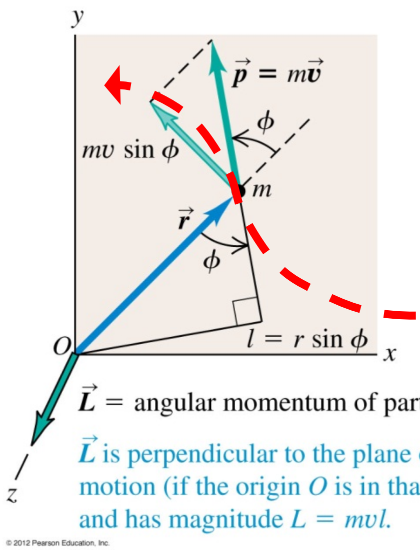
Angular Momentum

Intended Learning Outcomes – after this lecture you will learn: 1. the angular momentum of a system of particles and rigid body. 2. how to describe dynamics of a system using its angular momentum. 3. conservation of angular momentum. 4. precession of angular momentum vector in a gyroscope. Textbook Reference: 10.5 – 10.7 (excluding calculations starting Eq. 10.33)

For a point particle, define its **angular momentum** about the origin O by $\boxed{\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}}$



trajectory of m

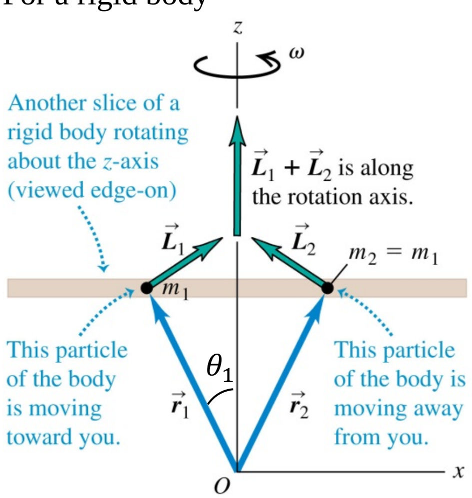
$$L = mvr\sin\phi = (mv\sin\phi)r = mv(r\sin\phi)$$

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \left(\frac{d\vec{\boldsymbol{r}}}{dt} \times \underbrace{\vec{\boldsymbol{p}}}_{m\frac{d\vec{\boldsymbol{r}}}{dt}}\right) + \left(\vec{\boldsymbol{r}} \times \underbrace{\frac{d\vec{\boldsymbol{p}}}{dt}}_{\vec{\boldsymbol{F}}}\right) = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} = \vec{\boldsymbol{\tau}}$$

i.e. $\boxed{\frac{d\vec{\boldsymbol{L}}}{dt} = \vec{\boldsymbol{\tau}}}$ *c.f.* $\frac{d\vec{\boldsymbol{P}}}{dt} = \vec{\boldsymbol{F}}$

⚠ the particle need not be rotating about any axis, can be travelling in a straight line

For a rigid body



Take the rotation axis as the z axis, m_1 is a small mass of the rigid body

$$L_1 = mv_1 r_1 = m(\omega r_1 \sin\theta_1)r_1$$

⚠ here r_1 is the distance from O, but *not* the ⊥ distance to the rotation axis as in the moment of inertia

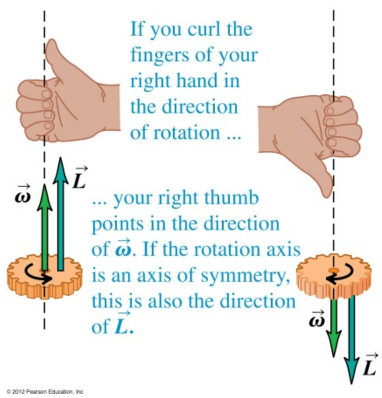
If rotation axis is a symmetry axis, then there exist m_2 on the opposite side whose x-y components of angular momentum cancel those of m_1. Therefore only z component of any $\vec{\boldsymbol{L}}_i$ is important.

Total angular momentum $\vec{\boldsymbol{L}} = \sum \vec{\boldsymbol{L}}_i = \sum L_i \sin\theta_i\, \hat{\boldsymbol{k}}$, points along rotation axis with magnitude

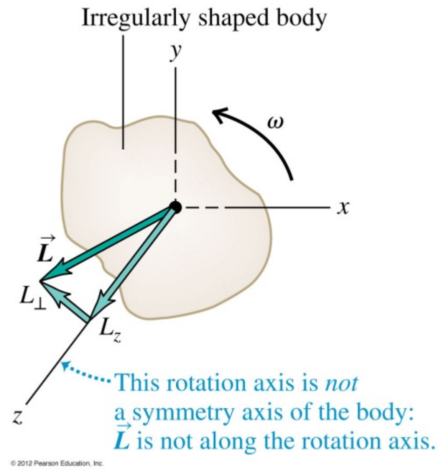
$$L = \sum [m_i(\omega r_i \sin\theta_i) r_i] \sin\theta_i = \left(\sum m_i (\underbrace{r_i \sin\theta_i}_{\perp \text{ distance of } m_i \text{ to rotation axis}})^2\right)\omega$$

Conclusion: if rotation axis is a symmetry axis, then $\boxed{\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\omega}}}$

⚠ while the linear momentum $\vec{\boldsymbol{p}}$ is always $m\vec{\boldsymbol{v}}$, the angular momentum $\vec{\boldsymbol{L}}$ may not be $I\vec{\boldsymbol{\omega}}$ *unless the rotation axis is a symmetry axis*



Rotation axis is a symmetry axis, $\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\omega}}$

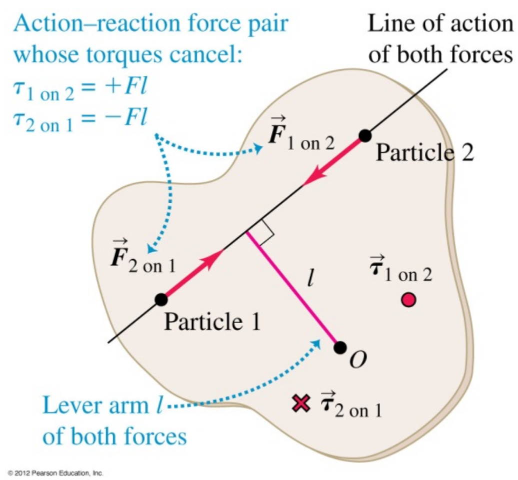


Rotation axis is not a symmetry axis, and $\vec{\boldsymbol{L}} \neq I\vec{\boldsymbol{\omega}}$

⚠ If the rotation axis (z axis, but not a symmetry axis) is fixed,

a) $\vec{\boldsymbol{L}}$ changes, i.e., there exist a finite torque to keep the body rotating.

b) $L_\perp$ not important since it does not produce physical motion, "angular momentum" may refer to the component of $\vec{\boldsymbol{L}}$ along the axis of rotation, i.e. $L_z = I\omega$, but not $\vec{\boldsymbol{L}}$ itself in this case.



Internal forces (action and reaction pairs) have the same line of action → no net torque. Therefore for a system of particles or a rigid body

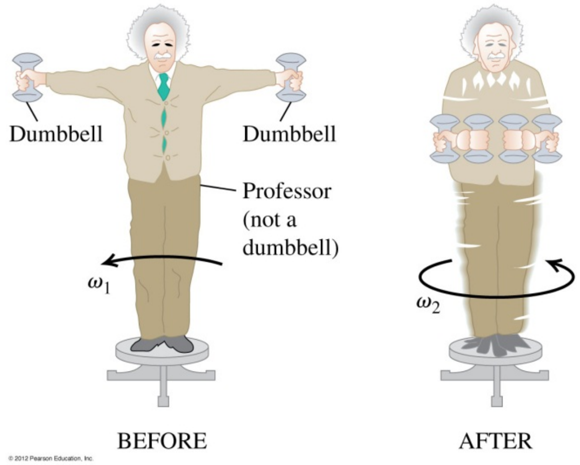
$$\boxed{\frac{d\vec{\boldsymbol{L}}}{dt} = \sum \vec{\boldsymbol{\tau}}_{\text{ext}}} \quad c.f. \quad \frac{d\vec{\boldsymbol{P}}}{dt} = \sum \vec{\boldsymbol{F}}_{\text{ext}}$$

Under no external torque (⚠ *not* force)

$$\frac{d\vec{\boldsymbol{L}}}{dt} = 0 \qquad \textbf{conservation of angular momentum}$$

Question: A particle is going around in uniform circular motion (constant speed). Does its linear momentum $\vec{\boldsymbol{p}}$ conserve? Does its angular momentum $\vec{\boldsymbol{L}}$ conserve?
Answer: see inverted text on P. 346 of textbook

Demonstration: A spinning physics professor Example 10.10 P. 348



Conservation of angular momentum

$$I_1\omega_1 = I_2\omega_2$$

If $I_2 = I_1/2$, then $\omega_2 = 2\omega_1$, and $K_2 = \frac{1}{2}I_2\omega_2^2 = \_\_\_\_ K_1$.

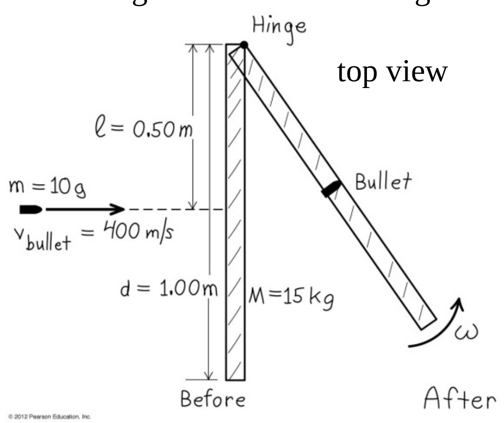
Where comes the extra energy?

And in the reverse process $I_2 \to I_1$, where goes the energy?

Example 10.12 P. 349

A bullet hits a door in a perpendicular direction, embeds in it and swings it open. During the collision:

1. linear momentum is not conserved because ___________________________________
2. angular momentum along the rotation axis is conserved because ___________________



$$\underbrace{mvl}_{\substack{\text{initial angular}\\\text{momentum of}\\\text{bullet about}\\\text{hinge}}} = \underbrace{\left(\frac{Md^2}{3}\right)}_{\substack{\text{moment}\\\text{of inertia}\\\text{of door}\\\text{about}\\\text{hinge}}}\omega + \underbrace{(ml^2)}_{\substack{\text{moment of}\\\text{inertia of bullet}\\\text{about hinge after}\\\text{embedded in}\\\text{door}}}\omega$$

$$\Rightarrow \quad \omega = \frac{mvl}{\frac{1}{3}Md^2 + ml^2}$$

Question: the hinge is not a symmetry axis! Why is the angular momentum $I\omega$?

Question: If the polar ice caps were to completely melt due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (increase / decrease / remain the same).
Answer: see inverted text on P. 349 of textbook

Example

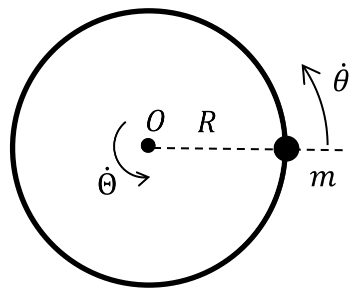
A circular turntable of mass M and radius R can freely turn about a vertical axis through its center O. An insect of mass m is on the edge of the turntable while both are at rest relative to the ground. The insect crawls along the circumference of the turntable for one round (relative to the turntable) and return to its original position. Find the angle through which the turntable turned relative to the ground during this period.

Let θ/ Θ be the angular displacements of the insect / turntable *relative to the ground* and in counter-clockwise sense.

Is angular momentum about O conserved? (Y / N)

$$(\qquad\qquad)\dot{\Theta}+(\qquad\qquad)\dot{\theta}=0 \;\Rightarrow\; \frac{1}{2}MR^2 d\Theta=-mR^2 d\theta$$

$$\Rightarrow\; \frac{1}{2}M\int_0^{\Theta} d\Theta=-m\int_0^{\theta} d\theta \quad\Rightarrow\quad \Theta=-\frac{2m}{M}\theta$$

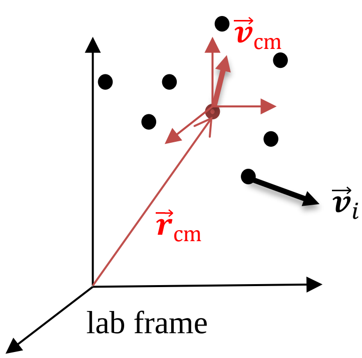


Angular displacement of insect relative to turntable is __________, therefore when the insect returns to its original point, $2\pi=\theta-\Theta$, we get $\Theta=-4m\pi/(2m+M)$.

⚠ Compare with the linear version of this problem on P. 6 of Lecture 6

Angular Momentum of a Moving Body First part of the **König Theorem**, *c.f.* lecture 9 P. 7

First consider a system consisting of particles



CM at $\vec{r}_{\rm cm}$ is moving with velocity $\vec{v}_{\rm cm}$ relative to the lab

m_i is the mass of particle i

$\vec{v}'_i$ its velocity relative to the CM, its velocity relative to the lab is $\vec{v}_i=\vec{v}_{\rm cm}+\vec{v}'_i$, likewise, $\vec{r}_i=\vec{r}_{\rm cm}+\vec{r}'_i$

$$\begin{aligned}\vec{L}&=\sum(\vec{r}_i\times m_i\vec{v}_i)\\&=\sum(\vec{r}_{\rm cm}+\vec{r}'_i)\times m_i(\vec{v}_{\rm cm}+\vec{v}'_i)\\&=\sum(\vec{r}_{\rm cm}\times m_i\vec{v}_{\rm cm})+\sum(\vec{r}'_i\times m_i\vec{v}'_i)\\&\quad+\vec{r}_{\rm cm}\times\underbrace{\Big(\sum m_i\vec{v}'_i\Big)}_{\text{CM velocity relative to CM – zero}}+\underbrace{\Big(\sum m_i\vec{r}'_i\Big)}_{\text{CM position relative to CM – zero}}\times\vec{v}_{\rm cm}\end{aligned}$$

$$\boxed{\vec{L}=\underbrace{\vec{r}_{\rm cm}\times M\vec{v}_{\rm cm}}_{\text{AM of CM}}+\underbrace{\vec{L}'}_{\text{AM in CM frame}}}$$

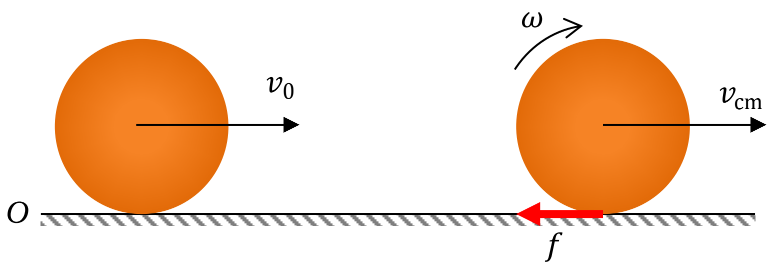
In words, the **AM in the lab frame is the AM of the CM plus the AM in the CM frame**

⚠ Holds even when CM frame is not an inertial frame

⚠ Also holds if the system is a rigid body

Example the same bowling ball problem from Lecture 10

A uniform bowling ball of radius R and mass M is tossed with speed v_0 without rolling on an alley with coefficient of kinetic friction μ. How far does the ball go before it starts rolling without slipping?



To find angular momentum, fix the origin O first (arbitrary point on the alley)

$$Mv_0R = \underbrace{Mv_{\text{cm}}R + I_{\text{cm}}\omega}$$

initial AM

König Theorem
Holds whether there is
slipping or not

When no slipping, $R\omega = v_{\text{cm}}$, we get, $v_{\text{cm}} = \frac{5}{7}v_0$. Therefore the distance travelled is

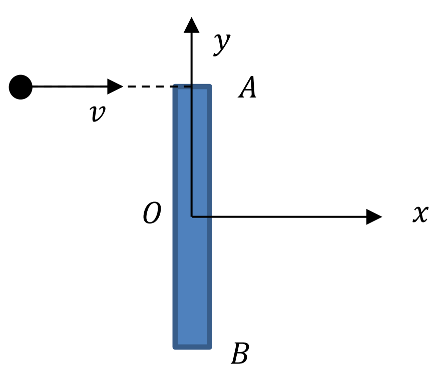
$$v_{\text{cm}}^2 = v_0^2 + 2(-\mu g)D^2 \quad \Rightarrow \quad D = \frac{12v_0^2}{49\mu g}$$

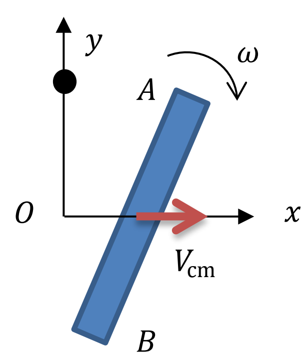
⚠ Why is angular momentum conserved? Are there external torques about O?

⚠ Shouldn't the AM about the CM be $-I_{\text{cm}}\omega$ because it is rotating clockwisely?

Example

A uniform rod of mass M and length L is free to slide on a smooth surface. Initially it is at rest along the y axis with its CM at the origin. A particle of mass m hits its end A with velocity $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$. The particle stops right after impact. Assuming elastic collision, find M in terms of m.





Unknowns: V_{cm}, ω, and M. Need three conservation laws and constraints.
Linear momentum: $mv = MV_{\text{cm}}$
Energy: $\frac{1}{2}mv^2 = \frac{1}{2}MV_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$
Angular momentum about O: $mv\frac{L}{2} = I_{\text{cm}}\omega +$ ______ (What is AM of CM about O?)
Final answer: $M = 4m$

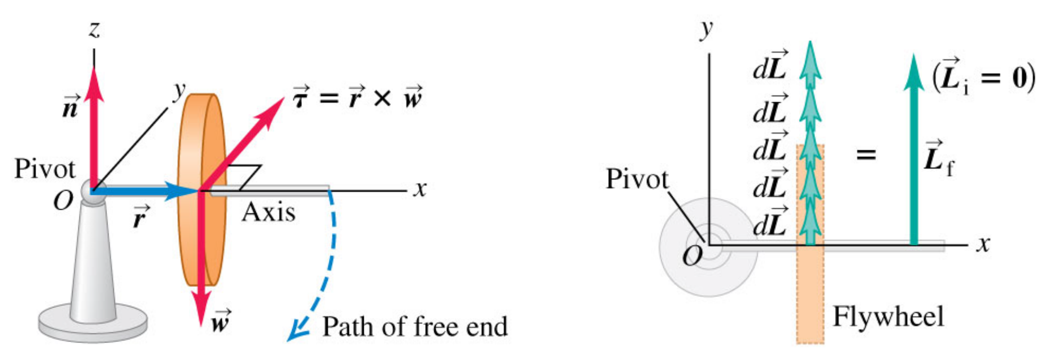
⚠ What if the particle bounces off? Can you solve it, i.e., is there a unique solution?
If the rod is rigid, there is no unique solution. Need to take into account secondary effects such as elasticity of the rod.

Gyroscopes and Precession (Optional)

Demonstration: [bicycle wheel gyroscope]()

Case 1: when the flywheel is not spinning – of course it falls down

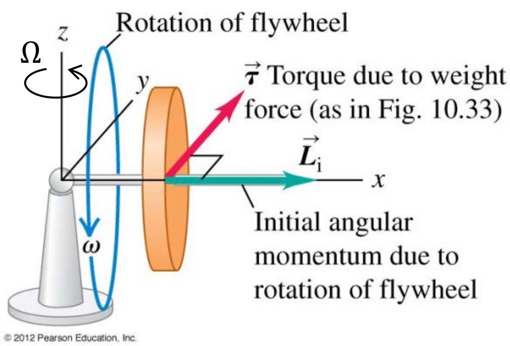
(a) Nonrotating flywheel falls (b) View from above as flywheel falls

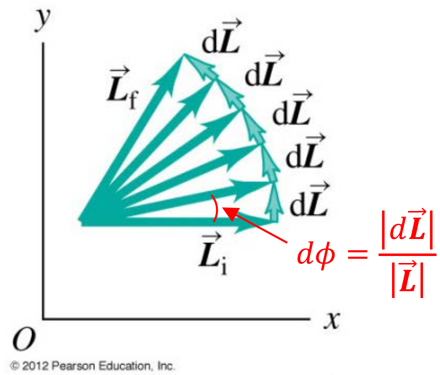


torque $\vec{\tau}$ due to weight of the flywheel $\vec{w}$ causes it to fall in the x-z plane

$\vec{L}$ increases as flywheel falls

Case 2: when flywheel spinning with initial angular moment $\vec{L}_i$ – it **precesses**





Since $\vec{L} \perp d\vec{L}$, flywheel axis execute circular motion called **precession**, $|\vec{L}|$ remains constant

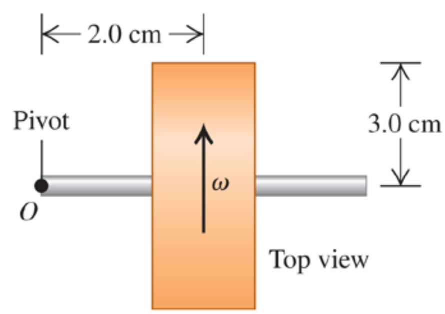
⚠ faster spinning ω → slower precession Ω

$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}|}{dt} = \frac{|\vec{\tau}|}{|\vec{L}|} = \frac{wr}{I\omega}$$

See the animation of the vectors $\vec{w}$, $\vec{\tau}$, and $\vec{L}$ at http://phys23p.sl.psu.edu/phys_anim/mech/gyro_s1_p.avi

Example 10.13 P. 352

For the gyroscopic motion as shown (top view), the precession is (clockwise / counterclockwise) as viewed from above. (*Hint*: compare with case 2 above.)



Suppose the precession speed is 4.0 s per revolution, i.e., $\Omega = 2\pi/4.0$ rad/s.

$$\omega = \frac{wr}{I\Omega} = \frac{mgr}{\left(\frac{1}{2}mR^2\right)\Omega} = \frac{2gr}{R^2\Omega} = 280 \text{ rad/s}$$
$$= 2600 \text{ rev/min}$$

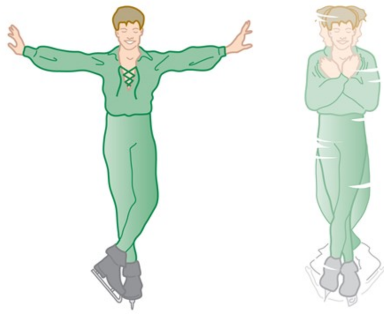
If $\omega \gg \Omega$, can ignore angular momentum due to precession. Otherwise there is *nutation* of the flywheel axis – it wobbles up and down. See animation at http://phys23p.sl.psu.edu/phys_anim/mech/gyro_s1_nu.avi

Demonstration: a formal gyroscope

Clicker Questions:

Q10.12

A spinning figure skater pulls his arms in as he rotates on the ice. As he pulls his arms in, what happens to his angular momentum L and kinetic energy K?



A. L and K both increase.

B. L stays the same; K increases.

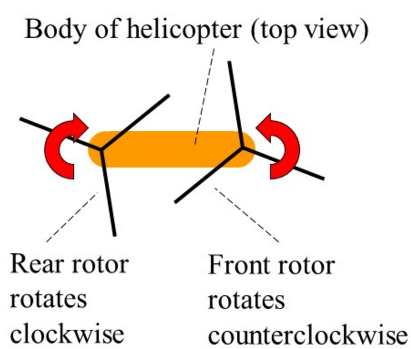
C. L increases; K stays the same.

D. L and K both stay the same.

E. None of the above.

Q10.13

Some helicopters have two large rotors that rotate in *opposite* directions as shown. If instead they *both* rotated in the clockwise direction as seen from above, what would happen to the body of the helicopter if the pilot increased the rotation speed of both rotors?



A. The body would rotate clockwise.

B. The body would rotate counterclockwise.

C. Nothing—there would be no effect on the body.

D. The answer depends on how fast the rotors rotate.

Ans: Q10.12) B, Q10.13) B