MATH 2001
Homework 1

Please write down the solutions to the problems below in full details and in full sentences. Unless otherwise specified, all claims have to be justified.

20% of the points will come the quality of your LaTeX typesetting (see the course syllabus for more detail). 80% of the points will come not only from the correctness and clarity of your solutions.

Problem 1 (10 points). Let $\{a_n\}_{n=0}^{\infty}$ be a sequence defined by the following recursive formula

$$\begin{cases} a_0 = 1, a_1 = 2 \\ a_{n+1} = 3a_n - 2a_{n-1} \end{cases}$$

Make a conjecture for the general closed form formula for a_n and then prove it using induction.

Problem 2 (10 points). Prove that the sum of the internal angles of an n-gon (a polygon with n sides) is equal to $(n-2) \cdot \pi$.

Problem 3 (10 points). Let $n > 2$ be an integer. Prove that if n is not a prime number, then n is divisible by a prime number. Note that by definition, we only know that if n is not a prime number, then n is divisible by a number that is neither 1 nor n.

Problem 4 (20 points).

(a) (10 points) Suppose that $f : A \to B$ is bijective. Prove that there exists a unique bijection $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$.

(b) (10 points) Suppose $f : A \to B$ such that there exists a $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$. Are f and g bijective? If not, what can you say about f and g?

Problem 5 (10 points). Let A and B be finite sets. Prove that $|A \times B| = |A| \cdot |B|$.

Problem 6 (10 points). Let A be a finite set. Then $|\mathcal{P}(A)| = 2^{|A|}$, where $\mathcal{P}(A)$ denotes the power set of A, i.e., the set of all subsets of A.

Problem 7 (10 points). Let A and B be finite sets. Then,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is known as the inclusion/exclusion principle.

Problem 8 (20 points). Suppose we would like to prove that the statements $A(n)$ are true for all integer $n \geq 1$. Then, we can use the following variant of the proof by induction technique:

- Prove that $A(1)$ is true.
- Prove that if $A(n)$ is true then $A(2n)$ is true.
- Prove that if $A(n)$ is true then $A(n-1)$ is true.

(a) (5 points) Explain intuitively why this technique works.

(b) (15 points) Use this technique to prove the general AM-GM inequality (inequality of arithmetic and geometric means), which states that for any non-negative real numbers $a_1, a_2, \ldots, a_n \geq 0$, we have

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n}.$$