PHYS1312 Homework, Part 1

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Due date: Fri, Sep 20

Problem 1.

Let's draw a diagram of the particle's motion and the forces acting on it:

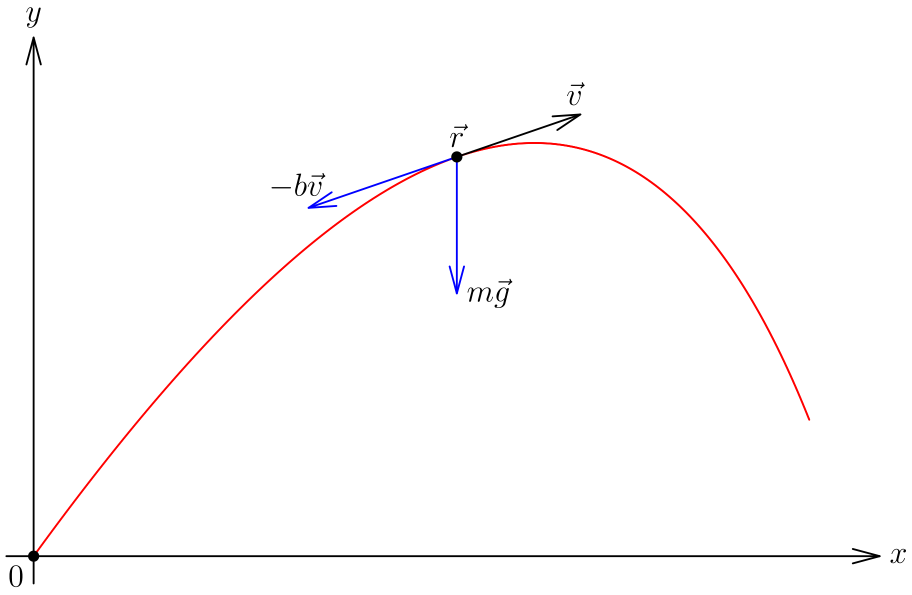

Figure 1: The trajectory of the projectile

The Newtonian equation of motion for this projectile will take the form

$$m\frac{d^2\vec{r}}{dt^2} = -b\frac{d\vec{r}}{dt} + m\vec{g}, \tag{1}$$

where $\vec{g}$ is defined as $(0, -g)$. Our goal is to solve this differential equation in order to obtain the expression of y in terms of x. We will now utilise the method of nondimensionalization, choosing characteristic values as follows:

1. $v_c = mg/b$ — characteristic speed.
2. $t_c = v_c/g = m/b$ — characteristic time.
3. $l_c = v_c t_c = (m^2 g)/b^2$ — characteristic length.

Variables with primes will denote the dimensionless counterparts of the corresponding variables, e.g.

$$t' = \frac{t}{t_c}, \qquad x' = \frac{x}{l_c}, \qquad \vec{v}' = \frac{\vec{v}'}{v_c}, \qquad v_{x'} = \frac{v_x}{(v_c)_x}, \qquad a' = \frac{a}{a_c} = \frac{a}{g}.$$

Dividing Equation 1 by mg and applying algebraig transformations yields a modified differential equation in terms of dimensionless quantities:

$$\frac{d^2\vec{r}'}{dt'^2} = -\frac{d\vec{r}'}{dt'} + \downarrow \qquad \text{or} \qquad \frac{d\vec{v}'}{dt'} = -\vec{v}' + \downarrow, \tag{2}$$

where $\downarrow$ denotes the vector $(0, -1)$. This is a separable differential equation in v', which we will solve separately by each coordinate. The result is

$$v_x'(t') = v_0' \cos\theta \cdot e^{-t'}, \qquad v_y'(t') = (v_0' \sin\theta + 1)e^{-t'} - 1. \tag{3}$$

Integrating Equation 3, we obtain the closed-form solutions for x' and y':

$$x'(t') = v_0' \cos\theta \left(1 - e^{-t'}\right), \qquad y'(t') = (v_0' \sin\theta + 1)\left(1 - e^{-t'}\right) - t'. \tag{4}$$

Finally, we write down the formula for $y'(x')$:

$$\begin{aligned} y' &= (v_0' \sin\theta + 1)\left(1 - e^{-t'}\right) - t' \\ &= v_0' \sin\theta \cdot \left(1 - e^{-t'}\right) + \left(1 - e^{-t'}\right) - t' \\ &= \tan\theta \cdot x' + \frac{x'}{v_0' \cos\theta} + \ln\left(1 - \frac{x'}{v_0' \cos\theta}\right) \\ &= x'\left(\tan\theta + \frac{1}{v_0' \cos\theta}\right) + \ln\left(1 - \frac{x'}{v_0' \cos\theta}\right). \end{aligned}$$

Problem 2.

We first examine the system from a laboratory frame. Let v_{px} and v_{py} denote the horizontal and vertical components of the particle, respectively, with v_p being the total velocity. By v_s we will denote the horizontal speed of the hemisphere (directed in the same direction as v_{px}, and thus having a negative value). Here is a sketch of the situation:

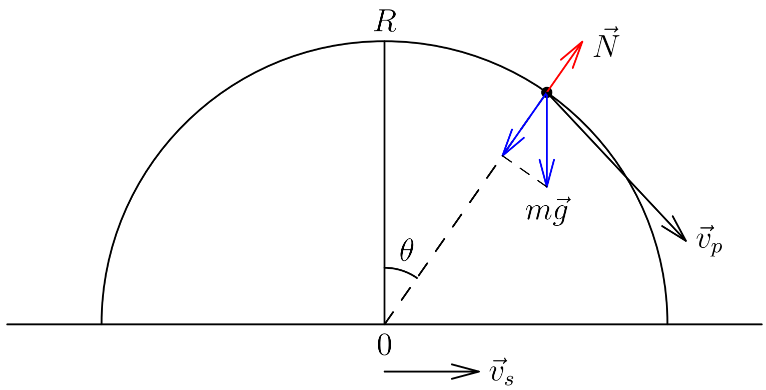

Figure 2: The free body diagram in the laboratory frame

We then make use of the laws of conservation of mechanical energy (since all interactions are frictionless, and thus elastic), and conservation of linear momentul along the horizontal direction, obtaining

$$mv_{bx} + Mv_s = 0, \tag{5.1}$$

$$\frac{Mv_s^2}{2} + \frac{mv_p^2}{2} = mgR(1 - \cos(\theta)). \tag{5.2}$$

Let $\vec{v}_p'$ denote the velocity of the particle relative to the hemisphere. We easily see that

$$\vec{v}_p' = \left(v_{px} - v_s, v_{py}\right). \tag{6}$$

At every moment in time, in the frame of the hemisphere, the particle is instantaneously moving in circular motion along a circle of radius R, and thus we can use Newton's second law to describe its centripetal acceleration:

$$\frac{m{v_p'}^2}{R} = mg\cos(\theta) - N. \tag{7}$$

Now, we shall derive a condition that is satisfied when the particle begins to detach from the hemisphere. We note two facts specific to this moment in time:

1. The velocity vector of the particle *in the reference frame of the hemisphere* is tangent to the hemisphere, as opposed to $\vec{v}_p$ shown on Figure 2 in the laboratory frame.
2. There is no longer any interaction between the hemisphere and the particle, meaning that $\vec{N} = \vec{0}$.

With $N = 0$, Equation 7 yields

$$v_p'^{\,2} = gR\cos(\theta). \tag{8}$$

On the other hand, by Equation 5.1 we have $v_s = -rv_{px}$, and by fact 1 we have

$$\frac{v_{py}}{v_{px} - v_s} = \tan(\theta),$$

since $\vec{v}_p'$ is tangent to the hemisphere. Combining these expressions, we write

$$v_p'^{\,2} = \left(v_{px} - v_s\right)^2 + v_{py}^2 = \left(v_{px} - v_s\right)^2\left(1 + \tan^2(\theta)\right) = \frac{v_{px}^2(1+r)^2}{\cos^2(\theta)}. \tag{9}$$

Further, from Equation 5.2 we have

$$\begin{aligned}
2mgR(1 - \cos(\theta)) &= rmv_{px}^2 + m\left(v_{px}^2 + v_{py}^2\right), \\
2gR(1 - \cos(\theta)) &= v_{px}^2(1 + r) + v_{py}^2 \\
&= v_{px}^2\left((r+1) + (r+1)^2\tan^2(\theta)\right), \\
v_{px}^2 &= \frac{2gR}{r+1} \cdot \frac{1 - \cos(\theta)}{1 + (r+1)\tan^2\theta}.
\end{aligned}$$

Substituting this into Equation 9 and using Equation 8, we obtain that

$$\begin{aligned}
gR\cos(\theta) &= \frac{2gR(1+r)^2}{(1+r)\cos^2(\theta)} \cdot \frac{1 - \cos(\theta)}{1 + (r+1)\tan^2\theta}, \\
\frac{\cos^3(\theta)}{r+1} &= \frac{2(1 - \cos(\theta))}{1 + (r+1)\tan^2(\theta)}.
\end{aligned}$$

Cross-multiplying this equation and cancelling like terms, we write

$$\begin{aligned}
2(1+r)(1 - \cos(\theta)) &= \cos^3(\theta)\left(1 + (r+1)\tan^2(\theta)\right), \\
2(1+r) - 2(1+r)\cos(\theta) &= \cos^3(\theta) + (1+r)\cos(\theta)\sin^2(\theta) \\
&= \cos^3(\theta) - r\cos^3(\theta) - \cos^3(\theta) + (1+r)\cos(\theta), \\
r\cos^3(\theta) - 3(1+r)\cos(\theta) + 2(1+r) &= 0,
\end{aligned}$$

q.e.d.