MATH1023 Homework, Part 5

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Due date: Fri, Oct 11

Exercise 1.5.1. Suppose the partial sum $s_n = \frac{n}{2n+1}$. Find the series $\sum x_n$ and its sum.
Solution: First, we find x_n:

$$x_n = s_n - s_{n-1} = \frac{n}{2n+1} - \frac{n-1}{2n-1} = \frac{1}{4n^2-1}.$$

Here, the derivation works for all $n \geq 1$, with s_0 being equal to 0.

Now, we find the sum of the series:

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

Exercise 1.5.2 (2). Compute the partial sum and the sum of the series:

$$\sum_{n=1}^{\infty} \frac{1}{(a+nd)(a+(n+1)d)}.$$

Solution: The partial sums are

$$s_k = \sum_{n=1}^{k} \frac{1}{(a+nd)(a+(n+1)d)} = \frac{1}{d}\sum_{n=1}^{k}\left(\frac{1}{a+nd} - \frac{1}{a+(n+1)d}\right) = \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+(k+1)d}\right).$$

The sum is

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+(k+1)d}\right) = \frac{1}{d(a+d)}.$$

Exercise 1.5.5. Find the partial sum of $\sum_{n=1}^{\infty} nx^n$ by multiplying $1-x$. Then find the sum.
Solution: Assume that $|x| < 1$. For the partial sum $s_n = x + 2x^2 + 3x^3 + \ldots + nx^n$ we write

$$(1-x)s_n = (x + 2x^2 + 3n^3 + \ldots + nx^n) - (x^2 + 2x^3 + \ldots + nx^{n+1}) = x + x^2 + \ldots + x^n - nx^{n+1} =$$
$$= x\left(\frac{1-x^n}{1-x} - nx^n\right),$$
$$s_n = \frac{x(1-(n+1)x^n + nx^{n+1})}{(1-x)^2}.$$

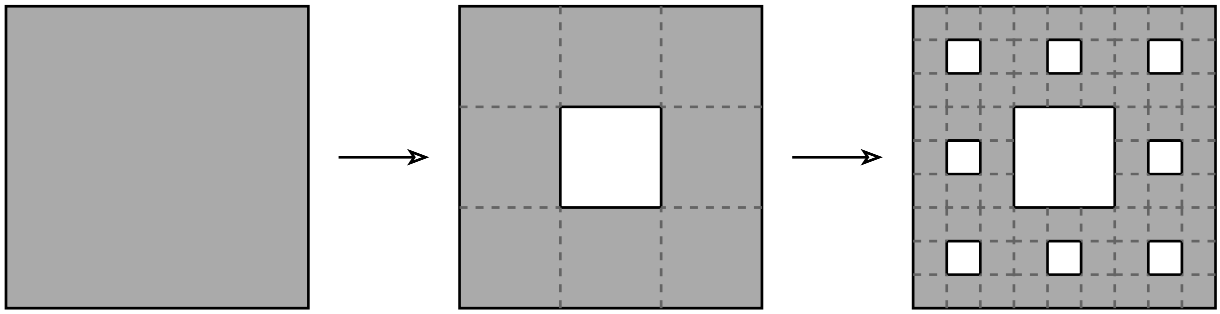
The sum will calculate as follows:

$$\sum_{n=1}^{\infty} nx^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{x(1-(n+1)x^n + nx^{n+1})}{(1-x)^2} = \frac{x}{(1-x)^2}.$$

If $|x| \geq 1$, then the derivation of the partial sums doesn't change, however, the series will diverge, since nx^n will not approach zero.

Exercise 1.5.7. Find the area of the Sierpinski carpet.
Solution: Let K denote the Sierpinski carpet and K_n $(n = 1, 2, \ldots)$ denote the intermediate shapes. For a

shape X let $S(X)$ denote its area. For K_n, we have $S(K_{n+1}) = \frac{8}{9} \cdot S(K_n)$, since every iteration removes one square out of nine:



$S = 1$ $\qquad$ $S = \frac{8}{9}$ $\qquad$ $S = \left(\frac{8}{9}\right)^2$

Figure 1: The areas of Sierpinski carpet iterations

Hence, we see that $S(K_n) = \left(\frac{8}{9}\right)^n \underset{n\to\infty}{\to} 0$. Now, since K is contained in every K_n, we see that $S(K) \leq S(K_n)$ for every n. Therefore, $S(K) = 0$.

<u>Exercise 1.5.11.</u> Prove that if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. Moreover, prove that the converse is not true.
<u>Solution:</u> Since $\sum a_n$ converges, we see that $a_n \underset{n\to\infty}{\to} 0$, meaning that $a_n < 1$ for sufficiently large n. For such n, we have $a_n^2 < a_n$. Hence, by the comparison test, we conclude that $\sum a_n^2$ converges as well.
The converse is not true, as shown by the example $a_n = \frac{1}{n}$. We have that $\sum a_n^2$ converges, while $\sum a_n$ diverges.

<u>Exercise 1.5.14 (2, 5).</u> Determine the convergence, $b, d, p, q > 0$:

2.
$$\sum \frac{3n^2 - 2n^3}{\sqrt{4n^5 + 5n^4}};$$

5.
$$\sum \frac{(c + nd)^q}{(a + nb)^p}.$$

<u>Solution:</u>

2. Consider the square of the common term:

$$x_n^2 = \frac{\left(2n^3 - 3n^2\right)^2}{4n^5 + 5n^4} = \frac{4n^6 - 12n^5 + 9n^4}{4n^5 + 5n^4}.$$

   This is a rational expression where the degree of the polynomial in the numerator is greater than the that of the polynomial in the denominator. Hence, x_n^2 diverges to infinity. As a consequence, x_n diverges as well, and we see that the series diverges since the necessary condition that $x_n \underset{n\to\infty}{\to} 0$ fails.

5. We manipulate the common term as follows:

$$x_n = \frac{(c + nd)^q}{(a + nb)^p} = n^{q-p} \frac{\left(\frac{c}{n} + d\right)^q}{\left(\frac{a}{n} + b\right)^p} = n^{q-p} \cdot A_n.$$

   The multiplier A_n converges to $\frac{d^q}{b^p}$, and can thus be bounded below and above by $\mu = \frac{1}{2}\frac{d^q}{b^p}$ and $\nu = 2\frac{d^q}{b^p}$, respectively (for sufficiently large n). Now, if $q - p < \alpha < -1$ for some α, then we have

$$\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} \frac{\nu}{n^{\alpha}},$$

and the series converges by the comparison test.

If, however, $q - p \geq 1$, then we have

$$\sum_{n=1}^{\infty} x_n \geq \sum_{n=1}^{\infty} \frac{\mu}{n},$$

and the series diverges by the converse of the comparison test.

Exercise 1.5.17 (3). Determine the convergence of

$$\sum \frac{5^{n-1} - n^2 2^n}{3^{n+1}}.$$

Solution: The common term can be expressed as

$$x_n = \frac{5^{n-1} - n^2 2^n}{3^{n+1}} = \frac{1}{15} \cdot \left(\frac{5}{3}\right)^n - \frac{1}{3} \cdot n^2 \cdot \left(\frac{2}{3}\right)^n.$$

The right part of the resulting difference converges to 0 (since $\frac{2}{3} < 1$), while the left part diverges to infinity (since $\frac{5}{3} > 1$). Hence, x_n diverges to infinity, and the series diverges due to the failure of the necessary condition that $x_n \underset{n\to\infty}{\to} 0$.

Exercise 1.5.18 (2). Determine the convergence of

$$\sum n x^{n^2}.$$

Solution: If $|x| \geq 1$, then the series obviously diverges, since the common term does not converge to 0. Otherwise, we have $|x| < 1$. We recall that the series $\sum n|x|^n$ converges, as was shown in Exercise 1.5.5. Now, since $|x| < 1$, we write

$$\left| n x^{n^2} \right| = n|x|^{n^2} \leq n|x|^n,$$

and conclude the convergence of the original series due to the convergence test.

Exercise 1.5.19 (1, 5). Determine the convergence, $a, b > 0$:

1. $$\sum_{n=0}^{\infty} (a^n + b^n)^p;$$

5. $$\sum_{n=1}^{\infty} \frac{n^p}{\left(a + \frac{b}{n}\right)^n}.$$

Solution:

1. If either of a, b equals or exceeds 1 (say, a), then we have

$$\sum (a^n + b^n)^p \geq \sum (a^p)^n = +\infty,$$

and the series diverges.

If both a and b lie in the interval $(0, 1)$, then we have

$$\sum_{n=0}^{\infty} (a^n + b^n)^p \leq \sum_{n=0}^{\infty} (2 \cdot \max(a,b)^n)^p = 2^p \cdot \sum_{n=0}^{\infty} (\max(a,b)^p)^n = 2^p \cdot \frac{1}{1 - \max(a,b)^p},$$

and the series converges by the comparison test.

5. Consider three cases for a:

- $0 < a < 1$. Then, for sufficiently large n, we have $\frac{b}{n} < \frac{1-a}{2}$ and $a + \frac{b}{n} < a + \frac{1-a}{2} = 1 - \frac{1-a}{2} = \alpha < 1$. Then, we have

$$\frac{n^p}{\left(a + \frac{b}{n}\right)^n} \geq n^p \left(\frac{1}{\alpha}\right)^n.$$

  Since the series of the right terms diverges (as $\frac{1}{\alpha} > 1$ and $n^p \left(\frac{1}{\alpha}\right)^n \underset{n\to\infty}{\to} +\infty$), the series of the left terms must diverge as well.

- $a = 1$. Then, if $p \geq -1$, then we have

$$\left(a + \frac{b}{n}\right)^n = \left(1 + \frac{b}{n}\right)^n = \left(\left(1 + \frac{1}{n/b}\right)^{n/b}\right)^b < e^b,$$

  and therefore

$$\sum \frac{n^p}{\left(a + \frac{b}{n}\right)^n} \geq \sum \frac{n^p}{e^b} \geq \sum \frac{1}{e^b \cdot n} = +\infty,$$

  and the series diverges. If, however, $p < -1$, then we write

$$\sum \frac{n^p}{\left(1 + \frac{n}{b}\right)^n} \leq \sum n^p,$$

  and the former series converges by the comparison test, since the latter series converges.

- $a > 1$. We again consider different cases for p. If $p \leq 1$, we write

$$\sum \frac{n^p}{\left(a + \frac{b}{n}\right)^n} \leq \sum \frac{n^p}{a^n} \leq \sum n \cdot \left(\frac{1}{a}\right)^n.$$

  The convergence of the last series has been established in Exercise 1.5.5, and so the original series converges as well, by the comparison test.

  If $p > 1$, we do a trick:

$$\frac{n^p}{a^n} = \left(\frac{n}{\left(a^{1/p}\right)^n}\right)^p.$$

  Since $\alpha = a^{1/p}$ is still greater than 1, we have $\frac{n}{\alpha^n} = n \cdot \left(\frac{1}{\alpha}\right)^n \underset{n\to\infty}{\to} 0$, and so $\frac{n}{\alpha^n} < 1$ for sufficiently large n. Since $p > 1$, for such n we also have $\left(\frac{n}{\alpha^n}\right)^p < \frac{n}{\alpha^n}$. Finally, we write

$$\frac{n^p}{\left(a + \frac{b}{n}\right)^n} \leq \frac{n^p}{a^n} = \left(\frac{n}{\alpha^n}\right)^p \leq \frac{n}{\alpha^n},$$

  and then apply the comparison test to conclude that the original series converges.

Exercise 1.5.20 (2). Determine convergence:

$$\frac{2}{4} + \frac{2 \cdot 6}{4 \cdot 7} + \frac{2 \cdot 6 \cdot 10}{4 \cdot 7 \cdot 10} + \cdots.$$

Solution: Seeing through the pattern, we recognize this as

$$\sum_{n=0}^{\infty} \frac{2 \cdot 6 \cdot \ldots \cdot (2+4n)}{4 \cdot 7 \cdot \ldots \cdot (4+3n)} = \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n} \frac{2+4n}{4+3n} \right)$$

Consider the ratio of consecutive terms (here x_n are the terms of the series):

$$\frac{x_n}{x_{n-1}} = \frac{2+4n}{4+3n}.$$

We see that $\frac{x_n}{x_{n-1}}$ converges to $\frac{4}{3} > 1$. As a consequence, there is a number α such that $1 < \alpha < \frac{4}{3}$ and for sufficiently large n (say, from $n = N$) we have

$$\frac{x_n}{x_{n-1}} > \alpha.$$

Now, for $n > N$

$$x_n = x_N \cdot \frac{x_{N+1}}{x_N} \cdot \ldots \cdot \frac{x_n}{x_{n-1}} > x_N \cdot \alpha^{n-N} \underset{n\to\infty}{\to} +\infty.$$

Hence x_n diverges to $+\infty$, and thus the series $\sum x_n$ diverges, since the necessary condition that $x_n \to 0$ fails.

Exercise 1.5.21 (4). Determine convergence:

$$\sum_{n=1}^{\infty} \frac{(a+c)(a+2c)^2 \cdots (a+nc)^n}{(b+d)(b+2d)^2 \cdots (b+nd)^n}.$$

Solution: Consider the ratio of consecutive terms:

$$\frac{x_n}{x_{n-1}} = \frac{(a+nc)^n}{(b+nd)^n} = \left(\frac{c + \frac{a}{n}}{d + \frac{b}{n}} \right)^n,$$

where x_n are the terms of the series. First, assume that $|c| < |d|$. Then we see that

$$\left| \frac{c + \frac{a}{n}}{d + \frac{b}{n}} \right| \underset{n\to\infty}{\to} \left| \frac{c}{d} \right| < 1.$$

Then there is α such that $|c/d| < \alpha < 1$ and for sufficiently large n (say, from $n > N$), we have

$$\left| \frac{x_n}{x_{n-1}} \right| = \left| \frac{c + \frac{a}{n}}{d + \frac{b}{n}} \right|^n < \alpha^n < \alpha.$$

As a result, we can bound x_n above by $x_N \cdot \alpha^{n-N}$, and the original series converges by the comparison test, since $\sum \alpha^n$ converges.

Now, let $|c| \geq |d|$. We will have to consider some more cases:

- $d = 0$. Then $b \neq 0$, and we consider another couple of cases:
  1. $c = 0$. Then the ratio x_n / x_{n-1} reduces to $(a/b)^n$. If $|a| < |b|$, then the series converges, by the same reason we provided in the case $|c| < |d|$. If $a = b$, then $x_n = 1$ for all n, and thus the series diverges to $+\infty$. If $|a| > |b|$, then

$$\left|\frac{x_n}{x_{n-1}}\right| = \left|\frac{a}{b}\right|^n > \left|\frac{a}{b}\right| > 1,$$

$$|x_n| = |x_1| \cdot \left|\frac{x_2}{x_1}\right| \cdot \ldots \cdot \left|\frac{x_n}{x_{n-1}}\right| > \left|\frac{a}{b}\right|^n \underset{n\to\infty}{\to} +\infty,$$

and therefore $|x_n|$ diverges to $+\infty$, which means that the series diverges as well.

2. $c \neq 0$. Then we have

$$\left|\frac{x_n}{x_{n-1}}\right| = \left|\frac{a+cn}{b}\right|^n \underset{n\to\infty}{\to} +\infty,$$

which again means that the series diverges, by logic similar to the previous case (where $|a| > |b|$).

- $d \neq 0$. Then $c \neq 0$, since $|c| \geq |d| > 0$. Another couple of cases:

1. $|c| > |d|$. Then, since

$$\left|\frac{c + \frac{a}{n}}{d + \frac{b}{n}}\right| \underset{n\to\infty}{\to} \left|\frac{c}{d}\right| > 1,$$

we have that $\left|\frac{x_n}{x_{n-1}}\right| > \alpha > 1$ for some α, for sufficiently large n. Again, the series diverges, by logic similar to previous cases.

2. $c = d$. Denote $f = c = d$. We will require the following lemma:

Lemma 1. *For all $x \in \mathbb{R}$, we have the convergence*

$$\left(1 + \frac{x}{n}\right)^n \underset{n\to\infty}{\to} e^x.$$

> *Proof:* If $x = 0$, then the result is obvious, a constant sequence converging to $1 = e^0$. If $x > 0$, then
>
> $$\left(1+\frac{x}{n}\right)^n = \left(\left(1 + \frac{1}{n/x}\right)^{n/x}\right)^x = \left(\left(1+\frac{1}{k}\right)^k\right)^x \underset{k\to\infty}{\to} e^x.$$
>
> Here we do a substitution $k = n/x$. This is justified since $n \to \infty$ if and only if $k \to \infty$.
>
> If $x < 0$, then we write
>
> $$\left(1+\frac{x}{n}\right)^n = \left(1 - \frac{-x}{n}\right)^n = \left(\left(1 - \frac{1}{-n/x}\right)^{-n/x}\right)^{-x} = \left(\left(1-\frac{1}{k}\right)^k\right)^{-x} \underset{k\to\infty}{\to} \left(\frac{1}{e}\right)^{-x} = e^x,$$
>
> substituting $-n/x$ with k. ■

Note 1. The formulation of the lemma is usually taken as the *definition* of the function $\exp(x) = e^x$, and subsequently for defining real number exponentiation. If the present course adopts the same approach, the lemma is unnecessary. Plus, prof. Yan Min claimed that putting this exercise at this stage of the course was a mistake, since we haven't learned the right tools yet. If my reasoning is insufficiently rigorous, all questions to him.

With this statement, we write

$$\left|\frac{x_n}{x_{n-1}}\right| = \left|\frac{f + \frac{a}{n}}{f + \frac{b}{n}}\right|^n = \left|\frac{1 + \frac{a/f}{n}}{1 + \frac{b/f}{n}}\right|^n \underset{n\to\infty}{\to} \frac{e^{a/f}}{e^{b/f}} = e^{\frac{a-b}{f}}$$

If $a = b$, then all of the terms x_n equal 1, and the series diverges.
If $(a - b)f < 0$, then we have

$$\left| \frac{x_n}{x_{n-1}} \right| < \alpha < 1$$

for some α for sufficiently large n, since $e^{\frac{a-b}{f}} < 1$. In this case, the series converges by logic already stated before.
If $(a - b)f > 0$, then for some α we will have

$$\left| \frac{x_n}{x_{n-1}} \right| > \alpha > 1$$

for sufficiently large n. This means that x_n does not converge to 0, and thus the series diverges.
This concludes the cases.

So, what's the result? The series converges in the following cases and only then:
1. $|c| < |d|$;
2. $c = d = 0$ and $|a| < |b|$;
3. $c = d$ and $(a - b)d < 0$.

Exercise 1.5.22 (4, 5). Determine convergence. There might be values of x for which no conclusion can yet be made.
4.
$$\sum \frac{n^n}{n!} x^n;$$
5.
$$\sum \frac{n!}{n^n} x^n.$$

Solution:
4. Consider the ratio of consecutive terms:

$$\frac{x_{n+1}}{x_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!} x^{n+1}}{\frac{n^n}{n!} x^n} = x\left(1 + \frac{1}{n}\right)^n \underset{n\to\infty}{\rightarrow} e \cdot x.$$

If $|x| < \frac{1}{e}$, then we see that

$$\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = e \cdot |x| < 1,$$

meaning that the series converges.
If $|x| = \frac{1}{e}$, then the series diverges, but the proof requires integration techniques we now lack.
If $|x| > \frac{1}{e}$, then for sufficiently large n

$$\left| \frac{x_{n+1}}{x_n} \right| > 1,$$

meaning that x_n does not converge to 0, and the series diverges.

Exercise 1.6.5. Prove that $\lim\limits_{n\to\infty} (x_n, y_n) = (k, l)$ with respect to the L^1-norm if and only if $\lim\limits_{n\to\infty} x_n = k$ and $\lim\limits_{n\to\infty} y_n = l$.
Solution:

$\Longrightarrow$: Assume $\lim\limits_{n\to\infty}(x_n, y_n) = (k, l)$. We will prove that $\lim\limits_{n\to\infty} x_n = k$. Let $\varepsilon > 0$ be arbitrary. Then there is N such that $n > N$ implies

$$\|(x_n, y_n) - (k, l)\|_1 = |x_n - k| + |y_n - l| < \varepsilon.$$

Consequently, $|x_n - k| < \varepsilon$ for $n > N$. Hence, x_n converges to k.
Similarly, y_n also converges to l, by the same logic (or by symmetry).

$\Longleftarrow$: Assume $\lim\limits_{n\to\infty} x_n = k$ and $\lim\limits_{n\to\infty} y_n = l$. Let $\varepsilon > 0$ be freely chosen. For $\delta = \frac{\varepsilon}{2}$, we have N_1 and N_2 such that

$$n > N_1 \Longrightarrow |x_n - k| < \delta \qquad \text{and} \qquad n > N_2 \Longrightarrow |y_n - l| < \delta.$$

Hence, for $n > N = \max(N_1, N_2)$ we have

$$\|(x_n, y_n) - (k, l)\|_1 = |x_n - k| + |y_n - l| < 2\delta = \varepsilon.$$

Therefore, (x_n, y_n) converges to (k, l).

Exercise 1.6.8. Extend the relation between the L^2-norm and the L^∞-norm on $\mathbb{R}^n$.
Solution: First of all, we see that

$$\|\vec{x}\|_\infty = \max_{1\le k\le n} |x_k| = \sqrt{\left(\max_{1\le k\le n} |x_k|\right)^2} \le \sqrt{\sum_{k=1}^{n} x_k^2} = \|\vec{x}\|_2.$$

On the other hand,

$$\|\vec{x}\|_2 = \sqrt{\sum_{k=1}^{n} x_k^2} \le \sqrt{n \cdot \left(\max_{1\le k\le n} x_k\right)^2} = \sqrt{n} \cdot \|\vec{x}\|_\infty.$$