Foundation of Mathematics

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Introduction

1. Sep. 3, '24 — Mathematics courses aimed toward mathematics majors, including even the introductory ones, usually assume that the students already know how to read and write proofs effectively in addition to (presumably) basic mathematical knowledge that (presumably) everyone should already know by the time they reach college. Neither of these assumptions is true, even for students who are good at mathematics in high school. This course thus aims to bridge this gap.

More precisely, the course's first aim is to teach students how to write and read proofs effectively. As the second aim, the course will go over interesting topics in mathematics that are not usually covered in a standard high school curriculum and use those as the context for the first aim. More generally, it will also try to cover skills needed in higher-level mathematics courses and to develop a solid foundation for future mathematical endeavors.

The course will closely follow this course note. Additional useful resources include

(i) [K; H];
(ii) Prof. Min Yan's course note for a previous version of this course;
(iii) Notes by various people that I find interesting/helpful.

The last two items are available on the course's webpage on Canvas. I believe that [K] is also available for download via the link given in the bibliography when you are on HKUST campus.

The term foundation of mathematics is normally used to refer to the study of the logical and set-theoretic underpinnings of mathematics. This course is *not* about that (though we will touch on those topics). Instead, it is about the foundation of mathematics in the sense of the basic skills and knowledge needed to study mathematics effectively. The fascinating subject of foundation of mathematics (in the original sense) is usually offered as a more advanced course.

CHAPTER 1

Definitions, Theorems, and Proofs

A mathematical textbook or paper does not read like a novel. Rather, it is usually organized into definitions, theorems/propositions/lemmas/corollaries, and proofs. This chapter will go over these concepts. Let us start with a brief overview.

- Definition: an explanation of a mathematical term.
- Theorem: a very important true statement.
- Proposition: a less important but nonetheless interesting and true statement.
- Lemma: a true statement that is usually used to prove a theorem or proposition.
- Corollary: a true statement that follows easily from a theorem or proposition.
- Proof: a logical argument that shows that a theorem/proposition/lemma is true.
- Conjecture: a statement believed to be true, but we have no proof.
- Axiom: a basic assumption about a mathematical situation.

1.1. Statements

Definition 1.1.1. A *statement* is a sentence that is either true or false.

Example 1.1.2. Below are a couple of examples.

(i) "$2+2=4$" is a statement because it is true.
(ii) "$2+2=5$" is also a statement, but it is false.
(iii) "$x+2=4$" is not a statement because it depends on the value of x.
(iv) "All cats are black" is a statement. It is false.
(v) "There exists a cat that is not black" is a statement. It is true.
(vi) "There are infinitely many prime numbers" is a statement. It is true, proved by Euclid in ∼300BC.

Example 1.1.3. "This statement is false" is not a statement. If it is true, then it is false. If it is false, then it is true. This is known as the *liar's paradox*.

As we saw from Example 1.1.2, there are sentences whose truth value depends on the value of some variables. We call such sentences *conditional statement*.

1.2. Truth tables

Truth tables are very useful in logic. They summarize all the possibilities for the truth or falsehood of a statement.

1.2.1. Equivalent statements.

Definition 1.2.2 (Equivalent statements)**.** Two statements P and Q are *equivalent* if they have the same truth value.

1.2.3. Negation.

Definition 1.2.4 (Negation)**.** The *negation* of a statement P is the statement "It is not the case that P." In particular, the negation of a statement P is false if P is true, and true if P is false. We denote the negation of P by $\neg P$.

Remark 1.2.5. Let P be a statement. Then $\neg(\neg P)$ is equivalent to P.

Example 1.2.6. (i) The negation of "Clear Water Bay is in Hong Kong" is "Clear Water Bay is not in Hong Kong."

(ii) The negation of "All cats are black" is *not* "All cats are not black." It is "There exists a cat that is not black."

The negation of "There exists a cat that is not black" is "There doesn't exist a cat that is not black" or "All cats are black."

P	$\neg P$
T	F
F	T

Example 1.2.7. The truth table for $\neg(\neg P)$ is given below

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

This shows that $\neg(\neg P)$ is equivalent to P.

1.2.8. "And" and "or".

Definition 1.2.9 (and)**.** $A \text{ and } B$ is true if both A and B are true. $A \wedge B$ is sometimes used to denote $A \text{ and } B$.

The truth table for $A \wedge B$ is given below.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.2.10 (or)**.** $A \text{ or } B$ (or $A \vee B$) is true if at least one of A and B is true.

The truth table for $A \vee B$ is given below.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.2.11. (i) The statement "$2+2=4$ and $2+3=5$" is true.

(ii) The statement "$2+2=5$ and $2+3=5$" is false.

(iii) The statement "$2+2=5$ or $2+3=5$" is true.

(iv) The statement "$2+2=5$ or $2+3=6$" is false.

Example 1.2.12. The truth table for $A \wedge (\neg B)$ is given below. For ease of reference, we also include $\neg B$.

A	B	$\neg B$	$A \wedge (\neg B)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Remark 1.2.13. What we call or is sometimes called the *inclusive or*. There is also the *exclusive or*, denoted by xor or $\oplus$, which is true if exactly one of A and B is true.

The truth table for xor is given below.

A	B	$A \operatorname{xor} B$
T	T	F
T	F	T
F	T	T
F	F	F

In everyday language, we don't usually distinguish between the inclusive and exclusive or. Consider the sentence

> "Assume one of a or b is odd."

For mathematicians, this means that at least one of a and b is odd, i.e., it can still be the case that both a and b are odd. This is unlike the usual usage in English!

1.3. Negation of "and" and "or"

Lemma 1.3.1. *The negation of $A \wedge B$ is $\neg A \vee \neg B$, i.e., $\neg(A \wedge B)$ is equivalent to $(\neg A) \vee (\neg B)$. Similarly, $\neg(A \vee B)$ is equivalent to $(\neg A) \wedge (\neg B)$.*

PROOF. This can be seen using truth tables. We will show the truth table for $\neg(A \wedge B)$ and $(\neg A) \vee (\neg B)$. The truth table for $\neg(A \vee B)$ and $(\neg A) \wedge (\neg B)$ is similar.

A	B	$\neg A$	$\neg B$	$A \wedge B$	$(\neg A) \vee (\neg B)$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	F	T

$\square$

1.4. Implications

Bertrand Russell made the following famous statement

> "Mathematics consists of propositions of the form: P implies Q, but you never ask whether P is true."

Strictly speaking, this is a bit of an exaggeration. But the majority of mathematics is indeed about implications.

Definition 1.4.1 (Implication)**.** An *implication* is a statement of the form "If statement P is true, then statement Q is true" (or simply, "If P then Q") or "P implies Q". It is usually written as $P \implies Q$ (or $P \Rightarrow Q$). It is only false if P is true and Q is false. In the statement $P \implies Q$, P is called the *hypothesis* or *assumption* and Q is called the *conclusion*.

The truth table for $P \implies Q$ is given below.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark 1.4.2. There are many ways to read $P \implies Q$. Some common ones are

- "If P, then Q."
- "P only if Q."
- "Given P, Q is true."
- "P implies Q."

Remark 1.4.3. From the table above, we see that the truth of $P \implies Q$ does not say anything about that of P or Q. For example, if P is false, then $P \implies Q$ is true regardless of the truth value of Q.

This might sound a bit counter-intuitive. But it just says that the statement "If P then Q" is only not correct when Q is false and P is true.

Consider the following example "If I am President George Washington, then I am the first President of the United States of America." This is true regardless of whether I am actually President George Washington or not.

Example 1.4.4.
- "Being divisible by 10 implies being even."
- "A number is even if it is divisible by 10."
- "If I am Winston Churchill, then I am English."

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Lemma 1.4.5. *$P \implies Q$ is equivalent to $\neg P \vee Q$.*

PROOF. This can be seen using the following truth table.

P	Q	$\neg P$	$\neg P \vee Q$	$P \implies Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

□

Corollary 1.4.6. *$P \implies Q$ is equivalent to $(\neg Q) \implies (\neg P)$.*

PROOF. From Lemma 1.4.5, we see that $P \implies Q$ is equivalent to $\neg P \vee Q$. But this is equivalent to $Q \vee \neg P$, and hence, to $\neg(\neg Q) \vee \neg P$, which is equivalent to $\neg Q \implies \neg P$, by Lemma 1.4.5. □

Remark 1.4.7. The statement $\neg Q \implies \neg P$ is known as the *contrapositive* of $P \implies Q$.

Remark 1.4.8. $P \implies Q$ is *not* equivalent to $(\neg P) \implies (\neg Q)$.

Corollary 1.4.9. *The negation of $P \implies Q$ is $P \wedge \neg Q$.*

PROOF. This can be seen using the following truth table.

P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg(P \implies Q)$
T	T	F	F	F
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

□

Remark 1.4.10. In the statement $P \implies Q$, P is called the *hypothesis* or *assumption* and Q is called the *conclusion*. We also say that P is a *sufficient condition* for Q and that Q is a *necessary condition* for P.

Definition 1.4.11 (if and only if)**.** The phrase *if and only if* is often used to express a bi-conditional relationship between two statements. It is denoted by $\iff$ or $\Leftrightarrow$. A statement of the form $P \iff Q$ means that P is true if and only if Q is true, or equivalently, P implies Q and Q implies P. In other words, $P \iff Q$ is equivalent to $(P \implies Q) \wedge (Q \implies P)$.

Remark 1.4.12. The bi-conditional relationship is a stronger condition than implication alone. While $P \implies Q$ allows for the possibility that P is true and Q is false, $P \iff Q$ requires that the truth values of P and Q are always the same. Therefore, if $P \iff Q$ is true, then P and Q are either both true or both false. In other words, P and Q are equivalent.

1.5. Quantifiers

Definition 1.5.1 (Universal quantifier)**.** The phrase "for all" is a *universal quantifier* and is denoted by $\forall$.

Definition 1.5.2 (Existential quantifier)**.** The phrase "there exists" is an *existential quantifier* and is denoted by $\exists$.

Example 1.5.3.

- "$\forall x \in \mathbb{R}, x^2 \geq 0$." This says that the square of any real number is non-negative.
- "$\exists x \in \mathbb{R}, x^2 = -1$."
- "For all even number n, n^2 is even."
- "For all $m, n > 0$, $mn > 0$."

We can combine quantifiers as in the following example.

Example 1.5.4. "For all integers n, there exists an integer m, such that $m > n$."

Remark 1.5.5. The order of quantifiers is important. For example, the statement "For all n, there exists m such that $m > n$" is different from "There exists m such that for all n, $m > n$."

1.6. Definitions

Mathematical definition gives the meaning of a term generally in terms of properties.

Example 1.6.1.

- An integer is *even* if it is the product of 2 and another integer.
- An integer is *odd* if it is not even.
- A natural number greater than 1 is called *prime* if it is only divisible by 1 and itself.

Remark 1.6.2. Definitions are usually written in the form "X is called Y if Z." However, it is to be understood as "X is called Y *if and only if* Z."

Remark 1.6.3. One important study skill in mathematics is finding examples. In particular, every time one sees a definition, one should try to come up with examples and non-examples. To go one step further, one should try to create new objects from old ones to see if they still fit the definition.

1.7. Theorems, propositions, and lemmas

High-school mathematics usually focuses on computations and processes one can apply to solve a problem (such as how to differentiate, how to solve a quadratic equation, etc.). Higher level mathematics is more about statements and proofs: what is true, why it is true, and how we know it is true.

Theorems, propositions, lemmas are statements that are true (and proofs are arguments that show that they are true, which will be the focus of the next section).[1] Normally, the statements themselves do not tell us how to apply it to solve a problem. Rather, they are usually about the structure of certain mathematical objects.

1.8. Proofs

This is one of the main topics of this class.

A proof is an explanation of why a statement is true. It has to be convincing *to mathematicians*. It has to start with obvious/accepted facts, proceeds in small logical steps, and concludes with the thing we are set out to prove. Proofs form the foundation of mathematics, without which, we cannot be certain what is correct or not.

Proofs can be hard to understand, even for experienced mathematicians. One of the reasons is that when we start working on a proof, we do not usually know how it will go (unless it is a very trivial example/something we have seen before). So, the discovery of a proof involves a lot of dead ends and failed attempts. In particular, it is not at all linear. But when the proof is written down, it is all linear.

The person who read the proof didn't go through the whole process of discovering the proof. So, it might be hard to understand why the proof goes in a certain direction. So, when writing a proof, it's good to keep this in mind to make it easy for the reader.

Broadly speaking, the main proof techniques are

- direct
- cases
- contradiction/contrapositive
- mathematical induction

1.8.1. Direct. A direct proof starts with the assumption and arrives at the conclusion using logical deductions.

Example 1.8.2. For any positive integer n, we prove

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

PROOF. Since the order of the terms does not affect the sum, we have

$$\begin{aligned} 2\sum_{k=1}^{n} k &= (1 + 2 + \cdots + (n-1) + n) + (n + (n-1) + \cdots + 2 + 1) \\ &= (1+n) + (2 + (n-1)) + \cdots + ((n-1)+2) + (n+1) \\ &= n(n+1). \end{aligned}$$

[1] For the difference between theorems, propositions, and lemmas, see the beginning of the current chapter.

Thus,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

□

1.8.3. Mathematical induction. Suppose we have a statement $A(n)$ for each positive integer n. The truthfulness of $A(n)$ can be established by *mathematical induction* (or just induction for short), which consists of the following steps

(i) Prove that $A(1)$ is true.
(ii) Prove that if $A(n-1)$ then $A(n)$ is also true.

There is another variant of proof by induction, which consists of the following steps

(i) Prove that $A(1)$ is true.
(ii) Prove that if $A(k)$ is true for all $1 \leq k \leq n-1$ then $A(n)$ is true.

Example 1.8.4. For any positive integer n, we will use mathematical induction to prove

$$\sum_{k=1}^{n} k = 1+2+\cdots+(n-1)+n = \frac{n(n+1)}{2}.$$

PROOF. We will use mathematical induction. Starting with the base case, for $n=1$, we have

$$\sum_{k=1}^{1} k = 1 = \frac{1 \cdot 2}{2}.$$

Suppose that we already know that

$$\sum_{k=1}^{n-1} k = \frac{(n-1)n}{2},$$

we will now show that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Indeed,

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n = \frac{(n-1)n}{2} + n = \frac{n(n+1)}{2}.$$

□

One can even do double induction.

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