as the convergence with respect to $\|\vec{x}\|'$.

We have seen that all L^p-norms are equivalent. In general, we have the following deep theorem.

Theorem 1.6.3. *In a finite dimensional vector space, all norms are equivalent.*

As a consequence of the theorem and Example 1.6.1, we have the following (which also holds in $\mathbb{R}^n$).

Theorem 1.6.4. *A sequence $\vec{x}_n = (x_n, y_n)$ converges to (k, l) if and only if x_n converges to k and y_n converges to l.*

The arithmetic rule can be interpreted as that the functions $f(x, y) = x + y, x - y, xy, \frac{x}{y}$ satisfy

$$\lim_{n\to\infty} f(x_n, y_n) = f\left(\lim_{n\to\infty}(x_n, y_n)\right).$$

The following example says the same is true for the exponential function $f(x,y) = x^y$.

Example 1.6.2. We need to show the following

$$\lim_{n\to\infty} x_n = k > 0 \text{ and } \lim_{n\to\infty} y_n = l \implies \lim_{n\to\infty} x_n^{y_n} = k^l.$$

By the examples and exercises leading to Exercise 1.1.64, we already know

$$\lim_{n\to\infty} x_n = k > 0 \implies \lim_{n\to\infty} x_n^l = k^l.$$

By Example 1.2.13 and Exercises 1.2.15 and 1.2.16, we also know

$$\lim_{n\to\infty} y_n = l \implies \lim_{n\to\infty} k^{y_n} = k^l.$$

We need to combine the ideas for proving the two limits.

We separate the two sequences by

$$x_n^{y_n} - k^l = (x_n^{y_n} - k^{y_n}) + (k^{y_n} - k^l) = k^{y_n}\left(\left(\frac{x_n}{k}\right)^{y_n} - 1\right) + (k^{y_n} - k^l).$$

We already have $\lim_{n\to\infty}(k^{y_n} - k^l) = 0$.

By $\lim_{n\to\infty} x_n = k > 0$, we have $\lim_{n\to\infty} \frac{x_n}{k} = 1$. On the other hand, by Theorem 1.3.1, the convergence of y_n implies y_n is bounded, and then k^{y_n} is also bounded. Then by Exercise 1.1.63 (also see Exercise 1.1.27), we get $\lim_{n\to\infty}\left(\frac{x_n}{k}\right)^{y_n} = 1$. This means $\lim_{n\to\infty}\left(\left(\frac{x_n}{k}\right)^{y_n} - 1\right) = 0$. Then by the boundedness of k^{y_n} and Exercise 1.3.4, we get $\lim_{n\to\infty} k^{y_n}\left(\left(\frac{x_n}{k}\right)^{y_n} - 1\right) = 0$.

Therefore both parts on the right converge to 0, and we get $\lim_{n\to\infty} x_n^{y_n} = k^l$.

Chapter 2

Constant Approximation

2.1 Continuity and Limit

We often use approximations in everyday life. For example, when we say a road is 25km long, the actual length can be 25.1km, or 24.83km. If we drive at 60km per hour, our actual speed can be sometimes 61km per hour, or sometimes 59.5km per hour. Moreover, the approximate length of 25km long and the approximate speed of 60km per hour is good enough for us to know that it takes about 25 minutes to complete the trip.

We can make two observations from the example. First, an approximation means the *error*, such as $|25.1\text{km}-25\text{km}| = 0.1\text{km}$, or $|61\text{km/hr}-60\text{km/hr}|$, is much smaller than the *basic unit*, such as 1km or 10km/hr. The choice of the basic unit and the amount of error we can tolerate depend on the problem. If the basic units are not refined enough, or the errors are too big, then we need to choose more refined approximations to solve our problem. On the other hand, more refined approximations mean more complicated data, and therefore more complicated calculation. Therefore we need to optimise, by finding the simplest approximations that are good enough for solving the problems.

The second observation is that, instead of the division by real data such as 25.1/61, we actually use the approximate data to calculate 25/60. We generally believe that combinations of approximate data are still approximations. The combinations can be arithmetic operations or even more sophisticated operations.

Our discussion about the sequence limit is based on our intuition of numerical approximations. Now we apply the approximation idea to functions, which are relations between quantities. Suppose we try to solve a problem, say whether a relation $y = f(x)$ between quantities x and y is increasing. We may approximate the relation by a simpler function $y = p(x)$, say a linear function $p(x) = a + bx$. In fact, we know $a + bx$ is increasing if and only if $b > 0$. In this case, we expect the original relation $y = f(x)$ to be also increasing. Then the increasing property becomes the calculation of the coefficient b (the so called derivative) in the linear approximation.

Our confidence that the solution to the approximation $y = p(x)$ is also the solution to the original $y = f(x)$ depends on how good the approximation is. Mathematically, we need to establish a theorem that gives precise condition for the increasing property of $p(x)$ implying the increasing property of $f(x)$.

To carry out the idea, we need to do two things. First, we need to choose the right family of approximation functions. This is similar to choosing km, m, or cm, etc., as the basic unit of approximating length. For approximating a single variable function $f(x)$ at a specific location $x = x_0$, the common choices are polynomials of various degrees:

- Degree 0 (constant): $1, -2, 3.5, \sqrt{5}, \frac{1}{\sqrt{2}} + 3$.
- Degree 1 (linear): $1 - 2x, 3.5 + \sqrt{5}x$.
- Degree 2 (quadratic): $1 - 2x + 3.5x^2, \sqrt{5} + (\frac{1}{\sqrt{2}} + 3)x^2$.
- Degree 3 (cubic): $1 - 2x^2 + 3.5x^3, \sqrt{5}x + (\frac{1}{\sqrt{2}} + 3)x^3$.
- $\cdots$.

The higher degree polynomials mean more refined approximations. We need to determine the optimal degree suitable for the problem (about f near x_0), and then calculate the approximation polynomial of the degree, and then use the approximation to solve the problem.

The approximation idea can be easily extended to multivariables. For example, we may approximate a two variable function $f(x, y)$ at (x_0, y_0) by two variable polynomials of various degrees:

- Degree 0 (constant): $1, -2, 3.5, \sqrt{5}, \frac{1}{\sqrt{2}} + 3$.
- Degree 1 (linear): $1 - 2x + 3.5y$, $\sqrt{5} + \frac{1}{\sqrt{2}}x + 3y$.
- Degree 2 (quadratic): $1 - 2x + 3.5y + \sqrt{5}x^2 + \frac{1}{\sqrt{2}}y^2 + 3xy$.
- $\cdots$.

For some problems, polynomials are not good families of functions for approximation. We need to use more suitable functions, such as the Fourier series, to solve the problem.

Differentiation is the theory of calculating the polynomial approximations, and using the polynomial approximations to solve problems. In this chapter, we discuss the simplest approximation, which is the constant approximation of a function.

2.1.1 Continuous Function

In the approximations 25.1km $\approx_{1\text{km}}$ 25km and 61km/hr $\approx_{10\text{km/hr}}$ 60km/hr, the error is much smaller than the basic units

$$|25.1\text{km} - 25\text{km}| = 0.1\text{km} \le \epsilon \cdot 1\text{km},$$
$$|61\text{km/hr} - 60\text{km/hr}| = 1\text{km/hr} \le \epsilon \cdot 10\text{km/hr}.$$

In general, an approximation $L \approx_u A$ means the error $L - A$ is much smaller than the basic unit u

$$|L - A| \le \epsilon u.$$

Here ϵ is a generic symbol for numbers much smaller than 1.

A function $f(x)$ is approximated by a constant function l at x_0, if

$$x \text{ close to } x_0 \implies f(x) \text{ close to } l.$$

The definition is similar to the limit of sequence. On the left, the closeness of x to x_0 (corresponding to the largeness of n, measured by $n > N$) is measured by

$$|x - x_0| \le \delta$$

for very small $\delta > 0$. The constant approximation is on the right (corresponding to the closeness of x_n to l, measured by $|x_n - l| < \epsilon$), where the closeness of $f(x)$ to l is measured by the *error* $|f(x) - l|$ being much smaller than the basic unit 1 for constant functions

$$|f(x) - l| \le \epsilon 1 = \epsilon.$$

Here $\epsilon > 0$ represents very small positive numbers.

Definition 2.1.1. A function $f(x)$ has *constant approximation* l at x_0, and denoted $f(x) \approx_1 l$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| \le \delta \implies |f(x) - l| \le \epsilon.$$

In this case, we say $f(x)$ is *continuous* at x_0.

Figure 2.1.1 shows the constant approximation. We see that, as x gets closer to x_0 (i.e., within smaller distance δ from x_0), the approximation gets better (i.e., within smaller distance ϵ from l).

Example 2.1.1. If the side of a square is approximately 2m, then we expect the area of the square to be approximately $2^2 = 4\text{m}^2$.

The intuition means that, if x is close to 2, then $f(x) = x^2$ is close to $f(2) = 2^2 = 4$. In other words, we have $x^2 \approx_1 4$ at $x_0 = 2$. Specifically, for any $\epsilon > 0$, we need to find $\delta > 0$, such that the following implication holds

$$|x - 2| \le \delta \implies |x^2 - 4| \le \epsilon.$$

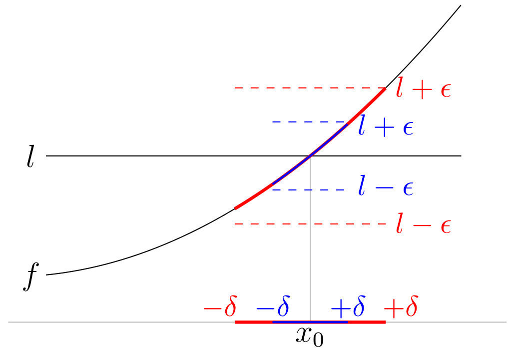


Figure 2.1.1: Constant approximation $f(x) \approx_1 l$ at x_0

We have

$$|x-2| \le \delta \implies |x^2-4| = |x+2||x-2| \le |x+2|\delta.$$

For x close to 2, we also have $|x+2| \le 5$. More precisely, we have

$$|x-2| \le 1 \implies |x+2| = |(x-2)+4| \le |x-2|+4 \le 5.$$

Therefore

$$|x-2| \le \delta \le 1 \implies |x^2-4| \le |x+2|\delta \le 5\delta.$$

If we further have $5\delta \le \epsilon$, then we establish the desired implication. This suggests us to choose δ to satisfy $\delta \le 1$ and $\delta \le \frac{1}{5}\epsilon$.

You may write the analysis above on scratch paper. Your formal writing should be the following: For any $\epsilon > 0$, take $\delta = \min\{1, \frac{1}{5}\epsilon\}$. Then for $|x-2| \le \delta$, we have

$$\begin{aligned} |x+2| &\le |x-2|+4 \le \delta+4 \le 5, \\ |x^2-4| &= |x+2||x-2| \le 5\delta \le \epsilon. \end{aligned}$$

Therefore x^2 is continuous at 2.

Example 2.1.2. It is almost a tautology that we should have $c \approx_1 c$. The rigorous proof of $\lim_{n\to\infty} c = c$ in Example 1.2.2 can be adopted. For any $\epsilon > 0$, take any δ (say $\delta = 1$). Then

$$|x-x_0| \le \delta \implies |c-c| = 0 \le \epsilon.$$

Again the right side is always true, regardless of the left side. Constant functions are continuous everywhere.

Example 2.1.3. We argue that x is continuous at any x_0. This reflects the tautology

$$x \text{ close to } x_0 \implies x \text{ close to } x_0.$$

The proof is rather trivial. For any $\epsilon > 0$, take $\delta = \epsilon$. Then

$$|x-x_0| \le \delta \implies |x-x_0| \le \delta = \epsilon.$$

Example 2.1.4. We argue that $|x|$ is continuous at any x_0. For any $\epsilon > 0$, take $\delta = \epsilon$. Then

$$|x - x_0| \le \delta \implies ||x| - |x_0|| \le |x - x_0| \le \delta = \epsilon.$$

Example 2.1.5. Polynomial functions are continuous. For example, we try to show $x^3 + 2x + 3 \approx_1 x_0^3 + 2x_0 + 3$ at x_0. We note that $|x - x_0| \le \delta \le 1$ implies $|x| \le |x_0| + \delta \le |x_0| + 1$, and

$$\begin{aligned} |(x^3 + 2x + 3) - (x_0^3 + 2x_0 + 3)| &= |(x^3 - x_0^3) + 2(x - x_0)| \\ &= |x^2 + xx_0 + x_0^2 + 2||x - x_0| \\ &\le [(|x_0| + 1)^2 + (|x_0| + 1)|x_0| + |x_0|^2 + 2]\delta. \end{aligned}$$

Therefore

$$|x - x_0| \le \delta = \min\left\{1, \frac{\epsilon}{(|x_0| + 1)^2 + (|x_0| + 1)|x_0| + |x_0|^2 + 2}\right\}$$

implies

$$|(x^3 + 2x + 3) - (x_0^3 + 2x_0 + 3)| \le \epsilon.$$

Exercise 2.1.1. Prove the continuity of $-|x|$ and $|x + a|$ at any x_0.

Exercise 2.1.2. Prove the continuity of $ax^2 + bx + c$ at any x_0.

Example 2.1.6. If the area of a square is approximately 4m^2, then the side of the square is approximately 2m.

We have (the estimation is similar to Example 1.2.11)

$$|x - 4| \le \delta \implies |\sqrt{x} - \sqrt{4}| = \frac{|x - 4|}{\sqrt{x} + \sqrt{4}} \le \frac{\delta}{2}.$$

Then we get

$$|x - 4| \le \delta = 2\epsilon \implies |\sqrt{x} - \sqrt{4}| \le \epsilon.$$

This proves $\sqrt{x}$ is continuous at 4.

Exercise 2.1.3. Prove the continuity of $\sqrt{x}$ at any $x_0 > 0$.

Example 2.1.7. If 1 dollar is approximately 2 yuan, then 1 yuan is approximately $\frac{1}{2}$ dollar.

For $|x - 2| \le 1$, we know $x \ge 1$, and

$$|x - 2| \le \delta \le 1 \implies \left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x - 2|}{2x} \le \frac{\delta}{2}.$$

Then we get

$$|x-2| \le \min\{1, 2\epsilon\} \implies \left|\frac{1}{x} - \frac{1}{2}\right| < \epsilon.$$

This proves $\frac{1}{x}$ is continuous at 2.

The argument is similar to Example 1.2.12.

Exercise 2.1.4. Prove the continuity.

1. $\frac{1}{x}$ at any $x_0 \neq 0$.
2. $\frac{x}{2x+1}$ at 0.
3. $\frac{1}{x^2}$ at 1.

Example 2.1.8. The sine and tangent functions are defined in Figure 2.1.2, at least for $0 < x < \frac{\pi}{2}$. We have

$$\text{Area(triangle } OBP) < \text{Area(fan } OBP) < \text{Area(triangle } OBQ).$$

This means

$$\tfrac{1}{2}\sin x < \tfrac{1}{2}x < \tfrac{1}{2}\tan x \text{ for } 0 < x < \tfrac{\pi}{2}.$$

Then we get

$$0 < \sin x < x, \text{ and } \cos x < \tfrac{\sin x}{x} < 1, \text{ for } 0 < x < \tfrac{\pi}{2}. \tag{2.1.1}$$

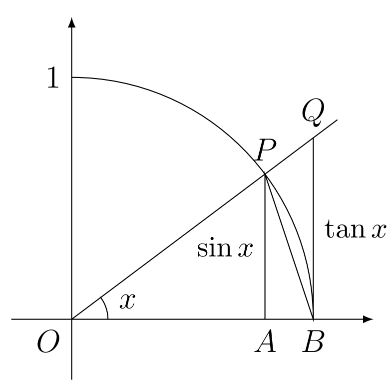


Figure 2.1.2: Trigonometric function

By (2.1.1) and $\sin(-x) = -\sin x$, we get

$$|\sin x - 0| = |\sin x| \le |x| \text{ for } |x| \le \tfrac{\pi}{2}.$$

This implies

$$|\cos x - 1| = 2\sin^2 \tfrac{1}{2}x \le \tfrac{1}{2}x^2 \text{ for } |x| \le \pi.$$

Then for any $\epsilon > 0$, we have

$$|x| \le \min\{\tfrac{\pi}{2}, \epsilon\} \implies |\sin x| \le |x| \le \epsilon,$$

and

$$|x| \le \min\{\pi, \sqrt{2\epsilon}\} \implies |\cos x - 1| = 2\sin^2 \tfrac{1}{2}x \le \tfrac{1}{2}x^2 \le \epsilon.$$

Therefore $\sin x$ and $\cos x$ are continuous at 0.

Note that we take the minimum with $\frac{1}{2}\pi$ or π in order for the inequalities (2.1.1) to hold. Alternatively, for any $1 > \epsilon > 0$, we have (note that $1 < \frac{\pi}{2}$)

$$|x| \le \epsilon \implies |\sin x| \le |x| \le \epsilon \text{ and } |\cos x - 1| = 2\sin^2 \tfrac{1}{2}x \le \tfrac{1}{2}x^2 \le \tfrac{1}{2}\epsilon^2 \le \epsilon.$$

By Example 1.2.19 and Exercise 1.2.22(1), this also shows the continuity of $\sin x$ and $\cos x$ at 0.

Exercise 2.1.5. Prove the continuity at 0.

1. $\sin 2x$.
2. $\cos 3x$.
3. $\sin^2 x$.
4. $\cos^2 x$.

Exercise 2.1.6. Prove the following are equivalent to the definition of the continuity of $f(x)$ at x_0.

1. For any $\epsilon > 0$, there is δ, such that $|x - x_0| < \delta$ implies $|f(x) - l| < \epsilon$.
2. For any $1 > \epsilon > 0$, there is δ, such that $|x - x_0| \le \delta$ implies $|f(x) - l| \le \epsilon$.
3. For any $\epsilon > 0$, there is a natural number n, such that $|x - x_0| \le \frac{1}{n}$ implies $|f(x) - l| < \epsilon$.
4. For any natural number k, there is $\delta > 0$, such that $|x - x_0| \le \delta$ implies $|f(x) - l| \le \frac{1}{k}$.
5. For any $\epsilon > 0$, there is $1 > \delta > 0$, such that $|x - x_0| \le \delta$ implies $|f(x) - l| \le \epsilon$.
6. For any $\epsilon > 0$, there is $\epsilon > \delta > 0$, such that $|x - x_0| < \delta$ implies $|f(x) - l| \le \epsilon^2$.
7. For any $\frac{1}{2} > \epsilon > 0$, there is $\delta > 0$, such that $|x - x_0| \le \delta$ implies $|f(x) - l| \le \frac{1}{2}\epsilon^2$.

2.1.2 Limit of Function

The constant approximation $f(x) \approx_1 l$ at x_0 means that, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| \le \delta \implies |f(x) - l| \le \epsilon.$$

We divide the implication into two parts.

1. $x = x_0$: Since $|x_0 - x_0| \le \delta$ for all $\delta > 0$, we get $|f(x_0) - l| \le \epsilon$ for all $\epsilon > 0$. This means $l = f(x_0)$.
2. $x \ne x_0$: The implication becomes the constant approximation *excluding* x_0

   $$|x - x_0| \le \delta \text{ and } x \ne x_0 \implies |f(x) - l| \le \epsilon.$$

   In other words, we have

   $$x \text{ close to } x_0 \text{ and } x \ne x_0 \implies f(x) \text{ close to } l.$$

The second part is the function limit.

Definition 2.1.2. A function $f(x)$ *converges* to l at x_0, and denoted $\lim_{x\to x_0} f(x) = l$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| \le \delta \text{ and } x \ne x_0 \implies |f(x) - l| \le \epsilon.$$

If $f(x)$ does not converge to any l, then we say $f(x)$ *diverges* at x_0, or the limit $\lim_{x\to x_0} f(x)$ diverges.

The implication in the definition is written as

$$0 < |x - x_0| \le \delta \implies |f(x) - l| \le \epsilon$$

in the usual textbook. The inequality $0 < |x - x_0|$ means exactly $x \ne x_0$. Moreover, Exercise 2.1.6 (and Example 1.2.19) shows that exchanging $<$ and $\le$ does not make any difference.

Combining the two parts together, we know the continuity of $f(x)$ at x_0 means the convergence to the value of the function

$$\lim_{x\to x_0} f(x) = f(x_0).$$

In particular, the constant approximations in the examples and exercises in Section 2.1.1 mean $\lim_{x\to x_0} p(x) = p(x_0)$ for polynomials $p(x)$, and

$$\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0},\ \lim_{x\to x_0} |x| = |x_0|,\ \lim_{x\to x_0} \frac{1}{x} = \frac{1}{x_0},\ \lim_{x\to 0} \sin x = 0,\ \lim_{x\to 0} \cos x = 1.$$

Since the limit $\lim_{x\to x_0} f(x) = l$ is the approximation excluding x_0, we do not require $l = f(x_0)$. In fact, we do not even require the function $f(x)$ to have value at x_0.

Example 2.1.9. The function $\frac{x^2-4}{x-2}$ is not defined for $x = 2$. Still, we have the limit

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2}(x + 2) = 4.$$

The first equality is by $\frac{x^2-4}{x-2} = x + 2$ for $x \ne 2$, and the second equality is proved below

$$|x - 2| \le \delta = \epsilon \implies |(x + 2) - 4| = |x - 2| \le \epsilon.$$

The limit can be interpreted as that the function

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \ne 2 \\ c, & \text{if } x = 2 \end{cases}$$

is continuous if and only if $c = 4$. In other words, the graph of the function is not broken. See Figure 2.1.9.

Exercise 2.1.7. Find the limits.

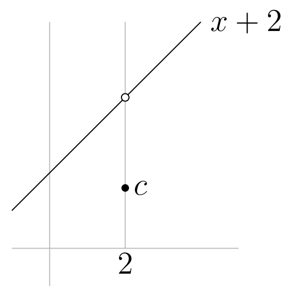


Figure 2.1.3: Function $\frac{x^2-4}{x-2}$, with added value at 2.

1. $\lim_{x\to 2}\frac{x^2-4x+4}{x-2}$.
2. $\lim_{x\to 1}\frac{x^3-1}{x-1}$.
3. $\lim_{x\to 2}\frac{x-2}{x^2-2x}$.
4. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$.

2.1.3 Properties of Constant Approximation

The constant approximations of functions and and sequences are very similar. In this section, we transfer some properties of sequence limit to functions.

Since the limit of a sequence x_n is about the trend for large n, changing finitely many terms does not change the limit (Proposition 1.1.2). Similarly, the constant approximation of a function $f(x)$ at x_0 is about the trend for x close to x_0. Therefore changing the function at some distance away from x_0 does not change the constant approximation.

Proposition 2.1.3. *If there is $\delta > 0$, such that $f(x) = g(x)$ on $[x_0 - \delta, x_0 + \delta]$, then $f(x) \approx_1 l$ at x_0 if and only $g(x) \approx_1 l$ at x_0.*

By on $[x_0 - \delta, x_0 + \delta]$, we mean for x near x_0. The proposition means that, if $|x - x_0| \le \delta$ implies $f(x) = g(x)$, then $f(x)$ is continuous at x_0 if and only if $g(x)$ is continuous at x_0.

The proof is quite simple. Suppose $f(x) \approx_1 l$ at x_0. Then for any $\epsilon > 0$, there is $\delta' > 0$, such that

$$|x - x_0| \le \delta' \implies |f(x) - l| \le \epsilon.$$

Then

$$|x - x_0| \le \min\{\delta, \delta'\} \implies |g(x) - l| = |f(x) - l| \le \epsilon.$$

On the right, the equality is due to $|x - x_0| \le \delta$, and the inequality is due to $|x - x_0| \le \delta'$. The implication means $g(x) \approx_1 l$ at x_0.

If we exclude x_0, then we get the limit version of Proposition 2.1.3.

Proposition 2.1.4. *If there is $\delta > 0$, such that $0 < |x-x_0| \le \delta$ implies $f(x) = g(x)$, then $\lim_{x\to x_0} f(x) = l$ if and only if $\lim_{x\to x_0} g(x) = l$.*

Exercise 2.1.8. Prove Proposition 2.1.4.

Proposition 2.1.5 (Arithmetic Rule). *If $f(x)$ and $g(x)$ are continuous x_0, then $f(x)+g(x)$, $cf(x)$, $f(x)g(x)$, and $\frac{f(x)}{g(x)}$ (assuming $g(x_0)\neq 0$) are continuous at x_0.*

Proposition 2.1.6 (Arithmetic Rule). *Suppose $\lim_{x\to x_0} f(x)=k$ and $\lim_{x\to x_0} g(x)=l$. Then*

$$\lim_{x\to a}(f(x)+g(x))=k+l,\quad \lim_{x\to a} cf(x)=ck,\quad \lim_{x\to a} f(x)g(x)=kl,\quad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{k}{l}\ (l\neq 0).$$

Proposition 2.1.5 is the continuity version of the arithmetic rule, where the constant approximation includes x_0. Proposition 2.1.6 is the limit version, where the constant approximation excludes x_0.

We may the proof of the arithmetic rule in Section 1.2.3. Consider

$$f(x)\approx_1 k \text{ and } g(x)\approx_1 l \text{ at } x_0 \implies f(x)+g(x)\approx_1 k+l.$$

The left side means that, for any $\epsilon>0$, there is $\delta>0$, such that

$$|x-x_0|\le\delta \implies |f(x)-k|\le\tfrac{\epsilon}{2},\ |g(x)-l|\le\tfrac{\epsilon}{2}.$$

Then

$$|x-x_0|\le\delta \implies |(f(x)+g(x))-(k+l)|\le|f(x)-k|+|g(x)-l|\le\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon.$$

This means $f(x)+g(x)\approx_1 k+l$.

If we replace $|x-x_0|\le\delta$ by $0<|x-x_0|\le\delta$ (i.e., adding $x\neq x_0$), then we get the proof of the addition of limit in Proposition 2.1.6.

Example 2.1.10. In Examples 2.1.2 and 2.1.3, we rigorously argued that the functions c and x are continuous. Then by Proposition 2.1.5, the power function x^p (p is a natural number) and polynomials are continuous. By dividing two polynomials, we know a *rational function*

$$f(x)=\frac{a_mx^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0}{b_nx^n+b_{n-1}x^{n-1}+\cdots+b_1x+b_0}$$

is continuous at x_0, as long as $f(x_0)$ is defined, i.e., the denominator is nonzero at x_0. In particular, for negative integers p, the power function x^p is continuous at any $x_0\neq 0$.

Exercise 2.1.9. Prove that the product of two functions that are continuous at x_0 is still continuous at x_0. This is the product part of Proposition 2.1.5.

Exercise 2.1.10. Suppose $\lim_{x\to x_0} f(x)=l\neq 0$. Prove $\lim_{x\to x_0}\frac{1}{f(x)}=\frac{1}{l}$.

Proposition 2.1.7 (Sandwich Rule). *Suppose there is $\delta > 0$, such that $f(x) \le g(x) \le h(x)$ on $[x_0 - \delta, x_0 + \delta]$. Then $f(x) \approx_1 l$ and $h(x) \approx_1 l$ at x_0 imply $g(x) \approx_1 l$ at x_0.*

In other words, if $f(x)$ and $h(x)$ are continuous at x_0, and $f(x_0) = h(x_0)$, then $g(x)$ is continuous at x_0.

Proposition 2.1.8 (Sandwich Rule). *Suppose there is $\delta > 0$, such that $0 < |x - x_0| \le \delta$ implies $f(x) \le g(x) \le h(x)$. Then $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l$ implies $\lim_{x\to x_0} g(x) = l$.*

Again we may copy the proof of the sandwich rule for sequence limit. In Proposition 2.1.7, for any $\epsilon > 0$, there is $\delta' > 0$, such that

$$|x - x_0| \le \delta' \implies |f(x) - l| \le \epsilon \text{ and } |h(x) - l| \le \epsilon.$$

Then

$$|x - x_0| \le \min\{\delta, \delta'\} \implies -\epsilon \le f(x) - l \le g(x) - l \le h(x) - l \le \epsilon \implies |g(x) - l| \le \epsilon.$$

Example 2.1.11. By $|\sin \frac{1}{x}| \le 1$, we get

$$-|x| \le x \sin \tfrac{1}{x} \le |x|.$$

By Example 2.1.4 (and Exercise 2.1.1), we know $\lim_{x\to 0} |x| = \lim_{x\to 0}(-|x|) = 0$. Then by the sandwich rule (Proposition 2.1.8 to be specific, because the function is not defined at 0), we get

$$\lim_{x\to 0} x \sin \tfrac{1}{x} = 0.$$

Example 2.1.12. In (2.1.1), we showed that

$$\cos x < \tfrac{\sin x}{x} < 1 \text{ for } 0 < x < \tfrac{1}{2}\pi.$$

By $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. The inequalities also hold for $-\frac{1}{2}\pi < x < 0$. Therefore the inequalities hold for $x \ne 0$ and near 0.

By Example 2.1.8, we know $\lim_{x\to 0} \cos x = 1$. Then by the sandwich rule, we get

$$\lim_{x\to 0} \tfrac{\sin x}{x} = 1.$$

Exercise 2.1.11. Show the limits converge to 0 at 0.

1. $x \cos x$.
2. $x^2 \cos \frac{1}{x}$.
3. $\frac{|x|}{x+1}$.
4. $\frac{x \cos x}{x+1}$.

Example 2.1.13. The functions $-x$ and x are continuous, and have equal value 0 at $x_0 = 0$. Then by the sandwich rule (Proposition 2.1.7 to be specific), any function

$f(x)$ satisfying $-x \le f(x) \le x$, i.e., $|f(x)| \le |x|$, is continuous at 0. The following are two such examples

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}, \quad f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}.$$

Exercise 2.1.12. Find c, such that the function is continuous at 0.

1. $\begin{cases} x \cos \frac{1}{x^2}, & \text{if } x \neq 0 \\ c, & \text{if } x = 0 \end{cases}$.

2. $\begin{cases} x, & \text{if } x \text{ is rational} \\ c, & \text{if } x \text{ is irrational} \end{cases}$.

The following is the limit version of the order rule.

Proposition 2.1.9 (Order Rule). *Suppose $\lim_{x\to a} f(x) = k$ and $\lim_{x\to a} g(x) = l$.*

1. *If there is $\delta > 0$, such that $0 < |x - x_0| \le \delta$ implies $f(x) \le g(x)$, then $k \le l$.*

2. *If $k < l$, then there is $\delta > 0$, such that $0 < |x - x_0| \le \delta$ implies $f(x) < g(x)$.*

The property for x satisfying $0 < |x - x_0| \le \delta$ means for x near x_0 and $x \neq x_0$. The following is the continuity version of the second part of the order rule.

Proposition 2.1.10 (Order Rule). *Suppose $f(x)$ and $g(x)$ are continuous at x_0. If $f(x_0) < g(x_0)$, then $f(x) < g(x)$ for x near x_0.*

The first part would be the following: If $f(x) \le g(x)$ for x near x_0, then $f(x_0) \le g(x_0)$. The statement is not included in the proposition because it is clearly trivial.

Exercise 2.1.13. Prove the second part of Proposition 2.1.9.

Exercise 2.1.14. Prove the first part of Proposition 2.1.9, either directly, or by using the second part.

The following is the function version of Proposition 1.3.1.

Proposition 2.1.11. *If $\lim_{x\to x_0} f(x)$ converges, then $f(x)$ is bounded near x_0.*

The bounded property means that there are $B, \delta > 0$, such that $0 < |x - x_0| \le \delta$ implies $|f(x)| \le B$. Note that we do not care whether x_0 is included or not, because a single values does not affect the boundedness.

If $\lim_{x\to x_0} f(x) = l$, then for $\epsilon = 1 > 0$, there is $\delta > 0$, such that

$$0 < |x - x_0| \le \delta \implies |f(x) - l| \le 1 \implies |f(x)| \le |l| + 1.$$

This means $f(x)$ is bounded near x_0.

Example 2.1.14. For $B > 0$, if $0 < |x| \le \delta = \frac{1}{B}$, then $|\frac{1}{x}| \ge \frac{1}{\delta} = B$. Therefore $\frac{1}{x}$ is not bounded near 0. By Proposition 2.1.11, the limit $\lim_{x\to 0} \frac{1}{x}$ diverges. This further implies that the function

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ c, & \text{if } x = 0 \end{cases}$$

is not continuous at 0 for any c.

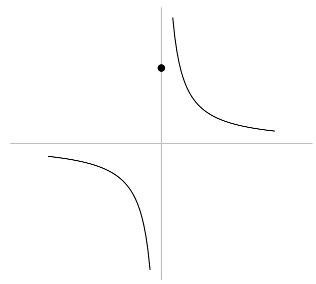


Figure 2.1.4: $\frac{1}{x}$ is not bounded near 0.

Exercise 2.1.15. Explain the divergence of limits.

1. $\lim_{x\to 0} \frac{1}{|x|}$.
2. $\lim_{x\to 2} \frac{x}{x-2}$.
3. $\lim_{x\to 0} \frac{1}{x} \cos \frac{1}{x}$.

2.1.4 Composition

Suppose three variables x, y, z are related by

$$x \xrightarrow{f} y = f(x) \xrightarrow{g} z = g(y).$$

Then x and z are related by $z = g(f(x))$, called the *composition* of two functions.

Suppose we have two constant approximations

$$\begin{aligned} f(x) \approx_1 y_0 \text{ at } x_0 &: \quad x \text{ close to } x_0 \implies y = f(x) \text{ close to } y_0, \\ g(y) \approx_1 z_0 \text{ at } y_0 &: \quad y \text{ close to } y_0 \implies z = g(y) \text{ close to } z_0. \end{aligned}$$

Then we can combine the two implications to get

$$g(f(y)) \approx_1 z_0 \text{ at } x_0 : \quad x \text{ close to } x_0 \implies z = g(f(y)) \text{ close to } z_0.$$

In the approximations above, we know the approximating constants $y_0 = f(x_0)$ and $z_0 = g(y_0) = g(f(x_0))$.

Proposition 2.1.12 (Composition Rule). *If $f(x)$ is continuous at x_0, and $g(y)$ is continuous at $y_0 = f(x_0)$, then $g(f(x))$ is continuous at x_0.*

The proof is the combination of the definitions of the two constant approximations. The argument goes backwards. For any $\epsilon > 0$, by $z = g(y) \approx_1 z_0$ at y_0, there is $\mu > 0$, such that

$$|y - y_0| \le \mu \implies |z - z_0| = |g(y) - z_0| \le \epsilon.$$

For this $\mu > 0$, by $y = f(x) \approx_1 y_0$ at x_0, there is $\delta > 0$, such that

$$|x - x_0| \le \delta \implies |y - y_0| = |f(x) - y_0| \le \mu.$$

Combining the two implications, we get

$$|x - x_0| \le \delta \implies |y - y_0| = |f(x) - y_0| \le \mu \implies |z - z_0| = |g(f(x)) - z_0| \le \epsilon.$$

This means $g(f(y)) \approx_1 z_0$ at x_0. The second $\implies$ is obtained by taking $y = f(x)$.

Example 2.1.15. In Example 2.1.8, we argued that $\sin x$ and $\cos x$ are continuous at 0. To get the continuity at any x_0, we "move" x_0 to 0 by taking $y = x - x_0$ to be the new variable. This means we write

$$\begin{aligned}
\sin x = \sin(x_0 + y) &= \cos x_0 \sin y + \sin x_0 \cos y \\
&= \cos x_0 \sin(x - x_0) + \sin x_0 \cos(x - x_0), \\
\cos x = \cos(x_0 + y) &= \cos x_0 \cos y - \sin x_0 \sin y \\
&= \cos x_0 \cos(x - x_0) - \sin x_0 \sin(x - x_0).
\end{aligned}$$

In the composition

$$x \to y = x - x_0 \to z = \sin y = \sin(x - x_0)$$

We know $y = x - x_0$ is continuous at x_0, and (by Example 2.1.8) $z = \sin y$ is continuous at $y_0 = x_0 - x_0 = 0$. Then by Proposition 2.1.12, we know $\sin(x - x_0)$ is continuous at x_0. Similarly, $\cos(x - x_0)$ is also continuous at x_0. We also note that $\sin x_0, \cos x_0$ are constants in the equalities above. Then by the arithmetic rule, we know $\sin x$ and $\cos x$ are continuous at x_0.

By the way, once we know the continuity of $\sin x$ at any x_0, we may also use $\cos x = \sin(\frac{\pi}{2} - x)$ and the composition rule to conclude $\cos x$ is also continuous at any x_0.

By the continuity of $\sin x$, $\cos x$, and the arithmetic rule, we know the other trigonometric functions $\tan x, \cot x, \sec x, \csc x$ are also continuous at wherever they are defined. We will also see that the power function x^p (Example 2.2.6) and the exponential function a^x (Example 2.2.7) are continuous. Moreover, we will define $\log x$ and inverse trigonometric functions, which are also continuous at wherever they are defined. The *elementary functions* are obtained from these by arithmetic operations and compositions. All elementary functions are continuous at wherever they are defined.

Next we consider the limit of composition. This means we assume two constant approximations excluding x_0 and y_0

$$\lim_{x\to x_0} f(x) = y_0\colon \quad x \text{ close to } x_0 \text{ and } x \neq x_0 \implies y = f(x) \text{ close to } y_0, \tag{2.1.2}$$

$$\lim_{y\to y_0} g(y) = z_0\colon \quad y \text{ close to } y_0 \text{ and } y \neq y_0 \implies z = g(y) \text{ close to } z_0. \tag{2.1.3}$$

Then we try to combine the two implications to get

$$\lim_{x\to x_0} g(f(x)) = z_0\colon \quad x \text{ close to } x_0 \text{ and } x \neq x_0 \implies z = g(f(x)) \text{ close to } z_0.$$

However the right of (2.1.2) does not imply the left of (2.1.3). We need to either strengthen the right of (2.1.2) to

$$x \text{ close to } x_0 \text{ and } x \neq x_0 \implies y = f(x) \text{ close to } y_0 \text{ and } y \neq y_0,$$

or weaken the left of (2.1.3) to

$$y \text{ close to } y_0 \implies z = g(y) \text{ close to } z_0.$$

We remark that the second choice means $g(y)$ is continuous at y_0.

With the extra condition, we get the limit of composition.

Proposition 2.1.13 (Composition Rule). *Suppose*

$$\lim_{x\to x_0} f(x) = y_0, \quad \lim_{y\to y_0} g(y) = z_0,$$

and one of the following extra conditions is satisfied

1. *There is $\delta > 0$, such that $0 < |x - x_0| \le \delta$ implies $f(x) \neq y_0$.*
2. *$g(y)$ is continuous at y_0.*

Then

$$\lim_{x\to x_0} g(f(x)) = z_0.$$

Note that in the second case, we have $z_0 = g(y_0)$, and the composition rule becomes

$$\lim_{x\to x_0} g(f(x)) = z_0 = g(y_0) = g\left(\lim_{x\to x_0} f(x)\right). \tag{2.1.4}$$

We see the continuity of a function is the same as that the function and the limit can be exchanged. For example, by the continuity of $\sqrt{y}$ at $y_0 > 0$, we know

$$\lim_{x\to x_0} f(x) = l > 0 \implies \lim_{x\to x_0} \sqrt{f(x)} = \sqrt{l}.$$

Of course after we know y^p is continuous at $y_0 > 0$, we also know $\lim_{x\to x_0} f(x)^p = l^p$.

Exercise 2.1.16. Find the limits, $a, b > 0$.

1. $\lim_{x\to 0}\sqrt{a+x}$.
2. $\lim_{x\to 0}\frac{1}{x}(\sqrt{a+x}-\sqrt{a-x})$.
3. $\lim_{x\to 0}\left(\sqrt{\frac{a+x}{a}}-\sqrt{\frac{a}{a+x}}\right)$.
4. $\lim_{x\to 0}\frac{1}{x}\left(\sqrt{\frac{a+x}{b}}-\sqrt{\frac{a}{b+x}}\right)$.

Exercise 2.1.17. Prove Proposition 2.1.13 under the first extra condition.

Exercise 2.1.18. Prove Proposition 2.1.13 under the second extra condition.

Example 2.1.16. A change of variable can often be applied to limits. For example, for $x_0\neq 0$, we have

$$\lim_{x\to x_0} f(x^2)=\lim_{y\to x_0^2} f(y).$$

The equality means that the left converges if and only if the right converges, and the two limit values are the same.

Suppose $\lim_{y\to x_0^2} f(y)=l$. We know $f(x^2)$ is the composition of $f(y)$ with $y=x^2$, and $\lim_{x\to x_0} x^2=x_0^2$. Moreover, by $x_0\neq 0$, the first condition is satisfied

$$0<|x-x_0|\le |x_0| \implies x, x_0 \text{ have the same sign and } x\neq x_0 \implies y=x^2\neq x_0^2.$$

Then the composition rule tells us $\lim_{x\to x_0} f(x^2)=l$.

Conversely, suppose $\lim_{x\to x_0} f(x^2)=l$. Assume $x_0>0$. Then x near x_0 are also positive, and $f(y)$ is the composition of $f(x^2)$ with $x=\sqrt{y}$. By Example 2.1.6 and Exercise 2.1.3, we know $\lim_{y\to x_0^2}\sqrt{y}=\sqrt{x_0^2}=x_0$. Moreover, the first condition is satisfied

$$0<|x-x_0|\le x_0 \implies 0\le x\neq x_0 \implies \sqrt{x}\neq\sqrt{x_0}.$$

Then the composition rule tells us $\lim_{y\to x_0^2} f(y)=l$. The case $x_0<0$ is left as Exercise 2.1.19.

The example illustrates the following common use of the composition rule under the first condition. Suppose $y=u(x)$ is continuous and invertible near x_0, and the inverse $x=v(y)$ is also continuous and invertible[1] near $y_0=u(x_0)$. Two functions $f(x)=g(u(x))$ and $g(y)=f(v(y))$ are related by the *continuous change of variable*.

The continuity of $u(x)$ at x_0 means $\lim_{x\to x_0} u(x)=u(x_0)=y_0$. The invertibility of $u(x)$ implies

$$x\neq x_0 \implies y=u(x)\neq y_0=u(x_0).$$

Then the first extra condition is satisfied, and we get

$$\lim_{y\to y_0} g(y)=l \implies \lim_{x\to x_0} g(u(x))=l.$$

[1]In fact, continuous and invertible $u(x)$ automatically implies that the inverse function $x=v(y)$ is also continuous and invertible. See Theorem 2.4.2.

Apply the same argument to $x = v(y)$, we get

$$\lim_{x\to x_0} f(x) = l \implies \lim_{y\to y_0} f(v(y)) = l.$$

Therefore we get the equality

$$\lim_{x\to x_0} f(x) = \lim_{y\to y_0} g(y) = l.$$

The left converges if and only if the right converges, and the two limits have the same value.

Exercise 2.1.19. Complete Example 2.1.16: For $x_0 < 0$, prove $\lim_{x\to x_0} f(x^2) = l$ implies $\lim_{y\to x_0^2} f(y) = l$.

Exercise 2.1.20. Prove $\lim_{x\to a} f(x) = \lim_{x\to 0} f(a+x)$. In particular, $f(x)$ is continuous at a if and only if $f(a+x)$ is continuous at 0.

Exercise 2.1.21. For $a \neq 0$, prove $\lim_{x\to a} f(x) = \lim_{x\to 1} f(ax)$. In particular, $f(x)$ is continuous at a if and only if $f(ax)$ is continuous at 1.

Exercise 2.1.22. Prove $\lim_{x\to a} f(x^3) = \lim_{x\to a^3} f(x)$.

Exercise 2.1.23. Prove that the function $x^2 + 2x$ is invertible near any number $\neq -\frac{1}{2}$, and the inverse is continuous. Then for $a \neq -\frac{1}{2}$, prove $\lim_{x\to a} f(x^2 + x) = \lim_{x\to a^2+a} f(x)$.

Example 2.1.17. In Example 2.1.12, we get

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

By the arithmetic rule and composition rule, we further get

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x - 1}{x^2} &= \lim_{x\to 0} \frac{-2\sin^2 \frac{x}{2}}{x^2} = \lim_{y\to 0} \frac{-2\sin^2 y}{(2y)^2} = -\frac{1}{2} \lim_{y\to 0} \left(\frac{\sin y}{y}\right)^2 = -\frac{1}{2}, \\
\lim_{x\to 0} \frac{\cos x - 1}{x} &= \lim_{x\to 0} \frac{\cos x - 1}{x^2} x = -\frac{1}{2} \cdot 0 = 0, \\
\lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{\sin x}{x} \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = 1, \\
\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} &= \lim_{y\to 0} \frac{\cos\left(y + \frac{\pi}{2}\right)}{y} = \lim_{y\to 0} \frac{-\sin y}{y} = -1, \\
\lim_{x\to x_0} \frac{\sin x - \sin x_0}{x - x_0} &= \lim_{y\to 0} \frac{\sin(x_0 + y) - \sin a}{y} \\
&= \lim_{y\to 0} \left(\sin x_0 \frac{\cos y - 1}{y} + \cos x_0 \frac{\sin y}{y}\right) = \cos x_0.
\end{aligned}$$

Exercise 2.1.24. Find the limits.

1. $\lim_{x\to -1} \frac{\sin \pi x}{x+1}$.
2. $\lim_{x\to \frac{\pi}{4}} \frac{\tan x - 1}{4x-\pi}$.
3. $\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{4x-\pi}$.
4. $\lim_{x\to \frac{1}{3}} \frac{\sin \pi x - \sqrt{3}\cos \pi x}{3x-1}$.
5. $\lim_{x\to x_0} \frac{\cos x - \cos x_0}{x-x_0}$.
6. $\lim_{x\to x_0} \frac{\tan x - \tan x_0}{x-x_0}$.
7. $\lim_{x\to 0} \frac{\sin(2x+1)+\sin(2x-1)}{x^2}$.
8. $\lim_{x\to 0} \frac{\cos(2x+1)-\cos(2x-1)}{x^2}$.

2.2 Partial Constant Approximation

We may restrict the approximation to a subset of the domain. For example, the right approximation at x_0 means the approximation of the restriction of the function on $[x_0, x_0+\delta)$, the right of x_0. Another example is the restriction to a sequence converging to x_0.

The restriction is part of the function, similar to subsequence. The limit of the restriction of function has properties similar to the limit of subsequence.

2.2.1 One Sided Approximation

The constant approximation on the right of x_0 is the approximation only for $x \geq x_0$

$$x \text{ close to } x_0 \text{ and } x \geq x_0 \implies f(x) \text{ close to } l.$$

Definition 2.2.1. A function $f(x)$ has constant approximation l on the right of x_0, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 \leq x - x_0 \leq \delta \implies |f(x) - l| \leq \epsilon.$$

In this case, we say $f(x)$ is *right continuous* at x_0.

We can similarly define the left constant approximation, which is the left continuity. The parts of the function on the right and left of x_0 can be compared with the odd and even subsequences of a sequence. As an analogue of Proposition 1.1.6, the following says the two sided constant approximation is the same as the left and right constant approximations by the same constant.

Proposition 2.2.2. *A function defined on both sides of x_0 is continuous at x_0 if and only if it is right and left continuous at x_0.*

Suppose we have the right and left approximations by the constant l. Then for any $\epsilon > 0$, there are $\delta, \delta' > 0$, such that

$$\begin{aligned} 0 \leq x - x_0 \leq \delta &\implies |f(x) - l| \leq \epsilon, \\ -\delta' \leq x - x_0 \leq 0 &\implies |f(x) - l| \leq \epsilon. \end{aligned}$$

Then $|x - x_0| \leq \min\{\delta, \delta'\}$ implies either $0 \leq x - x_0 \leq \delta$ or $-\delta' \leq x - x_0 \leq 0$, and further implies $|f(x) - l| \leq \epsilon$. This proves the two sided constant approximation.

Example 2.2.1. Consider the function

$$f(x) = \begin{cases} x^2, & \text{if } x > 1 \\ x, & \text{if } x \leq 1 \end{cases}.$$

We have $f(x) = x^2$ on $[1, +\infty)$, the right of 1. Since $f(x) = x^2$ is continuous on both sides of 1, by Proposition 2.2.2, it is right continuous at 1. Similarly, $f(x) = x$ on $(-\infty, 1]$, the left of 1, and is left continuous at 1. Therefore by Proposition 2.2.2, $f(x)$ is continuous at 1.

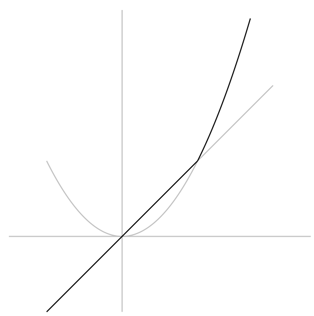


Figure 2.2.1: A continuous function.

Exercise 2.2.1. Prove that the two sided continuity implies the right and left continuities.

A function $f(x)$ on an open interval (a, b) is continuous if it is continuous at any $x_0 \in (a, b)$. A function on a closed interval $[a, b]$ is continuous if it is (two sided) continuous at any $x_0 \in (a, b)$, and right continuous at a, and left continuous at b. We may similarly define the continuity on half open and half closed intervals.

Example 2.2.2. The function $\sqrt{x}$ is defined for $x \geq 0$. In Example 2.1.6 and Exercise 2.1.3, we showed that $\sqrt{x}$ is continuous at any $x_0 \neq 0$. It remains to consider the right continuity at 0.

For any $\epsilon > 0$, we have

$$0 \leq x \leq \delta = \epsilon^2 \implies |\sqrt{x} - 0| = \sqrt{x} \leq \sqrt{\delta} = \epsilon.$$

This proves $\sqrt{x}$ is right continuous at 0. Therefore the function is continuous on $[0, +\infty)$, wherever the function is defined.

Exercise 2.2.2. For $p > 0$, prove x^p is right continuous at 0.

Similar to two-sided continuity, we may split the definition of the right continuity to two cases. First, for $x = x_0$, the approximation means $l = f(x_0)$. Second, for $x > x_0$, the approximation means the right constant approximation that excludes x_0. The second case is the definition of the right limit.

Definition 2.2.3. A function $f(x)$ *converges* to l at the right of x_0, and denoted $\lim_{x\to x_0^+} f(x) = l$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < x - x_0 \le \delta \implies |f(x) - l| \le \epsilon.$$

The right constant approximation requires $f(x)$ to be defined on $[x_0, x_0 + \delta)$, the right of x_0 and including x_0. The right limit requires $f(x)$ to be defined on $(x_0, x_0 + \delta)$, the right of x_0 and not necessarily including x_0. The right continuity at x_0 means

$$\lim_{x\to x_0^+} f(x) = f(x_0).$$

The right limit means

$$x > x_0 \text{ and } x \text{ close to } x_0 \implies f(x) \text{ close to } l.$$

The left limit can be similarly defined and means

$$x < x_0 \text{ and } x \text{ close to } x_0 \implies f(x) \text{ close to } l.$$

We have the excluding x_0 version of Proposition 2.2.2.

Proposition 2.2.4. $\lim_{x\to x_0} f(x) = l$ *if and only if* $\lim_{x\to x_0^+} f(x) = l$ *and* $\lim_{x\to x_0^-} f(x) = l$.

Exercise 2.2.3. Show the limit is 0, $p > 0$.

1. $\lim_{x\to 0^+} \sqrt{x}\cos x$.
2. $\lim_{x\to 0} \sqrt[3]{x}\cos\frac{1}{x}$.
3. $\lim_{x\to 0^+} x^p \sin\frac{1}{x^2}$.
4. $\lim_{x\to 0^+} \frac{x^p}{x+1}$.

Exercise 2.2.4. Find the limits, $a > 0$.

1. $\lim_{x\to a^+} \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}$.
2. $\lim_{x\to a^+} \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}$.

Example 2.2.3. The sign function is

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

We have

$$\lim_{x\to 0^+} \mathrm{sign}(x) = \lim_{x\to 0^+} 1 = 1,$$
$$\lim_{x\to 0^-} \mathrm{sign}(x) = \lim_{x\to 0^-} (-1) = -1.$$

Since the two limit values are different, by Proposition 2.2.4, we know the limit $\lim_{x\to 0} \mathrm{sign}(x)$ diverges, and $\mathrm{sign}(x)$ is not continuous at 0. The argument is very similar to the divergence of the sequence $(-1)^n$ in Example 1.1.21.

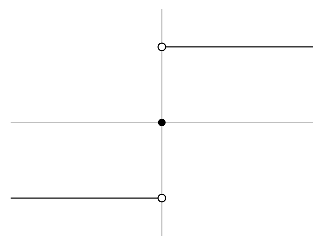


Figure 2.2.2: Sign function.

The discontinuity argument does not use the value $\mathrm{sign}(0) = 0$. The key point is actually the gap between the left and right limits. If a function $f(x)$ has a gap at x_0, then $\lim_{x\to x_0} f(x)$ diverges, and the function is not continuous at x_0, no matter what the value of $f(x_0)$ is.

Example 2.2.4. We change the function in Example 2.2.1 to

$$f(x) = \begin{cases} x^2, & \text{if } x > 1 \\ x + c, & \text{if } x \le 1 \end{cases}.$$

By the (two-sided) continuity of x^2 at 1, and Proposition 2.2.2, we get the right continuity, which means

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1.$$

Here the first equality is due to $f(x) = x^2$ for $x > 1$, and the limit does not include the value at 1. Similarly, by the continuity of $x + c$, we also get

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x + c) = 1 + c.$$

By Proposition 2.2.4, $\lim_{x\to 1} f(x)$ converges if and only if the right limit 1 and the left limit $1 + c$ are equal. This means $c = 0$. Example 2.2.1 shows that the function is continuous when $c = 0$.

Exercise 2.2.5. Can you find c, such that the function is continuous at 0?

1. $f(x) = \begin{cases} 2x, & \text{if } x \neq 0 \\ c, & \text{if } x = 0 \end{cases}$.
2. $f(x) = \begin{cases} x + c, & \text{if } x < 0 \\ \sin x, & \text{if } x \geq 0 \end{cases}$.
3. $f(x) = \begin{cases} c, & \text{if } x \geq 0 \\ x, & \text{if } x < 0 \end{cases}$.
4. $f(x) = \begin{cases} 1 + x, & \text{if } x > 0 \\ c, & \text{if } x = 0 \\ 1 - x, & \text{if } x < 0 \end{cases}$.

2.2.2 Property of One Sided Limit

All the usual properties of two sided limit are valid for one sided limit. We have the arithmetic rule, the sandwich rule, and the order rule. We also have the local nature of the limit, and convergence implying locally bounded. For example, by the arithmetic rule, we know arithmetic combinations of right continuous functions are still right continuous.

Exercise 2.2.6. State and prove the analogues of Propositions 2.1.3 and 2.1.4 for the right constant approximation.

Exercise 2.2.7. State and prove the analogue of Proposition 2.1.7 (sandwich rule for continuity) for the right continuity.

Exercise 2.2.8. State and prove the analogue of Proposition 2.1.9 (order rule for limit) for the left limit.

Exercise 2.2.9. State and prove the analogue of Proposition 2.1.11 (local boundedness) for functions with converging right limit.

Exercise 2.2.10. Explain the divergence of limits.

1. $\lim_{x\to 0^+} \frac{1}{x}$.
2. $\lim_{x\to 2^-} \frac{x}{x-2}$.
3. $\lim_{x\to 0^+} \frac{1}{x} \cos \frac{1}{x}$.

The extension of the composition rule is more varied. For example, if we only know the right limit $\lim_{y\to y_0^+} g(y)$, then in the composition rule, we need to require $f(x) \geq y_0$, or even $f(x) > y_0$. It is better to illustrate the one sided composition rule by examples.

Example 2.2.5. In Example 2.1.16 and Exercise 2.1.19, we argued that $\lim_{x\to x_0} f(x^2) = \lim_{y\to x_0^2} f(y)$ for $x_0 \neq 0$. For $x_0 = 0$, we use the continuous change of variable to argue that

$$\lim_{x\to 0^+} f(x^2) = \lim_{y\to 0^+} f(y).$$

The function $u(x) = x^2 \colon [0,1] \to [0,1]$ is continuous and invertible. The inverse function is $v(y) = \sqrt{y} \colon [0,1] \to [0,1]$. By Example 2.2.2, we know $v(y)$ is also continuous. We note that $x \to 0^+$ corresponds to $y \to 0^+$ under the continuous change of variable (this follows from the left continuity at 0). Then by the (one sided) composition rule, we get the equality of the two limits above.

We may also consider the continuous change of variable $u(x) = x^2 \colon [-1, 0] \to [0, 1]$, with continuous inverse $v(y) = -\sqrt{y} \colon [0, 1] \to [-1, 0]$. Then the same reason proves the equality

$$\lim_{x\to 0^-} f(x^2) = \lim_{y\to 0^+} f(y).$$

Combining the two equalities and using Proposition 2.2.4, we get

$$\lim_{x\to 0} f(x^2) = \lim_{y\to 0^+} f(y).$$

Exercise 2.2.11. Suppose $f(x)$ is right continuous at x_0 and $g(y)$ is right continuous at $y_0 = f(x_0)$. Find two extra conditions under which you can still conclude $g(f(x))$ is right continuous at x_0.

Exercise 2.2.12. Suppose $\lim_{x\to x_0^+} f(x) = y_0$ and $\lim_{y\to y_0} g(y) = z_0$, and one of the following extra conditions is satisfied.

1. There is $\delta > 0$, such that $f(x) \neq y_0$ on $(x_0, x_0 + \delta)$.
2. $g(y)$ is continuous at y_0.

Prove $\lim_{x\to x_0^+} g(f(x)) = z_0$. Moreover, under the second extra condition, extend the equality (2.1.4) to one sided limit.

Exercise 2.2.13. Suppose $\lim_{x\to x_0^+} f(x) = y_0$ and $\lim_{y\to y_0^-} g(y) = z_0$. Find suitable extra condition under which you can conclude $\lim_{x\to x_0^+} g(f(x)) = z_0$.

Exercise 2.2.14. Prove the equalities ($a > 0$ in the 4th equality). The equalities also identifies one sided continuities.

1. $\lim_{x\to 0^+} f(-x) = \lim_{x\to 0^-} f(x)$.
2. $\lim_{x\to a^+} f(-x) = \lim_{x\to (-a)^-} f(x)$.
3. $\lim_{x\to a^+} f(x) = \lim_{x\to 0^+} f(a + x)$.
4. $\lim_{x\to a^+} f(x) = \lim_{x\to 1^+} f(ax)$.
5. $\lim_{x\to 0} f(|x|) = \lim_{x\to 0^+} f(x)$.
6. $\lim_{x\to 1^-} f(x) = \lim_{x\to 0} f(1 - 2x^2)$.

2.2.3 Basic Limit

We establish some basic limits. Some of these limits mean the continuity of basic functions.

Example 2.2.6. We show the power function x^p is continuous.

In Example 2.1.3, we already know the function x is continuous everywhere. Then for integer p, by the arithmetic rule, we know x^p is continuous everywhere. For negative integer p, we know x^p is continuous everywhere except at 0. In particular, x^p is right and left continuous at 1, for integer p.

For any p, we have integers M and N satisfying $M < p < N$. Then for $x \geq 1$, we get

$$x^M \leq x^p \leq x^N.$$

By the right continuity of x^M, x^N, and $1^M = 1^N = 1$, and the sandwich rule, we know x^p is right continuous at 1.

Similarly, for $0 < x \leq 1$, we have

$$x^M \geq x^p \geq x^N.$$

Then the same reason shows x^p is left continuous at 1. Combining the left and right continuities, we know x^p is continuous at 1.

We remark that the proof of the continuity at 1 is essentially the same as the proof in Example 1.1.7 and Exercise 1.1.26.

Next we extend the continuity of x^p at 1 to the continuity at any $x_0 > 0$. The idea is to use the continuous change of variable $x = x_0 y$ to change the continuity of $f(x)$ at x_0 to the continuity of $f(x_0 y)$ at $y = 1$. See Exercise 2.1.21. We know $(x_0 y)^p = x_0^p y^p$, as a constant multiple of the function y^p that is continuous at $y = 1$, is continuous at 1. Then by the composition rule, we know x^p is continuous at any $x_0 > 0$.

Finally, for $p > 0$, by Example 2.2.2 and Exercise 2.2.2, we know x^p is right continuous at 0. Therefore x^p is continuous on $[0, +\infty)$.

For $p < 0$, the function x^p is not bounded on $(0, \delta)$, for any $\delta > 0$. Then by the right limit version of Proposition 2.1.11, we know x^p cannot be right continuous at 0. Therefore the continuity of x^p on $(0, +\infty)$ cannot be extended to 0.

Exercise 2.2.15. Suppose $p > 0$ is a fraction with odd number denominator (say $p = \frac{2}{3}, \frac{3}{7}$). Then x^p is also defined for $x \leq 0$. Prove that x^p is continuous at any x_0.

Exercise 2.2.16. Suppose $p < 0$ is a fraction with odd number denominator (say $p = -\frac{2}{3}, -\frac{3}{7}$). Then x^p is also defined for $x < 0$. Prove that x^p is continuous at any $x_0 < 0$.

Exercise 2.2.17. Combine the interpretation (2.1.4) of the continuity and Example 2.2.6 to explain that, if $\lim_{x\to x_0} f(x) = l > 0$, then $\lim_{x\to x_0} f(x)^p = l^p$. What can you say in case $l = 0$?

Example 2.2.7. For $a > 0$, we show the exponential function a^x is continuous.

First, we argue that a^x is right continuous at 1. This means $\lim_{x\to 0^+} a^x = 1$. In Examples 1.1.8 and 1.2.8, we already know the special case of the limit for $x = \frac{1}{n}$. Our idea is to compare the limit for general $x > 0$ with this special case.

Suppose $a > 1$. For any $\epsilon > 0$, there is natural number N satisfying $N \geq \frac{a}{\epsilon}$.

Then by the argument in Example 1.1.8, we get

$$a^{\frac{1}{N}} - 1 = \sqrt[N]{a} - 1 < \frac{a}{N} \le \epsilon.$$

Then

$$0 \le x \le \delta = \frac{1}{N} \implies 0 \le a^x - 1 \le a^{\frac{1}{N}} - 1 < \epsilon.$$

This proves the right continuity of a^x at 1.

If $0 < a < 1$, then $b = a^{-1} > 1$. We just proved b^x is right continuous at 0. By the arithmetic rule, we know $a^x = \frac{1}{b^x}$ is also right continuous at 0.

For the left continuity at 0, we use the continuous change of variable $x = -y$ to change the left continuity of $f(x)$ to the right continuity of $f(-x)$. See Exercise 2.2.14(1). Therefore the left continuity of a^x at 0 is the same as the right continuity of a^{-x} at 0. By $a^{-x} = \frac{1}{a^x}$, the right continuity of a^x at 0, and the arithmetic rule, we know a^{-x} is right continuous at 0.

By combining the left and right continuities (Proposition 2.2.2), we conclude x^p is continuous at 1.

Finally, we extend the continuity to any x_0. Using the continuous change of variable $x = x_0 + y$, the continuity of a^x at x_0 is the same as the continuity of a^{x_0+x} at 0. See Exercise 2.1.20. Since $a^{x_0+x} = a^{x_0}a^x$ is a constant multiple of a^x, and a^x is continuous at 0, we know a^{x_0+x} is indeed continuous at 0.

Exercise 2.2.18. Combine the interpretation (2.1.4) of the continuity and Example 2.2.7 to explain that, if $\lim_{x\to x_0} f(x) = l > 0$, then $\lim_{x\to x_0} a^{f(x)} = a^l$ for $a > 0$.

Exercise 2.2.19. Prove that if $0 < a \le f(x) \le b$ for x near 0 and some constants a, b, then $\lim_{x\to 0} f(x)^x = 1$. This is the analogue of Exercise 1.1.30.

Exercise 2.2.20. Suppose $0 < a \le f(x) \le b$ for some constants a, b and x near x_0, and $\lim_{x\to x_0} g(x) = 0$. Prove that $\lim_{x\to x_0} f(x)^{g(x)} = 1$.

Example 2.2.8. Example 2.2.7 is the extension of Example 1.1.8. We may rewrite the limit in Example 1.1.9 as

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = 1.$$

Then the natural extension is

$$\lim_{x\to 0^+} x^x = 1.$$

We try to prove this limit.

Similar to Example 2.2.7, we compare very small $x > 0$ with $\frac{1}{n}$. We need to use more complicated comparison.

Suppose $\frac{1}{n+1} \le x \le \frac{1}{n}$. Then

$$\frac{1}{(n+1)^{\frac{1}{n}}} = \left(\frac{1}{n+1}\right)^{\frac{1}{n}} \le x^{\frac{1}{n}} \le x^x \le x^{\frac{1}{n+1}} \le \left(\frac{1}{n}\right)^{\frac{1}{n+1}} = \frac{1}{n^{\frac{1}{n+1}}}.$$

By Example 1.1.10 that follows Example 1.1.9, we know

$$\lim_{n\to\infty} \frac{1}{(n+1)^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n+1}}} = 1.$$

Therefore we are in the sandwich situation. However, we have a function x^x sandwiched between two sequences. We have not yet stated such kind of sandwich rule. Therefore we adopt the proof of the sandwich rule.

By the limit of the two sequences, for any $\epsilon > 0$, there is a natural number N, such that

$$n \ge N \implies \left|\frac{1}{(n+1)^{\frac{1}{n}}} - 1\right| < \epsilon \text{ and } \left|\frac{1}{n^{\frac{1}{n+1}}} - 1\right| < \epsilon.$$

Then for $0 < x \le \delta = \frac{1}{N}$, we have $\frac{1}{n+1} \le x \le \frac{1}{n}$ for some $n \ge N$. Then we get

$$1 - \epsilon < \frac{1}{(n+1)^{\frac{1}{n}}} \le x^x \le \frac{1}{n^{\frac{1}{n+1}}} < 1 + \epsilon.$$

We proved that

$$0 < x \le \delta = \frac{1}{N} \implies |x^x - 1| < \epsilon.$$

This means $\lim_{x\to 0^+} x^x = 1$.

Exercise 2.2.21. Find the right limit at 0, $a > 0$.

1. $x^{2.3x}$.
2. x^{bx}.
3. $(ax)^x$.
4. $(x+1)^x$.
5. x^{x+1}.
6. $x^{(x^2)}$.
7. $(x^2)^x$.
8. $(x^2+2x)^x$.
9. $(x^2+2)^x$.
10. $(a+bx)^{cx}$.
11. $(a+bx+cx^2)^x$.
12. $(ax+bx^2)^{x+x^2}$.

Example 2.2.9. In Example 1.3.5, the constant e is defined as

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Extending $\frac{1}{n}$ to $x > 0$, we expect

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e.$$

We can use the idea of Example 2.2.8 to prove this.

Suppose $\frac{1}{n+1} \le x \le \frac{1}{n}$. Then

$$\left(1+\frac{1}{n+1}\right)^n \le (1+x)^n \le (1+x)^{\frac{1}{x}} \le (1+x)^{n+1} \le \left(1+\frac{1}{n}\right)^{n+1}.$$

By Exercise 1.3.20, we know

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e.$$

This means that, for any $\epsilon > 0$, there is a natural number N, such that

$$n \ge N \implies \left|\left(1+\frac{1}{n+1}\right)^n - e\right| < \epsilon \text{ and } \left|\left(1+\frac{1}{n}\right)^{n+1} - e\right| < \epsilon.$$

Then for $0 < x \le \delta = \frac{1}{N}$, we have $\frac{1}{n+1} \le x \le \frac{1}{n}$ for some $n \ge N$, and

$$e-\epsilon < \left(1+\frac{1}{n+1}\right)^n \le (1+x)^{\frac{1}{x}} \le \left(1+\frac{1}{n}\right)^{n+1} < e+\epsilon.$$

This means $|(1+x)^{\frac{1}{x}} - e| < \epsilon$, which completes the proof of $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$.

For the left limit $\lim_{x\to 0^-}(1+x)^{\frac{1}{x}}$, the analogous sequence limit is $\lim_{n\to\infty}(1-\frac{1}{n})^{-n}$ (see Exercise 1.3.20(3)). By $(m = n-1)$

$$\left(1-\frac{1}{n}\right)^{-n} = \left(\frac{n-1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^n = \left(1+\frac{1}{n-1}\right)^n = \left(1+\frac{1}{m}\right)^m\left(1+\frac{1}{m}\right).$$

we find the limit is still e. We replace $\frac{1}{n}$ by small $x > 0$ and get

$$(1-x)^{-\frac{1}{x}} = \left(\frac{1}{1-x}\right)^{\frac{1}{x}} = \left(1+\frac{x}{1-x}\right)^{\frac{1-x}{x}+1} = (1+y)^{\frac{1}{y}}\left(1+\frac{x}{1-x}\right).$$

The function $y = u(x) = \frac{x}{1-x} \colon [0, \frac{1}{2}) \to [0, 1)$ is a continuous and invertible change of variable, with the inverse $x = \frac{y}{1+y} \colon [0,1) \to [0, \frac{1}{2})$. Moreover, $x \to 0^+$ corresponds to $y \to 0^+$. Then by the composition rule, we get $\lim_{x\to 0^+}(1+\frac{x}{1-x})^{\frac{1-x}{x}} = \lim_{y\to 0^+}(1+y)^{\frac{1}{y}} = e$. Combined with $\lim_{x\to 0^+}(1+\frac{x}{1-x}) = 1$, we get $\lim_{x\to 0^+}(1-x)^{-\frac{1}{x}} = e$. By a further continuous change of x to $-x$, we get $\lim_{x\to 0^-}(1+x)^{\frac{1}{x}} = e$.

Combining the right and left limits, we get

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.$$

Exercise 2.2.22. Find the limits at 0.

1. $(1+2x)^{\frac{1}{x}}$.
2. $(1+x)^{\frac{1}{2x}}$.
3. $(1-x)^{\frac{1}{x}}$.
4. $(1-3x)^{\frac{2}{x}}$.
5. $(1+x+x^2)^{\frac{1}{x}}$.
6. $(1+x)^{\frac{1}{x+x^2}}$.

Exercise 2.2.23. We may directly prove $\lim_{x\to 0^-}(1+x)^{\frac{1}{x}} = e$, similar to the proof of the right limit. For x close to 0 and $x<0$, we have $-\frac{1}{n} \le x \le -\frac{1}{n+1}$. Then we compare $(1+x)^{\frac{1}{x}}$ with slight variations of $(1-\frac{1}{n})^{-n}$. Please complete such proof.

2.2.4 Approximation on Subsets

One sided approximations at x_0 are the restrictions of two sided approximation to the right and left of x_0. These are not the only subsets we can choose. For any subset A, the limit $\lim_{x\to x_0,\, x\in A} f(x) = l$ on A means

$$x \text{ close to } x_0 \text{ and } x \in A - x_0 \implies f(x) \text{ close to } l.$$

The notation $x \in A - x_0$ means $x \in A$ and $x \neq x_0$.

Definition 2.2.5. A function $f(x)$ converges to l at x_0 on a subset A, and denoted $\lim_{x\to x_0,\, x\in A} f(x) = l$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| \le \delta \text{ and } x \in A - x_0 \implies |f(x) - l| \le \epsilon.$$

The usual limit $\lim_{x\to x_0} f(x)$ is $\lim_{x\to x_0,\, x\in[x_0-\delta, x_0+\delta]} f(x)$, for small $\delta > 0$. The right limit $\lim_{x\to x_0^+} f(x)$ is $\lim_{x\to x_0,\, x\in[x_0, x_0+\delta]} f(x)$, for small $\delta > 0$.

The following is the function analogue of Proposition 1.1.6.

Proposition 2.2.6. *If $\lim_{x\to x_0,\, x\in A} f(x) = l$ and $B \subset A$, then $\lim_{x\to x_0,\, x\in B} f(x) = l$. Conversely, if A is the union of finitely many B_i, and $\lim_{x\to x_0,\, x\in B_i} f(x) = l$ have the same limit value l, then $\lim_{x\to x_0,\, x\in A} f(x) = l$.*

For $A = [x_0 - \delta, x_0 + \delta] = [x_0, x_0 + \delta] \cup [x_0 - \delta, x_0] = B_1 \cup B_2$, the proposition becomes Proposition 2.2.4. A common use of the proposition is to find B_1, B_2, such that $\lim_{x\to x_0,\, x\in B_1} f(x)$ and $\lim_{x\to x_0,\, x\in B_2} f(x)$ have distinct values. Then by the first part of the proposition, we conclude $\lim_{x\to x_0,\, x\in A} f(x)$ diverges.

We also have the continuity version (i.e., change $A - x_0$ to A) of the discussion above. We present the limit version because it is used more often.

Example 2.2.10. For the limit $\lim_{x\to 0} \sin\frac{1}{x}$, we consider

$$B_1 = \{\tfrac{1}{(2n+\frac{1}{2})\pi} : n \in \mathbb{N}\}, \quad B_2 = \{\tfrac{1}{(2n-\frac{1}{2})\pi} : n \in \mathbb{N}\}.$$

Then $\sin\frac{1}{x} = 1$ for all $x \in B_1$, and $\sin\frac{1}{x} = -1$ for all $x \in B_2$. Therefore $\lim_{x\to 0,\, x\in B_1} \sin\frac{1}{x} = 1$ and $\lim_{x\to 0,\, x\in B_2} \sin\frac{1}{x} = -1$. Since the two limit values are different, by the first part of Proposition 2.2.6, the whole limit $\lim_{x\to 0} \sin\frac{1}{x}$ diverges.

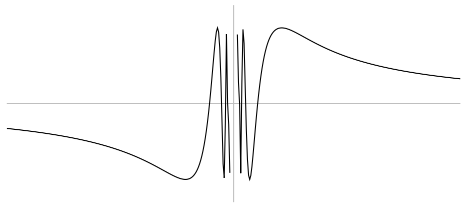


Figure 2.2.3: $\sin \frac{1}{x}$.

Exercise 2.2.24. Provide reason for divergence of $\lim_{x\to 0^+} \cos \frac{1}{x}$.

Example 2.2.11. The *Dirichlet function* is

$$D(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}.$$

All the real numbers $A = \mathbb{R}$ is the union of rational numbers $B_1 = \mathbb{Q}$ and irrational numbers $B_2 = \mathbb{R} - \mathbb{Q}$. For any x_0, we have

$$\lim_{x\to x_0,\, x\in\mathbb{Q}} D(x) = \lim_{x\to x_0,\, x\in\mathbb{Q}} 1 = 1, \qquad \lim_{x\to x_0,\, x\notin\mathbb{Q}} D(x) = \lim_{x\to x_0,\, x\notin\mathbb{Q}} 0 = 0.$$

Since the two limit values are different, we know $\lim_{x\to x_0} D(x)$ diverges. Therefore $D(x)$ is discontinuous everywhere.

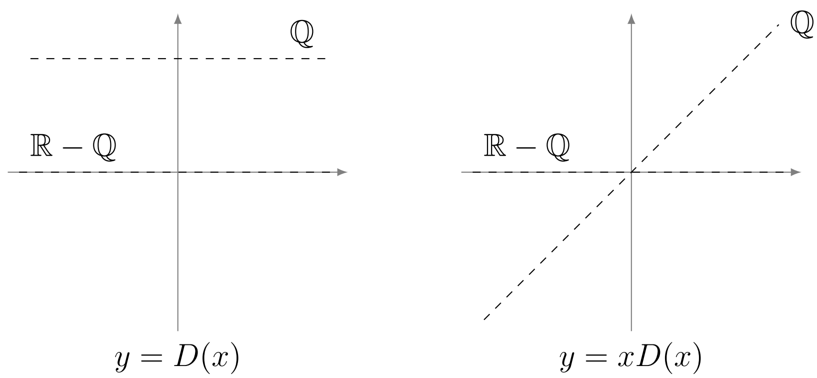


Figure 2.2.4: Dirichlet function.

Now consider

$$xD(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}.$$

We have

$$\lim_{x\to x_0,\, x\in\mathbb{Q}} xD(x) = \lim_{x\to x_0,\, x\in\mathbb{Q}} x = x_0, \qquad \lim_{x\to x_0,\, x\notin\mathbb{Q}} x(x) = \lim_{x\to x_0,\, x\notin\mathbb{Q}} 0 = 0.$$

By the second part of Proposition 2.2.6, $\lim_{x\to x_0} xD(x)$ converges if and only if $x_0 = 0$. Then by $0D(0) = 0$, we know $xD(x)$ is continuous only at 0.

Exercise 2.2.25. Determine all places where $D(x)\sin\frac{1}{x}$ is continuous.

Exercise 2.2.26. Let $f(x), g(x)$ be continuous functions. Find all the places where the following function is continuous

$$\begin{cases} f(x), & \text{if } x \text{ is rational} \\ g(x), & \text{if } x \text{ is irrational} \end{cases}.$$

Example 2.2.12. Consider (n is any nonzero integer)

$$f(x) = x\sin\frac{1}{x}, \quad g(y) = \begin{cases} y, & \text{if } y \neq 0 \\ c, & \text{if } y = 0 \end{cases}, \quad g(f(x)) = \begin{cases} x\sin\frac{1}{x}, & \text{if } x \neq \frac{1}{n\pi} \\ c, & \text{if } x = \frac{1}{n\pi} \end{cases}.$$

By Example 2.1.11, we have $\lim_{x\to 0} f(x) = 0$. We also have $\lim_{y\to 0} g(y) = \lim_{y\to 0} y = 0$.

Let $B_1 = \{\frac{1}{n\pi} : n \text{ is nonzero integer}\}$ and $B_2 = \mathbb{R} - B_1$. Then

$$\lim_{x\to 0,\, x\in B_1} g(f(x)) = \lim_{x\to 0,\, x\in B_1} c = c, \quad \lim_{x\to 0,\, x\in B_2} g(f(x)) = \lim_{x\to 0,\, x\in B_2} x\sin\frac{1}{x} = 0.$$

Therefore $\lim_{x\to 0} g(f(x))$ converges if and only if $c = 0$.

For the case $c \neq 0$, we get a counterexample showing that the extra condition in Proposition 2.1.13 is necessary.

2.2.5 Function Limit and Sequence Limit

Suppose $f(x)$ is continuous at x_0, and $\lim_{n\to\infty} a_n = x_0$. Then we may consider the sequence a_n as a subset $A = \{a_n\}$ near x_0, and Proposition 2.2.6 suggests $\lim_{n\to\infty} f(a_n) = f(x_0)$. Here we use a_n instead of x_n, in order to avoid confusion with x_0.

In fact, the more appropriate viewpoint is to consider a_n as a function $a(n) = a_n : \mathbb{N} \to \mathbb{R}$, and consider $f(a_n)$ as a composition

$$n \xrightarrow{a} x = a_n \xrightarrow{f} f(x) = f(a(n)) = f(a_n).$$

Then by the same argument for the composition rule ($f(x)$ and $g(y)$ in Proposition 2.1.12 are replaced by a_n and $f(x)$), we may prove

$$\lim_{n\to\infty} f(a_n) = f(x_0) = f\left(\lim_{n\to\infty} a_n\right). \tag{2.2.1}$$

In other words, the continuity implies that the function and the limit can be exchanged. The property can be compared with (2.1.4). The following says the converse is also true.

Proposition 2.2.7. *$f(x)$ is continuous at x_0, if and only if*

$$\lim_{n\to\infty} a_n = x_0 \implies \lim_{n\to\infty} f(a_n) = f(x_0).$$

If the constant approximation does not contain x_0, i.e., we have the limit $\lim_{x\to x_0} f(x) = l$, then a_n should not take the value x_0. This is similar to the first extra condition in Proposition 2.1.13, and we have the similar result.

Proposition 2.2.8. $\lim_{x\to x_0} f(x) = l$ *if and only if*

$$\lim_{n\to\infty} a_n = x_0 \text{ and } a_n \neq x_0 \text{ for sufficiently large } n \implies \lim_{n\to\infty} f(a_n) = l.$$

The assumption $a_n \neq x_0$ on the left can also be replaced by that a_n is a *non-repetitive* sequence: $m \neq n$ implies $a_m \neq a_n$.

Exercise 2.2.27. Prove the necessary part of Proposition 2.2.7. In other words, constant approximation implies the function and the limit can be exchanged.

Exercise 2.2.28. State the one sided versions of Propositions 2.2.7 and 2.2.8.

The restriction of function limit to sequences can be used to argue the sequence limit. For example, by the continuity of $(1+x)^p$, we get

$$\lim_{n\to\infty} (1+\sqrt[n]{a})^p = (1+1)^p = 2^p.$$

By $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (Example 2.1.12), we get

$$\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1, \quad \lim_{n\to\infty} \sqrt{n}\sin\frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = 1.$$

By $\lim_{x\to 0^+} x^x = 1$ (Example 2.2.8), we get

$$\lim_{n\to 0} (1+\sqrt{n})^{\frac{1}{1+\sqrt{n}}} = \frac{1}{\lim_{n\to 0}(\frac{1}{1+\sqrt{n}})^{\frac{1}{1+\sqrt{n}}}} = \frac{1}{1} = 1.$$

By $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$ (Example 2.2.9), we get

$$\lim_{n\to 0}\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = e.$$

The restriction of function limit to sequences can be used to argue the divergence like the examples in Section 2.2.4. For example, the divergence in Example 2.2.10

can be argued as follows. The sequences $x_n = \frac{1}{(2n+\frac{1}{2})\pi}$ and $y_n = \frac{1}{(2n-\frac{1}{2})\pi}$ satisfy $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, and $x_n, y_n \neq 0$. On the other hand, we have

$$\begin{aligned} \sin\frac{1}{x_n} &= 1, & \lim_{n\to\infty}\sin\frac{1}{x_n} &= 1, \\ \sin\frac{1}{y_n} &= -1, & \lim_{n\to\infty}\sin\frac{1}{y_n} &= -1. \end{aligned}$$

Since the two limit values are different, we know $\lim_{x\to 0}\sin\frac{1}{x}$ diverges.

Propositions 2.2.7 and 2.2.8 also says that many sequence limit properties obtained in Chapter 1 already imply the corresponding function limit. For example, we proved $\lim_{n\to\infty} x_n = l > 0$ implies $\lim_{n\to\infty}\sqrt{x_n} = \sqrt{x_n}$ Example 1.2.11. This means exactly $\sqrt{x}$ is continuous at any $x_0 > 0$. In fact, the argument in Example 2.1.6 and Exercise 2.1.3 is the same as the argument in Example 1.2.11.

We obtain the following in Chapter 1

$$\begin{aligned} \lim_{n\to\infty} x_n = l \neq 0 &\implies \lim_{n\to\infty}\frac{1}{x_n} = \frac{1}{l}, \text{ (Example 1.2.12)} \\ \lim_{n\to\infty} x_n = l &\implies \lim_{n\to\infty} a^{x_n} = a^l. \text{ (Example 1.2.13 and Exercises 1.2.15, 1.2.16)} \\ \lim_{n\to\infty} x_n = l > 0 &\implies \lim_{n\to\infty} x_n^p = l^p. \text{ (Example 1.1.23 and Exercises 1.1.28, 1.1.64)} \end{aligned}$$

By Proposition 2.2.7, these imply $\frac{1}{x}$, a^x, x^p are continuous wherever they are defined.

Finally, Example 1.6.2 says

$$\lim_{n\to\infty} x_n = k > 0 \text{ and } \lim_{n\to\infty} y_n = l \implies \lim_{n\to\infty} x_n^{y_n} = k^l.$$

The property suggests that x^y is a two variable continuous function wherever it is defined. In fact, you may follow the same argument to prove

$$\lim_{x\to x_0} f(x) = k > 0 \text{ and } \lim_{x\to x_0} g(x) = l \implies \lim_{x\to x_0} f(x)^{g(x)} = k^l.$$

2.3 Infinity

The constant approximation at x_0 is the behaviour of $f(x)$ for x near x_0, measured by $|x - x_0| \le \delta$. The constant approximation at the infinity is the behaviour for large x, measured by $|x| > B$.

Definition 2.3.1. A function $f(x)$ defined for sufficiently large x converges to l, and denoted $\lim_{x\to\infty} f(x) = l$, if for any $\epsilon > 0$, there is B, such that

$$|x| \ge B \implies |f(x) - l| \le \epsilon.$$

The function is not defined at the infinity, and it does not make sense to say the function is continuous at the infinity. We have only the limit version of the constant approximation at the infinity.

We can split the infinity into the positive and negative infinities, similar to the right and left limits.

Definition 2.3.2. A function $f(x)$ defined for sufficiently large $x > 0$ converges to l, and denoted $\lim_{x\to+\infty} f(x) = l$, if for any $\epsilon > 0$, there is B, such that

$$x \ge B \implies |f(x) - l| \le \epsilon.$$

A function $f(x)$ defined for sufficiently large $x < 0$ converges to l, and denoted $\lim_{x\to-\infty} f(x) = l$, if for any $\epsilon > 0$, there is B, such that

$$x \le B \implies |f(x) - l| \le \epsilon.$$

The limit at the infinity has all the usual properties of the limit at a finite x_0. The proofs are also similar. In particular, we have

$$\lim_{x\to\infty} f(x) = l \iff \lim_{x\to+\infty} f(x) = l \text{ and } \lim_{x\to-\infty} f(x) = l. \tag{2.3.1}$$

Moreover, the change of variable $y = \frac{1}{x}$ converts $\lim_{x\to\infty}$ to $\lim_{y\to 0}$, and we have

$$\lim_{x\to\infty} f(x) = l \iff \lim_{x\to 0} f(\tfrac{1}{x}) = l. \tag{2.3.2}$$

Similar to Section 1.4 for sequences, we may also define divergence to the infinity for functions, by replacing $|f(x) - l| \le \epsilon$ with $|f(x)| \ge B$.

Definition 2.3.3. A function $f(x)$ defined near x_0 diverges to ∞, and denoted $\lim_{x\to x_0} f(x) = \infty$, if for any B, there is $\delta > 0$, such that

$$0 < |x - x_0| < \delta \implies |f(x)| > B.$$

We may similarly define the variations such as $\lim_{x\to x_0} f(x) = +\infty$, $\lim_{x\to x_0^+} f(x) = \infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = +\infty$, etc. The properties in Section 1.4 are still valid. For example, the function version of Example 1.4.3 and Exercise 1.4.1 is

$$\lim_{x\to x_0} f(x) = 0 \iff \lim_{x\to x_0} \frac{1}{f(x)} = \infty.$$

Here x_0 can be a finite number, one side of a finite number, or infinity.

Exercise 2.3.1. State and prove the sandwich rule for $\lim_{x\to\infty} f(x)$.

Exercise 2.3.2. State and prove the order rule for $\lim_{x\to+\infty} f(x)$.

Exercise 2.3.3. Prove (2.3.2). Then explain

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e, \quad \lim_{x\to+\infty} x^{\frac{1}{x}} = 1, \quad \lim_{x\to\infty} x\sin\frac{1}{x} = 1.$$

Exercise 2.3.4. Prove $\lim_{x\to+\infty} f(x) = l$ if and only if $\lim_{x\to\infty} f(x^2) = l$.

Exercise 2.3.5. Determine limits.

1. $\lim_{x\to\infty} \frac{x}{x+2}$.
2. $\lim_{x\to 2} \frac{x}{x+2}$.
3. $\lim_{x\to -2} \frac{x}{x+2}$.
4. $\lim_{x\to\infty} \frac{x^2-1}{x^2+x-2}$.
5. $\lim_{x\to 0} \frac{x^2-1}{x^2+x-2}$.
6. $\lim_{x\to 1} \frac{x^2-1}{x^2+x-2}$.

Example 2.3.1. For any $\epsilon > 0$, we have

$$|x| \geq \frac{1}{\sqrt{\epsilon}} \implies \frac{1}{x^2} \leq \frac{1}{B^2} = \epsilon.$$

This proves $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

For any $B > 0$, we have

$$x \geq B^2 \implies \sqrt{x} \leq \sqrt{B^2} = B.$$

In general, we have

$$\lim_{x\to+\infty} x^p = \begin{cases} +\infty, & \text{if } p > 0 \\ 1, & \text{if } p = 0 \\ 0, & \text{if } p < 0 \end{cases}.$$

Exercise 2.3.6. Explain the limit

$$\lim_{x\to\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0} = \begin{cases} \infty, & \text{if } m > n, \\ \frac{a_m}{b_n}, & \text{if } m = n,\ b_n \neq 0, \\ 0, & \text{if } m < n,\ b_n \neq 0. \end{cases}$$

For the divergence to infinity, further discuss the sign of infinity for $\lim_{x\to+\infty}$ and $\lim_{x\to-\infty}$.

Exercise 2.3.7. Determine $\lim_{x\to\infty} f(x)$. Further determine $\lim_{x\to+\infty} f(x)$ and/or $\lim_{x\to-\infty} f(x)$ if there is ambiguity about $\lim_{x\to\infty} f(x)$.

1. $\frac{x}{x+2}$.
2. $\frac{|x|}{x+2}$.
3. $\frac{|x|}{|x+2|}$.
4. $\frac{|x-1|}{x^2-x+1}$.
5. $\frac{(|x|-1)^2(x-2)}{(x^2+1)(|x|+2)}$.
6. $\frac{x+\sin x}{x+\cos x}$.
7. $\frac{x^2+\sin x}{x+\cos x}$.
8. $\frac{x^2+\sin x}{(x+\cos x)^2}$.
9. $\frac{\cos x}{x}$.
10. $\frac{x\cos x}{x^2+2}$.
11. $\frac{x^2\cos x}{x+2}$.
12. $\frac{x^2+|x|\cos 2x}{x^3-x+\sin x}$.

Exercise 2.3.8. Determine $\lim_{x\to+\infty} f(x)$.

1. $\frac{|x+a|}{x+b}$.
2. $\frac{\sqrt{x}+2}{\sqrt{x}-3}$.
3. $\frac{\sqrt{x}+a}{x+b}$.
4. $\frac{2\sqrt{x}-3x+2}{3\sqrt{x}-4x+1}$.
5. $\frac{2}{\sqrt{x}+\sin\sqrt{x}}$.
6. $\frac{\sqrt{x}}{x-\sqrt{x}\sin x+3}$.
7. $\frac{\sqrt[3]{x}+b}{\sqrt{x}+a\sqrt[3]{x}+b\sqrt[4]{x}}$.
8. $\frac{(a\sqrt[3]{x}+b)^2}{(\sqrt{x}+d)^3}$.

Exercise 2.3.9. Determine $\lim_{x\to 0^+} x^p$ for all the possible p.

Exercise 2.3.10. Determine limits.

1. $\lim_{x\to\frac{\pi}{2}^-} \tan x$.
2. $\lim_{x\to-\frac{\pi}{2}^+} \tan x$.
3. $\lim_{x\to 0^-} \frac{\sin x}{x(\pi-x)}$.
4. $\lim_{x\to\pi^-} \frac{\sin x}{x(\pi-x)}$.
5. $\lim_{x\to 0^+} x^p \sin x^q$.
6. $\lim_{x\to\frac{\pi}{2}^-} (\tan x)^p$.

Example 2.3.2. For $0 < a < 1$, we know $\lim_{n\to\infty} na^n = 0$. The corresponding function version is $\lim_{x\to+\infty} xa^x = 0$. In fact, we also know $\lim_{n\to\infty}(n+1)a^n = 0$, and we may use this limit to prove $\lim_{x\to+\infty} xa^x = 0$.

Since $\lim_{n\to\infty}(n+1)a^n = 0$, for any $\epsilon > 0$, there is a natural number N, such that

$$n > N \implies (n+1)a^n < \epsilon.$$

Then for $x > N+1$, there is a natural number $n > N$ satisfying $n \le x \le n+1$. Then by $0 < a < 1$, we have

$$0 < xa^x \le (n+1)a^n < \epsilon.$$

This proves $\lim_{x\to+\infty} xa^x = 0$.

In general, we have $\lim_{n\to\infty} n^p a^n = 0$ and $\lim_{n\to\infty}(n+1)^p a^n = 0$ for any $0 < a < 1$ and any p. Then by the same argument, we get

$$\lim_{x\to+\infty} x^p a^x = 0 \text{ for } 0 < a < 1.$$

By taking the reciprocal (and changing p to $-p$, changing a to $\frac{1}{a}$), we get

$$\lim_{x\to+\infty} x^p a^x = +\infty \text{ for } a > 1.$$

Exercise 2.3.11. Use Example 2.3.2 to explain

$$\lim_{x\to+\infty} a^x = \begin{cases} 0, & \text{if } 0 < a < 1 \\ 1, & \text{if } a = 1 \\ +\infty, & \text{if } a > 1 \end{cases}, \quad \lim_{x\to-\infty} a^x = \begin{cases} +\infty, & \text{if } 0 < a < 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } a > 1 \end{cases}.$$

Example 2.3.3. Let a, b be constants. We know $x \ge 2|a|$ implies $x + a \ge \frac{x}{2}$. Then $x > \max\{2|a|, \frac{2|a-b|}{\epsilon}\}$ implies

$$|\sqrt{x+a} - \sqrt{x+b}| = \frac{|a-b|}{\sqrt{x+a}+\sqrt{x+b}} \le \frac{|a-b|}{\sqrt{\frac{x}{2}}} \le \epsilon.$$

This proves $\lim_{x\to+\infty}(\sqrt{x+a} - \sqrt{x+b}) = 0$. The proof is very similar to Example 1.2.6.

Exercise 2.3.12. Use the continuity of x^p to find limits.

1. $\lim_{x\to\infty}\sqrt{\frac{x+a}{x+b}}=1$.
2. $\lim_{x\to+\infty}(\sqrt{x^2+ax+b}-\sqrt{x^2+cx+d})$.
3. $\lim_{x\to-\infty}(\sqrt{2-x}-\sqrt{1-x})$.
4. $\lim_{x\to+\infty}(\sqrt[3]{x+a}-\sqrt[3]{x+b})$.

Example 2.3.4. To find $\lim_{x\to+\infty}(\sin\sqrt{x+a}-\sin\sqrt{x+b})$, we use trigonometric identity

$$\sin\sqrt{x+a}-\sin\sqrt{x+b}=2\sin\frac{\sqrt{x+a}-\sqrt{x+b}}{2}\cos\frac{\sqrt{x+a}+\sqrt{x+b}}{2}.$$

By (2.1.1), we have $|\sin y|\le|y|$. Then by $|\cos y|\le 1$, we get

$$|\sin\sqrt{x+a}-\sin\sqrt{x+b}|\le 2\frac{|\sqrt{x+a}-\sqrt{x+b}|}{2}=|\sqrt{x+a}-\sqrt{x+b}|.$$

Then by $\lim_{x\to+\infty}(\sqrt{x+a}-\sqrt{x+b})=0$ and the sandwich rule, we get

$$\lim_{x\to+\infty}(\sin\sqrt{x+a}-\sin\sqrt{x+b})=0.$$

Exercise 2.3.13. Determine limits as $x\to+\infty$.

1. $\cos\sqrt{x+a}-\cos\sqrt{x+b}$.
2. $\tan\sqrt{x+a}-\tan\sqrt{x+b}$.

Exercise 2.3.14. Determine limits.

1. $\lim_{x\to\infty}(1+\frac{a}{x})^x$.
2. $\lim_{x\to\infty}(1+\frac{a}{x})^{bx}$.
3. $\lim_{x\to\infty}(1+\frac{1}{x^2})^x$.
4. $\lim_{x\to+\infty}(1+\frac{1}{x})^{\sqrt{x}}$.
5. $\lim_{x\to-\infty}(1+\frac{1}{x})^{x^2}$.
6. $\lim_{x\to\infty}(\frac{x+a}{x+b})^x$.

Exercise 2.3.15. Determine limits as $x\to+\infty$.

1. $(x+a)^{\frac{1}{x}}$.
2. $(x+a)^{\frac{1}{x+b}}$.
3. $(x^2+ax+b)^{\frac{1}{x}}$.
4. $(x^2+a\sin x)^{\frac{1}{x}}$.
5. $(\sqrt{x}+a)^{\frac{b}{x}}$.
6. $(x+a)^{\frac{b}{\sqrt{x}}}$.

2.4 Property of Continuous Function

A function $f(x)$ is continuous at x_0 if $\lim_{x\to x_0}f(x)=f(x_0)$. In Figure 2.4.1, the function $f(x)$ satisfies $\lim_{x\to x_2}f(x)=f(x_2)$ and $\lim_{x\to x_5}f(x)=f(x_5)$. Therefore the function is continuous at x_2 and x_5. It is not continuous at the other x_i for various reasons. The function diverges at x_1 and x_7 because it has different left and right limits. The function converges at x_3 but the limit is not $f(x_3)$. The function diverges to infinity at x_4. The function diverges at x_6 because the left limit diverges. At all these places, the graph of the function is broken, and we can visibly see a gap. Therefore a function is continuous at x_0 if its graph is "not broken" at x_0.

Exercise 2.4.1. Determine the intervals on which the function is continuous. Is it possible to extend to a continuous function at more points?

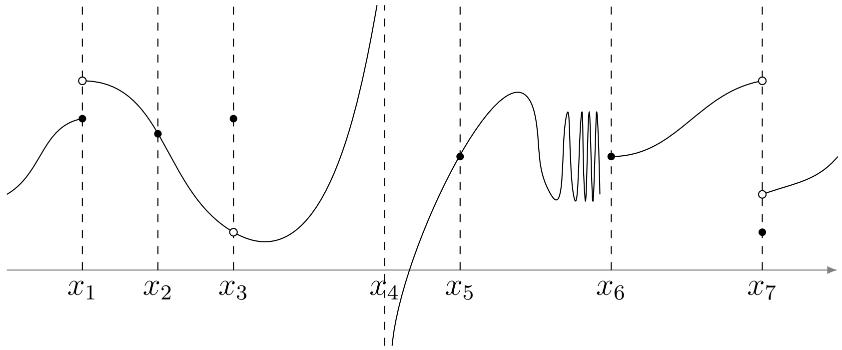


Figure 2.4.1: Continuity and discontinuity.

1. $\frac{x^2-3x+2}{x^2-1}$.
2. $\frac{x^2-1}{x-1}$.
3. $\text{sign}(x)$.
4. $x\sin\frac{1}{x}$.
5. x^x.
6. $\frac{\cos x}{2x-\pi}$.

Exercise 2.4.2. Find a function on $\mathbb{R}$ that is continuous at $1, 2, 3$ and is not continuous at all the other places.

Exercise 2.4.3. Find a function on $\mathbb{R}$ that is not continuous at $1, 2, 3$ and is continuous at all the other places.

Exercise 2.4.4. Find two continuous functions $f(x)$ and $g(x)$, such that $\lim_{x\to 0}\frac{1+f(0)g(x)}{1+f(x)g(0)}$ converges but the value is not 1.

2.4.1 Intermediate Value Theorem

If we start at the sea level and climb to the mountain top of 1000 meters, then we will be at 500 meters somewhere along the way, and will be at 700 meters some other place. This is the intuition behind the following result.

Theorem 2.4.1 (Intermediate Value Theorem). *If $f(x)$ is continuous on $[a, b]$, then for any number γ between $f(a)$ and $f(b)$, there is $c \in [a, b]$ satisfying $f(c) = \gamma$.*

Example 2.4.1. The polynomial $f(x) = x^3 - 3x + 1$ is continuous and satisfies $f(0) = 1, f(1) = -1$. Therefore $f(x)$ must attain value 0 somewhere on the interval $(0, 1)$. In other words, the polynomial has at least one root on the interval $(0, 1)$.

To find more precise location of the root, we may try to evaluate the function at $0.1, 0.2, \dots, 0.9$ and find $f(0.3) = 0.727, f(0.4) = -0.136$. This tells us that $f(x)$ has a root on the interval $(0.3, 0.4)$.

The discussion can be summarized as follows: If $f(x)$ is continuous on $[a, b]$, and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root in (a, b).

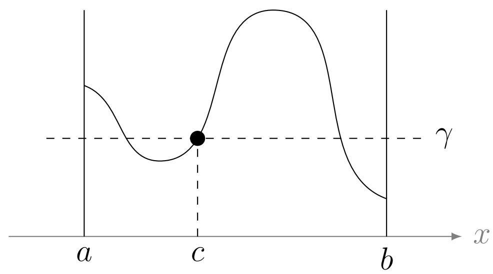


Figure 2.4.2: Intermediate value theorem.

Example 2.4.2. We have $\lim_{x\to+\infty}(x^3-3x+1)=+\infty$ and $\lim_{x\to-\infty}(x^3-3x+1)=-\infty$. Therefore for sufficiently large $b>0$, we have $f(b)>0$ and $f(-b)<0$. By applying the intermediate value theorem to the function on the interval $[-b,b]$, the polynomial has a root on $(-b,b)$.

In general, any odd order polynomial has at least one real root.

In Example 2.4.1, we already know that $f(x)=x^3-3x+1$ has at least one root on $(0,1)$. In fact, by $f(-b)<0, f(0)>0, f(1)<0, f(b)>0$, we know f has at least one root on each of the intervals $(-b,0)$, $(0,1)$, $(1,b)$. Since a polynomial of order 3 has at most three roots, we conclude that $f(x)$ has exactly one root on each of the three intervals.

Exercise 2.4.5. Show the equations have solution on the interval. Can you narrow down the solution to an interval of length 0.2?

1. $2x^3+x=3x^2+2$, on $[1,2]$.
2. $e^x=\cos x$, on $[-1,-2]$.
3. $x^2=\sin x$, on $[0.5,1.5]$.
4. $x^x=x^2+1$, on $[2,3]$.

Example 2.4.3. The function

$$f(x)=\begin{cases}x^2, & \text{if } x>1\\ x-1, & \text{if } x\le 1\end{cases}$$

satisfies $f(-1)=-2, f(1)=1$, but does not take any number in $(0,1]$ as value. The problem is that the function is not continuous at 1, where a jump in value misses the numbers in $(0,1]$. Therefore the intermediate value theorem cannot be applied.

Example 2.4.4. Since

$$\lim_{x\to-\frac{\pi}{2}^+}\tan x=-\infty,\quad \lim_{x\to\frac{\pi}{2}^-}\tan x=+\infty,$$

for any number γ, we can find $a>-\frac{\pi}{2}$ and very close to $-\frac{\pi}{2}$, such that $\tan a<\gamma$. We can also find $b<\frac{\pi}{2}$ and very close to $\frac{\pi}{2}$, such that $\tan b>\gamma$. Then $\tan x$ is continuous on $[a,b]$, and $\tan a < \gamma < \tan b$. This implies that $\gamma = \tan c$ for some $c \in (a,b)$. Therefore any number is the tangent of some angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The example shows that, if $f(x)$ is continuous on (a,b), and satisfies $\lim_{x\to a^+} f(x) = -\infty$ and $\lim_{x\to b^-} f(x) = +\infty$, then $f(x)$ can take any number as value on (a,b). Note that the interval (a,b) here does not even have to be bounded.

Exercise 2.4.6. Find all the possible values.

1. $\frac{x^2-3x+2}{x^2-1}$.
2. x^x.
3. $\sin x$.
4. $\sin\frac{1}{x}$.
5. $\begin{cases} 2^x, & \text{if } -1 \le x \le 0 \\ x^2+3, & \text{if } 0 < x \le 1 \end{cases}$.

Exercise 2.4.7. For $a > 1$ or $0 < a < 1$, show that the values of a^x form the interval $(0,+\infty)$.

Exercise 2.4.8. Explain the divergence.

1. $\lim_{x\to+\infty} \cos(\sqrt{x+2}+\sqrt{x})$.
2. $\lim_{x\to+\infty} \sqrt{x}(\sin\sqrt{x+2} - \sin\sqrt{x})$.

Exercise 2.4.9. Suppose a function $f(x)$ is continuous on $(a,b]$, and $\lim_{x\to a^+} f(x) = \alpha > f(b)$. Prove that for any number γ satisfying $f(b) \le \gamma < \alpha$, there is $a < c \le b$, such that $f(c) = \gamma$.

Exercise 2.4.10. Intervals are subsets I of $\mathbb{R}$ characterised by the property

$$a < c < b \text{ and } a,b \in I \implies c \in I.$$

Using this characterisation, prove that if $f(x)$ is continuous on an interval I, then the image $f(I) = \{f(x) \colon x \in I\}$ is an interval.

2.4.2 Inverse Continuous Function

A function $f(x)$ is *invertible* if $f(x) = y$ has unique solution $x = g(y)$. We call g the inverse of f, and denote $g = f^{-1}$.

Example 2.4.5. The function $y = f(x) = ax + b$ is invertible if and only if $a \ne 0$. By solving x for y, we get the inverse function $x = f^{-1}(y) = \frac{y-b}{a}$.

Example 2.4.6. To find the invertibility of $f(x) = x^2$, we try to solve $y = x^2$. For the equation to have solution, we need $y \ge 0$. This means we should restrict the *range* (i.e., the set of possible y) of f to the non-negative numbers $Y = [0,+\infty)$.

Next, for $y > 0$, there are exactly two x satisfying $x^2 = y$. In fact, the two solutions are negative of each other. In order to get the unique solution, we may restrict the *domain* (i.e., the set of possible x) of f to the non-negative numbers

$X = [0, +\infty)$. An alternative is to restrict the domain to the non-positive numbers $(-\infty, 0]$.

After the restrictions, we get the function with the specified domain and range

$$f(x) = x^2 \colon X = [0, +\infty) \to Y = [0, +\infty).$$

For any $y = Y$, the solution of $y = x^2$ for $x \in X$ is the unique $x = \sqrt{y}$. Then we get the inverse function

$$f^{-1}(y) = \sqrt{y} \colon Y = [0, +\infty) \to X = [0, +\infty).$$

Exercise 2.4.11. Explain $f(x) = x^2 \colon (-\infty, 0] \to [0, +\infty)$ is invertible. What is the inverse function?

The example shows that the invertibility depends on the choice of the domain and the range, and invertibility is equivalent to the following two properties:

- f is *onto*: For any $y \in Y$, there is $x \in X$ satisfying $f(x) = y$. In other words, the solution exists.
- f is *one-to-one*: If $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$, then $x_1 = x_2$. In other words, the solution is unique.

The onto property can always be achieved by taking Y to be all the values of f

$$Y = f(X) = \{f(x) \colon x \in X\}.$$

For continuous single variable function $f(x)$ on an interval I, by the intermediate value theorem (see Exercise 2.4.10), we know $J = f(I)$ is also an interval. Therefore we get an onto function $f \colon I \to J$ between two intervals.

For a single variable function $f(x)$, the one-to-one property is usually achieved by the strictly monotone property. A function $f(x)$ is *strictly increasing* if

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

For such a function, we have

$$\begin{aligned} x_1 \neq x_2 &\implies x_1 < x_2 \text{ or } x_1 > x_2 \\ &\implies f(x_1) < f(x_2) \text{ or } f(x_1) > f(x_2) \\ &\implies f(x_1) \neq f(x_2). \end{aligned}$$

The implication is equivalent to (the *contrapositive*)

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

This is the one-to-one property.

Similarly, a function $f(x)$ is *strictly decreasing* if

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

Then similar argument also shows $f(x)$ is one-to-one.

A function is *strictly monotone* if it is either strictly increasing or strictly decreasing. Strictly monotone functions are one-to-one.

Theorem 2.4.2. *Suppose f is a continuous and strictly monotone function on an interval I. Then $J = f(I)$ is also an interval, and $f\colon I \to J$ is invertible. Moreover, $f^{-1}\colon J \to I$ is also continuous and strictly monotone.*

The following explains the continuity of f^{-1}. The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are related by the exchange of x and y. In other words, one is the flip of the other with respect to the diagonal line of the cartesian plane. In particular, if $f(x)$ is continuous, then its graph is not broken. This implies the flip of the graph is also not broken. In other words, the inverse function $f^{-1}(x)$ is also continuous.

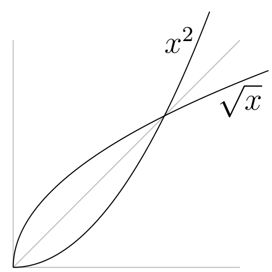


Figure 2.4.3: Graphs of x^2 and $\sqrt{x}$, for $x \ge 0$.

Example 2.4.7. The function $\sin\theta$ is the y-coordinate of the point of angle θ on the unit circle. The range of the y-coordinate is $[-1, 1]$. In other words, for any $y \in [-1, 1]$, the equation $y = \sin\theta$ has solution.

To get unique solution, we notice that each height $y \in [-1, 1]$ corresponds to a unique point on the right half of the circle. The right half circle corresponds to $\theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Therefore the function $\sin\theta\colon [-\frac{1}{2}\pi, \frac{1}{2}\pi] \to [-1, 1]$ is invertible. The inverse is

$$\theta = \arcsin y\colon [-1, 1] \to [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi].$$

By the continuity of $\sin\theta$ and Theorem 2.4.2, we know $\arcsin y$ is also continuous.

By $\cos\theta = \sin(\frac{1}{2}\pi - \theta)$, we have the composition

$$\cos\theta\colon \quad \theta \xrightarrow{\frac{1}{2}\pi-\theta} \rho = \tfrac{1}{2}\pi - \theta \xrightarrow{\sin\rho} y = \sin\rho = \cos\theta.$$

Here the range $\rho \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ corresponds to $\theta \in [0, \pi]$. Then we take the composition of inverses and get

$$\arccos y\colon \quad \theta = \tfrac{1}{2}\pi - \rho = \tfrac{1}{2}\pi - \arcsin y \xleftarrow{\frac{1}{2}\pi-\rho} \rho = \arcsin y \xleftarrow{\arcsin y} y.$$

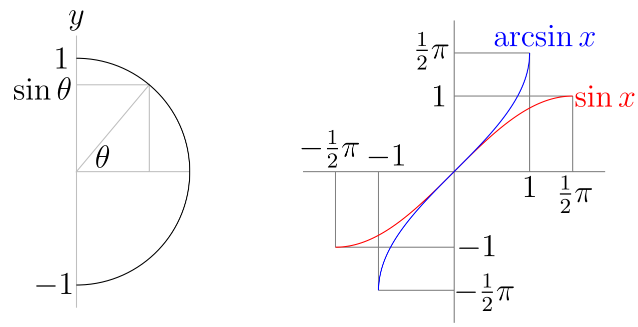


Figure 2.4.4: Inverse sine.

Therefore $\arccos y\colon [-1,1] \to [0,\pi]$ is the inverse of $\cos\theta\colon [0,\pi] \to [-1,1]$. Moreover, we have

$$\arccos y + \arcsin y = \tfrac{1}{2}\pi.$$

Exercise 2.4.12. Directly explain that $\cos\theta\colon [0,\pi] \to [-1,1]$ is onto and one-to-one. Therefore the function is invertible.

Example 2.4.8. In Example 2.4.4, we know the tangent function

$$y = \tan\theta\colon (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) \to (-\infty, +\infty)$$

is onto. It is also strictly increasing, and therefore one-to-one. Then by the continuity of $\tan\theta$ on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and Theorem 2.4.2, we get the inverse function

$$\theta = \arctan y\colon (-\infty, +\infty) \to (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi).$$

which is also continuous and strictly increasing.

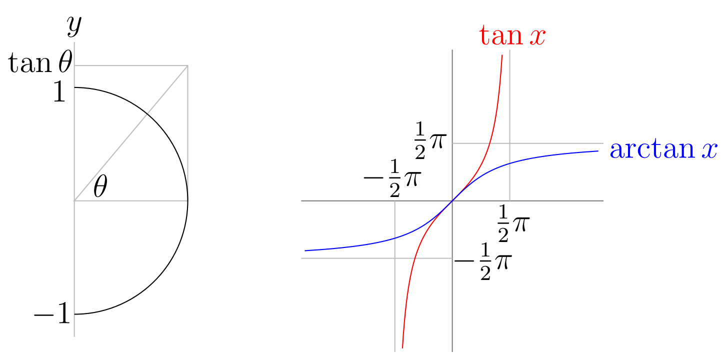


Figure 2.4.5: Inverse tangent.

We argue that

$$\lim_{y\to+\infty} \arctan y = \tfrac{\pi}{2}, \quad \lim_{y\to-\infty} \arctan y = -\tfrac{\pi}{2}.$$

Since $\arctan y$ is strictly increasing, for any $\epsilon > 0$, we have

$$y > B = \tan(\tfrac{\pi}{2} - \epsilon) \implies \tfrac{\pi}{2} > \arctan y > \tfrac{\pi}{2} - \epsilon.$$

This verifies $\lim_{y\to+\infty} \arctan y = \frac{\pi}{2}$. The other limit is similar.

We may also take the inverse of

$$\cot\theta\colon (0,\pi) \to (-\infty,+\infty),$$

and get

$$\text{arccot } y\colon (-\infty,+\infty) \to (0,\pi).$$

Similar to Example 2.4.7, we get

$$\text{arccot } y + \arctan y = \tfrac{1}{2}\pi.$$

Exercise 2.4.13. Find the inverse of $\sin x$ on $[\frac{1}{2}\pi, \frac{3}{2}\pi]$, and the inverse of $\tan x$ on $(-\frac{3}{2}\pi, -\frac{1}{2}\pi)$.

Exercise 2.4.14. Show that the function $f(x) = x^5 + 3x^3 + 1\colon \mathbb{R} \to \mathbb{R}$ is invertible.

Exercise 2.4.15. Suppose $f(x)\colon (a,b) \to (\alpha,\beta)$ is strictly increasing and onto.

1. Prove that $\lim_{x\to a^+} = \alpha$ and $\lim_{x\to b^-} = \beta$.
2. Prove that $\lim_{y\to\alpha^+} f^{-1}(y) = a$ and $\lim_{y\to\beta^-} f^{-1}(y) = b$.

What if $f(x)$ is strictly decreasing?

Example 2.4.9. For $a > 1$, the exponential function

$$a^x\colon \mathbb{R} \to (0,+\infty)$$

is strictly increasing and continuous. By Exercise 2.4.7, the function is also onto. Therefore by Theorem 2.4.2, the exponential function has an inverse

$$\log_a x : (0,+\infty) \to \mathbb{R},$$

called the *logarithmic function* with base a. Like the exponential function, the logarithmic function is also strictly increasing and continuous. Similar to Example 2.4.8 (see Exercise 2.4.15), we can also show

$$\lim_{x\to+\infty} \log_a x = +\infty, \quad \lim_{x\to 0^+} \log_a x = -\infty, \quad \text{for } a > 1.$$

The logarithm $\log_e x$ based on the special value e is called the *natural logarithm.* We will denote the natural logarithm simply by $\log x$.

For $0 < a < 1$, the exponential a^x is strictly decreasing and continuous. The corresponding logarithm can be similarly defined and is also strictly decreasing and continuous. Moreover, we have

$$\lim_{x\to+\infty} \log_a x = -\infty, \quad \lim_{x\to 0^+} \log_a x = +\infty, \quad \text{for } 0 < a < 1.$$

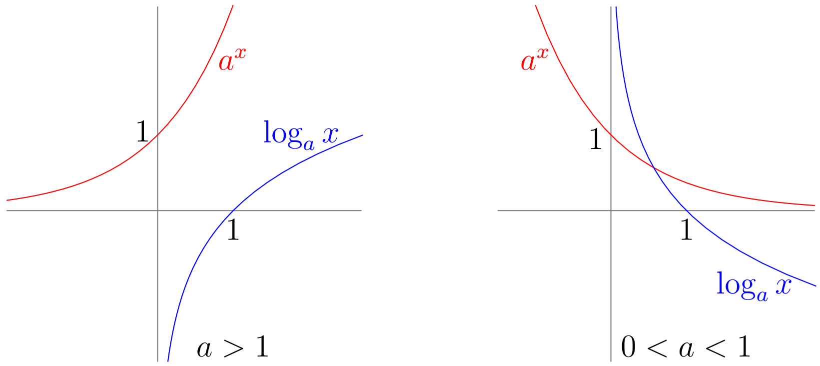


Figure 2.4.6: Exponential and logarithm.

Exercise 2.4.16. Let

$$x_n = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \log n, \quad y_n = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} - \log(n+1).$$

1. Use Exercise 1.3.21 to prove that $\frac{1}{n+1} < \log\left(1 + \frac{1}{n}\right) < \frac{1}{n}$.
2. Prove that x_n is strictly decreasing and y_n is strictly increasing.
3. Prove that both x_n and y_n converge to the same limit

$$\lim_{n\to\infty}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n} - \log n\right) = 0.577215669015328\cdots.$$

The number is called the *Euler*[2]*-Mascheroni*[3] *constant*.

Example 2.4.10. The continuity of the logarithm implies the continuity of the function $x^x = e^{x\log x}$ on $(0,\infty)$. In general, if $f(x)$ and $g(x)$ are continuous and $f(x) > 0$, then $f(x)^{g(x)} = e^{g(x)\log f(x)}$ is continuous.

The continuity of the exponential and the logarithm can also be used to prove that

$$\lim_{n\to\infty} x_n = k > 0,\ \lim_{n\to\infty} y_n = l \implies \lim_{n\to\infty} x_n^{y_n} = k^l.$$

The reason is that the continuity of log implies $\lim_{n\to\infty}\log x_n = \log k$. Then the arithmetic rule implies $\lim_{n\to\infty} y_n \log x_n = l\log k$. Finally, the continuity of the exponential implies

$$\lim_{n\to\infty} x_n^{y_n} = \lim_{n\to\infty} e^{y_n\log x_n} = e^{\lim_{n\to\infty} y_n \log x_n} = e^{l\log k} = k^l.$$

[2]Leonhard Paul Euler, born 1707 in Basel (Switzerland), died 1783 in St. Petersburg (Russia). Euler is one of the greatest mathematicians of all time. He made important discoveries in almost all areas of mathematics. Many theorems, quantities, and equations are named after Euler. He also introduced much of the modern mathematical terminology and notation, including $f(x)$, e, Σ (for summation), i (for $\sqrt{-1}$), and modern notations for trigonometric functions.

[3]Lorenzo Mascheroni, born 1750 in Lombardo-Veneto (now Italy), died 1800 in Paris (France). The Euler-Mascheroni constant first appeared in a paper by Euler in 1735. Euler calculated the constant to 6 decimal places in 1734, and to 16 decimal places in 1736. Mascheroni calculated the constant to 20 decimal places in 1790.

See Example 1.6.2.

2.4.3 Continuous Change of Variable

In Section 2.1.4, we showed that the limits can be translated under continuous change of variables. A continuous change of variable is the same as continuous and invertible function, and we know the inverse function is automatically continuous.

Example 2.4.11. In Example 2.1.12, we get

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

By the continuous change of variable $y=\sin x$, $x=\arcsin y$, at $x_0=0$ and $y_0=0$, we get

$$\lim_{y\to 0}\frac{y}{\arcsin y}=\lim_{x\to 0}\frac{\sin x}{x}=1.$$

This is the same as

$$\lim_{y\to 0}\frac{\arcsin y}{y}=1.$$

By the same argument, we also get

$$\lim_{y\to 0}\frac{\arctan y}{y}=1.$$

Exercise 2.4.17. Find the limits.

1. $\lim_{x\to a}\frac{\arcsin x-\arcsin a}{x-a}$.
2. $\lim_{x\to a}\frac{\arctan x-\arctan a}{x-a}$.

Example 2.4.12. In Example 2.2.9, we get

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}=e.$$

We apply the natural logarithm $\log y$ to the limit. Since the logarithm is continuous, we can exchange the limit and the logarithm

$$\lim_{x\to 0}\frac{\log(x+1)}{x}=\lim_{x\to 0}\log(1+x)^{\frac{1}{x}}=\log\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)=\log e=1.$$

Exercise 2.4.18. Find the limits.

1. $\lim_{x\to 0}\frac{\log_a(x+1)}{x}$.
2. $\lim_{x\to a}\frac{\log x-\log a}{x-a}$.
3. $\lim_{x\to a}\frac{\log_b x-\log_b a}{x-a}$.

Exercise 2.4.19. Find the limits.

1. $\lim_{x\to 2}\frac{\log(x^2-2x+1)}{x^2-4}$.
2. $\lim_{x\to 1}\frac{\log(x^2-2x+1)}{x^2-1}$.
3. $\lim_{x\to 0}\frac{\log(x^2-2x+1)}{x}$.

Exercise 2.4.20. Find the limits.

1. $\lim_{x\to 0}\frac{1}{x}\log\frac{ax+c}{bx+c}$.
2. $\lim_{x\to 0}\frac{1}{x}\log\frac{ax+b}{cx+d}$.
3. $\lim_{x\to 0}\frac{1}{x}\log\frac{a_2x^2+a_1x+a_0}{b_2x^2+b_1x+b_0}$.
4. $\lim_{x\to\infty}x\log\frac{ax+b}{ax+c}$.
5. $\lim_{x\to\infty}x\log\frac{ax+b}{cx+d}$.
6. $\lim_{x\to\infty}x\log\frac{a_2x^2+a_1x+a_0}{b_2x^2+b_1x+b_0}$.

Exercise 2.4.21. Find the limits.

1. $\lim_{x\to 0}\frac{x}{\log(ax+1)}$.
2. $\lim_{x\to 1}\frac{x^2-1}{\log x}$.
3. $\lim_{x\to 0}\frac{\log(x^2+3x+2)-\log 2}{x}$.
4. $\lim_{x\to 1}\frac{\log x}{\sin \pi x}$.

Example 2.4.13. We have the continuous change of variable $y = e^x - 1$ and $x = \log(y+1)$. We apply the change to the limit in Example 2.4.12 to get

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{y\to 0}\frac{y}{\log(y+1)} = 1.$$

Exercise 2.4.22. Find the limits.

1. $\lim_{x\to 0}\frac{e^{2x}-1}{x}$.
2. $\lim_{x\to 0}\frac{a^x-1}{x}$.
3. $\lim_{x\to a}\frac{e^x-e^a}{x-a}$.
4. $\lim_{x\to b}\frac{a^x-a^b}{x-b}$.
5. $\lim_{x\to\infty}\frac{e^x-1}{x}$.
6. $\lim_{x\to 0}\frac{a^x-b^x}{x}$.
7. $\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$.
8. $\lim_{x\to +\infty}\frac{e^{ax}-e^{bx}}{x}$.
9. $\lim_{x\to 0}\frac{a^x-b^x}{c^x-d^x}$.

Example 2.4.14. For $p \neq 0$, we have the continuous change of variable $y = p\log(x+1)$. Then $(x+1)^p = e^y$, and by Examples 2.4.12 and 2.4.13, we get

$$\begin{aligned}\lim_{x\to 0}\frac{(x+1)^p-1}{x} &= \lim_{x\to 0}\frac{e^y-1}{x} = \lim_{x\to 0}\frac{e^y-1}{y}\,\frac{p\log(x+1)}{x}\\ &= p\lim_{y\to 0}\frac{e^y-1}{y}\lim_{x\to 0}\frac{\log(x+1)}{x} = p.\end{aligned}$$

Exercise 2.4.23. Find the limits.

1. $\lim_{x\to 1}\frac{x^p-1}{x-1}$.
2. $\lim_{x\to 1}\frac{x^p-1}{x^q-1}$.
3. $\lim_{x\to a}\frac{x^p-a^p}{x-a}$.
4. $\lim_{x\to a}\frac{x^p-a^p}{x^q-a^q}$.
5. $\lim_{x\to a}\frac{\sin x^p-\sin a^p}{x-a}$.
6. $\lim_{x\to a}\frac{e^{x^p}-e^{a^p}}{b^x-b^a}$.

2.5 Multivariable

2.5.1 Multivariable Function

The constant approximation can be easily extended to multivariable functions. We simply need to change $|x - x_0|$ in the definition to $\|\vec{x} - \vec{x}_0\|$, where $\|\cdot\|$ is a norm in Section 1.6.

Definition 2.5.1. A multivariable function $f(\vec{x})$ defined around $\vec{x}_0$ has constant approximation l at $\vec{x}_0$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|\vec{x} - \vec{x}_0\| \le \delta \implies |f(\vec{x}) - l| \le \epsilon.$$

In this case, we say $f(\vec{x})$ is *continuous* at $\vec{x}_0$.

Similar to the single variable case, we may divide the constant approximation into the case of $\vec{x} = \vec{x}_0$ and $\vec{x} \ne \vec{x}_0$. The definition for the case $\vec{x} = \vec{x}_0$ means $l = f(\vec{x}_0)$, and the definition for the case $\vec{x} \ne \vec{x}_0$ means the limit of the function at $\vec{x}_0$.

Definition 2.5.2. Let $f(\vec{x})$ be a multivariable function defined around $\vec{x}_0$ but not necessarily including $\vec{x}_0$. Then $f(\vec{x})$ converges to l at $\vec{x}_0$, and denoted $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = l$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|\vec{x} - \vec{x}_0\| \le \delta \text{ and } \vec{x} \ne \vec{x}_0 \implies |f(\vec{x}) - l| \le \epsilon.$$

The continuity at $\vec{x}_0$ means $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$.

Strictly speaking, the definitions depend on the choice of the norm. It is easy to see that, if two norms are equivalent, then the definition in terms of one norm is equivalent to the definition in terms of the other norm. We know all L^p-norms are equivalent, and the discussion in this course only refers to the L^p-norm.

Example 2.5.1. Consider $f(x, y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ defined for $(x, y) \ne (0, 0)$. By $|x|, |y| \le \max\{|x|, |y|\} = \|(x, y)\|_\infty$, we get

$$\begin{aligned}\|(x, y)\|_\infty < \delta = \sqrt{\epsilon} &\implies |x|, |y| \le \sqrt{\epsilon} \\ &\implies |f(x, y)| = \frac{|xy||x^2 - y^2|}{x^2 + y^2} \le |x||y| \le (\sqrt{\epsilon})^2 = \epsilon.\end{aligned}$$

Therefore $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. We note that the convergence holds for all L^p-norms.

Exercise 2.5.1. Rigorously argue $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

1. $\frac{x(x^2-y^2)}{x^2+y^2}$.
2. $\frac{xy}{\sqrt{x^2+y^2}}$.
3. $\frac{xy}{\max\{|x|,|y|\}}$.
4. $\frac{xy}{(x^2+y^2)^p}$, $p<1$.
5. $x\cos(x+y)$.
6. $\frac{x^p\sin y}{|x|+|y|}$, $p>0$.

Example 2.5.2. Suppose a single variable function $f(x)$ has constant approximation l at x_0. Then for any $\epsilon>0$, there is $\delta>0$, such that

$$|x-x_0|\le\delta \implies |f(x)-l|\le\epsilon.$$

Consider the single variable function as a two variable function $f(x,y)=f(x)$. By $|x|\le\|(x,y)\|_\infty=\max\{|x|,|y|\}$, we get

$$\|(x-x_0,y-y_0)\|_\infty\le\delta \implies |x-x_0|\le\delta \implies |f(x,y)-l|=|f(x)-l|\le\epsilon.$$

Therefore the two variable function $f(x,y)=f(x)$ has constant approximation l at (x_0,y_0). In other words, a continuous single variable function is also a continuous multivariable function. For example, the basic functions x, y, x^p, y^p, a^x, a^y, $\sin x$, $\sin y$, $\dots$ are continuous (wherever they are defined) as functions of (x,y).

Similarly, if $\lim_{x\to x_0}f(x)=l$, then $\lim_{(x,y)\to(x_0,y_0)}f(x)=l$.

Exercise 2.5.2. If $\lim_{(x,y)\to(x_0,y_0)}f(x,y)=l$, prove that $\lim_{(x,y,z)\to(x_0,y_0,z_0)}f(x,y)=l$ and $\lim_{(x,y,z)\to(u_0,x_0,y_0)}f(y,z)=l$.

The multivariable constant approximation has similar properties as the single variable constant approximation. The following are the limit versions of the usual properties.

Arithmetic: Suppose $\lim_{\vec{x}\to\vec{x}_0}f(\vec{x})=k$ and $\lim_{\vec{x}\to\vec{x}_0}g(\vec{x})=l$. Then

$$\lim_{\vec{x}\to\vec{x}_0}(f(\vec{x})+g(\vec{x}))=k+l,\quad \lim_{\vec{x}\to\vec{x}_0}f(\vec{x})g(\vec{x})=kl,\quad \lim_{\vec{x}\to\vec{x}_0}\frac{f(\vec{x})}{g(\vec{x})}=\frac{k}{l}\ (l\ne 0).$$

Sandwich: Suppose $f(\vec{x})\le g(\vec{x})\le h(\vec{x})$ for sufficiently large n. If $\lim_{\vec{x}\to\vec{x}_0}f(\vec{x})=\lim_{\vec{x}\to\vec{x}_0}h(\vec{x})=l$, then $\lim_{\vec{x}\to\vec{x}_0}g(\vec{x})=l$.

Order: Suppose $\lim_{\vec{x}\to\vec{x}_0}f(\vec{x})=k$ and $\lim_{\vec{x}\to\vec{x}_0}g(\vec{x})=l$.

1. If there is $\delta>0$, such that $0<\|\vec{x}-\vec{x}_0\|\le\delta$ implies $f(\vec{x})\le g(\vec{x})$, then $k\le l$.
2. If $k<l$, then there is $\delta>0$, such that $0<\|\vec{x}-\vec{x}_0\|\le\delta$ implies $f(\vec{x})<g(\vec{x})$.

Bounded: If $\lim_{\vec{x}\to\vec{x}_0}f(\vec{x})$ converges, then $f(\vec{x})$ is bounded near $\vec{x}_0$.

Composition: If $\lim_{t\to t_0}x(t)=x_0$, $\lim_{t\to t_0}y(t)=y_0$, and $f(x,y)$ is continuous at (x_0,y_0), then $\lim_{t\to t_0}f(x(t),y(t))=f(x_0,y_0)$.

The composition rule as stated here corresponds to the second of Proposition 2.1.13. Basically, it says the continuity of $f(x,y)$ means

$$\lim_{t\to t_0} f(x(t),y(t)) = f\left(\lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t)\right).$$

Of course the function f can have more than two variables, and t can also be a multivariable $\vec{t}$. For the fuller version of the composition rule, we need to state the rule in terms of multivariable maps.

For the continuity version of the usual properties, we note that the arithmetic and composition combinations of continuous functions are continuous. Combined with Example 2.5.2, we know the multivariable elementary functions such as $x+y$, xy, $\frac{x}{y}$, $x^2+2xy+3y^3$, $x\sin(xy)$, $e^{xy}\cos(x^2+y^2)$ are continuous, wherever they are defined.

Exercise 2.5.3. State the continuity version of the sandwich rule.

Exercise 2.5.4. Prove the sandwich rule for two variable functions.

Exercise 2.5.5. Suppose $f(x,y)$ is continuous at (k,l), and $\lim_{n\to\infty}(x_n,y_n) = (k,l)$. Prove

$$\lim_{n\to\infty} f(x_n,y_n) = f(k,l) = f\left(\lim_{n\to\infty}(x_n,y_n)\right).$$

This can be considered as the composition rule for series.

Exercise 2.5.6. Suppose two norms $\|\cdot\|$ and $\|\cdot\|'$ satisfy $\|\cdot\| \le C\|\cdot\|'$ for some constant $C>0$. If $f(\vec{x})$ has constant approximation at $\vec{x}_0$ (including or excluding $\vec{x}_0$) with respect to $\|\cdot\|'$, then prove that $f(\vec{x})$ has constant approximation with respect to $\|\cdot\|$.

In particular, if two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, then the constant approximations with respect to the two norms are equivalent.

We may also define the limit of multivariable functions involving the infinity. For example, we define $\lim_{\vec{x}\to\infty} f(\vec{x}) = l$, if for any $\epsilon>0$, there is B, such that

$$\|\vec{x}\| \ge B \implies |f(\vec{x})-l| \le \epsilon.$$

We also define $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = \infty$, if for any B, there is $\delta>0$, such that

$$0 < \|\vec{x}-\vec{x}_0\| \le \delta \implies |f(\vec{x})| \ge B.$$

Example 2.5.3. Consider $f(x,y) = \frac{|x|+|y|}{x^2+y^2}$ defined for $(x,y)\ne(0,0)$. We have

$$(|x|+|y|)^2 = x^2+y^2+2|xy| \ge x^2+y^2, \quad |x|+|y| \ge \sqrt{x^2+y^2}.$$

Then for any $B > 0$, we have

$$\|(x,y)\|_2 = \sqrt{x^2+y^2} < \frac{1}{B} \implies f(x,y) \geq \frac{\sqrt{x^2+y^2}}{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}} > B.$$

This verifies $\lim_{(x,y)\to(0,0)} f(x,y) = +\infty$.

Exercise 2.5.7. Rigorously argue $\lim_{(x,y)\to(0,0)} f(x,y) = \infty$.

1. $\frac{x^2+x^2}{x^4+y^4}$.
2. $\frac{|x|^p+|y|^p}{|x|+|y|}$, $p < 1$.
3. $\frac{|x|+|y|}{(x^2+y^2)^p}$, $p < 1$.

Exercise 2.5.8. State the definitions.

1. $\lim_{\vec{x}\to\infty} f(\vec{x}) = -\infty$.
2. $\lim_{x\to+\infty,\, y\to-\infty,\, z\to z_0} f(x,y,z) = l$.
3. $\lim_{x\to\infty,\, y\to y_0^+} f(x,y) = \infty$.

Exercise 2.5.9. Suppose $\lim_{t\to t_0^+} x(t) = \infty$, $\lim_{t\to t_0^+} y(t) = y_0$, and $\lim_{x\to\infty,\, y\to y_0} f(x,y) = l$. Prove that $\lim_{t\to t_0^+} f(x(t), y(t)) = l$.

2.5.2 Limit on Subset

We extend Section 2.2.4 on partial constant approximation to multivariable functions.

A function $f(\vec{x})$ converges to l at $\vec{x}_0$ on a subset A, and denoted $\lim_{\vec{x}\to\vec{x}_0,\,\vec{x}\in A} f(\vec{x}) = l$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|\vec{x}-\vec{x}_0\| \leq \delta \text{ and } \vec{x} \in A - \vec{x}_0 \implies |f(\vec{x}) - l| \leq \epsilon.$$

The following is the extension of Proposition 2.2.6 to multivariable functions.

Proposition 2.5.3. *If $\lim_{\vec{x}\to\vec{x}_0,\,\vec{x}\in A} f(\vec{x}) = l$ and $B \subset A$, then $\lim_{\vec{x}\to\vec{x}_0,\,\vec{x}\in B} f(\vec{x}) = l$. Conversely, if A is the union of finitely many B_i, and $\lim_{\vec{x}\to\vec{x}_0,\,\vec{x}\in B_i} f(\vec{x})$ have the same limit value l, then $\lim_{\vec{x}\to\vec{x}_0,\,\vec{x}\in A} f(\vec{x}) = l$.*

We often restrict a multivariable function to a 1-dimensional subset A, and get the candidate limit value l. Then we try to argue the whole function converges to l. If the restrictions to two subsets give different limit values, then the limit diverges.

Example 2.5.4. Consider $f(x,y) = \frac{xy}{x^2+y^2}$ defined for $(x,y) \neq (0,0)$. Let A_c be the straight line of slope c that passes through the origin. Then A_c is defined by $y = cx$, and the points in A_c are (x, cx).

The restriction of the function to A_c is $f(x, cx) = \frac{c}{1+c^2}$. Therefore

$$\lim_{(x,y)\to(0,0),\,(x,y)\in A_c} f(x, y) = \frac{c}{1 + c^2}.$$

The limit value explicitly depends on c. By restricting to A_c for different c, we get different limit values. Therefore $\lim_{(x,y)\to(0,0)} f(x, y)$ diverges.

Example 2.5.5. Let R be the union of the y-axis and the two disks of radius 1 in Figure 2.5.5. Let

$$f(x, y) = \begin{cases} 0, & \text{if } (x, y) \in R \\ 1, & \text{if } (x, y) \notin R \end{cases}.$$

For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let L_θ be the straight line of angle θ and passing through the origin. Then points on L_θ are $(t\cos\theta, t\sin\theta)$, and the restriction of $f(x, y)$ on L_θ is

$$f(t\cos\theta, t\sin\theta) = \begin{cases} 0, & \text{if } |t| \le 2\cos\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 1, & \text{if } |t| > 2\cos\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0, & \text{if } \theta = \frac{\pi}{2} \end{cases}.$$

Then for any fixed θ, we have $f(t\cos\theta, t\sin\theta) = 0$ for sufficiently small t, and we get

$$\lim_{(x,y)\to(0,0),\,(x,y)\in L_\theta} f(x, y) = 0.$$

This means f convergence to 0 along all straight lines.

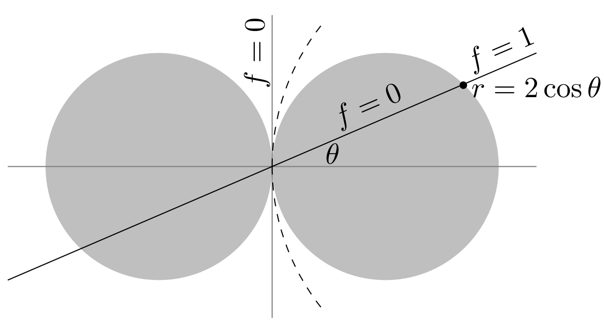


Figure 2.5.1: Converging to 1 along any straight edge, but diverges overall.

On the other hand, let B be the circle of radius 2, as indicated by the dashed lines. Then $f(x, y) = 1$ for $(x, y) \in B - (0, 0)$. This implies

$$\lim_{(x,y)\to(0,0),\,(x,y)\in B} f(x, y) = 1.$$

Since the limit on L_θ and B have different values, we know $\lim_{(x,y)\to(0,0)} f(x, y)$ diverges.

Exercise 2.5.10. Show that $\lim_{x,y\to 0}\frac{xy^2}{x^2+y^4}$ diverges, although the limit converges to 0 along any straight line leading to $(0,0)$.

Example 2.5.6. The function $f(x,y)=\frac{xy}{x^p+y^q}$, with $p,q>0$, is defined for $x>0$ and $y>0$. We want to determine when $\lim_{x,y\to 0^+}f(x,y)=0$.

The domain $(0,+\infty)\times(0,+\infty)$ of the function is the union of

$$A=\{(x,y)\colon x^p\le y^q,\ x>0,\ y>0\},\quad B=\{(x,y)\colon y^q\le x^p,\ x>0,\ y>0\}.$$

Then

$$\lim_{x,y\to 0^+}f(x,y)=0 \iff \lim_{x,y\to 0^+,\,(x,y)\in A}f(x,y)=\lim_{x,y\to 0^+,\,(x,y)\in B}f(x,y)=0.$$

For $(x,y)\in A$, we have $y^q\le x^p+y^q\le 2y^q$. This implies

$$\frac{1}{2}xy^{1-q}=\frac{xy}{2y^q}\le f(x,y)=\frac{xy}{x^p+y^q}\le\frac{xy}{y^q}=xy^{1-q}.$$

Therefore

$$\lim_{x,y\to 0^+,\,(x,y)\in A}f(x,y)=0 \iff \lim_{x,y\to 0^+,\,(x,y)\in A}xy^{1-q}=0.$$

We further restrict $\lim_{x,y\to 0^+,\,(x,y)\in A}xy^{1-q}=0$ to

$$A\cap B=\{(x,y)\colon x^p=y^q,\ x>0,\ y>0\}.$$

This means we substitute $x=y^{\frac{q}{p}}$ and get $\lim_{y\to 0^+}y^{\frac{q}{p}}y^{1-q}=\lim_{y\to 0^+}y^{1+\frac{q}{p}-q}=0$. We get the necessary condition $1+\frac{q}{p}-q>0$. The condition is the same as $\frac{1}{p}+\frac{1}{q}>1$.

Conversely, suppose the necessary condition is satisfied. Then

$$(x,y)\in A \implies x^p\le y^q \implies 0<xy^{1-q}\le y^{\frac{q}{p}}y^{1-q}=y^{1+\frac{q}{p}-q}.$$

By $1+\frac{q}{p}-q>0$, we have $\lim_{x,y\to 0^+,\,(x,y)\in A}y^{1+\frac{q}{p}-q}=0$. Then by the sandwich rule, we get $\lim_{x,y\to 0^+,\,(x,y)\in A}xy^{1-q}=0$.

We conclude that $\lim_{x,y\to 0^+,\,(x,y)\in A}f(x,y)=0$ if and only if $\frac{1}{p}+\frac{1}{q}>1$. By the similar argument, we also conclude that $\lim_{x,y\to 0^+,\,(x,y)\in B}f(x,y)=0$ if and only if $\frac{1}{p}+\frac{1}{q}>1$. Therefore $\lim_{x,y\to 0^+}f(x,y)=0$ if and only if $\frac{1}{p}+\frac{1}{q}>1$.

Exercise 2.5.11. If $\frac{1}{p}+\frac{1}{q}\ge 1$ in Example 2.5.6, show that $\lim_{x,y\to 0^+}f(x,y)$ diverges.

Exercise 2.5.12. Find the condition for $\lim_{x,y\to 0^+}f(x,y)=0$. Assume all the parameters are positive.

1. $(x^p+y^q)^r\log(x^m+y^n)$.
2. $x^p\log\left(\frac{1}{x^m}+\frac{1}{y^n}\right)$.
3. $\frac{x^py^q}{(x^m+y^n)^k}$.
4. $\frac{(x^p+y^q)^r}{(x^m+y^n)^k}$.

Exercise 2.5.13. Compute the convergent limits. All parameters are positive.

1. $\lim_{(x,y)\to(1,1)}\frac{1}{x-y}$.
2. $\lim_{x,y,z\to 0}\frac{xyz}{x^2+y^2+z^2}$.
3. $\lim_{(x,y)\to(0,0)}(x-y)\sin\frac{1}{x^2+y^2}$.
4. $\lim_{x,y\to\infty}(x-y)\sin\frac{1}{x^2+y^2}$.
5. $\lim_{x,y\to\infty}\frac{(x^2+y^2)^p}{(x^4+y^4)^q}$.
6. $\lim_{0<x<y^2,x\to 0,y\to 0}\frac{x^py}{x^2+y^2}$.
7. $\lim_{ax\le y\le bx,\ x,y\to\infty}\frac{1}{xy}$.
8. $\lim_{x,y\to 0^+}(x+y)^{xy}$.
9. $\lim_{x\to\infty,y\to 0}\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$.
10. $\lim_{x,y\to+\infty}\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$.
11. $\lim_{x,y\to+\infty}\frac{x^2+y^2}{e^{x+y}}$.

2.5.3 Repeated Limit

For a multivariable function, we may first take the limit in some variables and then take the limit in other variables. In this way, we get *repeated limits* such as $\lim_{x\to x_0}\lim_{y\to y_0}f(x,y)$. The following gives the relation between the repeated limits and the usual limit $\lim_{(x,y)\to(x_0,y_0)}f(x,y)$.

Proposition 2.5.4. *Suppose* $\lim_{(x,y)\to(x_0,y_0)}f(x,y)=l$ *converges. Suppose for each* x *near* x_0, $\lim_{y\to y_0}f(x,y)=g(x)$ *converges. Then*

$$\lim_{x\to x_0}g(x)=\lim_{x\to x_0}\lim_{y\to y_0}f(x,y)=l.$$

A consequence of the proposition is that, if the two variable limit $\lim_{(x,y)\to(x_0,y_0)}f(x,y)$ converges, and both single variable limits $\lim_{y\to y_0}f(x,y)$ and $\lim_{x\to x_0}f(x,y)$ converge, then both repeated limits converge and are equal

$$\lim_{x\to x_0}\lim_{y\to y_0}f(x,y)=\lim_{y\to y_0}\lim_{x\to x_0}f(x,y).$$

It is clear from the proof below that the proposition remains true if we change the variables x and y to multivariables $\vec{x}$ and $\vec{y}$.

The proof starts with the definition of $\lim_{(x,y)\to(x_0,y_0)}f(x,y)=l$. For any $\epsilon>0$, there is $\delta>0$, such that

$$0<\|(x-x_0,y-y_0)\|_\infty=\max\{|x-x_0|,|y-y_0|\}\le\delta\implies|f(x,y)-l|\le\epsilon.$$

Then for each fixed x near x_0, taking $\lim_{y\to y_0}$ in the inequality above, and using the order rule, we get

$$0<|x-x_0|\le\delta\implies|g(x)-l|=\lim_{y\to y_0}|f(x,y)-l|\le\epsilon.$$

This means $\lim_{x\to x_0} g(x) = l$.

Example 2.5.7. For $f(x,y) = \frac{x}{x+y}$, we have

$$\lim_{x\to\infty} \frac{x}{x+y} = 1, \qquad \lim_{y\to\infty}\lim_{x\to\infty} \frac{x}{x+y} = 1,$$
$$\lim_{y\to\infty} \frac{x}{x+y} = 0, \qquad \lim_{x\to\infty}\lim_{y\to\infty} \frac{x}{x+y} = 0.$$

Since the two repeated limit values are different, the two variable limit $\lim_{x,y\to\infty} \frac{x}{x+y}$ diverges.

Of course one may also show the divergence by restricting the two variable limit to $y = cx$.

Example 2.5.8. For $f(x,y) = \frac{xy}{x^2+y^2}$ in Example 2.5.4, we have

$$\lim_{x\to 0} f(x,y) = 0 \text{ for each } y, \quad \lim_{y\to 0} f(x,y) = 0 \text{ for each } x.$$

Therefore

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0.$$

However, the two variable limit $\lim_{(x,y)\to(0,0)} f(x,y)$ diverges.

Exercise 2.5.14. Study $\lim_{x\to 0, y\to 0}$, $\lim_{x\to 0}\lim_{y\to 0}$ and $\lim_{y\to 0}\lim_{x\to 0}$. Assume $p,q,r > 0$.

1. $\frac{x-y+x^2+y^2}{x+y}$.
2. $x\sin\frac{1}{x} + y\cos\frac{1}{x}$.
3. $\frac{|x|^p|y|^q}{(x^2+y^2)^r}$.
4. $\frac{x^2y^2}{|x|^3+|y|^3}$.
5. $(x+y)\sin\frac{1}{x}\sin\frac{1}{y}$.
6. $\frac{e^x-e^y}{\sin xy}$.

Exercise 2.5.15. Let

$$D(x,y) = D(x)D(y) = \begin{cases} 1, & \text{if } x \text{ and } y \text{ are rational} \\ 0, & \text{if } x \text{ or } y \text{ is irrational} \end{cases}.$$

1. Show that $D(x,y)$ is not continuous everywhere.
2. For which x does $\lim_{y\to y_0} D(x,y)$ converge?
3. Discuss the three limits $\lim_{x\to 0, y\to 0}$, $\lim_{x\to 0}\lim_{y\to 0}$ and $\lim_{y\to 0}\lim_{x\to 0}$.

Do the same for $xD(x,y)$ and $xyD(x,y)$.

2.5.4 Multivariable Map

A multivariable map has a multivariable input $\vec{x}$ and a multivariable output $\vec{y}$. For example, the polar coordinates to cartesian coordinates is a map of two variables to two variables

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

We may express the map as

$$(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)\colon \mathbb{R}^2 \to \mathbb{R}^2.$$

In fact, it is easier to follow the formula by writing the multivariables in vertical way

$$\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

In general, a map from two variables to three variables consists of three functions of two variables

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{pmatrix}.$$

A general map is $\vec{y} = F(\vec{x})$.

Definition 2.5.5. A multivariable map $F(\vec{x})$ defined around $\vec{x}_0$ has constant approximation $\vec{l}$ at $\vec{x}_0$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|\vec{x} - \vec{x}_0\| \le \delta \implies \|F(\vec{x}) - \vec{l}\| \le \epsilon.$$

In this case, we say $F(\vec{x})$ is *continuous* at $\vec{x}_0$.

Definition 2.5.6. A multivariable map $F(\vec{x})$ defined around $\vec{x}_0$ but not necessarily including $\vec{x}_0$, converges to $\vec{l}$ at $\vec{x}_0$, and denoted $\lim_{\vec{x}\to\vec{x}_0} F(\vec{x}) = \vec{l}$, if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|\vec{x} - \vec{x}_0\| \le \delta \text{ and } \vec{x} \ne \vec{x}_0 \implies \|F(\vec{x}) - \vec{l}\| \le \epsilon.$$

The continuity of $F(\vec{x}) = (f(\vec{x}), g(\vec{x}))$ at $\vec{x}_0$ means that, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|\vec{x} - \vec{x}_0\| \le \delta \implies \|(f(\vec{x}), g(\vec{x})) - (k, l)\|_\infty = \max\{|f(\vec{x}) - k|, |g(\vec{x}) - l|\} \le \epsilon.$$

The right side is the same as $|f(\vec{x}) - k| \le \epsilon$ and $|g(\vec{x}) - l| \le \epsilon$. Therefore the implication above means both $f(\vec{x})$ and $g(\vec{x})$ are continuous at $\vec{x}_0$.

If we change $\|\vec{x} - \vec{x}_0\| \le \delta$ to $0 < \|\vec{x} - \vec{x}_0\| \le \delta$, then we prove that $\lim_{\vec{x}\to\vec{x}_0}(f(\vec{x}), g(\vec{x})) = (k, l)$ if and only if $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = k$ and $\lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) = l$. This is comparable to Example 1.6.1.

Proposition 2.5.7. *Suppose* $F(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \dots, f_n(\vec{x}))$. *Then* $F(\vec{x})$ *is continuous at* $\vec{x}_0$ *if and only if each* $f_i(\vec{x})$ *is continuous at* $\vec{x}_0$. *Moreover,* $\lim_{\vec{x}\to\vec{x}_0} F(\vec{x}) = \vec{l} = (l_1, l_2, \dots, l_n)$ *if and only if each* $\lim_{\vec{x}\to\vec{x}_0} f_i(\vec{x}) = l_i$.

Many usual properties about constant approximations remain valid.

1. If $\lim_{\vec{x}\to\vec{x}_0} F(\vec{x}) = \vec{k}$ and $\lim_{\vec{x}\to\vec{x}_0} G(\vec{x}) = \vec{l}$, then $\lim_{\vec{x}\to\vec{x}_0}(F(\vec{x})+G(\vec{x})) = \vec{k}+\vec{l}$. However, we have many possible products such as the dot product $F(\vec{x})\cdot G(\vec{x})$ or the cross product $F(\vec{x}) \times G(\vec{x})$ (in case both $F, G \in \mathbb{R}^3$). In general, if $B(\cdot,\cdot)$ is a *bilinear map*, then we have $\lim_{\vec{x}\to\vec{x}_0} B(F(\vec{x}), G(\vec{x})) = B(\vec{k}, \vec{l})$.

2. There is no order among vectors. Therefore there is no monotone function or monotone map. There is no sandwich rule or order rule for multivariable maps.

3. If $\lim_{\vec{x}\to\vec{x}_0} F(\vec{x})$ converges, then $F(\vec{x})$ is bounded near $\vec{x}_0$. The proof is the same as before.

4. Composition of continuous maps is continuous. Moreover, Proposition 2.1.13 remains valid. Suppose $\lim_{\vec{x}\to\vec{x}_0} F(\vec{x}) = \vec{y}_0$ and $\lim_{\vec{y}\to\vec{y}_0} G(\vec{y}) = \vec{z}_0$, and one of the following extra conditions is satisfied.

   (a) There is $\delta > 0$, such that $0 < \|\vec{x} - \vec{x}_0\| \le \delta$ implies $F(\vec{x}) \ne \vec{y}_0$.

   (b) $G(\vec{y})$ is continuous at $\vec{y}_0$.

   Then $\lim_{\vec{x}\to\vec{x}_0} G(F(\vec{x})) = \vec{z}_0$.

We know the elementary two variable functions $\sigma(x, y) = x + y$, $\mu(x, y) = xy$ and $\delta(x, y) = \frac{x}{y}$ are continuous (wherever they are defined). Suppose $f(t)$ and $g(t)$ are continuous. Then by Proposition 2.5.7, we know $F(t) = (f(t), g(t))\colon \mathbb{R}^n \to \mathbb{R}^2$ is continuous. Therefore the compositions

$$(\sigma \circ F)(t) = f(t) + g(t), \quad (\mu \circ F)(t) = f(t)g(t), \quad (\delta \circ F)(t) = \frac{f(t)}{g(t)}$$

are also continuous.

If we change the continuity to the limits $\lim_{t\to t_0} f(t) = k$ and $\lim_{t\to t_0} g(t) = l$, then by the composition rule, we get the arithmetic rules for limits.

The intermediate value theorem essentially means that the image of an interval under a continuous function is still an interval. The key to the extension to multivariable maps is the extension of interval. Instead of just considering rectangles, we should consider path connected subsets. A subset $A \subset \mathbb{R}^n$ is *path connected* if for any $\vec{x}, \vec{y} \in A$, there is a continuous map $\gamma(t)\colon [0, 1] \to A \subset \mathbb{R}^n$, such that $\gamma(0) = \vec{x}$ and $\gamma(1) = \vec{y}$. We may regard γ as a continuous path inside A that connects $\vec{x}$ and $\vec{y}$.

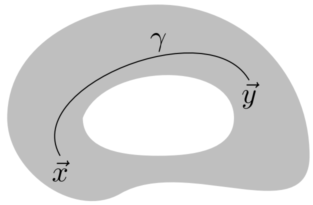


Figure 2.5.2: Path connected subset.

The multivariable version of the intermediate value theorem is that, if A is path connected, and $F(\vec{x})$ is a continuous map on A, then the image set $F(A) = \{F(\vec{x}) \colon \vec{x} \in A\}$ is also path connected.

Finally, the most important part of Theorem 2.4.2 is that the inverse of a continuous single variable function is also continuous. The result can be extended to multivariable maps, under different context. We remark that the monotone property does not make sense for multivariable.

Definition 2.5.8. A subset $A \subset \mathbb{R}^n$ is *closed*, if $\vec{x}_n \in A$ and $\lim_{\vec{x}_n \to \vec{x}_0} = \vec{l}$ implies $\vec{l} \in A$. In other words, if a sequence inside A converges, then the limit is still inside A.

Theorem 2.5.9. *Suppose A and B are subsets of Euclidean spaces, and $F \colon A \to B$ is an invertible continuous map. If A is closed and bounded, then $F^{-1} \colon B \to A$ is also continuous.*

Chapter 3

Linear Approximation

3.1 Linear Approximation

The basic idea of differentiation is solving problems by using simple functions to approximate general complicated functions. The simplest functions are the constant functions, which are usually too primitive to be useful. More effective approximations are given by linear functions $A + Bx$.

Definition 3.1.1. A *linear approximation* of a function $f(x)$ at x_0 is a linear function $L(x) = a + b(x - x_0)$, such that for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| < \delta \implies |f(x) - L(x)| = |f(x) - a - b(x - x_0)| \le \epsilon|x - x_0|.$$

A function is *differentiable* if it has a linear approximation.

The differentiability at x_0 requires the function to be defined on a neighborhood of x_0, and the linear approximation depends only on the function near x_0.

In everyday life, we use approximations all the time. For example, when we measure certain distance and get 7 meters and 5 centimeters, we really mean give or take some millimeters. So the real distance might be 7.052 meters or 7.046 meters. The function $f(x)$ is like the real distance (7.052 meters or 7.046 meters), and the linear function $L(x)$ is like the reading (7.05 meters) from the ruler.

The accuracy of the measurement depends on how refined the ruler is. We often use the rulers with two units m and cm. The centimeter cm is smaller among the two units and is therefore the “basic unit” that gives the accuracy of the ruler. The *error* $|7m5.2cm - 7m5cm| = 0.2cm$ between the real distance and the measurement should be significantly smaller than the basic unit $1cm$.

Analogously, the linear function $L(x) = a \cdot 1 + b \cdot (x - x_0)$ is a combination of two units 1 and $x - x_0$. Since the approximation happens for x near x_0, $x - x_0$ is much smaller than 1 and is therefore the “basic unit”. The *error* $|f(x) - L(x)|$ of the approximation should be significantly smaller than the size $|x - x_0|$ of the basic unit, which exactly means $\le \epsilon|x - x_0|$ on the right side of the definition.

3.1.1 Derivative

Geometrically, a function may be represented by its graph. The graph of a linear function is a straight line. Therefore a linear approximation at x_0 is a straight line that "best fits" the graph of the given function near x_0. This is the *tangent line* of the function.

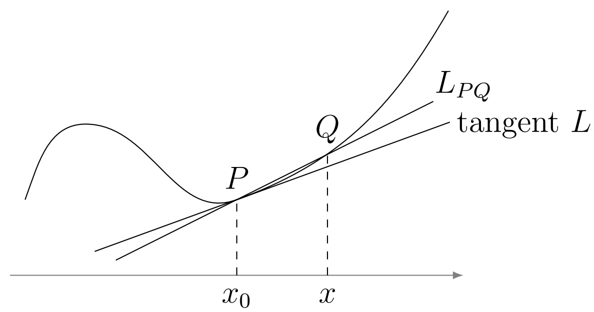


Figure 3.1.1: The linear approximation is the tangent line.

Specifically, the point P in Figure 3.1.1 is the point $(x_0, f(x_0))$ on the graph of $f(x)$. We pick a nearby point $Q = (x, f(x))$ on the graph, for x near x_0. The straight line connecting P and Q is the linear function (the variable in L_{PQ} is t because x is already used for Q)

$$L_{PQ}(t) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(t - x_0).$$

As $x \to x_0$, Q approaches P, and the linear function approaches $L(t) = a + b(t - x_0)$. Therefore we have $a = f(x_0)$, and b is given below.

Definition 3.1.2. The *derivative* of a function $f(x)$ at x_0 is

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We emphasize that the linear approximation is the concept. As the coefficient b of the first order term, the derivative $f'(x_0)$ is the computation of the concept. The following says that the concept and its computation are equivalent.

Proposition 3.1.3. *A function $f(x)$ is differentiable at x_0 if and only if the derivative $f'(x_0)$ exists. Moreover, the linear approximation is given by $f(x_0) + f'(x_0)(x - x_0)$.*

The notation f' for the derivative is due to Joseph Louis Lagrange. It is simple and convenient, but could become ambiguous when there are several variables related in more complicated ways. Another notation $\dfrac{df}{dx}$, due to Gottfried Wilhelm Leibniz, is less ambiguous. So we also write

$$\left.\frac{df}{dx}\right|_{x=x_0} = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}, \quad \Delta f = f(x)-f(x_0),\ \Delta x = x-x_0.$$

We emphasize that Leibniz's notation is not the "quotient" of two quantities df and dx. It is an integrated notation that alludes to the fact that the derivative is the limit of the quotient $\dfrac{\Delta f}{\Delta x}$ of differences.

Example 3.1.1. The function $f(x) = 3x-2$ is already linear. So its linear approximation must be $L(x) = f(x) = 3x-2$. This reflects the intuition that, if the distance is exactly $7m5cm$, then the measure by the ruler in centimeters should be $7m5cm$. In particular, the derivative $f'(x) = 3$, or $\dfrac{d(3x-2)}{dx} = 3$. In general, we have $(A+Bx)' = B$.

Example 3.1.2. To find the linear approximation of x^2 at 1, we rewrite the function in terms of $x-1$

$$x^2 = (1+(x-1))^2 = 1+2(x-1)+(x-1)^2.$$

Note that $L(x) = 1+2(x-1)$ is linear, and the error $|x^2-L(x)| = (x-1)^2$ is significantly smaller than $|x-1|$ when x is near 1

$$|x-1| < \delta = \epsilon \implies |x^2 - L(x)| \le \epsilon|x-1|.$$

Therefore $1+2(x-1)$ is the linear approximation of x^2 at 1, and the derivative $(x^2)'|_{x=1} = \left.\dfrac{d(x^2)}{dx}\right|_{x=1} = 2$.

Exercise 3.1.1. Find the linear approximations and then the derivatives.

1. $5x+3$ at x_0.
2. x^3-2x+1 at 1.
3. x^2 at x_0.
4. x^3 at x_0.
5. x^n at 1.
6. x^n at x_0.

Exercise 3.1.2. Interpret the limits as derivatives.

1. $\lim_{x\to 0}\dfrac{(1+x)^p-1}{x}$.
2. $\lim_{x\to 0}\dfrac{\sqrt{x+9}-3}{x}$.
3. $\lim_{x\to\frac{\pi}{2}}\dfrac{\cos x}{x-\frac{\pi}{2}}$.
4. $\lim_{x\to\pi}\dfrac{\sin x}{x-\pi}$.
5. $\lim_{x\to 0}\dfrac{\arcsin x}{x}$.
6. $\lim_{x\to 2}\dfrac{1}{x-2}\log\dfrac{x}{2}$.

3.1.2 Basic Derivative

We derive the derivatives of the power function, the exponential function, the logarithmic function, and the trigonometric functions.

Example 3.1.3. For $x_0 \neq 0$, we have

$$\left.\frac{d}{dx}\left(\frac{1}{x}\right)\right|_{x=x_0} = \lim_{x\to x_0} \frac{\frac{1}{x}-\frac{1}{x_0}}{x-x_0} = \lim_{x\to x_0} \frac{x_0-x}{x_0x(x-x_0)} = \lim_{x\to x_0} -\frac{1}{x_0x} = -\frac{1}{x_0^2}.$$

Therefore $\frac{1}{x}$ is differentiable at x_0, and the linear approximation is $\frac{1}{x_0}-\frac{1}{x_0^2}(x-x_0)$. We express the derivative as

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}.$$

Example 3.1.4. For $x_0 > 0$, we have

$$\left.\frac{d\sqrt{x}}{dx}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} = \lim_{x\to x_0} \frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})} = \frac{1}{2\sqrt{x_0}}.$$

Therefore $\sqrt{x}$ is differentiable, and the linear approximation is $\sqrt{x_0}-\frac{1}{2\sqrt{x_0}}(x-x_0)$. We express the derivative as

$$(\sqrt{x})' = \frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}.$$

Example 3.1.5. By Example **??**, we have

$$\left.\frac{d(x^p)}{dx}\right|_{x=1} = \lim_{h\to 0} \frac{(1+h)^p-1}{h} = p.$$

Therefore x^p is differentiable at 1 and has linear approximation $1+p(x-1)$.

Examples 3.1.3 and 3.1.4 are the derivatives of x^p for $p=-1$ and $\frac{1}{2}$ at general $x_0 > 0$. For general p, we take $h = x_0y$ and get

$$\begin{aligned}\left.\frac{d(x^p)}{dx}\right|_{x=x_0} &= \lim_{h\to 0} \frac{(x_0+h)^p-x_0^p}{h} = \lim_{y\to 0} \frac{(x_0+x_0y)^p-x_0^p}{x_0y} \\ &= \lim_{y\to 0} x_0^{p-1}\frac{(1+y)^p-1}{y} = px_0^{p-1}.\end{aligned}$$

We express the derivative as

$$(x^p)' = px^{p-1}.$$

Example 3.1.6. By Example **??**, we have

$$\left.\frac{d\log x}{dx}\right|_{x=1} = \lim_{x\to 1} \frac{\log x-\log 1}{x-1} = \lim_{y\to 0} \frac{\log(y+1)}{y} = 1.$$

Therefore $\log x$ is differentiable at 1 and has linear approximation $x-1$.

In general, at any $x_0>0$, by taking $h=x_0y$, we have

$$\left.\frac{d\log x}{dx}\right|_{x=x_0}=\lim_{h\to 0}\frac{\log(x_0+h)-\log x_0}{h}=\lim_{y\to 0}\frac{\log(y+1)}{x_0y}=\frac{1}{x_0}.$$

We express the derivative as

$$(\log x)'=\frac{1}{x}.$$

Example 3.1.7. By Example **??**, we have

$$\left.\frac{de^x}{dx}\right|_{x=x_0}=\lim_{h\to 0}\frac{e^{x_0+h}-e^{x_0}}{h}=\lim_{h\to 0}e^{x_0}\frac{e^h-1}{h}=e^{x_0}.$$

We express the derivative as

$$(e^x)'=e^x.$$

Example 3.1.8. In Section **??**, we find

$$\lim_{x\to 0}\frac{\sin x-\sin 0}{x-0}=\lim_{x\to 0}\frac{\sin x}{x}=1,$$

Therefore $\sin x$ is differentiable at $x=0$, and the linear approximation at 0 is x.

More generally, we have

$$\begin{aligned}\left.\frac{d\sin x}{dx}\right|_{x=x_0}&=\lim_{h\to 0}\frac{\sin(x_0+h)-\sin x_0}{h}\\&=\lim_{h\to 0}\frac{\sin h\cos x_0+\cos h\sin x_0-\sin x_0}{h}\\&=\lim_{h\to 0}\left(\frac{\cos h-1}{h}\sin x_0+\frac{\sin h}{h}\cos x_0\right)\\&=0\cdot\sin x_0+1\cdot\cos x_0=\cos x_0.\end{aligned}$$

We express the result as

$$(\sin x)'=\cos x.$$

By similar method, we have

$$(\cos x)'=-\sin x.$$

Example 3.1.9. In Section **??**, we find

$$\lim_{x\to 0}\frac{\sin x-\sin 0}{x-0}=\lim_{x\to 0}\frac{\sin x}{x}=1,$$

Therefore $\sin x$ is differentiable at $x=0$, and the linear approximation at 0 is x.

More generally, we have

$$\begin{aligned}
\left.\frac{d\sin x}{dx}\right|_{x=x_0} &= \lim_{h\to 0}\frac{\sin(x_0+h)-\sin x_0}{h} \\
&= \lim_{h\to 0}\frac{\sin h\cos x_0+\cos h\sin x_0-\sin x_0}{h} \\
&= \lim_{h\to 0}\left(\frac{\cos h-1}{h}\sin x_0+\frac{\sin h}{h}\cos x_0\right) \\
&= 0\cdot\sin x_0+1\cdot\cos x_0=\cos x_0.
\end{aligned}$$

We express the result as

$$(\sin x)'=\cos x.$$

By similar method, we have

$$(\cos x)'=-\sin x.$$

Exercise 3.1.3. Find the derivatives and then the linear approximations.

1. $\sqrt[3]{x}$ at 1.
2. $(\log x)^2$ at 1.
3. $\cos x^2$ at 0.
4. $\tan x$ at 0.
5. $\arcsin x$ at 0.
6. $\arctan x$ at 0.
7. $\sin\sin x$ at 0.
8. $x^2D(x)$ at 0.

Exercise 3.1.4. Find the derivatives, $a>0$.

1. $\log_a x$.
2. a^x.
3. $\tan x$.
4. $\arcsin x$.

Exercise 3.1.5. We have $\log|x|=\log(-x)$ for $x<0$. Show that the derivative of $\log(-x)$ at $x_0<0$ is $\dfrac{1}{x_0}$. The interpret your result as

$$(\log|x|)'=\frac{1}{x}.$$

Exercise 3.1.6. What is the derivative of $\log_a|x|$?

Exercise 3.1.7. Suppose p is an odd integer. Then x^p is defined for $x<0$. Do we still have $(x^p)'=px^{p-1}$ for $x<0$?

3.1.3 Constant Approximation

If a measurement of distance by a ruler in centimeters gives 7 meters and 5 centimeters, then the measurement by another (more primitive) ruler in meters should give 7 meters.

Analogously, if $a+b(x-x_0)$ is a linear approximation of $f(x)$ at x_0, then a is a *constant approximation* of $f(x)$ at x_0. Since the "basic unit" for constant functions is 1, the constant approximation means that, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| < \delta \implies |f(x) - a| \le \epsilon.$$

This means exactly that $f(x)$ is continuous at x_0, and the approximating constant is $a = f(x_0)$. Therefore the fact of linear approximation implying constant approximation means the following.

Theorem 3.1.4. *If a function is differentiable at a, then it is continuous at a.*

We do not expect the continuity to imply differentiability, because we do not expect the measurement in meters can tell us the measurement in centimeters.

Example 3.1.10. The sign function

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0, \end{cases}$$

is not continuous at 0, and is therefore not differentiable at 0. Of course, we have $(\mathrm{sign}(x))' = 0$ away from 0.

Example 3.1.11. The absolute value function $|x|$ is continuous everywhere. Yet the derivative

$$(|x|)'|_{x=0} = \lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \mathrm{sign}(x)$$

diverges. Therefore the continuous function is not differentiable at 0.

Example 3.1.12. The Dirichlet function $D(x)$ in Example **??** is not continuous anywhere and is therefore not differentiable anywhere.

On the other hand, the function $xD(x)$ is continuous at 0. Yet the derivative

$$(xD(x))'|_{x=0} = \lim_{x\to 0} \frac{xD(x)}{x} = \lim_{x\to 0} D(x)$$

diverges. Therefore $xD(x)$ is not differentiable at 0, despite the continuity.

Exercise 3.1.8. Find the derivative of $|x|$ at $x_0 \neq 0$.

Exercise 3.1.9. Determine the differentiability of $|x|^p$ at 0.

Exercise 3.1.10. Is $xD(x)$ differentiable at $x_0 \neq 0$?

Exercise 3.1.11. Determine the differentiability of $|x|^p D(x)$ at 0.

Exercise 3.1.12. Determine the differentiability of

$$f(x) = \begin{cases} |x|^p \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

at 0.

Exercise 3.1.13. Let $[x]$ be the greatest integer $\leq x$. Study the differentiability of $[x]$.

3.1.4 One Sided Derivative

Like one sided limits, we have *one sided derivatives*

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}, \quad f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

The derivative $f'(x_0)$ exists if and only if both $f'_+(x_0)$ and $f'_-(x_0)$ exist and are equal.

Example 3.1.13. We have

$$(|x|)'|_{\text{at } 0^+} = \lim_{x \to 0^+} \frac{|x|}{x} = 1, \quad (|x|)'|_{\text{at } 0^-} = \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Therefore $|x|$ has left and right derivatives. Since the two one sided derivatives are different, the function is not differentiable at 0.

Example 3.1.14. Consider the function

$$f(x) = \begin{cases} e^x, & \text{if } x \geq 0, \\ x + 1, & \text{if } x < 0. \end{cases}$$

We have (note that $f(0) = e^0 = 0 + 1$)

$$f'_+(0) = (e^x)'|_{x=0} = 1, \quad f'_-(0) = (x+1)'|_{x=0} = 1.$$

Therefore $f'(0) = 1$ and has linear approximation $1 + x$ at 0.

Exercise 3.1.14. Determine the differentiability.

1. $|x^2 - 3x + 2|$ at $0, 1, 2$.
2. $\sqrt{1 - \cos x}$ at 0.
3. $|\sin x|$ at 0.
4. $|\pi^2 - x^2| \sin x$ at π.

Exercise 3.1.15. Determine the differentiability at 0.

1. $\begin{cases} x^2, & \text{if } x \ge 0, \\ x, & \text{if } x < 0. \end{cases}$

2. $\begin{cases} \dfrac{1}{x+1}, & \text{if } x \ge 0, \\ x, & \text{if } x < 0. \end{cases}$

3. $\begin{cases} xe^{-\frac{1}{x}}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$

4. $\begin{cases} \log(1+x), & \text{if } x \ge 0, \\ e^x - 1, & \text{if } x < 0. \end{cases}$

Exercise 3.1.16. Determine the differentiability, $p, q > 0$.

1. $\begin{cases} (x-a)^p(b-x)^q, & \text{if } a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$

2. $\begin{cases} \arctan x, & \text{if } |x| \le 1, \\ \dfrac{\pi}{4}x, & \text{if } |x| > 1. \end{cases}$

3. $\begin{cases} x^2 e^{-x^2}, & \text{if } |x| \le 1, \\ e^{-1}, & \text{if } |x| > 1. \end{cases}$

4. $\begin{cases} \log |x|, & \text{if } |x| \ge 1, \\ x, & \text{if } |x| < 1. \end{cases}$

Exercise 3.1.17. Find a, b, c, such that the function

$$f(x) = \begin{cases} \dfrac{a}{x}, & \text{if } x > 1, \\ bx + c, & \text{if } x \le 1, \end{cases}$$

is differentiable at 1.

Exercise 3.1.18. For $p \ge 0$, x^p is defined on $[0, \delta)$. What is the right derivative of x^p at 0?

Exercise 3.1.19. For some p (see Exercises 3.1.7 and 3.1.18), x^p is defined on $(-\delta, \delta)$. What is the derivative of x^p at 0?

Exercise 3.1.20. Suppose $g(x)$ is continuous at x_0. Show that $f(x) = |x - x_0| g(x)$ is differentiable at x_0 if and only if $g(x_0) = 0$.

3.2 Property of Derivative

3.2.1 Arithmetic Combination of Linear Approximation

Suppose $f(x)$ and $g(x)$ are linearly approximated respectively by $a + b(x - x_0)$ and $c + d(x - x_0)$ at x_0. Then $f(x) + g(x)$ is approximated by

$$(a + b(x - x_0)) + (c + d(x - x_0)) = (a + c) + (b + d)(x - x_0).$$

Therefore $f + g$ is differentiable and $(f + g)'(x_0) = b + d$. Since $b = f'(x_0)$ and $d = g'(x_0)$, we conclude that

$$(f + g)'(x) = f'(x) + g'(x), \text{ or } \frac{d(f + g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}.$$

Similarly, $Cf(x)$ is approximated by

$$C(a+b(x-x_0)) = Ca + Cb(x-x_0).$$

Therefore $Cf(x)$ is differentiable and $(Cf)'(x_0) = Cb$, which means

$$(Cf)'(x) = Cf'(x), \text{ or } \frac{d(Cf)}{dx} = C\frac{df}{dx}.$$

We also have $f(x)g(x)$ approximated by

$$(a+b(x-x_0))(c+d(x-x_0)) = ac + (bc+ad)(x-x_0) + bd(x-x_0)^2.$$

Although the approximation is not linear, the square unit $(x-x_0)^2$ is much smaller than $x-x_0$ when x is close to x_0. Therefore $f(x)g(x)$ is differentiable and has linear approximation $ac+(bc+ad)(x-x_0)$, and we get $(fg)'(x_0) = bc+ad$. By $a = f(x_0), b = f'(x_0), c = g(x_0), d = g'(x_0)$, we get the *Leibniz rule*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x), \text{ or } \frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}.$$

The explanation above on the derivatives of arithmetic combinations are analogous to the arithmetic properties of limits.

Exercise 3.2.1. Find the derivative of the polynomial $p(x) = c_nx^n + \cdots + c_1x + c_0$.

Exercise 3.2.2. Compute the derivatives.

1. $e^x \sin x$.
2. $\sin^2 x$.
3. e^{2x}.
4. $\sin^2 x \cos x$.
5. $\sin x - x\cos x$.
6. $\cos x + x \sin x$.
7. $(x-1)e^x$.
8. x^2e^x.
9. $x\log x - x$.
10. $2x^2 \log x - x^2$.
11. $xe^x \cos x$.
12. $xe^x \cos x \log x$.

Exercise 3.2.3. Find a polynomial $p(x)$, such that $(p(x)e^x)' = x^2e^x$. In general, suppose $(p_n(x)e^x)' = x^ne^x$. Find the relation between polynomials $p_n(x)$.

Exercise 3.2.4. Find polynomials $p(x)$ and $q(x)$, such that $(p(x)\sin x + q(x)\cos x)' = x^2 \sin x$. Moreover, find a function with derivative $x^2 \cos x$?

Exercise 3.2.5. Find constants A and B, such that $(Ae^x \sin x + Be^x \cos x)' = e^x \sin x$. What about $(Ae^x \sin x + Be^x \cos x)' = e^x \cos x$?

3.2.2 Composition of Linear Approximation

Consider a composition $g \circ f$

$$x \mapsto y = f(x) \mapsto z = g(y) = g(f(x)).$$

Suppose $f(x)$ is linearly approximated by $a+b(x-x_0)$ at x_0 and $g(y)$ is linearly approximated by $c+d(y-y_0)$ at $y_0=f(x_0)$. Then

$$a=f(x_0)=y_0,\quad b=f'(x_0),\quad c=g(y_0)=g(f(x_0)),\quad d=g'(y_0)=g'(f(x_0)),$$

and the composition $g\circ f$ is approximated by the composition of linear approximations (recall $a=y_0$)

$$c+d[(a+b(x-x_0))-y_0]=c+db(x-x_0).$$

Therefore the composition is also differentiable, with

$$(g\circ f)'(x_0)=db=g'(y_0)f'(x_0)=g'(f(x_0))f'(x_0)$$

This gives us the *chain rule*

$$(g(f(x)))'=(g\circ f)'(x)=g'(f(x))f'(x)=g'(y)|_{y=f(x)}f'(x),$$

or

$$\frac{dz}{dx}=\frac{dz}{dy}\frac{dy}{dx}.$$

Example 3.2.1. We know $(\log x)'=\dfrac{1}{x}$ for $x>0$. For $x<0$, we have

$$(\log(-x))'=(\log y)'|_{y=-x}(-x)'=\left.\frac{1}{y}\right|_{y=-x}(-1)=\frac{1}{-x}(-1)=\frac{1}{x}.$$

Therefore we conclude

$$(\log|x|)'=\frac{1}{x},\ \text{for } x\neq 0.$$

Example 3.2.2. In Example 3.1.5, we use the definition to derive $x^p=px^{p-1}$. Alternatively, we may also derive the derivative of x^p at general $x_0>0$ from the derivative $(x^p)'_{x=1}=p$ at a special place.

To move from x_0 to 1, we introduce $y=\dfrac{x}{x_0}$. Then x^p is the composition

$$x\mapsto y=\frac{x}{x_0}\mapsto z=x^p=x_0^py^p.$$

Then $x=x_0$ corresponds to $y=1$, and we have

$$\begin{aligned}\left.\frac{d(x^p)}{dx}\right|_{x=x_0}&=\left.\frac{dz}{dy}\right|_{y=1}\left.\frac{dy}{dx}\right|_{x=x_0}=\left.\frac{d(x_0^py^p)}{dy}\right|_{y=1}\left.\frac{d}{dx}\left(\frac{x}{x_0}\right)\right|_{x=x_0}\\&=x_0^p\cdot\left.\frac{d(y^p)}{dy}\right|_{y=1}\cdot\frac{1}{x_0}=x_0^p\cdot p\cdot\frac{1}{x_0}=px_0^{p-1}.\end{aligned}$$

Exercise 3.2.6. Use the derivative at a special place to find the derivative at other places.

1. $\log x$.
2. e^x.
3. $\sin x$.
4. $\cos x$.

Exercise 3.2.7. Use $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and the derivative of sine to derive the derivative of cosine. Use the similar method to find the derivatives of $\cot x$ and $\csc x$.

Exercise 3.2.8. A function $f(x)$ is odd if $f(-x) = -f(x)$, and is even if $f(-x) = f(x)$. What can you say about the derivative of an odd function and the derivative of an even function?

Example 3.2.3. By Example 3.1.3 and the chain rule, we have

$$\left(\frac{1}{f(x)}\right)' = \left(\frac{1}{y}\right)'\bigg|_{y=f(x)} f'(x) = -\frac{1}{y^2}\bigg|_{y=f(x)} f'(x) = -\frac{f'(x)}{f(x)^2}.$$

Then we may use the Leibniz rule to get the derivative of quotient

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\frac{1}{g(x)}\right)' = f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)' \\ &= f'(x)\frac{1}{g(x)} - f(x)\frac{g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.\end{aligned}$$

Exercise 3.2.9. Derive the derivatives.

$$(\tan x)' = \sec^2 x, \quad (\sec x)' = \sec x \tan x, \quad (e^{-x})' = -e^{-x}.$$

Exercise 3.2.10. Compute the derivatives.

1. $\dfrac{1}{x+2}$.
2. $\dfrac{x+1}{x-2}$.
3. $\dfrac{x^2-x+1}{x^3+1}$.
4. $\dfrac{x^3+1}{x^2-x+1}$.
5. $\dfrac{1}{ax+b}$.
6. $\dfrac{ax+b}{cx+d}$.
7. $\dfrac{x}{x^2+ax+b}$.
8. $\dfrac{1}{(x+a)(x+b)}$.

Exercise 3.2.11. Compute the derivatives.

1. $\dfrac{\log x}{x}$.
2. $\dfrac{\log x}{x^p}$.
3. $\dfrac{x^p}{\log x}$.
4. $\dfrac{e^x}{x\log x}$.

Exercise 3.2.12. Compute the derivatives.

1. $\dfrac{\sin x}{a+\cos x}$.
2. $\dfrac{1}{a+\tan x}$.
3. $\dfrac{1+x\tan x}{\tan x - x}$.
4. $\dfrac{\cos x + x\sin x}{\sin x - x\cos x}$.

Exercise 3.2.13. The *hyperbolic trigonometric functions* are

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$

and

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \mathrm{sech} x = \frac{1}{\cosh x}, \quad \mathrm{csch} x = \frac{1}{\sinh x}.$$

Find their derivatives and express them in hyperbolic trigonometric functions.

Example 3.2.4. The function $(x^2-1)^{10}$ is the composition of $z = y^{10}$ and $y = x^2 - 1$. Therefore

$$\left((x^2-1)^{10}\right)' = \frac{d(y^{10})}{dy}\frac{d(x^2-1)}{dx} = 10y^9 \cdot 2x = 20x(x^2-1)^9.$$

Example 3.2.5. The function $a^x = e^{bx}$, $b = \log a$, is the composition of $z = e^y$ and $y = bx$. Therefore

$$(a^x)' = (e^y)'|_{y=bx}(bx)' = e^{bx}b = a^x \log a.$$

Exercise 3.2.14. Compute the derivatives.

1. $(1-x)^{10}$.
2. $(3x+2)^{10}$.
3. $(x^3-1)^{10}(1-x^2)^9$.
4. $(1+(1-x^2)^{10})^9$.
5. $((x^3-1)^8+(1-x^2)^{10})^9$.
6. $((1-(3x+2)^3)^8+1)^9$.
7. $\dfrac{(x+1)^9}{(3x+5)^8}$.
8. $\dfrac{x(x+1)}{(x+2)(x+3)}$.

Exercise 3.2.15. Compute the derivatives.

1. $\cos(x^5+3x^2+1)$.
2. $\tan^{10}\left(x(x+1)^9\right)$.
3. $\sin(\sqrt{x}+3)$.
4. $\sin(\sqrt{x-2}+3)$.
5. $(\sin x + \cos x)^{10}$.
6. $\sqrt{\sin x + \cos x}$.
7. $\left(\dfrac{\sin^3 x}{\cos^4 x}\right)^{10}$.
8. $\sin(\cos x)$.
9. $\sin(\cos(\tan x))$.
10. $\dfrac{\sin^2 x}{\sin x^2}$.
11. $\dfrac{\sin 2x + 2\cos 2x}{2\sin x - \cos 2x}$.
12. $\dfrac{\sin^8 \sqrt{x}}{1+\cos^{10}(\sin^9 x)}$.

Exercise 3.2.16. Compute the derivatives.

1. e^{x^2}.
2. $(x^2-1)e^{x^2}$.
3. $e^{(e^x)}$.
4. $e^{\log x}$.
5. $\log_x e$.
6. $\log(\log x)$.
7. $\log\left(\dfrac{1}{\log x}\right)$.
8. $\log(\log(\log x))$.
9. $\log|\cos x|$.
10. $\log|\tan x|$.
11. $\log|\sec x - \tan x|$.
12. $\log \dfrac{1-\sin x}{1+\sin x}$.

Exercise 3.2.17. Compute the derivatives.

1. $(ax+b)^p$.
2. $(ax^2+bx+c)^p$.
3. $(a+(bx^2+c)^p)^q$.
4. e^{ax}.
5. $\log(ax+b)$.
6. $\sin(ax+b)$.
7. $\cos(ax+b)$.
8. $\tan(ax+b)$.
9. $\sec(ax+b)$.

Exercise 3.2.18. Compute the derivatives.

1. $\dfrac{1}{\sqrt{a^2-x^2}}$.
2. $\dfrac{1}{\sqrt{a^2+x^2}}$.
3. $\dfrac{1}{\sqrt{x^2-a^2}}$.
4. $\dfrac{x}{\sqrt{a^2-x^2}}$.
5. $\dfrac{x}{\sqrt{a^2+x^2}}$.
6. $\dfrac{x}{\sqrt{x^2-a^2}}$.

Exercise 3.2.19. Compute the derivatives.

1. $\sqrt{1+\sqrt{x}}$.
2. $\dfrac{1}{\sqrt{1+\sqrt{x}}}$.
3. $\sqrt{1+\sqrt{1+\sqrt{x}}}$.
4. $\sqrt{x+\sqrt{x+\sqrt{x}}}$.
5. $(1+2\sqrt{x+1})^{10}$.
6. $(1+2\sqrt{x+1})^{-10}$.
7. $\dfrac{\sqrt{x}+1}{x-2}$.
8. $\dfrac{\sqrt{x}+1}{(1-\sqrt{x}+x)^3}$.
9. $\dfrac{(\sqrt{x}+1)^4}{(1-\sqrt{x}+x)^3}$.
10. $\sqrt{\dfrac{1+x^2}{1-x^2}}$.
11. $\sqrt[3]{\dfrac{1+x^2}{1-x^2}}$.
12. $\dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{1-\sqrt{x}}}$.
13. $\sqrt{\dfrac{1+\sqrt{x}}{1-\sqrt{x}}}$.
14. $\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)^{10}$.
15. $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$.
16. $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$.
17. $\left(\dfrac{1}{1+\sqrt{x}}+\dfrac{1}{1-\sqrt{x}}\right)^{10}$.
18. $\sqrt{1+\dfrac{1}{\sqrt{x^2+1}}}$.

Exercise 3.2.20. Compute the derivatives.

1. $|x^2(x+2)^3|$.
2. $|\sin^3 x|$.
3. $|x(e^x-1)|$.
4. $|(x-1)^2\log x|$.

Exercise 3.2.21. Compute the derivatives.

1. $\sqrt{x}-\log|\sqrt{x}+a|$.
2. $\dfrac{b}{a^2(ax+b)}+\dfrac{1}{a^2}\log|ax+b|$.
3. $-\dfrac{1}{b}\log\dfrac{ax+b}{x}$.
4. $\dfrac{1}{b(ax+b)}-\dfrac{1}{b^2}\log\dfrac{ax+b}{x}$.
5. $\dfrac{1}{2}\log(a^2+x^2)$.
6. $\sqrt{x(a+x)}-a\log(\sqrt{x}+\sqrt{x+a})$.
7. $\dfrac{1}{\sqrt{b}}\log\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}$.
8. $x\log(x+\sqrt{x^2+a})-\sqrt{x^2+a}$.
9. $-\dfrac{1}{a}\log\dfrac{a+\sqrt{a^2+x^2}}{x}$.

10. $-\dfrac{1}{a}\log\dfrac{a+\sqrt{a^2-x^2}}{x}$.

11. $\dfrac{b}{2a}x-\dfrac{1}{4}x^2+\dfrac{1}{2}\left(x^2-\dfrac{b^2}{a^2}\right)\log(ax+b)$.

12. $-\dfrac{1}{2}x^2+\dfrac{1}{2}\left(x^2-\dfrac{a^2}{b^2}\right)\log(a^2-b^2x^2)$.

Example 3.2.6. By the chain rule, we have

$$\begin{aligned}\left(\log\left|x+\sqrt{x^2+a}\right|\right)' &= (\log|y|)'|_{y=x+\sqrt{x^2+a}}\left[x'+(\sqrt{z})'|_{z=x^2+a}(x^2+a)'\right]\\ &= \frac{1}{x+\sqrt{x^2+a}}\left[1+\frac{1}{2\sqrt{x^2+a}}2x\right]=\frac{1}{\sqrt{x^2+a}}.\end{aligned}$$

Now suppose we wish to find a function $f(x)$ with derivative

$$f'(x)=\frac{1}{\sqrt{x^2+ax+b}}.$$

By

$$f(x)=\frac{1}{\sqrt{\left(x+\frac{a}{2}\right)^2+b-\frac{a^2}{4}}}=\frac{1}{\sqrt{y^2+c}},\quad y=x+\frac{a}{2},\ c=b-\frac{a^2}{4},$$

we may substitute x by $x+\dfrac{a}{2}$ and substitute a by $c=b-\dfrac{a^2}{4}$. Then we get

$$\begin{aligned}\left(\log\left|x+\frac{a}{2}+\sqrt{x^2+ax+b}\right|\right)' &= \left(\log\left|y+\sqrt{y^2+c}\right|\right)'\Big|_{y=x+\frac{a}{2}}\left(x+\frac{a}{2}\right)'\\ &= \frac{1}{\sqrt{y^2+c}}\Big|_{y=x+\frac{a}{2}}=\frac{1}{\sqrt{x^2+ax+b}}.\end{aligned}$$

Exercise 3.2.22. Find constants A and B, such that $(Ae^{ax}\sin bx+Be^{ax}\cos bx)'=e^{ax}\cos bx$. What about $(Ae^{ax}\sin bx+Be^{ax}\cos bx)'=e^{ax}\sin bx$?

Exercise 3.2.23. Use Example 3.2.6 to compute the derivatives.

1. $\log(e^x+\sqrt{1+e^{2x}})$.
2. $\log\left|x-\sqrt{x^2+a}\right|$.
3. $\log(\tan x+\sec x)$.

Exercise 3.2.24. Compute the derivative of $\log\dfrac{x}{x+1}$. Then find a function with derivative $\dfrac{1}{(x+a)(x+b)}$. In case $a^2\ge 4b$, can you find a function with derivative $\dfrac{1}{x^2+ax+b}$?

Exercise 3.2.25. Compute the derivative of $x\sqrt{x^2+a}+a\log(x+\sqrt{x^2+a})$. Then find a function with derivative $\sqrt{x^2+ax+b}$.

Exercise 3.2.26. Use Example 3.2.6 to compute the derivatives.

1. $\log(e^x+\sqrt{1+e^{2x}})$.
2. $\log(x-\sqrt{x^2+a})$.
3. $\log(\tan x+\sec x)$.

Example 3.2.7. By viewing $x^p = e^{p\log x}$ as a composition of $z = e^y$ and $y = p\log x$, we have

$$\frac{d(x^p)}{dx} = \frac{(e^{p\log x})}{dx} = \left.\frac{d(e^y)}{dy}\right|_{y=p\log x} \frac{d(p\log x)}{dx} = e^{p\log x}\frac{p}{x} = x^p\frac{p}{x} = px^{p-1}.$$

This derives the derivative of x^p by using the derivatives of e^x and $\log x$.

Example 3.2.8. Suppose $u(x)$ and $v(x)$ are differentiable and $u(x) > 0$. Then $u(x)^{v(x)} = e^{u(x)\log v(x)}$, and

$$(u(x)^{v(x)})' = (e^{v(x)\log u(x)})' = (e^y)'|_{y=v(x)\log u(x)}(v(x)\log u(x))'.$$

By

$$(e^y)'|_{y=v(x)\log u(x)} = e^y|_{y=v(x)\log u(x)} = e^{v(x)\log u(x)} = u(x)^{v(x)},$$

and

$$(v(x)\log u(x))' = v'(x)\log u(x)+v(x)(\log u)'|_{u=u(x)}u'(x) = v'(x)\log u(x)+\frac{v(x)u'(x)}{u(x)},$$

We get

$$\begin{aligned}(u(x)^{v(x)})' &= u(x)^{v(x)}\left(v'(x)\log u(x)+\frac{v(x)u'(x)}{u(x)}\right)\\ &= u(x)^{v(x)-1}(u(x)v'(x)\log u(x)+u'(x)v(x)).\end{aligned}$$

Exercise 3.2.27. Compute the derivatives.

1. x^x.
2. x^{x^2}.
3. $(x^2)^x$.
4. $(a^x)^x$.
5. $(x^a)^x$.
6. $(x^x)^a$.
7. $(x^x)^x$.
8. $a^{(x^x)}$.
9. $x^{(a^x)}$.
10. $x^{(x^a)}$.
11. $x^{(x^x)}$.
12. $(x^x)^{(x^x)}$.

Exercise 3.2.28. Compute the derivatives.

1. $x^{\sin x}$.
2. $(\sin x)^x$.
3. $(\sin x)^{\cos x}$.
4. $\sqrt[x]{x}$.
5. $\sqrt[x]{\log x}$.
6. $x^{\log x}$.
7. $(e^x+e^{-x})^x$.
8. $(\log|x^2-1|)^x$.

Exercise 3.2.29. Let $f(x) = u(x)^{v(x)}$. Then $\log f(x) = u(x)\log v(x)$. By taking the derivative on both sides of the equality, derive the formula for $f'(x)$.

Exercise 3.2.30. Use the idea of Exercise 3.2.29 to compute the derivatives.

1. $\dfrac{x+a}{x+b}$.
2. $\dfrac{1}{(x+a)(x+b)}$.
3. $\dfrac{(x+c)(x+d)}{(x+a)(x+b)}$.
4. $\dfrac{(x+3)^7\sqrt{2x-1}}{(2x+1)^3}$.
5. $\dfrac{(x^2+x+1)^7}{(x^2-x+1)^3}$.
6. $\dfrac{e^{x^2+1}\sqrt{\sin x}}{(x^2-x+1)^3\log x}$.

3.2.3 Implicit Linear Approximation

The chain rule can be used to compute the derivatives of functions that are "implicitly" given. Such functions often do not have explicit formula expressions.

Strictly speaking, we need to know that implicitly given functions are differentiable before taking their derivatives. There are general theorems confirming such differentiability. In the subsequent examples, we will always assume the differentiability of implicitly defined functions.

Example 3.2.9. The unit circle $x^2+y^2=1$ on the plane is made up of the graphs of two functions $y=\sqrt{1-x^2}$ and $y=-\sqrt{1-x^2}$. We may certainly compute the derivative of each one explicitly

$$(\sqrt{1-x^2})'=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)=\frac{-x}{\sqrt{1-x^2}},\quad (-\sqrt{1-x^2})'=\frac{x}{\sqrt{1-x^2}}.$$

On the other hand, we may use the fact that both functions $y=y(x)$ satisfy the equation $x^2+y(x)^2=1$. Taking the derivatives in x of both sides, we get $2x+2yy'=0$. Solving the equation, we get

$$y'=-\frac{x}{y}.$$

This is consistent with the two derivatives computed above.

There is yet another way of computing the derivative $y'(x)$. The circle can be parametrized as $x=\cos t$, $y=\sin t$. In this view, the function $y=y(x)$ satisfies $\sin t=y(\cos t)$. By the chain rule, we have

$$\cos t=y'(x)(-\sin t).$$

Therefore

$$y'(x)=-\frac{\cos t}{\sin t}=-\frac{x}{y}.$$

In general, the derivative of a function $y=y(x)$ given by a parametrized curve $x=x(t)$, $y=y(t)$ is

$$y'(x)=\frac{y'(t)}{x'(t)}.$$

Note that the formula is ambiguous, in that $y'(x)=-\cot t$ and $y'(t)=\cos t$ are not the same functions. The primes in the two functions refer to $\dfrac{d}{dx}$ and $\dfrac{d}{dt}$ respectively.

So it is better to keep track of the variables by using Leibniz's notation. The formula above becomes

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

This is just another way of expressing the chain rule $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$.

Exercise 3.2.31. Compute the derivatives of the functions $y = y(x)$ given by curves.

1. $x = \sin^2 t$, $y = \cos^2 t$.
2. $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
3. $x = e^t \cos 2t$, $y = e^t \sin 2t$.
4. $x = (1 + \cos t)\cos t$, $y = (1 + \cos t)\sin t$.

Example 3.2.10. Like the unit circle, the equation $2y - 2x^2 - \sin y + 1 = 0$ is a curve on the plane, made up of the graphs of several functions $y = y(x)$. Although we cannot find an explicit formula for the functions, we can still compute their derivatives.

Taking the derivative of both sides of the equation $2y - 2x^2 - \sin y + 1 = 0$ with respect to x and keeping in mind that y is a function of x, we get $2y' - 4x - y'\cos y = 0$. Therefore

$$y' = \frac{4x}{2 - \cos y}.$$

The point $P = \left(\sqrt{\frac{\pi}{2}}, \frac{\pi}{2}\right)$ satisfies the equation and lies on the curve. The tangent line of the curve at the point has slope

$$y'|_P = \frac{4\sqrt{\dfrac{\pi}{2}}}{2 - \cos\dfrac{\pi}{2}} = \sqrt{2\pi}.$$

Therefore the tangent line at P is given by the equation

$$y - \frac{\pi}{2} = \sqrt{2\pi}\left(x - \sqrt{\frac{\pi}{2}}\right),$$

or

$$y = \sqrt{2\pi}x - \frac{\pi}{2}.$$

Example 3.2.11. The equations $x^2+y^2+z^2=2$ and $x+y+z=0$ specify a circle in the Euclidean space $\mathbb{R}^3$ and define functions $y=y(x)$ and $z=z(x)$. To find the derivatives of the functions, we take the derivatives of the two equations in x

$$2x+2yy'+2zz'=0, \quad 1+y'+z'=0.$$

Solving for y' and z', we get

$$y'=\frac{z-x}{y-z}, \quad z'=\frac{y-x}{z-y}.$$

Exercise 3.2.32. Compute the derivatives of implicitly defined functions.

1. $y^2+3y^3+1=x$.
2. $\sin y=x$.
3. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.
4. $\sqrt{x}+\sqrt{y}=\sqrt{a}$.
5. $e^{x+y}=xy$.
6. $x^2+2xy-y^2-2x=0$.

Exercise 3.2.33. Find the derivative of the implicitly defined functions of x.

1. $x^p+y^p=2$ at $x=1, y=1$.
2. $xy=\sin(x+y)$ at $x=0, y=\pi$.
3. $\dfrac{x+y}{z}=\dfrac{y+z}{x}=\dfrac{z+x}{y}$ at $x=y=z=1$.

Exercise 3.2.34. If $f(\sin x)=x$, what can you say about the derivative of $f(x)$? What if $\sin f(x)=x$?

Example 3.2.12. In Example **??**, we argued that the function $f(x)=x^5+3x^3+1$ is invertible. The inverse $g(x)$ satisfies $g(x)^5+3g(x)^3+1=x$. Taking the derivative in x on both sides, we get $5g(x)^4g'(x)+9g(x)^2g'(x)=1$. This implies

$$g'(x)=\frac{1}{5g(x)+9g(x)^2}.$$

Example 3.2.13. In Example 3.2.12, we interpreted the derivative of an inverse function as an implicit differentiation problem. In general, the inverse function $g(x)=f^{-1}(x)$ satisfies $f(g(x))=x$. Taking the derivative of both sides, we get $f'(g(x))g'(x)=1$. Therefore

$$(g(x))'=\frac{1}{f'(g(x))}, \text{ or } (f^{-1}(x))'=\frac{1}{f'(f^{-1}(x))}.$$

For example, the derivative of $\arcsin x$ is

$$(\arcsin x)'=\frac{1}{(\sin y)'|_{y=\arcsin x}}=\frac{1}{(\cos y)|_{x=\sin y}}=\frac{1}{\sqrt{1-x^2}}.$$

In the last step, we have positive square root because $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The computation can also be explained by considering two variables related by $y = y(x)$ and $x = x(y)$, with $x(y)$ being the inverse function of $y(x)$. The chain rule tells us $\frac{dx}{dy}\frac{dy}{dx} = \frac{dx}{dx} = 1$. Then we get

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \text{ or } x'(y) = \frac{1}{y'(x)|_{x=x(y)}} = \frac{1}{y'(x(y))}.$$

Specifically, for $y = \arcsin x$, we have $x = \sin y$. then

$$\frac{d\arcsin x}{dx} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Exercise 3.2.35. Derive the derivatives of the inverse trigonometric functions

$$(\arctan x)' = \frac{1}{1+x^2}, \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad (\text{arcsec}x)' = \frac{1}{x\sqrt{x^2-1}}.$$

Exercise 3.2.36. Compute the derivatives.

1. $\arcsin\sqrt{x}$.
2. $\arcsin\sqrt{1-x^2}$.
3. $\arctan\frac{1+x}{1-x}$.
4. $\frac{\arccos x}{x} + \frac{1}{2}\log\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}$.
5. $\frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\log(1-x^2)$.
6. $\arctan\frac{1-x}{\sqrt{2x-x^2}}$.

Exercise 3.2.37. Compute the derivatives.

1. $2\arcsin\sqrt{\frac{x-a}{b-a}}$.
2. $\frac{1}{a}\arcsin\frac{a}{x}$.
3. $\frac{1}{a}\text{arcsec}\frac{x}{a}$.
4. $\frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}$.
5. $-\sqrt{x(a-x)} - a\arctan\frac{\sqrt{x(a-x)}}{x-a}$.
6. $\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\arctan\frac{x}{\sqrt{a^2-x^2}}$.
7. $x\log(x^2+a^2) + 2a\arctan\frac{x}{a} - 2x$.

Exercise 3.2.38. Compute the derivative of $\arcsin\frac{x}{a}$. Then use the idea and result of Example 3.2.6 to find a function with derivative $\frac{1}{\sqrt{ax^2+bx+c}}$.

Exercise 3.2.39. Compute the derivative of $x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}$. Then combine with the result of Exercise 3.2.25 to find a function with derivative $\sqrt{ax^2+bx+c}$.

Exercise 3.2.40. Compute the derivative of $\frac{1}{a}\arctan\frac{x}{a}$. Then for the case $a^2\le 4b$, find a function with derivative $\frac{1}{x^2+ax+b}$. This complements Exercise 3.2.24.

Exercise 3.2.41. Compute the derivative of $\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$ and $\arctan\sqrt{x}$. Then find a function with derivative $\frac{1}{x\sqrt{ax+b}}$.

Exercise 3.2.42. Suppose $f(x)$ is invertible, with $f(1)=1$, $f'(1)=a$. Find the derivative of the functions $f\left(\frac{1}{f^{-1}\left(\frac{x}{f(x)}\right)}\right)$ and $f^{-1}(f^{-1}(x))$ at 1.

Exercise 3.2.43. Explain the formula for the derivative of the inverse function by considering the inverse of the linear approximation.

Exercise 3.2.44. Find the place on the curve $y=x^2$ where the tangent line is parallel to the straight line $x+y=1$.

Exercise 3.2.45. Show that the area enclosed by the tangent line on the curve $xy=a^2$ and the coordinate axes is a constant.

Exercise 3.2.46. Let P be a point on the curve $y=x^3$. The tangent at P meets the curve again at Q. Prove that the slope of the curve at Q is four times the slope at P.

3.3 Application of Linear Approximation

The linear approximation can be used to determine behaviors of functions. The idea is that, if the linear approximation of a function has certain behavior, then the function is likely to have the similar behavior.

3.3.1 Monotone Property and Extrema

We say a function $f(x)$ has *local maximum* at x_0, if

$$x\in\text{domain},\ |x-x_0|<\delta \implies f(x)\le f(x_0).$$

Similarly, $f(x)$ has *local minimum* at x_0, if

$$x\in\text{domain},\ |x-x_0|<\delta \implies f(x)\ge f(x_0).$$

The function has a *(global) maximum* at x_0 if $f(x_0) \geq f(x)$ for *all* x in the domain

$$x \in \text{domain} \implies f(x) \leq f(x_0).$$

The concepts of *(global) minimum* can be similarly defined. The maximum and minimum are *extrema* of the function. A global extreme is also a local extreme.

The local maxima are like the peaks in a mountain, and the global maximum is like the highest peak.

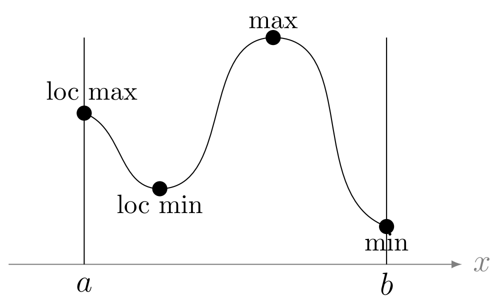


Figure 3.3.1: Local and global extrema.

The following result shows the existence of global extrema in certain case.

Theorem 3.3.1. *Any continuous function on a bounded closed interval has global maximum and global minimum.*

If a function f is increasing on $(x_0-\delta, x_0]$ (i.e., on the left of x_0 and including x_0), then $f(x_0)$ is the biggest value on $(x_0-\delta, x_0]$. If f is also decreasing on $[x_0, x_0+\delta)$ (i.e., on the right of x_0 and including x_0), then $f(x_0)$ is the biggest value on $[x_0, x_0+\delta)$. In other words, if f changes from increasing to decreasing as we pass x_0 from left to right, then x_0 is a local maximum of f. Similarly, if f changes from decreasing to increasing at x_0, then x_0 is a local minimum.

Example 3.3.1. The square function x^2 is strictly decreasing on $(-\infty, 0]$ because

$$x_1 < x_2 \leq 0 \implies x_1^2 > x_2^2.$$

By the same reason, the function is strictly increasing on $[0, +\infty)$. This leads to the local minimum at 0. In fact, by $x^2 \geq 0 = |0|$ for all x, we know x^2 has a global minimum at 0. The function has no local maximum and therefore no global maximum on $\mathbb{R}$.

On the other hand, if we restrict x^2 to $[-1, 1]$, then x^2 has global minimum at 0 and global maxima at -1 and 1. If we restrict to $[-1, 2]$, then x^2 has global minimum at 0, global maximum at 2, and local (but not global) maximum at -1. If we restrict to $(-1, 2)$, then x^2 has global minimum at 0, and has no local maximum.

Example 3.3.2. The sine function is strictly increasing on $\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$ and is strictly decreasing on $\left[2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$. This implies that $2n\pi + \frac{\pi}{2}$ are local maxima and $2n\pi - \frac{\pi}{2}$ are local minima. In fact, by $\sin\left(2n\pi - \frac{\pi}{2}\right) = -1 \le \sin x \le 1 = \sin\left(2n\pi + \frac{\pi}{2}\right)$, these local extrema are also global extrema.

Exercise 3.3.1. Determine the monotone property and find the extrema for $|x|$

1. on $[-1,1]$.
2. on $(-1,1]$.
3. on $[-2,1]$.
4. on $(-\infty,1]$.

Exercise 3.3.2. Determine the monotone property and find the extrema on $\mathbb{R}$.

1. $|x|$.
2. x^2+2x.
3. $|x^2+2x|$.
4. x^3.
5. x^6.
6. $\frac{1}{x}$.
7. $\sqrt{|x|}$.
8. $x+\frac{1}{x}$.
9. $\frac{1}{x^2+1}$.
10. $\cos x$.
11. $\sin^2 x$.
12. $\sin x^2$.
13. e^x.
14. e^{-x}.
15. $\log x$.
16. x^x.

Exercise 3.3.3. How are the extrema of the function related to the extrema of $f(x)$?

1. $f(x)+a$.
2. $af(x)$.
3. $f(x)^a$.
4. $a^{f(x)}$.

Exercise 3.3.4. How are the extrema of the function related to the extrema of $f(x)$?

1. $f(x+a)$.
2. $f(ax)$.
3. $f(x^2)$.
4. $f(\sin x)$.

Exercise 3.3.5. Is local maximum always the place where the function changes from increasing to decreasing? In other words, can you construct a function $f(x)$ with local maximum at 0, but $f(x)$ is *not* increasing on $(-\delta,0]$ for any $\delta>0$?

Exercise 3.3.6. Compare the global extrema on various intervals in Example 3.3.1 with Theorem 3.3.1.

3.3.2 Detect the Monotone Property

Suppose $f(x)$ is approximated by the linear function $L(x) = f(x_0) + f'(x_0)(x-x_0)$ near x_0. The linear function $L(x)$ is increasing if and only if the slope $f'(x_0) \ge 0$. Since $L(x)$ is very close to $f(x)$, we expect $f(x)$ to be also increasing. The expectation is true if the linear approximation is increasing *everywhere*.

Theorem 3.3.2. *If $f'(x)\ge 0$ on an interval, then $f(x)$ is increasing on the interval. If $f'(x)>0$ on the interval, then $f(x)$ is strictly increasing on the interval.*

Similar statements hold for decreasing functions. Moreover, for a function on a closed interval $[a, b]$, we just need the derivative criterion to be satisfied on (a, b) and the function to be continuous on $[a, b]$.

Example 3.3.3. We have $(x^2)' = 2x < 0$ on $(-\infty, 0)$ and x^2 continuous on $(-\infty, 0]$. Therefore x^2 is strictly decreasing on $(-\infty, 0]$. By the similar reason, x^2 is strictly increasing on $[0, +\infty)$. This implies that 0 is a local minimum. The conclusion is consistent with the observation in Example 3.3.1 obtained by direct inspection.

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f = x^2$	$\searrow$	loc min 0	$\nearrow$
$f' = 2x$	$-$	0	$+$

Example 3.3.4. The function $f(x) = x^3-3x+1$ has derivative $f'(x) = 3(x+1)(x-1)$. The sign of the derivative implies that the function is strictly increasing on $(-\infty, -1]$ and $[1, +\infty)$, and is strictly decreasing on $[-1, 1]$. This implies that -1 is a local maximum and 1 is a local minimum.

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, +\infty)$
$f = x^3 - 3x + 1$	$\nearrow$	loc max 3	$\searrow$	loc min -1	$\nearrow$
$f' = 3(x + 1)(x - 1)$	$+$	0	$-$	0	$+$

Example 3.3.5. The function $f(x) = \sin x - x\cos x$ has derivative $f'(x) = x\sin x$. The sign of the derivative determines the strict monotone property on the interval $[-5, 5]$ as described in the picture. The strict monotone property implies that $-\pi, 5$ are local minima, and $-5, \pi$ are local maxima.

x	-5	$(-5, -\pi)$	$-\pi$	$(-\pi, 0)$	0	$(0, \pi)$	π	$(\pi, 5)$	5
f	max	$\searrow$	min	$\nearrow$		$\nearrow$	max	$\searrow$	min
f'		$-$	0	$+$	0	$+$	0	$-$	

Example 3.3.6. The function $f(x) = \sqrt[3]{x^2}(x + 1)$ has derivative $f'(x) = \dfrac{(5x + 2)}{3\sqrt[3]{x}}$ for $x \neq 0$. Using the sign of the derivative and the continuity, we get the strict monotone property of the function, which implies that $-\dfrac{2}{5}$ is a local maximum, and 0 is a local minimum.

x	$(-\infty, -\frac{2}{5})$	$-\frac{2}{5}$	$(-\frac{2}{5}, 0)$	0	$(0, +\infty)$
f	$\nearrow$	max	$\searrow$	min	$\nearrow$
f'	$+$	0	$-$	no	$+$

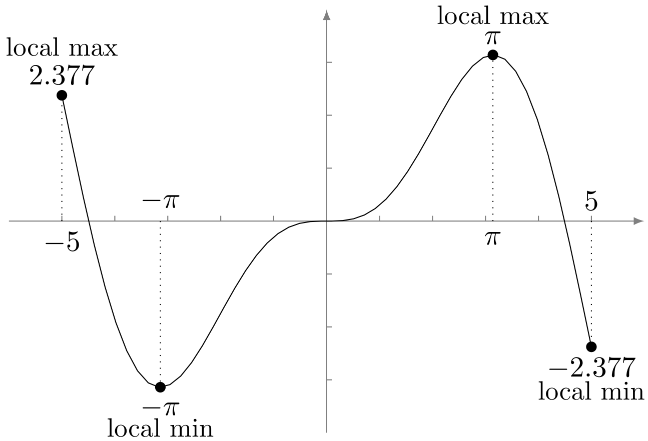


Figure 3.3.2: Graph of $\sin x - x\cos x$.

Example 3.3.7. The function $f(x) = \dfrac{x^3}{x^2-1}$ has derivative $f'(x) = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$ for $x \neq \pm 1$. The sign of the derivative determines the strict monotone property away from ± 1. The strict monotone property implies that $-\sqrt{3}$ is a local minimum, and $\sqrt{3}$ is a local maximum.

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f	$\nearrow$	max	$\searrow$		$\searrow$		$\searrow$		$\searrow$	min	$\nearrow$
f'	$+$	0	$-$	no	$-$	0	$-$	no	$-$	0	$+$

Exercise 3.3.7. Determine the monotone property and find extrema.

1. $x^3 - 3x + 2$ on $\mathbb{R}$.
2. $x^3 - 3x + 2$ on $[-1, 2]$.
3. $x^3 - 3x + 2$ on $(-1, 2)$.
4. $|x^3 - 3x + 2|$ on $\mathbb{R}$.
5. $|x^3 - 3x + 2|$ on $[-1, 2]$.
6. $|x^3 - 3x + 2|$ on $(-1, 2)$.
7. $\sqrt{|x^3 - 3x + 2|}$.
8. $\sqrt[7]{(x^3 - 3x + 2)^2}$.
9. $\dfrac{1}{x^3 - 3x + 2}$.

Exercise 3.3.8. Determine the monotone property and find extrema.

1. $\dfrac{x}{1+x^2}$ on $[-1, 1]$.
2. $\dfrac{\sin x}{1+\sin^2 x}$ on $[0, 2\pi]$.
3. $\dfrac{\cos x}{1+\cos^2 x}$ on $[0, 2\pi]$.
4. $\dfrac{1+x^2}{x}$ on $[-1, 0) \cup (0, 1]$.
5. $\tan x + \cot x$ on $\left[-\dfrac{\pi}{4}, 0\right) \cup \left(0, \dfrac{\pi}{4}\right]$.
6. $e^x + e^{-x}$ on $\mathbb{R}$.

Exercise 3.3.9. Determine the monotone property and find extrema.

1. x^4 on $[-1,1]$.
2. $\cos^4 x$ on $\mathbb{R}$.
3. $\sin^2 x$ on $\mathbb{R}$.

Exercise 3.3.10. Determine the monotone property and find extrema.

1. $-x^4+2x^2-1$ on $[-2,2]$.
2. $\sqrt{3+2x-x^2}$ on $(-1,3]$.
3. $|x|^p(x+1)$ on $\mathbb{R}$.
4. x^2e^x on $\mathbb{R}$.
5. x^pa^x on $(0,+\infty)$.
6. $|x|e^{-|x-1|}$ on $[-2,2]$.
7. $x\log x$ on $(0,+\infty)$.
8. $x^2\log^3 x$ on $(0,+\infty)$.
9. $x^p\log x$ on $(0,+\infty)$.
10. $x^3+3\log x$ on $(0,+\infty)$.
11. $x-\log(1+x)$ on $(-1,+\infty)$.
12. $e^{-x}\sin x$ on $\mathbb{R}$.
13. $x-\sin x$ on $[0,2\pi]$.
14. $|x-\sin x|$ on $[-\pi,\pi]$.
15. $|x-\sin x|$ on $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$.
16. $2\sin x+\sin 2x$ on $[0,2\pi]$.
17. $2x-4\sin x+\sin 2x$ on $[0,\pi]$.

Exercise 3.3.11. Show that $2x+\sin x=c$ has only one solution. Show that $x^4+x=c$ has at most two solutions.

Exercise 3.3.12. If f is differentiable and has 9 roots on (a,b), how many roots does f' have on (a,b)? If f also has second order derivative, how many roots does f'' have on (a,b)?

Exercise 3.3.13. Find smallest $A>0$, such that $\log x\le A\sqrt{x}$. Find smallest $B>0$, such that $\log x\ge -\dfrac{B}{\sqrt{x}}$.

Exercise 3.3.14. A quantity is measured n times, yielding the measurements $x_1,\dots,x_n$. Find the estimate value $\hat{x}$ of x that minimizes the squared error $(x-x_1)^2+\cdots+(x-x_n)^2$.

Exercise 3.3.15. Find the biggest term in the sequence $\sqrt[n]{n}$.

3.3.3 Compare Functions

If we apply Theorem 3.3.2 to $f(x)-g(x)$, then we get the following comparison of two functions.

Theorem 3.3.3. *Suppose $f(x)$ and $g(x)$ are continuous for $x\ge a$ and differentiable for $x>a$. If $f(a)\ge g(a)$ and $f'(x)\ge g'(x)$ for $x>a$, then $f(x)\ge g(x)$ for $x>a$. If $f(a)\ge g(a)$ and $f'(x)>g'(x)$ for $x>a$, then $f(x)>g(x)$ for $x>a$.*

There is a similar statement for the case $x<a$.

Example 3.3.8. We have $e^x > 1$ for $x > 0$ and $e^x < 1$ for $x < 0$. This is the comparison of e^x with the constant term of the Taylor expansion (or the 0th Taylor expansion) in Example 4.1.5. How do we compare e^x with the first order Taylor expansion $1 + x$?

We have $e^0 = 1+0$. For $x > 0$, we have $(e^x)' = e^x > (1+x)' = 1$. Therefore we get $e^x > 1+x$ for $x > 0$. On the other hand, for $x < 0$, we have $(e^x)' = e^x < (1+x)' = 1$. Therefore we also get $e^x > 1 + x$ for $x < 0$. We conclude that

$$e^x > 1 + x \text{ for } x \neq 0.$$

Example 3.3.9. We claim that

$$\frac{x}{1+x} < \log(1+x) < x \text{ for } x > -1,\ x \neq 0.$$

The three functions have the same value 0 at 0. Then we compare their derivatives

$$\left(\frac{x}{1+x}\right)' = \frac{1}{(1+x)^2}, \quad (\log(1+x))' = \frac{1}{1+x}, \quad (x)' = 1.$$

We have

$$\frac{1}{(1+x)^2} < \frac{1}{1+x} < 1 \text{ for } x > 0,$$

and

$$\frac{1}{(1+x)^2} > \frac{1}{1+x} > 1 \text{ for } -1 < x < 0.$$

The inequalities then follow from Theorem 3.3.3.

Example 3.3.10. For $a, b > 0$ and $p > q > 0$, we claim that

$$(a^p + b^p)^{\frac{1}{p}} < (a^q + b^q)^{\frac{1}{q}}.$$

By symmetry, we may assume $a \leq b$. Then $c = \dfrac{b}{a} > 1$, and the inequality means that $f(x) = (1 + c^x)^{\frac{1}{x}}$ is strictly decreasing for $x > 0$.

By Example 3.2.8, we have

$$\begin{aligned} f'(x) &= (1 + c^x)^{\frac{1}{x}-1}\left(-\frac{1+c^x}{x^2}\log(1+c^x) + c^x(\log c)\frac{1}{x}\right) \\ &= \frac{(1+c^x)^{\frac{1}{x}-1}}{x^2}(c^x \log c^x - (1+c^x)\log(1+c^x)). \end{aligned}$$

So we study the monotone property of the function $g(t) = t\log t$. By

$$g'(t) = \log t + 1 > 0, \text{ for } t > e^{-1},$$

we see that $g(t)$ is increasing for $t > e^{-1}$. Since $c \geq 1$ and $x > 0$ implies $1 + c^x > c^x \geq 1 > e^{-1}$, we get $c^x \log c^x \leq (1 + c^x)\log(1 + c^x)$. Therefore $f'(x) < 0$ and $f(x)$ is decreasing.

Exercise 3.3.16. State Theorem 3.3.3 for the case $x < a$.

Exercise 3.3.17. Prove the inequality.

1. $\sin x > \dfrac{2}{\pi}x$, for $0 < x < \dfrac{\pi}{2}$.
2. $\dfrac{1}{2^{p-1}} \le x^p + (1-x)^p \le 1$, for $0 \le x \le 1$, $p > 1$.
3. $\dfrac{\sqrt{3}}{6+2\sqrt{3}} \le \dfrac{1+x}{2+x^2} \le \dfrac{\sqrt{3}}{6-2\sqrt{3}}$.
4. $\left(1+\dfrac{1}{x}\right)^x < e < \left(1+\dfrac{1}{x}\right)^{x+1}$, for $x > 0$.
5. $\arctan x - \arctan y \le 2\arctan\dfrac{x-y}{2}$, for $x > y > 0$.
6. $\dfrac{x^2}{2(1+x)} < x - \log(1+x) < \dfrac{x^2}{2}$, for $x > 0$. What about $-1 < x < 0$?

Exercise 3.3.18. For natural number n and $0 < a < 1$, prove that the equation

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = ae^x$$

has only one solution on $(0, +\infty)$.

3.3.4 First Derivative Test

We saw that the local extrema are often the places where the function changes between increasing and decreasing. If the function is differentiable, then these are the places where the derivative changes the sign. In particular, we expect the derivatives at these places to become 0. This leads to the following criterion for the candidates of local extrema.

Theorem 3.3.4. *If $f(x)$ is differentiable at a local extreme x_0, then $f'(x_0) = 0$.*

If $f'(x_0) > 0$, then the linear approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$ of f near x_0 is strictly increasing. This means that $L(x) < L(x_0)$ for $x < x_0$ and $L(x) > L(x_0)$ for $x > x_0$. Since L is very close to f near x_0, we expect that f is also "lower" on the left of x_0 and "higher" on the right of x_0. In particular, this implies that x_0 is not a local extreme of f. By the similar argument, if $f'(x_0) < 0$, the x_0 is also not a local extreme. This is the reason behind the theorem.

Since our reason makes explicit use of both the left and right sides, the criterion does not work for one sided derivatives. Therefore for a function f defined on an interval, the candidates for the local extrema must be one of the following cases:

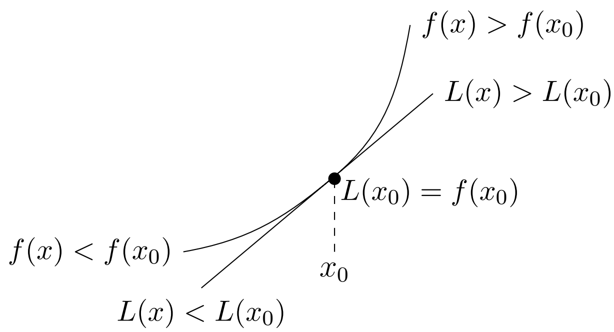


Figure 3.3.3: What happens near x_0 when $f'(x_0) > 0$.

1. End points of the interval.
2. Points inside the interval where f is not differentiable.
3. Points inside the interval where f is differentiable and has derivative 0.

Example 3.3.11. The derivative $(x^2)' = 2x$ vanishes only at 0. Therefore the only candidate for the local extrema of x^2 on $\mathbb{R}$ is 0. By $x^2 \geq 0^2$ for all x, 0 is a minimum.

If we restrict x^2 to the closed interval $[-1, 2]$, then the end points -1 and 2 are also candidates for the local extrema. By $x^2 \leq (-1)^2$ on $[-1, 0]$ and $x^2 \leq 2^2$ on $[-1, 2]$, -1 is a local maximum and 2 is a global maximum.

On the other hand, the restriction of x^2 on the open interval $(-1, 2)$ has no other candidates for local extrema besides 0. The function has global minimum at 0 and has no local maximum on $(-1, 2)$.

Example 3.3.12. Consider the function

$$f(x) = \begin{cases} x^2, & \text{if } x \neq 0, \\ 2, & \text{if } x = 0, \end{cases}$$

that modifies the square function by reassigning the value at 0. The function is not differentiable at 0 and has nonzero derivative away from 0. Therefore on $[-1, 2]$, the candidates for the local extrema are 0 and the end points -1 and 2. The end points are also local maxima, like the unmodified x^2. By $x^2 < f(0) = 2$ on $[-1, 1]$, 0 is a local maximum. The modified square function $f(x)$ has no local minimum on $[-1, 2]$.

Example 3.3.13. The function $f(x) = x^3 - 3x + 1$ in Example 3.3.4 has derivative $f'(x) = 3(x+1)(x-1)$. The possible local extrema on $\mathbb{R}$ are ± 1. These are not the global extrema on the whole line because $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$.

If we restrict the function to $[-2, 2]$, then ± 2 are also possible local extrema. By comparing the values

$$f(-2) = -1, \quad f(-1) = 3, \quad f(1) = -1, \quad f(2) = 3,$$

we get global minima at $-2, 1$, and global maxima at $2, -1$.

Example 3.3.14. By $(x^3)' = 3x^2$, 0 is the only candidate for the local extreme of x^3. However, we have $x^3 < 0^3$ for $x < 0$ and $x^3 > 0^3$ for $x > 0$. Therefore 0 is actually not a local extreme.

The example shows that the converse of Theorem 3.3.4 is not true.

Example 3.3.15. The function $f(x) = xe^{-x}$ has derivative $f'(x) = (1-x)e^{-x}$. The only possible local extreme on $\mathbb{R}$ is at 1. We have $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = 0$. We claim that the limits imply that $f(1) = e^{-1}$ is a global maximum.

Since the limits at both infinity are $< f(1)$, there is N, such that $f(x) < f(1)$ for $|x| \geq N$. In particular, we have $f(\pm N) < f(1)$. Then consider the function on $[-N, N]$. On the bounded and closed interval, Theorem 3.3.1 says that the continuous function must reach its maximum, and the candidates for the maximum on $[-N, N]$ are $-N, 1, N$. Since $f(\pm N) < f(1)$, we see that $f(1)$ is the maximum on $[-N, N]$. Combined with $f(x) < f(1)$ for $|x| \geq N$, we conclude that $f(1)$ is the maximum on the whole real line.

Exercise 3.3.19. Find the global extrema.

1. $x^2(x-1)^3$ on $\mathbb{R}$.
2. $x^2(x-1)^3$ on $[-1, 1]$.
3. $|x^2 - 1|$ on $[-2, 1]$.
4. $x^2 + bx + c$ on $\mathbb{R}$.
5. $1 - x^4 + x^5$ on $\mathbb{R}$.
6. $\sin x^2$ on $[-1, \sqrt{\pi}]$.
7. $x \log x$ on $(0, +\infty)$.
8. $x \log x$ on $(0, 1]$.
9. x^x on $(0, 1]$.
10. $(x^2 + 1)e^x$ on $\mathbb{R}$.

3.3.5 Optimization Problem

Example 3.3.16. Given the circumference a of a rectangle, which rectangle has the largest area?

Let one side of the rectangle be x. Then the other side is $\dfrac{a}{2} - x$, and the area

$$A(x) = x\left(\frac{a}{2} - x\right).$$

The problem is to find the maximum of $A(x)$ on $\left[0, \dfrac{a}{2}\right]$.

By $A'(x) = \dfrac{a}{2} - 2x$, the candidates for the local extrema are $0, \dfrac{a}{4}, \dfrac{a}{2}$. The values of A at the three points are $0, \dfrac{a^2}{16}, 0$. Therefore the maximum is reached when $x = \dfrac{a}{4}$, which means the rectangle is a square.

Example 3.3.17. The distance from a point $P = (x_0, y_0)$ on the plane to a straight line $ax + by + c = 0$ is the minimum of the distance from P to a point (x, y) on the line. The distance is minimum when the square of the distance

$$f(x) = (x - x_0)^2 + (y - y_0)^2$$

is minimum. Note that y is a function of x given by the equation $ax + by + c = 0$ and satisfies $a + by' = 0$.

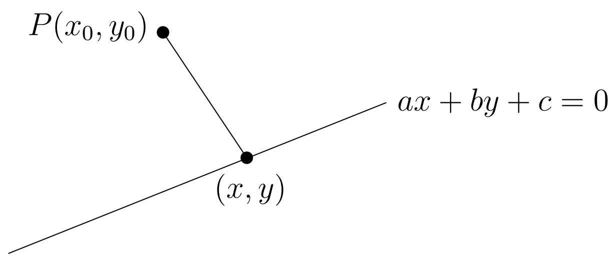


Figure 3.3.4: Distance from a point to a straight line.

From

$$f'(x) = 2(x - x_0) + 2(y - y_0)y' = \frac{2}{b}(b(x - x_0) - a(y - y_0)),$$

we know that $f(x)$ is minimized when

$$b(x - x_0) - a(y - y_0) = 0.$$

Moreover, recall that (x, y) must also be on the straight line

$$ax + by + c = 0.$$

Solving the system of two linear equations, we get

$$x - x_0 = -\frac{a(ax_0 + by_0 + c)}{a^2 + b^2}, \quad y - y_0 = -\frac{b(ax_0 + by_0 + c)}{a^2 + b^2}.$$

The minimum distance is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

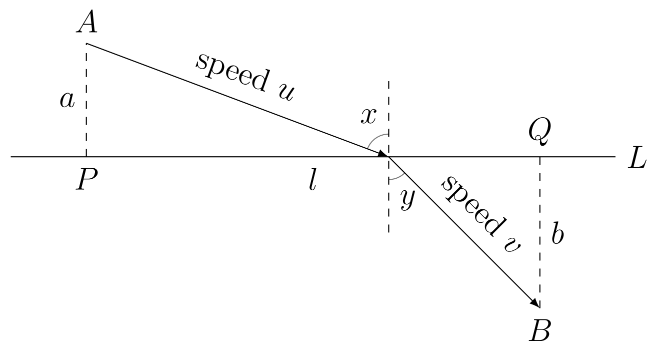


Figure 3.3.5: Snell's law.

Example 3.3.18. Consider light traveling from a point A in one medium to point B in another medium. Fermat's principle says that the path taken by the light is the path of shortest traveling time.

Let u and v be the speed of light in the respective medium. Let L be the place where two media meet. Draw lines AP and BQ perpendicular to L. Let the length of AP, BP, PQ be a, b, l. Let x be the angle by which the light from A hits L. Let y be the angle by which the light leaves L and reaches B.

The angles x and y are related by

$$a \tan x + b \tan y = l.$$

This can be considered as an equation that implicitly defines y as a function of x. The derivative of $y = y(x)$ can be obtained by implicit differentiation

$$y'(x) = -\frac{a \sec^2 x}{b \sec^2 y}.$$

The time it takes for the light to travel from A to B is

$$T = \frac{a \sec x}{u} + \frac{b \sec y}{v}.$$

By thinking of y as a function of x, the time T becomes a function of x. The time will be shortest when

$$\begin{aligned}
0 = \frac{dT}{dx} &= \frac{a \sec x \tan x}{u} + \frac{b \sec y \tan y}{v} y' \\
&= \frac{a \sec x \tan x}{u} - \frac{b \sec y \tan y}{v} \frac{a \sec^2 x}{b \sec^2 y} \\
&= a \sec^2 x \left(\frac{\sin x}{u} - \frac{\sin y}{v} \right).
\end{aligned}$$

This means that the ratio between the sine of the angles x and y is the same as the ratio between the speeds of light

$$\frac{\sin x}{\sin y} = \frac{u}{v}.$$

This is *Snell's law of refraction*.

Exercise 3.3.20. A rectangle is inscribed in an isosceles triangle. Show that the biggest area possible is half of the area of the triangle.

Exercise 3.3.21. Among all the rectangles with area A, which one has the smallest perimeter?

Exercise 3.3.22. Among all the rectangles with perimeter L, which one has the biggest area?

Exercise 3.3.23. A rectangle is inscribed in a circle of radius R. When does the rectangle have the biggest area?

Exercise 3.3.24. Determine the dimensions of the biggest rectangle inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Exercise 3.3.25. Find the volume of the biggest right circular cone with a given slant height l.

Exercise 3.3.26. What is the shortest distance from the point $(2,1)$ to the parabola $y = 2x^2$?

3.4 Mean Value Theorem

If one travels between two cities at the average speed of 100 kilometers per hour, then we expect that the speed reaches exactly 100 kilometers per hour somewhere during the trip. Mathematically, let $f(t)$ be the distance traveled by the time t. Then the average speed from the time a to time b is

$$\frac{f(b)-f(a)}{b-a}.$$

Our expectation can be interpreted as

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ for some } c \in (a,b).$$

3.4.1 Mean Value Theorem

Theorem 3.4.1 (Mean Value Theorem). *If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is $c \in (a,b)$, such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

The conclusion can also be expressed as

$$f(b) - f(a) = f'(c)(b - a), \text{ for some } a < c < b,$$

or

$$f(a + h) - f(a) = f'(a + \theta h)h, \text{ for some } 0 < \theta < 1.$$

We also note that the conclusion is symmetric in a, b. Therefore there is no need to insist $a < b$.

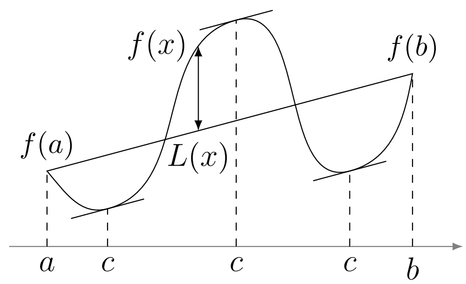


Figure 3.4.1: Mean value theorem.

Geometrically, the Mean Value Theorem means that the straight line L connecting the two ends $(a, f(a))$ and $(b, f(b))$ of the graph of f is parallel to the tangent of the function somewhere. Figure 3.4.1 suggests that c in the Mean Value Theorem is the place where the the distance between the graphs of f and L has local extrema. Since such local extrema for the distance $f(x) - L(x)$ always exists by Theorem 3.3.1, we get $(f - L)'(c) = 0$ for some c by Theorem 3.3.4. Therefore $f'(c) = L'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Example 3.4.1. We try to verify the Mean Value Theorem for $f(x) = x^3 - 3x + 1$ on $[-1, 1]$. This means finding c, such that

$$f'(c) = 3(c^2 - 1) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{-1 - 3}{1 - (-1)} = -2.$$

We get $c = \pm\dfrac{1}{\sqrt{3}}$.

Example 3.4.2. By the Mean Value Theorem, we have

$$\log(1 + x) = \log(1 + x) - \log 1 = \frac{1}{1 + \theta x}x \text{ for some } 0 < \theta < 1.$$

Since

$$\frac{x}{1 + x} \leq \frac{1}{1 + \theta x}x \leq x \text{ for } x > -1,$$

we conclude that

$$\frac{x}{1+x} \le \log(1+x) \le x.$$

The inequality already appeared in Example 3.3.9.

Example 3.4.3. For the function $|x|$ on $[-1,1]$, there is no $c \in (-1,1)$ satisfying

$$f(1) - f(-1) = 0 = f'(c)(1-(-1)).$$

The Mean Value Theorem does not apply because $|x|$ is not differentiable at 0.

Exercise 3.4.1. Is the conclusion of the Mean Value Theorem true? If true, find c. If not, explain why.

1. x^3 on $[-1,1]$.
2. 2^x on $[0,1]$.
3. $\dfrac{1}{x}$ on $[1,2]$.
4. $|x^3-3x+1|$ on $[-1,1]$.
5. $\sqrt{|x|}$ on $[-1,1]$.
6. $\cos x$ on $[-a,a]$.
7. $\log x$ on $[1,2]$.
8. $\log|x|$ on $[-1,1]$.
9. $\arcsin x$ on $[0,1]$.

Exercise 3.4.2. Suppose $f(1) = 2$ and $f'(x) \le 3$ on $\mathbb{R}$. How large and how small can $f(4)$ be? What happens when the largest or the smallest value is reached? How about $f(-4)$?

Exercise 3.4.3. Prove inequality.

1. $|\sin x - \sin y| \le |x-y|$.
2. $\dfrac{x-y}{x} < \log\dfrac{x}{y} < \dfrac{x-y}{y}$, for $x > y > 0$.
3. $|\arctan x - \arctan y| \le |x-y|$.

Exercise 3.4.4. Find the biggest interval on which $|e^x - e^y| > |x-y|$? What about $|e^x - e^y| < |x-y|$?

Exercise 3.4.5 (????? from analysis lecture note). Suppose $f(x)$ is continuous at x_0 and differentiable on $(x_0-\delta, x_0) \cup (x_0, x_0+\delta)$. Prove that if $\lim_{x\to x_0} f'(x) = l$ converges, then $f(x)$ is differentiable at x_0 and $f'(x_0) = l$.

3.4.2 Criterion for Constant Function

The Mean Value Theorem can be used to prove Theorem 3.3.2: If $f' \ge 0$ on an interval, then for any $x_1 < x_2$ on the interval, by the Mean Value Theorem, we have

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \ge 0.$$

For the special case $f' = 0$ throughout the interval, the argument gives the following result.

Theorem 3.4.2. *If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is a constant on (a, b).*

Applying the theorem to $f(x) - g(x)$, we get the following result.

Theorem 3.4.3. *If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there is a constant C, such that $f(x) = g(x) + C$ on (a, b).*

Example 3.4.4. The function e^x satisfies $f(x)' = f(x)$. Are there any other functions satisfying the equation?

If $f(x)' = f(x)$. Then

$$(e^{-x}f(x))' = (e^{-x})'f(x) + e^{-x}(f(x))' = e^x(-f(x) + f'(x)) = 0.$$

Therefore $e^{-x}f(x) = C$ is a constant, and $f(x) = Ce^x$.

Example 3.4.5. Suppose $f'(x) = x$ and $f(1) = 2$. Then $f'(x) = \left(\frac{x^2}{2}\right)'$ implies $f(x) = \frac{x^2}{2} + C$ for some constant C. By taking $x = 1$, we get $2 = \frac{1}{2} + C$. Therefore $C = \frac{3}{2}$ and $f(x) = \frac{x^2+3}{2}$.

Example 3.4.6. By

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}.$$

we have $(\arcsin x + \arccos x)' = 0$, and we have $\arcsin x + \arccos x = C$. The constant can be determined by taking a special value $x = 0$

$$C = \arcsin 0 + \arccos 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

Therefore we have

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

Exercise 3.4.6. Prove that a differentiable function is linear on an interval if and only if its derivative is a constant.

Exercise 3.4.7. Find all functions on an interval satisfying the following equations.

1. $f'(x) = -2f(x)$.
2. $f'(x) = xf(x)$.
3. $f'(x) = f(x)^2$
4. $f'(x)f(x) = 1$.

Exercise 3.4.8. Prove equality.

1. $\arctan x + \arctan x^{-1} = \dfrac{\pi}{2}$, for $x \neq 0$.

2. $3\arccos x - \arccos(3x - 4x^3) = \pi$, for $|x| \leq \dfrac{1}{2}$.

3. $\arctan \dfrac{x+a}{1-ax} - \arctan x = \arctan a$, for $ax < 1$.

4. $\arctan \dfrac{x+a}{1-ax} - \arctan x = \arctan a - \pi$, for $ax > 1$.

3.4.3 L'Hospital's Rule

The following limits cannot be computed by simple arithmetic rules.

$$\begin{aligned}
\frac{0}{0}&\colon \lim_{x\to 1}\frac{x^3-1}{x-1},\ \lim_{x\to 0}\frac{\sin x}{x},\ \lim_{x\to 0}\frac{\log(1+x)}{x};\\
\frac{\infty}{\infty}&\colon \lim_{x\to 0}\frac{\log x}{x^{-1}},\ \lim_{x\to\infty}\frac{\log x}{x},\ \lim_{x\to\infty}\frac{x^2}{e^x};\\
1^{\infty}&\colon \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x,\ \lim_{x\to 0}(1+\sin x)^{\log x};\\
\infty-\infty&\colon \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right).
\end{aligned}$$

We say these limits are *indeterminate*. Other indeterminates include $0\cdot\infty$, $\infty+\infty$, 0^0, ∞^0. The derivative can help us computing such limits.

Theorem 3.4.4 (L'Hospital's Rule). *Suppose $f(x)$ and $g(x)$ are differentiable functions on (a,b), with $g'(x) \neq 0$. Suppose*

1. *Either* $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$ *or* $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty$.

2. $\lim_{x\to a^+} \dfrac{f'(x)}{g'(x)} = l$ *converges.*

Then $\lim_{x\to a^+} \dfrac{f(x)}{g(x)} = l$.

The theorem computes the limits of the indeterminates of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. The conclusion is the equality

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

whenever the right side converges. It is possible that the left side converges but the right side diverges.

The theorem also has a similar left sided version, and the left and right sided versions may be combined to give the two sided version. Moreover, l'Hospital's rule also allows a or l to be any kind of infinity.

The reason behind l'Hospital's rule is the following version of the Mean Value Theorem, which can be proved similar to the Mean Value Theorem.

Theorem 3.4.5 (Cauchy's Mean Value Theorem). *If $f(x)$ and $g(x)$ are continuous on $[a,b]$ and differentiable on (a,b), such that $g'(x) \neq 0$ on (a,b), then there is $c \in (a,b)$, such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Consider the parametrized curve $(g(t), f(t))$ for $t \in [a,b]$. The theorem says that the straight line connecting the two ends $(g(a), f(a))$ and $(g(b), f(b))$ of the curve is parallel to the tangent of the curve somewhere. The slope of the tangent is $\dfrac{f'(c)}{g'(c)}$.

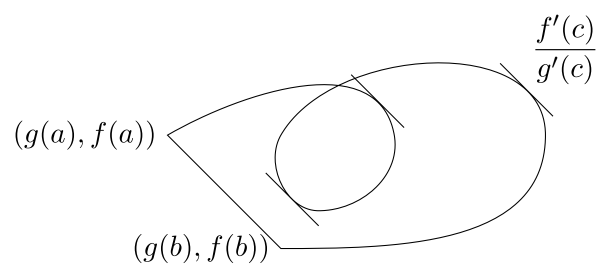


Figure 3.4.2: Cauchy's Mean Value Theorem.

For the case $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$ of l'Hospital's rule, we may extend f and g to continuous functions on $[a,b)$ by assigning $f(a) = g(a) = 0$. Then for any $a < x < b$, we may apply Cauchy's Mean Value Theorem to the functions on $[a,x]$ and get

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)} \text{ for some } c \in (a,x).$$

Since $x \to a^+$ implies $c \to a^+$, we conclude that $\lim_{c\to a^+} \dfrac{f'(c)}{g'(c)} = l$ implies $\lim_{x\to a^+} \dfrac{f(x)}{g(x)} = l$.

By changing the variable x to $\dfrac{1}{x}$, it is not difficult to extend the proof to the case $a = \pm\infty$. The proof for the $\dfrac{\infty}{\infty}$ type is must more complicated and is omitted here.

Example 3.4.7. In Example 1.2.9, we proved that $\lim_{x\to+\infty} a^x = 0$ for $0 < a < 1$. Exercise **??** extended the limit to $\lim_{x\to+\infty} x^2 a^x = 0$. We derive the second limit from the first one by using l'Hospital's rule.

We have $b = \dfrac{1}{a} > 1$, and $\lim_{x\to+\infty} a^x = 0$ is the same as $\lim_{x\to+\infty} b^x = \infty$. We also have $\lim_{x\to+\infty} x^2 = \infty$. Therefore $\lim_{x\to+\infty} x^2 a^x = \lim_{x\to+\infty} \dfrac{x^2}{b^x}$ is of type $\dfrac{\infty}{\infty}$, and we may apply l'Hospital's rule (twice)

$$\begin{aligned}\lim_{x\to+\infty}\frac{x^2}{b^x} &=_{(3)} \lim_{x\to+\infty}\frac{(x^2)'}{(b^x)'} = \lim_{x\to+\infty}\frac{2x}{b^x\log b}\\ &=_{(2)} \lim_{x\to+\infty}\frac{(2x)'}{(b^x\log b)'} = \lim_{x\to+\infty}\frac{2}{b^x(\log b)^2} =_{(1)} 0.\end{aligned}$$

Here is the precise reason behind the computation. The equality $=_{(1)}$ is from Example 1.2.9. Then by l'Hospital's rule, the convergence of the right side of $=_{(2)}$ implies the convergence of the left side of $=_{(2)}$ and the equality $=_{(2)}$ itself. The left of $=_{(2)}$ is the same as the right side of $=_{(3)}$. By l'Hospital's rule gain, the convergence of the right side of $=_{(3)}$ implies the convergence of the left side of $=_{(3)}$ and the equality $=_{(3)}$.

Example 3.4.8. Applying l'Hospital's rule to the limit $\lim_{x\to 0}\dfrac{\sin x}{x}$ of type $\dfrac{0}{0}$, we get

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{(\sin x)'}{(x)'} = \lim_{x\to 0}\cos x = 1.$$

However, this argument is logically circular because it makes use of the formula $(\sin x)' = \cos x$. A special case of this formula is

$$\lim_{x\to 0}\frac{\sin x}{x} = (\sin x)'|_{x=0} = 1,$$

which is exactly the conclusion we try to get.

Exercise 3.4.9. Are the application of l'Hospital's rule logically circular?

1. $\lim_{x\to 0}\dfrac{x}{\sin x}$.
2. $\lim_{x\to 0}\dfrac{\cos x - 1}{x}$.
3. $\lim_{x\to 0}\dfrac{\cos x - 1}{x^2}$.
4. $\lim_{x\to 1}\dfrac{x^2 - 1}{x - 1}$.
5. $\lim_{x\to 1}\dfrac{x^2 - 1}{x^3 - 1}$.
6. $\lim_{x\to 0}\dfrac{\log(1+x)}{x}$.

Example 3.4.9. By blindly using l'Hospital's rule four times, we have

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x-\sin x^2}{x^4}&=\lim_{x\to 0}\frac{2\sin x\cos x-2x\cos x^2}{4x^3}\\&=\lim_{x\to 0}\frac{\cos^2 x-\sin^2 x-\cos x^2+2x^2\sin x^2}{6x^2}\\&=\lim_{x\to 0}\frac{-4\cos x\sin x+6x\sin x^2+4x^3\cos x^2}{12x}\\&=\lim_{x\to 0}\frac{4\sin^2 x-4\cos^2 x+6\sin x^2+24x^2\cos x^2-8x^4\sin x^2}{12}\\&=-\frac{1}{3}.\end{aligned}$$

We find that it is increasingly difficult to calculate the derivatives. The following compute the the limit after calculating the derivatives twice.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x-\sin x^2}{x^4}&=\lim_{x\to 0}\frac{2\sin x\cos x-2x\cos x^2}{4x^3}\\&=\lim_{x\to 0}\frac{\cos^2 x-\sin^2 x-\cos x^2+2x^2\sin x^2}{6x^2}\\&=\lim_{x\to 0}\left(-\frac{1}{3}\left(\frac{\sin x}{x}\right)^2+\frac{1-\cos x^2}{6x^2}+\frac{1}{3}\sin x^2\right)\\&=-\frac{1}{3}\cdot 1^2+0+\frac{1}{3}\cdot 0=-\frac{1}{3}.\end{aligned}$$

In fact, the smartest way is not to calculate the derivatives at all. See Example 4.1.11.

Exercise 3.4.10. Use l'Hospital's rule to compute the limits.

1. $\lim_{x\to 0}\dfrac{\sin x-x}{x^3}$.
2. $\lim_{x\to 0}\dfrac{\sin x-\tan x}{x^3}$.
3. $\lim_{x\to 0}\dfrac{e^x-e^{\sin x}}{x^3}$.
4. $\lim_{x\to 0}\dfrac{\cos(\sin x)-\cos x}{x^4}$.
5. $\lim_{x\to 0}\dfrac{1-\cos x^2}{x^3\sin x}$.
6. $\lim_{x\to 1}\dfrac{x^x-x}{\log x-x+1}$.
7. $\lim_{x\to 0}\dfrac{x-\tan x}{x-\sin x}$.
8. $\lim_{x\to 1}\dfrac{(x-1)\log x}{1+\cos \pi x}$.

Example 3.4.10. The limit $\lim_{x\to 0^+}x\log x$ is of type $0\cdot\infty$. We convert into type $\dfrac{\infty}{\infty}$ and apply l'Hospital's rule (in the second equality)

$$\lim_{x\to 0^+}x\log x=\lim_{x\to 0^+}\frac{\log x}{x^{-1}}=\lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}}=\lim_{x\to 0^+}-x=0.$$

The similar argument gives

$$\lim_{x\to 0^+} x^p \log x = 0, \text{ for } p > 0.$$

Taking a positive power of the limit, we further get

$$\lim_{x\to 0^+} x^p(-\log x)^q = 0, \text{ for } p, q > 0.$$

By converting x to $\dfrac{1}{x}$, we also have

$$\lim_{x\to +\infty} \frac{(\log x)^q}{x^p} = 0, \text{ for } p, q > 0.$$

Example 3.4.11. We compute the limit in Example 4.1.13 by first converting it to type $\dfrac{0}{0}$ and then applying l'Hospital's rule

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + xe^x} = \lim_{x\to 0} \frac{e^x}{2e^x + xe^x} = \frac{1}{2}.$$

Example 3.4.12. If we apply l'Hospital's rule to the limit $\lim_{x\to\infty} \dfrac{x + \sin x}{x}$ of type $\dfrac{\infty}{\infty}$, then we get

$$\lim_{x\to\infty} \frac{x + \sin x}{x} = \lim_{x\to\infty} (1 + \cos x).$$

We find that the left converges and the right diverges. The reason for the l'Hospital's rule to fail is that the second condition is not satisfied.

Exercises needed ?????????

Chapter 4

Polynomial Approximation

4.1 High Order Approximation

Linear approximations can be used to solve many problems. When linear approximations are not enough, however, we may use high order approximations.

Definition 4.1.1. An *n-th order approximation* of $f(x)$ at x_0 is a degree n polynomial

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n,$$

such that for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| < \delta \implies |f(x) - P(x)| \le \epsilon |x - x_0|^n.$$

A function is *n-th order differentiable* if it has n-order approximation.

The error $R_n(x) = f(x) - P(x)$ of the approximation is called the *remainder*. The definition means that

$$\begin{aligned} f(x) &= P(x) + R_n(x) \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + o((x - x_0)^n), \end{aligned}$$

where the "small o" notation means that the remainder term satisfies

$$\lim_{x \to a} \frac{f(x) - P(x)}{(x - x_0)^n} = \lim_{x \to a} \frac{R_n(x)}{(x - x_0)^n} = 0.$$

The n-th order approximation of a function is unique. See Exercise 4.1.7. Moreover, if $m < n$, then the *truncation* $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_m(x - x_0)^m$ is the m-th order approximation of f at x_0. After all, if we have the 10th order approximation, then we should also have the 5th order approximation.

Example 4.1.1. We have

$$x^4 = (1 + (x-1))^4 = 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4.$$

For any $\epsilon > 0$, we have $|x-1| < \delta = \min\left\{1, \frac{\epsilon}{5}\right\}$ implying

$$\begin{aligned} |x^4 - 1 - 4(x-1) - 6(x-1)^2| &= |4(x-1)^3 + (x-1)^4| \le (4 + |x-1|)|x-1|^3 \\ &\le (4+1)\frac{\epsilon}{5}|x-1|^2 = \epsilon|x-1|^2. \end{aligned}$$

Therefore $1 + 4(x-1) + 6(x-1)^2$ is the quadratic approximation of x^4 at 1. By similar argument, we get approximations of other orders.

$$\begin{aligned} \text{linear}\colon & 1 + 4(x-1), \\ \text{quadratic}\colon & 1 + 4(x-1) + 6(x-1)^2, \\ \text{cubic}\colon & 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3, \\ \text{quartic}\colon & 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4, \\ \text{quintic}\colon & 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4, \\ & \vdots \end{aligned}$$

Example 4.1.2. The limit

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$$

in Example **??** can be interpreted as

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^2} = 0.$$

This means that $\cos x$ is second order differentiable at 0, with quadratic approximation $1 - \frac{1}{2}x^2$.

Exercise 4.1.1. Prove that $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \dfrac{x^{n+1}}{1-x}$. What does this tell you about the differentiability of $\dfrac{1}{1-x}$ at 0?

Exercise 4.1.2. Show that $\dfrac{1}{1+x}$ and $\dfrac{1}{1+x^2}$ are differentiable of arbitrary order. What are their high order approximations?

Exercise 4.1.3. What is the n-th order approximation of $1 + 2x + 3x^2 + \cdots + 100x^{100}$ at 0?

Exercise 4.1.4. Use l'Hospital's rule to compute the limits. Then interpret your results as high order approximations.

1. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$.
2. $\lim_{x\to 0} \dfrac{\sin x^2 - x^2}{x^6}$.
3. $\lim_{x\to 0} \dfrac{\sin^2 x - x^2}{x^4}$.
4. $\lim_{x\to 0} \dfrac{1}{x^5}\left(\sin x - x + \dfrac{1}{6}x^3\right)$.
5. $\lim_{x\to 0} \dfrac{\sin x - \tan x}{x^3}$.
6. $\lim_{x\to 0} \dfrac{1}{x^3}\left(e^x - 1 - x - \dfrac{1}{2}x^2\right)$.
7. $\lim_{x\to 0} \dfrac{1}{x^3}\left(\log(1+x) - x + \dfrac{1}{2}x^2\right)$.
8. $\lim_{x\to 1} \dfrac{2\log x + (x-1)(x-3)}{(x-1)^3}$.

Exercise 4.1.5. For what choice of a, b, c is the function

$$\begin{cases} x^4, & \text{if } x \geq 1, \\ a + bx + cx^2, & \text{if } x < 1, \end{cases}$$

second order differentiable at 1? Is it possible for the function to be third order differentiable?

Exercise 4.1.6. Suppose $f(x)$ is second order differentiable at 0. Show that

$$3f(x) - 3f(2x) + f(3x) = f(0) + o(x^2).$$

Exercise 4.1.7. Suppose $P(x) = a_0 + a_1(x-a) + a_2(x-a)^2$ satisfies $\lim_{x\to a} \dfrac{P(x)}{(x-a)^2} = 0$. Prove that $a_0 = a_1 = a_2 = 0$. Then explain that the result means the uniqueness of quadratic approximation. Moreover, extend the result to high order approximation.

Exercise 4.1.8. Suppose $P(x)$ is the n-order approximation of $f(x)$. What is the n-order approximation of $f(-x)$? Then use Exercise 4.1.7 to explain that the high order approximation of an even function has not odd power terms. What about the high order approximation of an odd function?

4.1.1 Taylor Expansion

The linear approximation may be computed by the derivative. The high order approximation may be computed by repeatedly taking the derivative. The idea is suggested by the following example. By applying l'Hospital's rule three times, we get a more precise limit than the one in Example 4.1.2

$$\lim_{x\to 0} \frac{\cos x - 1 + \dfrac{1}{2}x^2}{x^3} = \lim_{x\to 0} \frac{-\sin x - x}{3x^2} = \lim_{x\to 0} \frac{-\cos x - 1}{3\cdot 2x} = \lim_{x\to 0} \frac{\sin x}{3\cdot 2\cdot 1} = 0.$$

Each application of the l'Hospital's rule means taking derivative once. Therefore we get the third order approximation of $\cos x$ at 0 by taking derivative three times.

If $f(x)$ is differentiable everywhere on an open interval, then the derivative $f'(x)$ is a function on the open interval. If the derivative function $f'(x)$ is also differentiable, then we get the *second order derivative* $f''(x) = (f'(x))'$. If the function $f''(x)$ is yet again differentiable, then taking the derivative one more time gives the *third order derivative* $f'''(x) = (f''(x))'$. The process may continue and we have the *n-th order derivative* $f^{(n)}(x)$. The Leibniz notation for the high order derivative $f^{(n)}(x)$ is $\dfrac{d^n f}{dx^n}$.

Let $f(x) = \cos x$ and $P(x) = 1 - \dfrac{1}{2}x^2$. The key to the repeated application of the l'Hospital's rule is that the numerator is always 0 at $x_0 = 0$. This means that

$$f(x_0) = P(x_0),\ f'(x_0) = P'(x_0),\ f''(x_0) = P''(x_0),\ f'''(x_0) = P'''(x_0).$$

In general, if $P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3$, then the above equalities become

$$f(x_0) = a_0,\ f'(x_0) = a_1,\ f''(x_0) = 2a_2,\ f'''(x_0) = 3 \cdot 2a_3.$$

Theorem 4.1.2. *If $f(x)$ has n-th order derivative at x_0, then f is n-th order differentiable, with n-th order approximation*

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

The polynomial T_n is called the n-th order *Taylor expansion* of f.

Note that the existence of the derivative $f^{(n)}(x_0)$ implicitly assumes that $f^{(k)}(x)$ exists for all x near x_0 and all $k < n$. The theorem gives one way (but not the only way!) to compute the high order approximation in case the function has high order derivative. However, we will show in Example 4.1.8 that it is possible to have high order approximation without the existence of the high order derivative. Here the concept (high order differentiability) is strictly weaker than the computation (high order derivative).

Example 4.1.3. The high order derivatives of the power function x^p are

$$\begin{aligned}
(x^p)' &= px^{p-1}, \\
(x^p)'' &= p(p-1)x^{p-2}, \\
&\ \vdots \\
(x^p)^{(n)} &= p(p-1)\cdots(p-n+1)x^{p-n}.
\end{aligned}$$

More generally, we have

$$((a + bx)^p)^{(n)} = p(p-1)\cdots(p-n+1)b^n(a + bx)^{p-n}.$$

For $a = b = 1$, we get the Taylor expansion at 0

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + o(x^n).$$

For $a = 1$, $b = -1$ and $p = -1$, we get the Taylor expansion at 0

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n).$$

You may compare with Exercise 4.1.1.

Example 4.1.4. By $(\log x)' = x^{-1}$ and the derivatives from Example 4.1.3, we have $(\log x)^{(n)} = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$. This gives the Taylor expansion at 1

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n+1}\frac{1}{n}(x-1)^n + o((x-1)^n).$$

This can also be expressed as a Taylor expansion at 0

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + (-1)^{n+1}\frac{1}{n}x^n + o(x^n).$$

Example 4.1.5. By $(e^x)' = e^x$, it is easy to see that $(e^x)^{(n)} = e^x$ for all n. This gives the Taylor expansion at 0

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + o(x^n).$$

Example 4.1.6. The high order derivatives of $\sin x$ and $\cos x$ are 4-periodic in the sense that $\sin^{(n+4)} x = \sin^{(n)} x$ and $\cos^{(n+4)} x = \cos^{(n)} x$, and are given by

$$\begin{aligned}
(\sin x)' &= \cos x, & (\cos x)' &= -\sin x,\\
(\sin x)'' &= -\sin x, & (\cos x)'' &= -\cos x,\\
(\sin x)''' &= -\cos x, & (\cos x)''' &= \sin x,\\
(\sin x)'''' &= \sin x, & (\cos x)'''' &= \cos x.
\end{aligned}$$

This gives the Taylor expansions at 0

$$\begin{aligned}
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + (-1)^{n+1}\frac{1}{(2n-1)!}x^{2n-1} + o(x^{2n}),\\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^{n}\frac{1}{(2n)!}x^{2n} + o(x^{2n+1}).
\end{aligned}$$

Note that we have $o(x^{2n})$ for $\sin x$ at the end, which is more accurate than $o(x^{2n-1})$. The reason is that the $2n$-th term $0 \cdot x^{2n}$ is omitted from the expression, so that the approximation is actually of $2n$-th order. The similar remark applies to $\cos x$.

We also note that the Taylor expansions of e^x, $\sin x$, $\cos x$ are related by the equality

$$e^{ix} = \cos x + i\sin x, \quad i = \sqrt{-1}.$$

Exercise 4.1.9. Prove the following properties of high order derivative

$$\begin{aligned}(f+g)^{(n)} &= f^{(n)} + g^{(n)},\\ (cf)^{(n)} &= cf^{(n)},\\ (fg)^{(n)} &= \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} f^{(i)} g^{(n-i)}.\end{aligned}$$

Exercise 4.1.10. Prove the chain rule for second order derivative

$$(g(f(x)))'' = g''(f(x))f'(x)^2 + g'(f(x))f''(x).$$

Exercise 4.1.11. Compute derivatives of all order.

1. a^x.
2. e^{ax+b}.
3. $\sin(ax+b)$.
4. $\cos(ax+b)$.
5. $\log(ax+b)$.
6. $\log\dfrac{ax+b}{cx+d}$.
7. $\dfrac{ax+b}{cx+d}$.
8. $\dfrac{1}{(ax+b)(cx+d)}$.

Exercise 4.1.12. Compute high order derivatives.

1. $(\tan x)'''$.
2. $(\sec x)'''$.
3. $(\sin x^2)'''$.
4. $(\arcsin x)''$.
5. $(\arctan x)''$.
6. $(x^x)''$.
7. $\dfrac{d^2}{dx^2}\left(1+\dfrac{1}{x}\right)^x$.

Exercise 4.1.13. Use high order derivatives to find high order approximations.

1. a^x, $n=5$, at 0.
2. a^x, $n=5$, at 1.
3. $\sin^2 x$, $n=6$, at 0.
4. $\sin^2 x$, $n=6$, at π.
5. e^{x^2}, $n=6$, at 0.
6. e^{x^2}, $n=6$, at 1.
7. x^3e^x, $n=5$, at 0.
8. x^3e^x, $n=5$, at 1.
9. $e^x\sin x$, $n=5$ at 1.

Exercise 4.1.14. Compute high order derivatives.

1. $(x^2+1)e^x$.
2. $(x^2+1)\sin x$.
3. $x^2(x-1)^p$.
4. $x\log x$.

Exercise 4.1.15. Prove $(x^{n-1}e^{\frac{1}{x}})^{(n)} = \dfrac{(-1)^n}{x^{n+1}}e^{\frac{1}{x}}$.

Exercise 4.1.16. Prove $(e^{ax}\sin(bx+c))^{(n)} = (a^2+b^2)^{\frac{n}{2}}e^{ax}\sin(bx+c+n\theta)$, where $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$. What is the similar formula for $(e^{ax}\cos(bx+c))^{(n)}$?

Exercise 4.1.17. Suppose $f(x)$ has second order derivative near x_0. Prove that

$$f''(x_0) = \lim_{h\to 0}\frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}.$$

Exercise 4.1.18. Compare $e^x, \sin x, \cos x, \log(1+x)$ with their Taylor expansions. For example, is e^x bigger than or smaller than $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$?

4.1.2 High Order Approximation by Substitution

The functions (and their variations) in Examples 4.1.3 through 4.1.11 are the only ones that we can compute all the high order derivative functions. These give the basic examples of high order approximations. We get other high order approximations by combining the basic ones.

Example 4.1.7. Substituting x by $\frac{b}{a}x$ in the Taylor expansion of $(1+x)^p$, we get

$$\begin{aligned}
(a+bx)^p &= a^p \left(1 + \frac{b}{a}x\right)^p \\
&= a^p \left[1 + p\frac{b}{a}x + \frac{p(p-1)}{2!}\frac{b^2}{a^2}x^2 + \cdots \right. \\
&\quad \left. + \frac{p(p-1)\cdots(p-n+1)}{n!}\frac{b^n}{a^n}x^n + o\left(\frac{b^n}{a^n}x^n\right)\right] \\
&= a^p + pa^{p-1}bx + \frac{p(p-1)}{2!}a^{p-2}b^2x^2 + \cdots \\
&\quad + \frac{p(p-1)\cdots(p-n+1)}{n!}a^{p-n}b^nx^n + o(x^n).
\end{aligned}$$

Note that we used $a^p o\left(\frac{b^n}{a^n}x^n\right) = o(x^n)$ in the computation. The reason is that $o\left(\frac{b^n}{a^n}x^n\right)$ really means a function $R\left(\frac{b^n}{a^n}x^n\right)$, where $R(x)$ is the remainder of the n-th order Taylor expansion of $(1+x)^p$. Since $\lim_{x\to 0}\frac{R(x)}{x^n} = 0$, we get

$$\lim_{x\to 0}\frac{a^pR\left(\frac{b^n}{a^n}x^n\right)}{x^n} = \lim_{y\to 0}\frac{a^pR(y)}{\frac{a^n}{b^n}y^n} = a^{p-n}b^n\lim_{y\to 0}\frac{R(y)}{y^n} = 0.$$

This means $a^pR\left(\frac{b^n}{a^n}x^n\right) = o(x^n)$.

Further substitution of a, b, x by $x_0, 1, x - x_0$ gives the Taylor expansion of x^p at

x_0

$$\begin{aligned}
x^p &= (x_0 + (x - x_0))^p \\
&= x_0^p + p x_0^{p-1}(x - x_0) + \frac{p(p-1)}{2!} x_0^{p-2}(x - x_0)^2 + \cdots \\
&\quad + \frac{p(p-1)\cdots(p-n+1)}{n!} x_0^{p-n}(x - x_0)^n + o((x - x_0)^n).
\end{aligned}$$

The Taylor expansion can also be obtained from the high order derivative in Example 4.1.3.

Example 4.1.8. The Taylor expansion of $\dfrac{1}{1-x}$ at 0 in Example 4.1.3 induces the following approximations

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - \cdots + (-1)^n x^n + o(x^n), \\
\frac{1}{1+x^2} &= 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + o(x^{2n}).
\end{aligned}$$

Similar to the Taylor expansions of $\sin x$ and $\cos x$, we expect that the odd power terms vanish in the Taylor expansion of $\dfrac{1}{1+x^2}$. Therefore the remainder should be improved to $o(x^{2n+1})$. To get the improved remainder, we consider the $2(n+1)$-th order Taylor expansion of $\dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + (-1)^{n+1} x^{2(n+1)} + o(x^{2(n+1)}).$$

This shows that the remainder of the $2n$-th order Taylor expansion is $(-1)^{n+1}x^{2(n+1)} + R(x)$, where $R(x)$ satisfies $\lim_{x\to 0} \dfrac{R(x)}{x^{2(n+1)}} = 0$. By

$$\lim_{x\to 0} \frac{(-1)^{n+1} x^{2(n+1)} + R(x)}{x^{2n+1}} = 0 = \lim_{x\to 0} \left((-1)^{n+1} x + \frac{R(x)}{x^{2(n+1)}} x \right) = 0,$$

we get

$$(-1)^{n+1} x^{2(n+1)} + o(x^{2(n+1)}) = o(x^{2n+1}),$$

and the improved approximation

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + o(x^{2n+1}).$$

Finally, it is easy to see that $\dfrac{1}{1+x^2}$ has derivative of any order. From the coefficients in the Taylor expansion, we get

$$\left.\frac{d^n}{dx^n}\right|_{x=0} \left(\frac{1}{1+x^2} \right) = n! a_n = \begin{cases} 0, & \text{if } n = 2k-1, \\ (-1)^k (2k)!, & \text{if } n = 2k. \end{cases}$$

It is practically impossible to get this by directly computing the high order derivatives (i.e., by repeatedly taking derivatives).

Exercise 4.1.19. Explain and justify the following claims about remainders.

1. $o(x^5) = o(x^3)$.
2. $o(x^5) + o(x^5) = o(x^5)$.
3. $o(x^4) + o(x^5) = o(x^3)$.
4. $x^3 o(x^5) = o(x^8)$.
5. $o(x^3)o(x^5) = o(x^8)$.
6. $o(x^3) + x^5 = o(x^3)$.

Exercise 4.1.20. Find the Taylor expansion of $\dfrac{1}{1-x^3}$ at 0, and the high order derivatives of the function at 0.

Exercise 4.1.21. Use the high order derivatives in Example 4.1.8 to find the Taylor expansion of $\arctan x$ at 0.

Exercise 4.1.22. Find the Taylor expansion of $\dfrac{1}{\sqrt{1-x^2}}$ at 0. Find the high order derivatives of the function at 0. Then find the Taylor expansion of $\arcsin x$ at 0.

Example 4.1.9. The Taylor expansion of e^x at 0 induces

$$\begin{aligned}
e^{-x} &= 1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \cdots + (-1)^n \frac{1}{n!}x^n + o(x^n), \\
e^x = e^{x_0}e^{x-x_0} &= e^{x_0} - \frac{1}{1!}e^{x_0}(x-x_0) + \frac{1}{2!}e^{x_0}(x-x_0)^2 - \cdots \\
&\quad + (-1)^n \frac{1}{n!}e^{x_0}(x-x_0)^n + o((x-x_0)^n), \\
e^{x^2} &= 1 + \frac{1}{1!}x^2 + \frac{1}{2!}x^4 + \cdots + \frac{1}{n!}x^{2n} + o(x^{2n+1}).
\end{aligned}$$

Note that we have the more accurate remainder $o(x^{2n+1})$ for e^{x^2} for the reason similar to Example 4.1.8. Moreover, the Taylor expansion of e^{x^2} gives

$$(e^{x^2})^{(n)}|_{x=0} = \begin{cases} 0, & \text{if } n = 2k-1, \\ \dfrac{(2k)!}{k!}, & \text{if } n = 2k. \end{cases}$$

Example 4.1.10. The high order approximation of x^2e^x at 0 is

$$\begin{aligned}
x^2e^x &= x^2\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + o(x^n)\right) \\
&= x^2 + \frac{1}{1!}x^3 + \frac{1}{2!}x^4 + \cdots + \frac{1}{n!}x^{n+2} + o(x^{n+2}).
\end{aligned}$$

Here we use $x^2 o(x^n) = o(x^{n+2})$. The n-th order approximation is

$$x^2e^x = x^2 + \frac{1}{1!}x^3 + \frac{1}{2!}x^4 + \cdots + \frac{1}{(n-2)!}x^n + o(x^n).$$

Can you find the n-th order derivative of x^2e^x at 0?

On the other hand, to find the high order approximation of x^2e^x at 1, we express the variables in terms of $x-1$ and get

$$\begin{aligned}
x^2e^x &= ((x-1)+1)^2e^{(x-1)+1} = e((x-1)^2+2(x-1)+1)e^{x-1} \\
&= e((x-1)^2+2(x-1)+1)\left(\sum_{i=0}^{n}\frac{1}{i!}(x-1)^i+o((x-1)^n)\right) \\
&= e\sum_{i=0}^{n}\frac{1}{i!}(x-1)^{i+2}+o((x-1)^{n+2}) \\
&\quad+2e\sum_{i=0}^{n}\frac{1}{i!}(x-1)^{i+1}+o((x-1)^{n+1}) \\
&\quad+e\sum_{i=0}^{n}\frac{1}{i!}(x-1)^{i}+o((x-1)^{n}) \\
&= e\sum_{i=2}^{n}\frac{1}{(i-2)!}(x-1)^i+o((x-1)^n) \\
&\quad+2e(x-1)+2e\sum_{i=2}^{n}\frac{1}{(i-1)!}(x-1)^i+o((x-1)^n) \\
&\quad+e+e(x-1)+e\sum_{i=2}^{n}\frac{1}{i!}(x-1)^i+o((x-1)^n) \\
&= e+3e(x-1)+e\sum_{i=2}^{n}\left(\frac{1}{(i-2)!}+\frac{2}{(i-1)!}+\frac{1}{i!}\right)(x-1)^i+o((x-1)^n) \\
&= e+3e(x-1)+e\sum_{i=2}^{n}\frac{i^2+i+1}{i!}(x-1)^i+o((x-1)^n).
\end{aligned}$$

Exercise 4.1.23. Use the basic Taylor expansions to find the high order approximations and derivatives of functions in Exercise 4.1.11.

Exercise 4.1.24. Use the basic Taylor expansions to find the high order approximations and derivatives at 0.

1. $\dfrac{1}{x(x+1)(x+2)}$.
2. $\sqrt{1-x^2}$.
3. $\sqrt{1+x^3}$.
4. $\log(1+x^2)$.
5. $\log(1+3x+2x^2)$.
6. $\log\dfrac{1+x^2}{1-x^3}$.
7. e^{2x}.
8. a^{x^2}.
9. $\sin x\cos 2x$.
10. $\sin x\cos 2x\sin 3x$.
11. $\sin x^2$.
12. $\sin^2 x$.

Exercise 4.1.25. Use the basic Taylor expansions to find high order approximations and high order derivatives.

1. x^3+5x-1 at 1.
2. x^p at -3.
3. $\dfrac{x+3}{x+1}$ at 1.
4. $\sqrt{x+1}$ at 1.
5. e^{-2x} at 4.
6. $\log x$ at 2.
7. $\log(3-x)$ at 2.
8. $\sin x$ at $\dfrac{\pi}{2}$.
9. $\sin x$ at π.
10. $\cos x$ at π.
11. $\sin 2x$ at $\dfrac{\pi}{4}$.
12. $\sin^2 x$ at π.

Exercise 4.1.26. Use the basic Taylor expansions to find high order approximations and high order derivatives at x_0.

1. a^x.
2. x^2e^x.
3. $\log x$.
4. $\sin x$.
5. $\sin x\cos 2x$.
6. $\sin^2 x$.

4.1.3 Combination of High Order Approximations

So far we only used simple substitutions to get new approximations. In the subsequent examples, we compute more sophisticated combinations of approximations.

Example 4.1.11. By the Taylor expansion of $\sin x$, we have

$$\begin{aligned}\sin^2 x-\sin x^2 &= \left(x-\frac{1}{6}x^3+\frac{1}{120}x^5+o(x^6)\right)^2-\left(x^2-\frac{1}{6}x^6+o((x^2)^4)\right)\\ &= \left(x^2-\frac{1}{3}x^4+\frac{1}{36}x^6+\frac{1}{60}x^6+o(x^6)^2+2xo(x^6)+\cdots\right)\\ &\quad -\left(x^2-\frac{1}{6}x^6+o(x^8)\right)\\ &= -\frac{1}{3}x^4+\frac{19}{90}x^6+o(x^7).\end{aligned}$$

The term $o(x^7)$ at the end comes from

$$\lim_{x\to 0}\frac{R(x)}{x^6}=0 \implies \lim_{x\to 0}\frac{R(x)^2}{x^7}=0,\ \lim_{x\to 0}\frac{xR(x)}{x^7}=0,\ \dots .$$

In particular, we get the limit in Example 3.4.9

$$\lim_{x\to 0}\frac{\sin^2 x-\sin x^2}{x^4}=-\frac{1}{3}.$$

We also get the high order derivatives of $f(x)=\sin^2 x-\sin x^2$ at 0

$$f'(0)=f''(0)=f'''(0)=f^{(5)}(0)=0,\quad f^{(4)}(0)=-8,\quad f^{(6)}(0)=152.$$

Example 4.1.12. We may compute the Taylor expansions of $\tan x$ and $\sec x$ from the Taylor expansions of $\sin x$, $\cos x$ and $\dfrac{1}{1-x}$

$$\begin{aligned}
\sec x = \frac{1}{\cos x} &= \frac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^5)} \\
&= 1+\left(\frac{1}{2}x^2-\frac{1}{24}x^4+o(x^5)\right)+\left(\frac{1}{2}x^2-\frac{1}{24}x^4+o(x^5)\right)^2 \\
&\quad +\left(\frac{1}{2}x^2-\frac{1}{24}x^4+o(x^5)\right)^3+o(x^6) \\
&= 1+\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{4}x^4+o(x^5) \\
&= 1+\frac{1}{2}x^2+\frac{5}{24}x^4+o(x^5), \\
\tan x = \sin x\sec x &= \left(x-\frac{1}{6}x^3+\frac{1}{120}x^5+o(x^6)\right)\left(1+\frac{1}{2}x^2+\frac{5}{24}x^4+o(x^5)\right) \\
&= x-\frac{1}{6}x^3+\frac{1}{120}x^5+\frac{1}{2}x^3-\frac{1}{12}x^5+\frac{5}{24}x^5+o(x^6) \\
&= x+\frac{1}{3}x^3+\frac{2}{15}x^5+o(x^6).
\end{aligned}$$

The expansions give $(\sec x)^{(4)}_{x=0}=5$ and $(\tan x)^{(5)}_{x=0}=16$.

Example 4.1.13. We computed $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$ by using l'Hospital's rule in Example 3.4.11. Alternatively, we use the Taylor expansions of e^x and $\dfrac{1}{1-x}$

$$\begin{aligned}
\frac{1}{x}-\frac{1}{e^x-1} &= \frac{1}{x}-\frac{1}{x+\frac{x^2}{2}+o(x^2)} = \frac{1}{x}\left(1-\frac{1}{1+\frac{x}{2}+o(x)}\right) \\
&= \frac{1}{x}\left(1-1+\left(\frac{x}{2}+o(x)\right)+o\left(\frac{x}{2}+o(x)\right)\right) = \frac{1}{2}+\frac{o(x)}{x}.
\end{aligned}$$

This implies that the limit is $\dfrac{1}{2}$.

Example 4.1.14. We know $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$ from Example ??. The next question is what the difference $\left(1+\frac{1}{x}\right)^x-e$ looks like. As x goes to infinity, does the difference approach 0 like $\dfrac{1}{x}$?

The question is the same as the behavior of $(1+x)^{\frac{1}{x}}-e$ near 0. By the Taylor expansions of $\log(1+x)$ and e^x at 0, we get

$$\begin{aligned}(1+x)^{\frac{1}{x}}-e &= e^{\frac{1}{x}\log(1+x)}-e\\ &= e^{\frac{1}{x}\left(x-\frac{x^2}{2}+o(x^2)\right)}-e\\ &= e\left(e^{-\frac{x}{2}+o(x)}-1\right)\\ &= e\left[\left(-\frac{x}{2}+o(x)\right)+o\left(-\frac{x}{2}+o(x)\right)\right]\\ &= -\frac{e}{2}x+o(x).\end{aligned}$$

Translated back into x approaching infinity, we have

$$\left(1+\frac{1}{x}\right)^x-e=-\frac{e}{2x}+o\left(\frac{1}{x}\right).$$

Exercise 4.1.27. Find the 5-th order approximations at 0.

1. $e^x\sin x$.
2. $\sqrt{x+1}e^{x^2}\tan x$.
3. $(1+x)^x$.
4. $\log\dfrac{\sin x}{x}$.
5. $\log\cos x$.
6. $\dfrac{x}{e^x-1}$.

Exercise 4.1.28. Use approximations to compute limits.

1. $\lim_{x\to 0}\dfrac{1-\cos x^2}{x^3\sin x}$.
2. $\lim_{x\to 0}\dfrac{\sin x-\tan x}{x^3}$.
3. $\lim_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$.
4. $\lim_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{\tan^2 x}\right)$.
5. $\lim_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\log x}\right)$.
6. $\lim_{x\to 0}(\cos x+\sin x)^{\frac{1}{x(x+1)}}$.
7. $\lim_{x\to 0}\dfrac{e^x-e^{\sin x}}{x^3}$.
8. $\lim_{x\to 0}\log\dfrac{\cos ax}{\cos bx}$.
9. $\lim_{x\to 0}\dfrac{x-\tan x}{x-\sin x}$.
10. $\lim_{x\to\infty}x^2\left(e-\dfrac{1}{x}-\left(1+\dfrac{1}{x}\right)^x\right)$.
11. $\lim_{x\to\infty}x^2\log\left(x\sin\dfrac{1}{x}\right)$.
12. $\lim_{x\to 1}\dfrac{(x-1)\log x}{1+\cos\pi x}$.

Exercise 4.1.29. Use whatever method you prefer to compute limits, $p,q>0$.

1. $\lim_{x\to 0^+}x^pe^{-x^q}$.
2. $\lim_{x\to+\infty}x^pe^{-x^q}$.
3. $\lim_{x\to+\infty}x^p\log x$.
4. $\lim_{x\to 1^+}(x-1)^p\log x$.
5. $\lim_{x\to 1^+}(x-1)^p(\log x)^q$.
6. $\lim_{x\to+\infty}x^pe^{-x^q}\log x$.

7. $\lim_{x\to+\infty} x^p \log(\log x)$.
8. $\lim_{x\to+\infty} (\log x)^p (\log(\log x))^q$.
9. $\lim_{x\to e^+} (x-e)^p \log(\log x)$.

Exercise 4.1.30. Use whatever method you prefer to compute limits, $p, q > 0$.

1. $\lim_{x\to 0^+} \dfrac{\tan^p x - x^p}{\sin^p x - x^p}$.
2. $\lim_{x\to 0^+} \dfrac{\sin^p x - \tan^p x}{x^q}$.
3. $\lim_{x\to \frac{\pi}{4}} \dfrac{\tan x - \cot x}{4x - \pi}$.
4. $\lim_{x\to 0} \dfrac{a\tan bx - b\tan ax}{a\sin bx - b\sin ax}$.

Exercise 4.1.31. Use whatever method you prefer to compute limits.

1. $\lim_{x\to\infty} x^3 \left(\sin\dfrac{1}{x} - \dfrac{1}{2}\sin\dfrac{2}{x}\right)$.
2. $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{2}{x(2+x)} - \dfrac{1}{e^x - 1}\right)$.
3. $\lim_{x\to 0} \left(\dfrac{1}{\log(1+x)} - \dfrac{1}{x}\right)$.
4. $\lim_{x\to 0} \left(\dfrac{1}{\log(x+\sqrt{1+x^2})} - \dfrac{1}{\log(1+x)}\right)$.

Exercise 4.1.32. Use whatever method you prefer to compute limits.

1. $\lim_{x\to 1^-} \log x \log(1-x)$.
2. $\lim_{x\to 0^+} \dfrac{x^x - 1}{x\log x}$.
3. $\lim_{x\to 0^+} \dfrac{x^x - 1 - x\log x}{x^2(\log x)^2}$.
4. $\lim_{x\to 0^+} \dfrac{x^{(x^x)} - x}{x^2(\log x)^2}$.
5. $\lim_{x\to 1} \dfrac{x^{\log x} - 1}{(\log x)^2}$.
6. $\lim_{x\to 1} \dfrac{x^x - x}{\log x - x + 1}$.

Exercise 4.1.33. Use whatever method you prefer to compute limits.

1. $\lim_{x\to 0} \dfrac{(1+ax)^b - (1+bx)^a}{x^2}$.
2. $\lim_{x\to 0} \dfrac{(1+ax+cx^2)^b - (1+bx+dx^2)^a}{x^2}$.
3. $\lim_{x\to a} \dfrac{a^x - x^a}{x - a}$.
4. $\lim_{x\to 0} \dfrac{(a+x)^x - a^x}{x^2}$.
5. $\lim_{x\to 0^+} \dfrac{a^x - a^{\sin x}}{x^3}$, $a > 0$.
6. $\lim_{x\to 0^+} \dfrac{\log(\sin ax)}{\log(\sin bx)}$, $a, b > 0$.
7. $\lim_{x\to 0} \dfrac{\log(\cos ax)}{\log(\cos bx)}$.
8. $\lim_{x\to 0} \dfrac{\cos(\sin x) - \cos x}{x^4}$.
9. $\lim_{x\to 0} \dfrac{\arcsin 2x - 2\arcsin x}{x^3}$.
10. $\lim_{x\to 0} \dfrac{1 - \cos x^2}{x^2 \sin x^2}$.
11. $\lim_{x\to \frac{\pi}{6}} \dfrac{1 - 2\sin x}{\cos 3x}$.
12. $\lim_{x\to 1^+} \log x \tan\dfrac{\pi x}{2}$.
13. $\lim_{x\to 0^+} x^{x^x - 1}$.
14. $\lim_{x\to 0} x^{\sin x}$.
15. $\lim_{x\to 0^+} (-\log x)^x$.

16. $\lim_{x\to 0}\left(e^{-1}(1+x)^{\frac{1}{x}}\right)^{\frac{1}{x}}$.

17. $\lim_{x\to 0}(x^{-1}\arcsin x)^{\frac{1}{x^2}}$.

18. $\lim_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}}$.

19. $\lim_{x\to 0}\left(\frac{\cos x}{1+\sin x}\right)^{\frac{1}{x}}$.

Exercise 4.1.34. In Example 3.4.2, we applied the Mean Value Theorem to get $\log(1+x) = \dfrac{x}{1+\theta x}$ for some $0<\theta<1$.

1. Find explicit formula for $\theta = \theta(x)$.

2. Compute $\lim_{x\to 0}\theta$ by using l'Hospital's rule.

3. Compute $\lim_{x\to 0}\theta$ by using high order approximation.

You may try the same for other functions such as $e^x - 1 = e^{\theta x}$. What can you say about $\lim_{x\to 0}\theta$ in general?

Exercise 4.1.35. Show that the limits converge but cannot be computed by L'Hospital's rule.

1. $\lim_{x\to 0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}$.

2. $\lim_{x\to\infty}\dfrac{x-\sin x}{x+\sin x}$.

Exercise 4.1.36. Find a, b so that the following hold.

1. $x-(a+b\cos x)\sin x = o(x^4)$.

2. $x - a\sin x - b\tan x = o(x^4)$.

4.1.4 Implicit High Order Differentiation

Example 4.1.15. In Example 3.2.9, the function $y=y(x)$ implicitly given by the unit circle $x^2+y^2=1$ has derivative $y' = -\dfrac{x}{y}$. Then

$$y'' = -\frac{y-xy'}{y^2} = -\frac{y+x\dfrac{x}{y}}{y^2} = -\frac{x^2+y^2}{y^3} = -\frac{1}{y^3}.$$

You may verify the result by directly computing the second order derivative of $y = \pm\sqrt{1-x^2}$.

Example 4.1.16. In Example 3.2.10, we computed the derivative of the function $y=y(x)$ implicitly given by the equation $2y-2x^2-\sin y+1=0$ and then obtained the linear approximation at $P = \left(\sqrt{\dfrac{\pi}{2}}, \dfrac{\pi}{2}\right)$. We can certainly continue finding the formula for the second order derivative of $y(x)$ and then get the quadratic approximation at P.

Alternatively, we may compute the quadratic approximation at P by postulating the approximation to be

$$y = \frac{\pi}{2} + a_1\Delta x + a_2\Delta x^2 + o(\Delta x^2), \quad \Delta x = x - \sqrt{\frac{\pi}{2}}.$$

Substituting into the equation, we get

$$\begin{aligned} 0 &= 2\left(\frac{\pi}{2} + a_1\Delta x + a_2\Delta x^2 + o(\Delta x^2)\right) - 2\left(\sqrt{\frac{\pi}{2}} + \Delta x\right)^2 \\ &\quad - \sin\left(\frac{\pi}{2} + a_1\Delta x + a_2\Delta x^2 + o(\Delta x^2)\right) + 1 \\ &= 2a_1\Delta x + 2a_2\Delta x^2 - 4\sqrt{\frac{\pi}{2}}\Delta x - 2\Delta x^2 + o(\Delta x^2) - \cos(a_1\Delta x + a_2\Delta x^2 + o(\Delta x^2)) + 1 \\ &= 2a_1\Delta x + 2a_2\Delta x^2 - 2\sqrt{2\pi}\Delta x - 2\Delta x^2 + \frac{1}{2}(a_1\Delta x + a_2\Delta x^2)^2 + o(\Delta x^2) \end{aligned}$$

The coefficients of Δx and Δx^2 on the right must vanish. Therefore

$$2a_1 - 2\sqrt{2\pi} = 0, \quad 2a_2 - 2 + \frac{1}{2}a_1^2 = 0.$$

The solution is $a_1 = \sqrt{2\pi}$, $a_2 = \dfrac{2-\pi}{2}$, and the quadratic approximation is

$$y(x) = \frac{\pi}{2} + \sqrt{2\pi}\Delta x + \frac{2-\pi}{2}\Delta x^2 + o(\Delta x^2), \quad \Delta x = x - \sqrt{\frac{\pi}{2}}.$$

Exercise 4.1.37. Compute quadratic approximations of implicitly defined functions.

1. $y^2 + 3y^3 + 1 = x$.
2. $\sin y = x$.
3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
4. $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
5. $e^{x+y} = xy$.
6. $x^2 + 2xy - y^2 - 2x = 0$.

Exercise 4.1.38. Compute quadratic approximations of functions $y = y(x)$ given by the curves.

1. $x = \sin^2 t$, $y = \cos^2 t$.
2. $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
3. $x = e^t \cos 2t$, $y = e^t \sin 2t$.
4. $x = (1 + \cos t)\cos t$, $y = (1 + \cos t)\sin t$.

4.1.5 Two Theoretical Examples

Theorem 4.1.2 tells us that the existence of high order derivative implies the high order differentiability. The following example shows that the converse is not true.

Example 4.1.17. The function $x^3D(x)$ satisfies

$$|x| < \delta = \epsilon \implies |x^3D(x) - 0| \le |x|^3 \le \epsilon|x|^2.$$

Therefore the function is second order differentiable, with $0 = 0 + 0x + 0x^2$ as the quadratic approximation.

On the other hand, we have $(x^3D(x))'|_{x=0} = 0$ and $x^3D(x)$ is not differentiable (because not even continuous) away from 0. Therefore the second order derivative is not defined.

Exercise 4.1.39. Show that for any n, there is a function that is n-th order differentiable at 0 but has no second order derivative at 0.

Exercise 4.1.40. The lack of high order derivatives for the function in Example 4.1.17 is due to discontinuity away from 0. Can you find a function with the following properties?

1. f that has first order derivative everywhere on $(-1, 1)$.
2. f has no second order derivative at 0.
3. f is second order differentiable at 0.

The next example deals with the following intuition from everyday life. Suppose we try to measure a length by more and more refined rulers. If our readings from meter ruler, centimeter ruler, millimeter ruler, micrometer ruler, etc, are all 0, then the real length should be 0. Similarly, the Taylor expansion of a function at 0 is the measurement by "x^n-ruler". The following example shows that, even if the readings by all the "x^n-ruler" are 0, the function does not have to be 0.

Example 4.1.18. The function

$$f(x) = \begin{cases} e^{-\frac{1}{|x|}}, & \text{if } x \ne 0, \\ 0, & \text{if } x = 0, \end{cases}$$

has derivative

$$f'(x) = \begin{cases} \dfrac{1}{x^2}e^{-\frac{1}{|x|}}, & \text{if } x > 0, \\ -\dfrac{1}{x^2}e^{-\frac{1}{|x|}}, & \text{if } x < 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The derivative at $x \neq 0$ is computed by the usual chain rule, and the derivative at 0 is computed directly

$$f'(0) = \lim_{x \to 0} \frac{1}{x} e^{-\frac{1}{|x|}} = \lim_{y \to \infty} \frac{y}{e^{|y|}} = 0.$$

In general, it can be inductively proved that

$$f^{(n)}(x) = \begin{cases} p\left(\dfrac{1}{x}\right) e^{-\frac{1}{x}}, & \text{if } x > 0, \\ (-1)^n p\left(-\dfrac{1}{x}\right) e^{-\frac{1}{x}}, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $p(t)$ is a polynomial depending only on n.

The function has the special property that the derivative of any order vanishes at 0. Therefore the function is differentiable of any order, and all the high order approximations are 0

$$f(x) = 0 + 0x + 0x^2 + \cdots + 0x^n + o(x^n).$$

However, the function is not 0.

The example can be understood in two ways. The first is that, for some functions, even more refined ruler is needed in order to measure “beyond all orders”. The second is that the function above is not “measurable by polynomials”. The functions that are measurable by polynomials are call *analytic*, and the function above is not analytic.

Exercise 4.1.41. Directly show (i.e., without calculating the high order derivatives) that the function in Example 4.1.18 is differentiable of any order, with 0 as the approximation of any order.

4.2 Application of High Order Approximation

4.2.1 High Derivative Test

Theorem 3.3.4 gives the first order derivative condition for a (two sided) differentiable function to have local extreme. As the subsequent examples show, the theorem only provides candidates for the local extrema. To find out whether such candidates are indeed local extrema, high order approximations are needed.

Let us consider the first non-trivial high order approximation at x_0

$$f(x) = f(x_0) + c(x - x_0)^n + o((x - x_0)^n) = f(x_0) + c(x - x_0)^n(1 + o(1)), \quad c \neq 0.$$

When x is close to x_0, we have $1 + o(1) > 0$ and therefore $f(x) > f(x_0)$ when $c(x - x_0)^n > 0$ and $f(x) < f(x_0)$ when $c(x - x_0)^n < 0$. Specifically, we have the following signs of $c(x - x_0)^n$ for various cases.

- If n is odd and $c > 0$, then $c(x - x_0)^n > 0$ for $x > x_0$ and $c(x - x_0)^n < 0$ for $x < x_0$.
- If n is odd and $c < 0$, then $c(x - x_0)^n < 0$ for $x > x_0$ and $c(x - x_0)^n > 0$ for $x < x_0$.
- If n is even and $c > 0$, then $c(x - x_0)^n > 0$ for $x \neq x_0$.
- If n is even and $c < 0$, then $c(x - x_0)^n < 0$ for $x \neq x_0$.

The sign of $c(x-x_0)^n$ then further determines whether $f(x) < f(x_0)$ or $f(x) > f(x_0)$, and we get the following result.

Theorem 4.2.1. *Suppose $f(x)$ has high order approximation $f(x_0) + c(x - x_0)^n$ at x_0.*

1. *If n is odd and $c \neq 0$, then x_0 is not a local extreme.*
2. *If n is even and $c > 0$, then x_0 is a local minimum.*
3. *If n is even and $c < 0$, then x_0 is a local maximum.*

If f has n-th order derivative at x_0, the condition of the theorem means

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) = n!c \neq 0.$$

The special case $n = 1$ is Theorem 3.3.4. For the special case $n = 2$, the theorem gives the *second derivative test*: Suppose $f'(x_0) = 0$ (i.e., the criterion in Theorem 3.3.4 is satisfied), and f has second order derivative at x_0.

1. If $f''(x_0) > 0$, then x_0 is a local minimum.
2. If $f''(x_0) < 0$, then x_0 is a local maximum.

Example 4.2.1. In Example 3.3.13, we found the candidates ± 1 for the local extrema of $f(x) = x^3 - 3x + 1$. The second order derivative $f''(x) = 6x$ at the two candidates are

$$f''(1) = 6 > 0, \quad f''(-1) = -6 < 0.$$

Therefore 1 is a local minimum and -1 is a local maximum.

Example 4.2.2. Consider the function $y = y(x)$ implicitly defined in Example 3.2.10. By $y'(x) = \dfrac{4x}{2-\cos y}$, we find a candidate $x = 0$ for the local extreme of $y(x)$. Then we have

$$y''(x) = \frac{4}{2-\cos y} + 4x\, \frac{d}{dy}\left(\frac{1}{2-\cos y}\right)\bigg|_{y=y(x)} y'.$$

At the candidate $x = 0$, we already have $y'(0) = 0$. Therefore $y''(0) = \dfrac{4}{2-\cos y(0)} > 0$. This shows that $x = 0$ is a local minimum of the implicitly defined function.

Example 4.2.3. The function $f(x) = x^2 - x^3 D(x)$ has no second order derivative at 0, but still has the quadratic approximation $f(x) = x^2 + o(x^2)$. The quadratic approximation tells us that 0 is a local minimum of $f(x)$.

Example 4.2.4. Let $f(x) = \dfrac{\sin x}{6x - x^3}$ for $x \neq 0$ and $f(0) = \dfrac{1}{6}$. Then for $x \neq 0$ close to 0, we have

$$\begin{aligned} f(x) &= \frac{1}{6x-x^3}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6)\right) = \frac{1}{6-x^2}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^5)\right) \\ &= \frac{1}{6}\left(1 + \frac{x^2}{6} + \frac{x^4}{36} + o(x^5)\right)\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^5)\right) = \frac{1}{6} + \frac{x^4}{120} + o(x^5). \end{aligned}$$

We note that by $f(0) = \dfrac{1}{6}$, the 4-th order approximation also holds for $x = 0$. Then by Theorem 4.2.1, we find that $x = 0$ is a local minimum.

Alternatively, we may directly use the idea leading to Theorem 4.2.1. For $x \neq 0$ close to 0, we have

$$f(x) = \frac{1}{6x-x^3}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6)\right) = \frac{1}{6} + \frac{x^4(1+o(x))}{(6-x^2)\cdot 120}.$$

For small $x \neq 0$, we have $\dfrac{x^4(1+o(x))}{(6-x^2)\cdot 120} > 0$, which further implies $f(x) > \dfrac{1}{6} = f(0)$. Therefore 0 is a (strict) local minimum.

Exercise 4.2.1. Find the local extrema by using quadratic approximations.

1. $x^3 - 3x + 1$ on $\mathbb{R}$.
2. xe^{-x} on $\mathbb{R}$.
3. $x \log x$ on $(0, +\infty)$.
4. $(x^2+1)e^x$ on $\mathbb{R}$.

Exercise 4.2.2. Find the local extrema for the function $y = y(x)$ implicitly given by $x^3 + y^3 = 6xy$. ??????????? revision needed

Exercise 4.2.3. For $p>1$, determine whether 0 is a local extreme for the function

$$\begin{cases} x^2+|x|^p\sin x\sin\dfrac{1}{x}, & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

Exercise 4.2.4. Determine whether 0 is a local extreme.

1. x^3+x^4.
2. $\sin x - x\cos x$.
3. $\tan x-\sin x$.
4. $\left(1-x+\dfrac{1}{2!}x^2\right)e^x$.
5. $\left(1-x+\dfrac{1}{2!}x^2-\dfrac{1}{3!}x^3\right)e^x$.

Exercise 4.2.5. Let $f(0)=1$ and let $f(x)$ be given by the following for $x\neq 0$. Determine whether 0 is a local extreme.

1. $\dfrac{\sin x}{x+ax^2}$.
2. $\dfrac{\sin x}{x+bx^3}$.
3. $\dfrac{\sin x}{x+ax^2+bx^3}$.

4.2.2 Convex Function

A function f is *convex* on an interval if for any x,y in the interval, the straight line $L_{x,y}$ connecting points $(x,f(x))$ and $(y,f(y))$ lies above the part of the graph of f over $[x,y]$. This means

$$L_{x,y}(z)=f(x)+\frac{f(y)-f(x)}{y-x}(z-x)\geq f(z) \text{ for any } x\leq z\leq y.$$

The function is *concave* if $L_{x,y}$ lies below the graph of f, which means changing $\geq f(z)$ above to $\leq f(z)$. A function f is convex if and only if $-f$ is concave.

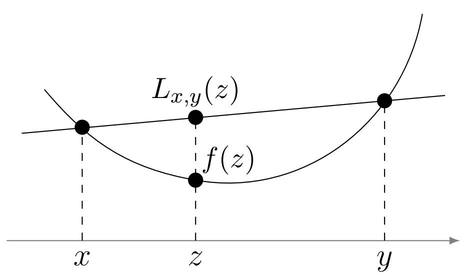


Figure 4.2.1: Convex function.

By the geometric intuition illustrated in Figure 4.2.2, the following are equivalent convexity conditions, for any $x\leq z\leq y$,

1. $L_{x,y}(z)\geq f(z)$.
2. $\text{slope}(L_{x,z})\leq\text{slope}(L_{x,y})$.

3. $\text{slope}(L_{z,y}) \geq \text{slope}(L_{x,y})$.

4. $\text{slope}(L_{x,z}) \leq \text{slope}(L_{z,y})$.

Algebraically, it is not difficult to verify the equivalence.

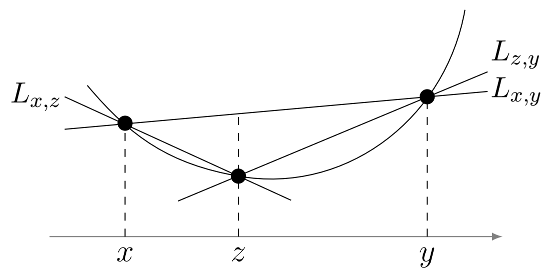


Figure 4.2.2: Convexity by comparing slopes.

If a convex function f is differentiable, then we may take $z \to x^+$ in the inequality $\text{slope}(L_{x,z}) \leq \text{slope}(L_{x,y})$, and get $f'(x) \leq \text{slope}(L_{x,y})$. Similarly, we may take $z \to y^-$ in the inequality $\text{slope}(L_{z,y}) \geq \text{slope}(L_{x,y})$, and get $f'(y) \geq \text{slope}(L_{x,y})$. Combining the two inequalities, we get

$$x < y \implies f'(x) \leq f'(y).$$

It turns out that the converse is also true.

Theorem 4.2.2. *A differentiable function f on an interval is convex if and only if f' is increasing. If f has second order derivative, then this is equivalent to $f'' \geq 0$.*

Similarly, a function f is concave if and only if f' is decreasing, and is also equivalent to $f'' \leq 0$ in case f'' exists.

The converse of Theorem 4.2.2 is explained by Figure 4.2.2. If f' is increasing, then by the Mean Value Theorem (for the two equalities), we have

$$\text{slope}(L_{x,z}) = f'(c) \leq f'(d) = \text{slope}(L_{z,y}).$$

A major application of the convexity is another interpretation of the convexity. In the setup above, a number z between x and y means $z = \lambda x + (1-\lambda)y$ for some $\lambda \in [0,1]$. Then the linear function $L_{x,y}(z) = az + b$ preserves the linear relation

$$\begin{aligned}
L_{x,y}(z) &= a(\lambda x + (1-\lambda)y) + b \\
&= \lambda(ax + b) + (1-\lambda)(ay + b) \\
&= \lambda L_{x,y}(x) + (1-\lambda)L_{x,y}(y) \\
&= \lambda f(x) + (1-\lambda)f(y).
\end{aligned}$$

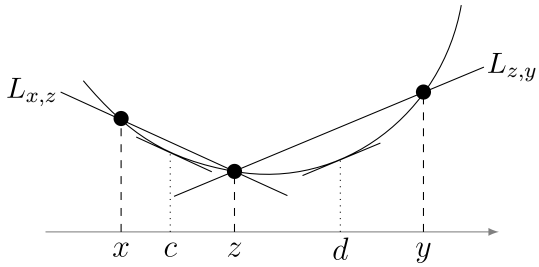


Figure 4.2.3: Increasing $f'(x)$ implies convexity.

Therefore the convexity means

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \text{ for any } 0 \le \lambda \le 1.$$

The following generalizes the inequality.

Theorem 4.2.3 (Jensen's inequality). *If f is convex, and*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1, \quad 0 \le \lambda_1, \lambda_2, \dots, \lambda_n \le 1,$$

then

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \le \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n).$$

By reversing the direction of inequality, we also get Jensen's inequality for concave functions.

In all the discussions about convexity, we may also consider the *strict* inequalities. So we have a concept of strict convexity, and a differentiable function is strictly convex on an interval if and only if its derivative is strictly increasing. Jensen's inequality can also be extended to the strict case.

Example 4.2.5. By $(x^p)'' = p(p-1)x^{p-2}$, we know x^p is convex on $(0, +\infty)$ when $p \ge 1$ or $p < 0$, and is concave when $0 < p \le 1$.

For $p \ge 1$, Jensen's inequality means that

$$(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)^p \le \lambda_1 x_1^p + \lambda_2 x_2^p + \cdots + \lambda_n x_n^p.$$

In particular, if all $\lambda_i = \dfrac{1}{n}$, then we get

$$\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^p \le \frac{x_1^p + x_2^p + \cdots + x_n^p}{n}, \text{ for } p \ge 1, x_i \ge 0.$$

This means that the p-th power of the average is smaller than the average of the p-th power.

We note that $(x^p)'' > 0$ for $p > 1$. Therefore all the inequalities are strict, provided some $x_i > 0$ and $0 < \lambda_i < 1$.

By replacing p with $\dfrac{p}{q}$ and replacing x_i with x_i^q, we get

$$\left(\frac{x_1^q + x_2^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}} \le \left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}}, \text{ for } p > q > 0.$$

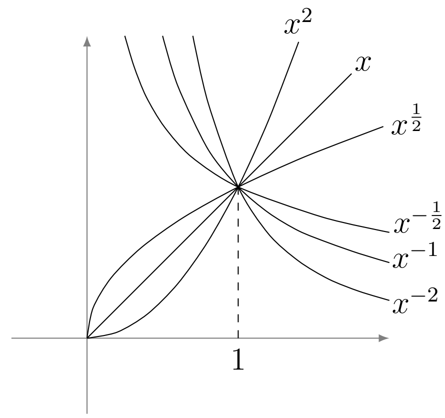


Figure 4.2.4: x^p for various p.

Example 4.2.6. By $(\log x)'' = -\dfrac{1}{x^2} < 0$, the logarithmic function is concave. Then Jensen's inequality tells us that

$$\log(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \ge \lambda_1 \log x_1 + \lambda_2 \log x_2 + \cdots + \lambda_n \log x_n.$$

This is the same as

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \ge x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

For the special case $\lambda_i = \dfrac{1}{n}$, we get

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \cdots x_n}.$$

In other words, the arithmetic mean is bigger than the geometric mean.

Exercise 4.2.6. Study the convexity.

1. $x^3 + ax + b$.
2. $x + x^{\frac{5}{3}}$.
3. $x + x^p$.
4. $(x^2 + 1)e^x$.
5. e^{-x^2}.
6. $\sqrt{x^2 + 1}$.
7. $\dfrac{1}{x^2 + 1}$.
8. $\dfrac{x^2 - 1}{x^2 + 1}$.
9. $\log(x^2 + 1)$.
10. x^x.
11. $x + 2\sin x$.
12. $x\sin(\log x)$.

Exercise 4.2.7. Find the condition on A and B so that the function

$$f(x) = \begin{cases} Ax^p, & \text{if } x \geq 0, \\ B(-x)^q, & \text{if } x < 0, \end{cases}$$

is convex. Is the function necessarily differentiable at 0?

Exercise 4.2.8. Let $p, q > 0$ satisfy $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Use the concavity of $\log x$ to prove Young's inequality

$$\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy.$$

Exercise 4.2.9. For the case $\lambda_1 \neq 1$, Find suitable $\mu_2, \dots, \mu_n$ satisfying

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \lambda_1 x_1 + (1 - \lambda_1)(\mu_2 x_2 + \cdots + \mu_n x_n).$$

Then prove Jensen's inequality by induction.

Exercise 4.2.10. Use the concavity of $\log x$ to prove that, for $x_i > 0$, we have

$$\frac{x_1 \log x_1 + x_2 \log x_2 + \cdots + x_n \log x_n}{x_1 + x_2 + \cdots + x_n} \leq \log \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} \leq \log(x_1 + x_2 + \cdots + x_n).$$

Exercise 4.2.11. Use Exercise 4.2.10 to show that $f(p) = \log(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}}$ is decreasing. Then explain

$$(x_1^q + x_2^q + \cdots + x_n^q)^{\frac{1}{q}} \geq (x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}}, \text{ for } p > q > 0.$$

Note that the similar inequality in Example 4.2.5 is in reverse direction.

Exercise 4.2.12. Verify the convexity of $x \log x$ and then use the property to prove the inequality $(x + y)^{x+y} \leq (2x)^x (2y)^y$. Can you extend the inequality to more variables?

4.2.3 Sketch of Graph

We have learned the increasing and decreasing properties, and the convex and concave properties. We may also pay attention to the symmetry properties such as even or odd function, and the periodic property. Moreover, we should pay attention to the following special points.

1. *Intercepts*, where the graph of function intersects the axes.

2. Disruptions, where the functions are not continuous, or not differentiable.

3. Local extrema, which is often (but not restricted to) the places where the function changes between increasing and decreasing.

4. *Points of inflection*, which is the place where the function changes between convex and concave.

5. Infinity, including the finite places where the function tends to infinity, and the behavior of the function at the infinity.

One infinity behavior is the *asymptotes* of a function. If a linear function $a + bx$ satisfies

$$\lim_{x\to+\infty}(f(x) - a - bx) = 0,$$

then the linear function is an asymptote at $+\infty$. If $b = 0$, then the line is a *horizontal asymptote*. We also have similar asymptote at $-\infty$ (perhaps with different a and b). Moreover, if $\lim_{x\to x_0} f(x) = \infty$ at a finite x_0, then the line $x = x_0$ is a *vertical asymptote*.

In subsequent examples, we sketch the graphs of functions and try to indicate the characteristics listed above as much as possible.

Example 4.2.7. In Example 3.3.4, we determined the monotone property and the local extrema of $f(x) = x^3 - 3x + 1$. The second order derivative $f''(x) = 6x$ also tells us that $f(x)$ is concave on $(-\infty, 0]$ and convex on $[0, +\infty)$, which makes 0 into a point of inflection. Moreover, the function has no asymptotes. The function is also symmetric with respect to the point $(0, 1)$ ($f(x) - 1$ is an odd function). Based on these information, we may sketch the graph.

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,+\infty)$
$f = x^3 - 3x + 1$	$-\infty \leftarrow$	3		0		-1	$\to +\infty$
$f' = 3(x+1)(x-1)$	$+$	0		$-$		0	$+$
	$\nearrow$	max		$\searrow$		min	$\nearrow$
$f'' = 6x$		$+$		0		$-$	
		$\cup$		infl		$\cap$	

Example 4.2.8. In Example 3.3.6, we determined the monotone property of $f(x) = \sqrt[3]{x^2}(x+1)$. The second order derivative $f''(x) = \dfrac{2(5x-1)}{9\sqrt[3]{x^4}}$ implies that the function is concave on $(-\infty, 0)$ and on $\left(0, \dfrac{1}{5}\right]$, convex on $\left[\dfrac{1}{5}, +\infty\right)$, with $\dfrac{1}{5}$ as a point of inflection. Moreover, we have $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$. The

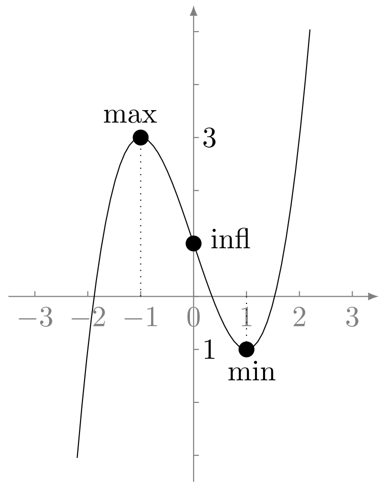


Figure 4.2.5: Graph of $x^3 - 3x + 1$.

function has no asymptote and no symmetry. We also note that $\lim_{x\to 0} \dfrac{f(x)}{x} = \lim_{x\to 0} \dfrac{x+1}{\sqrt[3]{x}} = \infty$. Therefore the tangent of $f(x)$ at 0 is vertical.

x	$(-\infty, -\frac{2}{5})$	$-\frac{2}{5}$	$(-\frac{2}{5}, 0)$	0	$(0, \frac{1}{5})$	$\frac{1}{5}$	$(\frac{1}{5}, +\infty)$
f	$-\infty \leftarrow$	0.1518		0		0.1073	$\to +\infty$
f'	$+$	0	$-$	no		$+$	
	$\nearrow$	max	$\searrow$	min		$\nearrow$	
f''		$-$		no	$-$	0	$+$
		$\frown$			$\frown$	infl	$\smile$

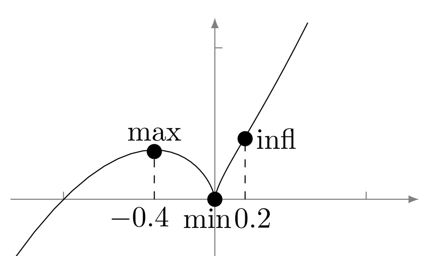


Figure 4.2.6: Graph of $\sqrt[3]{x^2}(x+1)$.

Example 4.2.9. In Example 3.3.15, we determined the monotone property of $f(x) = xe^{-x}$. From $f''(x) = (x-2)e^{-x}$, we also know the function is concave on $(-\infty, 2]$ and convex on $[2, +\infty)$, with $f(2) = e^{-2}$ as a point of inflection. Moreover, we have $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = 0$, so that the x-axis is a horizontal asymptote. The function has no symmetry.

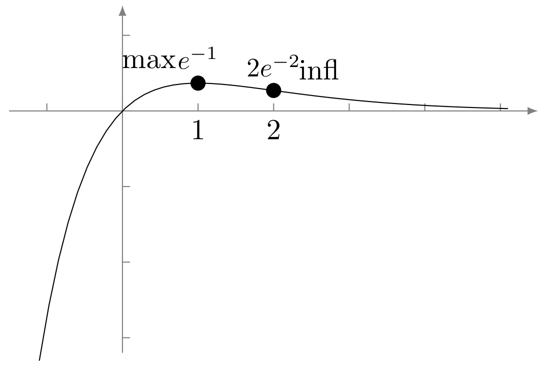


Figure 4.2.7: Graph of xe^{-x}.

Example 4.2.10. In Example 3.3.7, we determined the monotone property and the local extrema of $f(x) = \dfrac{x^3}{x^2-1}$. The function is not defined at ± 1, and has limits

$$\lim_{x\to 1^{\pm}} f(x) = \pm\infty, \quad \lim_{x\to -1^{\pm}} f(x) = \mp\infty.$$

These give vertical asymptotes at ± 1. Moreover, we have

$$\lim_{x\to\infty} (f(x) - x) = 0.$$

Therefore $y = x$ is a slant asymptote at ∞.

The second order derivative $f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3}$ shows that the function is convex on $(-1, 0)$, $(1, +\infty)$, and is concave at the other two intervals. Therefore 0 is a point of inflection.

We also know the function is odd. So the graph is symmetric with respect to the origin.

Exercise 4.2.13. Use the graph of $x^3 - 3x + 1$ in Example 4.2.7 to sketch the graphs.

1. $-x^3 + 3x^2 - 1$.
2. $x^3 - 3x + 2$.
3. $|x^3 - 3x + 1|$.
4. $x^3 - 3x$.
5. $x^3 - bx$, $b > 0$.
6. $ax^3 - bx + c$, $ab > 0$.

Exercise 4.2.14. Use the graph of xe^{-x} in Example 4.2.9 to sketch the graphs.

1. xe^x.
2. xe^{2x}.
3. $(x+1)e^x$.
4. $(ax+b)e^x$.
5. xa^x.
6. $|x|e^{-|x-1|}$.

Exercise 4.2.15. Sketch the graphs.

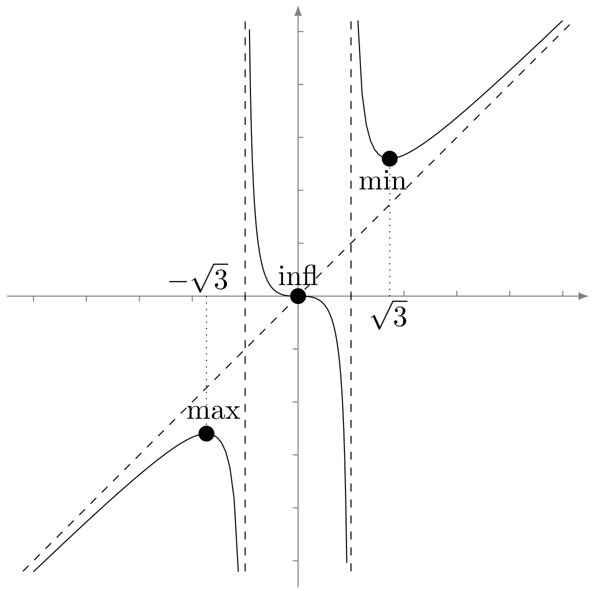


Figure 4.2.8: Graph of $\dfrac{x^3}{x^2-1}$.

1. $x - \sin x$.
2. $|x - \sin x|$.
3. $x - \cos x$.
4. $x + \sin x$.
5. $x + \cos x$.
6. $\dfrac{1}{2} + \sin x \cos x$.

Exercise 4.2.16. Sketch the graphs.

1. $\dfrac{1}{x^2}$.
2. $\dfrac{1}{x^2+1}$.
3. $\dfrac{1}{x^2-1}$.
4. $\dfrac{1}{x^2+bx+c}$.

Exercise 4.2.17. Sketch the graphs.

1. $(x^2+1)^p$.
2. $(x^2-1)^p$.
3. $\sqrt{1-x^2}$.
4. $\sqrt{ax^2+bx+c}$.

Exercise 4.2.18. Sketch the graph of $\dfrac{ax+b}{cx+d}$ by using the graph of $\dfrac{1}{x}$.

Exercise 4.2.19. Sketch the graphs on the natural domains of definitions.

1. $x^4 - 2x^2 + 1$.
2. $x + \dfrac{1}{x}$.
3. $\dfrac{x}{x^2+1}$.
4. $\dfrac{x^3}{(x-1)^2}$.
5. $x\sqrt{x-1}$.
6. $\sqrt{x^2+1} - x$.
7. $x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}$.
8. $x^2 e^{-x}$.
9. $\dfrac{1}{1+e^x}$.
10. $e^{\frac{1}{x}}$.
11. $x \log x$.
12. $x - \log(1+x)$.
13. $\log(1-\log x)$.
14. $\log(1+x^4)$.
15. $e^{-x}\sin x$.

16. $x \tan x$.

17. $\dfrac{1}{1+\sin^2 x}$.

18. $2\sin x + \sin 2x$.

19. $2x - 4\sin x + \sin 2x$.

4.3 Numerical Application

The linear approximation can be used to find approximate values of functions.

Example 4.3.1. The linear approximation of $\sqrt{x}$ at $x = 4$ is

$$L(x) = \sqrt{4} + (\sqrt{x})'|_{x=4}(x-4) = 2 + \frac{1}{4}(x-4).$$

Therefore the value of the square root near 4 can be approximately computed

$$\sqrt{3.96} \approx 2 + \frac{1}{4}(-0.04) = 1.99, \quad \sqrt{4.05} \approx 2 + \frac{1}{4}(0.05) = 2.0125.$$

Example 4.3.2. Assume some metal balls of radius $r = 10$ are selected to make a ball bearing. If the radius is allowed to have 1% relative error, what is the maximal relative error of the weight?

The weight of the ball is

$$W = \frac{4}{3}\rho\pi r^3.$$

where ρ is the density. The error ΔW of the weight caused by the error Δr of the radius is

$$\Delta W \approx \frac{dW}{dr}\Delta r = 4\rho\pi r^2 \Delta r.$$

Therefore the relative error is

$$\frac{\Delta W}{W} \approx 3\frac{\Delta r}{r}.$$

Given the relative error of the radius is no more than 1%, we have $\left|\dfrac{\Delta r}{r}\right| \le 1\%$, so that the relative error of the weight is $\left|\dfrac{\Delta W}{W}\right| \le 3\%$.

Example 4.3.3. In Example 3.2.11, we computed the derivatives of the functions $y = y(x)$ and $z = z(x)$ given by the equations $x^2 + y^2 + z^2 = 2$ and $x + y + z = 0$, which is really a circle in $\mathbb{R}^3$. The point $P = (1, 0, -1)$ lies on the circle, where we have

$$y(1) = 0, \quad z(1) = -1, \quad y'(1) = \frac{1-(-1)}{(-1)-0} = -2, \quad z'(1) = \frac{1-0}{0-(-1)} = 1.$$

Therefore

$$
\begin{aligned}
y(1.01) &\approx 0 - 2 \cdot 0.01 = -0.02, & z(1.01) &\approx -1 + 1 \cdot 0.01 = -0.99, \\
y(0.98) &\approx 0 - 2 \cdot (-0.02) = 0.04, & z(0.98) &\approx -1 + 1 \cdot (-0.02) = -1.02.
\end{aligned}
$$

In other words, the points $(1.01, -0.01, -0.99)$ and $(0.98, 0.04, -1.02)$ are near $(1, 0, -1)$ and almost on the circle.

Exercise 4.3.1. For $a > 0$ and small x, derive

$$\sqrt[n]{a^n + x} \approx a + \frac{x}{na^{n-1}}.$$

Then find the approximate values.

1. $\sqrt[4]{15}$.
2. $\sqrt{46}$.
3. $\sqrt[5]{39}$.
4. $\sqrt[7]{127}$.

Exercise 4.3.2. The period of a pendulum is $T = 2\pi\sqrt{\dfrac{L}{g}}$, where L the length of the pendulum and g is the gravitational constant. If the length of the pendulum is increased by 0.4%, what is the change in the period?

4.3.1 Remainder Formula

We may get more accurate values by using high order approximations. On the other hand, we have more confidence on the estimated values if we also know what the error is. The following result gives a formula for the error.

Theorem 4.3.1 (Lagrange Form of the Remainder)**.** *If $f(x)$ has $(n+1)$-st order derivative on (a, x), then the remainder of the n-th order Taylor expansion of $f(x)$ at a is*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \in (a, x).$$

We only illustrate the argument for the case $n = 2$. We know that the remainder satisfies $R(a) = R'(a) = R''(a) = 0$. Therefore by Cauchy's Means Value Theorem (Theorem 3.4.5), we have

$$
\begin{aligned}
\frac{R_2(x)}{(x-a)^3} &= \frac{R_2(x) - R_2(a)}{(x-a)^3 - (a-a)^3} = \frac{R_2'(c_1)}{3(c_1 - a)^2} && (a < c_1 < x) \\
&= \frac{R_2'(c_1) - R_2'(a)}{3[(c_1-a)^2 - (a-a)^2]} = \frac{R_2''(c_2)}{3 \cdot 2(c_2 - a)} && (a < c_2 < c_1) \\
&= \frac{R_2''(c_2) - R_2''(a)}{3 \cdot 2(c_2 - a)} = \frac{R_2'''(c_3)}{3 \cdot 2 \cdot 1} = \frac{f'''(c_3)}{3!}. && (a < c_3 < c_2)
\end{aligned}
$$

In the last step, we use $R_2''' = f'''$ because the $f - R_2$ is a quadratic function and has vanishing third order derivative.

A slight modification of the proof above actually gives a proof that the Taylor expansion is high order approximation (Theorem 4.1.2).

Example 4.3.4. The error for the linear approximation in Example 4.3.1 can be estimated by

$$R_1(x) = -\frac{\frac{1}{4c^{\frac{3}{2}}}}{2!}\Delta x^2 = \frac{1}{8c^{\frac{3}{2}}}\Delta x^3.$$

For both approximate values, we have

$$|R_1| \le \frac{1}{8\cdot 4^{\frac{3}{2}}}0.05^2 = 0.0000390625 < 4\times 10^{-5}.$$

If we use the quadratic approximation at 4

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2,$$

then we get better estimated values

$$\sqrt{3.96} \approx 2 + \frac{1}{4}(-0.04) - \frac{1}{64}(-0.04)^2 = 1.989975,$$
$$\sqrt{4.05} \approx 2 + \frac{1}{4}(0.05) - \frac{1}{64}(0.05)^2 = 2.0124609375.$$

The error can be estimated by

$$R_2(x) = \frac{\frac{3}{8c^{\frac{5}{2}}}}{3!}\Delta x^3 = \frac{1}{16c^{\frac{5}{2}}}\Delta x^3.$$

For both computations, we have

$$|R_2| \le \frac{1}{16\cdot 4^{\frac{5}{2}}}0.05^3 = 0.00000025 = 2.5\times 10^{-7}.$$

The true values are $\sqrt{3.96} = 1.9899748742 13\cdots$ and $\sqrt{4.05} = 2.01246117975\cdots$.

Example 4.3.5. The Taylor expansion of e^x tells us

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1).$$

By

$$|R_n(1)| = \frac{e^c}{(n+1)!}1^{n+1} \le \frac{e}{(n+1)!}, \quad 0<c<1,$$

we know

$$|R_{13}(1)| \le 0.000000000035 = 3.5\times 10^{-11}.$$

On the other hand,

$$1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{13!}=2.718281828447\cdots,$$

Therefore $e=2.7182818284\cdots$.

Exercise 4.3.3. Find approximate values by using Taylor expansions and estimate the errors.

1. $\sin 1$, 3rd order approximation.
2. $\log 2$, 5th order approximation.
3. e^{-1}, 5th order approximation.
4. $\arctan 2$, 3rd order approximation.

Exercise 4.3.4. Find approximate values accurate up to the 10-th digit.

1. $\sin 1$.
2. $\sqrt{4.05}$.
3. e^{-1}.
4. $\tan 46°$.

Exercise 4.3.5. Find the approximate value of $\tan 1$ accurate up to the 10-th digit by using the Taylor expansions of $\sin x$ and $\cos x$.

Exercise 4.3.6. If we use the Taylor expansion to calculate e accurate up to the 100-th digit, what is the order of the Taylor expansion we should use?

4.3.2 Newton's Method

The linear approximation can also be used to find approximate solutions of equations. To solve $f(x)=0$, we start with a rough estimation x_0 and consider the linear approximation at x_0

$$L_0(x)=f(x_0)+f'(x_0)(x-x_0).$$

We expect the solution of the linear equation $L_0(x)=0$ to be very close to the solution of $f(x)=0$. The solution of the linear equation is easy to find

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}.$$

Although x_1 is not the exact solution of $f(x)=0$, chances are it is an improvement of the initial estimation x_0.

To get an even better approximate solution, we repeat the process and consider the linear approximation at x_1

$$L_1(x)=f(x_1)+f'(x_1)(x-x_1)$$

The solution of the linear equation $L_1(x)=0$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

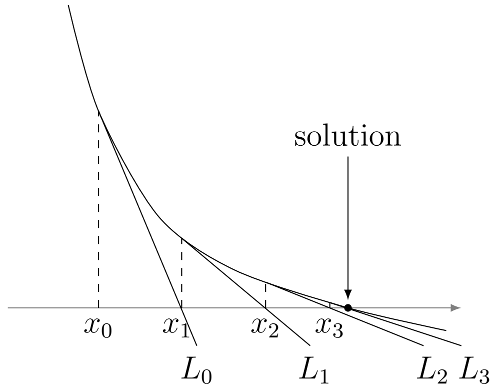


Figure 4.3.1: Newton's method.

is an even better estimation than x_1. The idea leads to an inductively constructed sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We expect the sequence to rapidly converge to the exact solution of $f(x) = 0$.

The scheme above for finding the approximate solution is called the *Newton's method*. The method may fail for various reasons. However, if the function is reasonably good and the initial estimation x_0 is sufficiently close to the exact solution, then the method indeed produces a sequence that rapidly converges to the exact solution. In fact, the error between x_n and the exact solution c satisfies

$$|x_{n+1} - c| \le M|x_n - c|^2$$

for some constant M that depends only on the function.

Example 4.3.6. By Example **??**, the equation $x^3 - 3x + 1 = 0$ should have a solution on $(0.3, 0.4)$. By Example **??**, the equation should also have a second solution > 1 and a third solution < 0. More precisely, by $f(-2) = -1$, $f(-1) = 3$, $f(1) = -1$, $f(2) = 3$, the second solution is on $(-2, -1)$ and the third solution is on $(1, 2)$. Taking -2, 0.3, 2 as initial estimations, we apply Newton's method and compute the sequence

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3(x_n^2 - 1)} = \frac{2}{3}x_n + \frac{2x_n - 1}{3(x_n^2 - 1)}.$$

We find the three solutions

$$-1.87938524157182\cdots, \quad 0.347296355333861\cdots, \quad 1.53208888623796\cdots.$$

n	$x_0 = -2$	$x_0 = 0.3$	$x_0 = 2$
1	-1.88888888888889	0.346520146520147	1.66666666666667
2	-1.87945156695157	0.347296117887934	1.54861111111111
3	-1.87938524483667	0.347296355333838	1.53239016186538
4	-1.87938524157182	0.347296355333861	1.53208898939722
5	-1.87938524157182	0.347296355333861	1.53208888623797
6			1.53208888623796
7			1.53208888623796

Note that the initial estimation cannot be 1 or -1, because the derivative vanishes at the points. Moreover, if we start from $0.88, 0.89, 0.90$, we get very different sequences that respectively converge to the three sequences. We see that Newton's method can be very sensitive to the initial estimation, especially when the estimation is close to where the derivative vanishes.

n	$x_0 = 0.88$	$x_0 = 0.89$	$x_0 = 0.90$
1	-0.5362647754137	-0.657267917268	-0.80350877192983
2	0.6122033746535	0.920119732577	1.91655789116111
3	0.2884916149262	-1.212642459862	1.63097998546252
4	0.3461342508923	-3.235117846394	1.54150263235725
5	0.3472958236620	-2.419800571908	1.53218794505509
6	0.3472963553337	-2.014098301161	1.53208889739446
7	0.3472963553339	-1.891076746708	1.53208888623796
8		-1.879485375060	
9		-1.879385249013	
10		-1.879385241572	

Example 4.3.7. We solve the equation $\sin x + x\cos x = 0$ by starting with the estimation $x_0 = 1$. After five steps, we find the exact solution should be $0.325639452727856\cdots$.

n	x_n
0	1.000000000000000
1	0.471924667505487
2	0.330968826345873
3	0.325645312076542
4	0.325639452734876
5	0.325639452727856
6	0.325639452727856

Exercise 4.3.7. Applying Newton's method to solve $x^3 - x - 1 = 0$ with the initial estimations 1, 0.6 and 0.57. What lesson can you draw from the conclusion?

Exercise 4.3.8. Use Newton's method to find the unique positive root of $f(x) = e^x - x - 2$.

Exercise 4.3.9. Use Newton's method to find all the solutions of $x^2 - \cos x = 0$.

Exercise 4.3.10. Use Newton's method to find the approximate values of $\sqrt{4.05}$ and e^{-1} accurate up to the 10-th digit.

Exercise 4.3.11. Use Newton's method to find all solutions accurate up to the 6-th digit.

1. $x^4 = x + 3$.
2. $e^x = 3 - 2x$.
3. $\cos^2 x = x$.
4. $x + \tan x = 1$.

Note that one may rewrite the equation into another equivalent form and derive a simpler recursive relation.

Exercise 4.3.12. The ancient Babylonians used the recursive relation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

to get more an more accurate approximate values of $\sqrt{a}$. Explain the scheme by Newton's method.

Exercise 4.3.13. What approximate values does the recursive relation $x_{n+1} = 2x_n - ax_n^2$ give you? Explain by Newton's method.

Exercise 4.3.14. Explain why Newton's method does not work if we try to solve $x^3 - 3x + 1 = 0$ by starting at the estimation 1.

Exercise 4.3.15. Newton's method fails to solve the following equations by starting at any $x_0 \neq 0$. Why?

1. $\sqrt[3]{x} = 0$.
2. $\operatorname{sign}(x)\sqrt{|x|} = 0$.

Chapter 5

Integration

5.1 Area and Definite Integral

5.1.1 Area below Non-negative Function

Let $f(x)$ be a non-negative function on $[a,b]$. We wish to find the area of the region

$$G_{[a,b]}(f) = \{(x,y)\colon a \le x \le b,\ 0 \le y \le f(x)\}$$

between the graph of the function and the x-axis.

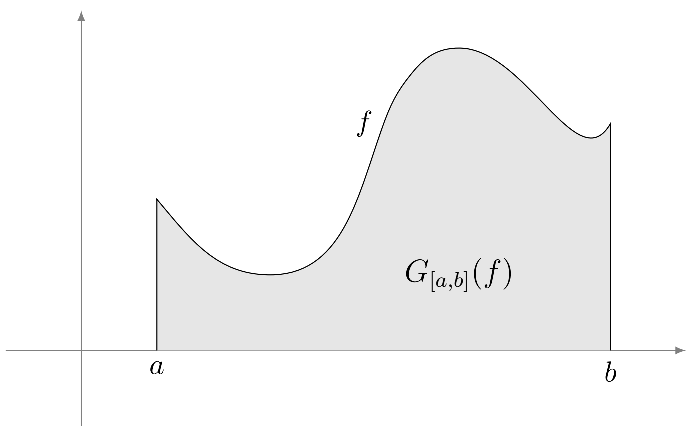


Figure 5.1.1: The region between the function and the x-axis.

Our strategy is the following. For any $x \in [a,b]$, let $A(x)$ be the area of $G_{[a,x]}(f)$, which is part of the region over $[a,x]$. We will find how the function $A(x)$ changes and recover $A(x)$ from its change. The area we wish to find is then the value $A(b)$. The subsequent argument assumes that $f(x)$ is continuous.

Consider an interval $[x, x+h] \subset [a,b]$, which implicitly assumes $h > 0$. Then the change $A(x+h) - A(x)$ is the area of $G_{[x,x+h]}(f)$. Note that $G_{[x,x+h]}(f)$ is sandwiched between two rectangles

$$[x, x+h] \times [0, m] \subset G_{[x,x+h]}(f) \subset [x, x+h] \times [0, M],$$

where

$$m = \min_{[x,x+h]} f, \quad M = \max_{[x,x+h]} f.$$

Since bigger region should have bigger area, we have

$$mh \leq A(x+h) - A(x) \leq Mh.$$

By $h > 0$, this is the same as

$$m \leq \frac{A(x+h) - A(x)}{h} \leq M. \tag{5.1.1}$$

Since the (right) continuity of $f(x)$ implies

$$\lim_{h\to 0^+} m = f(x) = \lim_{h\to 0^+} M,$$

by the sandwich rule, we further get the right derivative

$$A'_+(x) = \lim_{h\to 0^+} \frac{A(x+h) - A(x)}{h} = f(x). \tag{5.1.2}$$

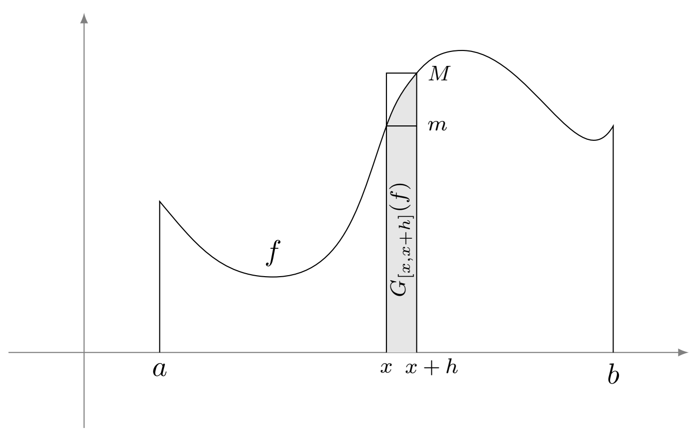


Figure 5.1.2: Estimate the change of area.

The argument above assumes $h > 0$. For $h < 0$, we consider $[x+h, x] \subset [a, b]$. Then the change $A(x+h) - A(x)$ is the *negative* of the area of $G_{[x+h,x]}(f)$, and the interval $[x+h, x]$ has length $-h$. By the same reason as before, we get

$$m(-h) \leq -(A(x+h) - A(x)) \leq M(-h).$$

By $-h > 0$, we still get the inequality (5.1.1), and further application of the sandwich rule gives the left derivative

$$A'_-(x) = \lim_{h\to 0^-} \frac{A(x+h) - A(x)}{h} = f(x).$$

We conclude that, for non-negative and continuous $f(x)$, we have

$$A'(x) = f(x). \tag{5.1.3}$$

Example 5.1.1. To find the area of the region below $f(x) = c$ over $[a, b]$, by (5.1.3), we have $A'(x) = c = (cx)'$. Then by Theorem 3.4.3, we get $A(x) = cx + C$. Further by $A(a) = 0$, we get $C = -ca$ and $A(x) = c(x - a)$. Therefore the area of the region is $A(b) = c(b - a)$.

The region $G_{[a,b]}(c)$ is actually a rectangle of base $b - a$ and height c. The computation of the area is consistent with the common sense.

Example 5.1.2. To find the area of the region below $f(x) = x$ over $[0, a]$, we start with $A'(x) = x = \left(\dfrac{1}{2}x^2\right)'$. This implies $A(x) = \dfrac{1}{2}x^2 + C$. By $A(0) = 0$, we further get $C = 0$ and $A(x) = \dfrac{1}{2}x^2$. Therefore the region has area $A(a) = \dfrac{1}{2}a^2$.

The region is actually a triangle, more precisely half of the square of side length a. The computation of the area is consistent with the common sense.

The pattern we see from the examples above is that, to find the area below a non-negative function and over an interval $[a, b]$, we first find a function $F(x)$ satisfying $f(x) = F'(x)$. Then by Theorem 3.4.3, $A'(x) = F'(x)$ implies $A(x) = F(x) + C$. Further, by $A(a) = 0$, we get $C = -F(a)$. Therefore $A(x) = F(x) - F(a)$, and the area we wish to find is

$$\text{Area}(G_{[a,b]}(f)) = F(b) - F(a).$$

This is the *Newton-Leibniz formula*. The function F is naturally called an *antiderivative* of f.

Example 5.1.3. To find the area of the region below x^2 and over $[0, a]$, we use $\left(\dfrac{1}{3}x^3\right)' = x^2$. The area is $\dfrac{1}{3}a^3 - \dfrac{1}{3}0^3 = \dfrac{1}{3}a^3$.

More generally, for any $p \neq -1$ and $0 < a < b$, by $\left(\dfrac{1}{p+1}x^p\right)' = x^p$, the area of the region below x^p and over $[a, b]$ is

$$\frac{1}{p+1}(b^{p+1} - a^{p+1}).$$

For example, the area of the region below $\sqrt{x}$ and over $[1, 2]$ is

$$\frac{2}{3}x^{\frac{3}{2}}\Big|_{x=1}^{x=2} = \frac{2}{3}(2^{\frac{3}{2}} - 1^{\frac{3}{2}}) = \frac{2}{3}(2\sqrt{2} - 1).$$

Exercise 5.1.1. Find the area of the region below the function over the given interval.

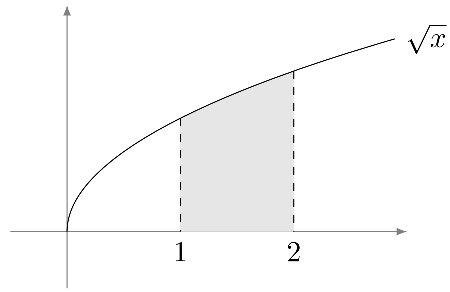


Figure 5.1.3: Area below parabola is $\dfrac{2}{3}(2\sqrt{2}-1)$.

1. x^p on $[0,1]$, $p>0$.
2. $\sin x$ on $[0,\frac{\pi}{2}]$.
3. e^x on $[0,a]$.
4. $\dfrac{1}{x}$ on $[1,a]$.
5. $\sqrt{1+x}$ on $[1,2]$.
6. $\log x$ on $[1,a]$.

Exercise 5.1.2. Find the area of the region bounded by $1-x^2$ and the x-axis.

5.1.2 Definite Integral of Continuous Function

What do we get if we apply the Newton-Leibniz formula to general continuous functions, which might become negative somewhere? The answer is the *signed area*. This means that we count the region between the non-negative part of f and the x-axis as having positive area and count the region between the non-positive part of f and the x-axis as having negative area. See Figure 5.1.4.

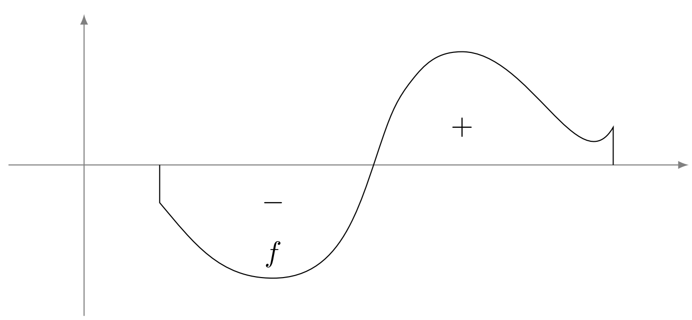


Figure 5.1.4: Computation by the Newton-Leibniz formula gives signed area.

To justify our claim, let $A(x)$ be the signed area for $f(x)$ over $[a,x]$. What we are really concerned with is the change of $A(x)$ where f is negative. So we consider $[x,x+h]\subset[a,b]$, with $h>0$ and $f<0$ on $[x,x+h]$. The change $A(x+h)-A(x)$ is the *negative* of the positive, "unsigned" area of the region

$$G_{[x,x+h]}(f)=\{(t,y)\colon x\le t\le x+h,\ 0\ge y\ge f(t)\}$$

between f and the x-axis along the interval $[x,x+h]$. We have the similar inclusion (see Figure 5.1.5, and note that $[M,0] \subset [m,0]$ because $0 \ge M \ge m$)

$$[x, x+h] \times [M,0] \subset G_{[x,x+h]}(f) \subset [x,x+h] \times [m,0], \quad m = \min_{[x,x+h]} f, M = \max_{[x,x+h]} f.$$

The heights of the rectangles are respectively $-M$ and $-m$, and we get (beware of the signs)

$$(-M)h \le -(A(x+h) - A(x)) \le (-m)h.$$

Again we get the inequality (5.1.1) and subsequently the limit (5.1.2).

The discussion for the case $h < 0$ is similar, and we conclude that $A'(x) = f(x)$.

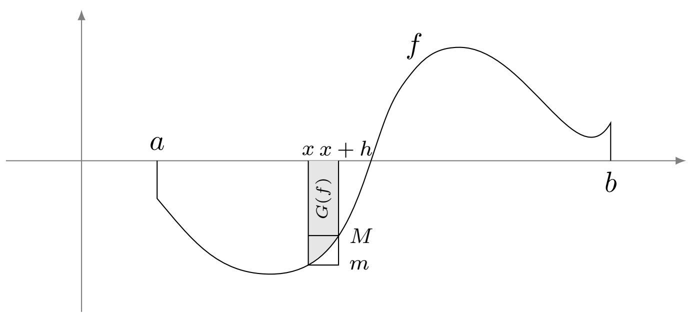


Figure 5.1.5: Estimate the change of negative area.

The signed area is the *definite integral* of $f(x)$ and is denoted $\int_a^b f(x)dx$. The function $f(x)$ is called the *integrand* and the ends a, b of the interval are called the *upper limit* and *lower limit*. The argument above and the explanation before Example 5.1.3 show that the definite integral can be computed by the Newton-Leibniz formula

$$\int_a^b f(x)dx = F(b) - F(a), \text{ where } F'(x) = f(x).$$

Example 5.1.4. By $(x)' = 1$ and $\left(\frac{1}{2}x^2\right)' = x$, we get

$$\int_a^b dx = b - a, \quad \int_a^b x dx = \frac{1}{2}(b^2 - a^2).$$

In general, for any integer $n \neq -1$, we have

$$\int_a^b x^n dx = \frac{1}{n+1}(b^{n+1} - a^{n+1}).$$

However, for $n < 0$, a and b need to have the same sign. The reason is that we derived the Newton-Leibniz formula under the assumption that the integrand is continuous. If a and b have different sign, then $0 \in [a,b]$, and x^n is not continuous on $[a,b]$ for $n<0$.

On the other hand, for any p, x^p is defined for $x>0$. Then for $b \geq a > 0$, we have

$$\int_a^b x^p dx = \frac{1}{p+1}(b^{p+1} - a^{p+1}).$$

We note that x^p is also defined at 0 for $p \geq 0$, and the formula above holds for $p \geq 0$ and $b \geq a \geq 0$. The reason is that x^p is right continuous at 0, and we derived Newton-Leibniz formula by one-sided derivatives.

Example 5.1.5. By $(e^x)' = e^x$ and $(\log x)' = \dfrac{1}{x}$, we get

$$\int_a^b e^x dx = e^b - e^a, \quad \int_a^b \frac{dx}{x} = \log b - \log a = \log \frac{b}{a}.$$

Note that the second integral requires $b \geq a > 0$.

Example 5.1.6. By $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, we get

$$\int_a^b \cos x dx = \sin b - \sin a, \quad \int_a^b \sin x dx = \cos a - \cos b.$$

For example, we have

$$\int_{-\pi}^0 \sin x dx = \cos(-\pi) - \cos 0 = -2.$$

Example 5.1.7. From the derivatives of $\arcsin x$ and $\arctan x$, we get

$$\int_a^b \frac{dx}{\sqrt{1-x^2}} = \arcsin b - \arcsin a, \quad \int_a^b \frac{dx}{1+x^2} = \arctan b - \arctan a.$$

Of course, we need $|a|, |b| < 1$ in the first equality because the integrand is defined only on the open interval $(-1,1)$. In particular, the area of the region below $\dfrac{1}{1+x^2}$ and over $[0,1]$ is

$$\int_0^1 \frac{dx}{1+x^2} = \arctan 1 - \arctan 0 = \frac{\pi}{4}.$$

We also note that

$$\lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b \frac{dx}{1+x^2} = \lim_{b\to+\infty} \arctan b - \lim_{a\to-\infty} \arctan a = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

So the area of the unbounded region between $\dfrac{1}{1+x^2}$ and the x-axis is π.

Exercise 5.1.3. Use the area meaning of definite integral to directly find the value.

1. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}dx.$
2. $\displaystyle\int_{0}^{3} (x-2)dx.$
3. $\displaystyle\int_{a}^{b} |x-1|dx.$

Exercise 5.1.4. Compute and compare

$$\int_0^2 x^3dx, \quad \int_0^2 x^5dx, \quad \int_0^2 4x^3dx, \quad \int_0^2 6x^5dx, \quad \int_0^2 (4x^3+6x^5)dx.$$

What can you observe about the relation between

$$\int_a^b f(x)dx, \quad \int_a^b g(x)dx, \quad \int_a^b cf(x)dx, \quad \int_a^b (f(x)+g(x))dx.$$

Exercise 5.1.5. Compute definite integral.

1. $\displaystyle\int_{-1}^{2} (x^2-3x-4)dx.$
2. $\displaystyle\int_{0}^{2} (3x+1)^2dx.$
3. $\displaystyle\int_{0}^{2} (3x+1)(x-3)dx.$
4. $\displaystyle\int_{0}^{8} \sqrt{3x+1}dx.$
5. $\displaystyle\int_{0}^{1} (3+x\sqrt{x})dx.$
6. $\displaystyle\int_{1}^{2} \left(x+\frac{1}{x}\right)^2 dx.$
7. $\displaystyle\int_{0}^{b} e^{x+a}dx.$
8. $\displaystyle\int_{0}^{1} (e^{-x}+\sin \pi x)dx.$
9. $\displaystyle\int_{0}^{\frac{\pi}{4}} \sec x \tan x dx.$

Exercise 5.1.6. For non-negative integers m and n, prove that

$$\int_0^{2\pi} \cos mx \sin nx dx = 0;$$

$$\int_0^{2\pi} \cos mx \cos nx dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n \neq 0, \\ 2\pi, & \text{if } m = n = 0; \end{cases}$$

$$\int_0^{2\pi} \sin mx \sin nx dx = \begin{cases} 0, & \text{if } m \neq n \text{ or } m = n = 0, \\ \pi, & \text{if } m = n \neq 0. \end{cases}$$

Exercise 5.1.7. Compute $\displaystyle\int_a^b \sqrt[3]{x}dx$ and $\displaystyle\int_a^b \frac{1}{\sqrt[3]{x}}dx$. Explain for what range of a, b are the formulae valid.

Exercise 5.1.8. What is wrong with the equality?

1. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}dx = -\frac{1}{x}\bigg|_{x=-1}^{x=1} = 2.$
2. $\displaystyle\int_{0}^{\pi} \sec^2 x dx = \tan \pi - \tan 0 = 0.$

Exercise 5.1.9. What is the area of the unbounded region between $\dfrac{1}{\sqrt{1-x^2}}$ and the x-axis, over the interval $(-1, 1)$?

5.1.3 Property of Area and Definite Integral

Since the definite integral is the signed area, the usual properties of area is reflected as properties of the definite integral. An important property of area is the additivity. Specifically, if $X \cap Y$ has zero area, then the area of $X \cup Y$ should be the area of X plus the area of Y. Translated to definite integral, this means

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx. \tag{5.1.4}$$

The equality can be used to calculate the definite integral of "piecewise continuous" functions.

Example 5.1.8. The definite integral of the function (which is not continuous at 0)

$$f(x) = \begin{cases} -2x, & \text{if } -1 \le x < 0, \\ e^x, & \text{if } 0 \le x \le 1, \end{cases}$$

on $[-1, 1]$ is

$$\begin{aligned} \int_{-1}^1 f(x)dx &= \int_{-1}^0 f(x)dx + \int_0^1 f(x)dx = \int_{-1}^0 (-2x)dx + \int_0^1 e^x dx \\ &= -x^2|_{-1}^0 + e^x|_0^1 = e. \end{aligned}$$

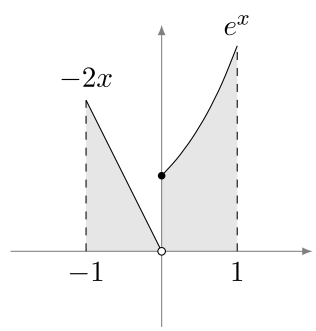


Figure 5.1.6: Definite integral of a piecewise continuous function.

We note that the computation of $\int_{-1}^0 f(x)dx$ actually reassigns the value $f(0) = 0$ to make the function continuous on $[-1, 0]$. The modification happens inside the vertical line $x = 0$. Since the vertical line has zero area, this does not affect the whole integral.

In general, changing the value of the integrand at finitely many places does not affect the integral.

Presumably, the definite integral $\int_a^b f(x)dx$ is defined only for the case $a \le b$, and the equality (5.1.4) implicitly assumes $a \le b \le c$. However, by

$$\int_a^a f(x)dx = 0,$$

and by taking $c = a$ in (5.1.4), we get

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

This extends the definite integral to the case the upper limit is smaller than the lower limit. With such extension, the equality (5.1.4) holds for any combination of a, b, c. Moreover, the extended definite integral is still computed by the antiderivative as before.

Another important property of area is positivity. Translated into definite integral, this means

$$f \ge 0 \implies \int_a^b f(x)dx \ge 0, \text{for } a < b. \tag{5.1.5}$$

Note that if $a > b$, then $\int_a^b f(x)dx \le 0$. The positivity is further extended to monotonicity in Example 5.5.5.

If we shift the graph under $f(x)$ over $[a, b]$ by d, we get the graph under $f(x-d)$ over $[a+d, b+d]$. Since the area is not changed by shifting, we get

$$\int_{a+d}^{b+d} f(x-d)dx = \int_a^b f(x)dx. \tag{5.1.6}$$

See Exercise for more examples of properties of area implying properties of definite integral.

In Section 5.5, we will introduce more properties from the the viewpoint of computation (i.e., Newton-Leibniz formula). Some of these properties cannot be easily explained by properties of area.

Exercise 5.1.10. Suppose $\int_0^2 f(x)dx = 3$, $\int_5^4 f(x)dx = 2$, $\int_5^0 f(x)dx = 0$. Find $\int_2^4 f(x)dx$.

Exercise 5.1.11. Use area to explain the equalities.

1. $\int_{-b}^{-a} f(-x)dx = \int_a^b f(x)dx$.

2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

3. $\int_a^b f(cx)dx = c^{-1}\int_{ca}^{cb} f(x)dx$.

Then express $\int_a^b f(Ax+B)dx$ as some multiple of the definite integral of $f(x)$ over some interval.

Exercise 5.1.12. Compute the integrals.

1. $\int_0^b a^x dx$.
2. $\int_a^b (1+2x)^n dx$.
3. $\int_{-1}^2 \text{sign}(x)dx$.
4. $\int_{-1}^2 x^2\text{sign}(x)dx$.
5. $\int_{-1}^2 |x|dx$.
6. $\int_2^{-1} |x|dx$.
7. $\int_0^2 |x^2-3x+2|dx$.
8. $\int_0^{2\pi} |\sin x|dx$.
9. $\int_{2\pi}^0 |\sin x|dx$.

Exercise 5.1.13. Compute the integrals.

1. $\int_0^2 f(x)dx,\ f(x) = \begin{cases} x^2, & \text{if } x < 1, \\ x^{-2}, & \text{if } x \geq 2. \end{cases}$
2. $\int_{-2}^2 f(x)dx,\ f(x) = \begin{cases} e^{2|x|}, & \text{if } |x| < 1, \\ e^{-x}, & \text{if } |x| \geq 1. \end{cases}$
3. $\int_0^{\pi} f(x)dx,\ f(x) = \begin{cases} \sin x, & \text{if } x < \frac{\pi}{2}, \\ \cos x, & \text{if } x \geq \frac{\pi}{2}. \end{cases}$
4. $\int_{-\pi}^{\pi} f(x)dx,\ f(x) = \begin{cases} \sin x, & \text{if } |x| < \frac{\pi}{2}, \\ \cos x, & \text{if } |x| \geq \frac{\pi}{2}. \end{cases}$

5.2 Rigorous Definition of Integral

5.2.1 What is Area?

The definite integral is defined as the signed area. Therefore the rigorous definition of integral relies on the rigorous definition of area. Any reasonable definition of area should have the following three properties (the area of a subset $X \subset \mathbb{R}^2$ is denoted $\mu(X)$):

1. Bigger subsets have bigger area: $X \subset Y$ implies $\mu(X) \leq \mu(Y)$.
2. Areas can be added: If $\mu(X \cap Y) = 0$, then $\mu(X \cup Y) = \mu(X) + \mu(Y)$.
3. Rectangles have the usual area: $\mu(\langle a, b\rangle \times \langle c, d\rangle) = (b-a)(d-c)$.

Here $\langle a, b\rangle$ can mean any of $[a, b]$, (a, b), $(a, b]$, or $[a, b)$. A carefully review of the argument in Section 5.1 shows that nothing beyond the three properties are used.

Suppose a plane region $A \subset \mathbb{R}^2$ is a union of finitely many rectangles, $A = \cup_{i=1}^n I_i$, such that the intersections between I_i are at most lines. Since lines have zero area by the third property, we may use the second property to further define $\mu(A) = \sum_{i=1}^n \mu(I_i)$. We give such a plane region the temporary name "good region", since we have definite idea about the area of a good region. (Strictly speaking, we still need to argue that $\sum_{i=1}^n \mu(I_i)$ is independent of the decomposition $A = \cup_{i=1}^n I_i$.)

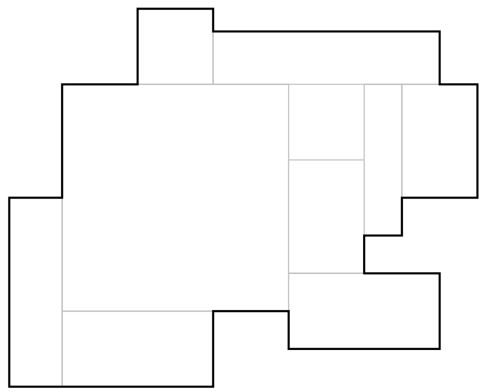


Figure 5.2.1: Good region.

For any (bounded) subset $X \subset \mathbb{R}^2$, we may try to approximate X by good regions, from inside as well as from outside. In other words, we consider good regions A and B satisfying $A \subset X \subset B$. Then by the first property of area, we must have

$$\mu(A) \subset \mu(X) \subset \mu(B).$$

Note that $\mu(A)$ and $\mu(B)$ have been defined, and $\mu(X)$ is yet to be defined. So we introduce the *inner area* (the maximum should really be the supremum)

$$\mu_*(X) = \max\{\mu(A) \colon A \subset X,\ A \text{ is a good region}\},$$

as the lower bound for $\mu(X)$, and the *outer area* (the minimum should really be the infimum)

$$\mu^*(X) = \min\{\mu(B) \colon B \supset X,\ B \text{ is a good region}\},$$

as the upper bound for $\mu(X)$. We say that the subset X *has area* (or *Jordan measurable*) if $\mu_*(X) = \mu^*(X)$, and the common value is the *area* $\mu(X)$ of X. If $\mu_*(X) \neq \mu^*(X)$, then we say X *has no area*.

The subset X has area if and only if for any $\epsilon > 0$, there are good regions A and B, such that $A \subset X \subset B$ and $\mu(B) - \mu(A) < \epsilon$. In other words, we can find good inner and outer approximations, such that the difference between the approximations can be arbitrarily small.

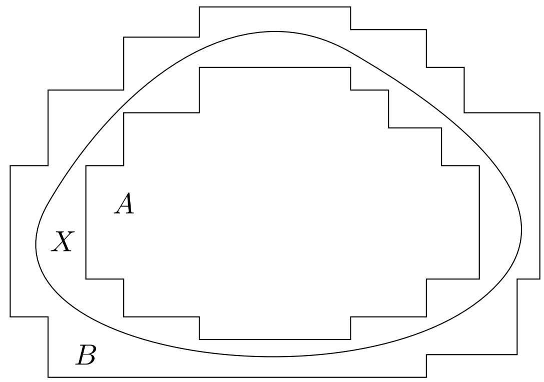


Figure 5.2.2: Approximation by good regions.

Example 5.2.1. A point can be considered as a reduced rectangle and has area 0. If X consists of finitely many points, then we can take $B = X$ be the union of all "point rectangles" in X. Since $\mu(B) = 0$, we get $\mu^*(X) = 0$. By $0 \le \mu_*(X) \le \mu^*(X)$, we also have $\mu_*(X) = 0$. Therefore finitely many points has area 0.

Example 5.2.2. Consider the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. We partition the interval $[0, 1]$ into n parts of equal length

$$[0,1] = \cup_{i=1}^n \left[\frac{i-1}{n}, \frac{i}{n}\right].$$

Correspondingly, we have the inner and outer approximations of the triangle

$$A_n = \cup_{i=1}^n \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[0, \frac{i-1}{n}\right], \quad B_n = \cup_{i=1}^n \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[0, \frac{i}{n}\right].$$

They have area

$$\mu(A_n) = \sum_{i=1}^n \frac{1}{n}\frac{i-1}{n} = \frac{1}{2n}(n-1), \quad \mu(B_n) = \sum_{i=1}^n \frac{1}{n}\frac{i}{n} = \frac{1}{2n}(n+1).$$

By taking sufficiently large n, the difference $\mu(B_n)-\mu(A_n) = \dfrac{1}{n}$ can be arbitrarily small. Therefore the triangle has area, and the area is given by $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \dfrac{1}{2}$. This justifies the conclusion of Example 5.1.2 for the case $a = 1$.

Example 5.2.3. For an example of subsets without area, i.e., satisfying $\mu_*(X) \neq \mu^*(X)$, let us consider the subset $X = (\mathbb{Q} \cap [0, 1])^2$ of all rational pairs in the unit square.

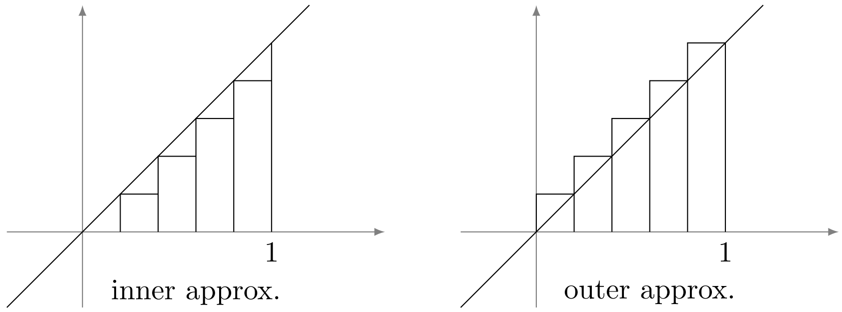


Figure 5.2.3: Approximating triangle.

Since the only rectangles contained in X are single points, any good region $A \subset X$ must be finitely many points. Therefore $\mu(A) = 0$ for any good region $A \subset X$, and $\mu_*(X) = 0$.

On the other hand, if B is a good region containing X, then B must almost contain the whole square $[0,1]^2$, with the only exception of finitely many horizontal or vertical (irrational) line segments. Therefore we have $\mu(B) \geq \mu([0,1]^2) = 1$. This implies $\mu^*(X) \geq 1$ (show that $\mu^*(X) = 1$!).

Exercise 5.2.1. Use inner and outer approximations to explain that any rectangle has area given by the multiplication of two sides. This justifies Example 5.1.1.

Exercise 5.2.2. Explain that a (not necessarily horizontal or vertical) straight line segment has area 0.

Exercise 5.2.3. Explain that the region between $y = x$ and the x-axis over $[0, a]$ has area $\frac{1}{2}a^2$. This fully justifies the computation in Example 5.1.2.

Exercise 5.2.4. Explain that the subset $X = (\mathbb{Q} \cap [0,1]) \times [0,1]$ of all vertical rational lines in the unit square has no area.

Exercise 5.2.5. Show that if $X \subset Y$, then $\mu_*(X) \leq \mu_*(Y)$ and $\mu^*(X) \leq \mu^*(Y)$. In particular, we have $\mu(X) \leq \mu(Y)$ in case both X and Y have areas. The property is used in deriving (5.1.1).

5.2.2 Darboux Sum

After the rigorous definition of area, we can give the rigorous definition of definite integral.

Definition 5.2.1. A function $f(x)$ is *Riemann integrable* if the region

$$G_{[a,b]}(f) = \{(x, y) \colon a \leq x \leq b,\ y \text{ is between } 0 \text{ and } f(x)\}$$

has area. Moreover, the *Riemann integral* is

$$\int_a^b f(x)dx = \mu\left(G_{[a,b]}(f) \cap H_+\right) - \mu\left(G_{[a,b]}(f) \cap H_-\right),$$

where

$$H_+ = \{(x,y)\colon y \geq 0\}, \quad H_- = \{(x,y)\colon y \leq 0\}$$

are the upper and lower half planes.

Suppose $f \geq 0$ on $[a,b]$. As indicated by Figure 5.2.4, for any inner approximation of $G_{[a,b]}(f)$, we can always choose "full vertical strips" to get a better approximation for $G_{[a,b]}(f)$. Here better means closer to the expected value of $\mu(G_{[a,b]}(f))$. The outer approximations have similar improvements by full vertical strips. Therefore we only need to consider the approximations by full vertical strips.

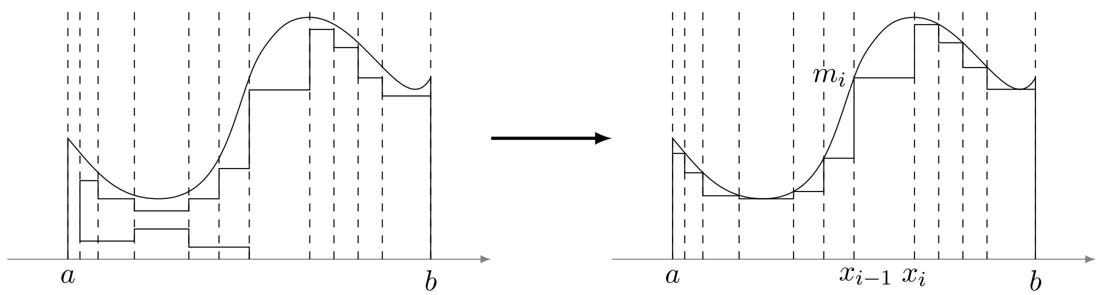


Figure 5.2.4: Better inner approximations by vertical strips.

An approximation by full vertical strips is determined by a *partition* of the interval

$$P\colon a = x_0 < x_1 < \cdots < x_n = b.$$

On the i-th interval $[x_{i-1}, x_i]$, the inner strip has height $m_i = \min_{[x_{i-1},x_i]} f$ (the minimum should really be the *infimum*), and the outer strip has height $M_i = \max_{[x_{i-1},x_i]} f$ (the maximum should really be the *supremum*). Therefore the inner and outer approximations are

$$A_P = \cup_{i=1}^n [x_{i-1}, x_i] \times [0, m_i) \subset X \subset B_P = \cup_{i=1}^n [x_{i-1}, x_i] \times [0, M_i].$$

The areas of the two approximations are the *lower and upper Darboux sums*

$$L(P,f) = \mu(A_P) = \sum_{i=1}^n m_i(x_i - x_{i-1}),$$

$$U(P,f) = \mu(B_P) = \sum_{i=1}^n M_i(x_i - x_{i-1}).$$

The Riemann integrability of f means that $G_{[a,b]}(f)$ has area, which further means that the difference between inner and outer approximations can be arbitrarily small. Therefore we get the following criterion for the integrability.

Theorem 5.2.2 (Riemann Criterion). *A bounded function f on $[a,b]$ is Riemann integrable, if and only if for any $\epsilon > 0$, there is a partition P, such that*

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} \left(\max_{[x_{i-1},x_i]} f - \min_{[x_{i-1},x_i]} f \right) (x_i - x_{i-1}) < \epsilon.$$

The quantity

$$\omega_{[x_{i-1},x_i]} f = \max_{[x_{i-1},x_i]} f - \min_{[x_{i-1},x_i]} f$$

measures how much the value of f fluctuates on $[x_{i-1}, x_i]$ and is called the *oscillation* of the function on the interval. Since the continuity of a function can imply that such oscillations are uniformly small, continuous functions are always Riemann integrable. The criterion also implies that monotone functions are Riemann integrable. On the other hand, there are functions that are not Riemann integrable.

Example 5.2.4. Consider the Dirichlet function $D(x)$ on $[0,1]$. We always have

$$\min_{[x_{i-1},x_i]} D = 0, \quad \max_{[x_{i-1},x_i]} D = 1.$$

Therefore $U(P,f) - L(P,f) = 1$ cannot be arbitrarily small. We conclude that the Dirichlet function is not Riemann integrable.

The example is closely related to Example 5.2.3. See Exercise 5.2.4.

We note that $U(P,f) - L(P,f)$ is the area of the good subset $B_P - A_P = \cup_{i=1}^{n} [x_{i-1}, x_i] \times [m_i, M_i]$. If we consider all the partitions P, the all such good subsets are essentially all the outer approximations of the graph $\{(x, f(x)) \colon a \le x \le b\}$ of f (a curve, not including the part below f). Therefore Theorem 5.2.2 basically says that a function is Riemann integrable if and only if the graph curve of the function has zero area.

The graph curve is part of the boundary of $G(f)$. In this viewpoint, Theorem 5.2.2 is a special case of the following.

Theorem 5.2.3. *A bounded subset $X \subset \mathbb{R}^2$ has area if and only if its boundary ∂X has zero area.*

We remark that the theory of area can be easily extended to the theory of volume for subsets in $\mathbb{R}^n$. We may then get the rigorous definition of multivariable Riemann integrals on subsets of Euclidean spaces, where the subsets should have volume themselves. The high dimensional versions of Theorems 5.2.2 and 5.2.3 are still valid.

Further extension of the area theory introduces countably many in place of finitely many. The result is the modern theory of Lebesgue measure and Lebesgue integral.

5.2.3 Riemann Sum

Suppose f is Riemann integrable. When the partition gets more and more refined, the upper and lower Darboux sums, as the areas of the outer and inner approximations, will become closer to the integral $\int_a^b f(x)dx$. Now choose ϕ_i satisfying

$$m_i \leq \phi_i \leq M_i.$$

Then we get the *Riemann sum*

$$S(P,f)=\sum_{i=1}^{n}\phi_i(x_i-x_{i-1})$$

sandwiched between the two Darboux sums

$$L(P,f)\leq S(P,f)\leq U(P,f).$$

We conclude that $S(P,f)$ will also become closer to the integral $\int_a^b f(x)dx$.

A useful case of the Riemann sum is obtained by taking $\phi_i = f(x_i^*)$ to be the values of some sample points in the partition intervals

$$S(P,f)=\sum_{i=1}^{n}f(x_i^*)(x_i-x_{i-1}),\quad x_i^*\in[x_{i-1},x_i].$$

This is what is usually called the Riemann sum in most textbooks.

Theorem 5.2.4. *Suppose f is Riemann integrable on $[a,b]$. Then for any $\epsilon > 0$, there is a partition P_0 of $[a,b]$, such that for any partition P obtained by adding more partition points to P_0 (we say P is a refinement of P_0), we have*

$$\left|S(P,f)-\int_a^b f(x)dx\right|<\epsilon.$$

The statement above is very similar to the limit of sequences and functions, and we may write

$$\int_a^b f(x)dx=\lim_P S(P,f).$$

The subtlety here is that refinement of partitions replaces $n > N$ or $|x-a| < \delta$.

5.3 Numerical Calculation of Integral

5.3.1 Left and Right Rule

Although Riemann integrals can be computed by the Newton-Leibniz formula, it is often impossible to find the exact formula of a function F satisfying $F' = f$.

Moreover, even if we can find a formula for F, it might be too complicated to evaluate. For many practical applications, it is sufficient to find an approximate value of the integration. Many efficient numerical schemes have been invented for this purpose.

All the schemes are the extensions of the Riemann sum in Section 5.2.3. Usually one starts with a partition that evenly divides the interval

$$P_n\colon a = x_0 < x_1 = a + h < \cdots < x_i = a + ih < \cdots < x_n = b = a + nh,$$

where

$$h = \frac{b-a}{n} = x_i - x_{i-1}$$

is the *step size* of the partition. By taking all the sample points to be the left of the partition intervals, we get $x_i^* = x_{i-1} = a + (i-1)h$ and the *left rule*

$$L_n = h(f(x_0) + f(x_1) + \cdots + f(x_{n-1})).$$

By taking all the sample points to be the right of the partition interval, we get $x_i^* = x_i = a + ih$ and the *right rule*

$$R_n = h(f(x_1) + f(x_2) + \cdots + f(x_n)).$$

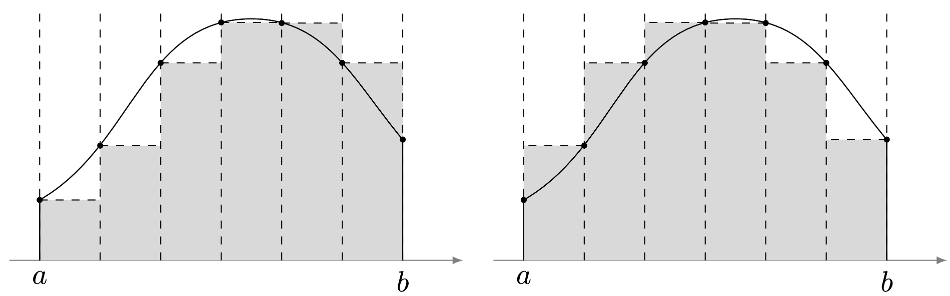


Figure 5.3.1: Left and right rules.

Example 5.3.1. For $f(x) = x$ on $[0, 1]$, we have

$$\begin{aligned}
L_n &= \frac{1}{n}\left(\frac{0}{n} + \frac{1}{n} + \cdots + \frac{n-1}{n}\right) \\
&= \frac{1}{n^2}(0 + 1 + \cdots + (n-1)) = \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{n-1}{2n}, \\
R_n &= \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right) \\
&= \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{1}{n^2}\frac{1}{2}n(n+1) = \frac{n+1}{2n}.
\end{aligned}$$

Both converge to $\dfrac{1}{2} = \displaystyle\int_0^1 x\,dx$ as $n \to \infty$.

Example 5.3.2. To compute the integral of $f(x) = \dfrac{1}{1+x^2}$ on $[0,1]$, we take $n = 4$. The partition is

$$P_4 \colon 0 < 0.25 < 0.5 < 0.75 < 1, \quad h = 0.25.$$

The values of $f(x)$ at the five partition points are

$$1.000000, \quad 0.941176, \quad 0.800000, \quad 0.640000, \quad 0.500000.$$

Then we get the following approximate values of $\displaystyle\int_0^1 \frac{dx}{1+x^2}$

$$\begin{aligned} L_4 &= 0.25 \times (1.000000 + 0.941176 + 0.800000 + 0.640000) \approx 0.845294, \\ R_4 &= 0.25 \times (0.941176 + 0.800000 + 0.640000 + 0.500000) \approx 0.720294. \end{aligned}$$

By Example 5.1.7, the actual value is $\dfrac{\pi}{4} = 0.7853981634\cdots$.

Exercise 5.3.1. Find L_n and R_n and confirm the value of related integral.

1. $f(x) = x$ on $[a, b]$.
2. $f(x) = x^2$ on $[0, 1]$.
3. $f(x) = 2^x$ on $[0, 1]$.
4. $f(x) = a^x$ on $[0, 1]$.

Exercise 5.3.2. Explain the identity.

1. $\lim_{n\to\infty} \left(\dfrac{1}{n^2} + \dfrac{2}{n^2} + \cdots + \dfrac{n-1}{n^2}\right) = \displaystyle\int_0^1 x dx$
2. $\lim_{n\to\infty} \left(\dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n}\right) = \displaystyle\int_0^1 \frac{dx}{1+x}$.
3. $\lim_{n\to\infty} \left(\dfrac{n}{n^2+1^2} + \dfrac{n}{n^2+2^2} + \cdots + \dfrac{n}{n^2+n^2}\right) = \displaystyle\int_0^1 \frac{dx}{1+x^2}$.
4. $\lim_{n\to\infty} \dfrac{1}{n}\left(\sin\dfrac{\pi}{n} + \sin\dfrac{2\pi}{n} + \cdots + \sin\dfrac{(n-1)\pi}{n}\right) = \displaystyle\int_0^1 \sin \pi x dx$.

Exercise 5.3.3. Interpret the limit as integration.

1. $\lim_{n\to\infty} \dfrac{1^p + 2^p + \cdots + n^p}{n^{p+1}}$.
2. $\lim_{n\to\infty} \dfrac{1}{n} \sum_{k=1}^n f\left(a + k\dfrac{b-a}{n}\right)$.

Exercise 5.3.4. By interpreting $\displaystyle\int_1^2 \log x dx$, find $\lim_{n\to\infty} \dfrac{1}{n}\sqrt[n]{\dfrac{(2n)!}{n!}}$.

5.3.2 Midpoint Rule and Trapezoidal Rule

The left and right rules are quite primitive approximations of the integral. A better choice is the middle points

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = a + \frac{2i-1}{2}h$$

and the corresponding Riemann sum

$$M_n = h(f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)).$$

Another choice is the average of the Riemann sums using the left and right points.

$$T_n = \frac{L_n + R_n}{2} = \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)).$$

The two approximation schemes are the *midpoint rule* and the *trapezoidal rule*.

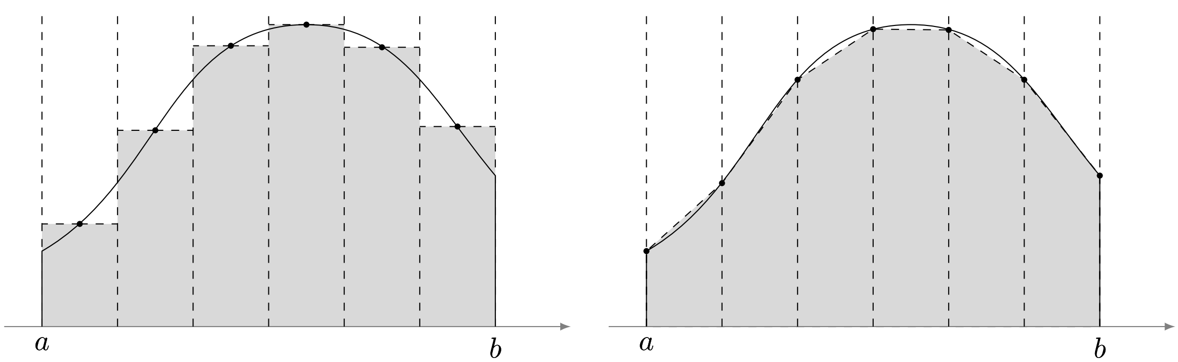


Figure 5.3.2: Midpoint and trapezoidal rules.

Example 5.3.3. For $f(x) = x$ on $[0,1]$, we have

$$\begin{aligned} M_n &= \frac{1}{n}\left(\frac{1}{2n} + \frac{3}{2n} + \cdots + \frac{2n-1}{2n}\right) \\ &= \frac{1}{2n^2}\left(1 + 3 + \cdots + (2n-1)\right) = \frac{1}{2n^2}n^2 = \frac{1}{2}, \\ T_n &= \frac{1}{2}\left(\frac{n-1}{2n} + \frac{n+1}{2n}\right) = \frac{1}{2}. \end{aligned}$$

Both happen to be equal to $\frac{1}{2} = \int_0^1 x dx$.

Example 5.3.4. For $f(x)=x^2$ on $[0,1]$, we have

$$\begin{aligned}
M_n &= \frac{1}{n}\left(\frac{1^2}{4n^2}+\frac{3^2}{4n^2}+\cdots+\frac{(2n-1)^2}{4n^2}\right)\\
&= \frac{1}{4n^3}[(1^2+2^2+\cdots+(2n)^2)-2^2(1^2+2^3+\cdots+n^2)]\\
&= \frac{1}{4n^3}\left(\frac{1}{6}2n(2n+1)(4n+1)-\frac{1}{6}n(n+1)(2n+1)\right)=\frac{4n^2-1}{12n^2},\\
T_n &= \frac{1}{2n}\left(\frac{0^2}{n^2}+2\frac{1^2}{n^2}+2\frac{2^2}{n^2}+\cdots+2\frac{(n-1)^2}{n^2}+\frac{n^2}{n^2}\right)\\
&= \frac{1}{2n^3}[2(1^2+2^2+\cdots+n^2)-n^2]\\
&= \frac{1}{2n^3}\left(2\frac{1}{6}n(n+1)(2n+1)-n^2\right)=\frac{2n^2+1}{6n^2}.
\end{aligned}$$

Compared with the actual value $\int_0^1 x^2dx=\frac{1}{3}$, the errors are $\frac{1}{12n^2}$ and $\frac{1}{6n^2}$.

Example 5.3.5. We apply the midpoint and trapezoidal rules to $\int_0^1 \frac{dx}{1+x^2}$. For $n=4$, we have

$$\begin{aligned}
M_4 &= 0.25\times(0.984615+0.876712+0.719101+0.566372)\approx 0.786700,\\
T_4 &= \frac{0.25}{2}\times(1.000000+2\times 0.941176+2\times 0.800000+2\times 0.640000+0.500000)\\
&\approx 0.782794.
\end{aligned}$$

For $n=8$, we have $h=0.125$ and the following values.

i	x_i	$\frac{1}{1+x_i^2}$	$\bar{x}_i$	$\frac{1}{1+\bar{x}_i^2}$
0	0	1.000000		
1	0.125	0.984615	0.0625	0.996109
2	0.25	0.941176	0.1875	0.966038
3	0.325	0.876712	0.3125	0.911032
4	0.5	0.800000	0.4375	0.839344
5	0.625	0.719101	0.5625	0.759644
6	0.75	0.640000	0.6875	0.679045
7	0.875	0.566372	0.8125	0.602353
8	1	0.500000	0.9375	0.532225

Then we get the approximations

$$M_8\approx 0.785721,\quad T_8\approx 0.784747.$$

Compared with the actual value $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0.7853981634\cdots$, the following are the errors of various schemes.

error	$n = 4$	$n = 8$
$\lvert R_n - I\rvert$	0.065104	0.031901
$\lvert L_n - I\rvert$	0.059896	0.030599
$\lvert M_n - I\rvert$	0.001302	0.000323
$\lvert T_n - I\rvert$	0.002604	0.000651

We observe that the midpoint and trapezoidal rules are much more accurate, and the error for the midpoint rule is about half of the error for the trapezoidal rule. Moreover, doubling the number of partition points improves the error by a factor of 4 for the two rules. The following gives an estimation of the errors.

Theorem 5.3.1. *Suppose $f''(x)$ is continuous and bounded by K_2 on $[a,b]$, then*

$$\left|\int_a^b f(x)dx - M_n\right| \le \frac{K_2(b-a)^3}{24n^2}, \quad \left|\int_a^b f(x)dx - T_n\right| \le \frac{K_2(b-a)^3}{12n^2}.$$

The estimations are derived in Exercises 5.5.9 and 5.5.17.

Exercise 5.3.5. To compute the integral $\int_a^b x^2dx$, for any partition of $[a,b]$, we take $x_i^* = \sqrt{\frac{1}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2)} \in [x_{i-1}, x_i]$. Show that the Riemann sum is exactly the value of the integral. How can you generalize this to $\int_a^b x^n dx$?

Exercise 5.3.6. For the integral $\int_a^b x^2dx$, we take any partition of $[a,b]$, in which the intervals may not have the same length. Estimate the error of the various schemes in terms of the size $\delta = \max_{i=1}^n(x_i - x_{i-1})$ of the partition.

Exercise 5.3.7. Apply the midpoint and trapezoidal rules to the integral and compare with the actual value.

1. $\int_1^2 \frac{dx}{x}$, $n = 6, 12$.
2. $\int_0^\pi \sin x dx$, $n = 4, 12$.

Exercise 5.3.8. Apply the midpoint and trapezoidal rules to the integral. Moreover, estimate the number of partition points needed for the approximation to be accurate up to 10^{-6}.

1. $\displaystyle\int_0^{\pi} \cos x^2 dx$, $n = 5, 10$.
2. $\displaystyle\int_0^{\pi} \frac{\sin x}{x} dx$, $n = 5, 10$.
3. $\displaystyle\int_0^{2} \frac{1}{\sqrt{1+x^3}} dx$, $n = 5, 10$.
4. $\displaystyle\int_0^{1} e^{x^2} dx$, $n = 10$.
5. $\displaystyle\int_1^{2} e^{\frac{1}{x}} dx$, $n = 10$.
6. $\displaystyle\int_1^{2} \frac{\log x}{1+x} dx$, $n = 10$.

Exercise 5.3.9. Show that $T_{2n} = \dfrac{1}{2}(M_n + T_n)$.

Exercise 5.3.10. Prove that if f is a concave positive function, then $T_n \le \displaystyle\int_a^b f(x)dx < M_n$.

5.3.3 Simpson's Rule

The midpoint rule is based on the constant approximation, and the trapezoidal rule is based on linear approximation (actually not quite, as average of two constant approximations). We may expect better approximation by using quadratic curves.

Since a quadratic curve is determined by three points, we try to approximate $f(x)$ on the interval $[x_{i-1}, x_{i+1}]$ by the quadratic function $Q(x) = A(x-x_i)^2 + B(x-x_i) + C$ satisfying

$$\begin{aligned} f(x_{i-1}) &= Q(x_{i-1}) = Ah^2 - Bh + C, \\ f(x_i) &= Q(x_i) = C, \\ f(x_{i+1}) &= Q(x_{i+1}) = Ah^2 + Bh + C. \end{aligned}$$

Then $\displaystyle\int_{x_{i-1}}^{x_{i+1}} f(x)dx$ is approximated by (the first equality uses (5.1.6))

$$\int_{x_{i-1}}^{x_{i+1}} Q(x)dx = \frac{2}{3}Ah^3 + 2Ch = \frac{h}{3}(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})).$$

Suppose n is even. Then we may apply the quadratic approximations to $[x_0, x_2]$, $[x_2, x_4]$, $\dots$, $[x_{n-2}, x_n]$. Adding such approximations together, we get an approximation of $\displaystyle\int_a^b f(x)dx$

$$S_n = \frac{h}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)).$$

This is *Simpson's rule*. Observe that $S_n = \dfrac{1}{3}(2T_n + M_{\frac{n}{2}})$ is the weighted average of the trapezoidal (with step size h) and midpoint (with step size $2h$) rules.

The errors in Simpson's rule can be estimated by the bound on the fourth order derivative.

Theorem 5.3.2. *Suppose $f^{(4)}(x)$ is continuous and bounded by K_4 on $[a,b]$, then*

$$\left|\int_a^b f(x)dx - S_n\right| \le \frac{K_4(b-a)^5}{180n^4}.$$

A consequence of the theorem is that doubling the partition improves the error by a factor of 16!

Example 5.3.6. Applying Simpson's rule to $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ for $n=4$, we use the same data from Example 5.3.1 to get

$$\begin{aligned} S_4 &= \frac{0.25}{3} \times (1.000000 + 4\times 0.941176 + 2\times 0.800000 + 4\times 0.640000 + 0.500000) \\ &\approx 0.785392. \end{aligned}$$

The error $|S_n - I| = 0.000540$ is comparable to the midpoint and trapezoidal rule for $n=8$.

How many partition points are needed in order to get the approximate value accurate up to the 6-th digit? To answer the question, we compute the derivatives

$$f^{(4)}(x) = \frac{24(5x^4-10x^2+1)}{(1+x^2)^5}, \quad f^{(5)}(x) = -\frac{240x(x^2-3)(3x^2-1)}{(1+x^2)^6}.$$

From $f^{(5)}(x)$, the extrema of $f^{(4)}(x)$ on $[0,1]$ can only be at 0, $\dfrac{1}{\sqrt{3}}$ or 1. By

$$|f^{(4)}(0)| = 24, \quad \left|f^{(4)}\left(\frac{1}{\sqrt{3}}\right)\right| = \frac{81}{8}, \quad |f^{(4)}(1)| = 3,$$

we get $K_4 = 24$. Then the question becomes

$$\frac{24}{180n^4} \le 10^{-6}.$$

Therefore we need $n \ge 19.1$. Since n should be an even integer, this means $n \ge 20$.

We may carry out the similar estimation for the midpoint and trapezoidal rules. We find $K_2 = |f''(0)| = 2$, so that the estimations become

$$\frac{2}{24n^2} \le 10^{-6}, \quad \frac{2}{12n^2} \le 10^{-6}.$$

The answers are respectively $n \ge 289$ and $n \ge 409$.

Exercise 5.3.11. Repeat Exercise 5.3.7 for the Simpson's rule.

Exercise 5.3.12. Repeat Exercise 5.3.8 for the Simpson's rule.

Exercise 5.3.13. If we apply Simpson's rule to $\int_0^1 \frac{dx}{1+x^2}$ to get an approximate value for π accurate up to 10^{-6}, how many partition points do we need?

Exercise 5.3.14. Simpson's 3/8 rule is obtained by using cubic instead of quadratic approximation. Derive the formula for this rule.

5.4 Indefinite Integral

5.4.1 Fundamental Theorem of Calculus

The Newton-Leibniz formula is derived from $A'(x) = f(x)$, where $A(x) = \int_a^x f(t)dt$ is the signed area over $[a, x]$. The equality is summarized below.

Theorem 5.4.1 (Fundamental Theorem of Calculus). *If $f(x)$ is continuous at x, then*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Note that the continuity was used critically in our argument for $A'(x) = f(x)$.

Example 5.4.1. Let $f(x)$ be a continuous function. To find the derivative of $\int_a^{x^2} f(t)dt$, we note that the integral is a composition

$$\int_a^{x^2} f(t)dt = A(x^2), \quad A(x) = \int_a^x f(t)dt.$$

By the chain rule and the Fundamental Theorem of Calculus, we have

$$\frac{d}{dx}\int_a^{x^2} f(t)dt = \frac{dA(x^2)}{dx} = A'(x^2)2x = 2xf(x^2).$$

The Fundamental Theorem also implies the following derivatives

$$\begin{aligned}
\frac{d}{dx}\int_a^x f(t^2)dt &= f(x^2),\\
\frac{d}{dx}\int_a^x f(t)^2dt &= f(x)^2,\\
\frac{d}{dx}\left(\int_a^x f(t)dt\right)^2 &= 2f(x)\int_a^x f(t)dt.
\end{aligned}$$

Example 5.4.2. The function $f(x) = \int_0^x e^{t^2} dt$ cannot be expressed as combinations of the usual elementary functions. Still, we know $f'(x) = e^{x^2}$. We also have

$$\frac{d}{dx}\int_x^{x^2} e^{t^2} dt = \frac{d}{dx}\left(\int_0^{x^2} e^{t^2} dt - \int_0^x e^{t^2} dt\right) = 2xe^{x^4} - e^{x^2}.$$

Example 5.4.3. For the sign function

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0, \end{cases}$$

we have

$$A(x) = \int_0^x \operatorname{sign}(x) dx = |x|, \quad A'(x) = \begin{cases} 1, & \text{if } x > 0, \\ \text{no}, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

We note that $A(x)$ is not differentiable at 0, exactly the place where the sign function is not continuous. The example shows that the continuity assumption cannot be dropped from the Fundamental Theorem.

Example 5.4.4. The *sine integral function* is

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt.$$

Since the integrand can be made continuous by assigning value 1 at 0, we know

$$\operatorname{Si}'(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Therefore the function $\operatorname{Si}(x)$ is strictly increasing on the following intervals

$$\dots,\ [-5\pi, -4\pi],\ [-3\pi, -2\pi],\ [-\pi, \pi],\ [2\pi, 3\pi],\ [4\pi, 5\pi],\ \dots,$$

and is strictly decreasing on the following intervals

$$\dots,\ [-4\pi, -3\pi],\ [-2\pi, -\pi],\ [\pi, 2\pi],\ [3\pi, 4\pi],\ \dots.$$

This implies that $\operatorname{Si}(x)$ has local maxima at $\dots, -6\pi, -4\pi, -2\pi, \pi, 3\pi, 5\pi, \dots$, and has local minima at $\dots, -5\pi, -3\pi, -\pi, 2\pi, 4\pi, 6\pi, \dots$. Moreover, we can also calculate the second order derivative

$$\operatorname{Si}''(x) = \begin{cases} \dfrac{x\cos x - \sin x}{x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and find the convexity property of $\operatorname{Si}(x)$.

Example 5.4.5. Suppose f is a continuous function satisfying

$$2\int_0^x tf(t)dt = x\int_0^x f(t)dt.$$

Then by taking derivative on both sides, we get

$$2xf(x) = xf(x) + \int_0^x f(t)dt.$$

Let $F(x) = \int_0^x f(t)dt$. Then the equation is the same as $xF'(x) = F(x)$. This implies

$$\left(\frac{F(x)}{x}\right)' = \frac{xF'(x) - F(x)}{x^2} = 0.$$

Therefore $\int_0^x f(t)dt = F(x) = Cx$ for a constant C, and we further find that $f(x) = C$ is a constant.

Exercise 5.4.1. Find the derivative of function.

1. $\int_0^x t^3 dt.$
2. $\int_0^{x^2} t^3 dt.$
3. $\int_0^{x^3} t^2 dt.$
4. $\int_{x^2}^{x^3} t^2 dt.$

Exercise 5.4.2. Find the derivative of function.

1. $\int_1^x \frac{dt}{1+t^3}.$
2. $\int_1^{x^2} \log(1+t^2)dt.$
3. $\int_x^{\pi} \cos t^2 dt.$
4. $\int_{x^2}^1 \sqrt{1+\sqrt{t}}dt.$
5. $\int_{\tan x}^{\pi} \arctan t dt.$
6. $\int_{\tan x}^{\cot x} (1+t^2)^{\frac{3}{2}} dt.$

Exercise 5.4.3. Let $f(x)$ be a continuous function. Find the derivative.

1. $\int_{x^2}^b f(t)dt.$
2. $\int_x^{x^2} f(t)dt.$
3. $\int_a^x f(t^2)dt.$
4. $\int_a^x e^{f(t)} dt.$
5. $\int_x^b f(\sin t)dt.$
6. $\int_{\sin x}^{\cos x} f(t)dt.$
7. $\int_0^{f(x)} f(t)dt.$
8. $\int_0^{f^{-1}(x)} f(t)dt.$

In the 7-th problem, f is differentiable. In the 8-th problem, f is invertible and differentiable.

Exercise 5.4.4. Study the monotone and convex properties, including the extrema and the points of inflection.

1. $\displaystyle\int_0^x \frac{dt}{1+t+t^2}$.
2. $\displaystyle\int_0^x \sin\frac{\pi t^2}{2}dt$.
3. $\displaystyle\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

Exercise 5.4.5. Find continuous functions $f(x)$ satisfying the equality.

1. $\displaystyle\int_0^x f(t)dt = \int_x^1 f(t)dt$ on $[0,1]$.
2. $\displaystyle A\int_0^x tf(t)dt = x\int_0^x f(t)dt$ on $(0,+\infty)$.
3. $\displaystyle (f(x))^2 = 2\int_0^x f(t)dt$ on $(-\infty,+\infty)$.
4. $\displaystyle\int_0^x f(t)dt = (2x-1)e^{2x} + \int_0^x e^{-t}f(t)dt$ on $(-\infty,+\infty)$.

Exercise 5.4.6. Find the limit.

1. $\displaystyle\lim_{x\to 0}\frac{1}{x}\int_0^x \frac{\sin t}{t}dt$.
2. $\displaystyle\lim_{x\to 0}\frac{1}{x^3}\int_0^x \sin t^2 dt$.
3. $\displaystyle\lim_{x\to+\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$.

Exercise 5.4.7. Prove that for a positive continuous function $f(x)$ on $(0,+\infty)$, the function

$$g(x) = \frac{\left(\int_0^x tf(t)dt\right)^2}{\int_0^x f(t)dt}$$

is strictly increasing on $(0,+\infty)$.

Exercise 5.4.8. Discuss where $f(x)$ is not continuous and where $\int_0^x f(t)dt$ is not differentiable.

1. $f(x) = \begin{cases} x, & \text{if } x\neq 0,\\ 1, & \text{if } x = 0,\end{cases}$.
2. $f(x) = \begin{cases} x, & \text{if } x > 0,\\ 1, & \text{if } x \le 0,\end{cases}$.
3. $f(x) = \begin{cases} \left(x^2\sin\dfrac{1}{x}\right)', & \text{if } x\neq 0,\\ 0, & \text{if } x = 0,\end{cases}$.

5.4.2 Indefinite Integral

A function F is an *antiderivative* of f if $F' = f$. By Theorem 3.4.3, the antiderivative is unique up to adding constants. Therefore we denote all the antiderivatives of f

by

$$\int f(x)dx = F(x) + C, \text{ for some } F(x) \text{ satisfying } F'(x) = f(x).$$

This is the *indefinite integral* of f.

The Newton-Leibniz formula says that the definite integral of a continuous function can be calculated from the antiderivative

$$\int_a^b f(x)dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus says that the signed area gives *one* antiderivative, and can be used as $F(x)$ above

$$\int f(x)dx = \int_a^x f(t)dt + C.$$

Example 5.4.6. By

$$(x^{p+1})' = (p+1)x^p, \quad (\log|x|)' = \frac{1}{x}, \quad (e^x)' = e^x,$$

we get

$$\int x^p dx = \begin{cases} \dfrac{x^{p+1}}{p+1} + C, & \text{for } p \neq -1, \\ \log|x| + C, & \text{for } p = -1; \end{cases} \quad \int e^x dx = e^x + C.$$

More generally, we have

$$\int (ax+b)^p dx \begin{cases} \dfrac{(ax+b)^{p+1}}{(p+1)a} + C, & \text{for } p \neq -1, \\ \dfrac{1}{a}\log|ax+b| + C, & \text{for } p = -1; \end{cases} \quad \int a^x dx = \frac{a^x}{\log a} + C.$$

Example 5.4.7. The antiderivative of the logarithmic function is more complicated

$$\int \log|x|dx = x\log|x| - x + C.$$

The equality can be verified by taking the derivative

$$(x\log|x| - x)' = \log|x| + x\frac{1}{x} - 1 = \log|x|.$$

Example 5.5.9 gives the systematic way of deriving $\int \log|x|dx$.

Example 5.4.8. The derivatives of the trigonometric functions give us

$$\int \cos x dx = \sin x + C, \qquad \int \sin x dx = -\cos x + C,$$
$$\int \sec^2 x dx = \tan x + C, \qquad \int \sec x \tan x dx = \sec x + C.$$

The antiderivatives of $\tan x$ and $\sec x$ are more complicated, and are given in Example 5.5.27.

Example 5.4.9. The derivative of the inverse sine function gives

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C, \quad a > 0.$$

Similarly, the derivative of the inverse tangent function gives

$$\int \frac{dx}{x^2+1} = \arctan x + C, \quad \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C.$$

The similar integrals $\int \frac{dx}{x^2-a^2}$ and $\int \frac{dx}{\sqrt{x^2+a}}$ are given in Exercise 5.4.10 and Examples 5.5.2, 5.5.31, 5.5.32.

Exercise 5.4.9. Compute the integrals.

1. $\int \sqrt[4]{1-x} dx$.
2. $\int \frac{1}{\sqrt[3]{2x+1}} dx$.
3. $\int a^x dx$.
4. $\int \csc^2 x dx$.
5. $\int \csc x \cot x dx$.
6. $\int \frac{dx}{\cos^2 x}$.

Exercise 5.4.10. Verify the antiderivatives.

1. $\int \frac{\log |x|}{x} dx = \frac{1}{2} (\log |x|)^2 + C$.
2. $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + C$.
3. $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$.
4. $\int \sqrt{a^2-x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2-x^2} + C$, $a > 0$.
5. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$.
6. $\int \frac{dx}{\sqrt{x^2+a}} = \log \left| x + \sqrt{x^2+a} \right| + C$.

7. $\displaystyle\int \sqrt{x^2+a}dx = \frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2}\log\left|x+\sqrt{x^2+a}\right| + C.$

Exercise 5.4.11. If $\int f(x)dx = F(x) + C$, then what is $\int f(ax+b)dx$? Apply your conclusion to compute the integrals.

1. $\displaystyle\int \log|a+bx|dx.$
2. $\displaystyle\int \sin(ax+b)dx.$
3. $\displaystyle\int \sec^2(3x-1)dx.$
4. $\displaystyle\int \frac{dx}{\sqrt{1-(x-1)^2}}.$
5. $\displaystyle\int \frac{dx}{\sqrt{x(1-x)}}.$
6. $\displaystyle\int \frac{dx}{x^2+2x+2}.$

Exercise 5.4.12. Find the antiderivative of $x(ax^2+b)^p$. Then compute the integrals.

1. $\displaystyle\int x\sqrt{x^2+3}dx.$
2. $\displaystyle\int \frac{xdx}{x^2+1}.$
3. $\displaystyle\int \frac{xdx}{\sqrt{4-x^2}}.$

Exercise 5.4.13. Compute the integrals.

1. $\displaystyle\int x\sin(ax^2+b)dx.$
2. $\displaystyle\int xe^{ax^2+b}dx.$
3. $\displaystyle\int x^2(ax^3+b)^p dx.$

One should not just mindlessly compute the antiderivative. Sometimes we need to consider the meaning of antiderivative and question whether the answer makes sense.

Example 5.4.10. Without much thinking, we may write

$$\int |x|dx = \begin{cases} \frac{1}{2}x^2 + C, & \text{if } x \ge 0, \\ -\frac{1}{2}x^2 + C, & \text{if } x < 0. \end{cases}$$

However, the constant C in the two cases cannot be independently chosen because the antiderivative must be differentiable and is therefore continuous at 0. The more sensible answer is

$$\int |x|dx = \begin{cases} \frac{1}{2}x^2, & \text{if } x \ge 0 \\ -\frac{1}{2}x^2, & \text{if } x < 0 \end{cases} + C.$$

In other words, the constant C is two cases must be equal.

For another example, instead of

$$f(x) = \begin{cases} e^x, & \text{if } x \ge 0, \\ 1, & \text{if } x < 0, \end{cases} \qquad \int f(x)dx = \begin{cases} e^x + C, & \text{if } x \ge 0, \\ x + C, & \text{if } x < 0, \end{cases}$$

we should have

$$\int f(x)dx = \begin{cases} e^x, & \text{if } x \ge 0 \\ x+1, & \text{if } x < 0 \end{cases} + C.$$

Exercise 5.4.14. Compute indefinite integral.

1. $\begin{cases} x^2, & \text{if } x \le 0, \\ \sin x, & \text{if } x > 0. \end{cases}$.
2. $\begin{cases} 1 - x^2, & \text{if } |x| \le 1, \\ \sin(1 - |x|), & \text{if } |x| > 1. \end{cases}$.

Exercise 5.4.15. Does the sign function

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0, \end{cases}$$

have antiderivative? Does the function have definite integral? What do you learn from the example?

5.5 Properties of Integration

Computationally, integration is the reverse of differentiation. Therefore properties of differentiation have corresponding properties of integration.

5.5.1 Linear Property

Suppose $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$. Then the linear property of the derivative

$$(F(x) + G(x))' = F'(x) + G'(x) = f(x) + g(x), \quad (cF(x))' = cF'(x) = cf(x),$$

implies the linear property of the antiderivative

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx, \quad \int cf(x)dx = c \int f(x)dx.$$

By the Newton-Leibniz formula, we get the linear property for the definite integral

$$\begin{aligned} \int_a^b (f(x) + g(x))dx &= (F(b) + G(b)) - (F(a) + G(a)) \\ &= (F(b) - F(a)) + (G(b) - G(a)) = \int_a^b f(x)dx + \int_a^b g(x)dx, \end{aligned}$$

$$\int_a^b cf(x)dx = cF(b) - cF(a) = c(F(b) - F(a)) = c \int_a^b f(x)dx.$$

Example 5.5.1. We have

$$\begin{aligned} \int x(1 + x)^2 dx &= \int (x + 2x^2 + x^3)dx = \int xdx + 2\int x^2 dx + \int x^3 dx \\ &= \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + C. \end{aligned}$$

This furthers gives the definite integral

$$\int_0^1 x(1+x)^2dx = \frac{1}{2}(1^2-0^2)+\frac{2}{3}(1^3-0^3)+\frac{1}{4}(1^4-0^4)=\frac{17}{12}.$$

On the other hand, it would be very complicated to compute $\int x(x+1)^{10}dx$ by the binomial expansion of $(1+x)^{10}$. The following is much simpler

$$\begin{aligned}\int x(x+1)^{10}dx &= \int((x+1)-1)(x+1)^{10}dx = \int(x+1)^{11}dx - \int(x+1)^{10}dx \\ &= \frac{1}{12}(x+1)^{12}-\frac{1}{11}(x+1)^{11}+C = \frac{1}{12\cdot 11}(11x-1)(x+1)^{11}+C.\end{aligned}$$

Exercise 5.5.1. Compute the integrals.

1. $\int x\sqrt{x+1}dx.$
2. $\int x(ax+b)^p dx.$
3. $\int x^2(ax+b)^p dx.$
4. $\int (x-1)(x+1)^{\frac{4}{3}}dx.$
5. $\int (x-1)(x+1)^p dx.$
6. $\int \frac{x}{(x+1)^{10}}dx.$
7. $\int \frac{x^2-x+1}{(x+1)^{10}}dx.$
8. $\int \frac{x-1}{\sqrt{x}}dx.$
9. $\int \left(\frac{x-1}{x}\right)^2 dx.$
10. $\int \left(\frac{x-1}{x^2}\right)^2 dx.$
11. $\int \left(\frac{x-1}{x+1}\right)^2 dx.$
12. $\int \frac{(x-1)^2}{(x+1)^4}dx.$

Exercise 5.5.2. Find A, B satisfying

$$\frac{ax+b}{cx+d} = A + \frac{B}{cx+d}.$$

Then compute the antiderivatives of $\frac{ax+b}{cx+d}$ and $\left(\frac{ax+b}{cx+d}\right)^2$.

Exercise 5.5.3. Compute the integrals.

1. $\int (e^x - e^{-x})^2 dx.$
2. $\int (2^x+3^x)^2 dx.$
3. $\int \frac{2^{x+1}-3^{x-1}}{6^x}dx.$

Example 5.5.2. To find the antiderivative of $\frac{1}{x^2-a^2}$, we use

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$$

to get

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log|x-a| - \frac{1}{2a}\log|x+a| + C = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C.$$

This furthers gives the definite integral

$$\int_2^3 \frac{1}{x^2-1}dx = \frac{1}{2}\log\left|\frac{3-1}{3+1}\right| - \frac{1}{2}\log\left|\frac{2-1}{2+1}\right| = \frac{1}{2}\log\frac{3}{2}.$$

As noted in Example 5.1.4, we should not blindly use the Newton-Leibniz formula in competing the definite integral. For example, we cannot get

$$\int_0^2 \frac{1}{x^2-1}dx = \frac{1}{2}\log\left|\frac{2-1}{2+1}\right| - \frac{1}{2}\log\left|\frac{0-1}{0+1}\right| = -\frac{1}{2}\log 3,$$

because the interval $[0,2]$ contains 1, where the integrand approaches infinity.

Example 5.5.3. The idea in Example 5.5.2 can be extended

$$\begin{aligned}
\int \frac{dx}{x(x+1)(x+2)} &= \int \frac{1}{x}\left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx = \int\left(\frac{1}{x(x+1)} - \frac{1}{x(x+2)}\right)dx \\
&= \int \left[\left(\frac{1}{x} - \frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)\right]dx \\
&= \frac{1}{2}\log|x| - \log|x+1| + \frac{1}{2}\log|x+2| + C \\
&= \frac{1}{2}\log\left|\frac{x(x+2)}{(x+1)^2}\right| + C, \\
\int \frac{dx}{(x^2-1)^2} &= \frac{1}{4}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)^2 dx \\
&= \frac{1}{4}\int\left(\frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} - \frac{2}{(x+1)(x-1)}\right)dx \\
&= \frac{1}{4}\int\left[\frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} - \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\right]dx \\
&= \frac{1}{4}\left(-\frac{1}{x-1} - \frac{1}{x+1} + \log\left|\frac{x+1}{x-1}\right|\right) + C \\
&= -\frac{x}{2(x^2-1)} + \frac{1}{4}\log\left|\frac{x+1}{x-1}\right| + C.
\end{aligned}$$

Exercise 5.5.4. Compute the integrals.

1. $\displaystyle\int \frac{xdx}{x^2-1}$.
2. $\displaystyle\int \frac{x^2dx}{x^2-1}$.
3. $\displaystyle\int \frac{dx}{x^2+3x+2}$.
4. $\displaystyle\int \frac{(2x+1)dx}{x^2+3x+2}$.
5. $\displaystyle\int_0^1 \frac{(2x+1)dx}{x^2+3x+2}$.
6. $\displaystyle\int \frac{x^2dx}{x^2+3x+2}$.
7. $\displaystyle\int \frac{x^2dx}{x^2+1}$.
8. $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)}$.
9. $\displaystyle\int \frac{x^2dx}{(x^2+1)(x^2+4)}$.

Exercise 5.5.5. Compute the integrals.

1. $\displaystyle\int \frac{dx}{(x+a)(x+b)}$.
2. $\displaystyle\int \frac{xdx}{(x+a)(x+b)}$.
3. $\displaystyle\int \frac{dx}{(x+a)(x+b)(x+c)}$.

Example 5.5.4. By trigonometric formula, we have

$$\begin{aligned}\int_0^\pi \sin^2 xdx &= \int_0^\pi \frac{1}{2}(1-\cos 2x)dx = \frac{1}{2}\int_0^\pi xdx - \frac{1}{2}\int_0^\pi \cos 2xdx \\ &= \frac{1}{4}(\pi^2 - 0^2) - \frac{1}{4}(\sin 2\pi - \sin 0) = \frac{1}{4}\pi^2.\end{aligned}$$

Similar idea gives

$$\begin{aligned}\int \sin^2 xdx &= \frac{1}{2}\int (1-\cos 2x)dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C, \\ \int \sin x\cos 2xdx &= \frac{1}{2}\int (\sin 3x - \sin x)dx = -\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C, \\ \int \tan^2 xdx &= \int (\sec^2 x - 1)dx = \tan x - x + C.\end{aligned}$$

Exercise 5.5.6. Compute the integrals.

1. $\displaystyle\int \cos x\sin xdx$.
2. $\displaystyle\int_0^\pi \sin^2 x\cos xdx$.
3. $\displaystyle\int \cos^2 xdx$.
4. $\displaystyle\int \sin^3 xdx$.
5. $\displaystyle\int \cos x\sin 2xdx$.
6. $\displaystyle\int \cot^2 xdx$.
7. $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\cos 2xdx$.
8. $\displaystyle\int_0^\pi |\sin x\cos 2x|dx$.
9. $\displaystyle\int_0^\pi |\sin x - \cos x|dx$.

Example 5.5.5. For $f \ge g$ and $a \le b$, by the inequality (5.1.5) and the linearity of definite integral, we have

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx \ge 0.$$

Therefore we have

$$f \ge g \implies \int_a^b f(x)dx \ge \int_a^b g(x)dx, \text{ for } a < b.$$

The inequality corresponds to Theorem 3.3.3 that uses the derivatives to compare functions. However, it is more direct to get the inequality by using the non-negativity of area.

If we apply the inequality to $-|f| \le f \le |f|$, then we get

$$\int_a^b |f(x)|dx \ge \left|\int_a^b f(x)dx\right|, \text{ for } a < b.$$

Example 5.5.6 (Average). The *average* of a function f on $[a,b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$. If $m \le f \le M$ on $[a,b]$, then

$$m(b-a) = \int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx = M(b-a).$$

This implies that the average of f lies between m and M, which is consistent with our intuition.

For continuous f, we may take m and M to be the minimum and maximum of f on $[a,b]$. By the Intermediate Value Theorem, any value between m and M can be reached by the function. Therefore the average

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c), \text{ for some } c \in (a,b).$$

This conclusion is the *Integral Mean Value Theorem*.

Example 5.5.7. Consider the function $F(x) = \displaystyle\int_0^x \frac{\sin t^2}{t} dt$. The 4-th order Taylor expansion $T(x) = x - \dfrac{x^3}{6}$ of $\sin x$ means that, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x| < \delta \implies |\sin x - T(x)| \le \epsilon |x|^4.$$

Then for t between 0 and x, we have

$$|x| < \sqrt{\delta} \implies |t^2| < \delta \implies |\sin t^2 - T(t^2)| \le \epsilon |t|^8,$$

so that

$$|x| < \sqrt{\delta} \implies \left| \frac{\sin t^2}{t} - t - \frac{t^5}{6} \right| = \left| \frac{\sin t^2}{t} - \frac{1}{t} T(t^2) \right| \le \epsilon |t|^7.$$

Therefore

$$|x| < \sqrt{\delta} \implies \left| F(x) - \frac{x^2}{2} - \frac{x^6}{36} \right| = \left| \int_0^x \left(\frac{\sin t^2}{t} - t - \frac{t^5}{6} \right) dt \right| \le \epsilon \int_0^x |t|^7 dt = \frac{\epsilon}{8} |x|^8.$$

This means exactly the 7-th order approximation of $F(x)$

$$F(x) = \frac{1}{2}x^2 + \frac{1}{36}x^6 + o(x^8).$$

Example 5.5.8. Suppose $f(x)$ has second order derivative on $[a,b]$. We may take the linear approximation at the middle point $c = \dfrac{a+b}{2}$. By the Lagrange form of the remainder (Theorem 4.3.1), we get

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(\bar{x})}{2}(x-c)^2,$$

where $\bar{x}$ depends on x and lies between x and c. Then we have

$$\left|\int_a^b (f(x) - f(c) - f'(c)(x-c))dx\right| = \left|\int_a^b \frac{f''(\bar{x})}{2}(x-c)^2 dx\right|$$

By the linear property of the integral, we have the left side

$$\begin{aligned}\int_a^b (f(x) - f(c) - f'(c)(x-c))dx &= \int_a^b f(x)dx - f(c)\int_a^b dx - f'(c)\int_a^b (x-c)dx \\ &= \int_a^b f(x)dx - f(c)(b-a).\end{aligned}$$

Let K_2 be the bound for the second order derivative. In other words, $|f''| \le K_2$ on $[a,b]$. Then the right side

$$\left|\int_a^b \frac{f''(\bar{x})}{2}(x-c)^2 dx\right| \le \int_a^b \frac{|f''(\bar{x})|}{2}(x-c)^2 dx \le \frac{K_2}{2}\int_a^b (x-c)^2 dx = \frac{K_2}{24}(b-a)^3.$$

We conclude the inequality

$$\left|\int_a^b f(x)dx - f(c)(b-a)\right| \le \frac{K_2}{24}(b-a)^3.$$

Exercise 5.5.7. Show that the integration of n-th order approximation is $(n+1)$-st order approximation. Specifically, find high order approximation of function at 0.

1. $\displaystyle\int_0^x \frac{\cos t - 1}{t}dt$, order 5.
2. $\displaystyle\int_0^{\sqrt{x}} \frac{\sin t - t}{t^2}dt$, order 4.
3. $\displaystyle\int_{-x^2}^0 \frac{e^t - 1}{t}dt$, order 5.
4. $\displaystyle\int_{-x}^x \frac{\log(1+t)}{t}dt$, order 7.

Exercise 5.5.8. Derive an estimation for $\left|\int_a^b f(x)dx - f(a)(b-a)\right|$ in terms of the bound K_1 of f on $[a,b]$.

Exercise 5.5.9. Apply the estimation in Example 5.5.8 to each interval of a partition and derive the error formula for the midpoint rule in Theorem 5.3.1.

5.5.2 Integration by Parts

The Leibniz rule says that, if $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, then

$$(F(x)G(x))' = F'(x)G(x) + F(x)G'(x) = f(x)G(x) + F(x)g(x).$$

In other words, $F(x)G(x)$ is an antiderivative of $f(x)G(x) + F(x)g(x)$, or

$$F(x)G(x) + C = \int f(x)G(x)dx + \int F(x)g(x)dx.$$

If we use the differential notation

$$dF(x) = F'(x)dx = f(x)dx, \quad dG(x) = G'(x)dx = g(x)dx,$$

then the equality becomes

$$\int F(x)dG(x) = F(x)G(x) - \int G(x)dF(x).$$

The equality can be used in the following way. To compute an integral $\int h(x)dx$, we separate the integrand into a product $h(x) = F(x)g(x)$ of two parts and integrate the second part to get $\int g(x)dx = G(x) + C$. Then $\int h(x)dx = \int F(x)dG(x)$, which by the equality above is converted into the computation of another integral $\int G(x)dF(x)$ that exchanges $F(x)$ and $G(x)$. This method of computing the integral is called the *integration by parts*.

By Newton-Leibniz formula, the integration by parts for indefinite integral implies the method for definite integral

$$\int_a^b F(x)dG(x) = F(b)G(b) - F(a)G(a) - \int_a^b G(x)dF(x).$$

The use of Newton-Leibniz formula requires that $f = F'$ and $g = G'$ to be continuous. Then it is not hard to extend the equality to the case that F and G are continuous on $[a, b]$ and have continuous derivatives at all but finitely many points on $[a, b]$.

Example 5.5.9. The antiderivative of the logarithmic function in Example 5.4.7 may be derived by using the integration by parts (taking $F(x) = \log|x|$ and $G(x) = x$)

$$\begin{aligned}\int \log|x|dx &= x\log|x| - \int x d\log|x| = x\log|x| - \int x(\log|x|)'dx \\ &= x\log|x| - \int dx = x\log|x| - x + C.\end{aligned}$$

The antiderivative $x\log|x| - x$ just obtained can be further used

$$\begin{aligned}\int x\log|x|dx &= \int x d(x\log|x| - x) = x(x\log|x| - x) - \int (x\log|x| - x)dx \\ &= x^2\log x - x^2 - \int x\log|x|dx + \frac{1}{2}x^2.\end{aligned}$$

Solving the equation, we get

$$\int x\log|x|dx = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C.$$

The following is an alternative way of applying the integration by parts to the same integral

$$\begin{aligned}\int x\log|x|dx &= \frac{1}{2}\int \log|x|d(x^2) = \frac{1}{2}x^2\log|x| - \frac{1}{2}\int x^2 d(\log|x|)\\ &= \frac{1}{2}x^2\log|x| - \frac{1}{2}\int x^2\frac{1}{x}dx = \frac{1}{2}x^2\log|x| - \frac{1}{4}x^2 + C.\end{aligned}$$

We may compute $\displaystyle\int x^p \log x dx$ by the similar idea.

Example 5.5.10. The integral in Example 5.5.1 can also be computed by using the integration by parts

$$\begin{aligned}\int x(x+1)^{10}dx &= \frac{1}{11}\int xd(x+1)^{11} && \text{(integrate } (x+1)^{10} \text{ part)}\\ &= \frac{1}{11}x(x+1)^{11} - \frac{1}{11}\int (x+1)^{11}dx && \text{(exchange two parts)}\\ &= \frac{1}{11}x(x+1)^{11} - \frac{1}{12\cdot 11}(x+1)^{12} + C.\end{aligned}$$

Example 5.5.11. Using integration by parts, we have

$$\begin{aligned}\int x^2e^{-x}dx &= -\int x^2de^{-x} = -x^2e^{-x} + 2\int xe^{-x}dx\\ &= -x^2e^{-x} - 2\int xde^{-x} = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}dx\\ &= -(x^2+2x+2)e^{-x} + C.\end{aligned}$$

In general, we have the recursive formula

$$\int x^na^xdx = \frac{1}{\log a}x^na^x - \frac{n}{\log a}\int x^{n-1}a^xdx.$$

Exercise 5.5.10. Compute the integral.

1. $\displaystyle\int x(ax+b)^pdx.$
2. $\displaystyle\int x^2(ax+b)^pdx.$
3. $\displaystyle\int (x-1)(x+1)^pdx.$

Exercise 5.5.11. Compute the integral.

1. $\displaystyle\int (x^2-1)a^x dx$.
2. $\displaystyle\int (x+a^x)^2 dx$.
3. $\displaystyle\int \frac{xe^x dx}{(x+1)^2}$.

Exercise 5.5.12. Compute the integral.

1. $\displaystyle\int x^2 \log|x| dx$.
2. $\displaystyle\int x^p \log x dx$.
3. $\displaystyle\int (\log|x|)^2 dx$.
4. $\displaystyle\int x^p (\log|x|)^2 dx$.
5. $\displaystyle\int x\log(x+1) dx$.
6. $\displaystyle\int x\log\frac{1+x}{1-x} dx$.

Exercise 5.5.13. Derive the recursive formula for $\displaystyle\int (\log|x|)^n dx$. How about $\displaystyle\int x^p(\log x)^n dx$?

Exercise 5.5.14. Compute $\displaystyle\int_0^1 x^n a^x dx$.

Exercise 5.5.15. For natural numbers m, n, show that $\displaystyle\int_0^1 x^m(1-x)^n dx = \frac{m!n!}{(m+n+1)!}$.

Exercise 5.5.16. Compute the integral.

1. $\displaystyle\int \log(\sqrt{x+a}+\sqrt{x-a}) dx$.
2. $\displaystyle\int \log(\sqrt{a+x}-\sqrt{a-x}) dx$.
3. $\displaystyle\int \left(\frac{\log(x+a)}{x+b}+\frac{\log(x+b)}{x+a}\right) dx$.
4. $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\log(x+\sqrt{1+x^2}) dx$.
5. $\displaystyle\int \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} dx$.
6. $\displaystyle\int \left(\log(x+\sqrt{1+x^2})\right)^2 dx$.

Exercise 5.5.17. Let f have second order derivative on $[a,b]$.

1. Show that for any constants A and B, we have
$$\int_a^b f(x)dx = \Big((x+A)f(x)-\frac{1}{2}((x+A)^2+B)f'(x)\Big)_a^b + \frac{1}{2}\int_a^b ((x+A)^2+B)f''(x)dx.$$

2. By choosing suitable A and B in the first part, show that
$$\int_a^b f(x)dx = \frac{f(a)+f(b)}{2}(b-a)+\int_a^b ((x+A)^2+2B)f''(x)dx,$$
and
$$\int_a^b |(x+A)^2+2B|dx = \frac{1}{12}(b-a)^3.$$

3. Use the second part to derive the error formula for the trapezoidal rule in Theorem 5.3.1.

Example 5.5.12. Using the integration by parts, we have

$$\int x\cos x dx = \int x d\sin x = x\sin x - \int \sin x dx = x\sin x + \cos x + C.$$

The idea can be extended to product of x^n, $\sin ax$ and $\cos bx$ for various a and b

$$\begin{aligned}\int x\sin x\sin 2x dx &= \frac{1}{2}\int x(\cos 3x - \cos x)dx = \frac{1}{2}\int xd\left(\frac{1}{3}\sin 3x - \sin x\right)\\ &= \frac{1}{6}x(\sin 3x - 3\sin x) - \frac{1}{6}\int(\sin 3x - 3\sin x)dx\\ &= \frac{1}{6}x\sin 3x - \frac{1}{2}x\sin x + \frac{1}{18}\cos 3x - \frac{1}{2}\cos x + C.\end{aligned}$$

An example of the definite integral is

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x^2\sin x dx &= -\int_0^{\frac{\pi}{2}} x^2 d\cos x = -\left(\frac{\pi}{2}\right)^2\cos\frac{\pi}{2} + 0^2\cos 0 + \int_0^{\frac{\pi}{2}} 2x\cos x dx\\ &= 2\int_0^{\frac{\pi}{2}} x d\sin x = 2\frac{\pi}{2}\sin\frac{\pi}{2} - 2\cdot 0\sin 0 - 2\int_0^{\frac{\pi}{2}}\sin x dx\\ &= \pi + 2\cos\frac{\pi}{2} - 2\cos 0 = \pi - 2.\end{aligned}$$

Example 5.5.13. Let

$$I_0 = \int e^{ax}\cos bx dx, \quad J_0 = \int e^{ax}\sin bx dx.$$

We have

$$\begin{aligned}I_0 &= a^{-1}\int\cos bx de^{ax} = a^{-1}e^{ax}\cos bx - a^{-1}\int e^{ax}d\cos bx\\ &= a^{-1}e^{ax}\cos bx + a^{-1}bJ_0,\\ J_0 &= a^{-1}\int\sin bx de^{ax} = a^{-1}e^{ax}\sin bx - a^{-1}\int e^{ax}d\sin bx\\ &= a^{-1}e^{ax}\sin bx - a^{-1}bI_0.\end{aligned}$$

Solving the system for I_0 and J_0, we get

$$\begin{aligned}\int e^{ax}\cos bx dx &= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C,\\ \int e^{ax}\sin bx dx &= \frac{e^{ax}}{a^2+b^2}(-b\cos bx + a\sin bx) + C.\end{aligned}$$

Let

$$I_1 = \int xe^x\cos x dx, \quad J_1 = \int xe^x\sin x dx.$$

Using the earlier computation of I_0 and J_0, we have

$$\begin{aligned}
I_1 &= \int x\cos x de^x = xe^x\cos x - \int(\cos x - x\sin x)e^x dx \\
&= xe^x\cos x - \frac{1}{2}e^x(\cos x + \sin x) + J_1.
\end{aligned}$$

Similarly, we have

$$J_1 = xe^x\sin x - \frac{1}{2}e^x(-\cos x + \sin x) - I_1.$$

Solving the two equations, we get

$$\begin{aligned}
\int xe^x\cos x dx &= \frac{1}{2}xe^x(\cos x + \sin x) - e^x\sin x + C, \\
\int xe^x\sin x dx &= \frac{1}{2}xe^x(-\cos x + \sin x) + e^x\cos x + C.
\end{aligned}$$

Example 5.5.14. Let

$$I_{m,n} = \int \cos^m x\sin^n x dx.$$

If $n \neq 0$, then we may integrate a copy of $\sin x$ to get

$$\begin{aligned}
I_{m,n} &= -\int \cos^m x\sin^{n-1} x d\cos x \\
&= -\cos^{m+1} x\sin^{n-1} x \\
&\quad + \int(-m\cos^{m-1} x\sin^n x + (n-1)\cos^{m+1} x\sin^{n-2} x)\cos x dx \\
&= -\cos^{m+1} x\sin^{n-1} x \\
&\quad + \int(-m\cos^m x\sin^n x + (n-1)\cos^m x(1-\sin^2 x)\sin^{n-2} x)dx \\
&= -\cos^{m+1} x\sin^{n-1} x - (m+n-1)I_{m,n} - (n-1)I_{m,n-2}.
\end{aligned}$$

Therefore (the formula can be directly verified for $n = 0$)

$$I_{m,n} = -\frac{1}{m+n}\cos^{m+1} x\sin^{n-1} x + \frac{n-1}{m+n}I_{m,n-2}, \quad m+n \neq 0.$$

The formula reduces the power of sine by 2. If we first integrate a copy of $\cos x$, then we get another recursive relation that reduces the power of cosine by 2

$$I_{m,n} = \frac{1}{m+n}\cos^{m-1} x\sin^{n+1} x + \frac{m-1}{m+n}I_{m-2,n}, \quad m+n \neq 0.$$

On the other hand, we can also express $I_{m,n-2}$ and $I_{m-2,n}$ in terms of $I_{m,n}$. After substituting n by $n+2$, we get recursive relations that increase the power by 2

$$\begin{aligned} I_{m,n} &= \frac{1}{n+1}\cos^{m+1}x\sin^{n+1}x + \frac{m+n+2}{n+1}I_{m,n+2}, \quad n\neq -1, \\ &= -\frac{1}{m+1}\cos^{m+1}x\sin^{n+1}x + \frac{m+n+2}{m+1}I_{m+2,n}, \quad m\neq -1. \end{aligned}$$

Here is a concrete example of using the recursive relation

$$\begin{aligned} \int \cos^4 x\sin^6 x dx = I_{4,6} &= -\frac{1}{4+6}\cos^{4+1}x\sin^{6-1}x + \frac{6-1}{4+6}I_{4,6-2} \\ &= -\frac{1}{10}\cos^5 x\sin^5 x + \frac{5}{10}I_{4,4} \\ &= -\frac{1}{10}\cos^5 x\sin^5 x + \frac{5}{10}\left(-\frac{1}{8}\cos^5 x\sin^3 x + \frac{3}{8}I_{4,2}\right) \\ &= -\cos^5 x\left(\frac{1}{10}\sin^5 x + \frac{5}{10\cdot 8}\sin^3 x\right) \\ &\quad + \frac{5\cdot 3}{10\cdot 8}\left(-\frac{1}{6}\cos^5 x\sin x + \frac{1}{6}I_{4,0}\right) \\ &= -\cos^5 x\left(\frac{1}{10}\sin^5 x + \frac{5}{10\cdot 8}\sin^3 x + \frac{5\cdot 3}{10\cdot 8\cdot 6}\sin x\right) \\ &\quad + \frac{5\cdot 3\cdot 1}{10\cdot 8\cdot 6}\left(\frac{1}{4}\cos^3 x\sin x + \frac{3}{4}I_{2,0}\right) \\ &= -\cos^5 x\left(\frac{1}{10}\sin^5 x + \frac{5}{10\cdot 8}\sin^3 x + \frac{5\cdot 3}{10\cdot 8\cdot 6}\sin x\right) \\ &\quad + \frac{5\cdot 3\cdot 1}{10\cdot 8\cdot 6}\frac{1}{4}\cos^3 x\sin x + + \frac{5\cdot 3\cdot 1}{10\cdot 8\cdot 6}\frac{3}{4}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}I_{0,0}\right) \\ &= -\cos^5 x\left(\frac{1}{10}\sin^5 x + \frac{5}{10\cdot 8}\sin^3 x + \frac{5\cdot 3}{10\cdot 8\cdot 6}\sin x\right) \\ &\quad + \frac{5\cdot 3\cdot 1}{10\cdot 8\cdot 6}\left(\frac{1}{4}\cos^3 x + \frac{3\cdot 1}{4\cdot 2}\cos x\right)\sin x + + \frac{5\cdot 3\cdot 1}{10\cdot 8\cdot 6}\frac{3\cdot 1}{4\cdot 2}x + C. \end{aligned}$$

Here is another example that requires increasing the power

$$\begin{aligned} \int \frac{\sin^2 x}{\cos^4 x}dx = I_{-4,2} &= -\frac{1}{-4+1}\cos^{-4+1}x\sin^{2+1}x + \frac{-4+2+2}{-4+1}I_{-4+2,2} \\ &= \frac{1}{3}\frac{\sin^3 x}{\cos^3 x} + C. \end{aligned}$$

Applying the recursive relation to the definite integral, we have

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = -\frac{1}{n}\cos x\sin^{n-1}x\Big|_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}x dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}x dx.$$

Then we get

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\int_0^{\frac{\pi}{2}} \sin^{0 \text{ or } 1} x dx = \begin{cases} \dfrac{(n-1)!!}{n!!}\dfrac{\pi}{2}, & \text{if } n \text{ is even}, \\ \dfrac{(n-1)!!}{n!!}, & \text{if } n \text{ is odd}. \end{cases}$$

Here $\sin^{0 \text{ or } 1} x$ takes power 0 for even n and takes power 1 for odd n. Moreover, we used the *double factorial*

$$n!! = n(n-2)(n-4)\cdots = \begin{cases} 2k(2k-2)(2k-4)\cdots 4\cdot 2, & \text{if } n = 2k, \\ (2k+1)(2k-1)(2k-3)\cdots 3\cdot 1, & \text{if } n = 2k+1. \end{cases}$$

Exercise 5.5.18. Find the recursive relations for $\int x^p \cos ax dx$ and $\int x^p \sin ax dx$. Then compute the integral.

1. $\int x \cos^2 x dx$.
2. $\int x^3 \cos^2 x dx$.
3. $\int x \cos^2 x \sin 2x dx$.
4. $\int x^3 \cos^2 x \sin 2x dx$.
5. $\int_0^{\pi} x^6 \cos x dx$.
6. $\int_0^{\frac{\pi}{2}} x^5 \sin 2x dx$.

Exercise 5.5.19. Find the recursive relations for $\int x^p e^{ax} \cos bx dx$ and $\int x^p e^{ax} \sin bx dx$. Then compute the integral.

1. $\int x^2 e^{-x} \sin 3x dx$.
2. $\int x^2 2^x \cos x dx$.
3. $\int x^3 e^x \cos^2 x dx$.

Exercise 5.5.20. Compute the integral.

1. $\int \sin^6 x dx$.
2. $\int \cos^8 x dx$.
3. $\int \cos^8 x \sin^6 x dx$.
4. $\int \cos^3 x \sin^2 x dx$.
5. $\int \cos^3 x \sin^5 x dx$.
6. $\int \cos^{-2} x \sin^2 x dx$.
7. $\int \dfrac{dx}{\cos^6 x}$.
8. $\int \dfrac{dx}{\sin^2 x \cos^2 x}$.

Exercise 5.5.21. Show that $\int_0^{\frac{\pi}{2}} \sin^{2m} x \cos^{2n} x dx = \dfrac{(2m)!(2n)!}{2^{2m+2n+1} m!n!(m+n)!}$ for natural numbers m, n. Can you find $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$?

Exercise 5.5.22. Use $(\tan x)' = \sec^2 x = \tan^2 x + 1$ to derive the recursive formula for $\int \sec^m x \tan^n x dx$ similar to Example 5.5.14 and then find the value of $\int_0^{\frac{\pi}{4}} \tan^{2n} x dx$.

Example 5.5.15. Let

$$I_p = \int (ax^2+bx+c)^p dx, \quad a \neq 0,\ b^2 \neq 4ac.$$

We have

$$\begin{aligned} I_p &= x(ax^2+bx+c)^p - \int xd(ax^2+bx+c)^p \\ &= x(ax^2+bx+c)^p - \int px(2ax+b)(ax^2+bx+c)^{p-1}dx. \end{aligned}$$

We try to express $px(2ax+b)$ as a combination of (ax^2+bx+c) and $(ax^2+bx+c)'$, up to adding a constant

$$2pax^2 + pbx = A(ax^2+bx+c) + B(ax^2+bx+c)' + C.$$

We get $A = 2p$, $B = -\dfrac{pb}{2a}$, $C = \dfrac{p(b^2-4ac)}{2a}$. Then

$$\begin{aligned} I_p &= x(ax^2+bx+c)^p - A\int (ax^2+bx+c)^p dx \\ &\quad - B\int (ax^2+bx+c)^{p-1}(ax^2+bx+c)'dx - C\int (ax^2+bx+c)^{p-1}dx \\ &= x(ax^2+bx+c)^p - AI_p - \frac{B}{p}(ax^2+bx+c)^p - CI_{p-1}. \end{aligned}$$

This gives us the recursive relation

$$I_p = \frac{1}{(2p+1)2a}(2ax+b)(ax^2+bx+c)^p - \frac{p(b^2-4ac)}{(2p+1)2a}I_{p-1}, \quad p \neq -\frac{1}{2}.$$

On the other hand, we may also express I_{p-1} in terms of I_p. After substituting p by $p+1$, we get

$$I_p = \frac{1}{(p+1)(b^2-4ac)}(2ax+b)(ax^2+bx+c)^{p+1} - \frac{(2p+3)2a}{(p+1)(b^2-4ac)}I_{p+1}, \quad p \neq -1.$$

For the special case

$$I_p = \int (ax^2+b)^p dx, \quad a, b \neq 0,$$

the recursive relations become

$$\begin{aligned} I_p &= \frac{1}{2p+1}x(ax^2+b)^p + \frac{2pb}{2p+1}I_{p-1}, \quad p \neq -\frac{1}{2}; \\ I_p &= -\frac{1}{2(p+1)b}x(ax^2+b)^{p+1} + \frac{2p+3}{2(p+1)b}I_{p+1}, \quad p \neq -1. \end{aligned}$$

For the special cases of $p = -\frac{1}{2}, -1$, I_p is given by Exercise 5.4.10 (and will be derived in Examples 5.5.30, 5.5.31, 5.5.32)

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2-x^2}} &= \arcsin\frac{x}{a} + C, \quad a>0,\\
\int \frac{dx}{\sqrt{x^2+a}} &= \log\left|x+\sqrt{x^2+a}\right| + C,\\
\int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C,\\
\int \frac{dx}{x^2+a^2} &= \frac{1}{a}\arctan\frac{x}{a} + C.
\end{aligned}$$

Then the recursive relations can be used to compute I_p when p is an integer or a half integer. For example, we have

$$\begin{aligned}
\int \sqrt{a^2-x^2}dx = I_{\frac{1}{2}} &= \frac{1}{2\cdot\frac{1}{2}+1}x(a^2-x^2)^{\frac{1}{2}} + \frac{2\cdot\frac{1}{2}a^2}{2\cdot\frac{1}{2}+1}I_{\frac{1}{2}-1}\\
&= \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C,\\
\int (a^2-x^2)^{\frac{3}{2}}dx = I_{\frac{3}{2}} &= \frac{1}{2\cdot\frac{3}{2}+1}x(a^2-x^2)^{\frac{3}{2}} + \frac{2\cdot\frac{3}{2}a^2}{2\cdot\frac{3}{2}+1}I_{\frac{3}{2}-1}\\
&= \frac{1}{4}x(a^2-x^2)^{\frac{3}{2}} + \frac{3a^2}{4}\left(\frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right) + C\\
&= -\frac{1}{8}x(2x^2-5a^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a} + C,\\
\int \frac{dx}{(x^2+a^2)^2} = I_{-2} &= -\frac{1}{2(-2+1)a^2}x(x^2+a^2)^{-2+1} + \frac{2(-2)+3}{2(-2+1)a^2}I_{-2+1}\\
&= \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\arctan\frac{x}{a} + C.
\end{aligned}$$

Exercise 5.5.23. Compute the integral.

1. $\int x^2\sqrt{a^2-x^2}dx$.
2. $\int x\sqrt{a^2-x^2}dx$.
3. $\int (x+b)^2\sqrt{a^2-x^2}dx$.
4. $\int \frac{x^2dx}{\sqrt{a^2-x^2}}$.
5. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$.
6. $\int \frac{xdx}{(a^2-x^2)^{\frac{3}{2}}}$.
7. $\int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}}$.
8. $\int \frac{dx}{(a^2-x^2)^3}$.
9. $\int \frac{xdx}{(a^2-x^2)^3}$.

Exercise 5.5.24. Compute the integral.

1. $\displaystyle\int x^2\sqrt{x^2+a}dx$.
2. $\displaystyle\int x\sqrt{x^2+a}dx$.
3. $\displaystyle\int \frac{x^2dx}{\sqrt{x^2+a}}$.
4. $\displaystyle\int \frac{xdx}{\sqrt{x^2+a}}$.
5. $\displaystyle\int (x^2+a)^{\frac{3}{2}}dx$.
6. $\displaystyle\int \frac{dx}{(x^2+a)^{\frac{3}{2}}}$.
7. $\displaystyle\int \frac{dx}{(x^2+a^2)^3}$.
8. $\displaystyle\int \frac{xdx}{(x^2+a^2)^3}$.
9. $\displaystyle\int \frac{x^2dx}{(x^2+a^2)^3}$.

Exercise 5.5.25. Combine the ideas of Exercise 5.4.11 and Example 5.5.15 to compute the integral.

1. $\displaystyle\int \frac{dx}{(x^2+2x+2)^2}$.
2. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+2}}$.
3. $\displaystyle\int \frac{dx}{(x^2+2x+2)^{\frac{3}{2}}}$.
4. $\displaystyle\int (x^2+2x+2)^{\frac{3}{2}}dx$.
5. $\displaystyle\int \sqrt{x(1-x)}dx$.
6. $\displaystyle\int \frac{dx}{(x(1-x))^{\frac{3}{2}}}$.

5.5.3 Change of Variable

The chain rule says that, if $\displaystyle\int f(y)dy = F(y)+C$ is the indefinite integral of $f(y)$, and $\phi(x)$ is a differentiable function, then

$$F(\phi(x))' = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x).$$

In other words, $F(\phi(x))$ is the antiderivative of $f(\phi(x))\phi'(x)$, or

$$\int f(\phi(x))\phi'(x)dx = F(\phi(x))+C = \int f(y)dy\bigg|_{y=\phi(x)}.$$

If we use the differential notation $d\phi(x)=\phi'(x)dx$, then the equality becomes

$$\int f(\phi(x))d\phi(x) = \int f(y)dy\bigg|_{y=\phi(x)}.$$

The right side means computing the antiderivative of the function of y first, and then substituting $y=\phi(x)$ into the antiderivative. This is the *change of variable* formula. By Newton-Leibniz formula, we further get the change of variable formula for definite integral

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_a^b f(\phi(x))d\phi(x) = \int_{\phi(a)}^{\phi(b)} f(y)dy.$$

Example 5.5.16. If $\displaystyle\int f(y)dy = F(y)+C$, then by letting $y=ax+b$, we have

$$\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b) = \frac{1}{a}F(ax+b)+C.$$

For example,

$$\begin{aligned}
\int (2x+1)^p dx &=_{y=2x+1} \frac{1}{2}\int y^p dy = \frac{y^{p+1}}{2(p+1)} + C = \frac{(2x+1)^{p+1}}{2(p+1)} + C,\\
\int \sin(3x-2)dx &=_{y=3x-2} \frac{1}{3}\int \sin y dy = -\frac{1}{3}\cos y + C = -\frac{1}{3}\cos(3x-2) + C,\\
\int \frac{dx}{x^2+2x+2} &= \int \frac{dx}{(x+1)^2+1} =_{y=x+1} \int \frac{dx}{y^2+1}\\
&= \arctan y + C = \arctan(x+1) + C.
\end{aligned}$$

Example 5.5.17. The following is a simple change of variable

$$\int xe^{x^2}dx =_{y=x^2} \frac{1}{2}\int e^y dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{x^2} + C.$$

The idea is a "mini-integration" of xdx that can be expressed more clearly by writing

$$\int xe^{x^2}dx = \int e^{x^2}\frac{1}{2}d(x^2) = \frac{1}{2}e^{x^2} + C.$$

After the mini-integration, we view x^2 as the new variable.

The following are more examples following the mini-integration idea

$$\begin{aligned}
\int \frac{dx}{x\log x} &= \int \frac{1}{\log x}\left(\frac{dx}{x}\right) = \int \frac{1}{\log x}d(\log x) = \log|\log x| + C,\\
\int \frac{dx}{x^2+a^2} &= \int \frac{d}{a^2\left(\left(\frac{x}{a}\right)^2+1\right)} = \int \frac{d\left(\frac{x}{a}\right)}{a\left(\left(\frac{x}{a}\right)^2+1\right)} = \frac{1}{a}\arctan\frac{x}{a} + C,\\
\int \frac{xdx}{x^4+a^4} &= \frac{1}{2}\int \frac{d(x^2)}{(x^2)^2+a^4} = \frac{1}{2a^2}\arctan\frac{x^2}{a^2} + C.
\end{aligned}$$

Example 5.5.18. The integral in Example 5.5.1 was computed in Example 5.5.10 again by using the integration by parts. The integral can also be computed by change of variable.

Let $y = x+1$. Then

$$\int x(x+1)^{10}dx = \int (y-1)y^{10}dy = \int (y^{10}-y^{11})dy = \frac{1}{11}y^{11} - \frac{1}{12}y^{12} + C.$$

Substituting $y = x+1$ back, we get

$$\int x(x+1)^{10}dx = \frac{1}{11}(x+1)^{11} - \frac{1}{12}(x+1)^{12} + C.$$

By the same change, we have

$$
\begin{aligned}
\int\left(\frac{x-1}{x+1}\right)^4 dx &=_{y=x+1} \int \frac{(y-2)^4}{y^4}dy \\
&= \int (1-4\cdot 2y^{-1}+6\cdot 2^2y^{-2}-4\cdot 2^3y^{-3}+2^4y^{-4})dy \\
&= x+1-8\log|x+1| \\
&\quad -24(x+1)^{-1}+16(x+1)^{-2}-\frac{16}{3}(x+1)^{-3}+C \\
&= x-\frac{8(9x^2+12x+5)}{3(x+1)^3}-8\log|x+1|+C.
\end{aligned}
$$

Note that the second C is the first C plus 1.

Compare the above with the computation of definite integral

$$
\begin{aligned}
\int_0^1\left(\frac{x-1}{x+1}\right)^4 dx &=_{y=x+1} \int_1^2 \frac{(y-2)^4}{y^4}dy \\
&= \int_1^2 (1-4\cdot 2y^{-1}+6\cdot 2^2y^{-2}-4\cdot 2^3y^{-3}+2^4y^{-4})dy \\
&= \left(y-8\log y-24y^{-1}+16y^{-2}-\frac{16}{3}y^{-3}\right)_1^2 \\
&= 1-8\log 2-24\left(\frac{1}{2}-1\right)+16\left(\frac{1}{4}-1\right)-\frac{16}{3}\left(\frac{1}{8}-1\right) \\
&= \frac{17}{3}-8\log 2.
\end{aligned}
$$

Note that the evaluation is done by using the new variable y instead of the old x.

Example 5.5.19. The integrals of inverse trigonometric functions can also be computed by combining integration by parts and change of variable

$$
\begin{aligned}
\int \arcsin x dx &= x\arcsin x-\int\frac{xdx}{\sqrt{1-x^2}} = x\arcsin x+\frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{1-x^2}} \\
&= x\arcsin x+\sqrt{1-x^2}+C.
\end{aligned}
$$

Alternatively, we may simply introduce the trigonometric function as the new variable. For example, by $y=\arcsin x$, $x=\sin y$, we have

$$
\begin{aligned}
\int \arcsin x dx &= \int yd(\sin y) = y\sin y-\int \sin y dy \\
&= y\sin y+\cos y+C = x\arcsin x+\sqrt{1-x^2}+C.
\end{aligned}
$$

Note that $\cos y=\sqrt{1-x^2}$ is non-negative because $x\in[-1,1]$ and $y\in[-\frac{\pi}{2},\frac{\pi}{2}]$. The integration by parts used in both computations are essentially the same.

The idea for integrating $\arcsin x$ can also be used to compute $\int x^m(\arcsin x)^n dx$ and $\int x^m(\arctan x)^n dx$.

Example 5.5.20. To compute $\int \frac{dx}{\sqrt{e^x+a}}$, we introduce

$$y=\sqrt{e^x+a},\quad y^2=e^x+a,\quad 2ydy=e^x dx.$$

Then

$$\int \frac{dx}{\sqrt{e^x+a}}=\int \frac{\frac{2y}{e^x}dy}{y}=\int \frac{2dy}{y^2-a}.$$

By Examples 5.4.9 and 5.5.2, we have

$$\int \frac{dx}{\sqrt{e^x+a}}=\begin{cases}\dfrac{1}{\sqrt{a}}\log\left|\dfrac{y-\sqrt{-a}}{y+\sqrt{-a}}\right|+C=\dfrac{1}{\sqrt{a}}\log\left|\dfrac{\sqrt{e^x+a}-\sqrt{a}}{\sqrt{e^x+a}+\sqrt{a}}\right|+C, & \text{if } a>0,\\ \dfrac{2}{\sqrt{-a}}\arctan\dfrac{y}{\sqrt{-a}}+C=\dfrac{2}{\sqrt{-a}}\arctan\sqrt{-\dfrac{e^x}{a}-1}+C, & \text{if } a<0.\end{cases}$$

Example 5.5.21. To compute

$$I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx,$$

we introduce $y=\pi-x$. Then

$$\begin{aligned}I&=-\int_\pi^0 \frac{(\pi-y)\sin y}{1+\cos^2 y}dy=\pi\int_0^\pi \frac{\sin y}{1+\cos^2 y}dy-\int_0^\pi \frac{y\sin y}{1+\cos^2 y}dy\\ &=\pi\int_0^\pi \frac{\sin y}{1+\cos^2 y}dy-I.\end{aligned}$$

Therefore

$$\begin{aligned}I&=\frac{\pi}{2}\int_0^\pi \frac{\sin y}{1+\cos^2 y}dy=-\frac{\pi}{2}\int_{\cos 0}^{\cos\pi}\frac{1}{1+z^2}dz\\ &=\frac{\pi}{2}\int_{-1}^1\frac{dz}{1+z^2}=\frac{\pi}{2}(\arctan 1-\arctan(-1))=\frac{\pi^2}{4}.\end{aligned}$$

Note that the computation of the definite integral makes use of the new variable z only. There is no need to go back to the original variable x.

Exercise 5.5.26. Compute the integral.

1. $\int (x^2+1)(2x-1)^{10}dx$.
2. $\int \log(2x-1)dx$.
3. $\int x\log(2x-1)dx$.
4. $\int \cos(2x-1)dx$.
5. $\int e^{3x}\cos(2x-1)dx$.
6. $\int \sin(2x+1)\cos(2x-1)dx$.

Exercise 5.5.27. Compute the integral.

1. $\int \frac{x}{x^2+1}dx$.
2. $\int \frac{bx+c}{x^2+a^2}dx$.
3. $\int \frac{x}{(x^2+1)^p}dx$.
4. $\int x(x^2+a^2)^p dx$.
5. $\int x^3\sqrt{x^2+1}dx$.
6. $\int \frac{x^3dx}{\sqrt[3]{x^2+a^2}}$.

Exercise 5.5.28. Compute the integral.

1. $\int \sin x\sin(\cos x)dx$.
2. $\int \sqrt{x}\sin(1+x^{\frac{3}{2}})dx$.
3. $\int \cot x dx$.
4. $\int x\tan x^2 dx$.
5. $\int \frac{1}{x}\tan(\log x)dx$.
6. $\int \sin(\log x)dx$.
7. $\int \frac{\sin x dx}{a+\cos^2 x}$.
8. $\int \frac{\sin 2x dx}{a+\cos^2 x}$.
9. $\int \sin 2x\sqrt{a+\cos^2 x}dx$.
10. $\int \frac{\cos x dx}{\sqrt{a+\cos^2 x}}$.
11. $\int \frac{\sin x\cos x dx}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}$.
12. $\int \frac{\sin x\cos x dx}{\sin^4 x+\cos^4 x}$.

Exercise 5.5.29. Compute the integral.

1. $\int \arccos x dx$.
2. $\int \frac{\arcsin x dx}{\sqrt{1-x^2}}$.
3. $\int \sqrt{1-x^2}\arcsin x dx$.
4. $\int x\sqrt{1-x^2}\arcsin x dx$.
5. $\int \frac{dx}{\sqrt{1-x^2}\arccos x}$.
6. $\int \frac{x^3\arccos x}{\sqrt{1-x^2}}dx$.
7. $\int x^2\arccos x dx$.
8. $\int \frac{\arcsin x}{x^2}dx$.
9. $\int (\arccos x)^2 dx$.
10. $\int x\arctan x dx$.
11. $\int \frac{(\arctan x)^2}{1+x^2}dx$.
12. $\int \frac{dx}{(1+x^2)\arctan x}$.

Exercise 5.5.30. Compute the integral.

1. $\int x^3e^{x^2}dx$.
2. $\int e^{\sqrt{x}}dx$.
3. $\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$.
4. $\int \sqrt{x}e^{\sqrt{x}}dx$.
5. $\int e^{\cos x}\sin x dx$.
6. $\int \frac{e^x dx}{1+e^x}$.

7. $\displaystyle\int \frac{dx}{1+e^x}$.

8. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$.

9. $\displaystyle\int \sqrt{e^x+a}dx$.

Exercise 5.5.31. Find a recursive relation for $\int (e^x+a)^p dx$. Then compute $\int (e^x+a)^{\frac{5}{2}}dx$ and $\displaystyle\int \frac{dx}{(e^x+a)^3}$.

Exercise 5.5.32. Compute the integral.

1. $\displaystyle\int \frac{dx}{x\sqrt{1+\log x}}$.
2. $\displaystyle\int \frac{dx}{x\log x\log(\log x)}$.
3. $\displaystyle\int \frac{1}{x^2-1}\log\frac{x+1}{x-1}dx$.
4. $\displaystyle\int \frac{\log(x+1)-\log x}{x(x+1)}dx$.

Exercise 5.5.33. Compute the integral.

1. $\displaystyle\int \frac{f'(x)}{f(x)^p}dx$.
2. $\displaystyle\int \frac{f'(x)}{1+f(x)^2}dx$.
3. $\displaystyle\int 2^{f(x)}f'(x)dx$.

Exercise 5.5.34. Prove the equalties

1. $\displaystyle\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$.
2. $\displaystyle\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

Exercise 5.5.35. Explain why we cannot use the change of variable $y=\dfrac{1}{x}$ to compute the integral $\displaystyle\int_{-1}^{1}\frac{dx}{1+x^2}$.

Exercise 5.5.36. Suppose f is continuous on an open interval containing $[a,b]$. Find the derivative $\displaystyle\frac{d}{dt}\int_a^b f(x+t)dx$.

Exercise 5.5.37. Explain the equalities in Exercise 5.1.11 by change of variable.

Exercise 5.5.38. Prove that $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ for even function f. Prove that $\displaystyle\int_{-a}^{a} f(x)dx=0$ for odd function f.

Example 5.5.22. To compute $\displaystyle\int \frac{dx}{1+\sqrt{x-1}}$, we simply let $y=\sqrt{x-1}$. Then $x=y^2+1$, $dx=2ydy$, and

$$\begin{aligned}\int \frac{dx}{1+\sqrt{x-1}} &= \int \frac{2ydy}{1+y} = 2\int\left(1-\frac{1}{1+y}\right)dy \\ &= 2y-2\log(1+y)+C = 2\sqrt{x-1}-2\log(1+\sqrt{x-1})+C.\end{aligned}$$

Example 5.5.23. By taking $y = x^6$, we get rid of the square root and cube root at the same time.

$$
\begin{aligned}
\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} &=_{x=y^6} \int \frac{6y^5dy}{y^3+y^2} = 6\int \frac{y^3dy}{y+1} \\
&= 6\int \frac{(y^3+1)-1dy}{y+1} = 6\int \left(y^2-y+1-\frac{1}{1+y}\right)dy \\
&= 2y^3-3y^2+6y-6\log(1+y)+C \\
&= 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log(1+\sqrt[6]{x})+C.
\end{aligned}
$$

Example 5.5.24. To compute $\displaystyle\int \frac{dx}{\sqrt{x+1}+\sqrt{x}+1}$, we introduce

$$
y = \sqrt{x+1}+\sqrt{x}.
$$

Then

$$
\frac{1}{y} = \sqrt{x+1}-\sqrt{x}, \quad y+\frac{1}{y} = 2\sqrt{x+1}, \quad y-\frac{1}{y} = 2\sqrt{x},
$$

and

$$
x = \frac{1}{4}\left(y-\frac{1}{y}\right)^2, \quad dx = \frac{1}{2}\left(y-\frac{1}{y^3}\right)dy.
$$

Therefore

$$
\begin{aligned}
\int \frac{dx}{\sqrt{x+1}+\sqrt{x}+1} &= \int \frac{\frac{1}{2}\left(y-\frac{1}{y^3}\right)dy}{y+1} = \frac{1}{2}\int \frac{(y-1)(y+1)(y^2+1)}{(y+1)y^3}dy \\
&= \frac{1}{2}\int \left(1-\frac{1}{y}+\frac{1}{y^2}-\frac{1}{y^3}\right)dy \\
&= \frac{1}{2}\left(y-\log|y|-\frac{1}{y}+\frac{1}{2y^2}\right)+C \\
&= -\frac{1}{2}\log(\sqrt{x+1}+\sqrt{x})+\sqrt{x}+\frac{1}{4}(\sqrt{x+1}+\sqrt{x})^2+C.
\end{aligned}
$$

Example 5.5.25. The change of variable in Example 5.5.24 can be used for integrating other functions involving $\sqrt{x+a}$ and $\sqrt{x+b}$. For example, to compute $\displaystyle\int \sqrt{\frac{x+1}{x-1}}dx$, which makes sense for $x>1$ or $x\le -1$, we introduce $y = \sqrt{x+1}+\sqrt{x-1}$ for $x>1$. Then

$$
\frac{2}{y} = \sqrt{x+1}-\sqrt{x-1}, \quad y+\frac{2}{y} = 2\sqrt{x+1}, \quad y-\frac{2}{y} = 2\sqrt{x-1},
$$

and

$$
x = \frac{1}{4}\left(y-\frac{2}{y}\right)^2+1, \quad dx = \frac{1}{2}\left(1-\frac{4}{y^4}\right)ydy.
$$

Therefore by $x > 1$, we have

$$\begin{aligned}\int \sqrt{\frac{x+1}{x-1}}dx &= \int \frac{y+\frac{2}{y}}{y-\frac{2}{y}}\frac{1}{2}\left(1-\frac{4}{y^4}\right)ydy = \frac{1}{2}\int\left(y+\frac{4}{y}+\frac{4}{y^3}\right)dy \\ &= \frac{1}{4}y^2+2\log|y|-\frac{1}{y^2}+C = 2\log|y|+\frac{1}{4}\left(y+\frac{2}{y}\right)\left(y-\frac{2}{y}\right)+C \\ &= \frac{1}{4}\log(\sqrt{x+1}+\sqrt{x-1})+\sqrt{x^2-1}+C.\end{aligned}$$

For $x \le -1$, we may introduce $y = \sqrt{-x+1}-\sqrt{-x-1}$. Then we have

$$\begin{aligned}\int \sqrt{\frac{x+1}{x-1}}dx &= \frac{1}{2}\int\left(y+\frac{4}{y}+\frac{4}{y^3}\right)dy = 2\log|y|+\frac{1}{4}\left(y+\frac{2}{y}\right)\left(y-\frac{2}{y}\right)+C \\ &= \frac{1}{4}\log(\sqrt{-x-1}-\sqrt{-x+1})-\sqrt{x^2-1}+C.\end{aligned}$$

Example 5.5.26. The integral $\int \sqrt{\frac{x}{1-x}}dx$ is comparable to the integral in Example 5.5.25. Yet the similar change of variable does not work, due to the requirement $0 \le x < 1$. So we introduce

$$y = \sqrt{\frac{x}{1-x}}, \quad x = \frac{y^2}{1+y^2}, \quad dx = \frac{2y}{(1+y^2)^2}dy,$$

and get

$$\begin{aligned}\int \sqrt{\frac{x}{1-x}}dx &= \int y\frac{2y}{(1+y^2)^2}dy = 2\int\left(\frac{1}{1+y^2}-\frac{1}{(1+y^2)^2}\right)dx \\ &= -\frac{y}{1+y^2}+\arctan y+C = -\sqrt{x(1-x)}+\arctan\sqrt{\frac{x}{1-x}}+C.\end{aligned}$$

The last computation in Example 5.5.15 is used here.

Note that the idea here can also be applied to the integral in Example 5.5.25, by introducing $y = \sqrt{\frac{x+1}{x-1}}$. The advantage of the approach is that we do not need to distinguish $x > 1$ and $x \le 1$.

Exercise 5.5.39. Compute the integrals in Example 5.5.24 and 5.5.25 by using change of variable similar to Example 5.5.26.

Exercise 5.5.40. Compute the integral.

1. $\int (1+\sqrt{x})^p dx$.
2. $\int (1+\sqrt[3]{x})^p dx$.
3. $\int (\sqrt{x}+\sqrt{x+1})^p dx$.
4. $\int \dfrac{1+\sqrt{x}}{1+\sqrt[3]{x}} dx$.
5. $\int \dfrac{\sqrt{1+x^2}}{x} dx$.
6. $\int \dfrac{dx}{x\sqrt{1+x^2}}$.

Example 5.5.27. We can use $(\cos x)' = -\sin x$, $(\sin x)' = \cos x$ and $\cos^2 x + \sin^2 x = 1$ to calculate the antiderivative of $\cos^m x \sin^n x$, in which either m or n is odd.

$$\begin{aligned}
\int \sin^3 x dx &= -\int \sin^2 x d\cos x = \int (\cos^2 x - 1) d\cos x \\
&= \frac{1}{3}\cos^3 x - \cos x + C, \\
\int \cos^4 x \sin^5 x dx &= -\int \cos^4 x \sin^4 x d\cos x = -\int \cos^4 x (1-\cos^2 x)^2 d\cos x \\
&= -\int (\cos^4 x - 2\cos^6 x + \cos^8 x) d\cos x \\
&= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C, \\
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx = -\int \frac{d\cos x}{\cos x} = -\log|\cos x| + C, \\
\int \sec x dx &= \int \frac{dx}{\cos x} = \int \frac{d\sin x}{\cos^2 x} = \int \frac{d\sin x}{1-\sin^2 x} \\
&= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} + C = \frac{1}{2}\log\frac{(1+\sin x)^2}{1-\sin^2 x} + C = \log\frac{1+\sin x}{|\cos x|} + C \\
&= \log|\sec x + \tan x| + C.
\end{aligned}$$

Example 5.5.28. Similar to Example 5.5.27, we can also use $(\tan x)' = \sec^2 x$, $(\sec x)' = \sec x \tan x$ and $\sec^2 x = 1 + \tan^2 x$ to calculate the antiderivative of $\sec^m x \tan^n x$.

$$\begin{aligned}
\int \sec x \tan^3 x dx &= \int \tan^2 x d\sec x = \int (\sec^2 x - 1) d\sec x = \frac{1}{3}\sec^3 x - \sec x + C, \\
\int \sec^4 x dx &= \int (\tan^2 x + 1) d\tan x = \frac{1}{3}\tan^3 x + \tan x + C, \\
\int \tan^4 x dx &= \int (\sec^2 x - 1)^2 dx = \int (\sec^4 x - 2\sec^2 x + 1) dx \\
&= \left(\frac{1}{3}\tan^3 x + \tan x\right) - 2\tan x + x + C \\
&= \frac{1}{3}\tan^3 x - \tan x + x + C,
\end{aligned}$$

The following is computed in Example 5.5.14 by more complicated method

$$\int \frac{\sin^2 x}{\cos^4 x} dx = \int \sec^2 x \tan^2 x dx = \int \tan^2 x d\tan x = \frac{1}{3}\tan^3 x + C.$$

Example 5.5.29. The method of Example 5.5.28 cannot be directly applied to the antiderivatives of $\sec^n x$ and $\tan^n x$ for odd n. Instead, the idea of Example 5.5.27 can be used.

Using the integration by parts, we have

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x d\tan x = \sec x \tan x - \int \tan x d\sec x \\ &= \sec x \tan x - \int \tan^2 x \sec x dx = \sec x \tan x - \int (\sec^2 x - 1)\sec x dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx.\end{aligned}$$

Then with the help of Example 5.5.27, we get

$$\begin{aligned}\int \sec^3 x dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\log|\sec x + \tan x| + C.\end{aligned}$$

In fact, the integral $\int \sec^3 x dx$ is $I_{-3,0}$ in Example 5.5.14, and the expression above in terms of $\int \sec x dx$ is the expression of $I_{-3,0}$ in terms of $I_{-1,0}$.

Example 5.5.27 also gives

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x(\sec^2 x - 1)dx = \int \tan x d\tan x - \int \tan x dx \\ &= \frac{1}{2}\tan^2 x + \log|\cos x| + C.\end{aligned}$$

Exercise 5.5.41. Compute the integral.

1. $\int \cos^3 x \sin^2 x dx$.
2. $\int \cos^3 x \sin^5 x dx$.
3. $\int \dfrac{dx}{\cos^6 x}$.
4. $\int \dfrac{dx}{\sin^2 x \cos^2 x}$.
5. $\int \dfrac{dx}{\sin x \cos x}$.
6. $\int \cos^{-3} x \sin^5 x dx$.

Exercise 5.5.42. Compute the integral.

1. $\int \csc x dx$.
2. $\int \tan^3 x dx$.
3. $\int \tan^6 x \sec^4 x dx$.
4. $\int \tan^n x \sec^4 x dx$.
5. $\int \cot^6 x \csc^4 x dx$.
6. $\int \tan^2 x \sec x dx$.
7. $\int \tan^3 x \sec x dx$.
8. $\int \tan^5 x \sec^7 x dx$.
9. $\int \tan^3 x \cos^2 x dx$.

10. $\displaystyle\int \cot^5 x \sin^4 x dx.$ 11. $\displaystyle\int \csc^4 x \cot^6 x dx.$ 12. $\displaystyle\int x \tan x \sec x dx.$

Exercise 5.5.43. Compute the integral.

1. $\displaystyle\int x^3 e^{-x^2} \cos x^2 dx.$
2. $\displaystyle\int x \sin(\log x) dx.$
3. $\displaystyle\int (\sin x)^p \cos^3 x dx.$
4. $\displaystyle\int \frac{\cos 2x dx}{\sin^2 x \cos^2 x}.$
5. $\displaystyle\int \frac{a\sin x + b\cos x}{\sin 2x} dx.$
6. $\displaystyle\int \frac{dx}{a\sin x + b\cos x}.$
7. $\displaystyle\int \frac{1+\sin x}{1+\cos x} dx.$
8. $\displaystyle\int \frac{dx}{1+\cos x}.$
9. $\displaystyle\int \frac{dx}{a+\tan x}.$
10. $\displaystyle\int \frac{A\sin x + B\cos x}{a\sin x + b\cos x} dx.$
11. $\displaystyle\int \frac{dx}{a\sin^2 x + b\cos^2 x}.$
12. $\displaystyle\int \frac{A\sin x + B\cos x}{a\sin^2 x + b\cos^2 x} dx.$
13. $\displaystyle\int \frac{dx}{\sqrt{2}+\sin x + \cos x}.$
14. $\displaystyle\int \frac{\sin x dx}{\sqrt{2}+\sin x + \cos x}.$
15. $\displaystyle\int \frac{x dx}{\cos^2 x}.$

Exercise 5.5.44. Show that there are constants A_n, B_n, C_n, such that

$$\int \frac{dx}{(a\sin x + b\cos x)^n} = \frac{A_n \sin x + B_n \cos x}{(a\sin x + b\cos x)^{n-1}} + C_n \int \frac{dx}{(a\sin x + b\cos x)^{n-2}}.$$

Exercise 5.5.45. For $|a| \neq |b|$, show that there are constants A_n, B_n, C_n, such that

$$\int \frac{dx}{(a+b\cos x)^n} = \frac{A\sin x}{(a+b\cos x)^{n-1}} + B\int \frac{dx}{(a+b\cos x)^{n-1}} + C\int \frac{dx}{(a+b\cos x)^{n-2}}.$$

Exercise 5.5.46. How to calculate $\displaystyle\int \frac{(A\sin x + B\cos x + C)dx}{(a\sin x + b\cos x + c)^n}$?

Exercise 5.5.47. Compute $\displaystyle\int \frac{dx}{\cos(x+a)\cos(x+b)}$ by using

$$\tan(x+a) - \tan(x+b) = \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}.$$

Use the similar idea to compute the following integral.

1. $\displaystyle\int \frac{dx}{\sin(x+a)\cos(x+b)}.$
2. $\displaystyle\int \tan(x+a)\tan(x+b) dx.$
3. $\displaystyle\int \frac{dx}{\sin x - \sin a}.$
4. $\displaystyle\int \frac{dx}{\cos x + \cos a}.$

Example 5.5.30. To integrate a function of $\sqrt{a^2-x^2}$, with $a>0$, we may introduce $x = a\sin y$, $dx = a\cos y dy$. Note that the function makes sense only for $|x| \le a$. Correspondingly, we take $y = \arcsin\frac{x}{a} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This implies $\cos y \ge 0$, and $\sqrt{a^2-x^2} = a\cos y$.

The following example is also computed in Example 5.5.15.

$$\begin{aligned}\int \sqrt{a^2-x^2}dx &= a^2\int \cos^2 y dy = \frac{a^2}{2}\int (1+\cos 2y)dy \\ &= a^2\left(\frac{1}{2}y+\frac{1}{4}\sin 2y\right)+C = \frac{a^2}{2}(y+\sin y\cos y)+C \\ &= \frac{a}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C.\end{aligned}$$

We may also use $x = a\cos y$ instead of $x = a\sin y$.

$$\begin{aligned}\int \frac{dx}{a+\sqrt{a^2-x^2}} &= \int \frac{a\sin y dy}{a+a\sin y} = \int \frac{\sin y(1-\sin y)dy}{(1+\sin y)(1-\sin y)} \\ &= \int \left(\frac{\sin y}{\cos^2 y}-\frac{\sin^2 y}{\cos^2 y}\right)dy = \int (\sec y\tan y-\sec^2 y+1)dy \\ &= \sec y-\tan y+y+C = \frac{a}{x}-\frac{a}{x}\sqrt{1-\frac{x^2}{a^2}}+\arccos\frac{x}{a}+C.\end{aligned}$$

The following is an example of definite integral.

$$\int_0^1 (1-x^2)^p dx = -\int_{\frac{\pi}{2}}^0 (1-\cos^2 y)^p \sin y dy = \int_0^{\frac{\pi}{2}} \sin^{2p+1} y dy.$$

By Example 5.5.15, we know the specific value when $2p+1$ is a natural number.

$$\int_0^1 (1-x^2)^n dx = \frac{(2n)!!}{(2n+1)!!}, \quad \int_0^1 (1-x^2)^{n-\frac{1}{2}} dx = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}.$$

Example 5.5.31. To integrate a function of $\sqrt{x^2+a^2}$, with $a > 0$, we may introduce $x = a\tan y$, $dx = a\sec^2 y dy$. We have $y = \arctan\frac{x}{a} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sqrt{x^2+a^2} = a\sec y$.

With the help of Example 5.5.27, we have

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+a^2}} &= \int \frac{a\sec^2 y dy}{a\sec y} = \int \sec y dy \\ &= \log|\sec y+\tan y|+C = \log(\sqrt{x^2+a^2}+x)+C.\end{aligned}$$

With the help of Example 5.5.29, we have

$$\begin{aligned}\int \sqrt{x^2+a^2}x dx &= \int a^2\sec^3 y dy = \frac{a^2}{2}\sec y\tan y+\frac{a^2}{2}\log|\sec y+\tan y|+C \\ &= \frac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}\log(\sqrt{x^2+a^2}+x)+C.\end{aligned}$$

One may verify that the two integrals satisfy the recursive relation in Example 5.5.15.

Example 5.5.32. To integrate a function of $\sqrt{x^2-a^2}$, with $a>0$, we may introduce $x=a\sec y$, $dx=a\sec y\tan y dy$. We have $y=\text{arcsec}\frac{x}{a}=\arccos\frac{a}{x}\in[0,\pi]$ and $\sqrt{x^2-a^2}=\pm a\tan y$, where the sign depends on whether $x\geq a$ or $x\leq -a$.

With the help of Example 5.5.28, we have

$$\begin{aligned}\int\frac{dx}{\sqrt{x^2-a^2}}&=\int\frac{a\sec y\tan y dy}{\pm a\tan y}=\int\sec y dy\\&=\pm\log|\sec y+\tan y|+C=\pm\log\left|x\pm\sqrt{x^2-a^2}\right|+C\\&=\log\left|x+\sqrt{x^2-a^2}\right|+C,\\\int\sqrt{x^2-a^2}dx&=\int(\pm a\tan y)a\sec y\tan y dy=\pm a^2\int(\sec^3 y-\sec y)dy\\&=\pm\frac{a^2}{2}\sec y\tan y\mp\frac{a^2}{2}\log|\sec y+\tan y|+C\\&=\frac{1}{2}x\sqrt{x^2-a^2}-\log\left|x+\sqrt{x^2-a^2}\right|+C.\end{aligned}$$

Combing with Example 5.5.31, for positive as well as negative a, we have

$$\begin{aligned}\int\frac{dx}{\sqrt{x^2+a}}&=\log\left|x+\sqrt{x^2+a}\right|+C,\\\int\sqrt{x^2+a}dx&=\frac{1}{2}x\sqrt{x^2+b}+\frac{a}{2}\log\left|x+\sqrt{x^2+a}\right|+C.\end{aligned}$$

Example 5.5.33. By completing the square, a quadratic function ax^2+bx+c can be changed to $a(y^2+d)$, where $y=x+\frac{b}{2a}$ and $d=\frac{4ac-b^2}{4a^2}$. For example, if $b^2<4ac$, then

$$\begin{aligned}\int\frac{dx}{ax^2+bx+c}&=\int\frac{dy}{a(y^2+(\sqrt{d})^2)}=\frac{1}{a\sqrt{d}}\arctan\frac{y}{\sqrt{d}}+C\\&=\frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}+C.\end{aligned}$$

Using the recursive relation in Example 5.5.15, we further get

$$\int\frac{dx}{(ax^2+bx+c)^2}=\frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)}-\frac{4a}{(4ac-b^2)^{\frac{3}{2}}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}+C.$$

If $b^2\geq 4ac$, then by the similar idea, we may get

$$\begin{aligned}\int\frac{dx}{ax^2+bx+c}&=\int\frac{dy}{a(y^2-(\sqrt{-d})^2)}=\frac{1}{2a\sqrt{-d}}\log\left|\frac{y-\sqrt{-d}}{y+\sqrt{-d}}\right|+C\\&=\frac{1}{\sqrt{b^2-4ac}}\arctan\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+C.\end{aligned}$$

In fact, the quadratic function has two real roots, and it is more direct to calculate the integral by using

$$ax^2 + bx + c = a(x - x_1)(x - x_2), \quad x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and the idea of Examples 5.5.2 and 5.5.3.

Example 5.5.34. We further use the idea of Example 5.5.33 to convert the antiderivatives of functions of $\sqrt{ax^2 + bx + c}$ to the computations in Examples 5.5.30, 5.5.31 and 5.5.32. For example, by

$$x(1 - x) = \frac{1}{2^2} - \left(x^2 - 2\frac{1}{2}x + \frac{1}{2^2}\right) = \frac{1}{2^2} - \left(x - \frac{1}{2}\right)^2 = \frac{1}{2^2}(1 - (2x - 1)^2),$$

we let $y = 2x - 1$ and get

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dy}{\sqrt{1 - y^2}} = \arcsin y + C = \arcsin(2x - 1) + C.$$

Moreover, for $0 \le x < 1$, we get

$$\begin{aligned}\int \sqrt{\frac{x}{1-x}}dx &= \int \frac{xdx}{\sqrt{x(1-x)}} = \int \frac{\frac{y+1}{2}dy}{\sqrt{1-y^2}} = -\int \frac{d(1-y^2)}{4\sqrt{1-y^2}} + \int \frac{dy}{2\sqrt{1-y^2}} \\ &= \frac{1}{4}\sqrt{1-y^2} + \frac{1}{2}\arcsin y + C = \frac{1}{2}\sqrt{x(1-x)} + \frac{1}{2}\arcsin(2x-1) + C.\end{aligned}$$

The reader is left to verify that the result is the same as the one in Example 5.5.26.

Exercise 5.5.48. Compute the integral.

1. $\displaystyle\int \frac{xdx}{\sqrt{1-x^2}}$.
2. $\displaystyle\int \frac{(ax^2+bx+c)dx}{\sqrt{1-x^2}}$.
3. $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}}$.
4. $\displaystyle\int \frac{x^2dx}{(a^2-x^2)^{\frac{3}{2}}}$.
5. $\displaystyle\int \frac{dx}{x(a^2-x^2)^{\frac{3}{2}}}$.
6. $\displaystyle\int (x^2+a^2)^{\frac{3}{2}}dx$.
7. $\displaystyle\int \frac{dx}{x(x^2+a^2)^{\frac{3}{2}}}$.
8. $\displaystyle\int \frac{x^3dx}{(x^2+a^2)^{\frac{3}{2}}}$.
9. $\displaystyle\int (x(x+1))^{\frac{3}{2}}dx$.
10. $\displaystyle\int x^2\sqrt{1-x^2}dx$.
11. $\displaystyle\int \frac{dx}{1-\sqrt{1-x^2}}$.
12. $\displaystyle\int \frac{dx}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.
13. $\displaystyle\int \frac{xdx}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.
14. $\displaystyle\int \frac{dx}{a+\sqrt{x^2+a^2}}$.
15. $\displaystyle\int \frac{xdx}{a+\sqrt{x^2+a^2}}$.

Exercise 5.5.49. Compute the integral.

1. $\displaystyle\int \sin x \log \tan x dx$.
2. $\displaystyle\int \arctan \sqrt{x} dx$.
3. $\displaystyle\int \frac{\log(\sin x)}{\sin^2 x} dx$.
4. $\displaystyle\int e^{\sin x}\left(\cos^2 x + \frac{1}{\cos^2 x}\right) dx$.

Exercise 5.5.50. Derive the formula

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}} \log\left(\dfrac{2ax+b}{\sqrt{a}} + \sqrt{ax^2+bx+c}\right) + C, & \text{if } a > 0, \\ \dfrac{1}{\sqrt{-a}} \arcsin \dfrac{-2ax-b}{\sqrt{b^2-4ac}} + C, & \text{if } a < 0. \end{cases}$$

Exercise 5.5.51. Use the change of variable $y = x \pm \dfrac{1}{x}$ to compute the integral.

1. $\displaystyle\int \frac{x^2+1}{x^4+1} dx$.
2. $\displaystyle\int \frac{x^2-1}{x^4+1} dx$.
3. $\displaystyle\int \frac{dx}{x^4+1} dx$.
4. $\displaystyle\int_{\frac{1}{2}}^{2} \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx$.

5.6 Integration of Rational Function

A rational function is the quotient of two polynomials. Examples 5.5.2, 5.5.3, 5.5.18, 5.5.33 are some typical examples of integrating rational functions. In this section, we systematically study how to integrate rational functions and how to convert some integrations into the integration of rational functions.

5.6.1 Rational Function

Example 5.6.1. The idea in Example 5.5.2 can be extended to the integral of rational functions whose denominator is a product of linear functions. For example, to integrate $\dfrac{x^2-2x+3}{x(x+1)(x+2)}$, we postulate

$$\frac{x^2-2x+3}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}.$$

The equality is the same as

$$x^2 - 2x + 3 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1).$$

Taking $x = 0, -1, -2$, we get

$$3 = 2A, \quad 6 = -B, \quad 11 = -2C.$$

Therefore

$$\begin{aligned}\int \frac{x^2-2x+3}{x(x+1)(x+2)}dx &= \int \left(\frac{3}{2x} - \frac{6}{x+1} - \frac{11}{2(x+2)}\right) dx \\ &= \frac{3}{2}\log|x| - 6\log|x+1| - \frac{11}{2}\log|x+2| + C.\end{aligned}$$

Example 5.6.2. If some real root of the denominator has multiplicity, then we need more sophisticated postulation. For example, to integrate $\frac{x^2-2x+3}{x(x+1)^3}$, we postulate

$$\frac{x^2-2x+3}{x(x+1)^3} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}.$$

This is the same as

$$x^2-2x+3 = A(x+1)^3 + Bx(x+1)^2 + Cx(x+1) + Dx.$$

Taking various values, we get

$$\begin{aligned} &x=0\colon 3=A, && x=-1\colon 6=-D, \\ &\text{(coefficient of) } x^3\colon 0=A+B, && \left.\frac{d}{dx}\right|_{x=-1}\colon\ -4=D-C. \end{aligned}$$

Therefore $A=3$, $B=-3$, $C=-2$, $D=-6$, and (C below means the general constant, and is different from the coefficient $C=-2$ above)

$$\begin{aligned}\int \frac{x^2-2x+3}{x(x+1)^3}dx &= \int \left(\frac{3}{x} - \frac{3}{x+1} - \frac{2}{(x+1)^2} - \frac{6}{(x+1)^3}\right) dx \\ &= 3\log\left|\frac{x}{x+1}\right| + \frac{2}{x+1} + \frac{3}{(x+1)^2} + C \\ &= 3\log\left|\frac{x}{x+1}\right| + \frac{2x+5}{(x+1)^2} + C.\end{aligned}$$

Example 5.6.3. In Examples 5.5.2, 5.6.1, 5.6.2, the numerator has lower degree than the denominator. In general, we need to divide polynomials for this to happen. For example, to integrate $\frac{x^5}{(x+1)^2(x-1)}$, we first divide x^5 by $(x+1)^2(x-1) = x^3+x^2-x-1$.

$$\begin{array}{r|rrrrrr} & & & x^2 & & -x & +2 \\ \hline x^3+x^2-x-1 & x^5 & & & & & \\ & -x^5 & -x^4 & +x^3 & +x^2 & & \\ \hline & & -x^4 & +x^3 & +x^2 & & \\ & & x^4 & +x^3 & -x^2 & -x & \\ \hline & & & 2x^3 & & -x & \\ & & & -2x^3 & -2x^2 & +2x & +2 \\ \hline & & & & -2x^2 & +x & +2 \end{array}$$

Then

$$\begin{aligned}\frac{x^5}{(x+1)^2(x-1)} &= x^2 - x + 2 + \frac{-2x^2+x+2}{(x+1)^2(x-1)} \\ &= x^2 - x + 2 - \frac{9}{4(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)},\end{aligned}$$

and

$$\int \frac{x^5}{(x+1)^2(x-1)}dx = \frac{x^3}{3} - \frac{x^2}{2} + 2x - \frac{9}{4}\log|x+1| - \frac{1}{2(x+1)} + \frac{1}{4}\log|x-1| + C.$$

Exercise 5.6.1. Compute the integral.

1. $\int \frac{x^2dx}{1+x}$.
2. $\int \frac{dx}{x^2+x-2}$.
3. $\int \frac{xdx}{x^2+x-2}$.
4. $\int \frac{x^5dx}{x^2+x-2}$.
5. $\int \frac{(2-x)^2dx}{2-x^2}$.
6. $\int \frac{x^4dx}{1-x^2}$.
7. $\int \frac{dx}{x(1+x)(2+x)}$.
8. $\int \frac{dx}{x^2(1+x)}$.
9. $\int \frac{dx}{(x+a)^2(x+b)^2}$.

The examples above illustrate how to integrate rational functions of the form $\frac{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0}{(x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_k)^{n_k}}$. This means exactly that all the roots of the denominator are real. In general, however, a real polynomial may have complex roots, and a conjugate pair of complex roots corresponds to a real quadratic factor.

Example 5.6.4. To integrate $\frac{1}{x^3-1}$, we note that $x^3 = (x-1)(x^2+x+1)$, where x^2+x+1 has a conjugate pair of complex roots. We postulate

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

This means $1 = A(x^2+x+1) + (Bx+C)(x-1)$ and gives

$$x=0\colon 1 = A - C; \quad x=1\colon 1 = 3A; \quad x^2\colon 0 = A+B.$$

Therefore $A = \frac{1}{3}$, $B = -\frac{1}{3}$, $C = -\frac{2}{3}$, and

$$\begin{aligned}
\int \frac{dx}{x^3-1} &= \frac{1}{3}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{-x-2}{x^2+x+1}dx \\
&= \frac{1}{3}\log|x-1| - \frac{1}{6}\int \frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int \frac{dx}{x^2+x+1} \\
&= \frac{1}{3}\log|x-1| - \frac{1}{6}\log(x^2+x+1) - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{6}\log\frac{(x-1)^2}{x^2+x+1} - \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C \\
&= \frac{1}{6}\log\frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.
\end{aligned}$$

Example 5.6.5. To integrate $\frac{x^2}{(x^4-1)^2}$, we postulate

$$\begin{aligned}
\frac{x^2}{(x^4-1)^2} &= \frac{x^2}{(x-1)^2(x+1)^2(x^2+1)^2} \\
&= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} + \frac{C_1x+D_1}{x^2+1} + \frac{C_2x+D_2}{(x^2+1)^2}.
\end{aligned}$$

Since changing x to $-x$ does not change the left side, we see that $A_1 = -B_1$, $A_2 = B_2$, $C_1 = C_2 = 0$, and the equality becomes

$$\frac{x^2}{(x^4-1)^2} = \frac{2A_1}{x^2-1} + 2A_2\frac{x^2+1}{(x^2-1)^2} + \frac{D_1}{x^2+1} + \frac{D_2}{(x^2+1)^2}.$$

It is then easy to find $A_1 = -\frac{1}{16}$, $A_2 = \frac{1}{16}$, $D_1 = 0$, $D_2 = -\frac{1}{4}$. Therefore with the help of Example 5.5.15,

$$\begin{aligned}
\int \frac{x^2}{(x^4-1)^2}dx &= \int \left(-\frac{1}{16(x-1)} + \frac{1}{16(x-1)^2}\right. \\
&\qquad \left. + \frac{1}{16(x+1)} + \frac{1}{16(x+1)^2} - \frac{1}{4(x^2+1)^2}\right)dx \\
&= \frac{1}{16}\log\left|\frac{x+1}{x-1}\right| - \frac{1}{16(x-1)} \\
&\qquad - \frac{1}{16(x+1)} - \frac{x}{8(x^2+1)} - \frac{1}{8}\arctan x + C \\
&= -\frac{x^3}{4(x^4-1)} + \frac{1}{16}\log\left|\frac{x+1}{x-1}\right| - \frac{1}{8}\arctan x + C.
\end{aligned}$$

Example 5.6.6. To integrate the rational function $\frac{(x^2+1)^4}{(x^3-1)^2}$, we first notice that the degree of the numerator is higher. Therefore we divide $(x^2+1)^4$ by $(x^3-1)^2$.

$$
\begin{array}{rl}
 & x^2+4 \\
x^6-2x^3+1 \big) & x^8+4x^6+6x^4+4x^2+1 \\
 & -x^8+2x^5-x^2 \\
\hline
 & 4x^6+2x^5+6x^4+3x^2+1 \\
 & -4x^6+8x^3-4 \\
\hline
 & 2x^5+6x^4+8x^3+3x^2-3
\end{array}
$$

This means that

$$\frac{(x^2+1)^4}{(x^3-1)^2} = x^2+4+\frac{2x^5+6x^4+8x^3+3x^2-3}{(x^3-1)^2}.$$

Since $x^3-1=(x-1)(x^2+x+1)$, we postulate

$$\frac{2x^5+6x^4+8x^3+3x^2-3}{(x^3-1)^2} = \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{B_1x+C_1}{x^2+x+1}+\frac{B_2x+C_2}{(x^2+x+1)^2}.$$

This can be interpreted as an expression for $2x^5+6x^4+8x^3+3x^2-3$, which gives

$$
\begin{aligned}
x=0&\colon\ -3=-A_1+A_2+C_1+C_2,\\
x=1&\colon\ 16=9A_2,\\
x=-1&\colon\ -4=-2A_1+A_2+4(-B_1+C_1)+4(-B_2+C_2),\\
x^5&\colon\ 2=A_1+B_1,\\
x^4&\colon\ 6=A_1+A_2-B_1+C_1,\\
\left.\frac{d}{dx}\right|_{x=1}&\colon\ 32=9A_1+18A_2.
\end{aligned}
$$

Solving the system, we get

$$A_2=\frac{32}{9},\quad A_1=\frac{16}{9},\quad B_1=-\frac{14}{9},\quad C_1=-\frac{8}{9},\quad B_2=0,\quad C_2=-\frac{1}{3}.$$

Then

$$\begin{aligned}\int \frac{(x^2+1)^4}{(x^3-1)^2}dx &= \frac{1}{3}x^3+4x+\int\frac{32}{9(x-1)}dx+\int\frac{16}{9(x-1)^2}dx \\ &\quad -\int\frac{14x+8}{9(x^2+x+1)}dx-\int\frac{1}{3(x^2+x+1)^2}dx \\ &= \frac{1}{3}x^3+4x+\frac{32}{9}\ln|x-1|-\frac{16}{9(x-1)} \\ &\quad -\int\frac{7d(x^2+x+1)}{9(x^2+x+1)}-\int\frac{dx}{9(x^2+x+1)}-\int\frac{dx}{3(x^2+x+1)^2} \\ &= \frac{1}{3}x^3+4x+\frac{32}{9}\log|x-1|-\frac{16}{9(x-1)}-\frac{7}{9}\log(x^2+x+1) \\ &\quad -\int\frac{dx}{9(x^2+x+1)}-\int\frac{dx}{3(x^2+x+1)^2}.\end{aligned}$$

By

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2,\quad \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{2x+1}{\sqrt{3}},$$

and Example 5.5.15, we have

$$\int\frac{dx}{x^2+x+1}=\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C,$$

and

$$\begin{aligned}\int\frac{dx}{(x^2+x+1)^2} &= \frac{x+\dfrac{1}{2}}{2\left(\frac{\sqrt{3}}{2}\right)^2(x^2+x+1)}+\frac{1}{2\left(\frac{\sqrt{3}}{2}\right)^3}\arctan\frac{2x+1}{\sqrt{3}}+C \\ &= \frac{2x+1}{3(x^2+x+1)}+\frac{4}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.\end{aligned}$$

Combining everything together, we get

$$\begin{aligned}\int\frac{(x^2+1)^4}{(x^3-1)^2}dx &= \frac{1}{3}x^3+4x-\frac{6x^2+5x+5}{3(x^3-1)}+\frac{13}{3}\log|x-1|-\frac{7}{9}\log|x^3-1| \\ &\quad -\frac{2}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.\end{aligned}$$

In general, the numerator of a rational function is a product of $(x+a)^m$ and $(x^2+bx+c)^n$, where $b^2<4c$ so that the factor x^2+bx+c has no real root. After dividing the numerator by the denominator, we can make sure the numerator has lower degree than the denominator. When the numerator has lower degree, the rational function can then be expressed as a sum: For each factor $(x+a)^m$ of the denominator, we have terms

$$\frac{A_1}{x+a}+\frac{A_2}{(x+a)^2}+\cdots+\frac{A_m}{(x+a)^m},$$

and for each factor $(x^2+bx+c)^n$ of the denominator, we have terms

$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

The computation is then reduced to the integration of the terms.

We have

$$\int\frac{A}{(x+a)^m}dx=\begin{cases}-\dfrac{1}{(m-1)(x+a)^{m-1}}+C, & \text{if } m>1,\\ \log|x+a|+C, & \text{if } m=1.\end{cases}$$

The quadratic term can be split into two parts

$$\int\frac{Bx+C}{(x^2+bx+c)^n}dx=\frac{B}{2}\int\frac{d(x^2+bx+c)}{(x^2+bx+c)^n}+\left(C-\frac{B}{2}\right)\int\frac{dx}{(x^2+bx+c)^n}.$$

The first part is easy to compute

$$\int\frac{d(x^2+bx+c)}{(x^2+bx+c)^n}=\begin{cases}-\dfrac{1}{(n-1)(x^2+bx+c)^{n-1}}+C, & \text{if } n>1,\\ \log|x^2+bx+c|+C, & \text{if } n=1.\end{cases}$$

The second part can be computed by the recursive relation in Example 5.5.15

$$\begin{aligned}&\int\frac{dx}{(x^2+bx+c)^n}\\ &=\frac{1}{(4c-b^2)(n-1)}\left(\frac{2x+b}{(x^2+bx+c)^{n-1}}+2(2n-3)\int\frac{dx}{(x^2+bx+c)^{n-1}}\right).\end{aligned}$$

For $n=1,2$, we have

$$\begin{aligned}\int\frac{dx}{x^2+bx+c}&=\frac{2}{\sqrt{4c-b^2}}\arctan\frac{2x+b}{\sqrt{4c-b^2}}+C,\\ \int\frac{dx}{(x^2+bx+c)^2}&=\frac{2x+b}{(4c-b^2)(x^2+bx+c)}+\frac{4}{(4c-b^2)^{\frac{3}{2}}}\arctan\frac{2x+b}{\sqrt{4c-b^2}}+C.\end{aligned}$$

Exercise 5.6.2. Compute the integral.

1. $\int \frac{(1+x)^2 dx}{1+x^2}$.
2. $\int \frac{(2x^2+3)dx}{x^3+x^2-2}$.
3. $\int \frac{dx}{x^3-1}$.
4. $\int \frac{(x+1)^3 dx}{x^3+1}$.
5. $\int \frac{dx}{(x+1)(x^2+1)}$.
6. $\int \frac{dx}{x(x+1)(x^2+x+1)}$.
7. $\int \frac{dx}{(x+1)(x^2+1)(x^3+1)}$.
8. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$.
9. $\int \frac{x^3 dx}{(x^2+1)^2}$.
10. $\int \frac{x^2 dx}{(x^2+4x+6)^2}$.
11. $\int \frac{dx}{x^4-1}$.
12. $\int \frac{x^2 dx}{x^4-1}$.
13. $\int \frac{dx}{(x^4-1)^2}$.
14. $\int \frac{dx}{x^4+4}$.
15. $\int \frac{dx}{x^4+x^2+1}$.
16. $\int \frac{dx}{x^6+1}$.
17. $\int \frac{x dx}{(x-1)^2(x^2+2x+2)}$.
18. $\int \frac{(x^4+4x^3+4x^2+4x+4)dx}{x(x+2)(x^2+2x+2)^2}$.

5.6.2 Rational Function of $\sqrt[n]{\dfrac{ax+b}{cx+d}}$

Using suitable changes of variables, some integrals can be changed to integrals of rational functions.

Example 5.6.7. To integrate $\sqrt{\frac{x-2}{x-1}}$, we introduce

$$y=\sqrt{\frac{x-2}{x-1}}, \quad x=\frac{y^2-2}{y^2-1}, \quad dx=\frac{2y}{(y^2-1)^2}dy.$$

Then for $x\ge 2$, we have

$$\begin{aligned}
\int \sqrt{\frac{x-2}{x-1}}dx &= \int y\frac{2y}{(y^2-1)^2}dy = \int \frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}+\frac{1}{(y-1)^2}+\frac{1}{(y+1)^2}\right) \\
&= \frac{1}{2}\log\left|\frac{y-1}{y+1}\right| - \frac{y}{y^2-1}+C \\
&= \frac{1}{2}\log\left|\frac{\sqrt{x-2}-\sqrt{x-1}}{\sqrt{x-2}+\sqrt{x-1}}\right| - \frac{\sqrt{\frac{x-2}{x-1}}}{\frac{x-2}{x-1}-1}+C \\
&= \log(\sqrt{x-1}-\sqrt{x-2})+\sqrt{(x-1)(x-2)}+C.
\end{aligned}$$

For $x \le 1$, the answer is $\log(\sqrt{2-x}-\sqrt{1-x})-\sqrt{(2-x)(1-x)}+C$.

The same substituting can be used to compute that, for $x>2$,

$$\begin{aligned}\int\frac{dx}{x\sqrt{x^2-3x+2}}&=\int\frac{dx}{x\sqrt{(x-1)(x-2)}}=\int\sqrt{\frac{x-2}{x-1}}\frac{dx}{x(x-2)}\\&=\int\sqrt{\frac{x-2}{x-1}}\frac{dx}{x(x-2)}=-\int\frac{dy}{y^2-2}\\&=\frac{1}{2\sqrt{2}}\log\left|\frac{y+\sqrt{2}}{y-\sqrt{2}}\right|+C=\frac{1}{2\sqrt{2}}\log\frac{\sqrt{2x-2}+\sqrt{x-2}}{\sqrt{2x-2}-\sqrt{x-2}}+C.\end{aligned}$$

For the case $x<1$, the answer is $-\dfrac{1}{2\sqrt{2}}\log\dfrac{\sqrt{2-2x}+\sqrt{2-x}}{\sqrt{2-2x}-\sqrt{2-x}}+C$. ??????

Example 5.6.8. For $y=\sqrt[3]{\dfrac{x}{x+1}}$, we have $\dfrac{1}{y^3}=1+\dfrac{1}{x}$ and

$$-\frac{3}{y^4}dy=-\frac{1}{x^2}dx=-\left(1-\frac{1}{y^3}\right)^2dx.$$

Therefore by Example 5.6.4, we have

$$\begin{aligned}\int\frac{dx}{\sqrt[3]{x^3+x^2}}&=\int\frac{1}{x}\sqrt[3]{\frac{x}{x+1}}dx=\int\left(1-\frac{1}{y^3}\right)y\frac{-\frac{3}{y^4}dy}{-\left(1-\frac{1}{y^3}\right)^2}=\int\frac{3dy}{y^3-1}\\&=\frac{1}{2}\log\frac{(y-1)^2}{y^2+y+1}-\sqrt{3}\arctan\frac{2y+1}{\sqrt{3}}+C\\&=\frac{1}{2}\log\frac{(y-1)^3}{y^3-1}-\sqrt{3}\arctan\frac{2y+1}{\sqrt{3}}+C\\&=\frac{3}{2}\log(\sqrt[3]{x+1}-\sqrt[3]{x})-\sqrt{3}\arctan\frac{1}{\sqrt{3}}\left(2\sqrt[3]{\frac{x}{x+1}}+1\right)+C.\end{aligned}$$

In general, a function involving $\sqrt[n]{\dfrac{ax+b}{cx+d}}$ can be integrated by introducing

$$y=\sqrt[n]{\frac{ax+b}{cx+d}},\quad x=\frac{dy^n-b}{-cy^n+a},\quad dx=n(ad-bc)\frac{y^{n-1}}{(cy^n-a)^2}dy.$$

Exercise 5.6.3. Compute the integral.

1. $\displaystyle\int\frac{dx}{1+\sqrt{x}}$.
2. $\displaystyle\int\frac{dx}{\sqrt{x}(1+x)}$.
3. $\displaystyle\int\frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx$.

4. $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}dx.$
5. $\displaystyle\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}}dx.$
6. $\displaystyle\int \frac{dx}{\sqrt[4]{x^3(a-x)}}.$
7. $\displaystyle\int \frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}dx.$
8. $\displaystyle\int \sqrt{\frac{1-x}{1+x}}dx.$
9. $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}dx.$
10. $\displaystyle\int \frac{1}{x^2}\sqrt[3]{\frac{1-x}{1+x}}dx.$
11. $\displaystyle\int \frac{dx}{2\sqrt{x}+\sqrt{x+1}+1}.$
12. $\displaystyle\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}+c}.$
13. $\displaystyle\int \frac{1+\sqrt{x+a}}{1+\sqrt{x+b}}dx.$
14. $\displaystyle\int \sqrt{\frac{x-a}{x-b}}dx.$
15. $\displaystyle\int \sqrt{\frac{x-a}{b-x}}dx.$
16. $\displaystyle\int \sqrt{(x-a)(x-b)}dx.$
17. $\displaystyle\int \sqrt{(x-a)(b-x)}dx.$
18. $\displaystyle\int x\sqrt{(x-a)(b-x)}dx.$
19. $\displaystyle\int x\sqrt{\frac{x-a}{x-b}}dx.$
20. $\displaystyle\int \frac{xdx}{\sqrt{(x-a)(x-b)}}.$

Exercise 5.6.4. Compute the integral.

1. $\displaystyle\int \sqrt{1+e^x}dx.$
2. $\displaystyle\int \frac{dx}{\sqrt{1+e^x}+\sqrt{1-e^x}}.$
3. $\displaystyle\int \frac{dx}{\sqrt{a^x+b}}.$

Exercise 5.6.5. Suppose R is a rational function. Suppose r, s are rational numbers such that $r+s$ is an integer. Find a suitable change of variable, such that $\int R(x,(ax+b)^r(cx+d)^s)dx$ is changed into the antiderivative of a rational function.

Exercise 5.6.6. Suppose r, s, t are rational numbers. For each of the following cases, find a suitable change of variable, such that $\int x^r(a+bx^s)^t dx$ is changed into the integral of a rational function.

1. t is an integer.
2. $\dfrac{r+1}{s}$ is an integer.
3. $\dfrac{r+1}{s}+t$ is an integer.

A theorem by Chebyshev[1] says that these are the only cases that the antiderivative can be changed to the integral of a rational function.

[1]Pafnuty Lvovich Chebyshev, born 1821 in Okatovo (Russia), died 1894 in St Petersburg (Russia). Chebyshev's work touches many fields of mathematics, including analysis, probability, number theory and mechanics. Chebyshev introduced his famous polynomials in 1854 and later generalized to the concept of orthogonal polynomials.

5.6.3 Rational Function of $\sin x$ and $\cos x$

A rational function of $\sin x$ and $\cos x$ can be integrated by introducing

$$y=\tan\frac{x}{2},\quad \sin x=\frac{2y}{1+y^2},\quad \cos x=\frac{1-y^2}{1+y^2},\quad dx=\frac{2}{1+y^2}dy.$$

Example 5.6.9. For $a\neq 0$, we have

$$\int\frac{dx}{a+\sin x}=\int\frac{2dy}{\left(a+\frac{2y}{1+y^2}\right)(1+y^2)}=\int\frac{2dy}{ay^2+2y+a}.$$

If $|a|>1$, then

$$\int\frac{dx}{a+\sin x}=\frac{1}{a\sqrt{1-\frac{1}{a^2}}}\arctan\frac{y+\frac{1}{a}}{\sqrt{1-\frac{1}{a^2}}}+C=\frac{2}{\mathrm{sign}(a)\sqrt{a^2-1}}\arctan\frac{a\tan\frac{x}{2}+1}{\sqrt{a^2-1}}+C.$$

If $|a|<1$, then

$$\begin{aligned}\int\frac{dx}{a+\sin x}&=\frac{1}{a\sqrt{\frac{1}{a^2}-1}}\log\left|\frac{y+\frac{1}{a}-\sqrt{\frac{1}{a^2}-1}}{y+\dfrac{1}{a}+\sqrt{\frac{1}{a^2}-1}}\right|+C\\&=\frac{1}{\mathrm{sign}(a)\sqrt{1-a^2}}\log\left|\frac{a\tan\frac{x}{2}+1-\sqrt{1-a^2}}{a\tan\frac{x}{2}+1+\sqrt{1-a^2}}\right|+C.\end{aligned}$$

If $|a|=1$, then

$$\int\frac{dx}{a+\sin x}=\frac{-2}{ay+1}+C=\frac{-2}{a\tan\frac{x}{2}+1}+C.$$

The example can be extended to $\displaystyle\int\frac{dx}{a+b\sin x+c\cos x}$. We have

$$b\sin x+c\cos x=\sqrt{b^2+c^2}\sin(x+\theta),$$

where θ is any fixed angle satisfying $\sin\theta=\dfrac{b}{\sqrt{b^2+c^2}}$ and $\cos\theta=\dfrac{c}{\sqrt{b^2+c^2}}$. Then

$$\int\frac{dx}{a+b\sin x+c\cos x}=\int\frac{dy}{a+\sqrt{b^2+c^2}\sin y},\quad y=x+\theta.$$

Example 5.6.10. Rational functions of $\sin x$ and $\cos x$ can be integrated by a simpler substitution if it has additional property. For example, to integrate the function $\dfrac{\sin x}{\sin x+\cos x}$, we introduce

$$y=\tan x,\quad x=\arctan y,\quad dx=\frac{dy}{y^2+1}.$$

Here we use the tangent of the full angle x, instead of half the angle in Example 5.6.9. Then

$$\int \frac{\sin x dx}{\sin x + \cos x} = \int \frac{\tan x dx}{\tan x + 1} = \int \frac{y \frac{dy}{y^2+1}}{y+1} = \frac{1}{4} \log \frac{y^2+1}{(y+1)^2} + \frac{1}{2} \arctan y + C$$
$$= \frac{1}{2}x - \frac{1}{2} \log |\sin x + \cos x| + C.$$

The key point here is that the integrand is a rational function $R(\sin x, \cos x)$ that satisfies $R(-u, -v) = R(u, v)$. In this case, the integrad can always be written as a rational function of $\tan x$, and the change of variable can be applied.

Example 5.6.11. Note that rational function $\dfrac{\cos x}{\cos x \sin x + \sin^3 x}$ of $\sin x$ and $\cos x$ is odd in the $\sin x$ variable. This is comparable to the function $\cos^m x \sin^n x$ for the case n is odd. We may introduce the same change of variable $y = \cos x$, $dy = -\sin x dx$ like the earlier example and get

$$\int \frac{\cos x dx}{\cos x \sin x + \sin^3 x} = \int \frac{\cos x \sin x dx}{\cos x \sin^2 x + (\sin^2 x)^2} = \int \frac{-y dy}{y(1-y^2) + (1-y^2)^2}$$
$$= \frac{1}{\sqrt{5}} \log \left| \frac{y - \frac{1+\sqrt{5}}{2}}{y - \frac{1-\sqrt{5}}{2}} \right| + \frac{1}{2} \log \left| \frac{y-1}{y+1} \right| + C$$
$$= \frac{1}{\sqrt{5}} \log \left| \frac{2\cos x - 1 - \sqrt{5}}{2\cos x - 1 + \sqrt{5}} \right| + \frac{1}{2} \log \frac{1-\cos x}{1+\cos x} + C.$$

Similarly, a rational function of $\sin x$ and $\cos x$ that is odd in the $\cos x$ variable can be integrated by introducing $x = \sin y$. If $R(-u, -v) = R(u, v)$, then we may introduce $y = \tan x$ to compute $\displaystyle\int R(\sin x, \cos x) dx$.

Exercise 5.6.7. Compute the integral.

1. $\displaystyle\int \frac{1-r^2}{1-2r\cos x + r^2} dx$, $|r| < 1$.
2. $\displaystyle\int \frac{dx}{a - \cos 2x}$.
3. $\displaystyle\int \frac{dx}{a + \tan x}$.
4. $\displaystyle\int \frac{dx}{\cos x + \tan x}$.
5. $\displaystyle\int \frac{dx}{\sin x + \tan x}$.
6. $\displaystyle\int \frac{dx}{2\sin x + \sin 2x}$.
7. $\displaystyle\int \frac{\sin^2 x}{1 + \sin^2 x} dx$.
8. $\displaystyle\int \frac{dx}{\sin(x+a)\ \sin(x+b)}$.
9. $\displaystyle\int \frac{dx}{(1+\cos^2 x)(2+\sin^2 x)}$.
10. $\displaystyle\int \frac{(1+\sin x)dx}{\sin x(1+\cos x)}$.

11. $\displaystyle\int \frac{dx}{(a+\cos x)\sin x}$.

12. $\displaystyle\int \frac{(\sin x+\cos x)dx}{\sin x(\sin x-\cos x)}$.

13. $\displaystyle\int \frac{dx}{(a+\cos^2 x)\sin x}$.

14. $\displaystyle\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$.

15. $\displaystyle\int \frac{1-\tan x}{1+\tan x}dx$.

16. $\displaystyle\int \sqrt{\tan x}dx$.

5.7 Improper Integral

The definition of Riemann integral requires both the function and the interval to be bounded. If either the function or the interval is unbounded, then the integral is *improper*. We may still make sense of an improper integral if it can be viewed as the limit of usual integral of bounded function on bounded interval.

5.7.1 Definition and Property

Example 5.7.1. The function e^{-x} is bounded on the unbounded interval $[0,+\infty)$. To make sense of the improper integral $\displaystyle\int_0^{+\infty} e^{-x}dx$, we consider the integral on any bounded interval

$$\int_0^b e^{-x}dx = 1-e^{-b}.$$

As the bounded interval approaches $[0,+\infty)$, we get

$$\lim_{b\to+\infty}\int_0^b e^{-x}dx = \lim_{b\to+\infty}(1-e^{-b}) = 1.$$

Therefore the improper integral $\displaystyle\int_0^{+\infty} e^{-x}dx$ has value 1. Geometrically, this means that the area of the unbounded region under the graph of the function e^{-x} and over the interval $[0,+\infty)$ is 1.

Example 5.7.2. The function $\log x$ is unbounded on the bounded interval $(0,1]$. Since the integral $\displaystyle\int_0^1 \log xdx$ is improper at 0^+, we consider the integral over $[\epsilon,1]$ for $\epsilon>0$

$$\int_\epsilon^1 \log xdx = (x\log x - x)\Big|_{x=\epsilon}^{x=1} = -1-\epsilon\log\epsilon+\epsilon.$$

Since the right side converges to -1 as $\epsilon\to 0^+$, the improper integral converges and has value

$$\int_0^1 \log xdx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 \log xdx = -1.$$

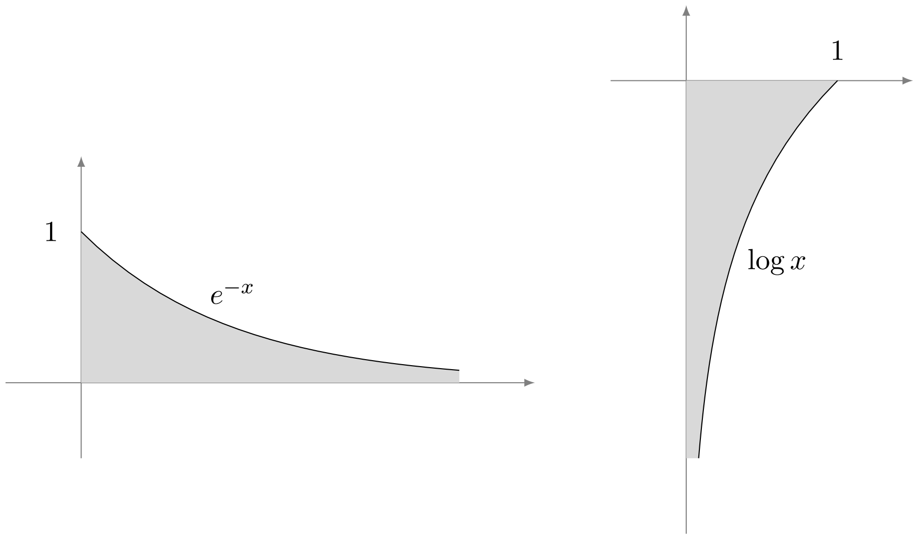


Figure 5.7.1: The unbounded region has area 1.

The unbounded region measured by this improper integral is actually the same as the one in Example 5.7.1, up to a rotation.

Example 5.7.3. Consider the improper integral $\displaystyle\int_a^{+\infty}\frac{dx}{x^p}$, where $a>0$. We have

$$\int_a^b \frac{dx}{x^p}=\begin{cases}\dfrac{b^{1-p}-a^{1-p}}{1-p}, & \text{if } p\neq 1,\\ \log b-\log a, & \text{if } p=1.\end{cases}$$

As $b\to+\infty$, we get

$$\int_a^{+\infty} \frac{dx}{x^p}=\begin{cases}\dfrac{a^{1-p}}{p-1}, & \text{if } p>1,\\ \text{diverge}, & \text{if } p\leq -1.\end{cases}$$

Example 5.7.4. The integral $\displaystyle\int_0^1\frac{dx}{x^p}$ is improper at 0^+ for $p>0$. For $\epsilon>0$, we have

$$\int_\epsilon^1 \frac{dx}{x^p}=\begin{cases}\dfrac{1-\epsilon^{1-p}}{1-p}, & \text{if } p\neq 1,\\ -\log\epsilon, & \text{if } p=1.\end{cases}$$

As $\epsilon\to 0^+$, we get

$$\int_0^1 \frac{dx}{x^p}=\begin{cases}\dfrac{1}{1-p}, & \text{if } p<1,\\ \text{diverge}, & \text{if } p\geq 1.\end{cases}$$

By the same argument, for $a < b$, the improper integrals $\int_a^b (x-a)^p dx$ and $\int_a^b (b-x)^p dx$ converge if and only if $p > -1$.

Example 5.7.5. The integral $\int_{-\infty}^{+\infty} \frac{dx}{x^2+1}$ is improper at $+\infty$ and $-\infty$. The integral on a bounded interval is

$$\int_a^b \frac{dx}{x^2+1} = \arctan b - \arctan a.$$

Then we get

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b \frac{dx}{x^2+1} = \lim_{b\to+\infty} \arctan b - \lim_{a\to-\infty} \arctan a = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Example 5.7.6. Since

$$\int_0^b \cos x dx = \sin b$$

diverges as $b \to +\infty$, the improper integral $\int_0^{+\infty} \cos x dx$ diverges.

In general, an integral may be improper at several places. We may divide the interval into several parts, such that each part contains exactly one improperness. If an integral has one improperness at $+\infty$ or $-\infty$, then we study the limit of the integral on bounded intervals. If an integral has one improperness at a^+ or a^-, then we study the limit of the integral on intervals $[a+\epsilon, b]$ or $[b, a-\epsilon]$.

Example 5.7.7. A naive application of the Newton-Leibniz formula would tell us

$$\int_{-1}^1 \frac{dx}{x} = (\log|x|)\Big|_{x=-1}^{x=1} = \log 1 - \log 1 = 0.$$

However, the computation is wrong since the integrand $\frac{1}{x}$ is not continuous on $[-1,1]$. In fact, the integral $\int_{-1}^1 \frac{dx}{x}$ is improper on both sides of 0, and we need both improper integrals $\int_{-1}^0 \frac{dx}{x}$ and $\int_0^1 \frac{dx}{x}$ to converge and then get

$$\int_{-1}^1 \frac{dx}{x} = \int_{-1}^0 \frac{dx}{x} + \int_0^1 \frac{dx}{x}.$$

Since

$$\int_0^1 \frac{dx}{x} = \lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{dx}{x} = \lim_{\epsilon\to 0^+} -\log\epsilon = +\infty$$

diverges, the improper integral $\displaystyle\int_0^1 \frac{dx}{x}$ diverges, so that $\displaystyle\int_{-1}^1 \frac{dx}{x}$ also diverges.

Example 5.7.8. To compute the improper integral $\displaystyle\int_{-\infty}^0 xe^x dx$, we start with integration by parts on a bounded interval

$$\int_b^0 xe^x dx = \int_b^0 xde^x = -be^b - \int_b^0 e^x dx = -be^b - 1 + e^b.$$

Taking $b \to -\infty$ on both sides, we get

$$\int_{-\infty}^0 xe^x dx = -1.$$

The example shows that the integration by parts can be extended to improper integrals, simply by taking the limit of the integration by parts formula for the usual proper integrals.

Example 5.7.9. For $a > 1$, consider the improper integral $\displaystyle\int_a^{+\infty} \frac{dx}{x(\log x)^p}$. We have

$$\int_a^b \frac{dx}{x(\log x)^p} = \int_a^b \frac{d(\log x)}{(\log x)^p} = \int_{\log a}^{\log b} \frac{dy}{y^p}.$$

Taking $b \to +\infty$ on both sides, we get

$$\int_a^{+\infty} \frac{dx}{x(\log x)^p} = \int_{\log a}^{+\infty} \frac{dy}{y^p}.$$

The equality means that the improper integral on the left converges if and only if the improper integral on the right converges, and the two values are the same. By Example 5.7.3, we see that the improper integral $\displaystyle\int_a^{+\infty} \frac{dx}{x(\log x)^p}$ converges if and only if $p < 1$, and

$$\int_a^{+\infty} \frac{dx}{x(\log x)^p} = -\frac{(\log a)^{p+1}}{p+1}, \quad \text{if } p < 1.$$

The example shows that the change of variable can also be extended to improper integrals, simply by taking the limit of the change of variable formula for the usual proper intervals.

Exercise 5.7.1. Determine the convergence of improper integrals and evaluate the convergent ones.

1. $\displaystyle\int_0^{+\infty} x^p dx.$
2. $\displaystyle\int_0^1 \frac{dx}{x(-\log x)^p}.$
3. $\displaystyle\int_0^{+\infty} a^x dx.$
4. $\displaystyle\int_{-\infty}^0 a^x dx.$
5. $\displaystyle\int_{-1}^1 \frac{dx}{1-x^2}.$
6. $\displaystyle\int_2^{+\infty} \frac{dx}{1-x^2}.$
7. $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}.$
8. $\displaystyle\int_0^{\frac{\pi}{2}} \tan x dx.$
9. $\displaystyle\int_0^{\pi} \sec x dx.$

Exercise 5.7.2. Determine the convergence of improper integrals and evaluate the convergent ones.

1. $\displaystyle\int_1^{+\infty} \frac{dx}{x+1}.$
2. $\displaystyle\int_{-\infty}^{+\infty} \frac{xdx}{x^2+1}.$
3. $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt[3]{x+1}}.$
4. $\displaystyle\int_0^{+\infty} \frac{x^2dx}{x^3+1}.$
5. $\displaystyle\int_0^{+\infty} \frac{x^2dx}{(x^3+1)^2}.$
6. $\displaystyle\int_0^{+\infty} \frac{dx}{x(x+1)(x+2)}.$
7. $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(1+x)}.$
8. $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-9}}.$
9. $\displaystyle\int_0^{+\infty} xe^x dx.$
10. $\displaystyle\int_0^{+\infty} xe^{-x^2} dx.$
11. $\displaystyle\int_0^{+\infty} e^{-\sqrt{x}} dx.$
12. $\displaystyle\int_0^1 x\log x dx.$
13. $\displaystyle\int_1^{+\infty} \frac{\log x}{x^2} dx.$
14. $\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}} dx.$
15. $\displaystyle\int_0^{+\infty} \frac{x\arctan x}{(1+x^2)^2} dx.$
16. $\displaystyle\int_0^{+\infty} e^{-ax}\cos bx dx.$
17. $\displaystyle\int_0^{+\infty} e^{-ax}\sin bx dx.$
18. $\displaystyle\int_0^{+\infty} e^{-x}|\sin x| dx.$

Exercise 5.7.3. Prove that

$$\lim_{n\to\infty}\frac{1}{n}\left(\log\frac{1}{n}+\log\frac{2}{n}+\cdots+\log\frac{n}{n}\right)=\int_0^1 \log x dx.$$

Note that the left side is a "Riemann sum" for the right side. However, since the integral is improper, we cannot directly use the fact that the Riemann sum converges to the integral. Moreover, the limit is the same as $\lim_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}=e^{-1}$.

5.7.2 Comparison Test

The improper integral is defined by taking limit. Therefore there is always the problem of convergence.

The convergence of the improper integral $\displaystyle\int_a^{+\infty} f(x)dx$ means the convergence of the function $I(b)=\displaystyle\int_a^b f(x)dx$ as $b\to+\infty$. The *Cauchy criterion* for the convergence is that, for any $\epsilon > 0$, there is N, such that

$$b, c > N \implies |I(c) - I(b)| = \left| \int_b^c f(x)dx \right| < \epsilon.$$

The Cauchy criterion for the convergence of other types of improper integrals is similar. For example, if the integral $\int_a^b f(x)dx$ is improper at a^+, then the integral converges if and only if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$c, d \in (a, a+\delta) \implies \left| \int_c^d f(x)dx \right| < \epsilon.$$

The Cauchy criterion shows that the convergence of an improper integral depends only on the behavior of the function near the improper place. Moreover, the Cauchy criterion also implies the following test for convergence.

Theorem 5.7.1 (Comparison Test). *If $|f(x)| \le g(x)$ on (a, b) and the integral $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges.*

Note that if $|f| \le g$, then $|\,|f|\,| \le g$. Therefore whenever we use the comparison test, we may always conclude that $\int_a^b |f(x)|dx$ also converges.

We say that $\int_a^b f(x)dx$ *absolutely converges* if $\int_a^b |f(x)|dx$ converges. The comparison test tells us that absolute convergence implies convergence.

Here we justify the comparison test for the integral $\int_a^{+\infty} f(x)dx$ that is improper at $+\infty$. The convergence of $\int_a^{+\infty} g(x)dx$ implies that for any $\epsilon > 0$, there is N, such that

$$c > b > N \implies \int_b^c g(x)dx < \epsilon.$$

The assumption $|f(x)| \le g(x)$ further implies that for $c > b$,

$$\left| \int_b^c f(x)dx \right| \le \int_b^c |f(x)|dx \le \int_b^c g(x)dx.$$

Combining the two implications, we get

$$c > b > N \implies \left| \int_b^c f(x)dx \right| \le \int_b^c g(x)dx < \epsilon.$$

This verifies the Cauchy criterion for the convergence of $\int_a^{+\infty} f(x)dx$.

Example 5.7.10. We know $\frac{1}{\sqrt{x^3+1}} < x^{-\frac{3}{2}}$ for $x \ge 1$. Since $\int_1^{+\infty} x^{-\frac{3}{2}} dx$ converges, by the comparison test, we know $\int_1^{+\infty} \frac{dx}{\sqrt{x^3+1}}$ also converges. We note that $\int_0^{+\infty} \frac{dx}{\sqrt{x^3+1}}$ converges too because only the behavior of the function for big x (i.e., near $+\infty$) is involved.

Example 5.7.11. To determine the convergence of $\int_0^{+\infty} e^{-x^2} dx$, we use $0 < e^{-x^2} \le e^{-x}$ for $x \ge 1$. By Example 5.7.1 and the comparison test, we know $\int_1^{+\infty} e^{-x^2} dx$ converges. Since $\int_0^1 e^{-x^2} dx$ is a proper integral, we know $\int_0^{+\infty} e^{-x^2} dx$ also converges.

Example 5.7.12. To determine the convergence of $\int_1^{+\infty} \frac{\log x}{x^p} dx$, $p > 0$, we use the comparison $\frac{\log x}{x^p} \ge \frac{1}{x^p} > 0$ for $x \ge e$. For $p \le 1$, the divergence of $\int_1^{+\infty} \frac{1}{x^p} dx$ implies the divergence of $\int_1^{+\infty} \frac{\log x}{x^p} dx$.

For $p > 1$, although we also know the convergence of $\int_1^{+\infty} \frac{1}{x^p} dx$, the comparison above cannot be used to conclude the convergence of $\int_1^{+\infty} \frac{\log x}{x^p} dx$. Instead, we choose q satisfying $p > q > 1$. Then by

$$\lim_{x\to+\infty} \frac{\frac{\log x}{x^p}}{\frac{1}{x^q}} = \lim_{x\to+\infty} \frac{\log x}{x^{p-q}} = 0,$$

we have $\left|\frac{\log x}{x^p}\right| \le \frac{1}{x^q}$ for sufficiently large x. By the convergence of $\int_1^{+\infty} \frac{dx}{x^q}$, therefore, we know the converges of $\int_1^{+\infty} \frac{\log x}{x^p} dx$.

We conclude that $\int_1^{+\infty} \frac{\log x}{x^p} dx$ converges if and only if $p > 1$.

Example 5.7.13. The integral $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ is improper at 0^+ and 1^-. By applying the idea of Example 5.7.12 to

$$\lim_{x\to 0^+}\frac{\dfrac{1}{\sqrt{x(1-x)}}}{\dfrac{1}{\sqrt{x}}}=1,\quad \lim_{x\to 1^-}\frac{\dfrac{1}{\sqrt{x(1-x)}}}{\dfrac{1}{\sqrt{1-x}}}=1,$$

and the convergence of $\displaystyle\int_0^1\frac{dx}{\sqrt{x}}$ and $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x}}$, we know that $\displaystyle\int_0^1\frac{dx}{\sqrt{x(1-x)}}$ converges.

Exercise 5.7.4. Compare the integrals $I=\displaystyle\int_a^{+\infty}f(x)dx$ and $J=\displaystyle\int_a^{+\infty}g(x)dx$ that are improper at $+\infty$.

1. Prove that if $\lim_{x\to+\infty}\dfrac{f(x)}{g(x)}$ converges, $g(x)\ge 0$ for sufficiently large x, and J converges, then I also converges.
2. Prove that if $\lim_{x\to+\infty}\dfrac{f(x)}{g(x)}$ converges to a nonzero number, and $g(x)\ge 0$ for sufficiently large x, then I converges if and only if J converges.

Exercise 5.7.5. Suppose $f\ge 0$, prove that the improper integral $\displaystyle\int_a^{+\infty}f(x)dx$ converges if and only if $\displaystyle\int_a^b f(x)dx$, for all $b\in[a,+\infty)$, is bounded. What about the integral $\displaystyle\int_a^b f(x)dx$ that is improper at a^+.

Exercise 5.7.6. Suppose $\displaystyle\int_a^{+\infty}f^2dx$ and $\displaystyle\int_a^{+\infty}g^2dx$ converge. Prove that $\displaystyle\int_a^{+\infty}fgdx$ and $\displaystyle\int_a^{+\infty}(f+g)^2dx$ converge.

Exercise 5.7.7. Determine convergence.

1. $\displaystyle\int_2^{+\infty}\frac{dx}{x^p(\log x)^q}$.
2. $\displaystyle\int_1^{2}\frac{dx}{x^p(\log x)^q}$.
3. $\displaystyle\int_0^{1}\frac{dx}{x^p|\log x|^q}$.
4. $\displaystyle\int_{-\infty}^{1}\frac{dx}{|x|^p|\log x|^q}$.
5. $\displaystyle\int_2^{+\infty}\frac{dx}{x^p+(\log x)^q}$.
6. $\displaystyle\int_0^{+\infty}\frac{x^pdx}{1+x^q}$.
7. $\displaystyle\int_0^{1}\frac{x^pdx}{\sqrt{1-x^q}}$, $q>0$.
8. $\displaystyle\int_0^{1}\frac{dx}{x^p(1-x^q)^r}$, $q>0$.
9. $\displaystyle\int_0^{1}x^pa^xdx$, $a>0$.
10. $\displaystyle\int_1^{+\infty}x^pa^xdx$, $a>0$.
11. $\displaystyle\int_1^{+\infty}x^p\log(1+x^q)dx$.
12. $\displaystyle\int_0^{1}x^p\log(1+x^q)dx$.

Exercise 5.7.8. Determine convergence.

1. $\displaystyle\int_2^{+\infty}\frac{xdx}{\sqrt{x^5-2x^2+1}}$.
2. $\displaystyle\int_0^{1}\frac{xdx}{\sqrt{x^5-2x^2+1}}$.
3. $\displaystyle\int_2^{+\infty}\frac{x\sin xdx}{\sqrt{x^5-2x^2+1}}$.
4. $\displaystyle\int_0^{1}\frac{x\sin xdx}{\sqrt{x^5-2x^2+1}}$.
5. $\displaystyle\int_2^{+\infty}\frac{x\arctan xdx}{\sqrt{x^5-2x^2+1}}$.
6. $\displaystyle\int_0^{1}\frac{x\arctan xdx}{\sqrt{x^5-2x^2+1}}$.

Exercise 5.7.9. Determine convergence.

1. $\displaystyle\int_1^{+\infty}\frac{x+1}{x^3-2x+3}dx$.
2. $\displaystyle\int_0^{10}\frac{dx}{\sqrt{|x^2-4x+3|}}$.
3. $\displaystyle\int_1^{+\infty}\frac{\log x}{\sqrt{x^p+1}}dx$.
4. $\displaystyle\int_0^{1}(1-x)^p|\log x|^qdx$.
5. $\displaystyle\int_1^{2}\frac{dx}{(3x-2-x^2)^p}$.
6. $\displaystyle\int_0^{+\infty}\frac{\log(1+x^2)dx}{1+x^q}$.
7. $\displaystyle\int_0^{1}\frac{dx}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.
8. $\displaystyle\int_1^{+\infty}\frac{dx}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.
9. $\displaystyle\int_0^{+\infty}\frac{dx}{\sqrt{1+\sqrt{1+\sqrt{x}}}}$.

Exercise 5.7.10. Determine convergence.

1. $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{\cos^p x}$.
2. $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{x^p\sin^q x}$.
3. $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^p x\cos^q x}$.
4. $\displaystyle\int_0^{\frac{\pi}{3}}\tan^p xdx$.
5. $\displaystyle\int_0^{\frac{\pi}{3}}\tan^p x\log^q xdx$.
6. $\displaystyle\int_0^{\frac{\pi}{4}}\frac{dx}{|\sin x-\cos x|^p}$.
7. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{(1-\sin x)^p}$.
8. $\displaystyle\int_0^{\frac{\pi}{2}}(-\log\sin x)^pdx$.
9. $\displaystyle\int_0^{\frac{\pi}{2}}x^p\log\sin xdx$.
10. $\displaystyle\int_0^{\frac{\pi}{2}}|\log\tan x|^pdx$.
11. $\displaystyle\int_0^{+\infty}e^{ax}\cos bxdx$.
12. $\displaystyle\int_0^{+\infty}e^{-\sqrt{x}}\cos bx^2dx$.
13. $\displaystyle\int_1^{+\infty}\frac{x+\cos x}{x^2-\sin x}dx$.
14. $\displaystyle\int_1^{+\infty}\frac{\log(x+\cos x)}{x^2-\sin x}dx$.
15. $\displaystyle\int_1^{+\infty}\frac{\log x+\cos x}{x^2-\sin x}dx$.

Exercise 5.7.11. Find a constant a, such that $\displaystyle\int_0^{+\infty}\left(\frac{1}{\sqrt{x^2+1}}+\frac{a}{x+1}\right)dx$ converges. Moreover, evaluate the integral for this a.

5.7.3 Conditional Convergence

Although the comparison test is very effective, some improper integrals needs to be modified before the comparison test can be applied.

Example 5.7.14. By the comparison test, we know $\int_1^{+\infty} \frac{\sin x}{x^p} dx$ converges for $p > 1$. However, the argument fails for the case $p = 1$. We will show that $\int_1^{+\infty} \frac{\sin x}{x} dx$ still converges. We will also show that, after taking the absolute value, the integral $\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx$ actually diverges. This means that the comparison test cannot be directly applied to $\int_1^{+\infty} \frac{\sin x}{x} dx$.

Using integration by parts, we have

$$\int_1^b \frac{\sin x}{x} dx = -\int_1^b \frac{1}{x} d\cos x = -\frac{\cos b}{b} + \cos 1 - \int_1^b \frac{\cos x}{x^2} dx.$$

By the comparison test, the improper integral $\int_1^{+\infty} \frac{\cos x}{x^2} dx$ converges. Therefore the right side converges as $b \to +\infty$, and we conclude that $\int_1^{+\infty} \frac{\sin x}{x} dx$ converges.

On the other hand, we have

$$\begin{aligned}\int_1^{n\pi} \left|\frac{\sin x}{x}\right| dx \ge \int_\pi^{n\pi} \left|\frac{\sin x}{x}\right| dx &= \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{x}\right| dx \ge \sum_{k=2}^n \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x| dx \\ &\ge \sum_{k=2}^n \frac{1}{k\pi} \ge \frac{1}{\pi}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).\end{aligned}$$

By Example 1.3.8, the right side diverges to $+\infty$. Therefore $\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx$ diverges.

Example 5.7.15. By a change of variable, we have

$$\int_0^{+\infty} \sin x^2 dx = \int_0^{+\infty} \sin y d(\sqrt{y}) = \int_0^{+\infty} \frac{\sin y}{2\sqrt{y}} dy.$$

The integral on the right is *proper* at 0^+ and improper at $+\infty$. It converges by an argument similar to Example 5.7.14. Therefore the integral $\int_0^{+\infty} \sin x^2 dx$ converges.

We first used the change of variable, then used the integration by parts, and finally used the comparison test to conclude the convergence of $\int_0^{+\infty} \sin x^2 dx$.

The reader can further use the idea of Example 5.7.14 to show that $\int_0^{+\infty} |\sin x^2| dx$ diverges.

Example 5.7.14 shows that it is possible for an improper integral $\int_a^b f(x)dx$ to converge but the corresponding absolute improper integral $\int_a^b |f(x)|dx$ to diverge. In this case, the integral converges but not absolutely, and we say $\int_a^b f(x)dx$ *conditionally converges*.

The idea in Example 5.7.14 can be elaborated to get the following useful tests.

Theorem 5.7.2 (Dirichlet Test). *Suppose* $\int_a^b f(x)dx$ *is bounded for all* $b \in [a, +\infty)$. *Suppose* $g(x)$ *is monotonic and* $\lim_{x\to+\infty} g(x) = 0$. *Then* $\int_a^{+\infty} f(x)g(x)dx$ *converges.*

Theorem 5.7.3 (Abel Test). *Suppose* $\int_a^{+\infty} f(x)dx$ *converges. Suppose* $g(x)$ *is monotonic and bounded on* $[a, +\infty)$. *Then* $\int_a^{+\infty} f(x)g(x)dx$ *converges.*

The tests basically replaces $\sin x$ and $\dfrac{1}{x}$ in the example by $f(x)$ and $g(x)$. In case $f(x)$ is continuous and $g(x)$ is continuously differentiable, we can justify the tests by repeating the argument in the example. Let $F(x) = \int_a^x f(t)dt$. Then $F(a) = 0$, and

$$\int_a^b f(x)g(x)dx = \int_a^b g(x)dF(x) = g(b)F(b) - \int_a^b F(x)g'(x)dx.$$

Under the assumption of the Dirichlet test, we have $\lim_{b\to+\infty} g(b)F(b) = 0$, and $|F(x)| < M$ for some constant M and all $x \geq a$. Assume the monotonic function $g(x)$ is increasing. Then $g'(x) \geq 0$, and

$$|F(x)g'(x)| \leq Mg'(x).$$

Since

$$\int_a^{+\infty} g'(x)dx = \lim_{b\to+\infty} \int_a^b g'(x)dx = \lim_{b\to+\infty} (g(b) - g(a)) = -g(a)$$

converges, by the comparison test, the improper integral

$$\int_a^{+\infty} F(x)g'(x)dx = \lim_{b\to+\infty} \int_a^b F(x)g'(x)dx$$

converges. Therefore $\int_a^{+\infty} f(x)g(x)dx = \lim_{b\to+\infty} \int_a^b f(x)g(x)dx$ also converges. The proof for decreasing $g(x)$ is similar.

Under the assumption of the Abel test, we know both $F(b)$ and $g(b)$ converge as $b \to +\infty$. Therefore $F(x)$ is bounded, and we may apply the comparison test as before. Moreover, the convergence of $\lim_{b\to+\infty} g(b)$ implies the convergence of $\int_a^{+\infty} g'(x)dx$. We conclude again that $\int_a^{+\infty} f(x)g(x)dx$ converges.

Exercise 5.7.12. Determine convergence. Is the convergence absolute or conditional?

1. $\displaystyle\int_0^{+\infty} \frac{\sin x^q}{x^p} dx.$
2. $\displaystyle\int_0^{+\infty} \frac{\cos x^q}{x^p} dx.$
3. $\displaystyle\int_0^{1} \frac{1}{x}\sin\frac{1}{x} dx.$
4. $\displaystyle\int_0^{1} \frac{1}{x^p}\sin\frac{1}{x} dx.$
5. $\displaystyle\int_0^{+\infty} \frac{\cos ax}{1+x^p} dx.$
6. $\displaystyle\int_0^{+\infty} \frac{x^p\sin ax}{1+x^q} dx.$
7. $\displaystyle\int_0^{+\infty} \frac{\sin^2 x}{x} dx.$
8. $\displaystyle\int_0^{+\infty} \frac{\sin^3 x}{x} dx.$
9. $\displaystyle\int_1^{+\infty} \frac{\sin x\arctan x}{x^p} dx.$

Exercise 5.7.13. Construct a function $f(x)$ such that $|f(x)| = 1$ and $\int_0^{+\infty} f(x)dx$ converges.

Finally, we show some examples of using the integration by parts and change of variable to compute improper integrals. We note that the convergence needs to be verified before applying the properties of integration.

Example 5.7.16. In Example 5.7.11, we know the convergence of $\int_0^{+\infty} e^{-x^2}dx$. By the similar idea, especially $\lim_{x\to+\infty} \dfrac{x^pe^{-x^2}}{e^{-x}} = 0$, we know that $\int_0^{+\infty} x^pe^{-x^2}dx$ converges for any $p \ge 0$.

Let

$$I_n = \int_0^{+\infty} x^ne^{-x^2}dx.$$

Then we may apply the integration by parts to get

$$\begin{aligned} I_n &= -\frac{1}{2}\int_0^{+\infty} x^{n-1}de^{-x^2} \\ &= -\frac{1}{2}x^{n-1}e^{-x^2}\Big|_{x=0}^{x=+\infty} + \frac{n-1}{2}\int_0^{+\infty} x^{n-2}e^{-x^2}dx = \frac{n-1}{2}I_{n-2}. \end{aligned}$$

It is known (by using integration of two variable function, for example) that

$$I_0 = \int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

We can also apply the change of variable to get

$$I_1 = \int_0^{+\infty} xe^{-x^2}dx = \frac{1}{2}\int_0^{+\infty} e^{-x}dx = \frac{1}{2}.$$

Then we can use the recursive relation to compute I_n for all natural number n.

Example 5.7.17. The integral $\int_0^{\frac{\pi}{2}} \log \sin x dx$ is improper at 0. By L'Hospital's rule, we have

$$\lim_{x\to 0^+} \frac{\log \sin x}{\log x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{x \cos x}{\sin x} = 1.$$

By the convergence of $\int_0^1 |\log x| dx = -\int_0^1 \log x dx$ in Example 5.7.2 and the comparison test, we see that $\int_0^{\frac{\pi}{2}} \log \sin x dx$ converges.

The value of the improper integral can be computed as follows

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \log \sin x dx &= \int_0^{\frac{\pi}{4}} \log \sin x dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log \sin x dx \\
&= \int_0^{\frac{\pi}{4}} \log \sin x dx - \int_{\frac{\pi}{4}}^{0} \log \cos x dx \\
&= \int_0^{\frac{\pi}{4}} (\log \sin x + \log \cos x) dx = \int_0^{\frac{\pi}{4}} \log \left(\frac{1}{2} \sin 2x\right) dx \\
&= \int_0^{\frac{\pi}{4}} \log \sin 2x dx - \frac{\pi}{4} \log 2 = \frac{1}{2} \int_0^{\frac{\pi}{2}} \log \sin x dx - \frac{\pi}{4} \log 2.
\end{aligned}$$

Note that all the deductions are legitimate because all the improper integrals involved converge. Therefore we conclude that

$$\int_0^{\frac{\pi}{2}} \log \sin x dx = -\frac{\pi}{2} \log 2.$$

Exercise 5.7.14. Compute improper integral.

1. $\int_0^1 (\log x)^n dx$.
2. $\int_0^{+\infty} x^n e^{-x} dx$.
3. $\int_0^{+\infty} \frac{dx}{(1+x^2)^n}$.
4. $\int_0^1 \frac{x^n dx}{\sqrt{1-x^2}}$.
5. $\int_0^{\frac{\pi}{2}} \log \cos x dx$.

Exercise 5.7.15. The *Gamma function* is

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

1. Show that the function is defined for $x > 0$.

2. Show the other formulae for the Gamma function

$$\Gamma(x) = 2\int_0^{\infty} t^{2x-1}e^{-t^2}dt = a^x\int_0^{\infty} t^{x-1}e^{-at}dt.$$

3. Show that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n) = (n-1)!$.

5.8 Application to Geometry

5.8.1 Length of Curve

Curves in a Euclidean space are often presented by parametrization. For example, the unit circle centered at the origin of $\mathbb{R}^2$ may be parametrized by the angle

$$x = \cos\theta, \quad y = \sin\theta, \quad 0 \le \theta \le 2\pi.$$

The *helix* in $\mathbb{R}^3$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \frac{h}{2\pi}\theta, \quad 0 \le \theta \le 2\pi,$$

moves along the circle of radius r from the viewpoint of the (x, y)-coordinates, and moves up in the z-direction in constant speed, such that each round moves up by height h.

In general, a *parametrized curve* in $\mathbb{R}^2$ is given by

$$x = x(t), \quad y = y(t), \quad a \le t \le b.$$

The *initial point* of the curve is $(x(a), y(a))$, and the *end point* is $(x(b), y(b))$. To compute the length of the curve between the two points, we consider the length $s(t)$ from the initial point $(x(a), y(a))$ to the point $(x(t), y(t))$. We find the change $s'(t)$ and then integrate the change to get $s(t)$. The length of the whole curve is $s(b)$.

Similar to the argument for the area of the region $G_{[a,b]}(f)$, we need to be careful about the sign. In the subsequent discussion, we pretend everything is positive (which at least gives you the right derivative), and further argument about the negative case is omitted. Moreover, we restrict the argument to the case $x(t)$ and $y(t)$ are nice. In fact, we will assume the two functions are continuously differentiable. In general, we may break the curve into finitely many continuously differentiable pieces and add the lengths of the pieces together.

As the parameter t is changed by Δt, the change $\Delta s = s(t+\Delta t) - s(t)$ of the length is the length of the curve segment from $(x(t), y(t))$ to $(x(t+\Delta t), y(t+\Delta t))$. The curve segment is approximated by the straight line connecting the two points. Therefore the length of the curve is approximated by the length of the straight line

$$\Delta s \approx \sqrt{(x(t+\Delta t) - x(t))^2 + (y(t+\Delta t) - y(t))^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

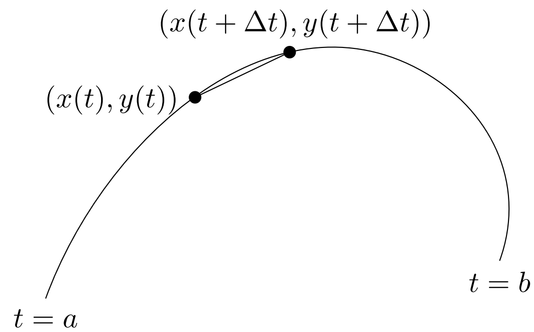


Figure 5.8.1: Length of curve.

Dividing the change Δt of parameter, we get

$$\frac{\Delta s}{\Delta t} \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}.$$

The approximation gets more refined as $\Delta t \to 0$. By taking the limit as $\Delta t \to 0$, the approximation becomes an equality

$$s'(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} = \sqrt{x'(t)^2 + y'(t)^2}.$$

Therefore the length function $s(t)$ is the antiderivative of $\sqrt{x'(t)^2 + y'(t)^2}$, or

$$ds = \sqrt{x'(t)^2 + y'(t)^2}dt,$$

and we have

$$\text{length of curve} = s(b) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt.$$

Example 5.8.1. The length of the unit circle is

$$\int_0^{2\pi} \sqrt{(-\sin\theta)^2 + (\cos\theta)^2}d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

More generally, an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be parametrized as

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi.$$

The length of the ellipse is the so called *elliptic integral*

$$\int_0^{2\pi} \sqrt{(-a\sin\theta)^2 + (b\cos\theta)^2}d\theta = a\int_0^{2\pi} \sqrt{1 + K\cos^2\theta}d\theta, \quad K = \frac{b^2}{a^2} - 1.$$

The integral cannot be computed as an elementary expression if $a \ne b$.

Example 5.8.2. The graph of a function $f(x)$ on $[a,b]$ is a curve

$$x=t,\quad y=f(t),\quad t\in[a,b].$$

The length of the graph is

$$\int_a^b \sqrt{1+f'(x)^2}dx,$$

where the variable t is substituted to the more familiar x.

For example, the parabola $y=x^2$ is cut by the diagonal $y=x$. With the help of Example 5.5.31, the finite segment corresponding to $x\in[0,1]$ has length

$$\begin{aligned}\int_0^1 \sqrt{1+(2x)^2}dx &= \frac{1}{2}\int_0^2 \sqrt{1+x^2}dx \\ &= \frac{1}{4}\left(x\sqrt{1+x^2}+\log(\sqrt{1+x^2}+x)\right)_0^2 = \frac{1}{2}\sqrt{5}+\frac{1}{4}\log(\sqrt{5}+2).\end{aligned}$$

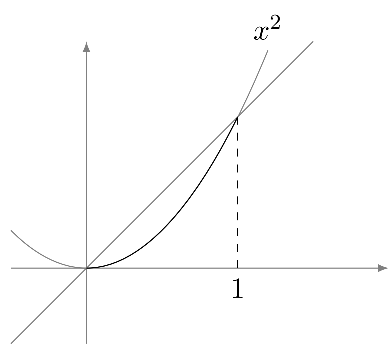


Figure 5.8.2: Parabola x^2 cut by the diagonal.

Example 5.8.3. The *astroid* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$ can be parametrized as

$$x=\cos^3 t,\quad y=\sin^3 t,\quad t\in[0,2\pi].$$

Note that the range $[0,2\pi]$ for t corresponds to moving around the astroid exactly once. Therefore the perimeter is

$$\int_0^{2\pi} \sqrt{(-3\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2}dt = \int_0^{2\pi} 3|\sin t\cos t|dt = 6.$$

Example 5.8.4. The argument about the length of curves also applies to curves in $\mathbb{R}^3$ and leads to

$$\text{length of curve} = \int_a^b \sqrt{x'(t)^2+y'(t)^2+z'(t)^2}dt.$$

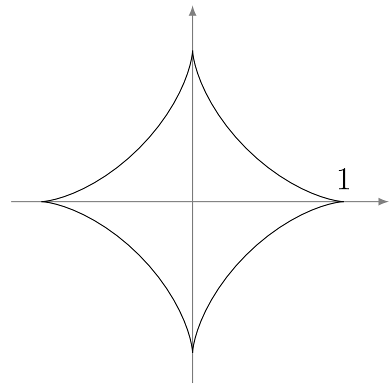


Figure 5.8.3: Astroid.

For example, the length of one round of the helix is

$$\begin{aligned}\int_0^{2\pi}\sqrt{(-r\sin\theta)^2+(r\cos\theta)^2+\left(\frac{h}{2\pi}\right)^2}d\theta&=\int_0^{2\pi}\sqrt{r^2+\left(\frac{h}{2\pi}\right)^2}d\theta\\&=2\pi\sqrt{r^2+\left(\frac{h}{2\pi}\right)^2}=\sqrt{(2\pi r)^2+h^2}.\end{aligned}$$

The result has a simple geometrical interpretation: By cutting along a vertical line, the cylinder can be "flattened" into a plane. Then the helix becomes the hypotenuse of a right triangle with horizontal length $2\pi r$ and vertical length h.

Example 5.8.5. When a circle rolls along a straight line, the track of one point on the circle is the *cycloid*. Let r be the radius of the circle, and assume the point is at the bottom at the beginning. After rotating angle t, the center of the circle is at (rt, r), and the point is at $(rt, r) + r(-\cos(t-\frac{\pi}{2}), \sin(t-\frac{\pi}{2}))$. Therefore the cycloid is parameterized by

$$x = rt - r\sin t, \quad y = r - r\cos t.$$

As the circle makes one complete rotation, we get one period of the cycloid, corresponding to $t \in [0, 2\pi]$. The length of this one period is

$$\int_0^{2\pi}\sqrt{(r-r\cos t)^2+(r\sin t)^2}dt = r\int_0^{2\pi}\sqrt{2(1-\cos t)}dt = r\int_0^{2\pi}2\left|\sin\frac{t}{2}\right|dt = 8r.$$

Exercise 5.8.1. Compute length.

1. $y^2 = 2x$, $x \in [0, a]$.
2. $x^2 = 2py$, $x \in [0, a]$.
3. $y = e^x$, $x \in [0, a]$.
4. $y = \log x$, $x \in [1, a]$.

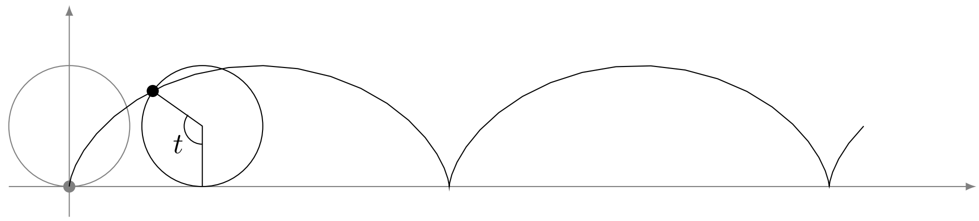


Figure 5.8.4: Cycloid.

5. $y = \log(4 - x^2)$, $x \in [-1, 1]$.
6. $y = \log\cos x$, $x \in \left[0, \frac{\pi}{4}\right]$.
7. $y = \log\sec x$, $x \in \left[0, \frac{\pi}{4}\right]$.
8. $y = \frac{e^x + e^{-x}}{2}$, $x \in [-a, a]$.
9. $y^2 = x^3$, $x \in [0, a]$.
10. $y^4 = x^3$, $x \in [0, a]$.
11. $y = \log \frac{e^x + 1}{e^x - 1}$, $x \in [1, 2]$.
12. $y = \int_0^x \sqrt{t^3 - 1}dt$, $x \in [1, 4]$.

Exercise 5.8.2. Compute length.

1. $\sqrt{x} + \sqrt{y} = 1$, $x, y \geq 0$.
2. $x = t$, $y = \log t$, $t \in [1, 2]$.
3. $x = e^t - t$, $y = e^t + t$, $t \in [0, 1]$.
4. $x = e^t \cos t$, $y = e^t \sin t$, $t \in [0, \pi]$.
5. $x = \cos^2 t$, $y = \cos t \sin t$, $t \in [0, \pi]$.
6. $x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$, $t \in [0, \pi]$.
7. $x = \cos t + \log\tan\frac{t}{2}$, $y = \sin t$, $t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.
8. $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, $z = t^2$, $t \in [0, 2\pi]$.
9. $x = a^2 e^t$, $y = b^2 e^{-t}$, $z = \sqrt{2}abt$, $t \in [0, 1]$.

Exercise 5.8.3. Compute the length of *Cornu's spiral*

$$x = \int_0^t \cos\frac{\pi u^2}{2}du, \quad y = \int_0^t \sin\frac{\pi u^2}{2}du.$$

Exercise 5.8.4. Think of the rolling circle that produces the cycloid as a disk. What is the track of a point on the disk that is not necessarily on the circle (i.e., the boundary of the disk)? Find the formula for computing the length of this track.

Exercise 5.8.5. Suppose a line is wrapped around a circle. When the line is unwrapped from the circle, the track of one point on the line is the *involute* of the circle. Let r be the radius of the circle and let t be the unwrapped angle.

1. Find the parameterized formula for the involute.

2. Find the length of the involute as the line is unwrapped by half of the circle.

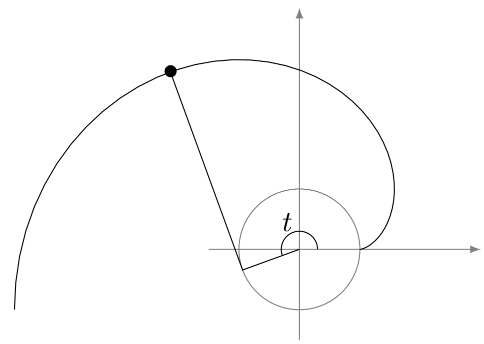


Figure 5.8.5: Involute of circle.

5.8.2 Area of Region

Being defined as area, the integration is naturally adapted to the computation of area. We start with the area of region bounded by two functions.

Example 5.8.6. The curve $y = x^2$ and the straight line $y = x$ enclose a region over $0 \le x \le 1$. The area of the region is the area below x subtracting the area below x^2, which is

$$\int_0^1 x dx - \int_0^1 x^2 dx = \int_0^1 (x - x^2) dx = \frac{1}{6}.$$

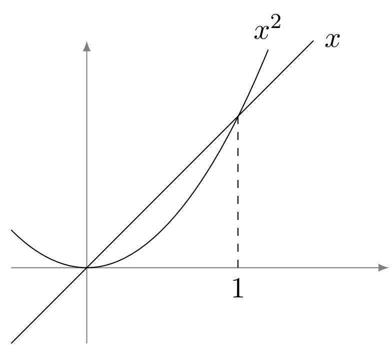


Figure 5.8.6: Region between x and x^2.

Example 5.8.7. To compute the area of the region bounded by $y = x^2 - 2x$ and $y = x$. We denote the (positive) areas of the four indicated regions by A_1, A_2, A_3, A_4. Then

$$\int_0^2 xdx = A_1, \quad \int_0^2 (x^2 - 2x)dx = -A_2, \quad \int_2^3 xdx = A_3 + A_4, \quad \int_2^3 (x^2 - 2x)dx = A_4.$$

The area we are interested in is

$$\begin{aligned} A_1 + A_2 + A_3 &= A_1 - (-A_2) + (A_3 + A_4) - A_4 \\ &= \int_0^2 xdx - \int_0^2 (x^2 - 2x)dx + \int_2^3 xdx - \int_2^3 (x^2 - 2x)dx \\ &= \int_0^3 xdx - \int_0^3 (x^2 - 2x)dx = \int_0^3 [x - (x^2 - 2x)]dx = \frac{9}{2}. \end{aligned}$$

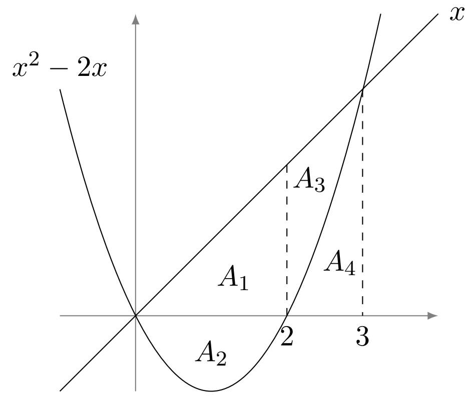


Figure 5.8.7: Region between x and $x^2 - 2x$.

The examples suggest that, if $f(x) \geq g(x)$ on $[a, b]$, then the area of the region between f and g over $[a, b]$ is

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx.$$

In general, we can divide $[a, b]$ into some intervals, such that on each interval, one of the following happens: $f(x) \geq 0 \geq g(x)$, $f(x) \geq g(x) \geq 0$, $0 \geq f(x) \geq g(x)$. Then an argument similar to Example 5.8.7 shows that the total area is indeed given by the formula above.

Example 5.8.8. The functions $\sin x$ and $\cos x$ intersect at many places and enclose many regions. One such region is over the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$, on which we have $\sin x \geq \cos x$. The area of the region is

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx = 2\sqrt{2}.$$

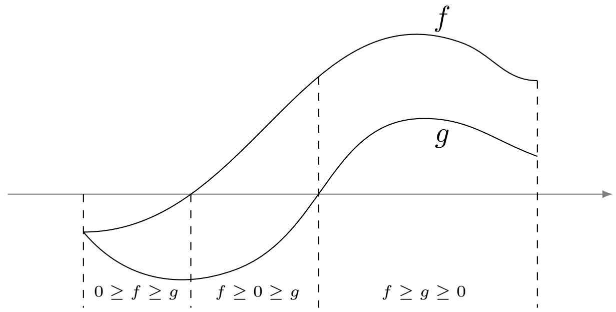


Figure 5.8.8: The area is $\int_a^b (f(x) - g(x))dx$.

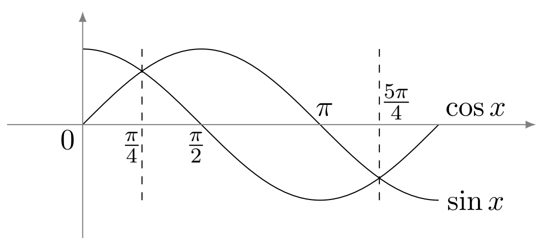


Figure 5.8.9: Region between $\sin x$ and $\cos x$.

Example 5.8.9. The region between the parabola $y^2 - x = 1$ and the straight line $x + y = 1$ is between the functions

$$f(x) = \begin{cases} \sqrt{x+1}, & \text{if } -1 \leq x \leq 0, \\ 1 - x, & \text{if } 0 \leq x \leq 3, \end{cases} \qquad g(x) = -\sqrt{x+1}.$$

The area is

$$\int_{-1}^{3} f(x)dx - \int_{-1}^{3} g(x)dx = \int_{-1}^{0} \sqrt{x+1}dx + \int_{0}^{3} (1-x)dx - \int_{-1}^{3} (-\sqrt{x+1})dx = \frac{9}{2}.$$

Note that the region is obtained by rotating the region in Example 5.8.7. Naturally the results are the same. The previous example actually suggests another way of computing the area, by exchanging the roles of x and y.

Example 5.8.10. Consider $f(x) = x^5 + 2x^2 - 2x - 3$ and $g(x) = x^5 - x^3 + x^2 - 3$. We have

$$f(x) - g(x) = x^3 + x^2 - 2x = x(x-1)(x+2).$$

Therefore the two functions intersect at $x = -2, 0, 1$ and enclose two regions. The first region is over $[-2, 0]$, on which $f(x) \geq g(x)$. The second region is over $[0, 1]$,

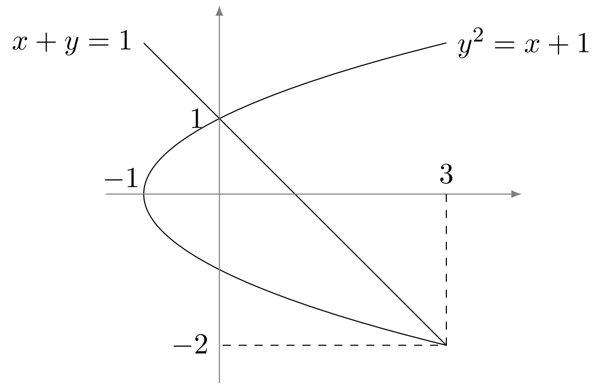


Figure 5.8.10: Region between a parabola and a straight line.

on which $f(x) \le g(x)$. The areas of the regions are

$$\text{Area over } [-2,0] = \int_{-2}^{0}(f(x)-g(x))dx = \frac{8}{3},$$
$$\text{Area over } [0,1] = \int_{0}^{1}(f(x)-g(x))dx = \frac{5}{12}.$$

Note that we may change the functions to $f(x) = x^5 + 2x^2 - 2x - 3 + e^{x^2}$ and $g(x) = x^5 - x^3 + x^2 - 3 + e^{x^2}$ and get the same result. Although it is hard (actually impossible) to compute the exact values of $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$. Yet we can still compute the area.

Exercise 5.8.6. Compute area of the region with the given bounds.

1. $y = \sqrt{x}$, y-axis, $y = 1$.
2. $y = e^x$, $y = x$, on $[0, 1]$.
3. $y = \log x$, $y = x$, $y = 0$, $y = 1$.
4. $y = x^2$, $y = 2x - x^2$.
5. $y = \sin x$, $y = \cos x$, on $[0, \frac{\pi}{4}]$.
6. $y = e^x$, $y = x^2 - 1$, on $[-1, 1]$.
7. $y = \log x$, $y^2 = x + 2$, $y = -1$, $y = 1$.
8. $x = y^2 - 4y$, $x = 2y - y^2$.
9. $y = 2x - x^2$, $x + y = 0$.
10. $y = x$, $y = x + \sin^2 x$, on $[0, \pi]$.

Exercise 5.8.7. Explain that, if $0 \ge f \ge g$ on $[a, b]$, then the area of the region between the graphs of f and g over $[a, b]$ is $\int_a^b (f(x) - g(x))dx$.

Exercise 5.8.8. Explain that, the area of the region between the graphs of f and g over $[a, b]$ is $\int_a^b |f(x) - g(x)|dx$, even when we may have $f > g$ some place and $f < g$ some other place.

In practise, a region is often enclosed by a closed boundary curve (or several closed curves if the region has holes). It is often more convenient to describe curves by their parameterisations. For example, the unit disk is enclosed by the unit circle, which can be conveniently parameterised as $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$.

A parameterisation of a curve can be considered as a movement along the curve, and therefore imposes a direction on the curve. We will always make the standard assumption that, as we move along a parameterised boundary curve, the region is always on the left of curve. Figure 5.8.2 illustrates the meaning of the assumption. For a region without holes, this means that the curve has *counterclockwise* direction. The unit circle parameterisation above is such an example. If the region has holes, then the "inside boundary components" should have *clockwise* direction.

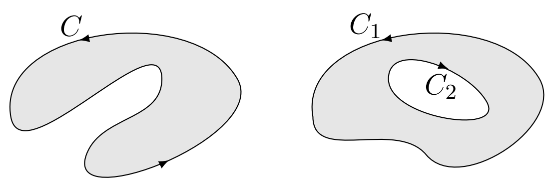


Figure 5.8.11: The region is always on the left of the boundary curve.

Consider a simple region in Figure 5.8.2, such that the boundary curve can be divided into the graphs of two functions $y = y_1(x)$ and $y = y_2(x)$ for $x \in [\alpha, \beta]$. Suppose the boundary curve has parameterisation $\phi(t) = (x(t), y(t))$, $t \in [a, b]$, such that y_1 and y_2 correspond respectively to $t \in [a, c]$ and $t \in [c, b]$. The direction of the parameterisation satisfies our assumption.

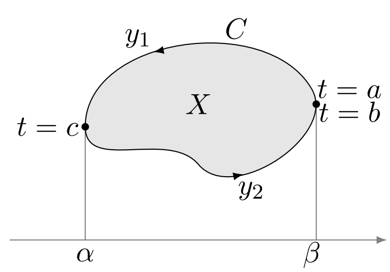


Figure 5.8.12: Calculate the area by integrating along the boundary curve.

The area of the region is $\int_\alpha^\beta (y_1(x) - y_2(x))dx$. We may use the parameterisation of the boundary curve as the change of variable to get the following formula for the area

$$\begin{aligned}\int_\alpha^\beta (y_1(x)-y_2(x))dx &= \int_\alpha^\beta y_1(x)dx - \int_\alpha^\beta y_2(x)dx \\ &= \int_c^a y(t)dx(t) - \int_c^b y(t)dx(t) \\ &= -\int_a^b y(t)dx(t) = -\int_C ydx.\end{aligned}$$

We note that the negative sign comes from our assumption of the direction. The upper part y_1 is supposed to positively contribute $\int_\alpha^\beta y_1(x)dx$ to the area of X. However, the direction of y_1 is leftward, opposite to the x-direction (the direction of dx). This introduces a negative sign. Similarly, the lower part y_2 should negatively contribute to the area, and yet has the rightward direction, the same as the x-direction. This also introduces a negative sign.

We may further use the integration by parts to get another formula for the area

$$-\int_a^b y(t)dx(t) = -y(b)x(b) + y(a)x(a) + \int_a^b x(t)dy(t) = \int_C xdy.$$

Here we have $x(a) = x(b)$ and $y(a) = y(b)$ because the boundary curve is closed. The positive sign on the right can be explained as follows. The area is supposed to be the contribution from the right boundary part subtracting the contribution from the left boundary part. From the picture, we see that the direction of the right part is upward, the same as the y-direction (the direction of dy), and the direction of the left part is downward, opposite to the y-direction.

Example 5.8.11. The boundary circle of the unit disk is parameterised by $x(t) = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$. Since the parameterisation satisfies our assumption, we may use it to calculate the area of the unit disk

$$-\int_0^{2\pi} y(t)dx(t) = \int_0^{2\pi} \sin^2 tdt = \pi.$$

Example 5.8.12. Consider the region enclosed by the Archimedean spiral $x = t\cos t$, $y = t\sin t$, $t \in [0, \pi]$, and the x-axis. The boundary of the region consists of the spiral and the interval $[-\pi, 0]$ on the x-axis. After checking that the direction of the boundary satisfies the assumption, we get the area

$$-\int_0^\pi (t\sin t)(t\cos t)'dt - \int_{-\pi}^0 0dx = -\int_0^\pi (t\sin t\cos t - t^2\sin^2 t)dt = \frac{1}{6}\pi^3.$$

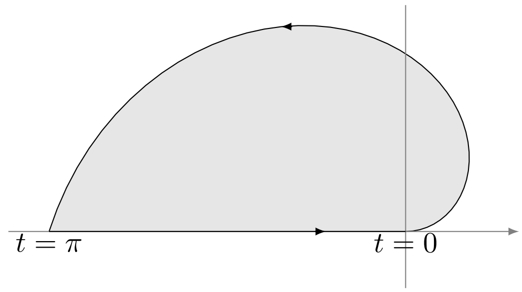


Figure 5.8.13: Archimedean spiral.

Exercise 5.8.9. Explain that the area of the region on the left of Figure 5.8.2 may be calculated by $-\int_C ydx$. You may need to break the boundary into four parts y_1, y_2, y_3, y_4. Moreover, show that the area may also be calculated by $\int_C xdy$.

Exercise 5.8.10. Explain that the area of the region on the right of Figure 5.8.2 may be calculated by $-\int_{C_1} ydx - \int_{C_2} ydx$ and $\int_{C_1} xdy + \int_{C_2} xdy$.

Exercise 5.8.11. Explain that, if the direction of the boundary curve C is opposite to our assumption, then the area is $\int_C ydx$.

Exercise 5.8.12. Compute the areas of the regions enclosed by the curves.

1. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
2. Astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.
3. $\sqrt{|x|} + \sqrt{|y|} = 1$.
4. Hyperbola $x^2 - y^2 = 1$ and $x = a$ $(a > 1)$.
5. Sprial $x = e^t \cos t$, $y = e^t \sin t$, $t \in [0, \pi]$, and the x-axis.
6. One period of the cycloid in Example 5.8.5 and the x-axis.

5.8.3 Surface of Revolution

If we revolve a curve on the plane with respect to a straight line, we get a surface. For example, the sphere is obtained by revolving a circle around any straight line passing through the center of the circle, and the torus is obtained by revolving a circle around any straight line not intersecting the circle.

Let $(x(t), y(t))$, $t \in [a, b]$, be a parametrized curve in the upper half of the (x, y)-plane (this means $y(t) \geq 0$). To find the area of the surface obtained by revolving the curve around the x-axis, we let $A(t)$ be the area of the surface obtained by revolving the $[a,t]$ segment of the curve around the x-axis. Again the subsequent argument ignores the sign.

As the parameter is changed by Δt, the change $\Delta A = A(t+\Delta t) - A(t)$ of the area is the area of surface obtained by revolving the curve segment from $(x(t), y(t))$ to $(x(t+\Delta t), y(t+\Delta t)) = (x,y) + (\Delta x, \Delta y)$. Since the curve segment is approximated by the straight line connecting the two points, the area ΔA is approximated by the area of the revolution of the straight line.

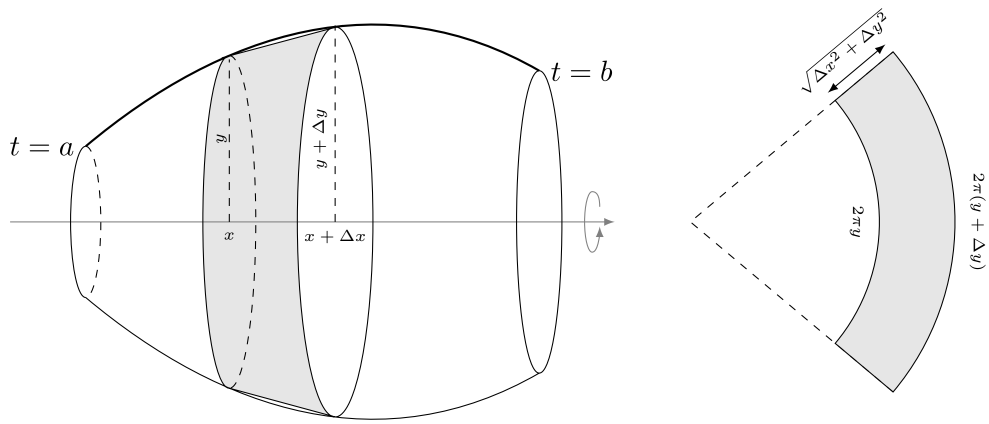


Figure 5.8.14: Area of surface of revolution.

The revolution of the straight line can be expanded to lie on the plane. It is part of the annulus of thickness $\sqrt{\Delta x^2 + \Delta y^2}$. Moreover, the inner arc has length $2\pi y(t)$ and the outer arc has length $2\pi y(t + \Delta t)$. Therefore the area of the partial annulus gives the approximation

$$\Delta A \approx \frac{1}{2}(2\pi y(t) + 2\pi y(t+\Delta t))\sqrt{\Delta x^2 + \Delta y^2} = \pi(y(t) + y(t+\Delta t))\sqrt{\Delta x^2 + \Delta y^2}.$$

Dividing the change Δt of the parameter, we get

$$\frac{\Delta A}{\Delta t} \approx \pi(y(t) + y(t+\Delta t))\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}.$$

The approximation gets more refined as $\Delta t \to 0$. By taking the limit as $\Delta t \to 0$, the approximation becomes an equality

$$\begin{aligned} A'(t) = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} &= \lim_{\Delta t \to 0} \pi(y(t) + y(t+\Delta t))\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \\ &= 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}. \end{aligned}$$

This leads to

$$\text{area of surface of revolution} = A(b) = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}dt.$$

We note that $ds = \sqrt{x'(t)^2 + y'(t)^2}dt$ is used for computing the length of curve, and we can write

$$\text{area of surface of revolution} = 2\pi \int_a^b y(t)ds.$$

Here y is really the distance from the curve to the axis of revolution.

Example 5.8.13. The 2-dimensional sphere of radius r is obtained by revolving the half circle

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \theta \in [0, \pi]$$

around the x-axis. Since the length of circular arc is given by $ds = rd\theta$, the area of the sphere is

$$2\pi \int_0^\pi (r\sin\theta)rd\theta = 4\pi r^2.$$

Example 5.8.14. The torus is obtained by revolving a circle on the upper half plane around the x-axis. Let the radius of the circle be a and let the center of the circle be $(0, b)$. Then $a < b$ and the circle may be parametrized as

$$x = a\cos\theta, \quad y = a\sin\theta + b, \quad \theta \in [0, 2\pi].$$

The length is given by $ds = ad\theta$, so that the area of the torus is

$$2\pi \int_0^{2\pi} (a\sin\theta + b)ad\theta = 4\pi^2 ab.$$

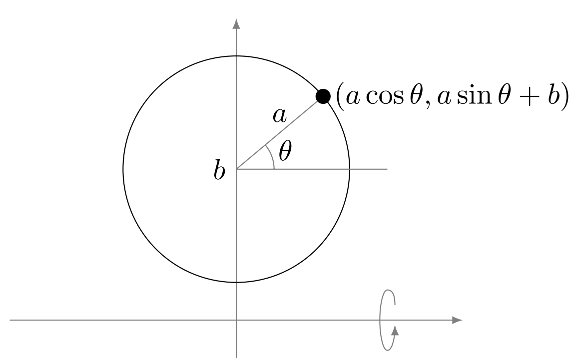


Figure 5.8.15: Torus.

Example 5.8.15. Take the segment $y = x^2$, $x \in [0, 1]$ of the parabola in Example 5.8.2. If we revolve the parabola around the x-axis, then $ds = \sqrt{1+4x^2}dx$, the distance from the curve to the axis of rotation (i.e., the x-axis) is x^2. Therefore the area of the surface of revolution is

$$2\pi \int_0^1 x^2\sqrt{1+4x^2}dx$$

With the help of Example 5.5.15 and the computation in Example 5.8.2, we have

$$\begin{aligned}
\int_0^1 x^2\sqrt{1+4x^2}dx &= \frac{1}{8}\int_0^2 x^2\sqrt{1+x^2}dx = \frac{1}{8}\int_0^2 \left((1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}}\right) dx \\
&= \frac{1}{8}\left(\frac{1}{2\cdot\frac{3}{2}+1}x(1+x^2)^{\frac{3}{2}}\Big|_0^2 + \left(\frac{2\cdot\frac{3}{2}}{2\cdot\frac{3}{2}+1} - 1\right)\int_0^2 (1+x^2)^{\frac{1}{2}}dx\right) \\
&= \frac{1}{8}\left(\frac{2}{5}5^{\frac{3}{2}} - \frac{2}{5}\left(\sqrt{5} + \frac{1}{2}\log(2+\sqrt{5})\right)\right) = \frac{1}{\sqrt{5}} - \frac{1}{40}\log(2+\sqrt{5}).
\end{aligned}$$

So the area is $\dfrac{2\pi}{\sqrt{5}} - \dfrac{\pi}{20}\log(2+\sqrt{5})$.

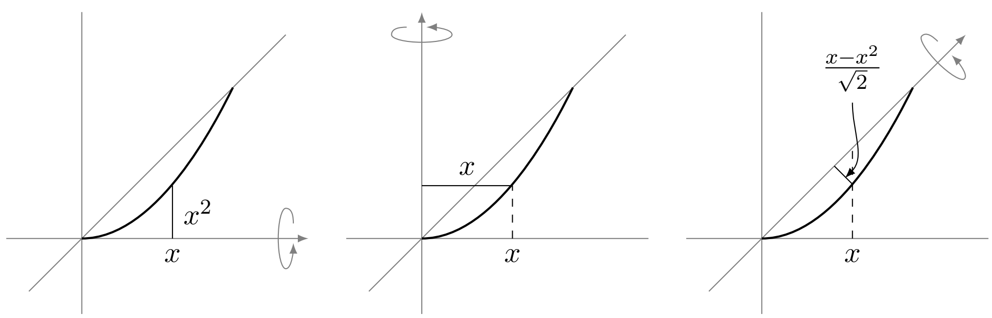


Figure 5.8.16: Revolving a parabola segment around different axes.

If we revolve around the y-axis, then we get a paraboloid. We still have $ds = \sqrt{1+4x^2}dx$, but the distance from the curve to the axis of rotation (i.e., the y-axis) is now x. Therefore the area of the paraboloid is

$$2\pi \int_0^1 x\sqrt{1+4x^2}dx = 2\pi \, \frac{2}{3\cdot 8}(1+4x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{\pi}{6}(5\sqrt{5}-1).$$

Finally, if we revolve around the diagonal $y = x$, then the distance from the curve to the axis of rotation is $\dfrac{x-x^2}{\sqrt{2}}$, and the area is

$$\begin{aligned}
2\pi \int_0^1 \frac{x-x^2}{\sqrt{2}}\sqrt{1+4x^2}dx &= \frac{1}{\sqrt{2}}\left(\frac{\pi}{6}(5\sqrt{5}-1) - \frac{2\pi}{\sqrt{5}} + \frac{\pi}{20}\log(2+\sqrt{5})\right) \\
&= \sqrt{2}\pi\left(\frac{13}{60}\sqrt{5} - \frac{1}{12} - \frac{1}{40}\log(2+\sqrt{5})\right).
\end{aligned}$$

Example 5.8.15 shows how to adapt the formula for the area of the surface of revolution to the more general case of any parametrized curve $(x(t), y(t))$ with respect to a straight line $\alpha x + \beta y + \gamma = 0$. Assume the curve is on the "positive side" of the straight line

$$\alpha x(t) + \beta y(t) + \gamma \geq 0, \quad \text{for all } t \in [a, b].$$

Then the distance $y(t)$ should be replaced by $\dfrac{\alpha x(t) + \beta y(t) + \gamma}{\sqrt{\alpha^2 + \beta^2}}$. We still have $ds = \sqrt{x'(t)^2 + y'(t)^2}dt$. Therefore we get the general formula

$$\text{area of surface of revolution} = 2\pi \int_a^b \frac{(\alpha x(t) + \beta y(t) + \gamma)\sqrt{x'(t)^2 + y'(t)^2}}{\sqrt{\alpha^2 + \beta^2}} dt.$$

Exercise 5.8.13. Find the formula for the area of the surface of revolution of the graph of a function $y = f(x)$ around the x-axis. What about revolving around the y-axis? What about revolving around the line $x = a$?

Exercise 5.8.14. Find the area of the surface of revolution.

1. $y = x^3$, $x \in [0, 2]$, around x-axis.
2. $x^2 = 2py$, $x \in [0, 1]$, around y-axis.
3. $y = e^x$, $x \in [0, 1]$, around x-axis.
4. $y = e^x$, $x \in [0, 1]$, around y-axis.
5. $y = e^x$, $x \in [0, 1]$, around $y = 1$.
6. $y = \tan x$, $x \in [0, \frac{\pi}{4}]$, around x-axis.
7. $y^2 = \dfrac{e^x + e^{-x}}{2}$, $x \in [-a, a]$, around x-axis.
8. $y^2 = x^3$, $x \in [0, 1]$, around x-axis.
9. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, around x-axis.
10. Astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, around x-axis.

Exercise 5.8.15. Find the area of the surface obtained by revolving one period of the cycloid in Example 5.8.5 around the x-axis.

Exercise 5.8.16. Find the area of the surface obtained by revolving the involute of the circle in Example 5.8.5 around the x-axis.

5.8.4 Solid of Revolution

If we revolve a region in the plane with respect to a straight line, we get a solid. For example, the ball is obtained by revolving a disk around any straight line passing through the center of the disk, and the solid torus is obtained by revolving a disk around any straight line not intersecting the disk.

Consider the region $G_{[a,b]}(f)$ for a function $f(x) \geq 0$ over the interval $[a, b]$. To find the volume of the solid obtained by revolving the region around the x-axis, we let $V(x)$ be the volume of the part of solid obtained by revolving $G_{[a,x]}(f)$ around the x-axis. Then the change $\Delta V = V(x+\Delta x) - V(x)$ is the volume of the solid obtained by revolving $G_{[x,x+\Delta x]}(f)$.

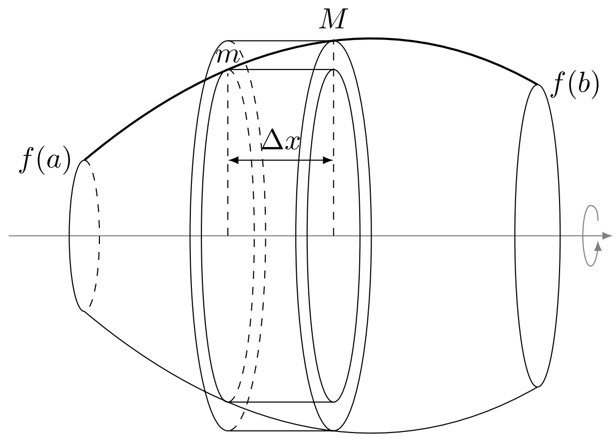


Figure 5.8.17: Volume of solid of revolution.

Let $m = \min_{[x,x+\Delta x]} f$ and $M = \max_{[x,x+\Delta x]} f$. Then $G_{[x,x+\Delta x]}(f)$ is sandwiched between the rectangles $[x, x+\Delta x] \times [0, m]$ and $[x, x+\Delta x] \times [0, M]$. Therefore the revolution of $G_{[x,x+\Delta x]}(f)$ is sandwiched between the revolutions of the two rectangles. The revolutions of rectangles are cylinders and have volumes $\pi m^2 \Delta x$ and $\pi M^2 \Delta x$. Therefore we get

$$\pi m^2 \Delta x \leq \Delta V \leq \pi M^2 \Delta x.$$

This implies

$$\pi m^2 \leq \frac{\Delta V}{\Delta x} \leq \pi M^2.$$

If f is continuous, then $\lim_{\Delta x \to 0} m = \lim_{\Delta x \to 0} M = f(x)$. By the sandwich rule, we get

$$V'(x) = \lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \pi f(x)^2.$$

This leads to

$$\text{volume of solid of revolution} = V(b) = \pi \int_a^b f(x)^2 dx.$$

Example 5.8.16. The 3-dimensional ball of radius r is obtained by revolving the half disk around the x-axis. The half disk is the region between $\sqrt{r^2 - x^2}$ and the x-axis over $[-r, r]$. Therefore the volume of the ball is

$$\pi \int_{-1}^{1} (\sqrt{r^2 - x^2})^2 dx = \frac{4}{3}\pi r^3.$$

Example 5.8.17. The solid torus is obtained by revolving a disk in the upper half plane around the x-axis. Let the radius of the disk be a and let the center of the disk be $(0, b)$. Then $a < b$ and the disk is the region between $y_1(x) = b + \sqrt{a^2 - x^2}$ and $y_2(x) = b - \sqrt{a^2 - x^2}$ over the interval $[-a, a]$. The torus is the solid obtained by revolving $G_{[-a,a]}(y_1)$ subtracting the solid obtained by revolving $G_{[-a,a]}(y_2)$. Therefore the volume of the torus is the volume of the first solid subtracting the second

$$\begin{aligned}
\pi\int_{-a}^{a} y_1(x)^2dx - \pi\int_{-a}^{a} y_2(x)^2dx &= \pi\int_{-a}^{a}(y_1(x)^2 - y_2(x)^2)dx \\
&= \pi\int_{-a}^{a}((b+\sqrt{a^2-x^2})^2 - (b-\sqrt{a^2-x^2})^2)dx \\
&= \pi\int_{-a}^{a} 4b\sqrt{a^2-x^2}dx \\
&= 4\pi b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2 t dt = 2\pi^2a^2b.
\end{aligned}$$

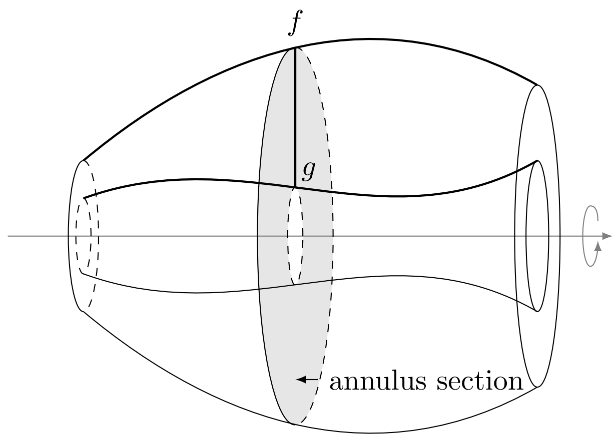


Figure 5.8.18: Solid of revolution of the region between two functions.

Example 5.8.17 shows that, if $f \geq g \geq 0$ on $[a, b]$, then the volume of the solid of revolution obtained by revolving the region between f and g over $[a, b]$ around the x-axis is

$$\pi\int_{a}^{b}(f(x)^2 - g(x)^2)dx.$$

Now we extend the discussion before Example 5.8.11 about calculating the area of a plan region by the integrating along the boundary curve. Suppose the region X in Figure 5.8.2 lies in the upper half plane. Then similar to the earlier discussion, the volume of the solid obtained by rotating X around the x-axis is

$$\begin{aligned}\pi\int_\alpha^\beta (y_1(x)^2 - y_2(x)^2)dx &= \pi\int_\alpha^\beta y_1(x)^2dx - \pi\int_\alpha^\beta y_2(x)^2dx \\ &= \pi\int_c^a y(t)^2dx(t) - \pi\int_c^b y(t)^2dx(t) \\ &= -\pi\int_a^b y(t)^2dx(t) = -\pi\int_C y^2dx.\end{aligned}$$

So all the earlier discussion about the area can be applied to the volume of the solid of revolution.

Example 5.8.18. The volume of the 3-dimensional ball of radius r is obtained by revolving the half disk around the x-axis. The boundary of the half disk consists of the half circle $x = \cos t$, $y = \cos t$, $t \in [0, \pi]$, and the interval $[-1, 1]$ on the x-axis. Moreover, the parameterisation of the boundary curve satisfies our assumption. Therefore the volume of the ball is

$$\pi\int_0^\pi (r\sin t)^2 d(r\cos t) + \pi\int_{-1}^1 0^2dx = \pi r^3\int_0^\pi (1-\cos^2 t)d(\cos t) = \frac{4}{3}\pi r^3.$$

Example 5.8.19. We use the parameterisation $x = a\cos t$, $y = b + a\sin t$, $t \in [0, 2\pi]$ of the circle to calculate the volume of the torus in Example 5.8.17

$$\begin{aligned}-\pi\int_0^{2\pi} (b + a\sin t)^2 d(a\cos t) &= \pi a\int_0^{2\pi} (b^2\sin t + 2ab\sin^2 t + a^2\sin^2 t)dt \\ &= 2\pi^2a^2b.\end{aligned}$$

Example 5.8.20. Consider the region enclosed by the Archimedean spiral and the x-axis in Example 5.8.20. The volume of the solid obtained by revolving the region around the x-axis is

$$-\int_0^\pi (t\sin t)^2 d(t\cos t) = -\int_0^\pi (t^2\sin^2 t\cos t - t^3\sin^3 t)dt = \frac{2}{3}\pi^3 - 4\pi.$$

Exercise 5.8.17. Find volume of the solid obtained by revolving the region in Exercise 5.8.12 around the x-axis.

Exercise 5.8.18. Use integration by parts to explain that the volume of the solid of revolution can also be calculated by $2\pi\int_C xydy$. Then use this formula to calculate the volumes of the ball, the torus, and the solids obtained by revolving the regions in Exercise 5.8.12 around the x-axis.

The formula will be the "shell method" in Example 5.8.27.

Exercise 5.8.19. Explain that, if the direction of the boundary curve C is opposite to our assumption, then the volume of the solid of revolution around the x-axis is $\pi \int_C y^2 dx$.

Exercise 5.8.20. For the solid obtained by revolving a region in the lower half plane around the x-axis, how should the formula $-\pi \int_C y^2 dx$ for the volume be modified?

Next we consider the general case of revolving a region X around a straight line $L\colon \alpha x + \beta y + \gamma = 0$. We assume X is on the "positive side" of L in the sense that the parameterisation $(x(t), y(t))$ of the boundary curve C of X satisfies

$$\alpha x(t) + \beta y(t) + \gamma \geq 0, \text{ for all } t \in [a, b].$$

Moreover, we still assume that the direction of C satisfies our assumption. Then in the formula $-\pi \int_C y^2 dx$, y should be understood as the distance $\dfrac{\alpha x(t) + \beta y(t) + \gamma}{\sqrt{\alpha^2 + \beta^2}}$ from C to L, and dx should be understood as the progression

$$\frac{\beta dx - \alpha dy}{\sqrt{\alpha^2 + \beta^2}} = \frac{\beta x'(t) - \alpha y'(t)}{\sqrt{\alpha^2 + \beta^2}} dt$$

along the direction $\dfrac{(\beta, -\alpha)}{\sqrt{\alpha^2 + \beta^2}}$ of L. Therefore the volume of the solid of revolution is

$$-\pi \int_a^b \frac{(\alpha x(t) + \beta y(t) + \gamma)^2 (\beta x'(t) - \alpha y'(t))}{(\sqrt{\alpha^2 + \beta^2})^3} dt$$

We note that the negative sign is due to the mismatch (See Figure 5.8.4) of the direction of the boundary curve and the direction of the progression along L. In general, we may determine the sign by comparing the direction of the parameter and the direction of progression.

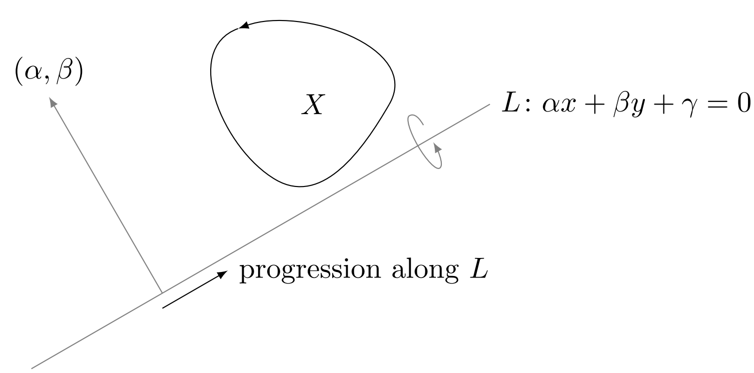


Figure 5.8.19: Revolving a region X around a line.

For the special case that X is above the horizontal line $y = b$ ($(\alpha,\beta) = (0,1)$), the volume of the solid of revolution around the line is $-\pi \int_C (y-b)^2 dx$. If X is on the *right* of the y-axis (i.e., the line $x = 0$, with $(\alpha,\beta) = (1,0)$), then the volume of the solid of revolution around the y-axis is

$$-\pi \int_a^b x(t)^2(-y'(t))dt = \pi \int_C x^2 dy.$$

The negative sign in front of y' comes from the fact that the progression for the line $x = 0$ goes *downwards*, the opposite of the y-direction. If X is on the *right* of the vertical line $x = a$, then the volume of the solid of revolution around the vertical line is $\pi \int_C (x-a)^2 dy$.

Example 5.8.21. Take the segment $y = x^2$, $x \in [0,1]$, of the parabola in Example 5.8.2. If we revolve the region X between the parabola and the x-axis around the x-axis, then the volume of the solid is

$$\pi \int_{x=0}^{x=1} y^2 dx = \pi \int_0^1 (x^2)^2 dx = \frac{1}{5}\pi.$$

If we revolve X around the y-axis, then the volume of the solid is

$$\pi \int_{x=0}^{x=1} x^2 dy = \pi \int_{y=0}^{y=1} y dy = \frac{1}{2}\pi.$$

Let Y be the region between the parabola and the vertical line $x = 1$. If we revolve Y around the vertical line $x = 1$, then the volume of the solid is

$$\pi \int_{x=0}^{x=1} (1-x)^2 dy = \pi \int_0^1 (1-x)^2 d(x^2) = \frac{7}{6}\pi.$$

If we revolve Y around the y-axis instead, then the volume of the solid is

$$\pi \int_{x=0}^{x=1} (1^2 - x^2) dy = \pi \int_{y=0}^{y=1} (1-y) dy = \frac{1}{2}\pi.$$

Let Z be the region between the parabola $y = x^2$ and the diagonal $y = x$. If we revolve Z around the x-axis, then the volume of the solid is

$$\pi \int_{x=0}^{x=1} (x^2 - (x^2)^2) dx = \frac{2}{15}\pi.$$

If we revolve Z around the line $x = -1$, then the volume of the solid is

$$\begin{aligned}\pi \int_{x=0}^{x=1} ((x+1)^2 d(x^2) - (x+1)^2 dx) &= \pi \int_0^1 (x+1)^2(2x-1)dx \\ &= \pi \int_1^2 z^2(2z-3)dz = \frac{1}{2}\pi.\end{aligned}$$

If we revolve Z around the diagonal $y = x$, then to make sure the region is on the positive side of the diagonal, we should write the diagonal as $x - y = 0$, with $(\alpha, \beta) = (1, -1)$. The distance between the parabola and the diagonal is $\dfrac{x - x^2}{\sqrt{2}}$. The progression of the parabola in the direction $\dfrac{(\beta, -\alpha)}{\sqrt{\alpha^2 + \beta^2}} = \dfrac{(-1, -1)}{\sqrt{2}}$ of the line is

$$\frac{-dx - dy}{\sqrt{2}} = \frac{-dx - d(x^2)}{\sqrt{2}} = \frac{-(1 + 2x)}{\sqrt{2}} dx.$$

Therefore the volume of the solid is

$$-\pi \int_{x=0}^{x=1} \left(\frac{x - x^2}{\sqrt{2}} \right)^2 \frac{-(1 + 2x)}{\sqrt{2}} dx = \frac{1}{30\sqrt{2}} \pi.$$

If we revolve Z around the line $y = x - 1$, we should write the line as $-x + y + 1 = 0$ for Z to be on the positive side, with $(\alpha, \beta) = (-1, 1)$. The distance from the diagonal and the parabola to the line are $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{2}} - \dfrac{x - x^2}{\sqrt{2}}$. The progressions of the diagonal and the parabola in the direction $\dfrac{(\beta, -\alpha)}{\sqrt{\alpha^2 + \beta^2}} = \dfrac{(1, 1)}{\sqrt{2}}$ of the line are

$$\frac{dx + dx}{\sqrt{2}} = \sqrt{2} dx \text{ and } \frac{dx + d(x^2)}{\sqrt{2}} = \frac{1 + 2x}{\sqrt{2}} dx.$$

Therefore the volume of the solid is

$$\pi \int_{x=0}^{x=1} \left(\left(\frac{1}{\sqrt{2}} \right)^2 \sqrt{2} dx - \left(\frac{1}{\sqrt{2}} - \frac{x - x^2}{\sqrt{2}} \right)^2 \frac{1 + 2x}{\sqrt{2}} dx \right) = \frac{3}{10\sqrt{2}} \pi.$$

Exercise 5.8.21. Let $A \le f(x) \le B$ for $x \in [a, b]$. Find the formula for the volume of the solid of revolution of the region between the graph of function f and $y = A$ around the line $y = C$, where $C \notin (A, B)$.

Exercise 5.8.22. Find the formula for the volume of the solid obtained by revolving a region X for which the parameterised boundary has the right direction.

1. X is on the left of the y-axis, around the y axis.

2. X is on the left of $x = a$, around $x = a$.

3. X is below $y = b$, around $y = b$.

4. X is on the negative side of $x + y = 0$, around $x + y = 0$.

5. X is on the negative side of $x + y = 1$, around $x + y = 1$.

Exercise 5.8.23. Find the volume of the solid obtained by revolving the region between the curve and the axis of revolution.

1. $y = x^3$, $x \in [0, 2]$, around x-axis.
2. $x^2 = 2py$, $x \in [0, 1]$, around y-axis.
3. $y = e^x$, $x \in [0, 1]$, around x-axis.
4. $y = e^x$, $x \in [0, 1]$, around y-axis.
5. $y = e^x$, $x \in [0, 1]$, around $x = 1$.
6. $y = \tan x$, $x \in \left[0, \dfrac{\pi}{4}\right]$, around x-axis.
7. $y^2 = \dfrac{e^x + e^{-x}}{2}$, $x \in [-a, a]$, around x-axis.
8. $y^2 = x^3$, $x \in [0, 1]$, around x-axis.
9. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, around x-axis.
10. Astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} \le 1$, around x-axis.

Exercise 5.8.24. Find the volume of the solid of revolution.

1. Region bounded by $y = x$ and $y = x^2$, around x-axis.
2. Region bounded by $y = x$ and $y = x^2$, around $x = 2$.
3. Region bounded by $y = x$ and $y = x^2$, around $x = y$.
4. Region bounded by $y = x$ and $y = x^2$, around $x + y = 0$.
5. Region bounded by $y^2 = x + 1$ and $x + y = 1$, around $x + y = 1$.
6. Region bounded by $y^2 = x + 1$ and $x + y = 1$, around $x = 3$.
7. Region bounded by $y^2 = x + 1$ and $x + y = 1$, around $y = 1$.
8. Region bounded by $y = \log x$, $y = 0$, $y = 1$, and y-axis, around y-axis.
9. Region bounded by $y = \cos x$ and $y = \sin x$, around $y = 1$.
10. Triangle with vertices $(0, 0), (1, 2), (2, 1)$, around x-axis.
11. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, around $y = b$.
12. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, around $bx + ay = 2ab$.
13. Region bounded by $y = \dfrac{1}{1 + |x|}$ and the x-axis, around x-axis.
14. Region bounded by $y = e^{-|x|}$ and the x-axis, around x-axis.

5.8.5 Cavalieri's Principle

The formulae for the area of surface of revolution and the volume of solid of revolution follow from a more general principle.

In general, an n-dimensional solid X has n-dimensional size. For $n=1$, X is a curve and the size is the length. For $n=2$, X is a region in $\mathbb{R}^2$ or more generally a surface, and the size is the area. For $n=3$, X is typically a region in $\mathbb{R}^3$ but can also be a "3-dimensional surface" such as the 3-dimensional sphere in $\mathbb{R}^4$, and the size is the volume.

To find the size of an n-dimensional solid X, we may decompose X into sections X_t of one lower dimension (i.e., X_t has dimension $n-1$). For 2-dimensional X, this means that X is decomposed into a one parameter family of curves. For 3-dimensional X, this means that X is decomposed into a one parameter family of surfaces. The decomposition is *equidistant* if the distance between two nearby pieces does not depend on the location where the distance is measured. In this case, we have the distance function $s(t)$, such that the distance between the sections X_t and $X_{t+\Delta t}$ is $\Delta s = s(t+\Delta t) - s(t)$. If X spans from distance $s=a$ to distance $s=b$, then

$$\text{size of } X = \int_{t=a}^{t=b} \text{size}(X_t)ds = \int_a^b \text{size}(X_t)s'(t)dt.$$

A consequence of the formula is the following *principle of Cavalieri*: If two solids X and Y have equidistant decompositions X_t and Y_t, such that X_t and Y_t have the same size, and the distance between X_t and $X_{t'}$ is the same as the distance between Y_t and $Y_{t'}$, then X and Y have the same size.

Example 5.8.22. Let X be a region inside the plane. We decompose X by intersecting with vertical lines $X_x = X \cap x \times \mathbb{R}$. The decomposition is equidistant, with the x-coordinate as the distance. Thus the area of X is $\displaystyle\int_a^b \text{length}(X_x)dx$. In the special case X is the region between $f(x)$ and $g(x)$, where $f(x) \geq g(x)$, the section X_x is the interval $[g(x), f(x)]$ and has length $f(x) - g(x)$. Then we recover the formula $\displaystyle\int_a^b (f(x) - g(x))dx$ in Section 5.8.2.

Example 5.8.23. Let X be a region inside $\mathbb{R}^3$. We decompose X by intersecting with vertical planes $X_x = X \cap x \times \mathbb{R}^2$. The decomposition is equidistant, with the x-coordinate as the distance. Thus the volume of X is $\displaystyle\int_a^b \text{area}(X_x)dx$. In the special case X is obtained by rotating the region between $f(x)$ and $g(x)$, where $f(x) \geq g(x) \geq 0$, around the x-axis, the section X_x is the annulus with outer radius $f(x)$ and inner radius $g(x)$. The section has area $\pi(f(x)^2 - g(x)^2)$, and we get the

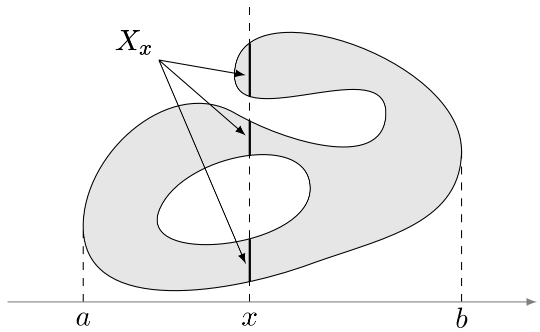


Figure 5.8.20: Area of a region in $\mathbb{R}^2$.

formula $\pi \int_a^b (f(x)^2 - g(x)^2)dx$ for the solid of revolution.

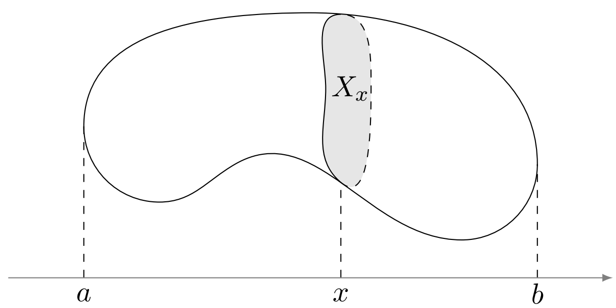


Figure 5.8.21: Volume of a solid in $\mathbb{R}^3$.

Example 5.8.24. Let R be a region in the plane. Let P be a point not in the plane. Connecting P to all points in R by straight lines produces the pyramid X with base R and apex P.

We may put R on the (x, y)-plane in $\mathbb{R}^3$ and assume that $P = (0, 0, h)$ lies in the positive z-axis, where h is the distance from P to the plane. Let A be the area of R. We decompose the pyramid by the horizontal planes, so that z is the distance. The section X_z is similar to R, so that the area of X_z is proportional to the square of its distance $h - z$ to P. We find the area of X_z to be $\left(\frac{h-z}{h}\right)^2 A$, and the volume of the pyramid is

$$\int_0^h \left(\frac{h-z}{h}\right)^2 A dx = \frac{1}{3}hA.$$

Example 5.8.25. Let X be the intersection of two round solid cylinders of radius 1 in

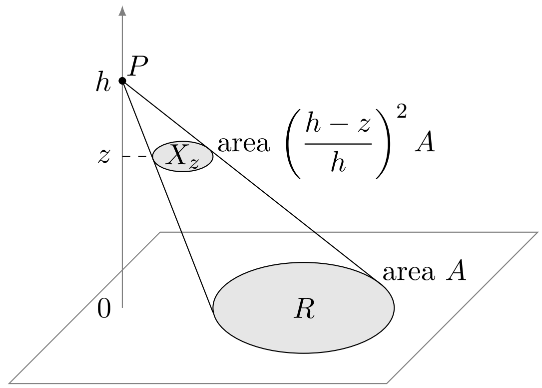


Figure 5.8.22: Pyramid.

orthogonal position. We put the two cylinders in $\mathbb{R}^3$, by assuming the two cylinders to be $x^2+y^2 \le 1$ and $x^2+z^2 \le 1$. Then we decompose the solid by intersecting with the planes perpendicular to the x-axis. The section X_x is a square of side length $2\sqrt{1-x^2}$ and therefore has area $4(1-x^2)$. The volume of the intersection solid X is

$$\int_{-1}^{1} 4(1-x^2)dx = \frac{16}{3}.$$

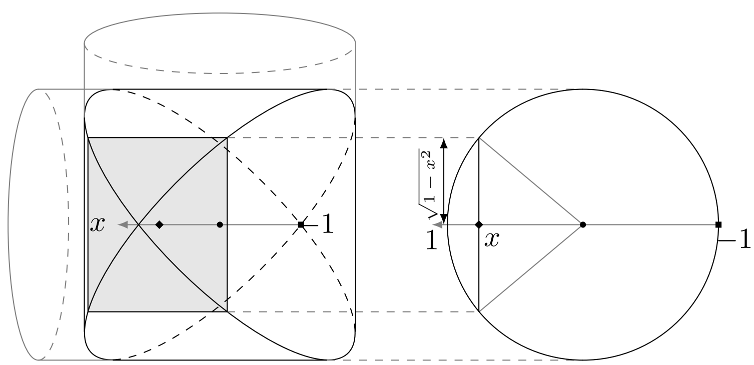


Figure 5.8.23: Orthogonal intersection of two cylinders.

Exercise 5.8.25. Explain the formula in Section 5.8.3 for the area of surface of revolution by using suitable equidistant decomposition.

Exercise 5.8.26. Explain that if a solid is stretched by a factor A in the x-direction, by B in the y-direction, and by C in the z-direction, then the volume of the solid is multiplied by the factor ABC.

Exercise 5.8.27. Find the volume of solid.

1. Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.
2. Solid bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and $z=\pm c$.
3. Intersection of the sphere $x^2+y^2+z^2\le 1$ and the cylinder $x^2+y^2\le x$.
4. Solid bounded by $x+y+z^2=1$ and inside the first quadrant.

Exercise 5.8.28. Find the volume of solid.

1. A solid with a disk as the base, and the parallel sections perpendicular with the base are equilateral triangles.
2. A solid with a disk as the base, and the parallel sections perpendicular with the base are squares.
3. Cylinder cut by two planes, one is perpendicular to the cylinder and the other form angle α with the cylinder. The two planes do not intersect inside the cylinder.
4. Cylinder cut by two planes forming respective angles α and β with the cylinder. The two planes do not intersect inside the cylinder.
5. A wedge cut out of a cylinder, by two planes forming respective angles α and β with the cylinder, such that the intersection of two planes is a diameter of the cylinder.

So far we used parallel lines and planes to construct the decomposition. We may also use equidistant curves and surfaces to construct the decomposition.

Example 5.8.26. We may decompose a region X in the plane by concentric circles. The decomposition is equidistant, with the radius r of the circles as the distance. The section X_r consists of the points in X of distance r from the origin and is typically an arc from angle $\phi(r)$ to angle $\psi(r)$. The length of the arc X_r is $(\psi(r)-\phi(r))r$, so that the area of X is $\int_a^b(\psi(r)-\phi(r))rdr$.

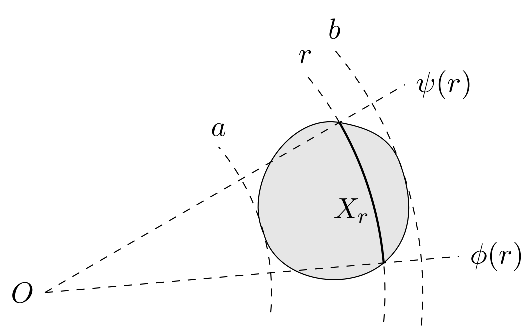


Figure 5.8.24: Equidistant decomposition by concentric circles.

For example, for the disk centered at $(1,0)$ and of radius 1, we have $\phi = -\arccos\dfrac{r}{2}$ and $\psi = \arccos\dfrac{r}{2}$, $r \in [0,2]$. The area of the disk of radius 1 is (taking $t = \arccos\dfrac{r}{2}$, $r = 2\cos t$)

$$\begin{aligned}\int_0^2 2r\arccos\frac{r}{2}dr &= \int_{\frac{\pi}{2}}^0 2t(2\cos t)d(2\cos t) = 8\int_0^{\frac{\pi}{2}} t\cos t\sin t dt\\ &= 4\int_0^{\frac{\pi}{2}} t\sin 2t dt = \int_0^{\pi} u\sin u du = \pi.\end{aligned}$$

Example 5.8.27. Let X be a region in the right plane (i.e., the right side of y-axis). Let Y be the solid obtained by revolving X around the y-axis. We may use the cylinders centered at the y-axis to decompose Y. The decomposition is equidistant, with x as the distance. Let X_x be the intersection of X with the vertical line $x \times \mathbb{R}$. Then the section Y_x is the cylinder obtained by revolving X_x around the y-axis. The area of the section is $2\pi x(\text{length of } X_x)$. Therefore the volume of the solid of revolution is $2\pi\displaystyle\int_a^b x(\text{length of } X_x)dx$. In particular, if X is the region between functions $f(x)$ and $g(x)$, where $f(x) \geq g(x)$ on $[a,b]$, then the volume is $2\pi\displaystyle\int_a^b x(f(x)-g(x))dx$.

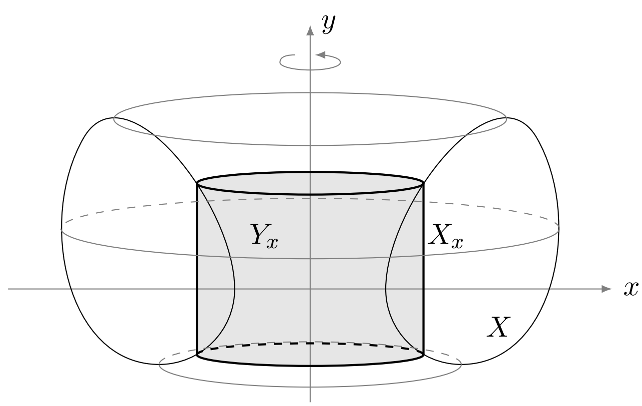


Figure 5.8.25: Equidistant decomposition by concentric cylinders.

For example, consider the solid torus in Example 5.8.17. The disk is the region between $x = \sqrt{a^2-(y-b)^2}$ and $x = -\sqrt{a^2-(y-b)^2}$, for $y \in [b-a, b+a]$. If we use the formula above (note that x and y are exchanged), we get the volume of the solid torus

$$\begin{aligned}2\pi\int_{b-a}^{b+a} y(2\sqrt{a^2-(y-b)^2})dy &= 4\pi\int_{-a}^{a}(t+b)\sqrt{a^2-t^2})dt\\ &= 8\pi\int_0^a b\sqrt{a^2-t^2}dt = 4\pi^2a^2b.\end{aligned}$$

Exercise 5.8.29. Compute the volumes of the solids of revolution in Example 5.8.21 by using the formula in Example 5.8.27.

Exercise 5.8.30. Compute the volumes of the solids of revolution in Exercise 5.8.23 by using the formula in Example 5.8.27.

Exercise 5.8.31. Compute the volumes of the solids of revolution in Exercise 5.8.24 by using the formula in Example 5.8.27.

Exercise 5.8.32. After Example 5.8.21, we presented the formula for computing the volume of a solid obtained by revolving a region in $\mathbb{R}^2$ bounded by a parameterized curve. Can you derive the similar formula by using the idea from Example 5.8.27?

Exercise 5.8.33. In Section 5.8.4 and Example 5.8.27, we have two ways of computing the volume of a solid of revolution. For the following simple case, explain that the two ways give the same result: Let $f(x)$ be and invertible non-negative function on $[0, a]$, such that $f(a) = 0$ and both $f(x)$ and $f^{-1}(y)$ are continuously differentiable. The solid is obtained by revolving the region between the graph of f and the two axis.

Example 5.8.28. Finally, we compute the size of high dimensional objects. Let α_n be the volume of the n-dimensional sphere S^n of radius 1. Then

$$\alpha_0 = 2, \quad \alpha_1 = 2\pi, \quad \alpha_2 = 4\pi.$$

Moreover, the n-dimensional sphere of radius r has volume $\alpha_n r^n$.

To compute α_n, we decompose S^n by intersecting with "horizontal hyperplanes". The hyperplanes are indexed by the angle t. The section at angle t is the $(n-1)$-dimensional sphere S^{n-1} of radius $\cos t$, and form an equidistant decomposition. In fact, the angle t can be used to measure the distance between the sections. Since the section at t has volume $\alpha_{n-1}\cos^{n-1} t$ and the range of t is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we conclude that

$$\alpha_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_{n-1}\cos^{n-1} t dt = 2\alpha_{n-1} I_{n-1},$$

where $I_{n-1} = \int_0^{\frac{\pi}{2}} \cos^{n-1} t dt = \int_0^{\frac{\pi}{2}} \sin^{n-1} t dt$ has been computed in Example 5.5.14

$$I_{2k} = \frac{(2k)!}{2^{2k+1}(k!)^2}\pi, \quad I_{2k+1} = \frac{2^{2k}(k!)^2}{(2k+1)!}.$$

Thus

$$\alpha_n = 2\alpha_{n-1} I_{n-1} = 4\alpha_{n-2} I_{n-1} I_{n-2} = 4\alpha_{n-2}\frac{\pi}{2(n-1)} = \frac{2\pi}{n-1}\alpha_{n-2}.$$

By the values of α_1 and α_2, we conclude that

$$\alpha_n = \begin{cases} \dfrac{(2\pi)^{\frac{n}{2}}}{2 \cdot 4 \cdots (n-2)}, & \text{if } n \text{ is even,} \\ \dfrac{2(2\pi)^{\frac{n-1}{2}}}{1 \cdot 3 \cdots (n-2)}, & \text{if } n \text{ is odd.} \end{cases}$$

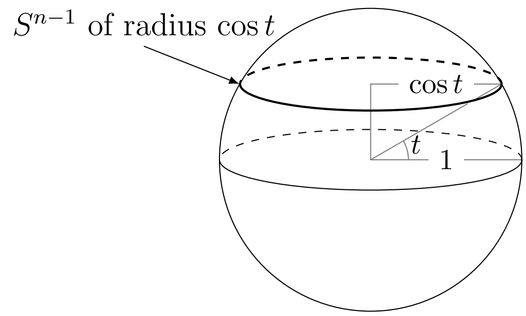


Figure 5.8.26: Decomposing n-dimensional sphere of radius 1.

Exercise 5.8.34. Let β_n be the volume of the ball B^n of radius 1.

1. Similar to Example 5.8.28, use the intersection with horizontal hyperplanes to derive the relation between β_n and β_{n-1}. Then use the special values β_1 and β_2, and Example 5.5.14 to compute β_n.

2. Use the decomposition of B^n by concentric $(n-1)$-dimensional spheres to derive the relation between β_n and α_{n-1}. Then use Example 5.8.28 to find β_n.

The two methods should give the same result.

Exercise 5.8.35. Suppose R is a region in $\mathbb{R}^{n-1}$ with volume. Suppose P is a point in $\mathbb{R}^n$ of distance h from $\mathbb{R}^{n-1}$. By connecting P to all points of R by straight lines, we get a pyramid X with base R and apex P. Find the relation between the volumes of X and R.

5.9 Polar Coordinate

The polar coordinate locates a point on the plane by its distance r to the origin and the angle θ indicating the direction from the viewpoint of origin. It is roughly related to the cartesian coordinates (x, y) by

$$x = r\cos\theta, \quad y = r\sin\theta; \quad r = \sqrt{x^2+y^2}, \quad \theta = \arctan\frac{y}{x}.$$

We say "roughly" because the relation between (x, y) and (r, θ) is not a one-to-one correspondence. For example, the last formula literally restricts θ, as the value of inverse tangent function, to be within $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In fact, the angle for a point in the plane is unique only up to adding an integer multiple of 2π, and is more precisely determined by

$$(\cos\theta, \sin\theta) = \frac{(x, y)}{\sqrt{x^2 + y^2}}.$$

Another way to say this is that θ is unique if we restrict to $[0, 2\pi)$ (or $[-\pi, \pi)$, etc.).

For the convenience of presenting polar equations, we also allow r to be negative, by specifying that $(-r, \theta)$ and $(r, \theta + \pi)$ represent the same point. In other words, $(-r, \theta)$ and (r, θ) are symmetric with respect to the origin. The cost of such extension is more ambiguity in the polar coordinates of a point because all the following represent the same point

$$(r, \theta),\ (-r, \theta \pm \pi),\ (r, \theta \pm 2\pi),\ (-r, \theta \pm 3\pi),\ \ldots.$$

5.9.1 Curves in Polar Coordinate

Example 5.9.1. The equation $r = c$ is the circle of radius $|c|$ centered at the origin. The equation $\theta = c$ is a straight line passing through the origin.

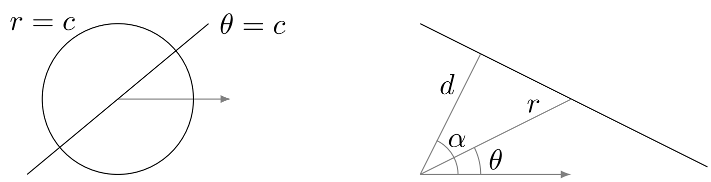


Figure 5.9.1: $r = c$, $\theta = c$, and polar equation for general straight line.

The equation for a general straight line is

$$r = \frac{d}{\cos(\alpha - \theta)}.$$

Moreover, $r = a\cos\theta$ is the circle of diameter a passing through the origin.

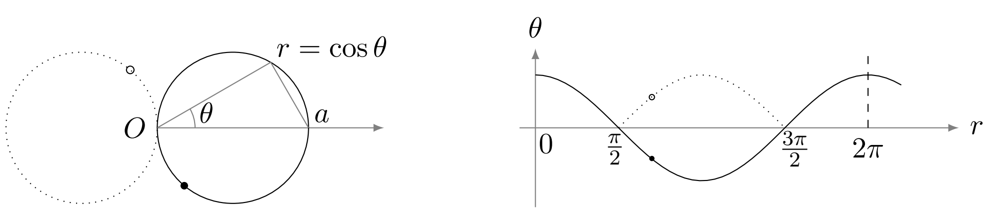


Figure 5.9.2: Circle $r = a\cos\theta$.

Exercise 5.9.1. Find the cartesian equation.

1. $r = 2$.
2. $r = -2$.
3. $r = \sin\theta$.
4. $r\sin\theta = 1$.
5. $r = \tan\theta\sec\theta$.
6. $r = \cos\theta + \sin\theta$.

Exercise 5.9.2. Find the polar equation.

1. $x = 1$.
2. $y = -1$.
3. $x + y = 1$.
4. $x = y^2$.
5. $x^2 + y^2 = x$.
6. $xy = 1$.

Exercise 5.9.3. What is the polar equation of the curve obtained by flipping $r = f(\theta)$ with respect to the origin? Then use your conclusion to find the curve $r = -\cos\theta$.

Exercise 5.9.4. What is the relation between the curves $r = f(\theta)$ and $r = -f(\theta + \pi)$?

Exercise 5.9.5. What is the polar equation of the curve obtained by rotating $r = f(\theta)$ by angle α? Then use your conclusion to answer the following.

1. What is the curve $r = \sin\theta$?
2. Find the polar equation for a general circle passing through the origin.

Exercise 5.9.6. Find the polar equation of a general circle.

Example 5.9.2. The *Archimedean spiral* is $r = \theta$. Note that $r < 0$ when $\theta < 0$, so that a flipping with respect to the origin is needed when we draw the part of the spiral corresponding to $\theta < 0$. The symmetry with respect to the y-axis is due to the fact that if (r, θ) satisfies $r = \theta$, then $(-r, -\theta)$ also satisfies $r = \theta$.

The *Fermat's spiral* is $r^2 = \theta$. The symmetry with respect to the origin is due to the fact that if (r, θ) satisfies $r^2 = \theta$, then $(-r, \theta)$ also satisfies $r^2 = \theta$.

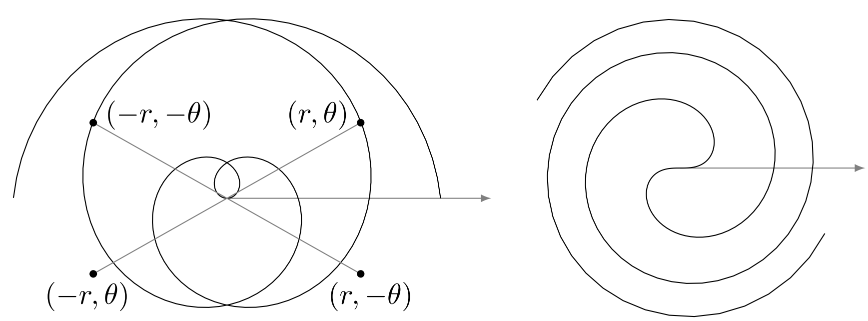


Figure 5.9.3: Spirals $r = \theta$ and $r^2 = \theta$.

Example 5.9.3. The curve $r = 1 + \cos\theta$ is a *cardioid*. Its clockwise rotation by 90° is another cardioid $r = 1 + \sin\theta$. More generally, the curve $r = a + \cos\theta$ is a *limaçon*.

The curve intersects itself when $|a| < 1$ and does not intersect itself when $|a| > 1$. The symmetry with respect to the x-axis is due to the fact that if (r, θ) satisfies $r = a + \cos\theta$, then $(r, -\theta)$ also satisfies the equation.

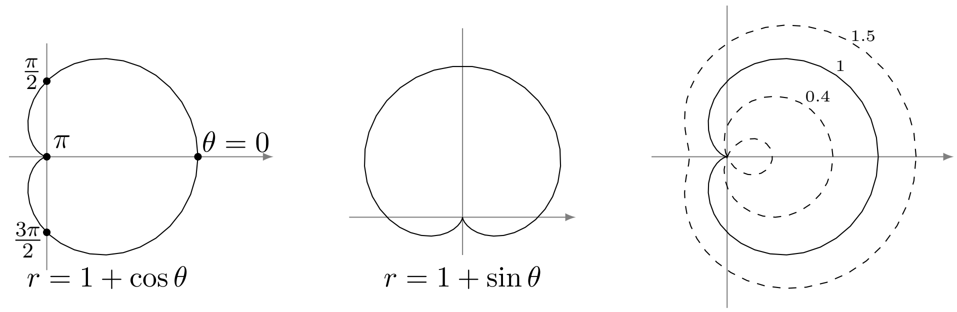


Figure 5.9.4: Cardioids and limaçons $r = a + \cos\theta$, $a = 0.4, 1, 1.5$.

The cardioid originates from the following geometrical construction. Consider a circle C of diameter 1 rolling outside of a circle A of equal diameter 1. This is the same as the circle rolling inside a big circle B of diameter 3. The track traced by a point on C is the cardioid. Note that the origin O of the polar coordinate should be a point on A, not the center of A.

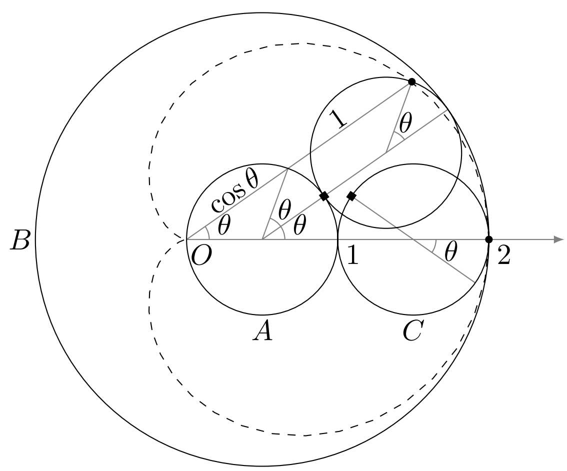


Figure 5.9.5: Origin of the cardioid.

If we imagine the rolling circle C as part of a rolling disk D, and we fix a point in D of distance d from the center of C. Then the track tranced by the point is the limaçon $r = 1 + 2d\cos\theta$, with the origin of the polar coordinate being a point of distance d form the center of A.

Example 5.9.4. The curve $r = \cos 2\theta$ is the *four-leaved rose*, and $r = \cos 3\theta$ is the *three-leaved rose*. The circle $r = \cos\theta$ can be considered as the one-leaved rose.

In general, the curve $r = \cos n\theta$ can be described as follows. For θ in the arc $I = \left[-\frac{\pi}{2n}, \frac{\pi}{2n}\right]$, the value of r goes from 0 to 1 and then back to 0, so that the corresponding curve is one leaf occupying $\frac{\pi}{n}$ angle of the whole circle. This is the leaf in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ for $n = 2$ and in $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ for $n = 3$. For θ in the second arc $I + \frac{\pi}{n}$, we need to rotate this first leaf by angle $\frac{\pi}{n}$ and then flipping with respect to the origin (because r becomes negative), which gives a leaf occupying $I + \frac{\pi}{n} + \pi$. This is the leaf in $\left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$ for $n = 2$ and in $\left[\frac{7\pi}{6}, \frac{9\pi}{6}\right]$ for $n = 3$. For θ in the third arc $I + \frac{2\pi}{n}$, we get the leaf obtained by rotating the first leaf by angle $\frac{2\pi}{n}$ (no flipping needed now because r becomes non-negative again), which gives a leaf occupying $I + \frac{2\pi}{n}$. This is the leaf in $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ for $n = 2$ and in $\left[\frac{3\pi}{6}, \frac{5\pi}{6}\right]$ for $n = 3$. Keep going, we see two distinct patterns depending on the parity of n.

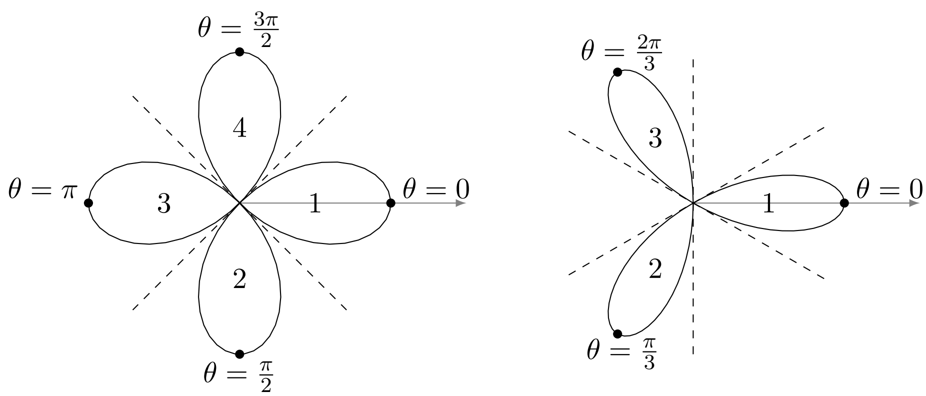


Figure 5.9.6: Four-leaved rose $r = \cos 2\theta$ and three-leaved rose $r = \cos 3\theta$.

More generally, we may consider $r = \cos p\theta$. Again we get first leaf occupying $I = \left[-\frac{\pi}{2p}, \frac{\pi}{2p}\right]$, the second leaf occupying $I + \frac{\pi}{p} + \pi$, the third leaf occupying $I + \frac{2\pi}{p}$, etc. The pattern could be very complicated, depending on whether p is rational or irrational, and in case p is rational, the parity of the numerator and denominator of p.

Finally, $r = \sin p\theta$ is obtained by rotating $r = \cos 2\theta$ by $\frac{\pi}{2p}$. We also get many leaved roses by other rotations.

Exercise 5.9.7. Describe the curve.

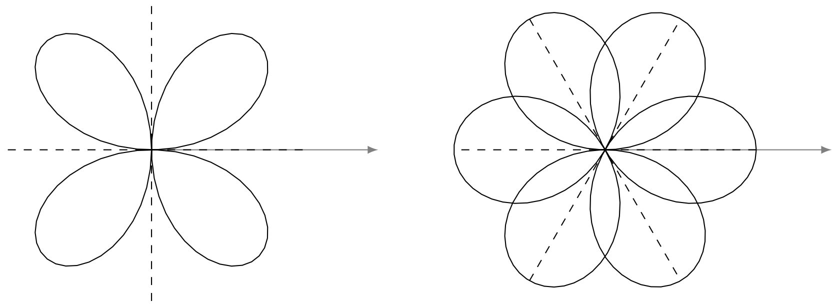


Figure 5.9.7: Many leaved roses $r = \sin 2\theta$ and $r = \sin \frac{3}{2}\theta$.

1. $r = -\theta$.
2. $r = \theta + \pi$.
3. $r = 2\theta$.
4. $r^2 = -\theta$.
5. $r^2 = 4\theta$.
6. $r^2 = \theta + \pi$.
7. $e^r = \theta$.
8. $r\theta = 1$.
9. $r = 2 + \cos\theta$.
10. $r = 2 + 3\cos\theta$.
11. $r = 2 - \cos\theta$.
12. $r = \cos\theta + \sin\theta$.
13. $r = 1 + \cos\theta + \sin\theta$.
14. $r = \cos 4\theta$.
15. $r = 2\sin 5\theta$.
16. $r = -3\sin 6\theta$.
17. $r = \sin 2\theta - \cos 2\theta$.
18. $r = \sin 2\theta + 2\cos 2\theta$.
19. $r = \cos \dfrac{4}{3}\theta$.
20. $r = \sin \dfrac{5}{3}\theta$.
21. $r = \cos \dfrac{1}{3}\theta$.
22. $r = \cos \dfrac{2}{3}\theta$.
23. $r = \cos \dfrac{2}{3}\theta + \sin \dfrac{2}{3}\theta$.
24. $r^2 = \sin 2\theta$.
25. $r^2 = -\cos 4\theta$.
26. $r = 1 + 2\cos \dfrac{1}{2}\theta$.
27. $r = 2 + \cos \dfrac{1}{2}\theta$.

5.9.2 Geometry in Polar Coordinate

The curve $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ is the parameterized curve

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta, \quad \theta \in [\alpha, \beta]$$

in the cartesian coordinate. The length of the curve is

$$\int_\alpha^\beta \sqrt{(f(\theta)\cos\theta)'^2 + (f(\theta)\sin\theta)'^2}d\theta = \int_\alpha^\beta \sqrt{f^2 + f'^2}d\theta.$$

For the area in terms of polar coordinate, assume $f \geq 0$ and consider the region $X_{[\alpha,\beta]}(f)$ bounded by $r = f(\theta)$, $\theta \in [\alpha, \beta]$, and the rays $\theta = \alpha$ and $\theta = \beta$. Using the idea of Section 5.1.1, let $A(\theta)$ be the area of the region $X_{[\alpha,\theta]}(f)$. Then the change $A(\theta + h) - A(\theta)$ is the area of $X_{[\theta,\theta+h]}(f)$. Since $X_{[\theta,\theta+h]}(f)$ is sandwiched between fans of angle between θ, $\theta + h$ and radii $m = \min_{[\theta,\theta+h]} f$, $M = \max_{[\theta,\theta+h]} f$, we get

$$\frac{1}{2}m^2h \leq A(\theta + h) - A(\theta) \leq \frac{1}{2}M^2h.$$

Here the left and right sides are the known areas of the fans. The inequality is the same as

$$\frac{1}{2}m^2 \le \frac{A(\theta+h)-A(\theta)}{h} \le \frac{1}{2}M^2.$$

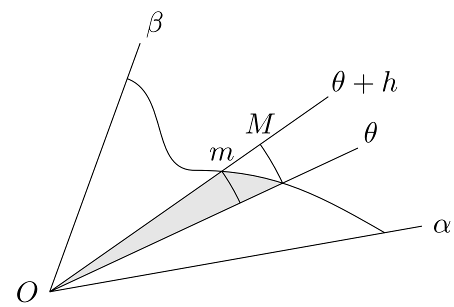


Figure 5.9.8: Estimate the change of area.

If f is continuous, then $\lim_{h\to 0} m = \lim_{h\to 0} M = f(\theta)$. By the sandwich rule, we get

$$A'(\theta) = \frac{1}{2}f(\theta)^2.$$

Therefore the area of $X_{[\alpha,\beta]}(f)$ is

$$A(\beta) = \frac{1}{2}\int_\alpha^\beta f(\theta)^2 d\theta.$$

Example 5.9.5. The cardioid $r = 1 + \sin\theta$ has length

$$\begin{aligned}\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (1+\sin\theta)'^2}d\theta &= \int_0^{2\pi} \sqrt{2(1+\sin\theta)}d\theta \\ &= \int_{\frac{1}{4}\pi}^{-\frac{3}{4}\pi} \sqrt{2(1+\cos 2t)}d\left(\frac{\pi}{2}-2t\right) \\ &= 4\int_{-\frac{3}{4}\pi}^{\frac{1}{4}\pi} |\cos t|dt = 4\int_0^\pi \cos t dt = 8.\end{aligned}$$

The region enclosed by the cardioid has area

$$\frac{1}{2}\int_0^{2\pi}(1+\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(1+2\sin\theta+\sin^2\theta)d\theta = \frac{3}{2}\pi.$$

Example 5.9.6. Let $p > \frac{1}{2}$. Then one leaf of the rose $r = \cos p\theta$ is from the angle

$-\dfrac{\pi}{2p}$ to the angle $\dfrac{\pi}{2p}$. The length of the leaf is

$$\begin{aligned}\int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}}\sqrt{(\cos p\theta)^2+(\cos p\theta)'^2}d\theta &= \int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}}\sqrt{\cos^2 p\theta+p^2\sin^2 p\theta}d\theta\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{p^{-2}\cos^2 t+\sin^2 t}dt\\ &= \frac{1}{2}\int_0^{2\pi}\sqrt{1+(p^{-2}-1)\cos^2 t}dt.\end{aligned}$$

This is the elliptic integral in Example 5.8.1. Moreover, the area of the leaf is

$$\frac{1}{2}\int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}}(\cos p\theta)^2 d\theta = \frac{\pi}{4p}.$$

Example 5.9.7. The cardioid $r = 1+\cos\theta$ and the circle $r = 3\cos\theta$ intersect at $\theta = \pm\dfrac{\pi}{3}$. The area of the region outside the cardioid and inside the circle is

$$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}((3\cos\theta)^2-(1+\cos\theta)^2)d\theta = \pi.$$

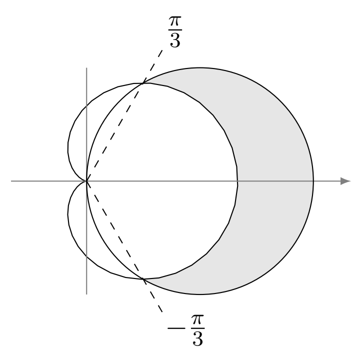


Figure 5.9.9: Ourside cardioid $r = 1+\cos\theta$ and inside circle $r = 3\cos\theta$.

Example 5.9.8. We try to find the volume of the solid of revolution obtained by revolving the region between the two leaves of the limaçon $r = a+\cos\theta$, $0 < a < 1$, around the x-axis. In the cartesian coordinate, the curve is parameterized by

$$x = (a+\cos\theta)\cos\theta, \quad y = (a+\cos\theta)\sin\theta, \quad \theta\in[0,\pi].$$

Let $\theta = \alpha$ at the origin O. Then the volume we are looking for is the volume of the solid of revolution from $\theta = 0$ to $\theta = \alpha$, subtracting the volume of the solid of revolution from $\theta=\alpha$ to $\theta=\pi$. As θ goes from 0 to α, we are moving opposite to the direction of the x-axis. Therefore the first volume is $-\pi\int_{\theta=0}^{\theta=\alpha} y^2dx$. As θ goes from α to π, we are moving in the direction of the x-axis. Therefore the second volume is $\pi\int_{\theta=\alpha}^{\theta=\pi} y^2dx$. We conclude that the volume we are looking for is

$$\begin{aligned}
-\pi\int_{\theta=0}^{\theta=\alpha} y^2dx-\pi\int_{\theta=\alpha}^{\theta=\pi} y^2dx &= -\pi\int_{\theta=0}^{\theta=\pi} y^2dx \\
&= -\pi\int_0^{\pi}(a+\cos\theta)^2\sin^2\theta d[(a+\cos\theta)\cos\theta] \\
&= -\pi\int_0^{\pi}(a+\cos\theta)^2(1-\cos^2\theta)(a+2\cos\theta)d(\cos\theta) \\
&= -\pi\int_1^{-1}(a+t)^2(1-t^2)(a+2t)dt=\frac{4}{3}\pi a(a^2+1).
\end{aligned}$$

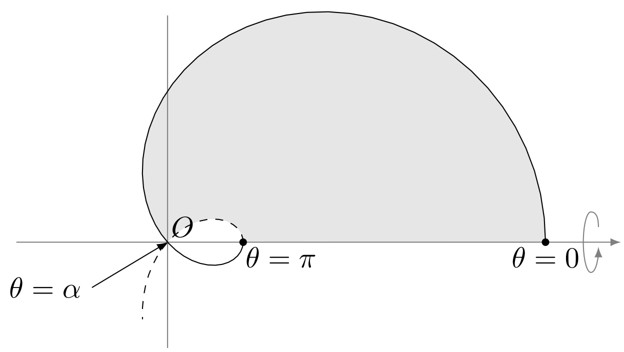


Figure 5.9.10: Revolving the region between two leaves of a limaçon.

Exercise 5.9.8. What is the length of lemiçon? What is the area of the region enclosed by lemiçon? Note that for $|c|>1$, we have two parts of the lemiçon and two regions.

Exercise 5.9.9. Find length of the part of the cardioid $r=1+\cos\theta$ in the first quadrant. Moreover, find the area of the region enclosed by this part and the two axes.

Exercise 5.9.10. Find the area of the region enclosed by *strophoid* $r=2\cos\theta-\sec\theta$.

Exercise 5.9.11. Find length.

1. $r=\theta$, $\theta\in[0,\pi]$.
2. $r=\theta^2$, $\theta\in[0,\pi]$.
3. $r=e^{\theta}$, $\theta\in[0,2\pi]$.

Exercise 5.9.12. Find area.

1. Bounded by $r=\theta$, $\theta\in[0,\pi]$ and the x-axis.

2. Outside $r = 1$ and inside $r = 2\cos\theta$.

3. Inside $r = 1$ and outside $r = 2\cos\theta$.

4. Inside both $r = 1$ and $r = 1 + \cos\theta$.

5. Outside $r = 3\sin\theta$ and inside $r = 2 - \sin\theta$.

6. Inside both $r = \cos 2\theta$ and $r = \sin 2\theta$.

7. Inside both $r = 1 + c\cos\theta$ and $r = 1 + c\sin\theta$, $|c| < 1$.

8. Inside both $r = 1 + c\cos\theta$ and $r = 1 - c\cos\theta$, $|c| < 1$.

9. Inside both $r^2 = \cos 2\theta$ and $r^2 = \sin 2\theta$.

10. Outside $r = 1$ and inside $r = 2\cos 3\theta$.

11. Between the two loops of $r = 1 + 2\cos 3\theta$.

5.10 Application to Physics

5.10.1 Work and Pressure

Integration is also widely used to compute physical quantities. If under a constant force F, an object moves by a distance d in the direction of the force, then the *work* done by the force is Fd. In general, however, the force may vary. For simplicity, assume the object moves along the x-axis, from $x = a$ to $x = b$, and a horizontal force $F(x)$ is applied when the object is at location x. Then the work done by the force when the objects moves a little bit from x to $x + \Delta x$ is approximately $\Delta W \approx F(x)\Delta x$.

Similar to the earlier argument, let $W(x)$ be the work done by the force when the object moves from a to x. Since the work is *additive*, we have $\Delta W = W(x + \Delta x) - W(x)$. The approximation $\dfrac{\Delta W}{\Delta x} \approx F(x)$ becomes more accurate as $\Delta x \to 0$, and we get an equality after taking the limit

$$W'(x) = \lim_{\Delta x \to 0} \frac{W(x + \Delta x) - W(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta W}{\Delta x} = F(x).$$

This implies that the work done for the whole trip from a to b is

$$W(b) = \int_a^b F(x)dx.$$

Example 5.10.1. Suppose one end of spring is fixed and the other end is attached to an object. In the natural position, when the spring is neither stretched nor compressed, no force is exercised on the object. When the position of the object deviate from the natural position by x, however, Hooke's law says that the spring exercises a force $F(x) = -kx$ on the object. Here k is the *spring constant*, and the negative sign indicates that the direction of the force is opposite to the direction of the deviation.

If the object starts at distance a from its natural position, then the work done by the spring in pulling the object to its natural position is

$$\int_0^a kxdx = \frac{k}{2}a^2.$$

Here we use the positive sign because the direction of movement is the same as the direction of the force.

The argument about the work done by a force is quite typical. In general, if a quantity is additive, then the quantity can be decomposed into small pieces. The estimation of each small piece tells us the change of the quantity. The whole quantity is then the integration of the change.

In the subsequent examples, we will only analyze a small piece of an additive quantity. We will omit the limit part of the argument and directly write down the corresponding integration.

Example 5.10.2. We want to find the work it takes to pump a bucket of liquid out of the top of the bucket.

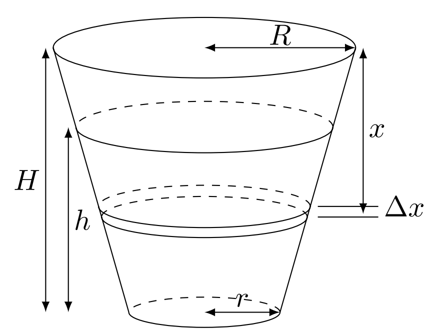


Figure 5.10.1: Bucket of liquid.

Suppose the bucket has base diameter r, top diameter R, and height H. Suppose the liquid has density ρ and depth h. We decompose liquid into horizontal sections. At distance x from the top, the section is a disk of radius $r(x)$ satisfying

$$\frac{r(x) - r}{R - r} = \frac{H - x}{H}.$$

The liquid of thickness Δx and at distance x from the top has (approximate) weight $g\rho\pi r(x)^2\Delta x$ (g is the gravitational constant). The work it takes to lift this piece of liquid to the top of bucket is $\Delta W \approx (g\rho\pi r(x)^2\Delta x)x = \pi g\rho x r(x)^2\Delta x$. Since the liquid spans from $x = H - h$ to $x = H$, the total work needed is

$$\begin{aligned} W &= \pi g\rho \int_{H-h}^{H} x r(x)^2 dx = \frac{\pi g\rho}{H^2}\int_{H-h}^{H} x\left[(R-r)(H-x)+rH\right]^2 dx \\ &= \pi g\rho H^2R^2\left(a^2b + \frac{1}{2}a(2a-3b)b^2 + \frac{1}{3}(1-a)(1-3a)b^3 - \frac{1}{4}(1-a)^2b^4\right), \end{aligned}$$

where $a = \dfrac{r}{R}$ and $b = \dfrac{h}{H}$.

Example 5.10.3. We want to find the force exercised by water on a dam.

Let ρ be the density of water. At the depth x, the pressure of water is ρx per unit area. Now suppose the dam is a vertical trapezoid with base length l, top length L, and height H. We decompose dam into horizontal sections. At distance x from the top, the section is a strip of height Δx and length $l(x)$ satisfying

$$\frac{l(x)-l}{L-l} = \frac{H-x}{H}.$$

The force exercised on the strip is $\Delta F \approx (\rho x)l(x)\Delta x$. Since the water spans from $x = 0$ to $x = H$, the total force

$$F = \rho\int_0^H xl(x)dx = \rho\int_0^H x\left(L - \frac{L-l}{H}x\right)dx = \frac{1}{6}\rho H^2(L+2l).$$

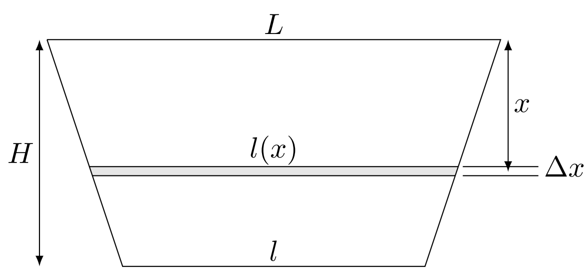


Figure 5.10.2: Hydraulic dam.

Exercise 5.10.1. A spring has natural length a. If the force F is needed to stretch the spring to length b, how muck work is needed to stretch the spring from the natural length to the length b?

Exercise 5.10.2. A ball of radius R is full of liquid of density ρ. Due to the gravity, the liquid leaks out of a hole at the bottom of the ball. How much work is done by the gravity in draining all the liquid?

Exercise 5.10.3. A circular disk of radius r is fully submerged in liquid of density ρ, such that the center of the disk is at depth h. What is the force exercised by the liquid on one side of the plate? Note that the plate may be inclined at some angle.

Exercise 5.10.4. A ball of radius r is fully submerged in liquid of density ρ, such that the center of the disk is at depth h. What is the force exercised by the liquid on the ball?

Exercise 5.10.5. A cable of mass m and length l has a mass M tied to the lower end. How much word is done in using the cable to lift the mass M to the top end of the cable?

Exercise 5.10.6. Newton's law of gravitation says that two bodies with masses m and M attract each other with a force $F = \dfrac{gmM}{d^2}$, where d is the distance between the bodies. Suppose the radius of the earth is R and the mass is M. How much work is needed to launch a satellite of mass m vertically to a circular orbit of height H? What is the minimal initial velocity needed for the satellite to escape the earth's gravity?

5.10.2 Center of Mass

Consider n masses $m_1, m_2, \dots, m_n$ distributed at the locations $x_1, x_2, \dots, x_n$ along a straight line. The *center of mass* is

$$\bar{x} = \frac{m_1x_2 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}.$$

The center has the physical meaning that the total moment of the system with respect to $\bar{x}$ is zero, or the system is balanced with respect to $\bar{x}$.

Now suppose we have masses distributed throughout an interval $[a, b]$, with the density $\rho(x)$ at location x. We partition the interval into small pieces

$$P\colon a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then the system is decomposed into n pieces. The i-th piece can be approximately considered as a mass $m_i = \rho(x_i^*)(x_i - x_{i-1})$ located at x_i^*, for some $x_i^* \in [x_{i-1}, x_i]$. The whole system is approximated by the system of n pieces, and has approximate center of mass

$$\bar{x}_P = \frac{\sum_{i=1}^n \rho(x_i^*)(x_i - x_{i-1})x_i^*}{\sum_{i=1}^n \rho(x_i^*)(x_i - x_{i-1})}.$$

The denominator is the Riemann sum of the function $x\rho(x)$ and the numerator is the Riemann sum of the function $\rho(x)$ (see the beginning of Section 5.3.1). Therefore as the partition gets more and more refined, the limit becomes the center of mass

$$\bar{x} = \frac{\displaystyle\int_a^b x\rho(x)dx}{\displaystyle\int_a^b \rho(x)dx}.$$

The center of mass can be extended to higher dimensions, simply by considering each coordinate separately. For example, the system of n masses $m_1, m_2, \ldots, m_n$ at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the plane has the center of mass $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}.$$

Now consider masses distributed along a curve $(x(t), y(t))$, $t \in [a, b]$, with the density $\rho(t)$ at location t. Take a partition P of $[a, b]$. The curve is approximated by straight line segments connecting $(x(t_{i-1}), y(t_{i-1}))$ to $(x(t_i), y(t_i))$. The i-th straight line segment has length $\Delta s_i = \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$ and can be approximately considered as a mass $m_i = \rho(t_i^*)\Delta s_i$ located at $(x(t_i^*), y(t_i^*))$, for some $t_i^* \in [t_{i-1}, t_i]$. The whole system is approximated by the system of n pieces, and has approximate center of mass

$$\bar{x}_P = \frac{\sum_{i=1}^n (\rho(t_i^*)\Delta s_i) x(t_i^*)}{\sum_{i=1}^n \rho(t_i^*)\Delta s_i}, \quad \bar{y}_P = \frac{\sum_{i=1}^n (\rho(t_i^*)\Delta s_i) y(t_i^*)}{\sum_{i=1}^n \rho(t_i^*)\Delta s_i}.$$

As the partition gets more and more refined, the limit becomes the center of mass

$$\bar{x} = \frac{\displaystyle\int_a^b x(t)\rho(t)ds}{\displaystyle\int_a^b \rho(t)ds}, \quad \bar{y} = \frac{\displaystyle\int_a^b y(t)\rho(t)ds}{\displaystyle\int_a^b \rho(t)ds}, \quad ds = \sqrt{x'(t)^2 + y'(t)^2}dt.$$

Example 5.10.4. For constant density $\rho(x) = \rho$ distributed on the interval, the center of mass is the middle point

$$\bar{x} = \frac{\displaystyle\int_a^b x\rho dx}{\displaystyle\int_a^b \rho dx} = \frac{\rho\frac{1}{2}(b^2 - a^2)}{\rho(b - a)} = \frac{a + b}{2}.$$

If the density is $\rho(x) = \lambda + \mu x$, which is linearly increasing, then the center of mass is

$$\bar{x} = \frac{\displaystyle\int_a^b x(\lambda + \mu x)dx}{\displaystyle\int_a^b (\lambda + \mu x)dx} = \frac{3\lambda(a + b) + 2\mu(a^2 + ab + b^2)}{3(2\lambda + \mu(a + b))}.$$

Example 5.10.5. Consider the semi-circular curve of radius r and constant density ρ. We have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad ds = rd\theta, \quad 0 \le \theta \le \pi,$$

and the center of mass is

$$\bar{x} = \frac{\displaystyle\int_0^{\pi} (r\cos\theta)\rho r d\theta}{\displaystyle\int_0^{\pi} \rho r d\theta} = 0, \quad \bar{y} = \frac{\displaystyle\int_0^{\pi} (r\sin\theta)\rho r d\theta}{\displaystyle\int_0^{\pi} \rho r d\theta} = \frac{2r}{\pi}.$$

Exercise 5.10.7. Find the center of mass of the parabola $y = x^2$, $x \in [0, 2]$, of constant density.

Exercise 5.10.8. Find the center of mass of a triangle of constant density and with vertices at $(-1, 0)$, $(0, \sqrt{15})$ and $(7, 0)$.

Exercise 5.10.9. Let $m_{[a,b]}$ and $\bar{x}_{[a,b]}$ be the mass and the center of mass of a distribution of masses on $[a, b]$ with the density $\rho(x)$. Let $[a, b] = [a, c] \cup [c, b]$ and similarly introduce $m_{[a,c]}$, $m_{[c,b]}$, $\bar{x}_{[a,c]}$, $\bar{x}_{[c,b]}$. Show that the center of mass has the distribution property

$$m_{[a,b]} = m_{[a,c]} + m_{[c,b]}, \quad \bar{x}_{[a,b]} = \frac{m_{[a,c]}\bar{x}_{[a,c]} + m_{[c,b]}\bar{x}_{[c,b]}}{m_{[a,c]} + m_{[c,b]}}.$$

Does the property extend to curves in $\mathbb{R}^2$?

Chapter 6

Series

6.1 Comparison Test

Series $\sum a_n$ are very much like improper integrals $\displaystyle\int_a^{+\infty} f(x)dx$. The two can be compared in two aspects. First the convergence of the two can be compared, through the integral test. Second all the convergence theorems for $\displaystyle\int_a^{+\infty} f(x)dx$, such as the comparison test, Dirichlet test and Abel test, have parallels for the convergence of series.

6.1.1 Integral Test

Theorem 6.1.1 (Integral Test). *Suppose $f(x)$ is a decreasing function on $[1, +\infty)$ satisfying $\lim_{x\to+\infty} f(x) = 0$. Then*

$$f(1) + f(2) + \cdots + f(n) = \int_1^n f(x)dx + \gamma + \epsilon_n,$$

for a constant $0 \le \gamma \le f(1)$ and a decreasing sequence ϵ_n converging to 0. In particular, the series $\sum f(n)$ converges if and only if the improper integral $\displaystyle\int_a^{+\infty} f(x)dx$ converges.

Let

$$x_n = f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)dx.$$

By f decreasing, we get

$$\begin{aligned} x_n - x_{n-1} &= f(n) - \int_{n-1}^{n} f(x)dx \le 0, \\ x_n - f(n) &= f(1) + \cdots + f(n-1) - \int_1^n f(x)dx \ge 0 \\ &= \sum_{k=1}^{n-1} \left(f(k) - \int_k^{k+1} f(x)dx \right) \ge 0. \end{aligned}$$

The first inequality implies x_n is decreasing, and the second inequality implies $x_n \ge f(n) \ge 0$. Therefore $\lim x_n = \gamma$ converges, and the theorem follows. We have $0 \le \gamma \le x_1 = f(1)$.

Example 6.1.1. For $p > 0$, the function $\dfrac{1}{x^p}$ is decreasing and converges to 0 as $x \to +\infty$. By Theorem 6.1.1, therefore, the series $\sum \dfrac{1}{n^p}$ converges if and only if the improper integral $\displaystyle\int_1^{+\infty} \frac{dx}{x^p}$ converges. By Example 5.7.3, this happens if and only if $p > 1$.

Although the harmonic series $\sum \dfrac{1}{n}$ diverges, Theorem 6.1.1 estimates the partial sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \log n + \gamma + \epsilon_n,$$

where ϵ_n decreases and converges to 0, and

$$\gamma = 0.57721566490153286060651209008 2\cdots$$

is the *Euler-Mascheroni constant.*

Example 6.1.2. For $p > 0$ and $x > e$, the integral test can be applied to the function $\dfrac{1}{x(\log x)^p}$. We conclude that $\sum \dfrac{1}{n(\log n)^p}$ converges if and only if the improper integral $\displaystyle\int_a^{+\infty} \frac{dx}{x(\log x)^p}$ converges. By Example 5.7.9, this means $p > 1$.

Example 6.1.3. We will show that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$ in Example 6.4.13. In fact, for even k, $\sum_{n=1}^{\infty} \dfrac{1}{n^k}$ can be calculated as a rational multiple of π^k. However, very little is known about the sum for odd k. Still, we may use the idea of Theorem 6.1.1 to estimate the remainder

$$\int_{n+1}^{+\infty} f(x)dx = \sum_{k=n+1}^{\infty} \int_k^{k+1} f(x)dx \le \sum_{k=n+1}^{\infty} f(k) \le \sum_{k=n}^{\infty} \int_k^{k+1} f(x)dx = \int_n^{+\infty} f(x)dx.$$

For example, the 10-th partial sum of $\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ is

$$s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{10^3} = 1.197532\cdots .$$

By

$$\int_{10}^{\infty}\frac{dx}{x^3} = \frac{1}{2(10)^2} = 0.005, \quad \int_{11}^{\infty}\frac{dx}{x^3} = \frac{1}{2(11)^2} = 0.004132\cdots ,$$

we get

$$\begin{aligned} 1.201664\cdots &= 1.197532\cdots + 0.004132\cdots \\ &\le \sum_{n=1}^{\infty}\frac{1}{n^3} \le 1.197532\cdots + 0.005 = 1.202532\cdots . \end{aligned}$$

If we want to get the approximate value of $\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ up to the 6-th digit, then we my try to find n satisfying

$$\int_{n}^{n+1}\frac{dx}{x^3} = \frac{2n+1}{2n^2(n+1)^2} < \frac{1}{n^3} < 0.000001.$$

So we may take $n = 100$ and get

$$\sum_{n=1}^{100}\frac{1}{n^3} + \int_{101}^{\infty}\frac{dx}{x^3} \le \sum_{n=1}^{\infty}\frac{1}{n^3} \le \sum_{n=1}^{100}\frac{1}{n^3} + \int_{100}^{\infty}\frac{dx}{x^3}.$$

Exercise 6.1.1. Determine the convergence of $\sum \dfrac{1}{n(\log n)(\log(\log n))^p}$.

Exercise 6.1.2. Find suitable function $f(n)$, such that the sequence $1+\dfrac{1}{\sqrt{2}}+\cdots+\dfrac{1}{\sqrt{n}}-f(n)$ converges to a limit γ. Then express the sum of the series $\sum_{n=0}^{\infty}(-1)^n\dfrac{1}{\sqrt{n}}$ in terms of γ.

Exercise 6.1.3. Estimate $\sum_{n=1}^{\infty}\dfrac{1}{n^{\frac{3}{2}}}$ to within 0.01.

6.1.2 Comparison Test

Theorem 6.1.2 (Comparison Test). *Suppose $|a_n| \le b_n$ for sufficiently large n. If $\sum b_n$ converges, then $\sum a_n$ also converges.*

The test is completely parallel to the similar test (Theorem 5.7.1) for the convergence of improper integrals, and can be proved similarly by using the Cauchy criterion (Theorem 1.5.4).

For the special case $b_n = |a_n|$, the test says that if $\sum |a_n|$ converges, then $\sum a_n$ converges. In other words, absolute convergence implies convergence. We note that the conclusion of the comparison test is always absolute convergence.

Example 6.1.4. Consider the series $\sum \frac{\log n}{n^p}$. If $p \le 1$, then $\frac{\log n}{n^p} \ge \frac{1}{n}$. By the comparison test, the divergence of $\sum \frac{1}{n}$ implies the divergence of $\sum \frac{\log n}{n^p}$.

If $p > 1$, then choose q satisfying $p > q > 1$. We have

$$\frac{\log n}{n^p} = \frac{\log n}{n^{p-q}} \frac{1}{n^q} < \frac{1}{n^q} \text{ for large } n.$$

Here the inequality is due to the fact that $p - q > 0$ implies $\lim \frac{\log n}{n^{p-q}} = 0$. By Example 6.1.1, $\sum \frac{1}{n^q}$ converges. Then by the comparison test, we conclude that $\sum \frac{\log n}{n^p}$ converges.

The key idea of the example above is to compare $a_n = \frac{\log n}{n^p}$ with $b_n = \frac{1}{n^q}$ by using the limit of their quotient. By

$$\lim \frac{a_n}{b_n} = \lim \frac{\log n}{n^{p-q}} = 0,$$

we get $\frac{a_n}{b_n} < 1$ for sufficiently large n. Since both a_n and b_n are positive, we may apply the comparison test to conclude that the convergence of $\sum b_n$ implies the convergence of $\sum a_n$.

In general, if $a_n, b_n > 0$ and $\lim \frac{a_n}{b_n} = l$ converges, then by the comparison test, the convergence of $\sum b_n$ implies the convergence of $\sum a_n$. Moreover, if $l \neq 0$, then we also have $\lim \frac{b_n}{a_n} = \frac{1}{l}$, and we conclude that $\sum a_n$ converges if and only if $\sum b_n$ converges.

Example 6.1.5. For $\sum \frac{n + \sin n}{n^3 + n + 2}$, we make the following comparison

$$\lim_{n\to\infty} \frac{\dfrac{n + \sin n}{n^3 + n + 2}}{\dfrac{1}{n^2}} = 1.$$

By the convergence of $\sum \frac{1}{n^2}$, we get the convergence of $\sum \frac{n + \sin n}{n^3 + n + 2}$.

Similarly, by the comparison

$$\lim_{n\to\infty} \frac{\dfrac{2^n+n^2}{\sqrt{5^{n-1}-n^4 3^n}}}{\left(\dfrac{2}{\sqrt{5}}\right)^n} = \sqrt{5},$$

and the convergence of $\sum \left(\dfrac{2}{\sqrt{5}}\right)^n$, the series $\sum \dfrac{2^n+n^2}{\sqrt{5^{n-1}-n^4 3^n}}$ converges.

Example 6.1.6. By Example 4.1.14, we know $\left(1+\dfrac{1}{x}\right)^x - e = -\dfrac{e}{2x} + o\left(\dfrac{1}{x}\right)$. This implies that for sufficiently large n, $\left(1+\dfrac{1}{n}\right)^n - e$ is negative and comparable to $\dfrac{1}{n}$. Since the harmonic series $\sum \dfrac{1}{n}$ diverges, we conclude that $\sum \left[\left(1+\dfrac{1}{n}\right)^n - e\right]$ diverges.

Example 6.1.7. By Example 5.7.14, we know that $\displaystyle\int_1^{+\infty} \frac{|\sin x|}{x^p}dx$ converges if and only if $p>1$. By a change of variable, we also know that, for $a \neq 0$, $\displaystyle\int_1^{+\infty} \frac{|\sin ax|}{x^p}dx$ converges if and only if $p>1$. However, we cannot use the integral test (Theorem 6.1.1) to get the similar conclusion for $\sum \dfrac{|\sin na|}{n^p}$. The problem is that $\dfrac{|\sin ax|}{x^p}$ is not a decreasing function.

By $\dfrac{|\sin na|}{n^p} \le \dfrac{1}{n^p}$ and the comparison test, we know $\sum \dfrac{|\sin na|}{n^p}$ converges for $p>1$. The series also converges if a is a multiple of π, because all the terms are 0. It remains to consider the case $p \le 1$ and a is not a multiple of π.

First assume $0 < a \le \dfrac{\pi}{2}$. For any natural number k, the interval $\left[k\pi+\dfrac{\pi}{4}, k\pi+\dfrac{3\pi}{4}\right]$ has length $\dfrac{\pi}{2}$ and therefore must contain $n_k a$ for some natural number n_k. Then $|\sin n_k a| \ge \dfrac{1}{\sqrt{2}}$, and for $p \le 1$,

$$\sum_{n=1}^{\infty} \frac{|\sin na|}{n^p} \ge \sum_{n=1}^{\infty} \frac{|\sin na|}{n} \ge \sum_{k=1}^{\infty} \frac{|\sin n_k a|}{n_k} \ge \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

By $n_k \le k\pi + \dfrac{3\pi}{4}$, we get $\dfrac{1}{n_k} \ge \dfrac{4}{4k+3}\dfrac{a}{\pi} > \dfrac{a}{4k}$. Then by $\sum \dfrac{1}{k} = +\infty$, we get $\sum \dfrac{1}{n_k} = +\infty$ and $\sum \dfrac{|\sin na|}{n^p} = +\infty$.

In general, if a is not an integer multiple of π, then there is b, such that $0 < b \le \frac{\pi}{2}$ and either $a+b$ or $a-b$ is an integer multiple of π. Then we have $|\sin na| = |\sin nb|$, and we still conclude that $\sum \frac{|\sin na|}{n^p}$ diverges for $p \le 1$.

Exercise 6.1.4. Show that if $a_n > 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges. Moreover, show that the converse is not true.

Exercise 6.1.5. Show that if $\sum a_n^2$, then $\sum \frac{a_n}{n}$ converges.

Exercise 6.1.6. Show that if $\sum a_n^2$ and $\sum b_n^2$ converge, then $\sum a_n b_n$ and $\sum (a_n + b_n)^2$ converge.

Exercise 6.1.7. Determine the convergence.

1. $\sum \frac{\sqrt{4n^5+5n^4}}{3n^2-2n^3}$.
2. $\sum \frac{3n^2-2n^3}{\sqrt{4n^5+5n^4}}$.
3. $\sum \frac{3n^2+(-1)^n 2n^3}{4n^5+5n^4}$.

Exercise 6.1.8. Determine the convergence, $p, q, r, s > 0$.

1. $\sum \frac{1}{n^p+(\log n)^q}$.
2. $\sum \frac{1}{n^p(\log n)^q}$.
3. $\sum \frac{n^r+(\log n)^s}{n^p+(\log n)^q}$.
4. $\sum \frac{n^r(\log n)^s}{n^p+(\log n)^q}$.
5. $\sum \frac{n^r+(\log n)^s}{n^p(\log n)^q}$.
6. $\sum \frac{1}{n^p(\log n)^q(\log(\log n))^r}$.

Exercise 6.1.9. Determine the convergence, $b, d, p, q > 0$.

1. $\sum \frac{1}{(a+nb)^p}$.
2. $\sum \frac{(c+nd)^q}{(a+nb)^p}$.
3. $\sum \frac{1}{(a+nb)^p(c+nd)^q}$.
4. $\sum \frac{(\log(c+nd))^q}{(a+nb)^p}$.
5. $\sum \frac{1}{(a+nb)^p(\log(c+nd))^q}$.
6. $\sum \frac{(\log(c+nd))^q}{(a+nb)^p}$.

Exercise 6.1.10. Determine the convergence, $p, q > 0$.

1. $\sum((n^p+a)^r - (n^p+b)^r)$.
2. $\sum \left[\left(\frac{n^p+a}{n^p+b}\right)^q - 1\right]$.

Exercise 6.1.11. Determine the convergence.

1. $\sum \frac{1}{n\sqrt{n}}$.
2. $\sum \frac{1}{n^{1+\frac{1}{n}}}$.
3. $\sum \frac{1}{n^{1+\frac{1}{\log n}}}$.
4. $\sum \frac{1}{(\log n)^n}$.
5. $\sum \frac{n^2}{(\log n)^n}$.
6. $\sum \frac{1}{\sqrt[n]{\log n}}$.
7. $\sum \frac{(\log n)^n}{n^n}$.
8. $\sum \frac{n^{\log n}}{(\log n)^n}$.

Exercise 6.1.12. Determine the convergence, $p, q > 0$.

1. $\sum \sin \frac{1}{n}$.
2. $\sum \frac{1}{n^p} \sin \frac{1}{n^q}$.
3. $\sum \frac{n^2 - n\sin n}{n^3 + \cos n}$.
4. $\sum \frac{n^2 - n\sin n}{n^3 + \cos n} \sin \frac{1}{n}$.
5. $\sum \left(\cos \frac{1}{n^p} - 1\right)$.
6. $\sum \cos \frac{1}{n^p} \sin \frac{1}{n^q}$.

Exercise 6.1.13. Determine the convergence.

1. $\sum \frac{1}{5^n - 1}$.
2. $\sum \frac{3^{n+1}}{5^{n-1} - n^2 2^n}$.
3. $\sum \frac{5^{n-1} - n^2 2^n}{3^{n+1}}$.

Exercise 6.1.14. Determine the convergence, $a, b > 0$.

1. $\sum \sqrt{a^n + b^n}$.
2. $\sum \frac{1}{\sqrt{a^n + b^n}}$.
3. $\sum \frac{1}{a^n + b^n}$.
4. $\sum (a^n + b^n)^p$.
5. $\sum \frac{n^2}{na^n + b^n}$.
6. $\sum \frac{1}{\sqrt[n]{a^n + b^n}}$.

Exercise 6.1.15. Determine the convergence.

1. $\sum x^{n^2}$.
2. $\sum nx^{n^2}$.
3. $\sum x^{\sqrt{n}}$.
4. $\sum nx^{\sqrt{n}}$.
5. $\sum n^2 x^{n^2}$.
6. $\sum n^p x^{n^q}$.

Exercise 6.1.16. Determine the convergence.

1. $\sum a^{n^p}$.
2. $\sum (a^{\frac{1}{n}} - 1)$.
3. $\sum \left(e^{\frac{1}{n}} - 1 - \frac{1}{n}\right)$.
4. $\sum \left(n^{\frac{1}{n^p}} - 1\right)$.
5. $\sum \left(1 + \frac{a \log n}{n}\right)^n$.
6. $\sum \left(\frac{an + b}{cn + d}\right)^n$.
7. $\sum n^3 \left(\frac{a + (-1)^n}{b + (-1)^n}\right)^n$.
8. $\sum \left(\frac{1}{\sqrt{n}} - \sqrt{\log \frac{n+1}{n}}\right)$.
9. $\sum \left(1 - \frac{1}{n}\right)^{n^2}$.
10. $\sum \left(1 + \frac{1}{n}\right)^{2n - n^2}$.
11. $\sum \frac{n^2}{\left(a + \frac{1}{n}\right)^n}$.
12. $\sum \frac{n^{2n}}{(n + a)^{n+b}(n + b)^{n+a}}$.
13. $\sum \left(2\sqrt[n]{a} - \sqrt[n]{b} - \sqrt[n]{c}\right)$.
14. $\sum \left(\cos \frac{a}{n}\right)^{n^2}$.
15. $\sum \left(-\log \cos \frac{1}{n}\right)^p$.
16. $\sum \log \left(n^p \sin \frac{a}{n^q}\right)$.
17. $\sum \frac{|\cos na|}{n^p}$.

Exercise 6.1.17. Determine the convergence.

1. $\sum \int_n^{n+1} e^{-\sqrt{x}} \sin x dx$.
2. $\sum \int_n^{n+1} \frac{\sin x}{x^p} dx$.
3. $\sum \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\log x}{x^p} dx$.
4. $\sum \int_0^{\frac{1}{n}} \frac{x^p}{1 + x^2} dx$.
5. $\sum \int_0^{\frac{1}{n}} |\sin x|^p dx$.
6. $\sum \int_0^1 \sin x^n dx$.

Exercise 6.1.18. Suppose a_n is a bounded sequence. Show that $\sum \frac{1}{n}(a_n - a_{n+1})$ converges.

Exercise 6.1.19. The decimal representations of positive real numbers are actually the sum of series. For example,

$$\pi = 3.1415926\cdots = 3+0.1+0.04+0.001+0.0005+0.00009+0.000002+0.0000006+\cdots.$$

Explain why the expression always converges.

6.1.3 Special Comparison Test

We compare a series $\sum a_n$ with the geometric series $\sum r^n$, which we know converges if and only if $|r| < 1$. If $|a_n| \le r^n$ for some $r < 1$, then the comparison test implies that $\sum a_n$ converges. We note that the condition $|a_n| \le r^n$ for some $r < 1$ is the same as $\sqrt[n]{|a_n|} \le r < 1$.

Theorem 6.1.3 (Root Test). *Suppose $|a_n| \le r^n$ for some $r < 1$ and sufficiently large n. Then $\sum a_n$ converges.*

Example 6.1.8. To determine the convergence of $\sum(n^5 + 2n + 3)x^n$, we note that $\lim_{n\to\infty} \sqrt[n]{|(n^5 + 2n + 3)x^n|} = |x|$. If $|x| < 1$, then we can pick r satisfying $|x| < r < 1$. By $\lim_{n\to\infty} \sqrt[n]{|(n^5 + 2n + 3)x^n|} < r$ and the order rule, we get $\sqrt[n]{|(n^5 + 2n + 3)x^n|} < r$ for sufficiently large n. Then by the root test, we conclude that $\sum(n^5+2n+3)x^n$ converges for $|x| < 1$.

If $|x| \ge 1$, then the term $(n^5+2n+3)x^n$ of the series does not converge to 0. By Theorem 1.5.1, the series diverges for $|x| \ge 1$.

The example suggests that, in practice, it is often more convenient to use the limit version of the root test. Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then fix r satisfying $\lim_{n\to\infty} \sqrt[n]{|a_n|} < r < 1$. By the order rule, we have $\sqrt[n]{|a_n|} < r$ for sufficiently large n. Then the root test shows that $\sum a_n$ converges. On the other hand, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then we have $\sqrt[n]{|a_n|} > 1$ for sufficiently large n. This implies $|a_n| > 1$, and $\sum a_n$ diverges by Theorem 1.5.1.

Exercise 6.1.20. Determine the convergence, $a, b > 0$.

1. $\sum \frac{(\log n)^n}{n^2}$.
2. $\sum \frac{n^p}{(\log n)^n}$.
3. $\sum \frac{1}{a^n + b^n}$.
4. $\sum(a^n + b^n)^p$.
5. $\sum n^p x^n$.
6. $\sum n^p x^{n^q}$.
7. $\sum \left(\frac{an+b}{cn+d}\right)^n$.
8. $\sum \left(1+\frac{a}{n}\right)^{n^2}$.
9. $\sum \left(1+\frac{a}{n}\right)^{-n^2}$.
10. $\sum \left(1+\frac{a}{n}\right)^{2n-n^2}$.
11. $\sum \frac{n^p}{\left(a+\frac{b}{n}\right)^n}$.

12. $\sum n^3 \left(\frac{a+(-1)^n}{b+(-1)^n}\right)^n$.

Next we turn to another way of comparing series. Theorem 3.3.3 compares two functions by comparing their derivatives (i.e., the changes of functions). Similarly, we may compare two sequences a_n and b_n by either comparing the differences $a_{n+1} - a_n$ and $b_{n+1} - b_n$, or the ratios $\frac{a_{n+1}}{a_n}$ and $\frac{b_{n+1}}{b_n}$. The comparison of ratio is especially suitable for the comparison of series.

Suppose $a_n, b_n > 0$, and $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$ for $n \ge N$. Then for $c = \frac{a_N}{b_N}$, the two sequences a_n and cb_n are equal at $n = N$, and $\frac{a_{n+1}}{a_n} \le \frac{cb_{n+1}}{cb_n}$ implies that the second sequence has bigger change than the first one, at least for $n \ge N$. This should imply $a_n \le cb_n$ for $n \ge N$. The following is the rigorous argument

$$a_n = a_N \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_n}{a_{n-1}} \le cb_N \frac{b_{N+1}}{b_N} \frac{b_{N+2}}{b_{N+1}} \cdots \frac{b_n}{b_{n-1}} = cb_n.$$

By the comparison test, if $\sum b_n$ converges, then $\sum a_n$ converges.

Theorem 6.1.4 (Ratio Test). *Suppose* $\left|\frac{a_{n+1}}{a_n}\right| \le \frac{b_{n+1}}{b_n}$ *for sufficiently large* n. *If* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*

We note that the assumption implies that the terms b_n have the same sign for sufficiently large n. By changing all b_n to $-b_n$ if necessary, we may assume that $b_n > 0$ for sufficiently large n.

Example 6.1.9. The series $\sum \frac{(2n)!}{(n!)^2} x^n$ satisfies

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n-1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{(2n)!}{(n!)^2} x^n}{\frac{(2n-2)!}{((n-1)!)^2} x^{n-1}}\right| = \lim_{n\to\infty} \frac{2n(2n-1)}{n^2} |x| = 4|x|.$$

If $4|x| < 1$, then we fix r satisfying $4|x| < r < 1$. By the order rule, we have

$$\left|\frac{a_n}{a_{n-1}}\right| < r = \frac{r^n}{r^{n-1}} \text{ for large } n.$$

By comparing with the power series $\sum b_n = \sum r^n$, Theorem 6.1.4 implies that $\sum a_n$ converges. If $4|x| > 1$, then we get

$$\left|\frac{a_n}{a_{n-1}}\right| > 1 \text{ for large } n.$$

Therefore $|a_n|$ is increasing and does not converge to 0. By Theorem 1.5.1, $\sum a_n$ diverges.

We conclude that $\sum \frac{(2n)!}{(n!)^2}x^n$ converges for $|x| < \frac{1}{4}$ and diverges for $|x| > \frac{1}{4}$. For $x = \frac{1}{4}$, we cannot compare with the geometric series $\sum r^n$. Instead, we may try to compare with $\sum \frac{1}{n^p}$. The terms $a_n = \frac{(2n)!}{(n!)^2 4^n} > 0$, and

$$\frac{a_n}{a_{n-1}} = \frac{2n(2n-1)}{4n^2} = 1 - \frac{1}{2n}, \quad \frac{\frac{1}{n^p}}{\frac{1}{(n-1)^p}} = 1 - \frac{p}{n} + o\left(\frac{1}{n}\right).$$

So we expect a_n to be comparable to $\frac{1}{n^{\frac{1}{2}}}$. Since $\sum \frac{1}{n^{\frac{1}{2}}}$ diverges, we expect $\sum a_n$ diverges. For a rigorous argument, we wish to show that

$$\frac{a_n}{a_{n-1}} \geq \frac{\frac{1}{n^p}}{\frac{1}{(n-1)^p}} \text{ for some } p \leq 1 \text{ and large } n.$$

Of course this holds for $p = 1 > \frac{1}{2}$. Therefore by the ratio test, we conclude that $\sum \frac{(2n)!}{(n!)^2 4^n}$ diverges.

We will show in Example 6.2.3 that, for $r = -\frac{1}{4}$, the series $\sum (-1)^n \frac{(2n)!}{(n!)^2 4^n}$ converges.

There are several generalisations we can make from the example. First, if we apply the ratio test to the case $\sum b_n = \sum r^n$ is the geometric series, we find that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq r < 1 \text{ for large } n \implies \sum a_n \text{ converges.}$$

The limit version of this specialised ratio test is

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \implies \sum a_n \text{ converges.}$$

On the other hand, the example also shows that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \implies \sum a_n \text{ diverges.}$$

Second, when the comparison with the geometric series does not work, we may compare with $\sum \frac{1}{n^p}$. Suppose

$$\left|\frac{a_{n+1}}{a_n}\right| \le 1 - \frac{p}{n} \text{ for some } p > 1 \text{ and large } n. \tag{6.1.1}$$

We find q satisfying $p > q > 1$. Then the property above implies

$$\left|\frac{a_{n+1}}{a_n}\right| \le 1 - \frac{q}{n} + o\left(\frac{1}{n}\right) = \frac{\dfrac{1}{n^q}}{\dfrac{1}{(n-1)^q}} \text{ for large } n.$$

By applying the ratio test to $b_n = \frac{1}{n^p}$, we conclude that $\sum a_n$ converges. The use of criterion (6.1.1) for the convergence of series is the *Raabe test*.

The Raabe test also has the limit version. We note that (6.1.1) is equivalent to

$$n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) \ge p > 1 \text{ for large } n.$$

This will be satisfied if we can verify

$$\lim_{n\to\infty} n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) > 1.$$

Exercise 6.1.21. Determine convergence.

1. $\frac{4}{2} + \frac{4\cdot 7}{2\cdot 6} + \frac{4\cdot 7\cdot 10}{2\cdot 6\cdot 10} + \cdots$.
2. $\frac{2}{4} + \frac{2\cdot 6}{4\cdot 7} + \frac{2\cdot 6\cdot 10}{4\cdot 7\cdot 10} + \cdots$.
3. $\frac{2}{4} + \frac{2\cdot 5}{4\cdot 7} + \frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10} + \cdots$.
4. $\frac{2}{4\cdot 7} + \frac{2\cdot 5}{4\cdot 7\cdot 10} + \frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10\cdot 13} + \cdots$.

Exercise 6.1.22. Determine convergence.

1. $\sum \frac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)}$.
2. $\sum \frac{a(a+1^p)\cdots(a+n^p)}{b(b+1^p)\cdots(b+n^p)}$.
3. $\sum \frac{a^p(a+c)^p\cdots(a+nc)^p}{b^q(b+d)^q\cdots(b+nd)^q}\frac{1}{n^r}$.
4. $\sum \frac{(a+c)(a+2c)^2\cdots(a+nc)^n}{(b+d)(b+2d)^2\cdots(b+nd)^n}$.
5. $\sum \frac{(a_1+b_1 1+c_1 1^2)\cdots(a_1+b_1 n+c_1 n^2)}{(a_2+b_2 1+c_2 1^2)\cdots(a_2+b_2 n+c_2 n^2)}$.
6. $\sum \frac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)}\frac{c(c+1)\cdots(c+n)}{d(d+1)\cdots(d+n)}$.

Exercise 6.1.23. Determine convergence. There might be come special values of r for which you cannot yet make conclusion.

1. $\sum \dfrac{(n!)^2}{(2n)!} r^n$.
2. $\sum \dfrac{(3n)!}{(n!)^3} r^n$.
3. $\sum \dfrac{n!(2n)!}{(3n)!} r^n$.
4. $\sum \dfrac{n^n}{n!} r^n$.
5. $\sum \dfrac{n!}{n^n} r^n$.
6. $\sum \dfrac{n!}{(n+1)^n} r^n$.
7. $\sum \dfrac{n^{n+1}}{(n+1)!} r^n$.
8. $\sum \dfrac{(2n)!}{n^{2n}} r^n$.

Exercise 6.1.24. Prove the divergent part of the Raabe test.

1. If $\dfrac{a_{n+1}}{a_n} \geq 1 - \dfrac{1}{n}$ for sufficiently large n, then $\sum a_n$ diverges.
2. If $a_n > 0$ and $\lim_{n\to\infty} n\left(1 - \left|\dfrac{a_{n+1}}{a_n}\right|\right) < 1$, then $\sum a_n$ diverges.

6.2 Conditional Convergence

Like improper integrals, the comparison test implies that a series can have three mutually exclusive possibilities:

- *Absolute Convergence*: $\sum |a_n|$ converges ($\implies \sum a_n$ converges).
- *Conditional Convergence*: $\sum |a_n|$ diverges and $\sum a_n$ converges.
- *Divergence*: $\sum a_n$ diverges ($\implies \sum |a_n|$ diverges).

6.2.1 Test for Conditional Convergence

The series $\sum \dfrac{(-1)^{n+1}}{n}$ in Example **??** is a typical conditionally convergent series. Its absolute value series is the harmonic series $\sum \dfrac{1}{n}$, which we know diverges. We cannot apply the comparison test to the whole series because the conclusion of comparison test is always absolute convergence. In fact, we applied Theorem 1.5.2 to the even partial sum of the series in Example **??**. The following is an elaboration of the idea of Example **??**.

Example 6.2.1. Consider the series $\sum n^a b^n$. By $\lim_{n\to\infty} \sqrt[n]{|n^a b^n|} = |b|$ and the root test, $\sum n^a b^n$ absolutely converges for $|b| < 1$ and diverges for $|b| > 1$.

If $b = 1$, then the series is $\sum n^a$, which converges if and only if $a < -1$. If $b = -1$, then the series is $\sum (-1)^n n^a$, which by Example **??** converges if and only if $a < 0$.

In conclusion, the series $\sum n^a b^n$ absolutely converges for either $|b| < 1$, or $a < -1$ and $|b| = 1$, conditionally converges for $-1 \leq a < 0$ and $b = -1$, and diverges otherwise.

Example 6.2.2. Consider the alternating series $\sum (-1)^n \dfrac{n^2 + a}{n^3 + b}$. The corresponding absolute value series is comparable to the harmonic series and therefore diverges.

If we can show that $f(x)=\dfrac{x^2+a}{x^3+b}$ is decreasing, therefore, then the Leibniz test implies the conditional convergence. By

$$f'(x)=\frac{-x^4-3ax^2+2xb}{(x^3+b)^2},$$

the function indeed decreases for sufficiently large x.

Example 6.2.3. In Example 6.1.9, we determined the convergence of $\sum\dfrac{(2n)!}{(n!)^2}x^n$ for all x except $x=-\dfrac{1}{4}$. For $x=-\dfrac{1}{4}$, the series is alternating, and $|a_n|$ is decreasing by $\left|\dfrac{a_n}{a_{n-1}}\right|=1-\dfrac{1}{2n}<1$. If we can show that a_n converges to 0, then we can apply the Leibniz test.

We compare with the ratio of $|a_n|$ with the ratio of $\dfrac{1}{n^p}$

$$\left|\frac{a_n}{a_{n-1}}\right|=1-\frac{1}{2n}\le\frac{\dfrac{1}{n^p}}{\dfrac{1}{(n-1)^p}}=1-\frac{p}{n}+o\left(\frac{1}{n}\right).$$

This happens if we pick $p=0.4$ and n is sufficiently large. The comparison of the ratio implies $|a_n|<\dfrac{c}{n^{0.4}}$ for a constant a and sufficiently large n. This further implies that $\lim|a_n|=0$. By the Leibniz test, we conclude that $\sum(-1)^n\dfrac{(2n)!}{(n!)^2 4^n}$ converges.

Combined with Examples 6.1.9, we conclude that $\sum\dfrac{(2n)!}{(n!)^2}x^n$ absolutely converges for $|x|<\dfrac{1}{4}$, conditionally converges for $x=-\dfrac{1}{4}$, and diverges otherwise.

Exercise 6.2.1. Suppose $a_n>0$ and $\dfrac{a_n}{a_{n-1}}=1-\dfrac{p}{n}+o\left(\dfrac{1}{n}\right)$ for some $p>0$. Prove that $\lim a_n=0$ and $\sum(-1)^n a_n$ converges.

Exercise 6.2.2. Determine absolute or conditional convergence.

1. $\dfrac{4}{2}-\dfrac{4\cdot 7}{2\cdot 6}+\dfrac{4\cdot 7\cdot 10}{2\cdot 6\cdot 10}-\cdots$.
2. $\dfrac{2}{4}-\dfrac{2\cdot 6}{4\cdot 7}+\dfrac{2\cdot 6\cdot 10}{4\cdot 7\cdot 10}-\cdots$.
3. $\dfrac{2}{4}-\dfrac{2\cdot 5}{4\cdot 7}+\dfrac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10}-\cdots$.
4. $\dfrac{2}{4\cdot 7}-\dfrac{2\cdot 5}{4\cdot 7\cdot 10}+\dfrac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10\cdot 13}-\cdots$.

Exercise 6.2.3. Determine absolute or conditional convergence.

1. $\sum(-1)^n\dfrac{a(a+1^r)\cdots(a+n^r)}{b(b+1^r)\cdots(b+n^r)}$.
2. $\sum(-1)^n\dfrac{(a+1)(a+2)^2\cdots(a+n)^n}{(b+1)(b+2)^2\cdots(b+n)^n}$.

Exercise 6.2.4. Determine absolute or conditional convergence.

1. $\sum\dfrac{(-1)^n n^2}{n^3+n+2}$.
2. $\sum\dfrac{n^2+\sin n}{(-1)^n n^3+n+2}$.
3. $\sum\dfrac{(-1)^n}{(a+nb)^p}$.
4. $\sum\dfrac{r^n}{n^p}$.
5. $\sum\dfrac{r^n}{n^p(\log n)^q}$.
6. $\sum(-1)^{\frac{n(n-1)}{2}}\dfrac{1}{n^p}$.
7. $\sum(-1)^n n^q a^{n^p}$.
8. $\sum\dfrac{(-1)^n}{n^{p+\frac{q}{\log n}}}$.
9. $\sum\dfrac{(-1)^n n^{n+p}}{(an^2+bn+c)^{\frac{n}{2}+q}}$.

Exercise 6.2.5. Determine the absolute or conditional convergence for the undecided cases in Exercise 6.1.23.

Like the convergence of improper integrals, we also have the analogues of the Dirichlet and Abel tests.

Proposition 6.2.1 (Dirichlet Test). *Suppose the partial sum of $\sum a_n$ is bounded. Suppose b_n is monotonic and $\lim_{n\to\infty} b_n = 0$. Then $\sum a_nb_n$ converges.*

Proposition 6.2.2 (Abel Test). *Suppose $\sum a_n$ converges. Suppose b_n is monotonic and bounded. Then $\sum a_nb_n$ converges.*

Example 6.2.4. In Example 6.1.7, we showed that $\sum\dfrac{|\sin na|}{n^p}$ diverges when a is not an integer multiple of π. Then Example 5.7.14 suggests that the series should converge conditionally.

By the Dirichlet test, if we can show that the partial sum

$$s_n = \sin a + \sin 2a + \cdots + \sin na$$

is bounded, then the series converges. By

$$\begin{aligned}2s_n\sin\frac{a}{2} &= \left(\cos\left(a-\frac{a}{2}\right)-\cos\left(a+\frac{a}{2}\right)\right)+\left(\cos\left(2a-\frac{a}{2}\right)-\cos\left(2a+\frac{a}{2}\right)\right)\\ &\quad+\cdots+\left(\cos\left(na-\frac{a}{2}\right)-\cos\left(na+\frac{a}{2}\right)\right)\\ &=\cos\frac{a}{2}-\cos\left(na+\frac{a}{2}\right),\end{aligned}$$

we get

$$|s_n|\le\frac{1}{\left|\sin\dfrac{a}{2}\right|}.$$

The right side is a bound for the partial sums in case a is not a multiple of π.

Exercise 6.2.6. Derive the Leibniz test and the Abel test from the Dirichlet test.

Exercise 6.2.7. Prove that if $\sum \frac{a_n}{n^p}$ converges, then $\sum \frac{a_n}{n^q}$ converges for any $q > p$.

Exercise 6.2.8. Determine the absolute and conditional convergence.

1. $\sum \dfrac{\cos na}{(n+b)^p}$.
2. $\sum (-1)^n \dfrac{\cos na}{n+b}$.
3. $\sum \dfrac{\sin na}{n^p(\log n)^q}$.
4. $\sum (-1)^n \dfrac{\sin^2 na}{n^p}$.
5. $\sum \dfrac{\sin^3 na}{n^p(\log n)^q}$.
6. $\sum (-1)^{\frac{n(n-1)}{2}} \dfrac{\sin na}{n^p}$.

Exercise 6.2.9. Determine absolute or conditional convergence.

1. $\sum \dfrac{1}{n+(-1)^n n^2}$.
2. $\sum \dfrac{(-1)^n}{(\sqrt{n}+(-1)^n)^p}$.
3. $\sum \dfrac{(-1)^n}{(n+(-1)^n)^p}$.
4. $\sum \dfrac{(-1)^{\frac{n(n-1)}{2}}}{\sqrt{n}+(-1)^n}$.
5. $\sum \dfrac{(-1)^n}{n^p+(-1)^n}$.

Exercise 6.2.10. Determine the convergence.

1. $\sum \sin \sqrt{n^2+a}\pi$.
2. $\sum (-1)^n \left(1 - \dfrac{a \log n}{n}\right)^n$.
3. $\sum \left(\log \dfrac{an+b}{cn+d}\right)^n$.

Exercise 6.2.11. Let $[x]$ be the biggest integer $\le x$. Determine the convergence of $\sum \dfrac{(-1)^{[\sqrt{n}]}}{n^p}$ and $\sum \dfrac{(-1)^{[\log n]}}{n^p}$.

6.2.2 Absolute v.s. Conditional

The distinction between absolute and conditional convergence has implications on how we can manipulate series. For example, we have $a+b+c+d = c+b+a+d$. However, we need to be more careful in rearranging orders in an infinite sum.

Theorem 6.2.3. *The sum of an absolutely convergent series does not depend on the order. On the other hand, given any conditionally convergent series and any number s, it is possible to rearrange the order so that the sum of the rearranged series is s.*

Example 6.2.5. We know from Example **??** that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally. The partial sum can be estimated from the partial sum of the harmonic series in Example 6.1.1

$$\begin{aligned}
&1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\\
&=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)\\
&=(\log 2n+\gamma+\epsilon_{2n})-(\log n+\gamma+\epsilon_n)=\log 2+(\epsilon_{2n}-\epsilon_n).
\end{aligned}$$

This implies

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log 2.$$

If the terms are rearranged, so that one positive term is followed by two negative terms, then the partial sum is

$$\begin{aligned}
&1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots+\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\\
&=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)\\
&\quad-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{4n}\right)\\
&=(\log 2n+\gamma+\epsilon_{2n})-\frac{1}{2}(\log n+\gamma+\epsilon_n)-\frac{1}{2}(\log 2n+\gamma+\epsilon_{2n})\\
&=\frac{1}{2}\log 2+\frac{1}{2}(\epsilon_{2n}-\epsilon_n).
\end{aligned}$$

If two positive terms are followed by one negative term, then the partial sum is

$$\begin{aligned}
&1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots+\frac{1}{4n-1}+\frac{1}{4n-3}-\frac{1}{2n}\\
&=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{4n}\right)\\
&\quad-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{4n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)\\
&=(\log 4n+\gamma+\epsilon_{4n})-\frac{1}{2}(\log 2n+\gamma+\epsilon_{2n})-\frac{1}{2}(\log n+\gamma+\epsilon_n)\\
&=\frac{3}{2}\log 2+\frac{1}{2}(\epsilon_{4n}-\epsilon_{2n}-\epsilon_n).
\end{aligned}$$

We get

$$\begin{aligned}
1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots&=\frac{1}{2}\log 2,\\
1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots&=\frac{3}{2}\log 2.
\end{aligned}$$

Exercise 6.2.12. Rearrange the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ so that p positive terms are followed by q negative terms and the pattern repeated. Show that the sum of new series is $\log 2+\frac{1}{2}\log\frac{p}{q}$.

Exercise 6.2.13. Show that $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$ converges, but $1-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{10}}-\frac{1}{\sqrt{12}}+\cdots$ diverges.

Another distinction between absolute and conditional convergence is reflected on the product of two series.

Theorem 6.2.4. *Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely. Then $\sum_{i,j=1}^{\infty} a_ib_j$ also converge absolutely, and*

$$\sum_{i,j=1}^{\infty} a_ib_j=\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right).$$

Note that the infinite sum $\sum_{i,j=1}^{\infty} a_ib_j$ is a "double series" with two indices i and j. There are many ways of arranging this series into a single series. For example, the following is the "diagonal arrangement"

$$\begin{aligned}\sum (ab)_k &= a_1b_1+a_1b_2+a_2b_1+\cdots\\ &\quad + a_1b_{n-1}+a_2b_{n-2}+\cdots+a_{n-1}b_1+\cdots,\end{aligned}$$

and the following is the "square arrangement"

$$\begin{aligned}\sum (ab)_k &= a_1b_1+a_1b_2+a_2b_2+a_2b_1+\cdots\\ &\quad + a_1b_n+a_2b_n+\cdots+a_nb_{n-1}+a_nb_n+a_nb_{n-1}+\cdots+a_nb_1+\cdots.\end{aligned}$$

Under the condition of the theorem, the series is supposed to converge absolutely. Then by Theorem 6.2.3, all arrangements give the same sum.

Example 6.2.6. We know from Example **??** that $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ absolutely converges to e^x. By Theorem 6.2.4, we have

$$\begin{aligned}e^xe^y &= \left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{y^n}{n!}\right)=\sum_{i,j=0}^{\infty}\frac{x^iy^j}{i!j!}=\sum_{n=0}^{\infty}\sum_{i+j=n}\frac{x^iy^j}{i!j!}\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}x^iy^j=\sum_{n=0}^{\infty}\frac{1}{n!}(x+y)^n=e^{x+y}.\end{aligned}$$

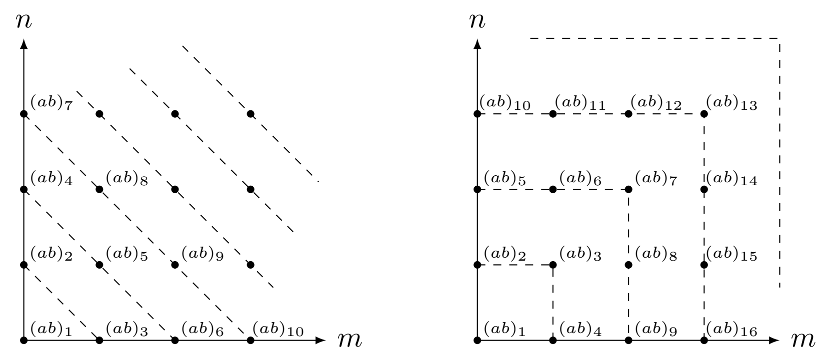


Figure 6.2.1: Diagonal and square arrangements.

In the second to the last equality, we used the binomial expansion.

Exercise 6.2.14. If you take the product of a geometric series with itself, what conclusion can you make?

Exercise 6.2.15. Suppose $\sum \frac{(-1)^n}{\sqrt{n}} = l$. Show that the square arrangement of the product of the series with itself converges to l^2. What is the sum of the diagonal arrangement?

6.3 Power Series

A *power series* at x_0 is

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

By a simple change of variable, it is sufficient to consider power series at 0

$$\sum_{n=0}^{\infty} a_n x^n.$$

6.3.1 Convergence of Taylor Series

If $f(x)$ has derivatives of arbitrary order at x_0, then the high order approximations of the function gives us the *Taylor series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\cdots .$$

The partial sum of the series is the n-th order Taylor expansion $T_n(x)$ of $f(x)$ at x_0.

In Example **??**, we used the Lagrange form of the remainder $R_n(x) = f(x)-T_n(x)$ (Theorem 4.3.1) to show that the Taylor series of e^x converges to e^x. Exercise **??**

further showed that the Taylor series of $\sin x$ and $\cos x$ converge to the trigonometric functions.

Example 6.3.1. Consider the Taylor series $\sum_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{n}x^n$ of $\log(1+x)$. We have

$$\frac{d^n}{dx^n}\log(1+x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n},$$

and the Lagrange form of the remainder gives

$$|R_n(x)|=\frac{1}{(n+1)!}\frac{n!}{|1+c|^{n+1}}|x|^{n+1}=\frac{|x|^{n+1}}{(n+1)|1+c|^{n+1}},$$

where c lies between 0 and x. If $-\dfrac{1}{2}\le x\le 1$, then $|x|\le|1+c|$, and we get $|R_n(x)|<\dfrac{1}{n+1}$. Therefore $\lim_{n\to\infty}R_n(x)=0$, and the Taylor series converges to $\log(1+x)$.

The Taylor series is the harmonic series at $x=-1$ and therefore diverges. For $|x|>1$, the terms of the Taylor series diverges to ∞, and therefore the Taylor series also diverges. The remaining case is $-1<x<-\dfrac{1}{2}$.

In Exercise 6.3.2, a new form of the remainder is used to show that the Taylor series actually converges to $\log(1+x)$ for all $-1<x\le 1$.

Example 6.3.2. The Taylor series of $(1-x)^{-1}$ is the geometric series $\sum_{n=0}^{\infty}x^n$. Example **??** shows that the Taylor series converges to the function for $|x|<1$ and diverges for $|x|\ge 1$.

The Taylor series of $(1+x)^p$ is $\sum_{n=0}^{\infty}\dfrac{p(p-1)\cdots(p-n+1)}{n!}x^n$. The Lagrange form of the remainder gives

$$\begin{aligned}|R_n(x)|&=\frac{|p(p-1)\cdots(p-n)|}{(n+1)!}|1+c|^{p-n-1}|x|^n\\&=\frac{|p(p-1)\cdots(p-n)|}{(n+1)!}\frac{|1+c|^{p-1}|x|^n}{|1+c|^n},\end{aligned}$$

where c lies between 0 and x. For $-\dfrac{1}{2}<x\le 1$, we have $|x|\le|1+c|$ and $|1+c|^{p-1}$ is bounded. Moreover, by Exercise 6.2.1, for $p>-1$, we have

$$\lim_{n\to\infty}\frac{|p(p-1)\cdots(p-n)|}{(n+1)!}=0.$$

Then we conclude that $\lim_{n\to\infty}R_n(x)=0$ for $-\dfrac{1}{2}<x\le 1$ and $p>-1$, and the Taylor series converges to $(1+x)^p$.

The Taylor series diverges for $|x| > 1$. Using a new form of the remainder, Exercise 6.3.2 shows that the Taylor series actually converges to $(1+x)^p$ for $|x| < 1$ and any p.

Example 6.3.3. In Example 4.1.18, we showed that the function

$$f(x) = \begin{cases} e^{-\frac{1}{|x|}}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

has all the high derivatives equal to 0. Therefore the Taylor series of the function is $\sum 0 = 0$, although the function is not 0.

A smooth function is *analytic* if it is always equal to its Taylor series. As pointed out after Example 4.1.18, the analytic property means that the function can be measured by polynomials.

Exercise 6.3.1. Use the Lagrange form of the remainder to show that *Cauchy form* of the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-x_0),$$

where c lies between x_0 and x. Use this to show that the Taylor series of $\log(1+x)$ and $(1+x)^p$ at $x_0 = 0$ converge to the respective functions for any $|x| < 1$.

Exercise 6.3.2. The *Cauchy form* of the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-x_0),$$

where c lies between x_0 and x. Use this to show that the Taylor series of $\log(1+x)$ and $(1+x)^p$ at $x_0 = 0$ converge to the respective functions for any $|x| < 1$.

6.3.2 Radius of Convergence

We regard a power series $\sum a_n x^n$ as a function with variable x. The domain of the function consists of those x, such that the series converges.

We may use the root test to find the domain. Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ converges. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Let

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}.$$

By the limit version of the root test (see the discussion after Theorem 6.1.3), the power series converges for $|x| < R$ and diverges for $|x| > R$.

Example 6.3.4. For the geometric series $\sum x^n = 1+x+x^2+\cdots$, we have $\lim_{n\to\infty} \sqrt[n]{|1|} = 1$. Therefore the geometric series converges for $|x| < 1$ and diverges for $|x| > 1$. Moreover, the series also diverges for $|x| = 1$.

For the Taylor series $\sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}}{n} x^n$ of $\log(1+x)$, we have $\lim_{n\to\infty} \sqrt[n]{\left|\dfrac{(-1)^{n+1}}{n}\right|} = \lim_{n\to\infty} \dfrac{1}{\sqrt[n]{n}} = 1$. Therefore the series converges for $|x| < 1$ and diverges for $|x| > 1$. We also know that the series converges at $x = 1$ and diverges at $x = -1$.

Example 6.3.5. The series $\sum x^{2n}$ has $a_n = 1$ for even n and $a_n = 0$ for odd n. The sequence $\sqrt[n]{|a_n|}$ diverges because it has two limits: $\lim_{\text{even}} \sqrt[n]{|a_n|} = 1$ and $\lim_{\text{odd}} \sqrt[n]{|a_n|} = 0$.

On the other hand, as a function, we have $\sum x^{2n} = f(x^2)$, where $f(x) = \sum x^n$ is the geometric series. Since the domain of f is $|x| < 1$, the domain of $f(x^2)$ is $|x^2| < 1$, which is equivalent to $|x| < 1$. Therefore $\sum x^{2n}$ converges for $|x| < 1$ and diverges for $|x| > 1$.

Theorem 6.3.1. *For any power series $\sum a_n x^n$, there is $R \geq 0$, such that the series absolutely converges for $|x| < R$ and diverges for $|x| > R$.*

Using the root test, we have proved the theorem in case $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ converges. Example 6.3.11 suggests that the theorem is true in general. The theorem is a consequence of the fact that, if $\sum a_n r^n$ converges and $|x| < |r|$, then $\sum a_n x^n$ absolutely converges. Specifically, the convergence of $\sum a_n r^n$ implies $\lim_{n\to\infty} a_n r^n = 0$. This further implies that $|a_n r^n| < 1$ for sufficiently large n. Then for any fixed x satisfying $|x| < |r|$, we have

$$|a_n x^n| = |a_n r^n| \cdot \left|\frac{x}{r}\right|^n \leq \left|\frac{x}{r}\right|^n.$$

Since $\left|\dfrac{x}{r}\right| < 1$ implies the convergence of $\sum \left|\dfrac{x}{r}\right|^n$, by the comparison test, the series $\sum a_n x^n$ absolutely converges.

The number R is the *radius of convergence* of power series. If $R = 0$, then the power series converges only for $x = 0$. If $R = +\infty$, then the power series converges for all x.

The same radius of convergence applies to $\sum a_n (x - x_0)^n$. The power series converges on $(x_0 - R, x_0 + R)$, and diverges on $(-\infty, x_0 - R)$ and on $(x_0 + R, +\infty)$.

We had the formula for radius in case $\sqrt[n]{|a_n|}$ converges. In general, the sequence may have many possible limits (for various subsequences). Let the *upper limit* $\overline{\lim}\, \sqrt[n]{|a_n|}$ be the maximum of all the possible limits. Then the radius is

$$R = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}}.$$

One can verify the formula for Example 6.3.11.

Example 6.3.6. For any p, we have $\lim_{n\to\infty}\sqrt[n]{n^p}=1$. Therefore the radius of convergence for the power series $\sum n^p x^n$ is 1. The example already appeared in Example 6.2.1.

Example 6.3.7. By $\lim_{n\to\infty}\sqrt[n]{2^n+3^n}=3$, the series $\sum(2^n+3^n)x^n$ converges for $|x|<\dfrac{1}{3}$ and diverges for $|x|>\dfrac{1}{3}$. The series also diverges for $|x|=\dfrac{1}{3}$ because the terms do not converge to 0.

We note that the radius of convergence is $\dfrac{1}{2}$ for $\sum 2^n x^n$ and $\dfrac{1}{3}$ for $\sum 3^n x^n$. The radius for the sum of the two series is the smaller one.

Example 6.3.8. By $\lim_{n\to\infty}\sqrt[n]{n^n}=+\infty$, the series $\sum n^n x^n$ diverges for all $x\neq 0$.

By $\lim_{n\to\infty}\sqrt[n]{\dfrac{1}{n^n}}=0$, the series $\sum\dfrac{(-1)^n}{n^n}x^n$ converges for all x.

Example 6.3.9. In Example 6.1.9, we use the ratio test to show that the radius of convergence of $\sum\dfrac{(2n)!}{(n!)^2}x^n$ is $\dfrac{1}{4}$. The idea can be used to show that the radius of convergence for $\sum a_n x^n$ is $R=\lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$, provided that the limit converges.

For example, by

$$\lim_{n\to\infty}\frac{n^p}{(n+1)^p}=\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^p=1,$$

the radius of convergence for $\sum n^p x^n$ is 1. Moreover, the radius of convergence for the Taylor series of e^x is

$$\lim_{n\to\infty}\frac{\dfrac{1}{n!}}{\dfrac{1}{(n+1)!}}=\lim_{n\to\infty}(n+1)=+\infty.$$

Example 6.3.10. The *Bessel function* of order 0 is

$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

The radius of convergence is the square root of the radius of convergence of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{2n}(n!)^2}.$$

By

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^n}{2^{2n}(n!)^2}}{\dfrac{(-1)^{n+1}}{2^{2n+2}((n+1)!)^2}}\right| = \lim_{n\to\infty} 4(n+1)^2 = +\infty,$$

the later series converges for all x. Therefore the Bessel function is defined for all x.

Note that we cannot calculate $\lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$ directly for the Bessel function. In fact, the limit diverges.

Exercise 6.3.3. Suppose a_n can be divided into two subsequences $a_{n'}$ and $a_{n''}$. Suppose $\lim_{n'\to\infty}\sqrt[n']{|a_{n'}|} = l'$ and $\lim_{n''\to\infty}\sqrt[n'']{|a_{n''}|} = l''$ converge. Prove that $\dfrac{1}{\max\{l', l''\}}$ is the radius of convergence for $\sum a_n x^n$.

Exercise 6.3.4. Suppose $\lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = R$ converges. Prove that R is the radius of convergence for $\sum a_n x^n$.

Exercise 6.3.5. Determine the radius of convergence.

1. $\sum(-1)^n\dfrac{x^n}{n^p}$.
2. $\sum n^p(x-1)^n$.
3. $\sum n^p(2x-1)^n$.
4. $\sum n^p(2x+3)^n$.
5. $\sum\left(\dfrac{a^n}{n}+\dfrac{b^n}{n^2}\right)x^n$.
6. $\sum\dfrac{x^n}{a^n+b^n}$.
7. $\sum n^{\sqrt{n}}x^n$.
8. $\sum n!x^n$.
9. $\sum\dfrac{(-1)^{n+1}}{\sqrt{n!}}x^n$.
10. $\sum\dfrac{(n!)^2}{(2n)!}x^n$.
11. $\sum\dfrac{(3n)!}{n!(2n)!}x^n$.
12. $\sum a^{n^2}x^n$.
13. $\sum a^{n^2}x^{n^2}$.
14. $\sum 2^n x^{n^2-1}$.
15. $\sum(2+(-1)^n)^n x^n$.
16. $\sum\dfrac{(2+(-1)^n)^n}{\log n}x^n$.

Exercise 6.3.6. Find the radius of convergence.

1. $\sum\left(\dfrac{n+1}{n}\right)^n x^n$.
2. $\sum\left(\dfrac{n+1}{n}\right)^{n^2} x^n$.
3. $\sum\left(\dfrac{n+a}{n+b}\right)^{n^2} x^{n+2}$.
4. $\sum(-1)^n\left(\dfrac{n+1}{n}\right)^{n^2} x^n$.
5. $\sum(-1)^n\left(\dfrac{n+1}{n}\right)^{n^2} x^{n^2}$.
6. $\sum\left(\dfrac{an+b}{cn+d}\right)^n x^{n-2}$.

Exercise 6.3.7. Find the domain of the *Bessel function* of order 1

$$J_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}.$$

Exercise 6.3.8. Find the domain of the *Airy function*

$$A(x) = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots .$$

Exercise 6.3.9. Suppose the radii of convergence for $\sum a_n x^n$ and $\sum b_n x^n$ are R and R'. What can you say about the radii of convergence for the following power series?

$$\sum (a_n + b_n)x^n, \quad \sum (a_n - b_n)x^n, \quad \sum (-1)^n a_n x^n, \quad \sum a_n(2x-1)^n,$$

$$\sum a_n x^{2n}, \quad \sum a_n x^{n+2}, \quad \sum a_{2n} x^n, \quad \sum a_{n+2} x^n,$$
$$\sum a_n x^{n^2}, \quad \sum a_{n^2} x^n, \quad \sum a_{2n} x^{2n}, \quad \sum a_{n^2} x^{n^2}.$$

6.3.3 Function Defined by Power Series

Examples 4.1.18 and 6.3.3 suggest that if a function is the sum of a power series, then the function is particularly nice. In fact, the function should be nicer than functions with derivatives of any order.

Because power series converge absolutely within the radius of convergence, by Theorem 6.2.4, we can multiply two power series together within the common radius of convergence.

Theorem 6.3.2. *Suppose $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$ have radii of convergence R and R'. Then for $|x| < \min\{R, R'\}$, we have*

$$f(x)g(x) = \sum c_n x^n, \quad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

The product should be the sum of $a_i b_j x^{i+j}$. We get the power series $\sum c_n x^n$ by gathering all the terms with power x^n.

The power series can also be differentiated or integrated term by term within the radius of convergence.

Theorem 6.3.3. *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$. Then*

$$f'(x) = \sum_{n=1}^{\infty} (a_n x^n)' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots ,$$

and

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1} + \cdots$$

for $|x| < R$.

Example 6.3.11. Taking the derivative of $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we get

$$\begin{aligned}
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots, \\
\frac{2}{(1-x)^3} &= 2\cdot 1 + 3\cdot 2x + 4\cdot 3x^2 + 5\cdot 4x^3 + \cdots + n(n-1)x^{n-2} + \cdots.
\end{aligned}$$

Therefore

$$\begin{aligned}
1^2x + 2^2x^2 + \cdots + n^2x^n + \cdots &= \sum_{n=1}^{\infty} n^2x^n = x\sum_{n=1}^{\infty} nx^{n-1} + x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} \\
&= x\frac{1}{(1-x)^2} + x^2\frac{2}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}.
\end{aligned}$$

If we integrate instead, then we get

$$\log(1-x) = -\int_0^x \frac{dx}{1-x} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n!} - \cdots, \quad \text{for } |x| < 1.$$

Substituting $-x$ for x, we get the Taylor series of $\log(1+x)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n!} + \cdots, \quad \text{for } |x| < 1.$$

Note that in Example 6.3.1, by estimating the remainder, we were able to prove the equality rigorously only for $-\frac{1}{2} < x < 1$. Here by using term wise integration, we get the equality for all x within the radius of convergence.

Example 6.3.12. By integrating $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, we get the Taylor series of $\arctan x$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \cdots \quad \text{for } |x| < 1.$$

Exercise 6.3.10. Use the product of power series to verify the identity $\sin 2x = 2\sin x\cos x$.

Exercise 6.3.11. Find Taylor series and determine the radius of convergence.

1. $\dfrac{1}{(x-1)(x-2)}$, at 0.
2. $\sqrt{x}$, at $x = 2$.
3. $\sin x^2$, at 0.
4. $\sin^2 x$, at 0.
5. $\sin x$, at $\dfrac{\pi}{2}$.
6. $\sin 2x$, at $\dfrac{\pi}{2}$.
7. $\arcsin x$, at 0.
8. $\arctan x$, at 0.
9. $\displaystyle\int_0^x \frac{\sin t}{t}dt$, at 0.

Exercise 6.3.12. Given the Taylor series $\sum_{n=0}^{\infty} a_n x^n$ of $f(x)$, find the Taylor series of $\dfrac{f(x)}{1+x}$.

Exercise 6.3.13. Show that the function $f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{(n!)^2}$ satisfies $xf'' + f' - f = 0$.

Exercise 6.3.14. Show that the Airy function in Exercise 6.3.8 satisfies $f'' - xf = 0$.

Exercise 6.3.15. Show that the Bessel functions in Example 6.3.10 and Exercise 6.3.7 satisfy

$$xJ_0'' + J_0' + xJ_0 = 0, \quad x^2 J_1'' + xJ_1' + (x^2 - 1)J_1 = 0.$$

A power series may or may not converge at the radius of convergence (i.e., at $\pm R$). If it converges, then the following gives the value of the sum.

Theorem 6.3.4. *Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$ and $x = R$. Then $\sum_{n=0}^{\infty} a_n R^n = \lim_{x \to R^-} \sum_{n=0}^{\infty} a_n x^n$.*

The theorem says that, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is also defined at R, then $f(x)$ is left continuous at R. We also have the similar statement at the other end $-R$.

Example 6.3.13. By Examples 6.3.11 and **??**, we know $\log(1 + x) = \sum_{n=1}^{n} \dfrac{(-1)^{n+1}}{n} x^n$ converges for $|x| < 1$, and the series converges at $x = 1$. Since $\log(1 + x)$ is continuous at $x = 1$, by Theorem 6.3.4, we get

$$\sum_{n=1}^{n} \frac{(-1)^{n+1}}{n} = \lim_{x \to 1^-} \sum_{n=1}^{n} \frac{(-1)^{n+1}}{n} x^n = \lim_{x \to 1^-} \log(1 + x) = \log(1 + 1) = \log 2.$$

We computed the sum in Example 6.2.5 by another way.

Exercise 6.3.16. Find the sum. Discuss what happens at the radius of convergence.

1. $\sum_{n=1}^{\infty} n^2 x^n$.
2. $\sum_{n=1}^{\infty} n^3 x^n$.
3. $\sum_{n=2}^{\infty} \dfrac{(x-1)^n}{n(n-1)}$.
4. $\sum_{n=1}^{\infty} \dfrac{x^n}{n(n+1)(n+2)}$.
5. $\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{2n+1}$.
6. $\sum_{n=0}^{\infty} \dfrac{x^n}{2n+1}$.

Exercise 6.3.17. Find the sum. Discuss what happens at the radius of convergence.

1. $\sum_{n=1}^{\infty} (-1)^n \dfrac{x^{2n+1}}{n!}$.
2. $\sum_{n=1}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n+1)!}$.
3. $\sum_{n=1}^{\infty} \dfrac{x^n}{2^n (2n-1)!}$.

Exercise 6.3.18. Find the Taylor series of the function and the radius of convergence. Then explain why the sum of the Taylor series is the given function.

1. $\arcsin x$.
2. $\displaystyle\int_0^x \frac{\sin t}{t}dt$.
3. $\arctan x$.
4. $\displaystyle\int_0^x e^{-t^2}dt$.
5. $\log(x+\sqrt{1+x^2})$.
6. $\displaystyle\int_0^x \frac{\log(1-t)}{t}dt$.

6.4 Fourier Series

If $f(x+p) = f(x)$ for all x and a constant p, then we say $f(x)$ is a *periodic function* of *period* p. For example, the functions $\sin x$ and $\cos x$ have period 2π, and $\tan x$ has period π.

A periodic function of period p is also a periodic function of period kp for any integer k. For example, $\cos nx$ and $\sin nx$ have the period $p = \dfrac{2\pi}{n}$ as well as the period $np = 2\pi$.

If $f(x)$ has period p, then $f(x+a)$ still has period p, and $f(ax)$ is periodic with period $\dfrac{p}{a}$.

A combination of periodic functions of the same period p is still periodic of period p. For example, $\sin x + \cos x$, $\sin 3x \cos^2 x$, $\sqrt{2\sin^2 x + \cos^4 x}$ are periodic of period 2π.

The Taylor series approximates a function near a point by linear combinations of power functions. Similarly, we wish to approximate a periodic function by linear combinations of simple periodic functions such as sine and cosine. Specifically, we wish a periodic function $f(x)$ of period 2π to be approximated as

$$\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots . \end{aligned}$$

Note that there is no b_0 because $a_0 = a_0 \cos 0x + b_0 \sin 0x$. Moreover, like the Taylor series, we use $\sim$ instead of $=$ to indicate that the equality is yet to be established.

The approximation of a periodic function by (linear combinations of) trigonometric functions is not measured by the values at single points, but rather the overall approximation in terms of the integral of the difference function. This means that we can only expect that the sum of the trigonometric series to be equal to the function "almost everywhere".

6.4.1 Fourier Coefficient

Let $f(x)$ be a periodic function of period 2π. Our first problem is to find the coefficients a_n and b_n in the trigonometric series. By Exercise 5.1.6, the trigonometric functions are "orthogonal" in the sense that

$$\int_0^{2\pi} \cos mx \sin nx dx = 0;$$

$$\int_0^{2\pi} \cos mx \cos nx dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n \neq 0, \\ 2\pi, & \text{if } m = n = 0; \end{cases}$$

$$\int_0^{2\pi} \sin mx \sin nx dx = \begin{cases} 0, & \text{if } m \neq n \text{ or } m = n = 0, \\ \pi, & \text{if } m = n \neq 0. \end{cases}$$

We expect the sum of the trigonometric series to be equal to $f(x)$ as far as integrations are concerned. We also assume that the integration of infinite series can be calculated term by term. Then we get

$$\begin{aligned} \int_0^{2\pi} f(x) \cos nx dx &= a_0 \int_0^{2\pi} \cos nx dx + \sum_{k=1}^{\infty} a_m \int_0^{2\pi} \cos mx \cos nx dx \\ &\quad + \sum_{k=1}^{\infty} b_m \int_0^{2\pi} \sin mx \cos nx dx = \begin{cases} \pi a_n, & \text{if } n \neq 0, \\ 2\pi a_0, & \text{if } n = 0; \end{cases} \\ \int_0^{2\pi} f(x) \sin nx dx &= a_0 \int_0^{2\pi} \sin nx dx + \sum_{m=1}^{\infty} a_m \int_0^{2\pi} \cos mx \sin nx dx \\ &\quad + \sum_{m=1}^{\infty} b_m \int_0^{2\pi} \sin mx \sin nx dx = \pi b_n. \end{aligned}$$

Definition 6.4.1. The *Fourier series* of a periodic function $f(x)$ of period 2π is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

with the *Fourier coefficients*

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) dx, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \quad n \neq 0, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx, \quad n \neq 0. \end{aligned}$$

Example 6.4.1. Similar to Example 4.1.1, by

$$\begin{aligned}\sin^4 x &= \frac{1}{4}(1-\cos 2x)^2 = \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}(1-\cos 4x)\right)\\ &= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x,\end{aligned}$$

the right side is the Fourier series of $\sin^4 x$. The coefficients give

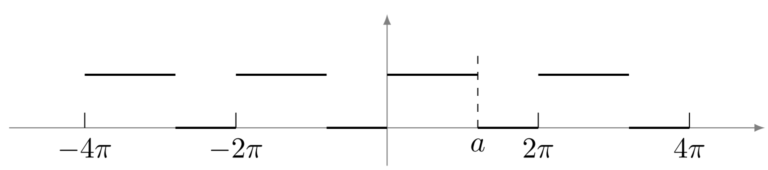
$$\int_0^{2\pi} \sin^4 x dx = \frac{3\pi}{4}, \quad \int_0^{2\pi} \sin^4 x \cos 2x dx = -\frac{\pi}{2}, \quad \int_0^{2\pi} \sin^4 x \cos 4x dx = \frac{\pi}{8}.$$

Example 6.4.2. A periodic function is determined by its value on one interval of period length. For example, if $f(x)$ is a periodic function of period 2π and satisfies

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < a, \\ 0, & \text{if } a \le x < 2\pi, \end{cases}$$

then

$$f(x) = \begin{cases} 0, & \text{if } 2k\pi \le x < 2k\pi + a, \\ 1, & \text{if } 2k\pi + a \le x < 2(k+1)\pi. \end{cases}$$



The Fourier coefficients are

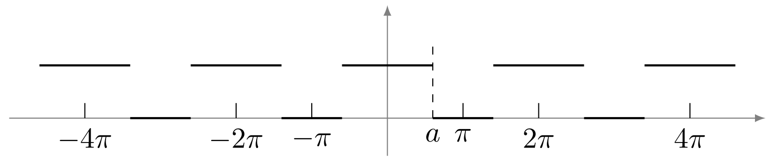
$$\begin{aligned}a_0 &= \frac{1}{2\pi}\int_0^a 1 dx = \frac{a}{2\pi},\\ a_n &= \frac{1}{\pi}\int_0^a \cos nx dx = \frac{\sin na}{n\pi},\\ b_n &= \frac{1}{\pi}\int_0^a \sin nx dx = \frac{1-\cos na}{n\pi}.\end{aligned}$$

and the Fourier series is

$$f(x) \sim \frac{a}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(\sin na \cos nx + (1-\cos na)\sin nx).$$

Example 6.4.3. Let $f(x)$ be the even periodic function of period 2π satisfying

$$f(x) = \begin{cases} 1, & \text{if } |x| \le a, \\ 0, & \text{if } a < |x| \le \pi. \end{cases}$$



We have

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{a} 1dx = \frac{a}{\pi},$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_0^{a} \cos nxdx = \frac{2a}{n\pi}.$$

The calculation used the fact that $\displaystyle\int_0^p f(x)dx = \int_a^{a+p} f(x)dx$ for any periodic function of period p. We may also calculate b_n and find $b_n = 0$. In fact, for even function, we expect that all the odd terms $b_n \sin nx$ to vanish.

Exercise 6.4.1. Suppose $f(x)$ is an even periodic function of period 2π. Prove that

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx, \quad b_n = 0.$$

Exercise 6.4.2. Suppose $f(x)$ is an odd periodic function of period 2π. Prove that

$$a_0 = a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx.$$

Exercise 6.4.3. Extend $f(x)$ on $\left(0, \frac{\pi}{2}\right)$ to a periodic function of period 2π, such that its Fourier series is of the form $\sum_{n=1}^{\infty} a_n \cos(2n-1)x$? How about $\sum_{n=1}^{\infty} b_n \sin(2n-1)x$?

Exercise 6.4.4. Given the Fourier series of $f(x)$ and $g(x)$, what is the Fourier series of $af(x) + bg(x)$? Use the idea and Example 6.4.2 to find the Fourier series of the periodic function $f(x)$ of period 2π satisfying

$$f(x) = \begin{cases} 1, & \text{if } a \le x < b, \\ 0, & \text{if } 0 \le x < a \text{ or } b \le x < 2\pi. \end{cases}$$

Exercise 6.4.5. Suppose $f(x)$ is a periodic function of period 2π. What is the relation between the Fourier series of $f(x)$ and $f(x+a)$? Use the idea and Example 6.4.3 to derive Example 6.4.2.

A periodic function $f(x)$ of period p may be converted to a periodic function $f\left(\frac{p}{2\pi}x\right)$ of period 2π. Then the Fourier series of $f\left(\frac{p}{2\pi}x\right)$ gives the Fourier series of $f(x)$. Alternatively, the basic periodic functions $\cos nx$ and $\sin nx$ of period 2π

give the basic periodic functions $\cos \dfrac{2n\pi}{p}x$ and $\sin \dfrac{2n\pi}{p}x$ of period p, and we expect the Fourier series of $f(x)$ to be

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{p}x + b_n \sin \frac{2n\pi}{p}x \right).$$

By an argument similar to the case of period 2π, we get the Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{p}\int_0^p f(x)dx, \\ a_n &= \frac{2}{p}\int_0^p f(x)\cos\frac{2n\pi}{p}xdx, \quad n \neq 0, \\ b_n &= \frac{2}{p}\int_0^p f(x)\sin\frac{2n\pi}{p}xdx, \quad n \neq 0. \end{aligned}$$

Exercise 6.4.6. Suppose $f(x)$ is a periodic function of period p.

1. Write down the Fourier series for $f\left(\dfrac{p}{2\pi}x\right)$, together with the formulae for its coefficients.

2. Convert the first part to statements about the original $f(x)$.

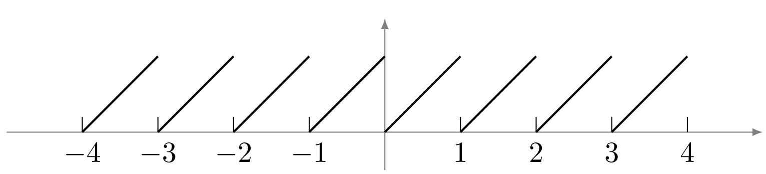
Exercise 6.4.7. Use Example 6.4.2 to derive the Fourier series of the periodic function of period 1 satisfying

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < a, \\ 0, & \text{if } a \le x < 1. \end{cases}$$

Exercise 6.4.8. Derive the formulae for the Fourier coefficients of periodic even or odd functions of period p, similar to Exercises 6.4.1 and 6.4.2.

Example 6.4.4. The function x on $(0,1)$ extends to a periodic function of period 1

$$f(x) = x - k, \quad k < x < k+1.$$



Note that we do not care about the value at the integer points because it does not affect the Fourier coefficients, which are

$$
\begin{aligned}
a_0 &= \frac{1}{1}\int_0^1 x dx = \frac{1}{2}, \\
a_n &= \frac{2}{1}\int_0^1 x\cos 2n\pi x dx = \frac{1}{n\pi}\int_0^1 x d\sin 2n\pi x = -\frac{1}{n\pi}\int_0^1 \sin 2n\pi x dx = 0, \\
b_n &= \frac{2}{1}\int_0^1 x\sin 2n\pi x dx = -\frac{1}{n\pi}\int_0^1 x d\cos 2n\pi x \\
&= -\frac{1}{n\pi}\left(1 - \int_0^1 \cos 2n\pi x dx\right) = -\frac{1}{n\pi}.
\end{aligned}
$$

We note that the reason for $a_n = 0$ for $n \neq 0$ is that $f(x) - \dfrac{1}{2}$ is an odd function.

The Fourier series is

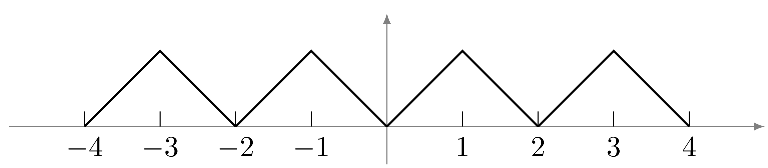
$$x \sim \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi x, \quad x \in (0,1).$$

We indicate $x \in (0,1)$ because the function equals x only on the interval. The function is $x-1$ instead of x on the interval $(1,2)$.

Example 6.4.5. The function x on $(0,1)$ extends to an even periodic function of period 2

$$f(x) = |x - 2k|, \quad 2k-1 < x < 2k+1.$$

This is also the extension of the function $|x|$ on $(-1,1)$ to a periodic function of period 2.



Using Exercises 6.4.1 and 6.4.8, we get $b_n = 0$ and

$$
\begin{aligned}
a_0 &= \frac{2}{2}\int_0^1 f(x)dx = \int_0^1 x dx = \frac{1}{2}, \\
a_n &= \frac{2}{1}\int_0^1 f(x)\cos n\pi x dx = 2\int_0^1 x\cos n\pi x dx = \begin{cases} -\dfrac{4}{n^2\pi^2}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}
\end{aligned}
$$

Replacing n by $2n+1$, the Fourier series is

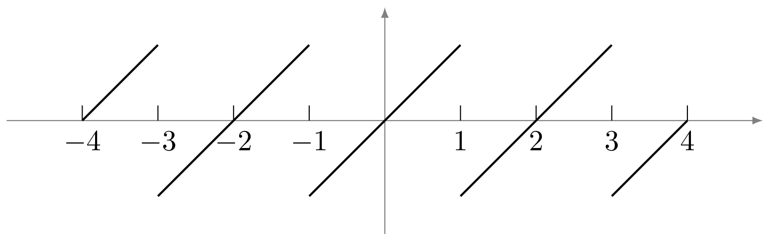
$$|x| \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x, \quad x \in (-1,1).$$

We may also extend the function to an odd periodic function of period 2. This is also the extension of the function x on $(-1,1)$ to a periodic function of period 2. The Fourier coefficients $a_n = 0$ for the odd function, and

$$b_n = 2\int_0^1 x \sin n\pi x dx = \frac{(-1)^{n+1}2}{n\pi}.$$

The Fourier series is

$$x \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n\pi} \sin n\pi x, \quad x \in (-1,1).$$

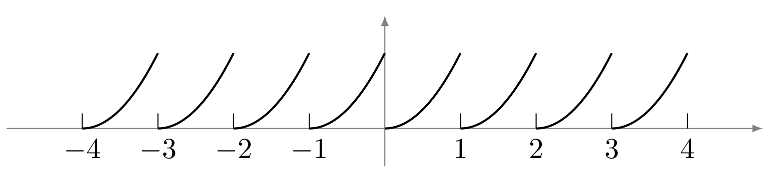


Example 6.4.6. Let $f(x)$ be the periodic function of period 1 extending the function x^2 on $(0,1)$. Then

$$a_0 = \int_0^1 x^2 dx = \frac{1}{3},$$
$$a_n = 2\int_0^1 x^2 \cos 2n\pi x dx = \frac{1}{n^2\pi^2},$$
$$b_n = 2\int_0^1 x^2 \sin 2n\pi x dx = -\frac{1}{n\pi}.$$

The Fourier series is

$$x^2 \sim \frac{1}{3} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2\pi^2} \cos 2n\pi x - \frac{1}{n\pi} \sin 2n\pi x \right), \quad x \in (0,1).$$



Exercise 6.4.9. The periodic function is given on one interval of period length. Find the Fourier series.

1. $\sin^2 x$ on $(-\pi,\pi)$.
2. $\sin x$ on $(0,\pi)$.
3. $|\sin x|$ on $(0,2\pi)$.
4. $\sin x$ on $(0,p)$.
5. $\cos x$ on $(0,p)$.
6. $|x|$ on $(-p,p)$.
7. x on (a,b).
8. $x\sin x$ on $(-\pi,\pi)$.
9. e^x on $(0,1)$.

Exercise 6.4.10. Use the Fourier coefficient to calculate the integral.

1. $\displaystyle\int_0^{\pi} \sin^2 x\cos 2x dx$.
2. $\displaystyle\int_0^{2\pi} \sin^6 x dx$.
3. $\displaystyle\int_0^{2\pi} \sin x\cos 2x\sin 3x dx$.
4. $\displaystyle\int_0^{2\pi} \sin x\cos 2x\cos 3x dx$.
5. $\displaystyle\int_0^{2\pi} \sin^3 x\sin 3x dx$.
6. $\displaystyle\int_0^{2\pi} \sin^3 x\cos 3x dx$.

Exercise 6.4.11. Write the formula for the Fourier coefficients of the even periodic (of period $2p$) extension of a function $f(x)$ on $(0,p)$. What about the odd extension?

Exercise 6.4.12. Extend the function on $(0,p)$ to even and odd functions of period $2p$ and compute the Fourier series.

1. x^2 on $(0,1)$.
2. $\sin x$ on $(0,\pi)$.
3. $\cos x$ on $(0,p)$.

Exercise 6.4.13. Given the Fourier series of functions $f(x)$ and $g(x)$ of period p. Find the Fourier series of the following periodic functions.

1. $f(ax)$.
2. $f(x)\cos\dfrac{2\pi x}{p}$.
3. $f(x)\sin\dfrac{2\pi x}{p}$.
4. $\displaystyle\frac{1}{2h}\int_{x-h}^{x+h} f(t)dt$.
5. $\displaystyle\frac{1}{h}\int_{x}^{x+h} f(t)dt$.
6. $\displaystyle\int_0^{p} f(t)g(x-t)dt$.

Exercise 6.4.14. Use Examples 6.4.4 and 6.4.5 to find the Fourier series.

1. x on $(0,p)$.
2. $-x$ on $(-p,0)$.
3. $|x|$ on $(-p,p)$.
4. 0 on $(-1,0)$ and x on $(0,1)$.
5. ax on $(-1,0)$ and bx on $(0,1)$.
6. ax on $(-p,0)$ and bx on $(0,p)$.

6.4.2 Complex Form of Fourier Series

The sine and cosine functions are related to the exponential function via the use of complex numbers

$$e^{ix} = \cos x + i\sin x,\quad \cos x = \frac{e^{ix}+e^{-ix}}{2},\quad \sin x = \frac{e^{ix}-e^{-ix}}{2i},\quad i=\sqrt{-1}.$$

Correspondingly, the Fourier series may be rewritten

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_0 = a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2},$$

where complex Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx.$$

In this course, $f(x)$ is a real valued function, and a_n, b_n are real numbers. Therefore $c_{-n}e^{i(-n)x}$ is the complex conjugation of $c_n e^{inx}$, and the usual Fourier series is given by

$$a_0 = c_0, \quad a_n \cos nx + b_n \sin nx = 2\mathrm{Re}(c_n e^{inx}).$$

Example 6.4.7. For the function in Example 6.4.2, the complex Fourier coefficient is

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_0^a dx = \frac{a}{2\pi}, \\ c_n &= \frac{1}{2\pi}\int_0^a e^{-inx}dx = \frac{1}{-2in\pi}e^{-inx}|_0^a = \frac{i}{2n\pi}(e^{-ina} - 1). \end{aligned}$$

The complex Fourier series is

$$f(x) \sim \frac{a}{2\pi} + \sum_{n\neq 0} \frac{i}{2n\pi}(e^{-ina} - 1)e^{inx}, \quad x \in (0, 2\pi).$$

By

$$\begin{aligned} 2\mathrm{Re}\left(\frac{i}{2n\pi}(e^{-ina} - 1)e^{inx}\right) &= \mathrm{Re}\left(\frac{i}{n\pi}((\cos na - 1) - i\sin na)(\cos nx + i\sin nx)\right) \\ &= \frac{1}{n\pi}(\sin na \cos nx - (\cos na - 1)\sin nx), \end{aligned}$$

we recover the Fourier series in terms of trigonometric functions in Example 6.4.2.

Example 6.4.8. Consider the periodic function of period 2π given by e^x on $(0, 2\pi)$. The complex Fourier coefficient is (recall that $e^{2ik\pi} = 1$)

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^x e^{-inx}dx = \frac{1}{2\pi(1 - in)}e^{(1-in)x}|_0^{2\pi} = \frac{e^{2\pi} - 1}{2n\pi(1 - in)}.$$

The complex Fourier series is

$$e^x \sim \frac{e^{2\pi} - 1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{1}{1 - in}e^{inx}, \quad x \in (0, 2\pi).$$

By

$$\mathrm{Re}\left(\frac{1}{1-in}e^{inx}\right)=\mathrm{Re}\left(\frac{1+in}{1+n^2}(\cos nx+i\sin nx)\right)=\frac{\cos nx-n\sin nx}{1+n^2},$$

we get the Fourier series in terms of trigonometric functions

$$e^x\sim\frac{e^{2\pi}-1}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos nx-n\sin nx}{1+n^2}\right),\quad x\in(0,2\pi).$$

By changing x to $2\pi x$, we get the Fourier series for the periodic function of period 1 given by $e^{2\pi x}$ on $(0,1)$

$$e^{2\pi x}\sim\frac{e^{2\pi}-1}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos 2n\pi x-n\sin 2n\pi x}{1+n^2}\right),\quad x\in(0,1).$$

Example 6.4.9. Consider the function $f(x)=\dfrac{a\sin x}{1-2a\cos x+a^2}$, with $|a|<1$. We rewrite the function and take the Taylor expansion in terms of $e^{inx}=z^n$, $z=e^{ix}$,

$$\begin{aligned}\frac{a\sin x}{1-2a\cos x+a^2}&=\frac{a\dfrac{e^{ix}-e^{-ix}}{2i}}{1-2a\dfrac{e^{ix}+e^{-ix}}{2}+a^2}\\&=\frac{a}{2i}\frac{e^{ix}-e^{-ix}}{(1-ae^{ix})(1-ae^{-ix})}=\frac{1}{2i}\left(\frac{1}{1-ae^{ix}}-\frac{1}{1-ae^{-ix}}\right)\\&=\frac{1}{2i}\left(\sum_{n=0}^{\infty}(ae^{ix})^n-\sum_{n=0}^{\infty}(ae^{-ix})^n\right)=\frac{1}{2i}\sum_{n\neq 0}a^{|n|}e^{inx}\\&=\frac{1}{2i}\sum_{n=1}^{\infty}a^n(e^{inx}-e^{-inx})=\sum_{n=1}^{\infty}a^n\sin nx.\end{aligned}$$

Note that we have geometric series because $|ae^{ix}|=|ae^{-ix}|=|a|<1$. Moreover, the Fourier series is actually equal to the function. The example also shows the connection between the Fourier series and the power series.

Exercise 6.4.15. Find the complex form of the Fourier series of a^x on $(0,2\pi)$. Then by taking $a=e^{\frac{1}{2\pi}}$, derive the Fourier series of the function e^x on $(0,1)$.

Exercise 6.4.16. Find the complex form of the Fourier series of $e^{\frac{ix}{2}}$ on $(0,2\pi)$. Then derive the Fourier series of $\cos x$ and $\sin x$ on $(0,\pi)$ by taking the real and imaginary parts.

Exercise 6.4.17. What is the complex form of the Fourier series for a periodic function of period p?

Exercise 6.4.18. Find complex form of Fourier series.

1. $\sin^2 x$ on $(-\pi,\pi)$.
2. e^{ix} on $(0,p)$.
3. $|x|$ on $(-p,p)$.
4. x^2 on $(0,1)$.
5. $x\sin x$ on $(-\pi,\pi)$.
6. $x\cos x$ on $(-\pi,\pi)$.

Exercise 6.4.19. Find Fourier series, where $|a|<1$.

1. $\dfrac{1-a\cos x}{1-2a\cos x+a^2}$.
2. $\dfrac{1-a^2}{1-2a\cos x+a^2}$.
3. $\log(1-2a\cos x+a^2)$.

6.4.3 Derivative and Integration of Fourier Series

The derivative of a periodic function is still periodic. We may use $f(p)=f(0)$ (which is the periodic property) and the integration by parts to compute the Fourier coefficients A_n, B_n of $f'(x)$

$$\begin{aligned}
A_0 &= \frac{1}{p}\int_0^p f'(x)dx = \frac{1}{p}(f(p)-f(0)) = 0,\\
A_n &= \frac{2}{p}\int_0^p f'(x)\cos\frac{2n\pi}{p}xdx = \frac{2}{p}\int_0^p \cos\frac{2n\pi}{p}xdf(x)\\
&= \frac{2}{p}\left(f(p)-f(0)+\int_0^p f(x)\frac{2n\pi}{p}\sin\frac{2n\pi}{p}xdx\right) = \frac{2n\pi}{p}b_n,\\
B_n &= \frac{2}{p}\int_0^p f'(x)\sin\frac{2n\pi}{p}xdx = \frac{2}{p}\int_0^p \sin\frac{2n\pi}{p}xdf(x)\\
&= -\frac{2}{p}\int_0^p f(x)\frac{2n\pi}{p}\cos\frac{2n\pi}{p}xdx = -\frac{2n\pi}{p}a_n.
\end{aligned}$$

This shows that we may differentiate the Fourier series term by term.

Proposition 6.4.2. *Suppose $f(x)$ is a periodic function of period p that is continuous and piecewise continuously differentiable. If the Fourier coefficients of $f(x)$ are a_n, b_n, then the Fourier series of $f'(x)$ is*

$$f'(x) \sim \frac{2\pi}{p}\sum_{n=1}^{\infty} n\left(b_n\cos\frac{2n\pi}{p}x - a_n\sin\frac{2n\pi}{p}x\right).$$

The integration $F(x)=\displaystyle\int_0^x f(t)dt$ of a periodic function of period p is still periodic if $a_0=\dfrac{1}{p}\displaystyle\int_0^p f(x)dx=0$. Then for $n\neq 0$, the other Fourier coefficients of

$F(x)$ are

$$\begin{aligned}
A_n &= \frac{2}{p}\int_0^p F(x)\cos\frac{2n\pi}{p}xdx = \frac{1}{n\pi}\int_0^p F(x)d\sin\frac{2n\pi}{p}x \\
&= -\frac{1}{n\pi}\int_0^p f(x)\sin\frac{2n\pi}{p}xdx = -\frac{p}{2n\pi}b_n, \\
B_n &= \frac{2}{p}\int_0^p F(x)\sin\frac{2n\pi}{p}xdx = -\frac{1}{n\pi}\int_0^p F(x)d\cos\frac{2n\pi}{p}x \\
&= -\frac{1}{n\pi}\left(\int_0^p f(x)dx - \int_0^p f(x)\cos\frac{2n\pi}{p}xdx\right) = \frac{p}{2n\pi}a_n.
\end{aligned}$$

The 0-th coefficient is

$$A_0 = \frac{1}{p}\int_0^p F(x)dx = \frac{1}{p}\left(xF(x)|_{x=0}^{x=p} - \int_0^p xf(x)dx\right) = -\frac{1}{p}\int_0^p xf(x)dx.$$

This shows that, in case $a_0 = 0$, we may almost integrate the Fourier series term by term.

Proposition 6.4.3. *Suppose $f(x)$ is a periodic function of period p. If the Fourier coefficients of $f(x)$ are a_n, b_n and $a_0 = 0$, then the Fourier series of $F(x) = \int_0^x f(t)dt$ is*

$$\int_0^x f(t)dt \sim A_0 + \frac{p}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(-b_n\cos\frac{2n\pi}{p}x + a_n\sin\frac{2n\pi}{p}x\right).$$

Example 6.4.10. In Example 6.4.4, we found the Fourier series

$$x \sim \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi x, \quad x \in (0,1).$$

We wish to integrate to get the Fourier series of x^2 on $(0,1)$. However, to satisfy the condition of Proposition 6.4.3, we should consider the Fourier series of $x - \frac{1}{2}$, which has vanishing 0-th coefficient. Then $F(x) = \int_0^x \left(t - \frac{1}{2}\right)dt = \frac{1}{2}x^2 - \frac{1}{2}x$ has Fourier series

$$\frac{1}{2}x^2 - \frac{1}{2}x \sim -\frac{1}{12} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2n\pi x, \quad x \in (0,1).$$

Here the 0-th coefficient is

$$A_0 = -\int_0^1 x\left(x - \frac{1}{2}\right)dx = -\frac{1}{12}.$$

Then the Fourier series of $x^2 = 2\left(\frac{1}{2}x^2 - \frac{1}{2}x\right) + x$ on $(0,1)$ is

$$\begin{aligned}
x^2 &\sim 2\left(-\frac{1}{12} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2n\right) + \left(\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi x\right) \\
&= \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}\cos 2n\pi x - \frac{1}{n\pi}\sin 2n\pi x\right), \quad x \in (0,1).
\end{aligned}$$

We obtained this Fourier series in Example 6.4.6 by direct computation.

Exercise 6.4.20. Derive the Fourier series of x^3 on $(0,1)$ from the Fourier series of x^2.

Exercise 6.4.21. Derive the Fourier series of $|x|$ on $(-1,1)$ from the Fourier series of its derivative.

Exercise 6.4.22. Suppose $f(x)$ is continuously differentiable on $[0,p]$, with perhaps different $f(0^+)$ and $f(p^-)$. Then we can extend both $f(x)$ and $f'(x)$ to periodic functions of period p. If the Fourier coefficients of the extended $f(x)$ are a_n, b_n, prove that

$$f'(x) \sim \frac{f(p^-) - f(0^+)}{p} + \frac{2}{p}\sum_{n=1}^{\infty}\left((f(p^-) - f(0^+) + n\pi b_n)\cos\frac{2n\pi}{p}x + n\pi a_n \sin\frac{2n\pi}{p}x\right).$$

6.4.4 Sum of Fourier Series

Like the Taylor series, the Fourier series may not always converge to the function. The following is one good case when the Fourier series converges.

Theorem 6.4.4. *Suppose $f(x)$ is a periodic function. If $f(x)$ has one sided limits $f(x_0^-)$ and $f(x_0^+)$ at x_0, and there is M, such that*

$$x < x_0 \text{ and close to } x_0 \implies |f(x) - f(x_0^-)| \le M|x - x_0|,$$

and

$$x > x_0 \text{ and close to } x_0 \implies |f(x) - f(x_0^+)| \le M|x - x_0|.$$

Then the Fourier series of $f(x)$ converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$ at $x = x_0$.

The condition means that the value of $f(x)$ lies in two "corners" on the two sides of x_0.

Example 6.4.11. In Example 6.4.4, we find the Fourier series of x on $(0,1)$. The function satisfies the condition of Theorem 6.4.4 at $x_0 = 0$. We conclude that

$$\frac{0+1}{2} = \frac{f(0^-) + f(0^+)}{2} = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi 0.$$

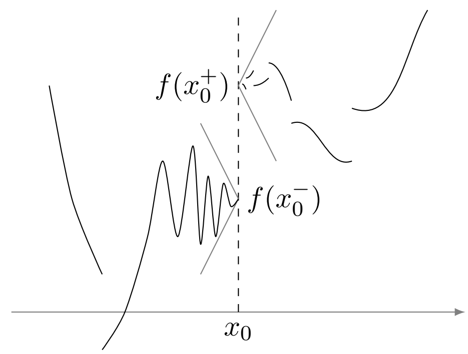


Figure 6.4.1: Condition for the convergence of Fourier series.

This is trivially true. We get similar trivial equalities at $x_0 = 1$ and $x_0 = \frac{1}{2}$.

If we take $x_0 = \frac{1}{3}$, then

$$\frac{1}{3} = \frac{1}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{3k+1}\sin\frac{2(3k+1)\pi}{3} + \frac{1}{3k+2}\sin\frac{2(3k+2)\pi}{3} + \frac{1}{3k+3}\sin\frac{2(3k+3)\pi}{3}\right)$$
$$= \frac{1}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{3k+1}\frac{\sqrt{3}}{2} - \frac{1}{3k+2}\frac{\sqrt{3}}{2}\right).$$

This means that

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \frac{1}{13} - \frac{1}{14} + \cdots = \frac{\pi}{3\sqrt{3}}.$$

Similarly, by evaluating the Fourier series at $x_0 = \frac{1}{4}$ and $x_0 = \frac{1}{8}$, we get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \cdots = \frac{\pi}{4},$$
$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} - \cdots = \frac{\pi}{2\sqrt{2}}.$$

Example 6.4.12. The periodic function of period 2 given by $|x|$ on $(-1, 1)$ satisfies the condition of Theorem 6.4.4 everywhere. Evaluating the Fourier series in Example 6.4.4 at $x = 0$, we get

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

This means that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

If the sum also includes the even terms, then we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

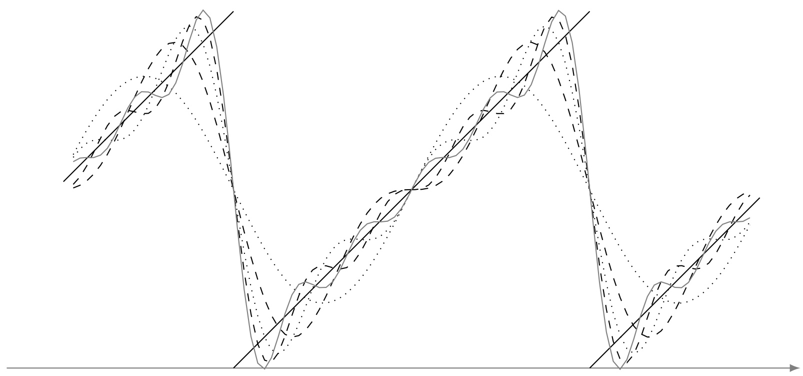


Figure 6.4.2: Partial sums of the Fourier series for x on $(0,1)$.

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{4}{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{6}.$$

If we evaluate at $x=\frac{1}{4}$, then we get

$$\frac{1}{4}=\frac{1}{2}-\frac{4}{\sqrt{2}\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}-\frac{1}{13^2}-\frac{1}{15^2}+\cdots\right).$$

Combined with the sum above, we get

$$\begin{aligned}\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{17^2}+\frac{1}{19^2}+\cdots&=\left(\frac{1}{12}+\frac{1}{16\sqrt{2}}\right)\pi^2,\\ \frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{13^2}+\frac{1}{15^2}+\frac{1}{21^2}+\frac{1}{23^2}+\cdots&=\left(\frac{1}{12}-\frac{1}{16\sqrt{2}}\right)\pi^2.\end{aligned}$$

Exercise 6.4.23. Use the Fourier series of x^2 on $(0,1)$ to compute $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Exercise 6.4.24. Use the Fourier series of x^2 on $(-1,1)$ to get the Fourier series of x^3 and x^4 on $(-1,1)$. Then evaluate the Fourier series of x^4 to get $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

6.4.5 Parseval's Identity

In Example 6.4.1, we had the Fourier series

$$\sin^4 x=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x.$$

Then

$$\int_0^{2\pi}\sin^8 x dx=\int_0^{2\pi}\left(\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)^2 dx.$$

We may expand the square on the right. We get square terms such as $\dfrac{1}{2^2}\cos^2 2x$ and cross terms such as $2\dfrac{1}{2}\cdot\dfrac{1}{8}\cos 2x\cos 4x$. The square term has nonzero integral

$$\int_0^{2\pi} \cos^2 2x dx = \pi.$$

The cross terms has vanishing integral

$$\int_0^{2\pi} \cos 2x \cos 4x dx = 0.$$

Therefore

$$\int_0^{2\pi} \sin^8 x dx = \left(\frac{3}{8}\right)^2 2\pi + \left(\frac{1}{2}\right)^2 \pi + \left(\frac{1}{8}\right)^2 \pi = \frac{35}{64}\pi.$$

The idea (which is essentially Pythagorean theorem) leads to the following formula.

Theorem 6.4.5 (Parseval's Identity). *Suppose $f(x)$ is a periodic function of period p. Then its Fourier coefficients satisfy*

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{2}{p}\int_0^p |f(x)|^2 dx.$$

The identity means that the Fourier series, considered as a conversion between periodic functions and sequences of numbers, preserves the "Euclidean length". The complex form of Parseval's identity is

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx.$$

Example 6.4.13. Applying Parseval's identity to the Fourier series of x on $(0,1)$, we get

$$2 + \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} = 2\int_0^1 x^2 dx = \frac{2}{3}.$$

This is the same as

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

Applying the identity to the Fourier series of x^2 on $(0,1)$, we get

$$\frac{2}{9} + \sum_{n=1}^{\infty}\frac{1}{n^4\pi^4} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} = 2\int_0^1 x^4 dx = \frac{2}{5}.$$

Using $\sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, we get

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Example 6.4.14. Applying Parseval's identity to the complex form of the Fourier series of e^x on $(0, 2\pi)$ in Example 6.4.8, we get

$$\frac{(e^{2\pi}-1)^2}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2}=\frac{1}{2\pi}\int_0^{2\pi}e^{2x}dx=\frac{e^{4\pi}-1}{4\pi}.$$

The equality leads to

$$1+2\sum_{n=1}^{\infty}\frac{1}{1+n^2}=\frac{e^{2\pi}+1}{e^{2\pi}-1}\pi,$$

or

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2}=\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}+\cdots=\frac{\pi(e^{2\pi}+1)}{2(e^{2\pi}-1)}-\frac{1}{2}.$$

Exercise 6.4.25. Apply Parseval's identity to the Fourier series in Example 6.4.2 to find $\sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}$ and $\sum_{n=1}^{\infty}\frac{\cos^2 na}{n^2}$.

Exercise 6.4.26. Apply Parseval's identiy to the Fourier series of x^3 and x^4 on $(0, 1)$ to find $\sum_{n=1}^{\infty}\frac{1}{n^6}$ and $\sum_{n=1}^{\infty}\frac{1}{n^8}$.

Exercise 6.4.27. Find $\sum_{n=1}^{\infty}\frac{1}{1+n^2}$ by evaluating the Fourier series in Example 6.4.8.