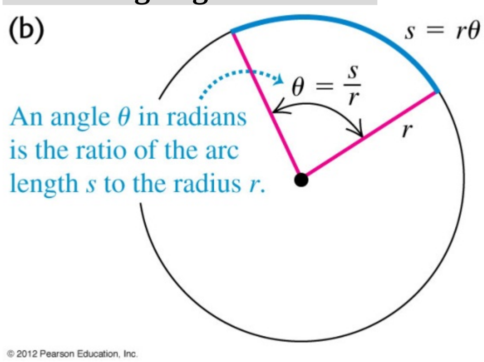
DYNAMICS OF RIGID BODIES I

Intended Learning Outcomes – after this lecture you will learn:
1. radian as a measure of angle
2. angular displacement, velocity and acceleration and their vector representation
3. angular motion as compared to rectilinear motion
4. rotational kinetic energy and moment of inertia
5. how to calculate the moment of inertia of simple symmetric rigid bodies

Textbook Reference: Ch 9.1 – 9.6

Measuring angles in radian



Define the value of an angle θ in **radian** as $\theta = \frac{s}{r}$, or arc length $s = r\theta$

- ⚠ a pure number, without dimension
- ⚠ independent of radius r of the circle
- ⚠ one complete circle

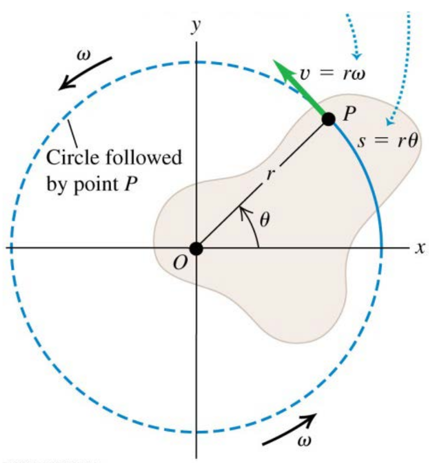
$$\theta = \frac{2\pi r}{r} = 2\pi \text{ (in radian)} \leftrightarrow 360°$$

$$\pi \text{ (in radian)} \leftrightarrow 180°$$

$$\pi/2 \text{ (in radian)} \leftrightarrow 90°$$

Consider a rigid body rotating about a fixed axis

<u>Convention</u>: θ measured from *x* axis in <u>counterclockwise</u> direction



angular displacement: $\Delta\theta = \theta_2 - \theta_1$

angular velocity:

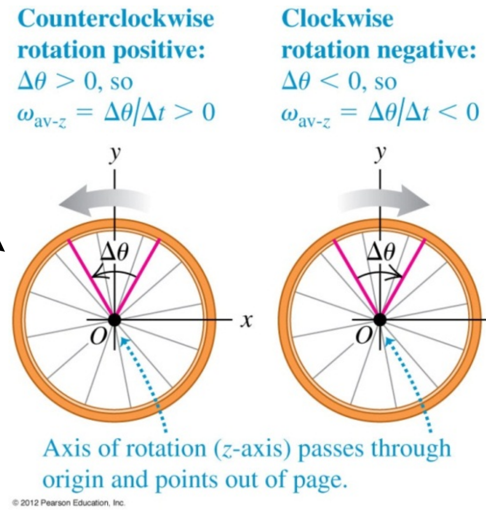
$$\omega = \underset{\text{(average)}}{\frac{\Delta\theta}{\Delta t}} \xrightarrow{\Delta t \to 0} \underset{\text{(instantaneous)}}{\frac{d\theta}{dt}}$$

angular acceleration:

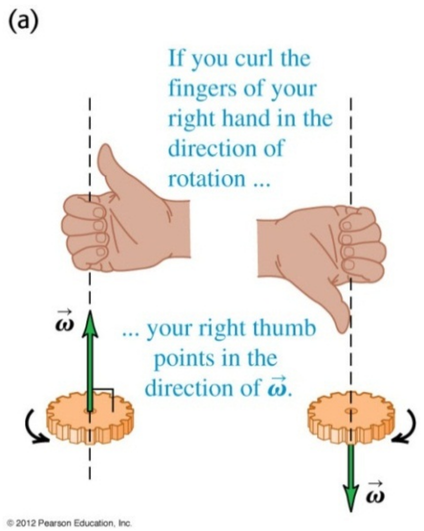
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$\Delta\theta = \theta_f - \theta_i > 0$

$\Delta\theta = \theta_f - \theta_i < 0$



Angular velocity is a vector, direction defined by the **right hand rule**

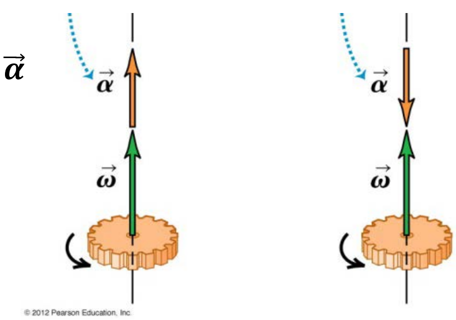


direction of $\vec{\boldsymbol{\omega}}$ represents sense of rotation

Angular acceleration is defined as $\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{\omega}}/dt$

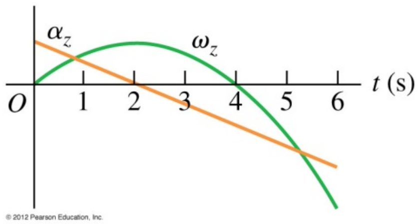
⚠ if rotation axis is fixed, $\vec{\boldsymbol{\alpha}}$ along the direction of $\vec{\boldsymbol{\omega}}$

Rotation speeding up, $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\omega}}$ in the same direction



Rotation slowing down, $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\omega}}$ in the opposite direction

Question:



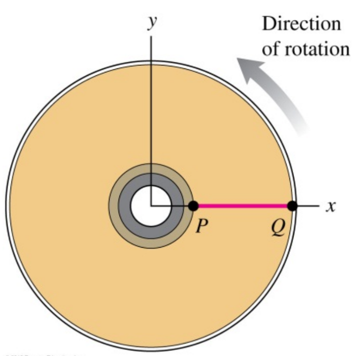
The figure shows a graph of ω and α versus time. During which time intervals is the rotation speeding up?
(i) $0 < t < 2$ s; (ii) 2 s $< t <$ 4 s; (iii) 4 s $< t <$ 6 s.

Answer: see inverted text on P. 305 of textbook

Rotation with constant angular acceleration

Straight-Line Motion with Constant Linear Acceleration		**Fixed-Axis Rotation with Constant Angular Acceleration**	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$	(9.10)

Example 9.3 P. 306 Rotation with constant angular acceleration
A Blu-ray disc is slowing down to a stop with constant angular acceleration $\alpha = -10.0\ \mathrm{rad/s^2}$. At $t = 0$, $\omega_0 = 27.5\ \mathrm{rad/s}$, and a line *PQ* marked on the disc surface is along the *x* axis.



angular velocity at $t = 0.300$ s:

$$\omega = \omega_0 + \alpha t = 27.5\ \mathrm{rad/s} + (-10.0\ \mathrm{rad/s^2})(0.300\ \mathrm{s})$$
$$= 24.5\ \mathrm{rad/s}$$

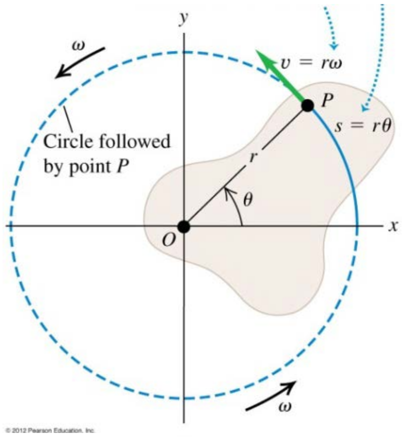
Suppose θ is the angular position of *PQ* at $t = 0.300$ s

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 7.80\ \mathrm{rad} = (7.8\ \mathrm{rad})\left(\frac{360°}{2\pi\ \mathrm{rad}}\right) = 447°$$
$$= 87°$$

What are the directions of $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$?

Question:
In the above example, suppose the initial angular velocity is doubled to $2\omega_0$, and the angular acceleration (deceleration) is also doubled to 2α, it will take (more / less / the same amount of) time for the disc to come to a stop compared to the original problem.
Answer: see inverted text on P. 307 of textbook

Relation to linear velocity and acceleration

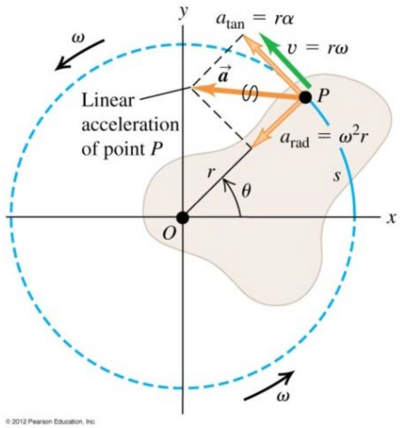


In time Δt, angular displacement is $\Delta\theta$,

tangential displacement is $\Delta s = r\Delta\theta$

$$\therefore \text{ tangential velocity} \quad v = \frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \to r\frac{d\theta}{dt} = r\omega$$

Linear velocity of point P, $\vec{\boldsymbol{v}}$, is tangential and has magnitude $v = |r\omega|$

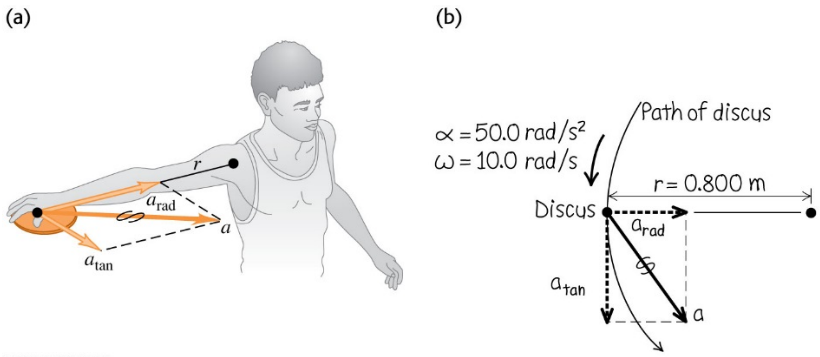


tangential acceleration $\boxed{a_{\tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha}$

radial acceleration $\boxed{a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r}$

⚠ every point of the rigid body has identical $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$, but different $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$

Example 9.4 P. 309 Throwing a discus



An athlete whirls a discus in a circle of radius 80.0 cm. At some instant $\omega = 10.0$ rad/s, and $\alpha =$ 50.0 rad/s^2. Then

$$a_{\tan} = r\alpha = (0.800\text{ m})(50.0\ \text{rad/s}^2) = 40.0\text{ m/s}^2$$
$$a_{\text{rad}} = \omega^2 r = (10.0\ \text{rad/s})^2(0.800\text{ m}) = 80.0\text{ m/s}^2$$

Magnitude of the linear acceleration is

$$a = \sqrt{a_{\tan}^2 + a_{\text{rad}}^2} = 89.4\text{ m/s}^2$$

Example

A uniform rope of mass M and length L is pivoted at one end O and whirls in a circle. Find the tension in the rope.

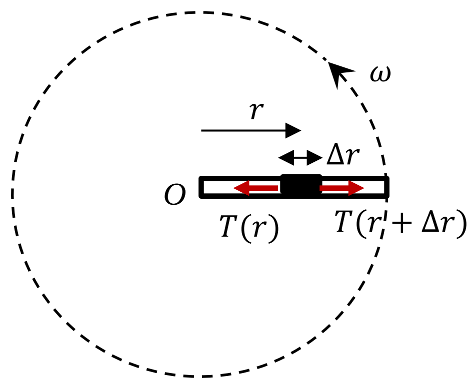
Write down Newton's law for a segment Δr

$$-T(r+\Delta r)+T(r)=\left(\frac{\Delta r}{L}M\right)\omega^2 r$$

$$\Rightarrow \frac{dT}{dr}=-\frac{M\omega^2 r}{L}$$

Since $T(L)=0$, $T(r)=\frac{M\omega^2}{2L}(L^2-r^2)$

If $T(L)\neq 0$, an infinitesimal segment ΔL will be subjected to a non-zero inward tension in the limit $\Delta L\rightarrow 0$ (its mass also $\rightarrow 0$), a contradiction.



Rotational kinetic energy of a rigid body

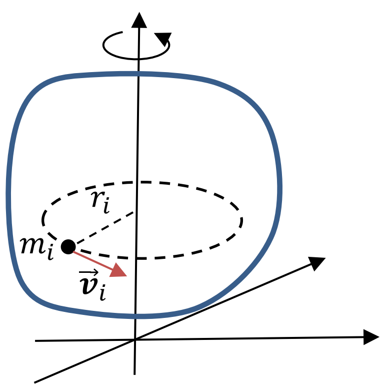
Consider a rigid body as a collection of particles, the kinetic energy due to rotation is

$$K=\sum \tfrac{1}{2}m_i v_i^2=\sum \tfrac{1}{2}m_i r_i^2\omega^2=\frac{1}{2}\underbrace{\left(\sum m_i r_i^2\right)}_{}\omega^2$$

c.f. in rectilinear motion, $K=\frac{1}{2}mv^2$

moment of inertia I, analogous to *mass* in rectilinear motion

$$\boxed{K=\tfrac{1}{2}I\omega^2, I=\sum m_i r_i^2}$$



⚠ When defining I, must specify a rotation axis. r_i is the $\perp$ distance to the rotation axis, not the distance from the origin.

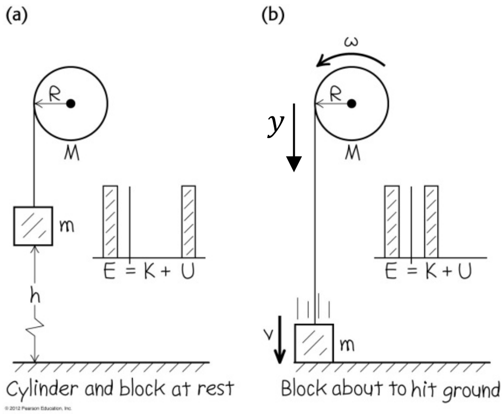
Gravitational potential energy of a rigid body

$$U=m_1gy_1+m_2gy_2+\cdots=(m_1y_1+m_2y_2+\cdots)g=Mgy_{\text{cm}}$$

Gravitational PE is as if all the mass is concentrated at the CM.

Demonstration: [Euler's disk] to demonstrate the conservation of energy – the lower the CM of the disk, the faster it spin.

Example 9.8 P. 314 An unwinding cable



Assumption: rotation of cylinder is frictionless
no slipping between cylinder and cable

At the moment the block hits the ground, speed of block is v, angular speed of cylinder is ω

$$v = R\omega$$

$$0 + \underbrace{mgh}_{\text{initial PE of block}} = \frac{1}{2}mv^2 + \underbrace{\frac{1}{2}I\omega^2}_{\text{rotational KE}}$$

rotational KE, $I = \frac{1}{2}MR^2$ (refer to next lecture)

Constraint: no slipping

$$y = R\theta \ \Rightarrow \ \dot{y} = R\dot{\theta} \ \Rightarrow \ \ddot{y} = R\ddot{\theta}$$

$$\Rightarrow \quad v = \sqrt{\frac{2gh}{1 + M/2m}}$$

⚠ if $M = 0$, $v = \sqrt{2gh}$, same as free falling

Question: Is there friction between the string and pulley? Does it dissipate energy?

Questions:
Suppose the cylinder and block have the same mass, $m = M$. Just before the block hits the floor, its KE is (larger than / less than / the same as) the KE of the cylinder.
Answer: see inverted text on P. 315 of textbook

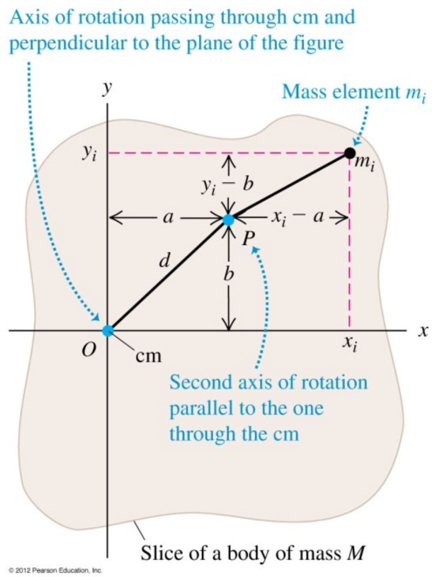
Parallel axis theorem

I_{cm}: moment of inertia about an axis through its CM

I_p: moment of inertia about another axis ∥ to the original one and at ⊥ distance d

$$\boxed{I_p = I_{\text{cm}} + Md^2}$$

Proof: take CM as the origin, rotation axis as the z axis. A point mass m_i in the solid has coordinates (x_i, y_i, z_i).



⚠ not $\sum m_i(x_i^2 + y_i^2 + z_i^2)$

$$I_{\text{cm}} = \sum m_i \overbrace{(x_i^2 + y_i^2)}^{\text{square of } \perp \text{ distance of } m_i \text{ to rotation axis}}$$

$$I_p = \sum m_i[(x_i - a)^2 + (y_i - b)^2]$$

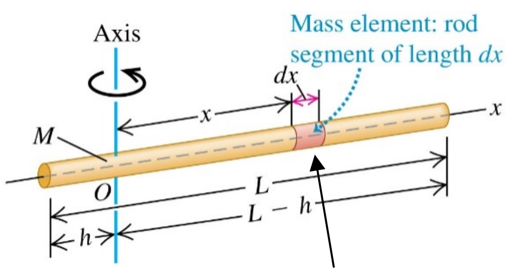
$$= \underbrace{\sum m_i(x_i^2 + y_i^2)}_{I_{\text{cm}}} - 2a\underbrace{\sum m_i x_i}_{Mx_{\text{cm}} = 0} - 2b\underbrace{\sum m_i y_i}_{My_{\text{cm}} = 0}$$

$$+(a^2 + b^2)\underbrace{\sum m_i}_{M}$$

Question: A pool cue is a wooden rod with a uniform composition and tapered with a larger diameter at one end than at the other end. Does it have a larger moment of inertia (i) for an axis through the thicker end of the rod and perpendicular to the length of the rod, or (ii) for an axis through the thinner end of the rod and perpendicular to the length of the rod? Answer: see inverted text on P. 316 of textbook

Significance of the parallel axis theorem: need formula for I_{cm} only.

Example A thin rod with uniform linear density $\rho = M/L$



⚠ Before calculating moment of inertia, must specify rotation axis

$$I = \sum m_i \underset{\uparrow}{r_i^2} \longrightarrow \int r^2\, dm$$

$\perp$ distance of m_i to rotation axis

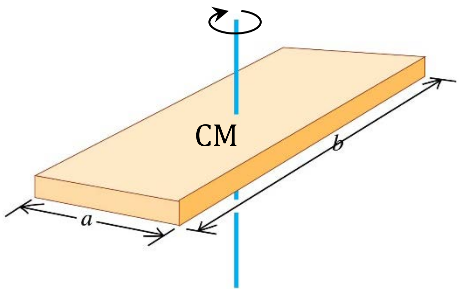
Choose a convenient mass element dm – a segment of length dx at a $\perp$ distance x from the axis, and mass $dm = \rho dx$

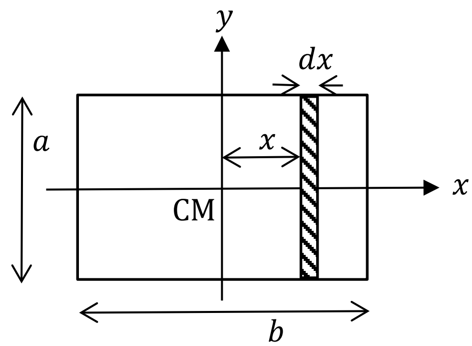
$$I_O = \int_{-h}^{L-h} x^2\,(\rho dx) = \frac{\rho}{3}[(L-h)^3 + h^3] = \frac{M}{3}(L^2 - 3Lh + 3h^2)$$

⚠ Put $h = L/2$, we get $I_{\text{cm}} = ML^2/12$.

⚠ Check the parallel axis theorem $I_O = I_{\text{cm}} + M(\qquad\qquad)^2$

Example A rectangular plate





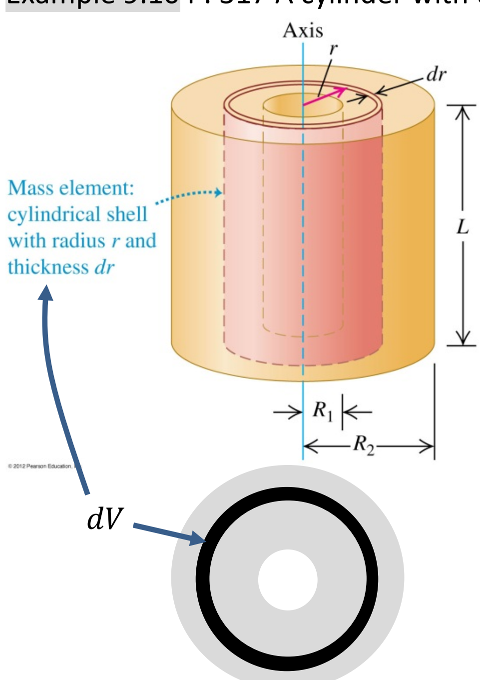
Choose the mass element dm to be a rod at $\perp$ distance x from the axis. *Why? Because you know its moment of inertia!*

$$dI = \underbrace{\frac{(dm)a^2}{12}}_{\text{about CM \textit{of the rod}, not of the plate}} + \underbrace{(dm)x^2}_{\text{parallel axis theorem}}$$

Since $dm = \left(\frac{M}{b}\right) dx$

$$I = \int dI = \frac{M}{b}\int_{-b/2}^{b/2}\left[\frac{a^2}{12} + x^2\right]dx = \frac{1}{12}M(a^2 + b^2)$$

Example 9.10 P. 317 A cylinder with uniform density



⚠ Before calculating moment of inertia, must specify rotation axis

CM along axis of symmetry

$$I = \sum m_i r_i^2 \longrightarrow \int r^2\, dm = \int r^2 \underbrace{\rho}_{\text{uniform density}} dV$$

$\perp$ distance of m_i to rotation axis

Key: choose dV (the volume element) wisely, as symmetric as possible

$$dV = (2\pi r)(dr)L$$

$$I = 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr = \frac{\pi\rho L}{2}(R_2^4 - R_1^4)$$

$$= \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

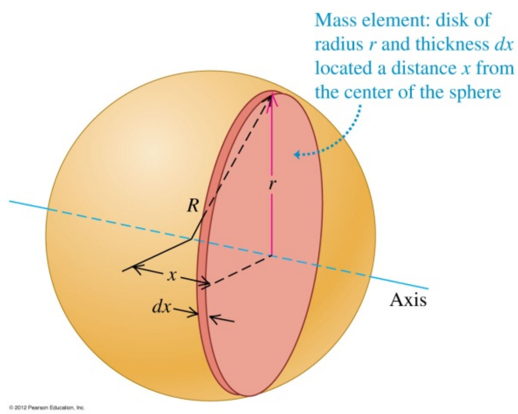
But $M = \rho(\pi R_2^2 L - \pi R_1^2 L) = \pi\rho L(R_2^2 - R_1^2)$

$$\boxed{I = \frac{1}{2}M(R_2^2 + R_1^2)}$$

⚠ independent of length

Question:
Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of wood and the other of lead. The wooden cylinder has (larger / smaller / the same) moment of inertia about the symmetry axis than the lead one.
Ans. See inverted text on P. 318.

Example 9.11 P. 318 A uniform sphere



Choose dV to be a disk of radius $r = \sqrt{R^2 - x^2}$ and thickness dx

From the last example, moment of inertia of this disk is $\frac{1}{2}(dm)r^2 = \frac{1}{2}(\rho\pi r^2 dx)r^2 = \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$

Therefore

$$I = \int \tfrac{1}{2}(dm)r^2 = \frac{\rho\pi}{2}\int_{-R}^{R}(R^2 - x^2)^2 dx = \frac{8\pi\rho R^5}{15}$$

Since $\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$

$$\boxed{I = \tfrac{2}{5}MR^2}$$

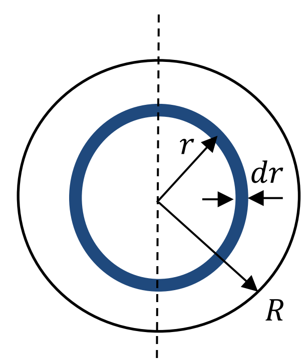
Alternate approach to a uniform sphere:
Instead of slicing up a sphere into circular disk as in Example 9.11, let's consider the sphere to be composed of infinitely many thin shells, just like an onion.

Given: moment of inertia of a thin spherical shell is $\frac{2}{3}mr^2$ (see below). Therefore moment of inertia of the solid sphere is

$$I_{\text{sphere}} = \int_0^R \frac{2}{3}(4\pi r^2 dr)\rho r^2 = \frac{8\pi\rho R^5}{15} = \frac{2}{5}MR^2$$

You can even find the moment of inertial of a hollow sphere with inner and outer radii R_1 and R_2

$$I_{\text{hollow sphere}} \int_{R_1}^{R_2} \frac{2}{3}(4\pi r^2 dr)\rho r^2 = \frac{2}{5}M\frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)}$$

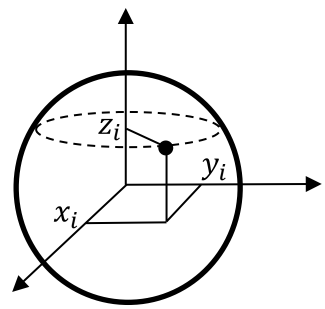


It remains to show that the moment of inertia of a thin shell is $\frac{2}{3}mr^2$. You can do it by integration, such as at https://www.miniphysics.com/uy1-calculation-of-moment-of-inertia-of-thin-spherical-shell.html. But an easier way, without integration, is to use symmetry argument.

Symmetry argument in calculating moment of inertia

Since the shell has spherical symmetry,

$$\sum m_i x_i^2 = \sum m_i y_i^2 = \sum m_i z_i^2 = \frac{1}{3}\sum m_i \underbrace{(x_i^2 + y_i^2 + z_i^2)}_{r^2} = \frac{1}{3}mr^2$$



(Otherwise the x, y, and z axis are not equivalent, inconsistent with spherical symmetry.)

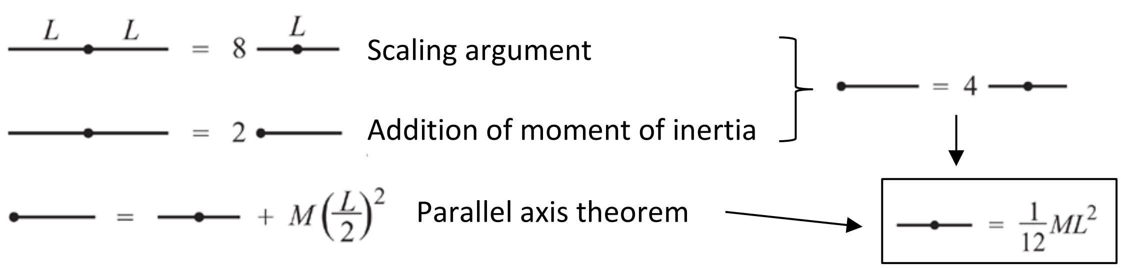
Therefore

$$I_{\text{shell}} = \sum m_i (x_i^2 + y_i^2) = \frac{2}{3}mr^2$$

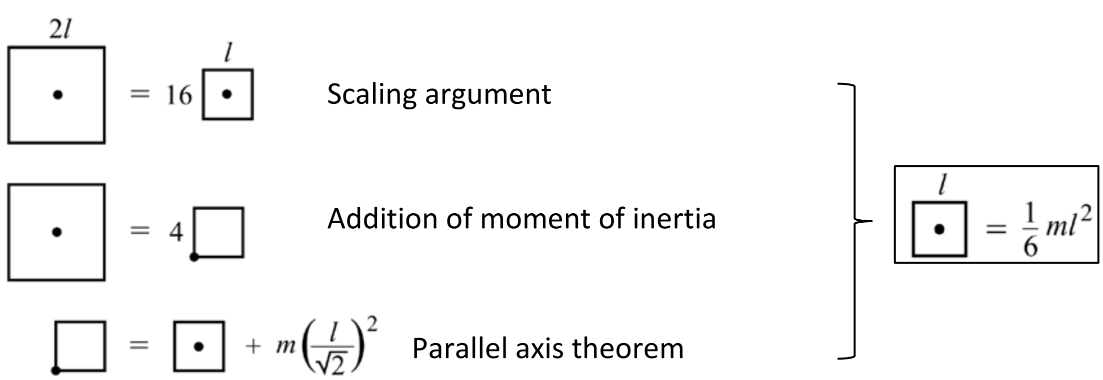
Question: A uniform sphere has the same spherical symmetry $\sum m_i x_i^2 = \sum m_i y_i^2 = \sum m_i z_i^2$. Why this symmetry argument cannot be applied to a uniform sphere?

Scaling argument in calculating moment of inertia

For a uniform rod, $\int x^2 dm = \int x^2 \rho dx$. If we scale up the length of the rod (at constant ρ, i.e., mass also scaled up) by a factor of 2, the momentum of inertia increases by a factor of $2^3 = 8$. (In the following diagrams a dot indicates a rotational axis perpendicular to the plane and passing through that point.)



In 2D, $\int r^2 dm = \int r^2 \rho dx dy$. If we scale up both x and y dimensions by a factor of 2, we get a factor 16



Exercise:
Use scaling argument and the parallel axis theorem to show that the moment of inertia of a uniform plate in the form of an equilateral triangle of sides l about an axis perpendicular to the plate and passing through its center of mass (centroid) is $\frac{1}{12}ml^2$.

Challenge Exercise: moment of inertia of an object with fractal dimension

A **Sierpinski gasket** is an object of fractal dimension $\log_2 3 \approx 1.58$. Starting from an equilateral triangle of sides l, remove the central inverted triangle. Do the same to each remaining upright equilateral triangle, then repeat the same process forever. Show that the moment of inertia about a perpendicular axis through the center of mass (centroid) is $\frac{1}{9}ml^2$.

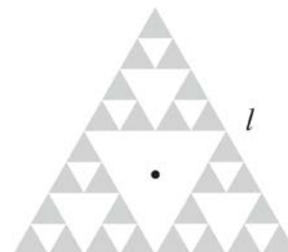
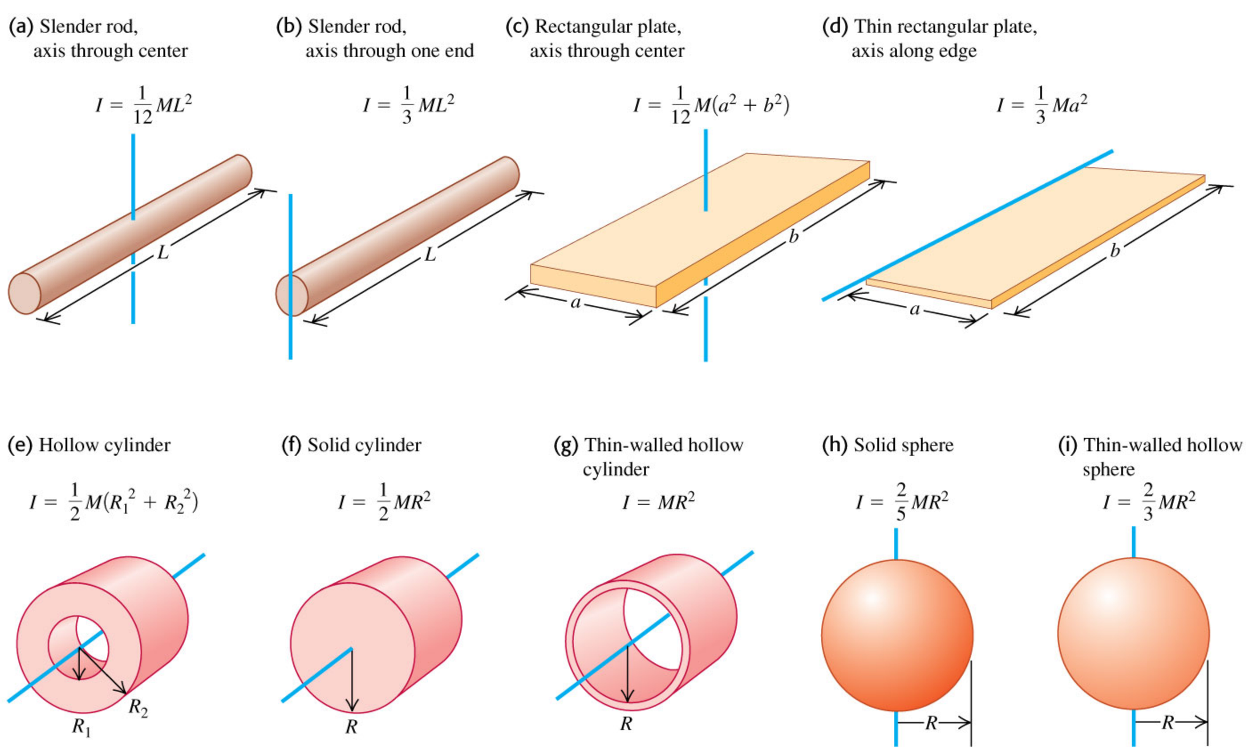


Table 9.2 Moments of Inertia of Various Bodies



Clicker Questions:

Q9.2

A DVD is initially at rest so that the line *PQ* on the disc's surface is along the $+x$-axis. The disc begins to turn with a constant $\alpha_z = 5.0\ \text{rad/s}^2$. At $t = 0.40$ s, what is the angle between the line *PQ* and the $+x$-axis?

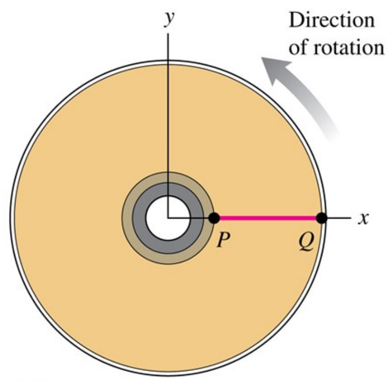
A. 0.40 rad

B. 0.80 rad

C. 1.0 rad

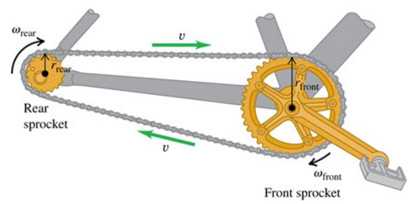
D. 1.6 rad

E. 2.0 rad



Q9.5

Compared to a gear tooth on the rear sprocket (on the left, of small radius) of a bicycle, a gear tooth on the *front* sprocket (on the right, of large radius) has



A. a faster linear speed and a faster angular speed.

B. the same linear speed and a faster angular speed.

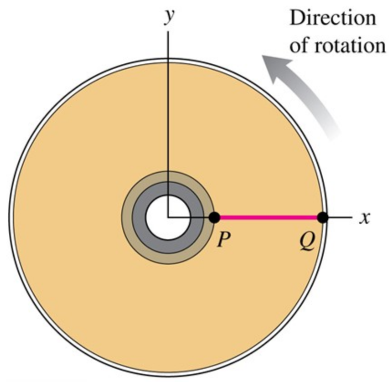
C. a slower linear speed and the same angular speed.

D. the same linear speed and a slower angular speed.

E. none of the above.

Q9.4

A DVD is rotating with an ever-increasing speed. How do the centripetal acceleration a_{rad} and tangential acceleration a_{tan} compare at points P and Q?



A. P and Q have the same a_{rad} and a_{tan}.

B. Q has a greater a_{rad} and a greater a_{tan} than P.

C. Q has a smaller a_{rad} and a greater a_{tan} than P.

D. Q has a greater a_{rad} and a smaller a_{tan} than P.

E. P and Q have the same a_{rad}, but Q has a greater a_{tan} than P.

Q9.6

You want to double the radius of a rotating solid sphere while keeping its kinetic energy constant. (The mass does not change.) To do this, the final angular velocity of the sphere must be

A. four times its initial value.

B. twice its initial value.

C. the same as its initial value.

D. half of its initial value.

E. one-quarter of its initial value.

Q9.7

The three objects shown here all have the same mass and the same outer radius. Each object is rotating about its axis of symmetry (shown in blue). All three objects have the *same* rotational kinetic energy. Which object is rotating *fastest*?

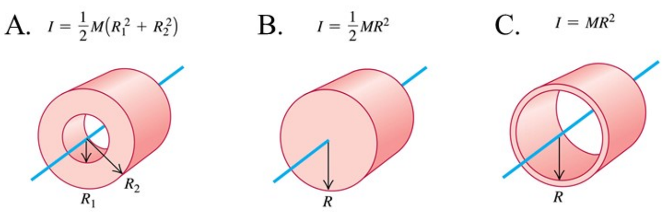
A. Object A is rotating fastest.

B. Object B is rotating fastest.

C. Object C is rotating fastest.

D. Two of these are tied for fastest.

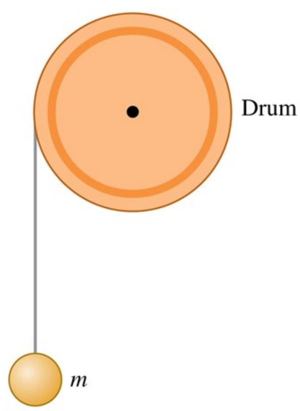
E. All three rotate at the same speed.



Q9.8

A thin, very light wire is wrapped around a drum that is free to rotate. The free end of the wire is attached to a ball of mass m. The drum has the same mass m. Its radius is R and its moment of inertia is $I = (1/2)mR^2$. As the ball falls, the drum spins.

At an instant that the ball has translational kinetic energy K, what is the rotational kinetic energy of the drum?



A. K B. $2K$ C. $K/2$ D. $K/4$

E. none of these

Ans: Q9.2) A, Q9.5) D, Q9.4) B, Q9.6) D, Q9.7) B, Q9.8) C