WORK AND KINETIC ENERGY

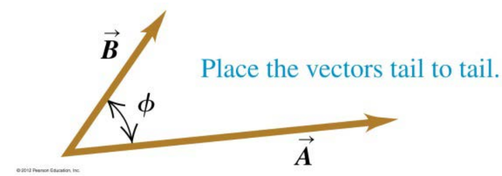
Intended Learning Outcomes – after this lecture you will learn:

1. Scalar product of vectors
2. the meanings of +ve and –ve work done.
3. the Hooke's law as an example of a variable force.
4. the work-energy theorem in the general case.

Textbook Reference: Ch 6

Work

$\boxed{\text{Scalar product}}$: $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB\cos\phi$



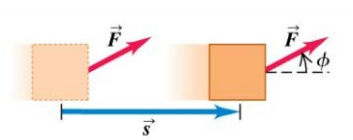
Special cases:

(i) if $\vec{\boldsymbol{A}} \parallel \vec{\boldsymbol{B}}$, $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB$, in particular, $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$

(ii) if $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$, $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = 0$, in particular, $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$

In analytical form, $\boxed{\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_x B_x + A_y B_y + A_z B_z}$

From high school,



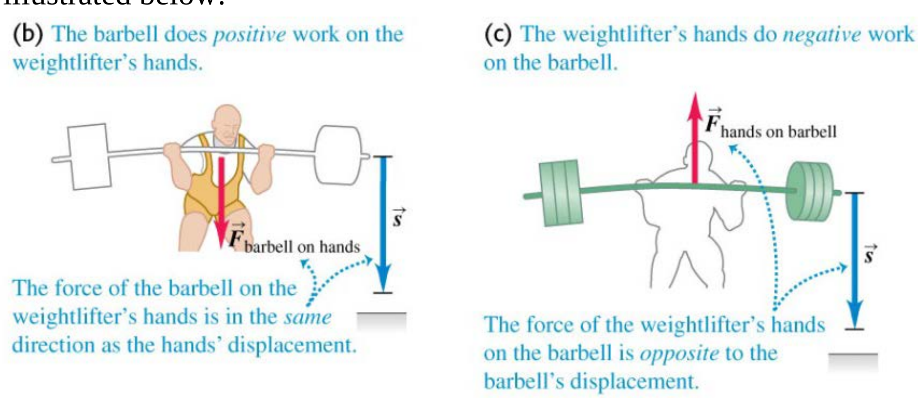
work done $W = Fs\cos\phi$ SI unit: joule 1 J = 1 N·m

$\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{s}}$, see how useful vector notation is!!

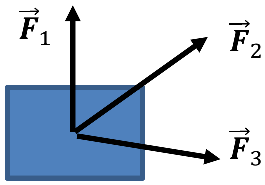
In general, $\boxed{W = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{s}} = F_x s_x + F_y s_y + F_z s_z}$

⚠ W must refer to the work done by a **specific force** on a **body**, otherwise you may be confused by the sign as illustrated below:

Action and reaction – a body does work on a second body, the second body does an equal and opposite amount of work on the first.



Workdone by multiple forces:

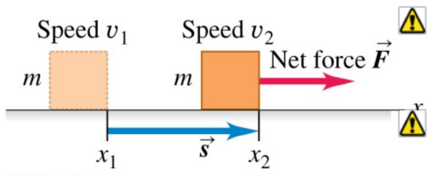


$$W = \underbrace{\left(\sum \vec{F}\right) \cdot \vec{s}}_{\text{work done by resultant force}} = \underbrace{\sum \vec{F} \cdot \vec{s}}_{\text{sum of work done by individual forces}}$$

Question: An electron moving in a straight line with a constant speed of 8 × 10^7 m/s. You are told that it has electric, magnetic, and gravitational forces acting on it. During a 1 m displacement, the total work done on the electron is (i) +ve, (ii) –ve, (iii) zero, (iv) not enough information given to decide. Answer: see inverted text on P. 204 of the textbook

Also from high school:

1) Definition of **kinetic energy**, $K = \frac{1}{2}mv^2$
2) **Work-energy theorem**
   Work done by the net external force = change in KE of the particle

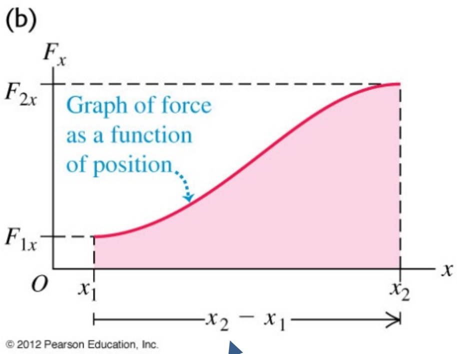


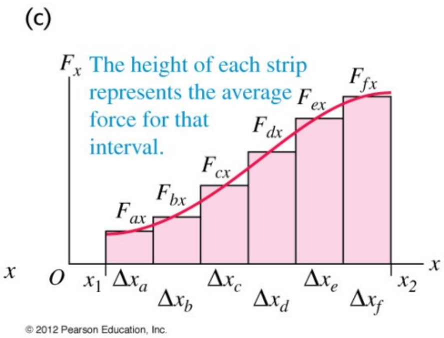
⚠ When accelerating a particle, work done by the external net force $W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 > 0$

⚠ When decelerating a particle, $W < 0$

The above results are easy to prove if you consider 1D motion under a constant external force. You have done it high school. If you have forgotten, see Textbook P. 205.

<u>Question</u>: What if the force is not constant (but still in 1D)? Need a magical mathematical tool called **calculus**.

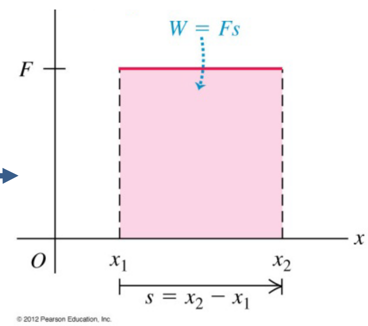




approximate each sub-interval by a constant force
$W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \cdots$

as $\Delta x \to 0$
$W =$ area under curve
$= \int_{x_1}^{x_2} F_x \, dx$

c.f. constant force



Example An ideal spring

Hooke's law (Robert Hooke, 1678)
– restoring force (i.e., tension in the spring) $= -kx$

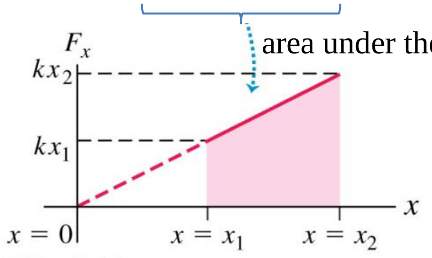
displacement from natural length (unstretched position)

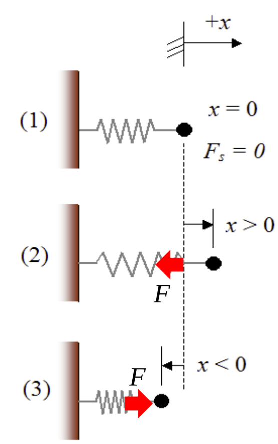
direction opposite to displacement

force constant
unit: N/m

Work done by an external force (⚠ not tension in the spring) in *stretching* a spring from x_1 to x_2, ($x_2 > x_1 > 0$)

$$W = \int_{x_1}^{x_2} F \, dx = k \int_{x_1}^{x_2} x \, dx = \underbrace{\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2}_{\text{area under the line}}$$





In *compressing* from $-x_1$ to $-x_2$, same formula holds, $W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$

1D motion with variable force, $a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$, i.e., $F = ma = mv\frac{dv}{dx}$

$\therefore$ work done by the net external force

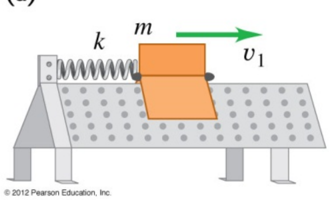
$$\int_{x_1}^{x_2} F\,dx = m\int_{x_1}^{x_2} v\frac{dv}{dx}dx = m\int_{v_1}^{v_2} v\,dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \text{change in KE}$$

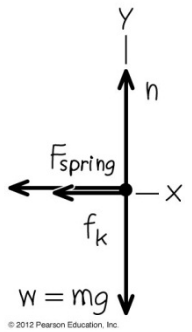
Work-energy theorem works for variable force!

Example 6.7 P. 212 Motion with a varying force

A glider of mass m, and a spring with force constant k. Initially the spring is unstretched and the glider is moving with speed v_1. What is the maximum displacement d to the right if the frictional coefficient is μ_k?

(a)





By the work-energy theorem

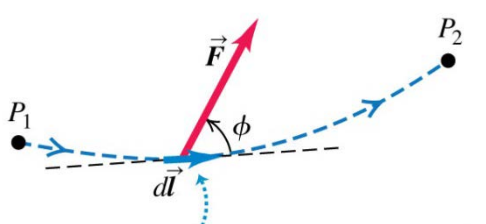
$$\underbrace{(-\mu_k mgd)}_{\text{work done by } f_k} + \underbrace{\left(-\int_0^d kxdx\right)}_{\text{work done by } F_{\text{spring}}} = \underbrace{0 - \frac{1}{2}mv_1^2}_{\text{change in KE}}$$

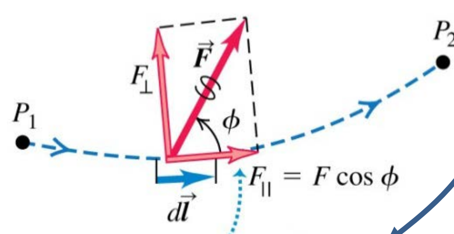
$$\frac{1}{2}kd^2 + \mu_k mgd - \frac{1}{2}mv_1^2 = 0$$

$$\Rightarrow \quad d = -\frac{\mu_k mg}{k} \pm \sqrt{\left(\frac{\mu_k mg}{k}\right)^2 + \frac{mv_1^2}{k}}$$

3D motion with variable force

Idea: break up the path into very short segments so that in each segment, $\vec{\boldsymbol{F}}$ is approximately constant





⚠ v along tangential direction (why?)

work done in this small segment $dW = \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{l}} = F_\parallel dl = mv\frac{dv}{dl}dl = mvdv$

total work done = sum over all segments

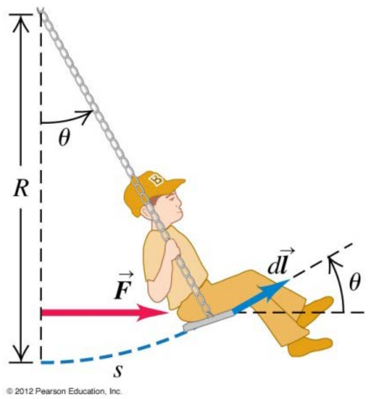
$$W_{tot} = \sum \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{l}} \rightarrow \int_{P_1}^{P_2} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{l}} = \int_{P_1}^{P_2} mvdv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

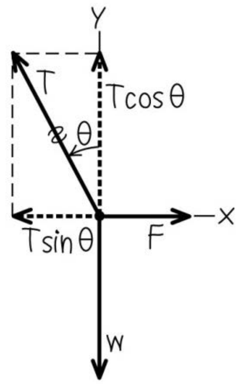
Conclusion: work-energy theorem holds for motion along a curve under variable force.

Example 6.8 P. 214 Motion on a curved path

Apply a horizontal force $\vec{\boldsymbol{F}}$ to push the swing up from $\theta = 0$ to θ_0

Assumption: $\vec{\boldsymbol{F}}$ is just enough to push it up so that the swing is in equilibrium any time





$$\sum F_x = F - T\sin\theta = 0$$
$$\sum F_y = T\cos\theta - w = 0$$
$$\Rightarrow\ T = w\sec\theta$$
$$F = w\tan\theta$$

Work done by net force, W_{net} = _____

Work done by $\vec{\boldsymbol{T}}$,

W_T = ______ ($\because \vec{\boldsymbol{T}} \perp d\vec{\boldsymbol{l}}$)

Work done by $\vec{\boldsymbol{F}}$,

$$W_F = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = \int_0^{\theta_0} F\cos\theta\, dl = \int_0^{\theta_0} w\tan\theta\cos\theta\, Rd\theta = wR(1-\cos\theta_0)$$

Work done by $\vec{\boldsymbol{w}}$,

$$W_w = \int \vec{\boldsymbol{w}} \cdot d\vec{\boldsymbol{l}} = \int_0^{\theta_0} w\cos\left(\frac{\pi}{2}+\theta\right) dl = -\int_0^{\theta_0} w\sin\theta\, Rd\theta = -wR(1-\cos\theta_0)$$

Check that $W_{\text{net}} = W_T + W_F + W_w$

Power

Average over a period Δt, $P_{av} = \frac{\Delta W}{\Delta t}$

Instantaneous power ($\Delta t \to 0$), $P = \lim_{\Delta t\to 0}\frac{\Delta W}{\Delta t} = \frac{dW}{dt}$ SI unit: watt, 1 W = 1 J/s

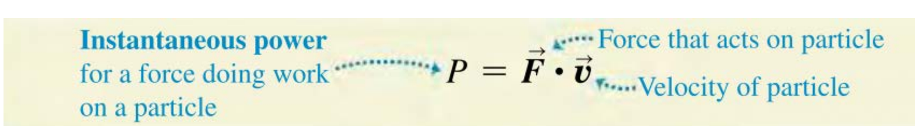
Another unit of *energy* besides J – kilowatt hour, common in electric bills

1 KW·h = (10^3 J/s)(3600 s) = 3.6×10^6 J

If $\vec{\boldsymbol{F}}$ is the force that do work (can be constant or variable), workdone during Δt is $\Delta W = \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{s}}$

$$P = \lim_{\Delta t\to 0}\frac{\Delta W}{\Delta t} = \lim_{\Delta t\to 0}\frac{\vec{\boldsymbol{F}}\cdot\Delta\vec{\boldsymbol{s}}}{\Delta t} = \vec{\boldsymbol{F}}\cdot\frac{d\vec{\boldsymbol{s}}}{dt} = \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}}$$

Conclusion



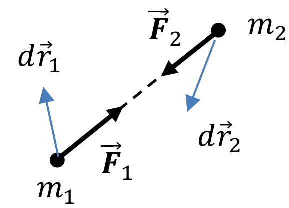
Work Done by an Action-reaction Pair

$\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ are action-reaction pair, $\vec{\boldsymbol{F}}_1 = -\vec{\boldsymbol{F}}_2$

Total work done on the two masses:

$$dW = \vec{\boldsymbol{F}}_1 \cdot d\vec{\boldsymbol{r}}_1 + \vec{\boldsymbol{F}}_2 \cdot d\vec{\boldsymbol{r}}_2 = \vec{\boldsymbol{F}}_1 \cdot (d\vec{\boldsymbol{r}}_1 - d\vec{\boldsymbol{r}}_2) = \vec{\boldsymbol{F}}_1 \cdot d\vec{\boldsymbol{r}}_{12}$$

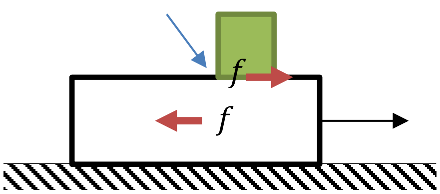
$$\boxed{W = \int \vec{\boldsymbol{F}}_1 \cdot d\vec{\boldsymbol{r}}_{12}}$$



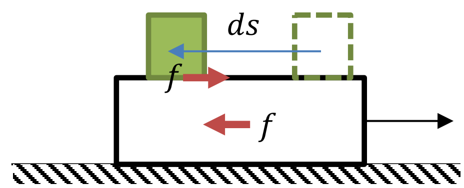
⚠ Work done by an action-reaction pair is the work done on one body as it moves along its path relative to the other body, i.e., sit on m_2 and find the workdone by $\vec{\boldsymbol{F}}_1$ on m_1.

⚠ If the two bodies do not move relative to each other, work done is zero.

If no relative displacement, static friction, no work done



With relative motion, kinetic friction, work done $= -f\,ds$



Example (Problem 6.71 P. 225)

A mass of 0.0600 kg going around a circular path on a smooth table.

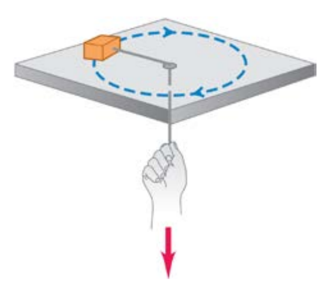
Original: radius 0.40 m, speed 0.70 m/s

Final: radius 0.10 m, speed 2.80 m/s

Workdone by you in pulling the string downwards

$= \frac{1}{2}(0.0600\text{ kg})[(2.80\text{ m/s})^2 - (0.70\text{ m/s})^2] =$ ________J

Average force exerted by you in the process is ______________ N



Example – effective spring constant of two springs

Two springs of constants k_1 and k_2 in series:

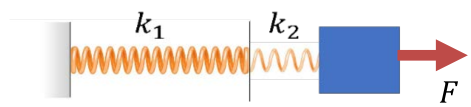
Suppose their extensions are x_1 and x_2, the effective spring constant is K

Set up equations of motion at:

At the block: $F = K(x_1 + x_2) = k_2 x_2$;

At the two springs: $k_2 x_2 - k_1 x_1 = 0$

by eliminating x_1 and x_2 $\Rightarrow$ $\boxed{\dfrac{1}{K} = \dfrac{1}{k_1} + \dfrac{1}{k_2}}$



Mathematical Digression

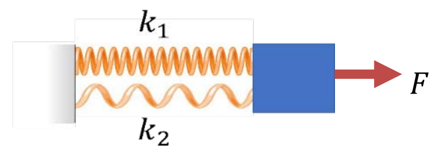
In solving the two simultaneous *homogenous* equations

$$\begin{cases} Kx_1 + (K - k_2)x_2 = 0 \\ k_1x_1 - k_2x_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} K & K-k_2 \\ k_1 & -k_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K & K-k_2 \\ k_1 & -k_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Only solution is $x_1 = x_2 = 0$. This *mathematical* solution is not *physical* because it means the springs are inextensible. To avoid getting this conclusion, the only possibility is that the inverse of the matrix $\begin{pmatrix} K & K-k_2 \\ k_1 & -k_2 \end{pmatrix}$ does not exist, i.e., $\begin{vmatrix} K & K-k_2 \\ k_1 & -k_2 \end{vmatrix} = 0 \Rightarrow \frac{1}{K} = \frac{1}{k_1} + \frac{1}{k_2}$

Two springs in parallel: obviously $\boxed{K = k_1 + k_2}$

c.f. two electrical resistors in series and in parallel



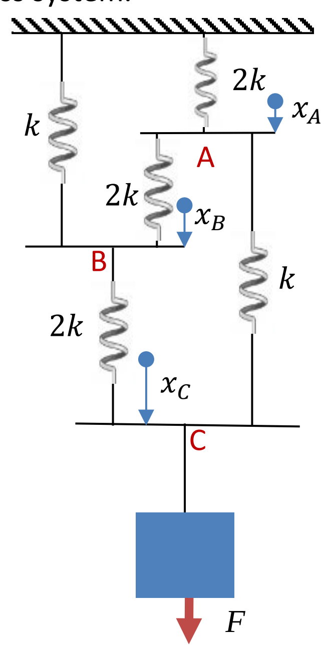
Scaling Analysis

Let's apply the scaling analysis in Lecture 1 P. 9 to analyze the parallel case with $k_1 = k_2 = k$. Using Galileo's argument, two identical systems of mass m and spring constant k hanging vertically side by side with the same extension, $mg = k\Delta y$. If tie them together, the mass becomes $2m$ with no change in the extension, $(2m)g = K\Delta y$. Therefore effective spring constant must be $K = 2k$.

Challenge: use a similar argument for the case $k_1 \neq k_2$.

Challenge Example

Find the effective spring constant of the following spring and mass system.



Taking downwards as positive, x_A, x_B, and x_C are the extensions of the corresponding springs.

$$F = K(x_A + x_B + x_C)$$

Set up equations of motion at the junctions A, B, and C respectively:

$$-2kx_A + 2kx_B + k(x_B + x_C) = 0$$
$$-k(x_A + x_B) - 2kx_B + 2kx_C = 0$$
$$-k(x_B + x_C) - 2kx_C + K(x_A + x_B + x_C) = 0$$

By eliminating x_A, x_B, and x_C, or by setting the corresponding determinant to zero, we get the effective spring constant

$$K = \frac{10}{7}k$$

Clicker Questions

Q1.4

Consider the two vectors

$$\vec{A} = 3\hat{i} + 4\hat{j}$$
$$\vec{B} = -8\hat{i} + 6\hat{j}$$

$$\vec{A} \bullet \vec{B}\,?$$

A. zero

B. 14

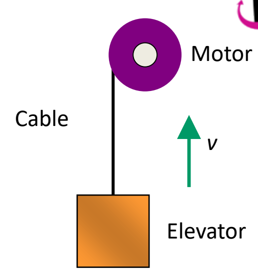
C. 48

D. 50

E. none of these

Q6.1

An elevator is being *lifted* at a constant speed by a steel cable attached to an electric motor. Which statement is correct?



A. The cable does positive work on the elevator, and the elevator does positive work on the cable.

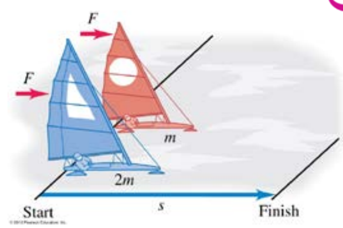
B. The cable does positive work on the elevator, and the elevator does negative work on the cable.

C. The cable does negative work on the elevator, and the elevator does positive work on the cable.

D. The cable does negative work on the elevator, and the elevator does negative work on the cable.

Q6.3

Two iceboats (one of mass *m,* one of mass 2*m*) hold a race on a frictionless, horizontal, frozen lake. Both iceboats start at rest, and the wind exerts the same constant force on both iceboats.



Which iceboat crosses the finish line with more kinetic energy (KE)?

A. The iceboat of mass *m*: it has twice as much KE as the other.

B. The iceboat of mass *m*: it has 4 times as much KE as the other.

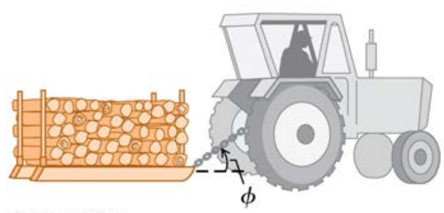
C. The iceboat of mass 2*m*: it has twice as much KE as the other.

D. The iceboat of mass 2*m*: it has 4 times as much KE as the other.

E. They both cross the finish line with the same kinetic energy.

Q6.4

A tractor driving at a constant speed pulls a sled loaded with firewood. There is friction between the sled and the road.



The total work done on the sled after it has moved a distance *d* is

A. positive.

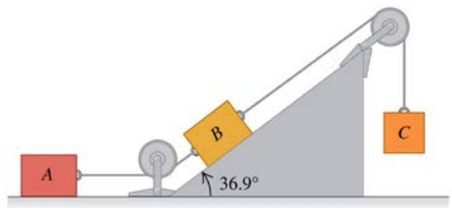
B. negative.

C. zero.

D. not enough information given to decide

Q6.8

Three blocks are connected as shown. The ropes and pulleys are of negligible mass. When released, block *C* moves downward, block *B* moves up the ramp, and block *A* moves to the right.



After each block has moved a distance *d*, the force of gravity has done

A. positive work on *A*, *B*, and *C.*

B. zero work on *A,* positive work on *B*, and negative work on *C.*

C. zero work on *A,* negative work on *B*, and positive work on *C.*

D. none of these

Q6.10

An object is initially at rest. A net force (which always points in the same direction) is applied to the object so that the *power* of the net force is constant. As the object gains speed,

A. the magnitude of the net force remains constant.

B. the magnitude of the net force increases.

C. the magnitude of the net force decreases.

D. not enough information given to decide

Ans: Q1.4) A, Q6.1) B, Q6.3) E, Q6.4) C, Q6.8) C, Q6.10) C

Robert Hooke

From Wikipedia, the free encyclopedia

Robert Hooke FRS (28 July [O.S. 18 July] 1635 – 3 March 1703) was an English natural philosopher, architect and polymath.

His adult life comprised three distinct periods: as a scientific inquirer lacking money; achieving great wealth and standing through his reputation for hard work and scrupulous honesty following the great fire of 1666, but eventually becoming ill and party to jealous intellectual disputes. These issues may have contributed to his relative historical obscurity.

He was at one time simultaneously the curator of experiments of the Royal Society and a member of its council, Gresham Professor of Geometry and a Surveyor to the City of London after the Great Fire of London, in which capacity he appears to have performed more than half of all the surveys after the fire. He was also an important architect of his time, though few of his buildings now survive and some of those are generally misattributed, and was instrumental in devising a set of planning controls for London whose influence remains today. Allan Chapman has characterised him as "England's Leonardo".[1]

Hooke studied at Wadham College during the Protectorate where he became one of a tightly knit group of ardent Royalists centred around John Wilkins. Here he was employed as an assistant to Thomas Willis and to Robert Boyle, for whom he built the vacuum pumps used in Boyle's gas law experiments. He built some of the earliest Gregorian telescopes, observed the rotations of Mars and Jupiter and, based on his observations of fossils, was an early proponent of biological evolution.[2][3] He investigated the phenomenon of refraction, deducing the wave theory of light, and was the first to suggest that matter expands when heated and that air is made of small particles separated by relatively large distances. He performed pioneering work in the field of surveying and map-making and was involved in the work that led to the first modern plan-form map, though his plan for London on a grid system was rejected in favour of rebuilding along the existing routes. He also came near to deducing that gravity follows an inverse square law, and that such a relation governs the motions of the planets, an idea which was subsequently developed by Newton.[4] Much of Hooke's scientific work was conducted in his capacity as curator of experiments of the Royal Society, a post he held from 1662, or as part of the household of Robert Boyle.

Robert Hooke

An artist's impression of Robert Hooke. No authenticated contemporary likenesses of Hooke survive.

Born	28 July [O.S. 18 July] 1635 Freshwater, Isle of Wight, England
Died	3 March 1703 (aged 67) London, England
Fields	Physics and chemistry
Institutions	Oxford University
Alma mater	Christ Church, Oxford
Academic advisors	Robert Boyle
Known for	Hooke's Law Microscopy applied the word 'cell'
Influences	Richard Busby

See http://en.wikipedia.org/wiki/Robert_Hooke for more information.