MOMENTUM, IMPULSE, AND COLLISIONS II

Intended Learning Outcomes – after this lecture you will learn:
1. characteristics of elastic collisions.
2. center of mass and its relation to center of gravity.
3. the dynamics of the center of mass of a system or a body.

Textbook Reference: Ch 8.4, 8.5

Elastic collision

This is a good place to introduce the idea of ***invariants*** which relate a system at different instances. In this case the invariants are energy and momentum.

Conservation of energy: $\frac{1}{2}m_A v_{A1x}^2 + \frac{1}{2}m_B v_{B1x}^2 = \frac{1}{2}m_A v_{A2x}^2 + \frac{1}{2}m_B v_{B2x}^2$

Conservation of momentum: $m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$

Want to solve for v_{A2x} and v_{B2x}. Trick:

$$m_A(v_{A1x}^2 - v_{A2x}^2) = m_B(v_{B2x}^2 - v_{B1x}^2)$$

$$m_A(v_{A1x} + v_{A2x})(v_{A1x} - v_{A2x}) = m_B(v_{B2x} + v_{B1x})(v_{B2x} - v_{B1x})$$

From momentum conservation we have

$$m_A(v_{A1x} - v_{A2x}) = m_B(v_{B2x} - v_{B1x})$$

$$\Rightarrow \quad v_{A1x} + v_{A2x} = v_{B1x} + v_{B2x}$$

Physical meaning: $v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x})$

relative velocity *after* collision = – (relative velocity *before* collision)

⚠ In an elastic collision we can write down three equations:

1. conservation of energy
2. conservation of momentum.
3. relative velocity *after* collision = – (relative velocity *before* collision)

But only two of them are independent. Usually 2 and 3 are preferred because they are first order.

$$\begin{cases} m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x} \\ v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x}) \end{cases}$$

$$\Rightarrow v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x} + \frac{2m_B}{m_A + m_B} v_{B1x}, \quad v_{B2x} = \frac{m_B - m_A}{m_A + m_B} v_{B1x} + \frac{2m_A}{m_A + m_B} v_{A1x}$$

⚠ Symmetric in A and B.

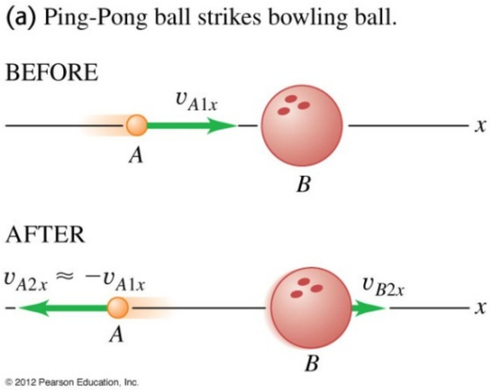
Special case: Elastic collision with one body initially at rest

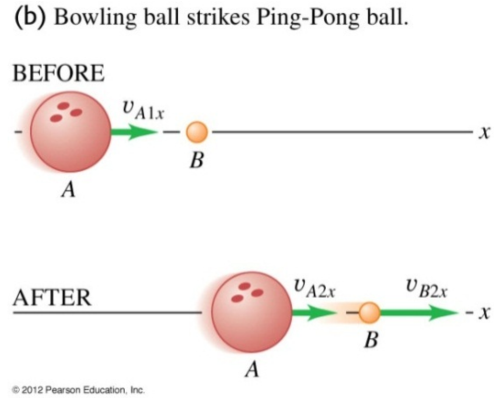
B initially at rest

$$\Rightarrow v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x}, \quad v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x}$$

⚠ Symmetry in A and B is *broken*

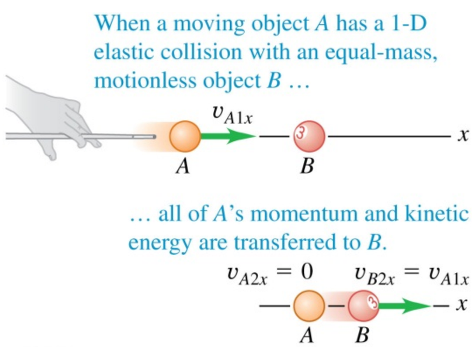
⚠ v_{B2x} same direction (same sign) as v_{A1x}, but direction of v_{A2x} depends on $m_A - m_B$





$m_B > m_A$, A reflected back

$m_B < m_A$, A continue forward and $v_{A2x} < v_{B2x}$



$$m_A = m_B,\ v_{A2x} = 0,\ v_{B2x} = v_{A1x}$$

What if the collision is not 1D?

From conservation of energy and momentum:

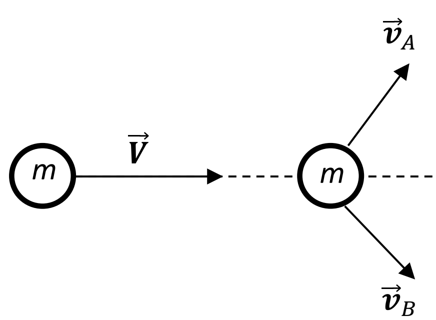
$$\left|\vec{V}\right|^2 = |\vec{v}_A|^2 + |\vec{v}_B|^2$$

$$\vec{V} = \vec{v}_A + \vec{v}_B$$

Then

$$2\vec{v}_A \cdot \vec{v}_B = |\vec{v}_A + \vec{v}_B|^2 - |\vec{v}_A|^2 - |\vec{v}_B|^2 = 0$$

i.e., **the two velocities after collision are at right angle**.

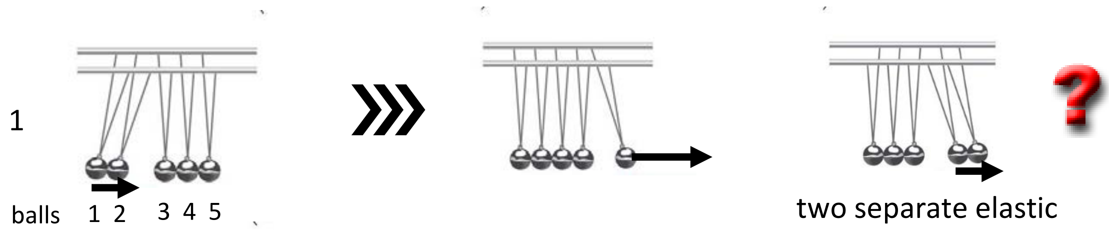


Demonstrations:

Newton's cradle	Gaussian gun – where comes the extra energy when there is a magnet?

More on Newton's cradle

Case 1

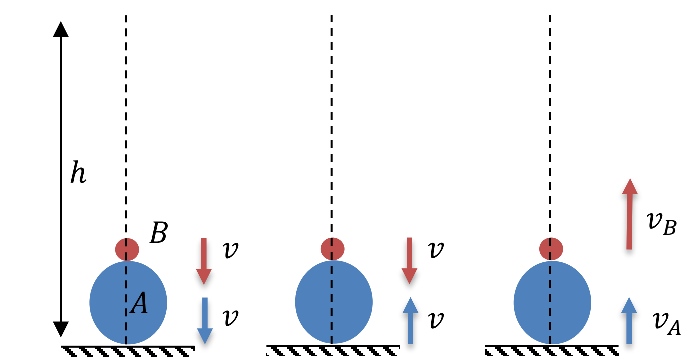


two separate elastic collisions by balls 1 and 2

Case 2

elastic collision of balls 1, 2, 3, they stop there; inelastic collision of balls 4, 5, energy is lost

Example – velocity amplification revisit



before collision during collision after collision

$$v = \sqrt{2gh}$$

Assume all collisions are elastic

$$v_B = \frac{2m_A}{m_A + m_B} v + \frac{m_B - m_A}{m_A + m_B}(-v)$$
$$= \frac{3m_A - m_B}{m_A + m_B} v$$

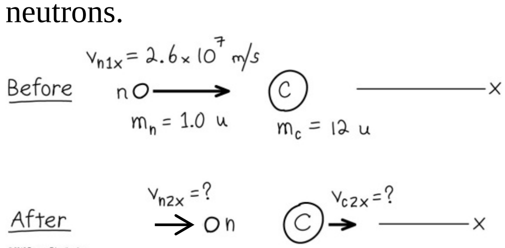
m_B will bounce to height

$$h' = \frac{v_B^2}{2g} = \left(\frac{3m_A - m_B}{m_A + m_B}\right)^2 h$$

Upper bound is $h' = 9h$ when $m_A \gg m_B$

Example 8.11 P. 279 Moderating fission neutrons in a nuclear reactor

Fission of uranium produces high speed neutrons which must be slowed down before it can initiate another fission process. Suppose graphite (carbon) is used as moderator to slow down neutrons.



assuming elastic collision

relative velocities: $v_{C2x} - v_{n2x} = -(0 - v_{n1x})$

conservation of momentum:

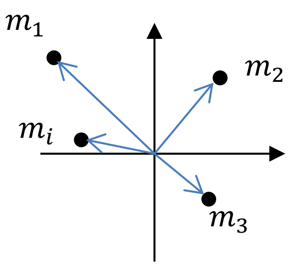
$$m_{\mathrm{n}} v_{\mathrm{n}1x} = m_{\mathrm{C}} v_{\mathrm{C}2x} + m_{\mathrm{n}} v_{\mathrm{n}2x}$$

get: $v_{\mathrm{n}2x} = -2.2 \times 10^7$ m/s, $v_{\mathrm{C}2x} = 0.4 \times 10^7$ m/s

⚠ Don't worry about the direction (forward or backward) of neutron after collision. Assume all v are +ve, *c.f.*, figure in textbook.

Center of Mass (CM)

The CM of a system of particles is defined as the *weighted average of the particle positions*

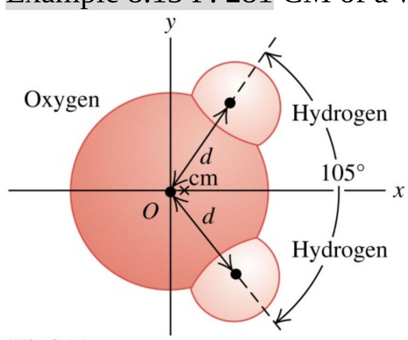


$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m_i\vec{\boldsymbol{r}}_i}{\sum m_i}$$

i.e.

$$x_{\text{cm}} = \frac{\sum m_i x_i}{\sum m_i}, \quad y_{\text{cm}} = \frac{\sum m_i y_i}{\sum m_i}, \quad z_{\text{cm}} = \frac{\sum m_i z_i}{\sum m_i}$$

Example 8.13 P. 281 CM of a water molecule



Need to worry about mass of nuclei only (why not e^-?)

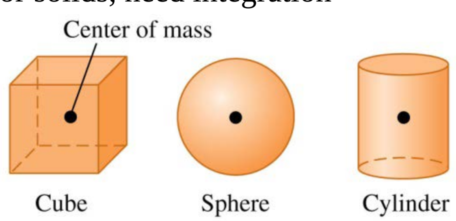
⚠ choose symmetry axis of the molecule as the *x*, *y*, and *z* directions

meaning under rotation about that axis, the molecule looks the same

$$x_{\text{cm}} = \frac{(1.0\text{ u})(d\cos 52.5°) + (1.0\text{ u})(d\cos 52.5°) + (16.0\text{ u})(0)}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0.068d$$
$$= 6.5 \times 10^{-12}\text{ m}$$

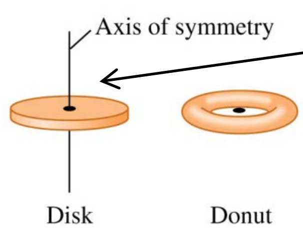
$$y_{\text{cm}} = \frac{(1.0\text{ u})(d\sin 52.5°) + (1.0\text{ u})(-d\sin 52.5°) + (16.0\text{ u})(0)}{1.0\text{ u} + 1.0\text{ u} + 16.0\text{ u}} = 0$$

For solids, need integration



Make use of *symmetry* rules to find CM of homogeneous (constant density) solids:

1) at the geometric center of regular solids
2) lies on symmetry axis
3) CM can lie outside a body



c.f. center of gravity (CG) which you have learned in high school.

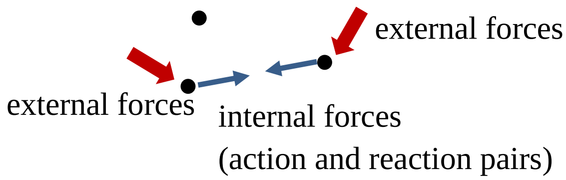
⚠ If *g* is the same at all points on a body, its CG is identical to its CM.

⚠ You know how to determine the CG of a rigid body experimentally.

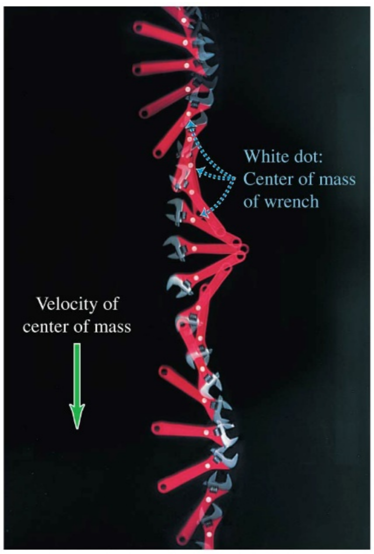
From definition of $\vec{\boldsymbol{r}}_{\text{cm}}$, (by differentiation)

$$\vec{\boldsymbol{v}}_{\text{cm}} = \frac{m_1\vec{\boldsymbol{v}}_1 + \cdots}{m_1 + \cdots} \quad \Longrightarrow \quad M\vec{\boldsymbol{v}}_{\text{cm}} = m_1\vec{\boldsymbol{v}}_1 + \cdots = \vec{\boldsymbol{P}} \quad \text{total linear momentum}$$

$$M\vec{\boldsymbol{a}}_{\text{cm}} = m_1\vec{\boldsymbol{a}}_1 + \cdots = \sum \vec{\boldsymbol{F}}_{\text{ext}} + \underbrace{\sum \vec{\boldsymbol{F}}_{\text{int}}} = \sum \vec{\boldsymbol{F}}_{\text{ext}}$$



in equal and opposite pairs, add up to zero (⚠ didn't we say action and reaction do not cancel?)

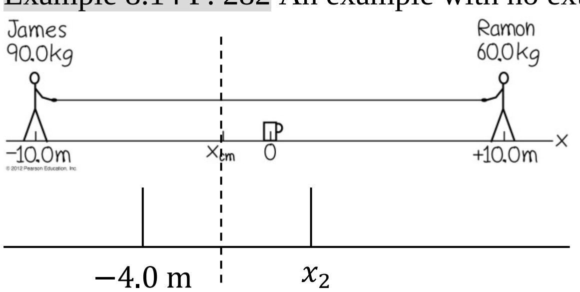


Conclusion: $M\vec{\boldsymbol{a}}_{\text{cm}} = \sum \vec{\boldsymbol{F}}_{\text{ext}} = d\vec{\boldsymbol{P}}/dt$
When a body or a collection of particles is acted on by external forces, the CM moves just as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of the external forces on the system.

e.g. a falling rotating wrench

See also the MIT demo at
https://www.youtube.com/watch?v=DY3LYQv22qY

Example 8.14 P. 282 An example with no external force – tug of war on ice

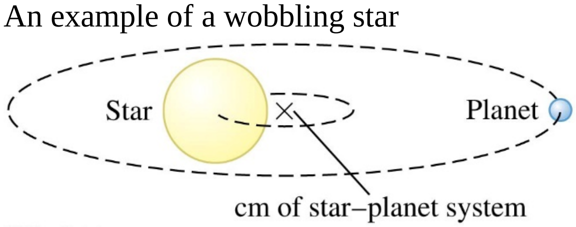


$$x_{\text{cm}} = \frac{(90.0\ \text{kg})(-10.0\ \text{m}) + (60.0\ \text{kg})(10.0\ \text{m})}{90.0\ \text{kg} + 60.0\ \text{kg}}$$

$$x_{\text{cm}} = \frac{(90.0\ \text{kg})(-4.0\ \text{m}) + (60.0\ \text{kg})x_2}{90.0\ \text{kg} + 60.0\ \text{kg}}$$

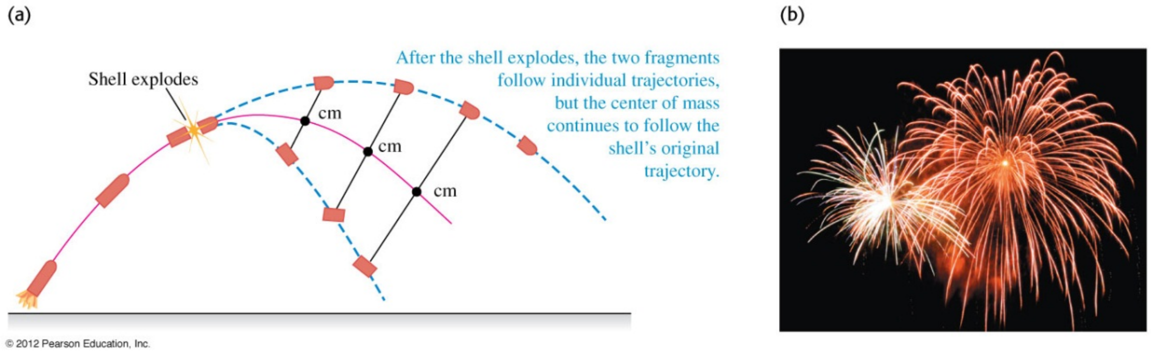
$$\Rightarrow \quad x_2 = 1.0\ \text{m}$$

An example of a wobbling star



A wobbling star shows the presence of an accompanying planet which is too dim to be seen

An example with external force – A shell explodes into two fragments in flight.



Question: Will the CM in the above problem continue on the same parabolic trajectory even after one of the fragments hits the ground? Answer: see the inverted text on P. 284 of textbook

Example Problem 8.92, P. 297 Walking on a canoe in still water

⚠ Here the invariant is not energy nor momentum, but the location of the CM

Canoe is 5.00 m long. CM of canoe is at its center

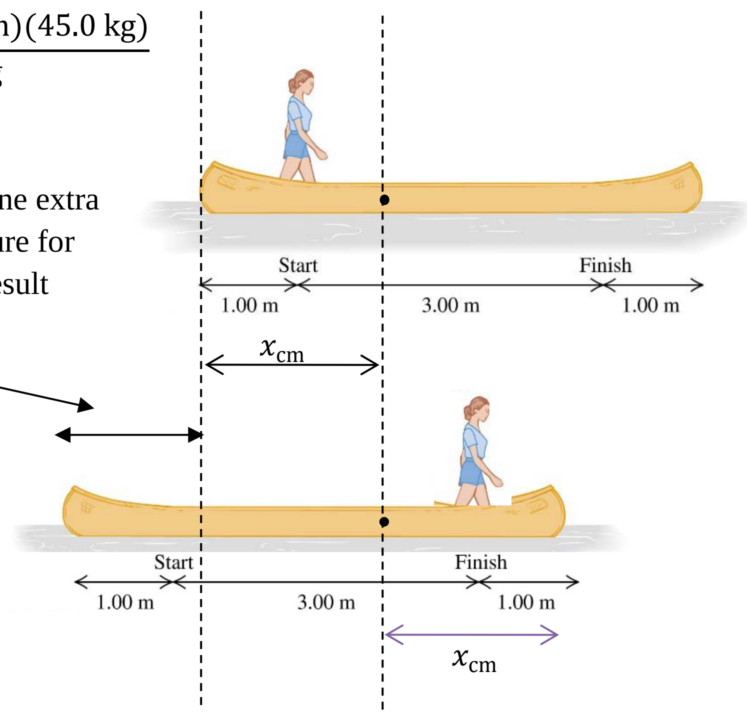
CM of the system (canoe + girl) doesn't move. Girl's mass 45.0 kg, canoe's mass 60.0 kg

$$x_{\text{cm}} = \frac{(2.50\ \text{m})(60.0\ \text{kg}) + (1.00\ \text{m})(45.0\ \text{kg})}{45.0\ \text{kg} + 60.0\ \text{kg}}$$
$$= 1.857\ \text{m}$$

⚠ may keep one extra significant figure for intermediate result

Canoe moves to the left by

$$5.00\ \text{m} - 2x_{\text{cm}} = 1.29\ \text{m}$$

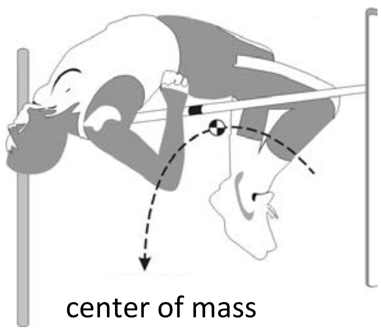


CM does not move

Alternate method:
In shore frame: v and V are velocities of girl and canoe, $mv + MV = 0$
In canoe's frame: v' velocity of girl, $v' = v - V$
Invariant: time taken for the process is the same measured in shore and canoe's frame

$$t = \left|\frac{l}{v'}\right| = \left|\frac{x}{V}\right| \Rightarrow x = \left|\frac{l}{v/V - 1}\right| = \frac{l}{M/m + 1} = \frac{3.00\text{ m}}{(60.0\text{ kg})/(45.0\text{ kg}) + 1} = 1.29\text{ m}$$

Example The Fosbury Flop in high jump

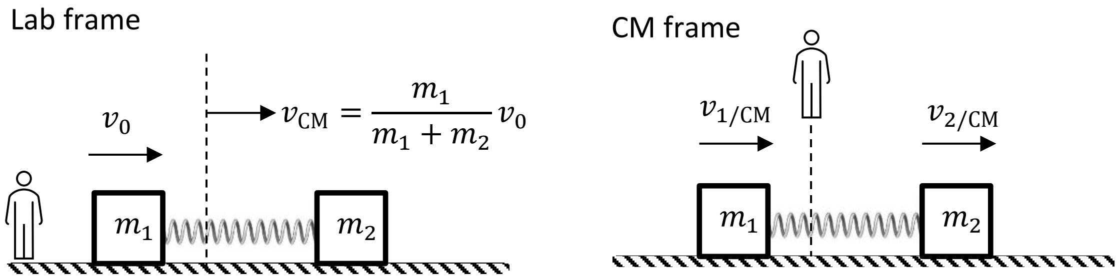


Introduced by Dick Fosbury at the 1968 Summer Olympics, CM always below the bar in order to minimize the gain in potential energy

Example motion in CM frame
Two masses m_1 and m_2 are attached to the ends of a light spring of force constant k. The masses are at rest on a frictionless floor and the spring is at its natural length l_0. $m_1$is given a speed v_0 to the right. What is the minimum length $l_{\min}$ of the spring in subsequent motion?

Note: no external force, CM moves with constant velocity, ∴ CM frame is an inertial frame
Physical picture: In the CM frame, the masses are oscillating (compress, expand, …). Obviously at minimum length $l_{\min}$ the masses are instantaneously at rest (this corresponds to both masses moving with the same velocity in the lab frame). Total energy of the system at that instant is $\frac{1}{2}k(\Delta l)^2$.

To find the initial energy in the CM frame, transform the velocities from lab to CM frame:



$$v_{1/\text{CM}} = v_0 - v_{\text{CM}} = \frac{m_2}{m_1 + m_2} v_0, \quad v_{2/\text{CM}} = -v_{\text{CM}} = \frac{-m_1}{m_1 + m_2} v_0$$

Conservation of energy in CM frame:

$$\underbrace{\frac{1}{2} k (\Delta l)^2}_{\text{max compression, no KE}} = \underbrace{\frac{1}{2} m_1 v_{1/\text{CM}}^2 + \frac{1}{2} m_2 v_{2/\text{CM}}^2}_{\text{initial total energy, no PE}} \quad \Rightarrow \quad \Delta l = v_0 \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}$$

Ans: $l_{\min} = l_0 - \Delta l$

⚠ this $l_{\min}$ is measured in CM frame. Is it the same as measured in the lab frame?

Clicker Questions:

Q8.11

A yellow block and a red rod are joined together. Each object is of uniform density. The center of mass of the *combined* object is at the position shown by the black "X." Which has the *greater mass*, the yellow block or the red rod?

A. The yellow block has the greater mass.

B. The red rod has the greater mass.

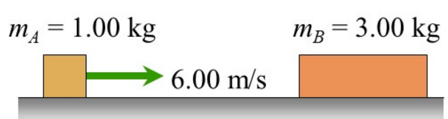
C. They both have the same mass.

D. Either A or B is possible.

E. A, B, or C is possible.

Q8.13

Block A on the left has mass 1.00 kg. Block B on the right has mass 3.00 kg.



Block A is initially moving to the right at 6.00 m/s, while block B is initially at rest. The surface they move on is level and frictionless. What is the velocity of the center of mass of the two blocks *after* the blocks collide?

A. 6.00 m/s, to the right
B. 3.00 m/s, to the right
C. 1.50 m/s, to the right
D. zero
E. Not enough information is given to decide.

Ans: Q8.11) A, Q8.13) C