Potential Energy & Energy Conservation

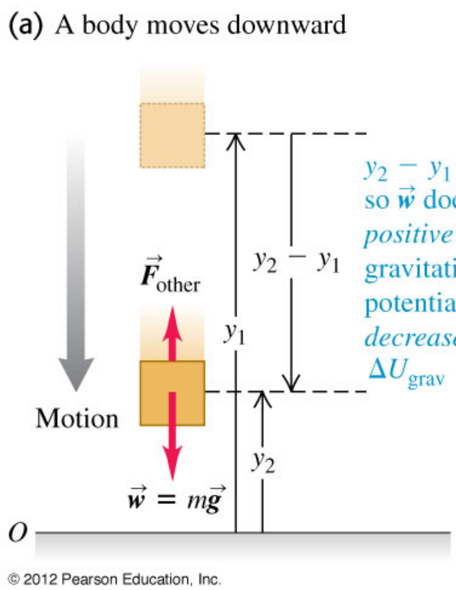
Intended Learning Outcomes – after this lecture you will learn:

1. gravitational and elastic forces as examples of conservative force
2. properties of the potential energy function of a conservative force
3. to derive the force from the potential energy function

Textbook Reference: Ch 7

Potential energy – energy associated with the position of bodies in a system

Gravitational PE Defined by $U_{\text{grav}} = mgy$

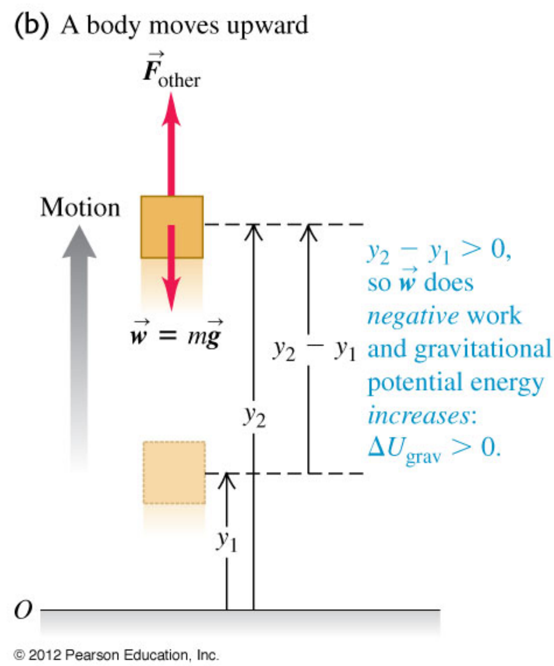


Work done by the weight of the body

$W_{\text{grav}} = mg(y_1 - y_2) > 0$, $\vec{\boldsymbol{w}}$ does +ve work

$\Delta U_{\text{grav}} = mg(y_2 - y_1) = -W_{\text{grav}} < 0$

gravitational PE *decreases*

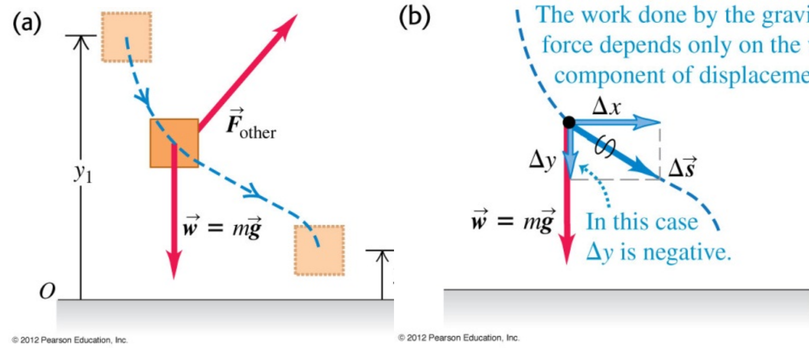


$W_{\text{grav}} = -mg(y_2 - y_1) < 0$, $\vec{\boldsymbol{w}}$ does –ve work

$\Delta U_{\text{grav}} = mg(y_2 - y_1) = -W_{\text{grav}} > 0$

gravitational PE *increases*

Along a curved path



work done by the weight

$$W_{\text{grav}} = \vec{\boldsymbol{w}} \cdot \overrightarrow{\Delta \boldsymbol{s}} = -mg\Delta y$$
$$= -\Delta U_{\text{grav}}$$

same as vertical motion!

<u>Conclusion</u>: $W_{\text{grav}} = -\Delta U_{\text{grav}}$

Note: gravitational PE acts like a bank to store workdone for later use

if $W_{\text{grav}} < 0$, $\Delta U_{\text{grav}} > 0$, U_{grav} increases, *c.f.* deposit money into a bank

if $W_{\text{grav}} > 0$, $\Delta U_{\text{grav}} < 0$, U_{grav} decreases *c.f.* draw money from the bank and spend it

⚠ gravitational PE does not belong to the body only, it belongs to both the body and the earth

By **work-energy theorem**

$$\Delta K = -\Delta U_{\text{grav}} \quad \Rightarrow \quad \Delta K + \Delta U_{\text{grav}} = 0 \text{ , or } \boxed{K_{\text{initial}} + U_{\text{grav,initial}} = K_{\text{final}} + U_{\text{grav,final}}}$$

conservation of mechanical energy

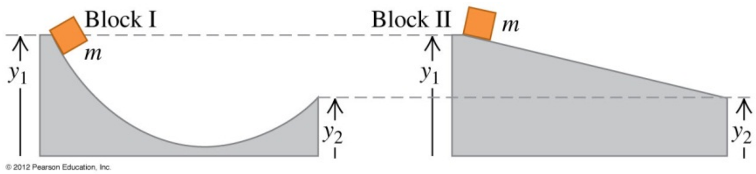
What if other forces also do work?

Work-energy theorem $\quad W_{\text{other}} + W_{\text{grav}} = \Delta K \quad \Rightarrow \quad \boxed{W_{\text{other}} = \Delta K + \Delta U_{\text{grav}}}$

Digression Energy storage using potential energy

See [How gravity batteries could change the world (youtube.com)]

Question: The figure shows two different frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed? (i) block I; (ii) block II; (iii) the speed is the same for both blocks.

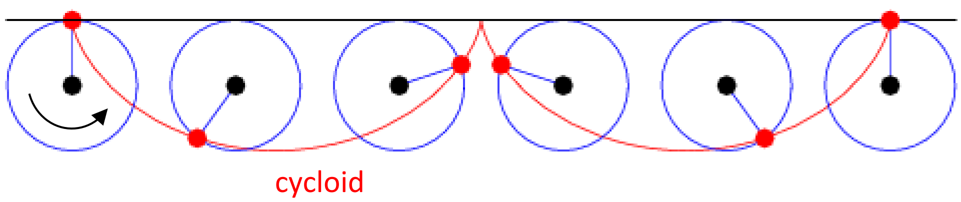


Answer: see inverted text on P. 238 of textbook

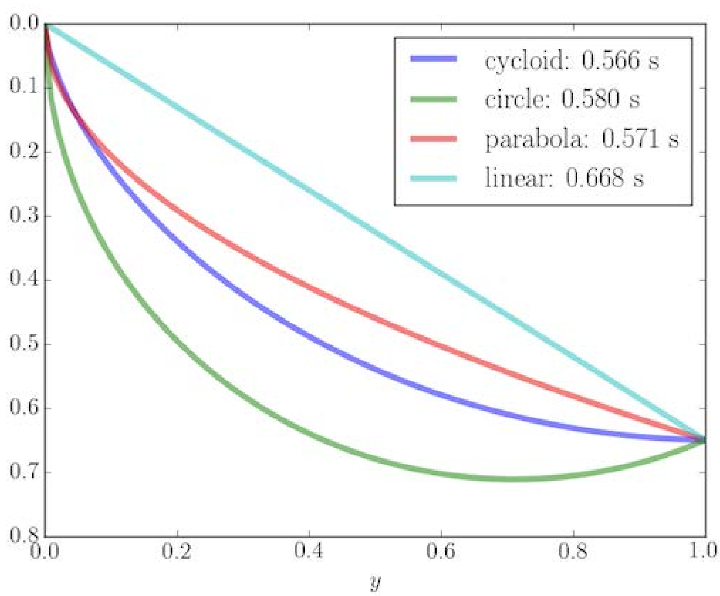
Mathematical Digression the curve of fastest descent, or **brachistochrone curve**

Posted by Johann Bernoulli in 1696, solved independently by, among others, Isaac Newton, Jakob Bernoulli, Gottfried Leibniz, and Guillaume de l'Hôpital

Solution is a **cycloid**, which is the curve traced out by a point on the circumference of a circle as it rolls:



The following is simulation result at scipython.com for different curves

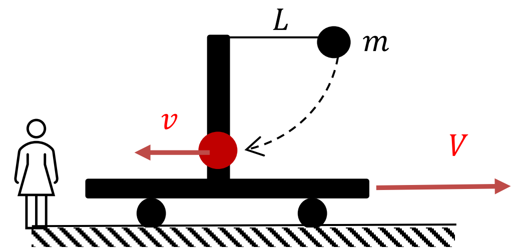


https://scipython.com/blog/the-brachistochrone-problem/

[Youtube version]()

Example

A ball of mass m is mounted on a trolley by a sting of length L. The trolley moves on a smooth floor and the ball is free to swing. Initially the trolley is at rest and the string is in a horizontal position as shown. Find the speeds (relative to the ground) of the ball and trolley when the ball swings down and the string is vertical.



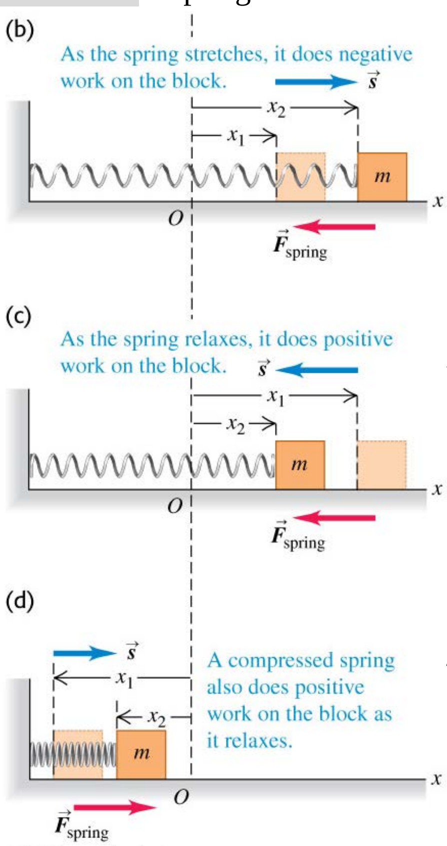
No other force except gravitation does work (why?). By conservation of energy:

$$\frac{1}{2}mv^2 + \frac{1}{2}MV^2 = mgL$$

Conservation of linear momentum: $mv = MV$ (no external forces?)

$$v = \sqrt{\frac{2MgL}{M+m}}, \quad V = \sqrt{\frac{2m^2gL}{M(M+m)}}$$

Elastic PE – spring



Check that it works in all 3 cases

Work done by restoring force in spring

$$W_{\text{el}} = \int_{x_1}^{x_2}(-kx)dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$

Define elastic PE of spring $U_{\text{el}} = \frac{1}{2}kx^2$

$$W_{\text{el}} = -\Delta U_{\text{el}}$$

c.f. gravitational PE

⚠ U_{grav} free to choose zero level position, but for U_{el}, zero level position must correspond to unstretched position.

In the presence of gravitational, elastic, and other forces

Work-energy theorem $W_{\text{grav}} + W_{\text{el}} + W_{\text{other}} = \Delta K$

$$\Rightarrow \qquad W_{\text{other}} = \Delta K + \Delta\left(U_{\text{grav}} + U_{\text{el}}\right) = \Delta K + \Delta PE$$

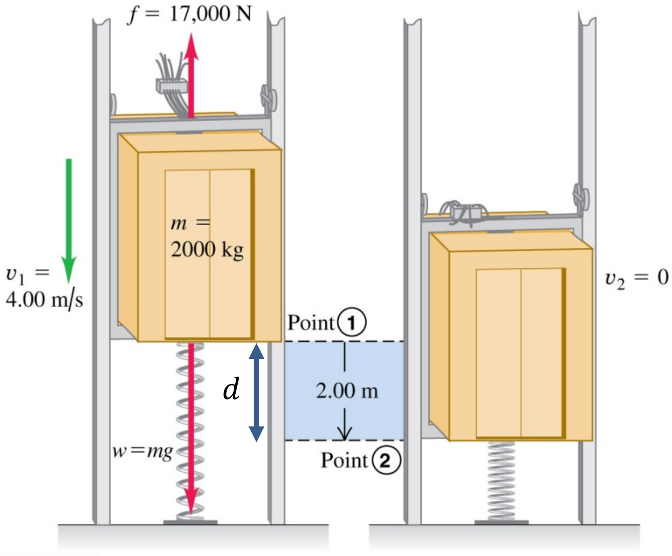
If $W_{\text{other}} = 0$, $\Delta K + \Delta PE = 0$, or $\boxed{K_{\text{initial}} + PE_{\text{initial}} = K_{\text{final}} + PE_{\text{final}}}$

conservation of mechanical energy

Demonstration: Maxwell's wheel

Example 7.9 P. 242 Motion with gravitational, elastic, and friction forces

An elevator with a broken cable. A safety clamp provides a constant friction f between the rail and the elevator. What is the spring constant k if the elevator has initial speed v_1 when it just touches the spring, and comes to rest at a distance $d = 2.00$ m?



work done by friction $W_{\text{other}} = -fd$

$\Delta K = 0 - \frac{1}{2}mv_1^2$

$\Delta PE = -mgd + \frac{1}{2}kd^2$

$W_{\text{other}} = \Delta K + \Delta PE$

$$\Rightarrow \quad -fd = -\tfrac{1}{2}mv_1^2 - mgd + \tfrac{1}{2}kd^2$$

$$\Rightarrow \quad k = \frac{2\left(mgd + \frac{1}{2}mv_1^2 - fd\right)}{d^2}$$

Condition for no rebound at Point ②:

Assume the same f above is the max friction provided by the safety clamp,

$$kd - mg = f_{\text{static}} \le f$$

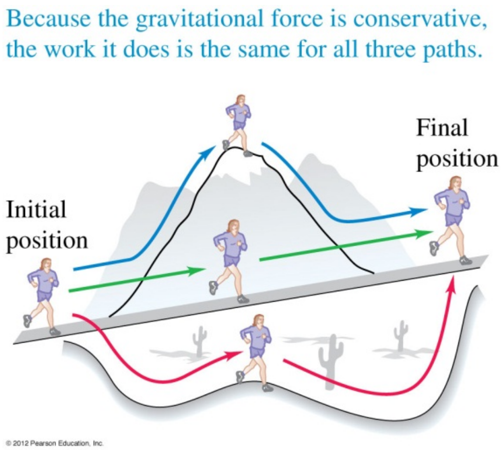
$$\Rightarrow \quad mg + \frac{mv_1^2}{d} \le 3f$$

Conservative Forces

-ve work done on a system can be "reclaimed" as KE, e.g. gravitation, spring

These are called **conservative forces**

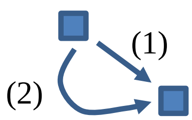
Demonstration: energy stored in a spring



<u>Properties of the work done by conservative forces</u>:

1. It can be expressed as the difference between the initial and final values of a potential energy function.
2. It is reversible, i.e., if path is reversed, workdone changes sign.
3. It depends on the starting and ending point only, not on the path.
4. When the starting and ending points are the same (path forms a close loop), the total work is zero.

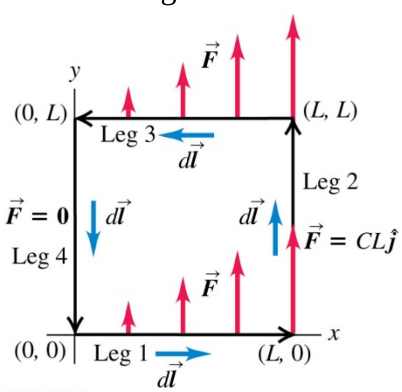
c.f. –ve work done by friction cannot be "reclaimed", called **non-conservative forces**.
⚠ work done by non-conservative force is <u>path dependent</u>



work done by friction in path (2) is more negative than in path (1).

⚠ The term PE is reserved for conservative forces only
To test whether a force is conservative – check if the work done is zero around a close loop.

Example 7.11 P. 245 Conservative or nonconservative?
An electron goes counter clockwise around a square loop under a force $\vec{\boldsymbol{F}} = Cx\hat{j}$, C constant



Leg 1, $W_1 = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = 0$
Leg 2, $W_2 = CL^2$
Leg 3, $W_3 = 0$
Leg 4, $W_4 = 0$
$\vec{\boldsymbol{F}}$ is (conservative / non-conservative)

To derive a conservative force $\vec{F}$ from its potential energy function U:

Work done by a conservative force $W = -\Delta U(x)$ in 1D

$F\Delta x$

$$\Rightarrow F = -\frac{\Delta U}{\Delta x} \xrightarrow{\Delta x \to 0} \boxed{F = -\frac{dU}{dx}}$$

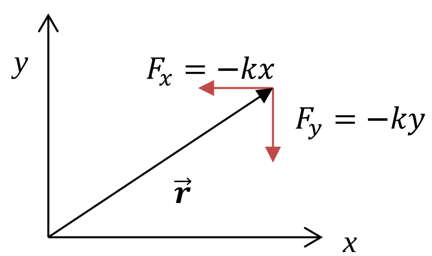
⚠ Free to add a constant to $U(x)$ without changing the force

Check: $U_{\text{grav}} = mgh$, $F = -mg$

$U_{\text{el}} = \frac{1}{2}kx^2$, $F = -kx$

In 3D, $F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, $F_z = -\frac{\partial U}{\partial z}$

Example 7.14 P. 249 Force and potential energy in 2D



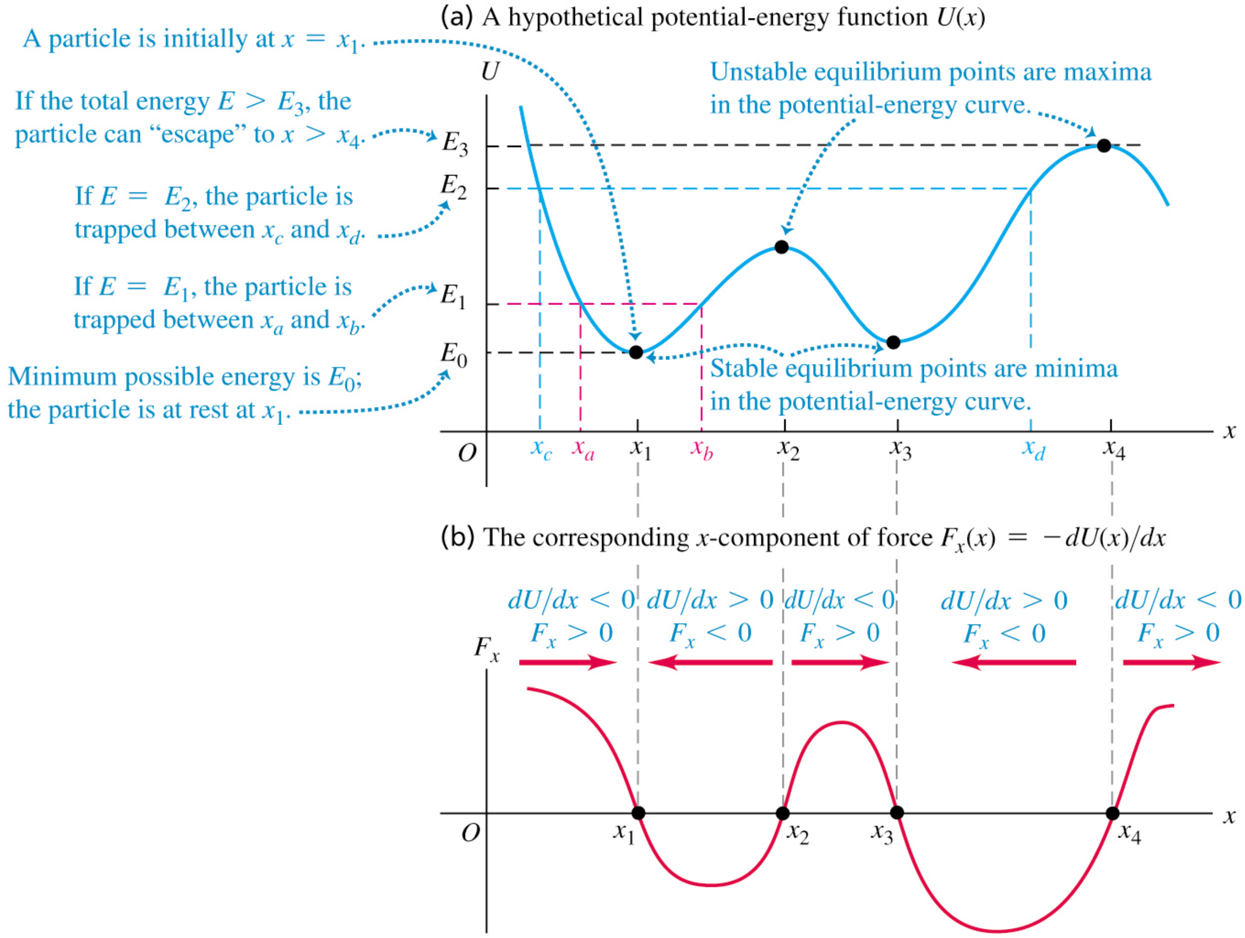
$$U(x, y) = \frac{1}{2}k(x^2 + y^2)$$

$$F_x = -\frac{\partial U}{\partial x} = -kx, \; F_y = -\frac{\partial U}{\partial y} = -ky$$

Question: A particle moving along the x-axis is acted upon by a conservative force F_x. At a certain point, the force is zero. At that point the value of the potential energy function $U(x)$ is (= 0 / > 0 / < 0 / not enough information to decide), and dU/dx is (= 0 / > 0 / < 0 / not enough information to decide).

Answer: see inverted text on P. 250 of textbook

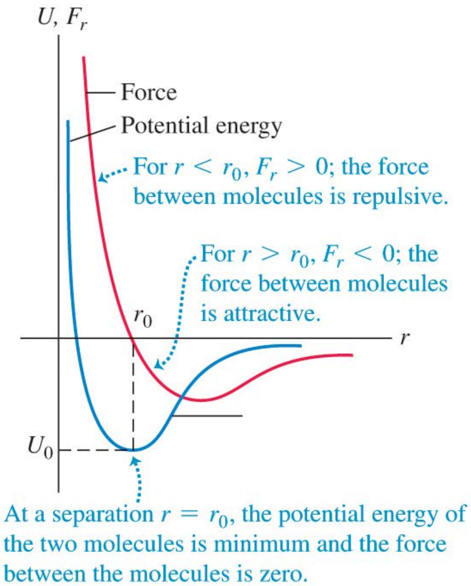
Interpretation of an **energy diagram**:
Note the meanings of stable and unstable equilibrium.



Example Inter-atomic force

For an inert gas (He, Ar, Ne, …), potential energy between two atoms at distance r can be approximated by the Lennard-Jones potential

$$U(r) = U_0\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right]$$



$$F(r) = -\frac{dU}{dr} = \frac{12U_0}{r}\left[\left(\frac{r_0}{r}\right)^{12} - \left(\frac{r_0}{r}\right)^{6}\right]$$

Obviously $F(r_0) = 0$, and r_0 is a stable equilibrium point.

Clicker Questions

Q7.1

A piece of fruit falls straight down. As it falls,

A. the gravitational force does positive work on it and the gravitational potential energy increases.

B. the gravitational force does positive work on it and the gravitational potential energy decreases.

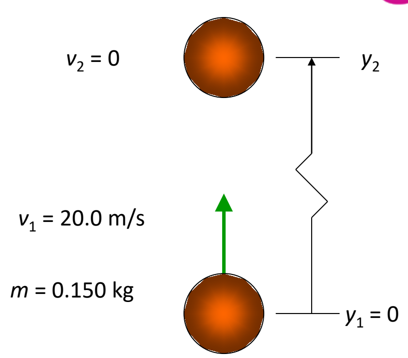
C. the gravitational force does negative work on it and the gravitational potential energy increases.

D. the gravitational force does negative work on it and the gravitational potential energy decreases.

Q7.2

You toss a 0.150-kg baseball straight upward so that it leaves your hand moving at 20.0 m/s. The ball reaches a maximum height y_2.

What is the speed of the ball when it is at a height of $y_2/2$? Ignore air resistance.



A. 10.0 m/s

B. less than 10.0 m/s but greater than zero

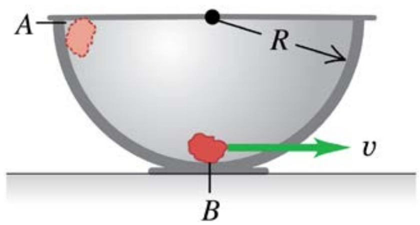
C. greater than 10.0 m/s

D. not enough information given to decide

Q7.3

As a rock slides from *A* to *B* along the inside of this frictionless hemispherical bowl, mechanical energy is conserved. Why?

(Ignore air resistance.)



A. The bowl is hemispherical.

B. The normal force is balanced by centrifugal force.

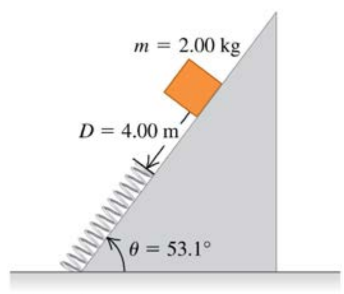
C. The normal force is balanced by centripetal force.

D. The normal force acts perpendicular to the bowl's surface.

E. The rock's acceleration is perpendicular to the bowl's surface.

Q7.5

A block is released from rest on a frictionless incline as shown. When the moving block is in contact with the spring and compressing it, what is happening to the gravitational potential energy U_{grav} and the elastic potential energy U_{el}?



A. U_{grav} and U_{el} are both increasing.

B. U_{grav} and U_{el} are both decreasing.

C. U_{grav} is increasing; U_{el} is decreasing.

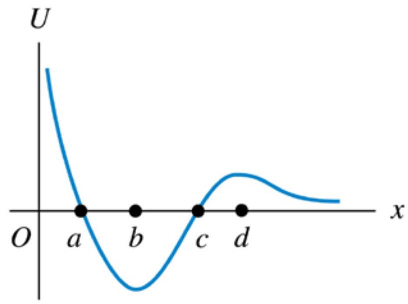
D. U_{grav} is decreasing; U_{el} is increasing.

E. The answer depends on how the block's speed is changing.

Q7.6

The graph shows the potential energy U for a particle that moves along the x-axis.

The particle is initially at $x = d$ and moves in the negative x-direction. At which of the labeled x-coordinates does the particle have the greatest *speed*?



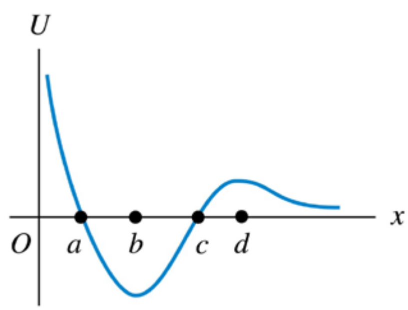
A. at $x = a$ B. at $x = b$ C. at $x = c$ D. at $x = d$

E. more than one of the above

Q7.7

The graph shows the potential energy U for a particle that moves along the x-axis.

The particle is initially at $x = d$ and moves in the negative x-direction. At which of the labeled x-coordinates is the particle *slowing down*?

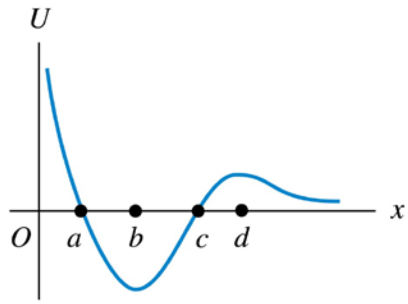


A. at $x = a$ B. at $x = b$ C. at $x = c$ D. at $x = d$

E. more than one of the above

Q7.8

The graph shows the potential energy U for a particle that moves along the x-axis. At which of the labeled x-coordinates is there *zero* force on the particle?



A. at $x = a$ and $x = c$

B. at $x = b$ only

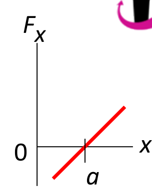
C. at $x = d$ only

D. at $x = b$ and d

E. misleading question—there is a force at all values of x

Q7.9

The graph shows a conservative force F_x as a function of x in the vicinity of $x = a$. As the graph shows, $F_x = 0$ at $x = a$. Which statement about the associated *potential energy* function U at $x = a$ is correct?



A. $U = 0$ at $x = a$

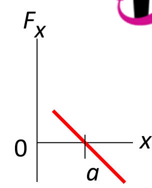
B. U is a maximum at $x = a$.

C. U is a minimum at $x = a$.

D. U is neither a minimum or a maximum at $x = a$, and its value at $x = a$ need not be zero.

Q7.10

The graph shows a conservative force F_x as a function of x in the vicinity of $x = a$. As the graph shows, $F_x = 0$ at $x = a$. Which statement about the associated *potential energy* function U at $x = a$ is correct?



A. $U = 0$ at $x = a$

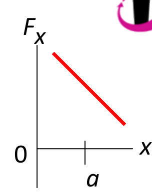
B. U is a maximum at $x = a$.

C. U is a minimum at $x = a$.

D. U is neither a minimum or a maximum at $x = a$, and its value at $x = a$ need not be zero.

Q7.11

The graph shows a conservative force F_x as a function of x in the vicinity of $x = a$. As the graph shows, $F_x > 0$ and $dF_x/dx < 0$ at $x = a$. Which statement about the associated *potential energy* function U at $x = a$ is correct?



A. $dU/dx > 0$ at $x = a$

B. $dU/dx < 0$ at $x = a$

C. $dU/dx = 0$ at $x = a$

D. Any of the above could be correct.

Ans: Q7.1) B, Q7.2) C, Q7.3) D, Q7.5) D, Q7.6) B, Q7.7) A, Q7.8) D, Q7.9) B, Q7.10) C, Q7.11) B