MATH1023 Homework, Part 1

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Exercise 1.1.3. Explain that, if $\lim_{n\to\infty} x_n = l$ and p is a positive integer, then $\lim_{n\to\infty} x_n^p = l^p$. *Solution:* We will prove this statement by induction.

1. If $p = 1$, then the result follows trivially.
2. Suppose that the result holds for $p - 1$. Now, with reference to the arithmetic rule, we can write

$$\lim_{n\to\infty} x_n^p = \lim_{n\to\infty} \left(x_n^{p-1} \cdot x_n\right) = \left(\lim_{n\to\infty} x_n^{p-1}\right) \cdot \left(\lim_{n\to\infty} x_n\right) = l^{p-1} \cdot l = l^p.$$

Exercise 1.1.5 (6). Find the limit ($n \to \infty$) of

$$\frac{(n^2+1)(n+2)}{(n+1)(n^2+2)}.$$

Solution:

$$\frac{(n^2+1)(n+2)}{(n+1)(n^2+2)} = \frac{n^3 + p_1 n^2 + q_1 n + s_1}{n^3 + p_2 n^2 + q_2 n + s_2} = \frac{1 + \frac{p_1}{n} + \frac{q_1}{n^2} + \frac{s_1}{n^3}}{1 + \frac{p_2}{n} + \frac{q_2}{n^2} + \frac{s_2}{n^3}} \underset{n\to\infty}{\to} 1.$$

Exercise 1.1.7 (5). Find the limit ($n \to \infty$) of

$$\frac{cn+d}{(\sqrt{n}+a)(\sqrt{n}+b)}.$$

Solution:

$$\frac{cn+d}{(\sqrt{n}+a)(\sqrt{n}+b)} = \frac{cn+d}{n+p\sqrt{n}+q} = \frac{c+\frac{d}{n}}{1+\frac{p}{\sqrt{n}}+\frac{q}{n}} \underset{n\to\infty}{\to} c.$$

Exercise 1.1.9 (3). Find the limit ($n \to \infty$) of

$$\frac{5^5(2\sqrt{n}+1)^2 - 10^{10}}{10n-5}.$$

Solution:

$$\frac{5^5(2\sqrt{n}+1)^2 - 10^{10}}{10n-5} = \frac{5^5 \cdot 4n + p\sqrt{n} + q}{10n-5} \underset{n\to\infty}{\to} \frac{5^5 \cdot 4}{10} = 1250.$$

Exercise 1.1.11 (1). Find the limit ($n \to \infty$) of

$$\frac{n^2 + a_1 n + a_0}{n+b} - \frac{n^2 + c_1 n + c_0}{n+d}.$$

Solution:

$$\frac{n^2+a_1n+a_0}{n+b}-\frac{n^2+c_1n+c_0}{n+d}=\frac{(n+d)(n^2+a_1n+a_0)-(n+b)(n^2+c_1n+c_0)}{(n+d)(n+b)}=$$
$$=\frac{(d-b+a_1-c_1)n^2+p_1n+q_1}{n^2+p_2n+q_2}\underset{n\to\infty}{\to}d-b+a_1-c_1.$$

Exercise 1.1.15 (8). Find the limit ($n\to\infty$) of

$$\frac{\cos(n)}{\sqrt{n+\sin(\sqrt{n})}}.$$

Solution: It is trivial to see that

$$0\underset{n\to\infty}{\leftarrow}\frac{-1}{\sqrt{n-1}}\leq\frac{\cos(n)}{\sqrt{n+\sin(\sqrt{n})}}\leq\frac{1}{\sqrt{n-1}}\underset{n\to\infty}{\to}0,$$

which implies that the the limit in question is equal to 0, by the sandwich rule.

Exercise 1.1.15 (16). Find the limit ($n\to\infty$) of

$$\frac{n+\sin(n)}{n-\cos(n)}.$$

Solution: The answer is 1 by the sandwich rule, since

$$1\underset{n\to\infty}{\leftarrow}\frac{n-1}{n+1}\leq\frac{n+\sin(n)}{n-\cos(n)}\leq\frac{n+1}{n-1}\underset{n\to\infty}{\to}1.$$

Exercise 1.1.21 (11). Find the limit ($n\to\infty$) of $\sqrt{n^2+an+b}-\sqrt{n^2+cn+d}$.
Solution: Consider two real numbers $\alpha,\beta>0$. From the algebraic rule $x^2-y^2=(x-y)(x+y)$, it follows that

$$\sqrt{\alpha}-\sqrt{\beta}=\frac{\alpha-\beta}{\sqrt{\alpha}+\sqrt{\beta}}.$$

Now assume that $\alpha\geq\beta$. Then, since $2\sqrt{\beta}\leq\sqrt{\alpha}+\sqrt{\beta}\leq2\sqrt{\alpha}$, we can bound the difference $\sqrt{\alpha}-\sqrt{\beta}$ in the following way:

$$\frac{\alpha-\beta}{2\sqrt{\alpha}}\leq\sqrt{\alpha}-\sqrt{\beta}\leq\frac{\alpha-\beta}{2\sqrt{\beta}}. \tag{1}$$

With respect to the original problem, we first note that for all $a,b\in\mathbb{R}$, the expression n^2+an+b will become positive as n goes to infinity, and thus the square roots can be considered to be well-defined. Further, it is clear by exhausting the relative positions of a, b, c, and d that one of the expressions n^2+an+b and n^2+cn+d will not exceed the other, starting at some point. This is explained by the fact that their difference is an affine function of n, and it acquires a constant sign as n approaches infinity. In other words, without loss of generality, we can assume that

$$(n^2+an+b)\geq(n^2+cn+d)\text{ as }n\to\infty. \tag{2}$$

(In case the sign is $\leq$, we switch the expressions, moving into an analogous situation)
Now, in view of Formula 1 and Formula 2, we can write that

$$\frac{(a-c)n+(b-d)}{2\sqrt{n^2+an+b}} \leq \sqrt{n^2+an+b} - \sqrt{n^2+cn+d} \leq \frac{(a-c)n+(b-d)}{2\sqrt{n^2+cn+d}}.$$

By squaring the non-negative sequences on the left and the right, we see that they both approach $\frac{a-c}{2}$, meaning that, by the sandwich rule, the sequence in the middle also approaches $\frac{a-c}{2}$.

Exercise 1.1.24 (3). Find the limit ($n \to \infty$) of

$$\left(\frac{n-2}{n+1}\right)^{-\sqrt{2}}.$$

Solution:

$$1 \leq \left(\frac{n-2}{n+1}\right)^{-\sqrt{2}} = \left(\frac{n+1}{n-2}\right)^{\sqrt{2}} \leq \left(\frac{n+1}{n-2}\right)^{2} = \left(1+\frac{3}{n-2}\right)^{2} \underset{n\to\infty}{\to} 1,$$

and therefore the limit is 1.

Exercise 1.1.27. Suppose that $\lim\limits_{n\to\infty} x_n = 1$, $x_n \geq 1$, and y_n is bounded. Prove that $\lim\limits_{n\to\infty} x_n^{y_n} = 1$.
Solution: Let $a \geq 1$. Then, if $b_1 \leq b_2$, we have $a^{b_1} \leq a^{b_2}$. Now, since y_n is bounded, there exist numbers $\mu, \nu \in \mathbb{R}$ such that $\mu \leq y_n \leq \nu$ for all n. Consequently, we have

$$1 \underset{n\to\infty}{\leftarrow} x_n^{\mu} \leq x_n^{y_n} \leq x_n^{\nu} \underset{n\to\infty}{\to} 1,$$

and we are done.