MATH 2001
Homework 2

Please write down the solutions to the problems below in full details and in full sentences. Unless otherwise specified, all claims have to be justified.

20% of the points will come the quality of your $\LaTeX$ typesetting (see the course syllabus for more detail). 80% of the points will come not only from the correctness and clarity of your solutions.

Problem 1 (30 points). For any sets A and B, we use $\mathrm{Hom}(A,B)$ to denote the set of functions from A to B. Let S be a set and $f : A \to B$ a map from A to B. Then, function composition defines a map $f_S : \mathrm{Hom}(B,S) \to \mathrm{Hom}(A,S)$ given by $\alpha \mapsto \alpha \circ f$, i.e., $f_S(\alpha) = \alpha \circ f$ for all $\alpha \in \mathrm{Hom}(B,S)$.

(a) (10 points) Show that if f is injective, then f_S is surjective for any non-empty S.
(b) (10 points) Show that if f surjective, then f_S is injective for any S.
(c) (10 points) Are the converses of the above statements true? Please make sure to justify your answer.

Problem 2 (10 points). Let $f : X \to Y$ be a function. Define a relation on X given by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$.

(a) (5 points) Show that $\sim$ is an equivalence relation on X.
(b) (5 points) Construct a bijection between the quotient set $X/\sim$ and the image $\mathrm{Im}\, f$ of f.

Problem 3 (20 points). For each fixed $n \in \mathbb{Z}$, consider the equivalence relation $\sim$ on $\mathbb{Z}$ given by $a \sim b$ if and only if $a - b$ is a multiple of n. Let $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/\sim$ denote the quotient set and $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ the quotient map.

(a) (10 points) Show that $\sim$ is an equivalence relation and describe $\mathbb{Z}/n\mathbb{Z}$ when $n = 0$, $n = 1$, and $n = 2$.
(b) (10 points) Define operations $+$ and $\cdot$ on $\mathbb{Z}/n\mathbb{Z}$ such that the quotient map $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ satisfies $\pi(a+b) = \pi(a) + \pi(b)$ and $\pi(a \cdot b) = \pi(a) \cdot \pi(b)$ for all $a, b \in \mathbb{Z}$.

    You need to verify that the operations you define are well-defined and indeed satisfy the properties above.

Problem 4 (10 points). Let $m, n \in \mathbb{N}$ such that $m + n = 0$. Prove that $m = n = 0$ (from the definition of natural numbers via Peano's axioms).

Problem 5 (10 points). The multiplication operation on the set of integers is defined as follows: $[a,b] \cdot [c,d] = [ac + bd, ad + bc]$ for any natural numbers a, b, c, d. Show that this is well-defined, i.e., if $[a,b] = [a',b']$ and $[c,d] = [c',d']$, then $[a,b] \cdot [c,d] = [a',b'] \cdot [c',d']$.

Problem 6 (20 points). Prove that the operation $\cdot$ (multiplication) on $\mathbb{Z}$ satisfies the following properties (4 points each):

(a) Distributivity: $m \cdot (n + p) = m \cdot n + m \cdot p$ for all $m, n, p \in \mathbb{Z}$.
(b) Associativity: $(m \cdot n) \cdot p = m \cdot (n \cdot p)$ for all $m, n, p \in \mathbb{Z}$.
(c) Commutativity: $m \cdot n = n \cdot m$ for all $m, n \in \mathbb{Z}$.

(d) Multiplicative unit (i.e. one): the element 1 satisfies the property that $m \cdot 1 = m$ for all $m \in \mathbb{Z}$. Moreover, any element with this property has to be 1 itself. The element 1 is called the multiplicative unit.

(e) Cancelation: $m \cdot k = n \cdot k$ implies $m = n$ for all $m, n, k \in \mathbb{Z}$ such that $k \neq 0$.