

Зам "f ∘ g" существует

Диаграмма $\rightarrow f \circ g$

Англ. языч. запись $\rightarrow g \circ f$

$a \in A$ $f(g(a))$ — не一定是单射
 b 也是单射

$(g(f(a)))$

Def A, B — мн-ва. $A \cup B$ \leftrightarrow равносильны, если $\exists f: A \rightarrow B$:
 f — биективна

Пример $A = \{1\}$, $B = \{2\}$

$f: A \rightarrow B$

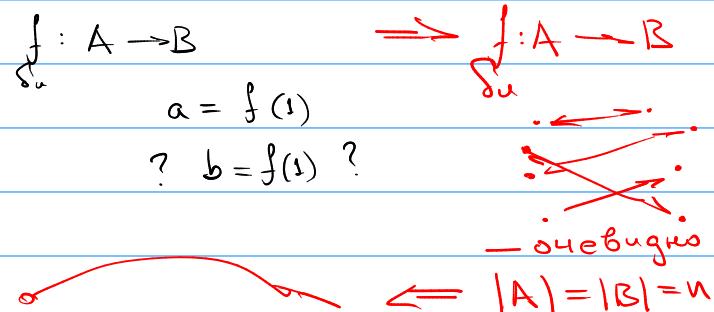
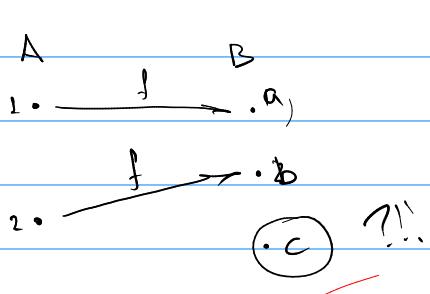
$f(1) = 2$ — ин.

— сюрп. ($\forall b \in B \exists a \in A: b = f(a)$)

$(f(A) = B)$

$\Rightarrow f$ — биект. $\Rightarrow |A| = |B|$

$A = \{1, 2\}$, $B = \{a, b, c\}$ $|A| \neq |B|$



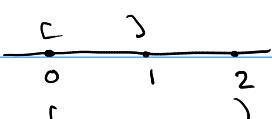
Те. A, B — мн. A равносильна $B \Leftrightarrow |A| = |B|$

$\exists f: A \rightarrow B$

$\underbrace{\quad}_{\text{числа}}$ $\} n$
— очевидно

Л-бо Упрощение

Пример $[0, 1]$ биJECT. $[0, 2]$ ($[0, 1] \cong [0, 2]$)



$f: [0, 1] \rightarrow [0, 2]$

$$f(x) = 2x$$

$$x_1, x_2 \in [0, 1] \quad f(x_1) = f(x_2)$$

$$x_1 = x_2?$$

$$\cancel{x_1 = 2x_2}$$

$$\Rightarrow f(x_1) = x_2$$

$$y \in [0, 2] \quad x = y/2 \in [0, 1] \quad f(x) = f(y/2) = 2 \cdot y/2 = y \Rightarrow f - \text{снф.} \\ \Rightarrow f - \text{снф.}$$

$$g: [0, 2] \rightarrow [0, 1] \quad f \circ g - \text{обратное} \\ g(y) = y/2 \quad g = f^{-1}$$

Пример: $(0, 1) \cong (1, +\infty)$

$$f: (0, 1) \rightarrow (1, +\infty) \\ f(x) = \frac{1}{x}$$

Чтобы: $f - \text{снф.твна}$ (через обратную)

$$g: (1, +\infty) \rightarrow (0, 1) \\ g(y) = \frac{1}{y}$$

Одн $A, B - \text{мн-ва.}$ $A^B := \{f \mid f: B \rightarrow A\}$

$$\begin{matrix} & f \\ \swarrow & \downarrow \\ A & B \end{matrix}$$

$$A, B - \text{кон.} \quad |A^B| = |A|^{|\mathbb{B}|}$$

$$\begin{array}{ll} |\mathbb{B}| = n & b_1, \dots, b_n \\ |A| = m & a_1, \dots, a_m \\ & \vdots \\ & b_m \end{array}$$

$$\underbrace{m \cdot m \cdot m \cdot \dots \cdot m}_{n \text{ множ.}} = m^n = |A|^{\mathbb{B}} \quad |A^B|$$

$$\begin{aligned} & g \in (1, +\infty) \quad f(g(y)) = \\ & g(f(x)) = g(1/x) = \\ & = 1/(1/x) = x \end{aligned}$$

$$\begin{aligned} & y \in (1, +\infty) \quad f(g(y)) = \\ & f(1/y) = y \end{aligned}$$

$$\Rightarrow f \circ g - \text{обратные}$$

$$\Rightarrow f - \text{снф.твна}$$

$$\Rightarrow (0, 1) \cong (1, +\infty)$$

Пример $\{0, 1, 2, 3\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$

$$0, 1, 2, 0, 2, 3, 3, 3, \dots \in \{0, 1, 2, 3\}^{\mathbb{N}}$$

$$f \in A^{\mathbb{N}} \iff \text{нек-ть} \text{ из } \text{з-тв} \text{ } A$$

$$0, 1, 1, 1, 0, 0, \dots \in \{0, 1\}^{\mathbb{N}}$$

$$(\in \{0, 1, 2, 3\}^{\mathbb{N}})$$

$$\varphi: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1, 2, 3\}^{\mathbb{N}}$$

$$\underbrace{0, 1, 0, 0, 1, 1, 0, 1, 0, 0, \dots}_{1 \ 0 \ 3 \ 1 \ 0} \rightarrow 1, 0, 3, 1, 0, \dots$$

$$\varphi^{-1}: \{0, 1, 2, 3\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$$

$$\underbrace{2, 0, 1, 3, 3, 0, 0, 1, 2, \dots}_{1, 0, 0, 0, 1, 1, 1, 1, \dots} \rightarrow 1, 0, 0, 0, 0, 1, 1, 1, 1, \dots$$

$$\varphi \circ \varphi^{-1} - \text{обратное}$$

$$\Rightarrow \{0, 1\}^{\mathbb{N}} \cong \{0, 1, 2, 3\}^{\mathbb{N}}$$

Одн $A_{\text{нн-го.}} \text{ @ } \text{снф.твна, если } A \cong \mathbb{N}$

$$A = \{a_1, a_2, a_3, \dots\}$$

Оп: $A \cap B$ не более чем счётное, если A конечно или счётно

Чт: $A, B - \text{нбчс} \Rightarrow A \cap B - \text{нбчс}$
 $A \cup B - \text{нбчс}$

Теорема 1

1) $A - \text{нбчс} \Rightarrow (B \subset A \Rightarrow B - \text{нбчс})$

2) $A - \text{сект.} \Rightarrow \exists B \subset A : B \text{ счётно}$

3) $f = \{A_i\}, f \text{ нбчс}, A_i - \text{нбчс} \Rightarrow \bigcup A_i - \text{нбчс}$

д-бо 1) $A: a_1, a_2, a_3, a_4, \dots$

$\sqrt{\quad} \quad \sqrt{\quad} \quad \times \quad \sqrt{\quad} \quad \dots$

$A = \{2, 4, 6, 8, \dots\}$
 $B = \{4, 6, 8\} \subset A$

$\cancel{2, 4, 6, 8, \dots} \quad \cancel{\times \times \times \dots}$

$B = \{8, 10, 12, \dots\}$

$\cancel{2, 4, 6, 8, 10, 12, \dots}$

B

2) $b_1 \in A$

$b_2 \in A \setminus \{b_1\} - \text{сект.}$

$b_3 \in A \setminus \{b_1, b_2\} - \text{сект.}$

$b_4 \in A \setminus \{b_1, b_2, b_3\}$

:

$b_{n+1} \in A \setminus \{b_1, b_2, \dots, b_n\} - \text{сект.}$

:

$b_1, b_2, b_3, \dots, b_k \in A, \forall k \in \mathbb{N}$

$B = \{b_1, b_2, \dots\} - \text{счётное}$

$\subset A$

3) $f - \text{нбчс}, \forall A_i \in f \quad A_i - \text{нбчс} \quad (!) \quad \bigcup A_i = \bigcup f - \text{нбчс.}$

$A_1 = a_{11} a_{12} a_{13} \dots$

$A_2 = a_{21} a_{22} a_{23} \dots$

$A_3 = a_{31} a_{32} a_{33} \dots$

A_4

~~$a_{11} a_{12} a_{13} \dots$~~

~~$a_{21} a_{22} a_{23} \dots$~~

~~$a_{31} a_{32} a_{33} \dots$~~

~~$a_{41} a_{42} a_{43} \dots$~~

~~$a_{51} a_{52} a_{53} \dots$~~

~~$a_{61} a_{62} a_{63} \dots$~~

~~$a_{71} a_{72} a_{73} \dots$~~

~~$a_{81} a_{82} a_{83} \dots$~~

~~$a_{91} a_{92} a_{93} \dots$~~

~~$a_{101} a_{102} a_{103} \dots$~~

~~$a_{111} a_{112} a_{113} \dots$~~

~~$a_{121} a_{122} a_{123} \dots$~~

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~~$a_{181} a_{182} a_{183} \dots$~~

~~$a_{191} a_{192} a_{193} \dots$~~

~~$a_{201} a_{202} a_{203} \dots$~~

~~$a_{211} a_{212} a_{213} \dots$~~

~~$a_{221} a_{222} a_{223} \dots$~~

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~~$a_{461} a_{462} a_{463} \dots$~~

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~~$a_{491} a_{492} a_{493} \dots$~~

~~$a_{501} a_{502} a_{503} \dots$~~

~~$a_{511} a_{512} a_{513} \dots$~~

~~$a_{521} a_{522} a_{523} \dots$~~

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~~$a_{581} a_{582} a_{583} \dots$~~

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~~$a_{621} a_{622} a_{623} \dots$~~

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~~$a_{641} a_{642} a_{643} \dots$~~

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~~$a_{681} a_{682} a_{683} \dots$~~

~~$a_{691} a_{692} a_{693} \dots$~~

~~$a_{701} a_{702} a_{703} \dots$~~

~~$a_{711} a_{712} a_{713} \dots$~~

~~$a_{721} a_{722} a_{723} \dots$~~

~~$a_{731} a_{732} a_{733} \dots$~~

~~$a_{741} a_{742} a_{743} \dots$~~

~~$a_{751} a_{752} a_{753} \dots$~~

~~$a_{761} a_{762} a_{763} \dots$~~

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~~$a_{811} a_{812} a_{813} \dots$~~

~~$a_{821} a_{822} a_{823} \dots$~~

~~$a_{831} a_{832} a_{833} \dots$~~

~~$a_{841} a_{842} a_{843} \dots$~~

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~~$a_{881} a_{882} a_{883} \dots$~~

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~~$a_{921} a_{922} a_{923} \dots$~~

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~~$a_{941} a_{942} a_{943} \dots$~~

~~$a_{951} a_{952} a_{953} \dots$~~

~~$a_{961} a_{962} a_{963} \dots$~~

~~$a_{971} a_{972} a_{973} \dots$~~

~~$a_{981} a_{982} a_{983} \dots$~~

~~$a_{991} a_{992} a_{993} \dots$~~

~~$a_{1001} a_{1002} a_{1003} \dots$~~

1. Не берём повторения

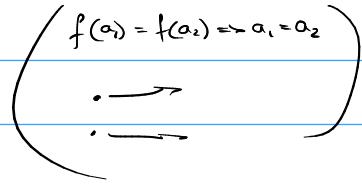
2. Пропускаем нечётные элементы

$\rightarrow b_1, b_2, b_3, \dots$

$B = \{b_1, b_2, b_3, \dots\} = \bigcup A_i$

нбчс

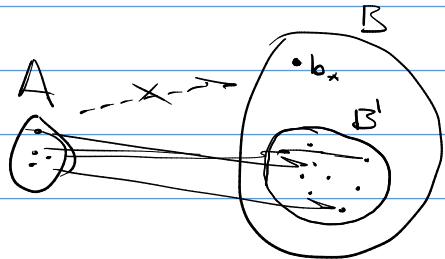
Сл-пое A, B — мн-ва, B — нбчс, $f: A \rightarrow B$ — инъективна (вложение)



$\Rightarrow A$ — нбчс

$$\text{Д-по } B' = f(A) = \{f(a) \mid a \in A\}$$

$f: A \rightarrow B'$ — сюръективна



1) f — инъективна \checkmark (но узарено)

2) f — сюръективна $\forall b \in B' \exists a \in A : b = f(a)$ — но не пр. B'

$$\Rightarrow |A| = |B'|$$

B' — нбчс (но T_1)

$\Rightarrow A$ — нбчс

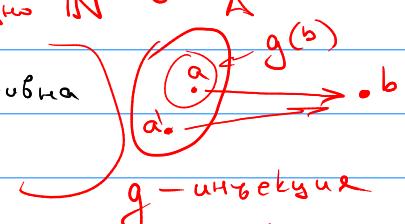
$$f(g(b)) = f(a) = b$$

$\Delta\text{-по } g: B \rightarrow A$

$$g(b) = a \in A \mid f(a) = b$$

конечно или равнозначно N

Уп: A, B — мн-ва, A — нбчс, $f: A \rightarrow B$ — сюръективна
 \Rightarrow (!) B — нбчс.



Теорема 2 A — фин., B — нбчс. \Rightarrow (!) $|A \cup B| = |A|$

Д-по: 1) Можно считать, что $A \cap B = \emptyset$



$$f(g(b_1)) = f(g(b_2))$$

$$\Rightarrow b_1 = b_2$$

$A: \{2, 4, 6, 8, 10, \dots\}$ \checkmark — фин.

$B: \{a, b, c, d, e\}$ \checkmark — фин.

$$B' = B \setminus (A \cap B)$$

$$|A \cup B| = |A| \Rightarrow |A \cup B| = |A|$$

Задача $A \cup B = A \sqcup B$ — доказательство

$$A \sqcup B$$

$$g(B) = \{g(b) \mid b \in B\}$$

П.о. T_1 , $P \subseteq A$, P — счётное

$$Q = A \setminus P$$

$$\begin{matrix} \parallel \\ A \end{matrix}$$

$$A \sqcup B = P \sqcup Q \sqcup B = Q \sqcup (P \sqcup B)$$

$$A = P \sqcup Q = Q \sqcup P$$

$$\{a \mid a = g(b), b \in B\}$$

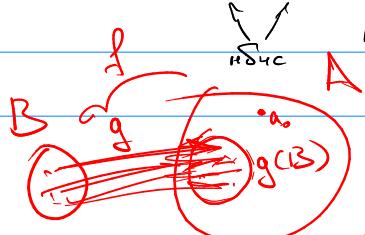
A — нбчс $\Rightarrow g(B)$ — нбчс
 $g(B) \subseteq A$

П.о. T_1 , $P \sqcup B$ — нбчс (утв. 3)

$\Rightarrow P \sqcup B$ — счётное

P — счётное

$$|P \sqcup B| = |B| = |P| \Rightarrow |P \sqcup B| = |P|$$



$\Rightarrow B \cong g(B)$ — нбчс

счётное

$$g: B \rightarrow g(B)$$

ин-глобально
 суп-псеч. $g(B)$ покрыта

$\exists \varphi: P \cup B \rightarrow P$ — биект.

A -бдк. \Rightarrow удобн.

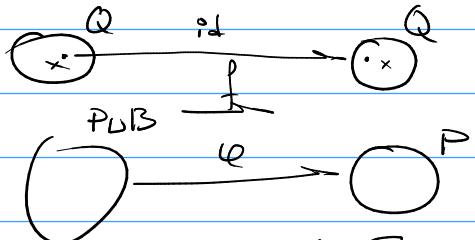
$$a_0 \in A \quad B = A \setminus \{a_0\}$$

$$B \subset A \quad \infty - 1 = \infty$$

$$B \neq A \quad |A| = |B| = |A \setminus \{a_0\}|$$

$f: Q \cup (P \cup B) \rightarrow Q \cup P$

$$f(x) = \begin{cases} x, & x \in Q \\ \varphi(x), & x \notin Q \end{cases}$$



Упр f — биективна

$$\Rightarrow |A \cup B| = |A|$$

A конечн. \Rightarrow не удобн.

$B \subset A, B \neq A \quad B$ меньше
эк-точ., чем в A

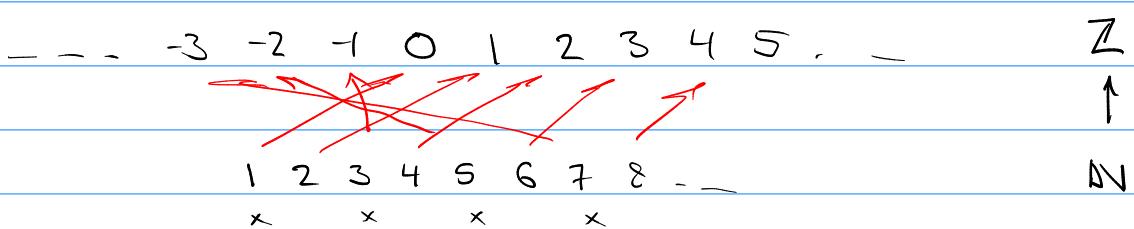
$$|A| \neq |B|$$

Упр $A \subset$ бесконечн., если $|A \cup \mathbb{N}| = |A|$

$A \subset$ бесконечн., если $\exists B \subset A: B \neq A, |A| = |B|$

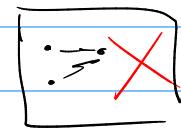
Упр $\exists k - \text{точ.},$ что второе упр. выполняется ($\text{беск. мн-ва удобн., а конечные нет}$)

Пример $\mathbb{Z} \cong \mathbb{N}$ ($f: \mathbb{N} \rightarrow \mathbb{Z}$ f -биективн.)



$$g: \mathbb{N} \rightarrow \mathbb{Z}$$

1) g инъективна?
(бюджетное)



— инъективность

2) g суръективна?
(однородные)



— суръективность

НЕТ $\Leftrightarrow \exists 0 \in \mathbb{Z}: \nexists n \in \mathbb{N}: g(n) = 0$

$$g: \mathbb{N} \xrightarrow{\text{бюджетное}} \mathbb{Z} \Rightarrow |\mathbb{N}| \leq |\mathbb{Z}|$$

“ $\infty \leq \infty + 1$ ($\infty = \omega + 1$)

но и $|\mathbb{N}| < |\mathbb{R}|$

свяжимые числа

$\omega, \omega + 1, \omega + 2, \dots, 2\omega, 2\omega + 1, \dots, 3\omega, 4\omega, \dots$
 $\omega: \omega > n \quad \forall n \in \mathbb{N} \quad \omega \approx \infty$

$$\omega^2 \cong \omega$$

$$\frac{p}{q} \cdot \mathbb{Q} \cong \mathbb{N} \times \mathbb{N} \cong \mathbb{N} \quad (\text{из т. 1})$$

$$\mathbb{N} \left(\begin{array}{c} \mathbb{N} \\ \mathbb{N} \\ \mathbb{N} \end{array} \right) \cup \mathbb{N} \cong \mathbb{N} \times \mathbb{N}$$

$$f: \mathbb{N} \rightarrow \mathbb{Z}$$

$$f(n) = \begin{cases} -\frac{n-1}{2}, & n \text{ нечетное} \\ \frac{n}{2}, & n \text{ четное} \end{cases}$$

— инъективна

n_1, n_2 i) n_1, n_2 парные $n -$ ти
 $(n_1 \neq n_2) \Rightarrow f(n_1), f(n_2)$ unequal
 $f(n_1) \neq f(n_2)$

— сюръективна

 $(\forall a \in \mathbb{Z} \exists n \in \mathbb{N}: f(n) = a)$

$\forall a \in \mathbb{Z}$ 1) $a \leq 0$. $n = \underbrace{-2a+1}_{\substack{\in \mathbb{N}}} > 0$ 2) n_1, n_2 не парные $n -$ ти
 $n_1 \neq n_2 \Rightarrow \frac{n_1}{2} \neq \frac{n_2}{2}$
 $f(n) = -\frac{n-1}{2} = -\frac{-2a+1-1}{2} = -\frac{-2a}{2} = a \Rightarrow -\frac{n-1}{2} \neq -\frac{n_2-1}{2}$
 $f(n) = a \checkmark \quad f(n_1) \neq f(n_2)$

2) $a > 0 \quad n = 2a > 0 \quad \in \mathbb{N}$

$$f(n) = \frac{n}{2} = \frac{2a}{2} = a \quad f(n) = a \quad \checkmark \Rightarrow f \text{ — сюръективна}$$

$\Rightarrow f$ биективна $\Rightarrow \mathbb{N} \cong \mathbb{Z} \quad (\infty = 2 \cdot \omega + 1)$

$\mathbb{Q} \cong \mathbb{N} \quad (\infty^2 = \infty)$
 $(2^\infty > \infty)$
 \parallel
 $|\mathbb{R}|$

\mathbb{R}

Теорема $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$

Бесшаблонность: $\{0,1\}^{\mathbb{N}} = \{f: \mathbb{N} \rightarrow \{0,1\}\}$ $\in \mathbb{N}$ -го беск. функц.

$f: \{0,1\}$ симметрическое $\{(0,1,1,0,\dots), (\dots), \dots\}$
 бесшаблонное, бесконечное, бесконечно-многоточие, $\{(a_1, a_2, a_3, a_4, \dots) \mid a_i \in \{0,1\}\}$

$$g: \mathbb{N} \rightarrow \{0,1,2\} \leftrightarrow 0, 2, 1, 1, 0, 0, 0, 2, 0, \dots$$

$$g: \mathbb{N} \rightarrow \mathbb{R} \leftrightarrow x_1, x_2, x_3, \dots \in \mathbb{R}$$

Доказательство X отом $f: \{0,1\}^{\mathbb{N}} \rightarrow \{0,1\}^{\mathbb{N}}$

$$x = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

$$x = 0 + \frac{1}{4} + 0 + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots = \sum_{k=1}^{\infty} \frac{a_k}{2^k}, \quad a_k \in \{0,1\}$$

$(a_1, a_2, a_3, \dots) \in \{0,1\}^{\mathbb{N}} \overset{f(x)}{\sim}$

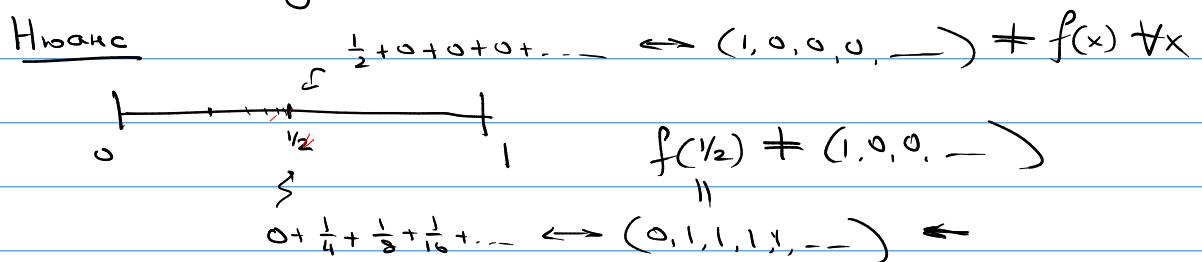
1) Универсальность $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (1)

$$\begin{aligned} & \stackrel{\text{"}}{(a_1, a_2, \dots)} \quad \stackrel{\text{"}}{(b_1, b_2, \dots)} \\ \rightarrow & (a_1, a_2, \dots) = (b_1, b_2, \dots) \Rightarrow a_k = b_k \forall k \\ \Rightarrow \sum_{k=1}^{\infty} \frac{a_k}{2^k} &= \sum_{k=1}^{\infty} \frac{b_k}{2^k} \Rightarrow x_1 = x_2 \quad \blacksquare \\ & \stackrel{\text{"}}{x_1} \quad \stackrel{\text{"}}{x_2} \end{aligned}$$

2) Сюръективность $(a_1, a_2, a_3, \dots) \in \{0, 1\}^{\mathbb{N}}$ $\exists x \in [0, 1] :$

$$f(x) = (a_1, a_2, \dots)$$

$$x = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \stackrel{?}{\in} [0, 1], \quad f(x) = (a_1, a_2, \dots) \times$$



"Левые" числа-точка имеют 0 в первом (0 начальная с чет. места)

$$(0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \Leftrightarrow (0, 0, 1, 1, 0, 1, 0, 1)$$

$$A \subset \{0, 1\}^{\mathbb{N}} \quad A = \{ \text{левые числа-точка} \} \cong \{ \text{лев. числа-точка из } 0 \text{ и } 1 \} \underbrace{\}_{B}$$

$$\begin{aligned} (1) \quad B &\cong \mathbb{N} \quad (2) \quad B - \text{бесконечно} \\ &\Rightarrow |B| \geq |\mathbb{N}| \end{aligned}$$

$$\begin{aligned} (2) \quad B &\hookrightarrow \mathbb{Q} \quad (a_1, a_2, \dots, a_n) \Leftrightarrow \frac{a_1}{2} + \frac{a_2}{4} + \dots + \frac{a_n}{2^n} \in \mathbb{Q} \\ &\text{вложение} \Rightarrow |B| \leq |\mathbb{Q}| = |\mathbb{N}| \leq |B| \Rightarrow |B| = |\mathbb{N}| \end{aligned}$$

B-счетно

$$\{0, 1\}^{\mathbb{N}} \setminus A \cong \{0, 1\}$$

$\overset{f}{\leftarrow}$

$1/2 \leftarrow \omega \tau 3$

$A \cong B$ — счетно

Теорема 3: $A \subseteq B$, B -счетно
 $\Rightarrow |A \cup B| = |A|$

$$(\{0, 1\}^{\mathbb{N}} \setminus A) \cup A = \{0, 1\}^{\mathbb{N}}$$

$$\Rightarrow \{0, 1\} \cong \{0, 1\}^{\mathbb{N}} \quad \blacksquare$$

Чт $\{0, 1\}^{\mathbb{N}}$ не счетно

Теорема $|N| < |R|$ ($\varphi_N : N \rightarrow R$ — инъекция, $b^0 : N \rightarrow \{0, 1\}^N$ — сюръекция)

1-го Лемма (доказательство методом Кантора): $|N| \leq |\{0, 1\}^N|$

2-го $\varphi_N : N \rightarrow \{0, 1\}^N$ — инъекция.

$$n \mapsto (0, 0, 0, \dots, 0, 1, 0, 0, \dots)$$

↑
n-ная
ногука

Допустим, что $b^0 : N \rightarrow \{0, 1\}^N$ — сюръекция

$$b^0(1) \quad a_{11} \ a_{12} \ a_{13} \ a_{14} \ \dots$$

$$b^0(2) \quad a_{21} \ a_{22} \ a_{23} \ a_{24} \ \dots$$

$$a \in \{0, 1\}$$

$$b^0(3) \quad a_{31} \ a_{32} \ a_{33} \ a_{34} \ \dots$$

$$\bar{a} = 1 - a$$

$$\left. \begin{array}{l} b_k = \bar{a}_{kk} \\ b \in \{0, 1\}^N \end{array} \right\}$$

$$b : \bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}, \bar{a}_{44}, \dots$$

$$b \neq a_1$$

$$\left. \begin{array}{l} b_1 = \bar{a}_{11} \neq a_{11} \\ b \neq a_2 \end{array} \right\}$$

$$\left. \begin{array}{l} b_2 = \bar{a}_{22} \neq a_{22} \\ b \neq a_n \end{array} \right\}$$

$$b_n = \bar{a}_{nn} \neq a_{nn}$$

Противоречие с определением b^0 .

(не находимся инъекции b^0)

(Было бы нарушение к теореме)

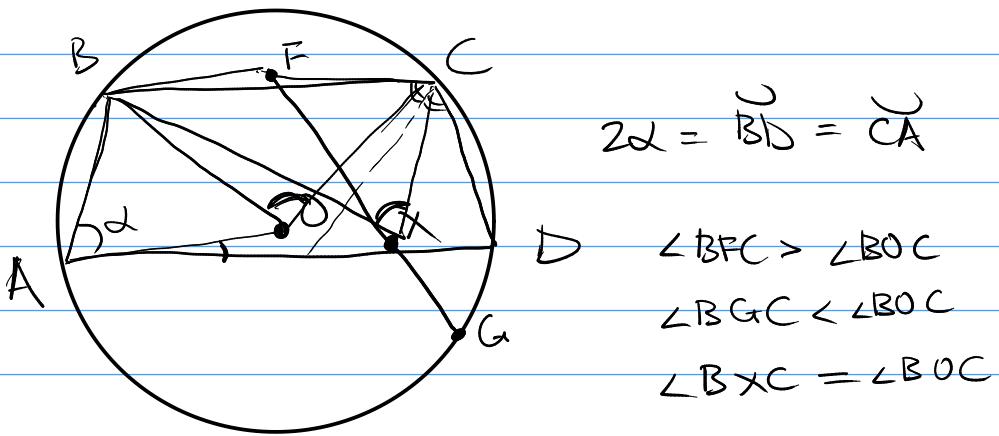
$$|N| < |\{0, 1\}^N| = |\{0, 1\}| \leq |R| \Rightarrow |N| < |R|$$

но нечно

$\{0, 1\} \subset R$



Следовательно $N \subset \{0, 1\}^N \subset \{0, 1\}^{(\{0, 1\}^N)}$ $\subset \dots$

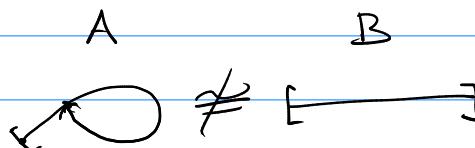


Топология

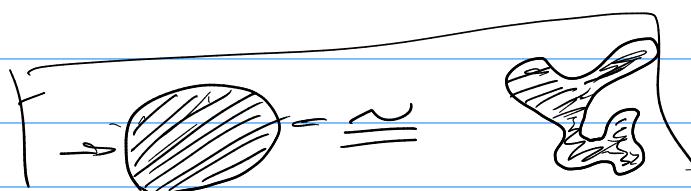
Утверждение 1: Равнодоминантные множества имеют одинаковые топологии.

$$\xrightarrow[0]{1} \cong \xrightarrow[0]{2} (\text{Top})$$

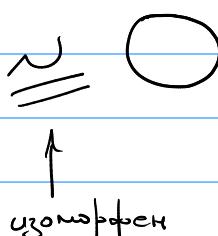
Утверждение 2: Непрерывность



$f: A \rightarrow B$ не непрерывно



? НЕ?



A^B

$$|2^X| = 2^{|X|} (X - \text{множн.})$$

Определение X - множество, $X \neq \emptyset$. $\Sigma \subset 2^X$ (Булевы - мн-во всех подмн-б)

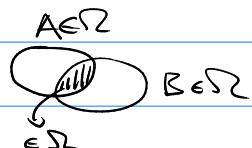
Σ @ топологией на X, если:

$$\textcircled{1} \emptyset, X \in \Sigma \quad \textcircled{2} A, B \in \Sigma \Rightarrow A \cap B \in \Sigma$$

$$\textcircled{3} f \subset \Sigma \Rightarrow \bigcup_{A \in \Sigma} f(A) \in \Sigma$$

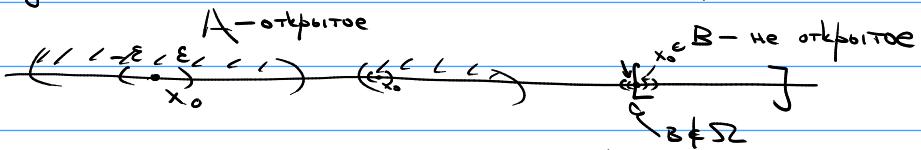
(в частности, $A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$)

$$\bigcup_{A \in \Sigma} A$$



В таком случае наз-ва $A \in \Sigma$ @ открытыми.

Пример: \mathbb{R}



$$\Sigma = \{A \subset \mathbb{R} \mid \forall x_0 \in A \exists \varepsilon > 0: (x_0 - \varepsilon, x_0 + \varepsilon) \subset A\}$$

$$\Leftrightarrow O_\varepsilon(x_0) = \text{---} \varepsilon \text{- окрестность точки } x_0$$

Задача $X = \mathbb{R}$, $\Sigma^1 = \{\emptyset\} \cup \{A \subset \mathbb{R} \mid A \text{ связочечно}\} \subset 2^{\mathbb{R}}$

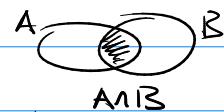
(1) Правда ли, что Σ^1 — топология?

$$\textcircled{1} \emptyset, \mathbb{R} \in \Sigma^1$$

$$\textcircled{2} A, B \in \Sigma^1 \Rightarrow A \cap B \in \Sigma^1 \quad A \cap B = \{x \mid x \in A \& x \in B\}$$

$$\textcircled{3} f \in \Sigma^1 \Rightarrow \bigcup f \in \Sigma^1$$

$$\textcircled{4} A, B \in \Sigma^1$$



$$1) A = \emptyset \vee B = \emptyset \Rightarrow A \cap B = \emptyset \in \Sigma^1 \quad \checkmark$$

$$2) A, B - связочечные \quad 2.1) A \cap B = \emptyset \in \Sigma^1 \quad \checkmark$$

$$2.2) A \cap B \neq \emptyset$$

$$\Sigma^1 - \text{не топология}$$

$$A = (-\infty, 1]$$

$$B = [1, +\infty)$$

$$A \cap B = \{1\}$$



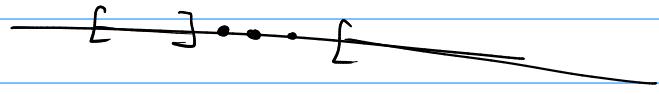
Одн X — мн-бо, $\Sigma \subset 2^X$

(1) система замкнутых мн-б, если

$$\textcircled{1} \emptyset, X \in \Sigma$$

$$\mathbb{R}$$

$$\textcircled{2} A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$$



$$\textcircled{3} f \subset \Sigma \Rightarrow \bigcup f \in \Sigma$$

Свойство ($\Sigma \leftrightarrow \Sigma'$)

$\nexists (X, \Sigma)$. $\Sigma := \{X \setminus A \mid A \in \Sigma'\}$ — сист. замк. мн-б

$\nexists (X, \Sigma)$. $\Sigma := \{X \setminus A \mid A \in \Sigma'\}$ — топология

1-бо утверждение
(уровень аксиомы)



Задача $X = [0, +\infty)$ $\Sigma = \{(a, +\infty) \mid a \geq 0\} \cup \{\emptyset\} \cup \{X\}$

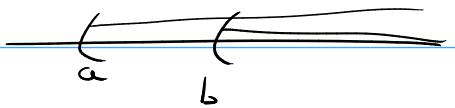
Правда ли, что Σ — топология?

① $\emptyset, X \in \Sigma \quad \checkmark$ (из оп.)

② $A, B \in \Sigma \Rightarrow A \cap B \in \Sigma?$ \checkmark

$$A = (a, +\infty), \quad B = (b, +\infty)$$

$$A \cap B = (\max(a, b), +\infty)$$



③ $f \in \Sigma \Rightarrow \cup f \in \Sigma$ — вып.

$$1. \text{ a)} \quad 2 \sin 2x + 2\sqrt{3} \sin x = 2 \cos x + \sqrt{3} \quad \rightarrow$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(2x) = \sin(x+x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$$

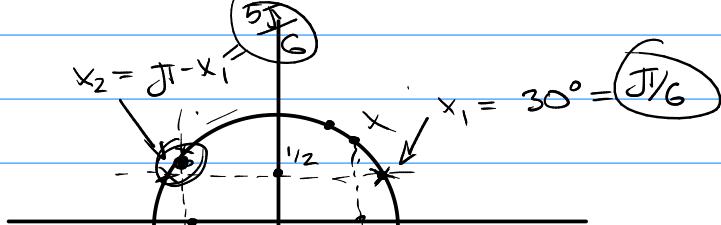
$$\underbrace{4 \sin x \cos x}_{A(x)} + \underbrace{2\sqrt{3} \sin x}_{B(x)} - \underbrace{2 \cos x}_{A(x)} - \underbrace{\sqrt{3}}_{B(x)} = 0$$

$$A(x) \cdot B(x) = 0 \Leftrightarrow \begin{cases} A(x) = 0 \\ B(x) = 0 \end{cases}$$

$$2 \cos x (2 \sin x - 1) + \sqrt{3} (2 \sin x - 1) = 0$$

$$= (2 \sin x - 1)(2 \cos x + \sqrt{3}) = 0$$

$$\Leftrightarrow \begin{cases} 2 \sin x = 1 \Leftrightarrow \sin x = \frac{1}{2} \\ 2 \cos x = -\sqrt{3} \Leftrightarrow \cos x = -\frac{\sqrt{3}}{2} \end{cases}$$



$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\cos(180^\circ - 30^\circ) = -\frac{\sqrt{3}}{2}$$

$$\frac{5\pi}{6}$$

$$x_3 = -x_2 = -\frac{5\pi}{6} + 2\pi$$

$$\frac{12\pi}{6} - \frac{5\pi}{6} = \frac{7\pi}{6}$$

$$[0, 2\pi)$$

$$\forall x \in \mathbb{R} \quad \cos(x) = \cos(-x)$$

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

$$x \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \right\} + 2\pi k, \quad k \in \mathbb{Z}$$

$$\textcircled{1} \quad A + \alpha := \{x + \alpha \mid x \in A\}$$

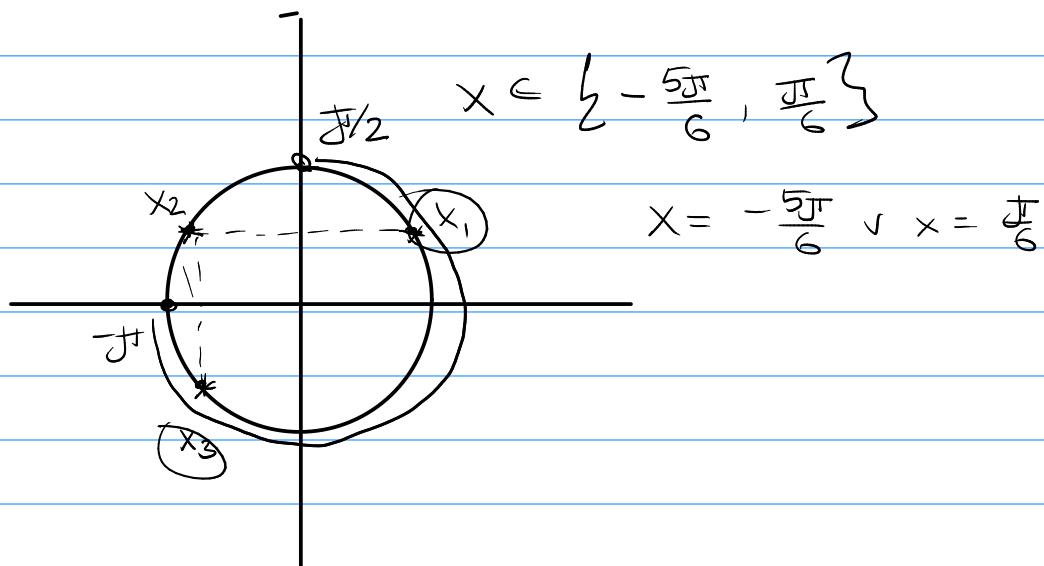
$$[0, 1) + 6 = [6, 7)$$

$$\textcircled{2} \quad A + 2\pi k, \quad k \in \mathbb{Z} = \bigcup_{k \in \mathbb{Z}} (A + 2\pi k)$$

$$x \in \bigcup_{k \in \mathbb{Z}} \left(\left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \right\} + 2\pi k \right) \quad - \text{nat. ganach}$$

$$x = \frac{\pi}{6} + 2\pi k \quad \vee \quad x = \frac{5\pi}{6} + 2\pi k \quad \vee \quad x = \frac{7\pi}{6} + 2\pi k \quad - \exists \Gamma \exists$$

5)



$$2 \log_2(2 \cos x) - 3 \log_2(2 \cos x) + 4 = 0$$

$$f_a(x) = a^x, \quad a > 0. \quad g_a(x) = \log_a x = f_a^{-1}(x)$$

$$a^{\log_a x} = x$$

$$\log_a(a^x) = x$$

Часть 1а задача

$$1. a^0 = 1 \quad \forall a > 0$$

$$2. a^1 = a \quad \forall a > 0$$

$$3. a^{x+y} = a^x \cdot a^y$$

$$4. a^x \cdot b^x = (ab)^x$$

$$5. (a^x)^y = a^{xy}$$

$0^0 = ?$ Не определено

e^x нечетно опред.
Через эту 3 сб-ка

Часть 1а корректна $\log_a a^x = x$ $a^{\log_a x} = x$

$$1. 0 = \log_a 1 \quad \forall a > 0$$

$$2. 1 = \log_a a \quad \forall a > 0$$

$$3. x+y = \log_a(a^x \cdot a^y)$$

$$\log_a(u \cdot v) = \log_a u + \log_a v$$

$\forall u, v > 0$

4. Задача

5. Задача

$$2 \underbrace{\log_2^2(2 \cos x)}_{\text{II (3)}} - 9 \underbrace{\log_2(2 \cos x)}_{\text{II (3)}} + 4 = 0$$

$$\underbrace{(\log_2 2 + \log_2(\cos x))^2}_{\text{II (2)}}$$

$$\underbrace{\log_2 2 + \log_2(\cos x)}_{\text{II (2)}}$$

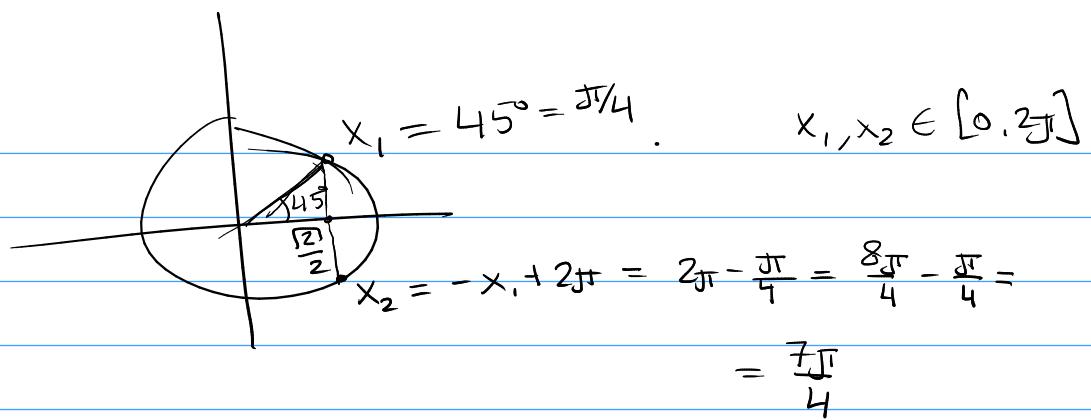
$$2 (\log_2(\cos x) + 1)^2 - 9 (\log_2(\cos x) + 1) + 4 = 0$$

$$2 \log_2^2(\cos x) + 4 \log_2(\cos x) + 2 - 9 \log_2(\cos x) - 9 + 4 = 0$$

$$2 \log_2^2(\cos x) - 5 \log_2(\cos x) - 3 = 0$$

$$(2 \log_2(\cos x) + 1) \cdot (\log_2(\cos x) - 3) = 0$$

$$\Leftrightarrow \begin{cases} \log_2(\cos x) = -\frac{1}{2} \iff \cos x = 2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \log_2(\cos x) = 3 \iff \cos x = 2^3 > 1 \end{cases} \times$$



$$x_1, x_2 \in [0, 2\pi]$$

$$= \frac{7\pi}{4}$$

Orts: $x \in \{\frac{\pi}{4}, \frac{7\pi}{4}\} + 2\pi k, k \in \mathbb{Z}$

und

$$x = \frac{\pi}{4} + 2\pi k \vee x = \frac{7\pi}{4} + 2\pi k, k \in \mathbb{Z}.$$

II

$$-\frac{\pi}{4} + 2\pi k$$

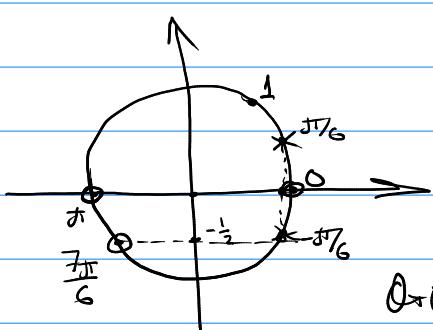
II

$$-\frac{9\pi}{4} + 2\pi k = -$$

3. $\frac{2\sin^2 x + \sin x}{2\cos x - \sqrt{3}} = 0 \quad (\Leftrightarrow)$

$$\frac{a}{b} = 0 \Leftrightarrow \begin{cases} a = 0 \\ b \neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2\sin^2 x + \sin x = 0 \Leftrightarrow \sin x (2\sin x + 1) = 0 \\ 2\cos x - \sqrt{3} \neq 0 \end{cases}$$



$$\cos x \neq \frac{\sqrt{3}}{2}$$

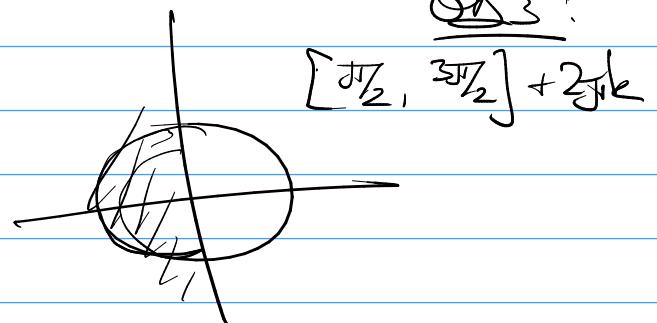
$$x \neq \frac{\pi}{6}, -\frac{\pi}{6} + 2\pi k$$

$$\begin{cases} \sin x = 0 \\ 2\sin x = -1 \\ \sin x = -\frac{1}{2} \end{cases}$$

Orts: $x \in \{0, \pi, \frac{7\pi}{6}\} + 2\pi k, k \in \mathbb{Z}$

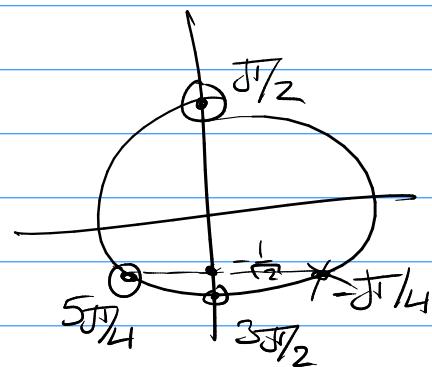
$$4. (\sqrt{2} \sin x + 1) \sqrt{-5 \cos x} = 0 \quad (\Leftrightarrow)$$

OK3: $-5 \cos x \geq 0 \Leftrightarrow \cos x \leq 0$



\Leftrightarrow

$$\begin{cases} \sin x = -\frac{1}{\sqrt{2}} \\ \cos x = 0 \end{cases}$$



OK4x: $x \in \{\frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}\} + 2\pi k, k \in \mathbb{Z}$

$$5. (\cos x - \sin 2x)(1 + \sqrt{1 + \tan^2 x}) = 0 \quad (\Leftrightarrow)$$

OK3: $\tan x \geq 0 \Leftrightarrow \frac{\sin x}{\cos x} \geq 0 \Leftrightarrow \begin{cases} \sin x \cdot \cos x \geq 0 \\ \cos x \neq 0 \end{cases}$

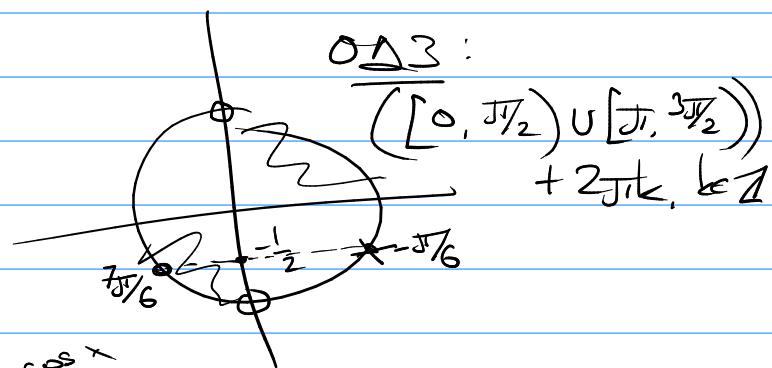
\Leftrightarrow

$$\begin{cases} \cos x = \sin 2x \\ 1 + \sqrt{1 + \tan^2 x} = 0 \quad X \\ \sqrt{1} = 1 \end{cases}$$

$$\cos x = \sin 2x = 2 \sin x \cos x$$

$$1 = 2 \sin x$$

$$\sin x = -\frac{1}{2}$$



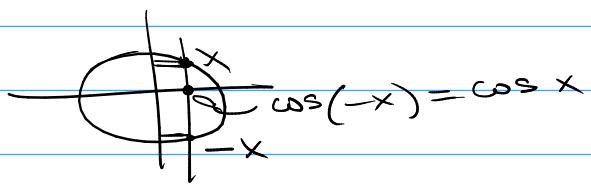
$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(2x) = 2 \sin x \cos x$$

OK4x: $x = \frac{7\pi}{6} + 2\pi k, k \in \mathbb{Z}$

$$6. \sin \frac{5x}{2} \cos \frac{3x}{2} = \frac{\sqrt{2}}{2} \cdot \sin 2x + \sin \frac{3x}{2} \cos \frac{5x}{2} \quad \text{↔}$$

ΩΔ3: R

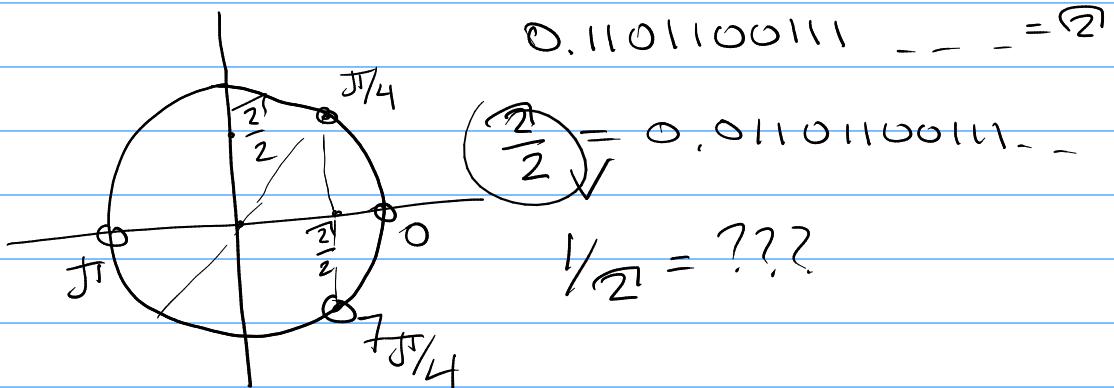


$$\begin{aligned}
 & \sin \alpha + \sin \beta = & \bar{\alpha} = \frac{\alpha + \beta}{2} \\
 & = \sin(\bar{\alpha} + \bar{\beta}) + \sin(\bar{\alpha} - \bar{\beta}) = & \bar{\beta} = \frac{\alpha - \beta}{2} \\
 & = \sin \bar{\alpha} \cos \bar{\beta} + \sin \bar{\beta} \cos \bar{\alpha} + & \alpha = \bar{\alpha} + \bar{\beta} \\
 & + \sin \bar{\alpha} \cos \bar{\beta} - \sin \bar{\beta} \cos \bar{\alpha} = & \beta = \bar{\alpha} - \bar{\beta} \\
 & = 2 \sin \frac{\alpha + \beta}{2} \cdot \cos \frac{\alpha - \beta}{2} &
 \end{aligned}$$

$$\Rightarrow \frac{1}{2} (\cancel{\sin 4x + \sin x}) = \frac{\sqrt{2}}{2} \sin 2x + \frac{1}{2} (\cancel{\sin 4x - \sin x})$$

$$\Leftrightarrow \sin x = \frac{\sqrt{2}}{2} \sin 2x = \frac{\sqrt{2}}{2} \cdot 2 \sin x \cos x \quad \text{↔}$$

$$\begin{cases} \sin x = 0 \\ 1 = \sqrt{2} \cdot \cos x \Leftrightarrow \cos x = \frac{\sqrt{2}}{2} \end{cases}$$



$$\text{Orts. } x \in \{0, \frac{\pi}{4}, \pi, \frac{3\pi}{4}\} + 2\pi k, k \in \mathbb{Z}$$

$$7. 36^{\sin 2x} = 6^{2 \sin x}$$

$\underbrace{\hspace{10em}}$

$$(6^2)^{\sin x}$$

||

$$36^{\sin x}$$

$a^p = b^q$
 $c^p = c^q$

$$(a > 0) \quad a^{f(x)} = a^{g(x)} \Leftrightarrow f(x) = g(x)$$

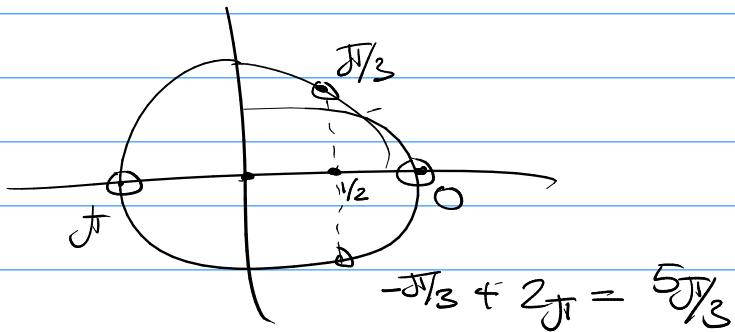
$a \neq 1$

$$\sin 2x = \sin x$$

||

$$2 \sin x \cos x$$

$\Leftrightarrow \begin{cases} \sin x = 0 \\ 2 \cos x = 1 \Leftrightarrow \cos x = \frac{1}{2} \end{cases}$

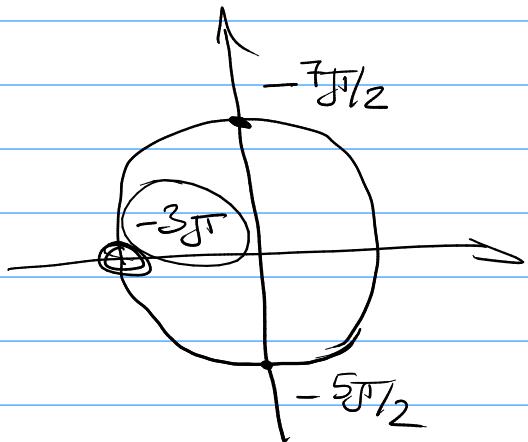


Observe: $x \in \{-\frac{7\pi}{2}, \pi, \frac{5\pi}{3}\} + 2\pi k, k \in \mathbb{Z}$.

$$8) \quad x \in \left[-\frac{7\pi}{2}, -\frac{5\pi}{2} \right]$$

$\frac{1}{2}$

$$-\frac{7\pi}{2} + 4\pi = \frac{\pi}{2}$$



$$8. \quad 2\sin^2 x - 2\cos 2x - \sin 2x = 0$$

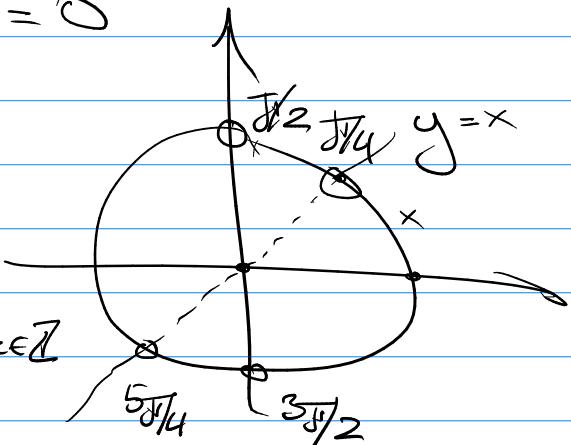
II

$$\cos(x+x) = \cos x \cos x - \sin x \sin x = \\ = \cos^2 x - \sin^2 x$$

~~(2)~~ $\cancel{2\cos^2 x} - \cancel{2\sin x \cos x} = 0$

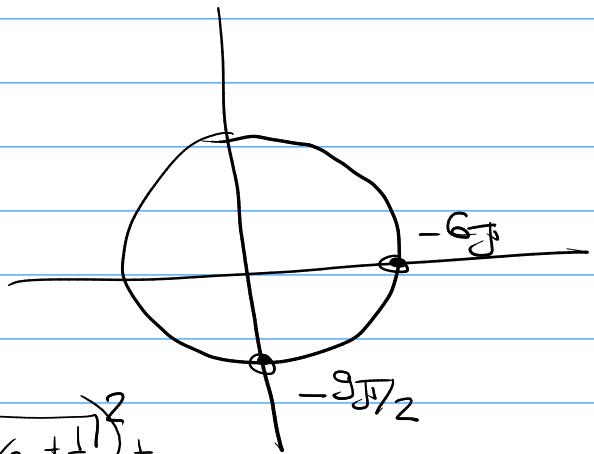
$$\cos x (\cos x - \sin x) = 0$$

$$\begin{cases} \cos x = 0 \\ \cos x = \sin x \end{cases}$$



Abez: $x \in \{\pi/4, \pi/2\} + \pi k, k \in \mathbb{Z}$

$$8) \quad x \in [-\pi, -\frac{9\pi}{2}]$$



$$9. \quad 3^{2x+1} - 4 \cdot 3^x + 4 = \left(\sqrt{-x^2 - x/2 + 1/2} \right)^2 + \\ x^2 + x/2 + 5/2$$

O13: $-x^2 - x/2 + 1/2 \geq 0$

$$2x^2 + x - 1 \leq 0$$

O13 $[-1, 1/2]$

$$2(x - \frac{1}{2})(x + 1) \leq 0$$



$$3^{2x+1} - 4 \cdot 3^x + 4 = 3$$

$$3^{2x+1} - 4 \cdot 3^x + 1 = 0$$

$$3^1 \cdot 3^{2x} - 4 \cdot 3^x + 1 \quad \leftarrow \quad a^p \cdot a^q = a^{p+q}$$

$$3 \cdot (3^x)^2 - 4 \cdot 3^x + 1 \quad \leftarrow \quad a^{pq} = (a^p)^q$$

$$y = 3^x$$

$$3y^2 - 4y + 1 = 0, \quad y > 0$$

$$(3y^2 - 3y) - (y - 1) = 0$$

$$3y(y-1) - (y-1) = 0$$

$$(3y-1)(y-1) = 0$$

$$y = 1 \quad \vee \quad y = \frac{1}{3}$$

$$3^x = 1$$

$$3^x = \frac{1}{3}$$

$$\boxed{x = 0}$$

$$\boxed{x = -1.}$$

✓

$$8) \quad x \in \left[\log_2 \frac{1}{6}, \log_2 \frac{2}{3} \right] ?$$

↑	↑	↓	-	✓
-1	0	-1	0	X

$$10. \quad 2 \log_4^2(\sin x) - x^2 + 21 = (\sqrt{25-x^2})^2 + 7 \log_4(\sin x)$$

$$\underline{\text{OBZ}}: \quad -5 \leq x \leq 5 \quad x \in [-5, 5]$$

~~$$2 \log_4^2(\sin x) - x^2 + 21 = 25 - x^2 + 7 \log_4(\sin x)$$~~

$$y = \log_4(\sin x)$$

$$2y^2 - 7y - 4 = 0$$

$$y = 4 \quad \vee \quad y = -\frac{1}{2}$$

$$1) y = 4 \Leftrightarrow \log_4(\sin x) = 4$$

$$\sin x = 4^4 = 256$$

X

$$2) \log_4(\sin x) = -\frac{1}{2}$$

$$4^{\log_4(\sin x)} = 4^{-\frac{1}{2}}$$

" " "

$$\sin x = \frac{1}{2}$$

$$\log_4 a = b :$$

$$4^b = a$$

$$x = \frac{\pi}{6} + 2\pi k \quad x = \frac{5\pi}{6} + 2\pi k$$

$$\frac{\pi}{3} > 3 \quad 2\pi > 6$$

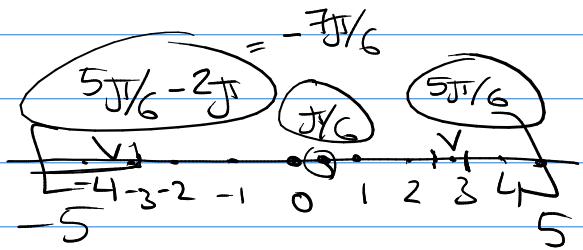
$$3 < \frac{\pi}{6} < 4$$

$$\frac{1}{2} < \frac{\pi}{6} < \frac{2}{3}$$

$$\frac{5\pi}{2} < \frac{5\pi}{6} < \frac{10}{3}$$

$$\frac{11}{2} < \frac{1}{2}$$

$$\frac{4}{3} < \frac{1}{3}$$



$$\cancel{x} \quad 2\pi < 7$$

$$-2\pi > -7$$

$$\frac{5\pi}{6} > 2$$

$$\frac{5\pi}{6} - 2\pi > -5$$

Пределы

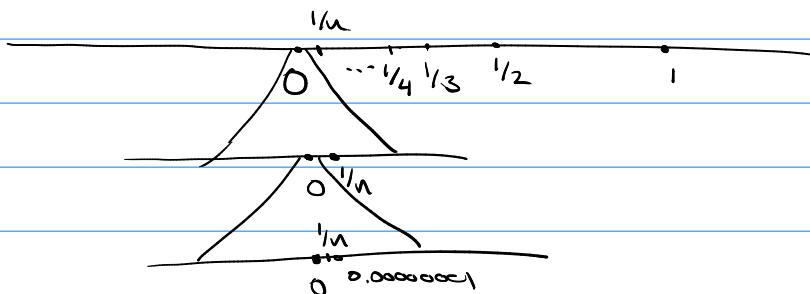
1. Пределы последовательностей

$$(a_n) \in \mathbb{R}$$

$$a_1, a_2, a_3, a_4, \dots \in \mathbb{R}$$

$$a: \mathbb{N} \rightarrow \mathbb{R}$$

$$a_n = a(n)$$

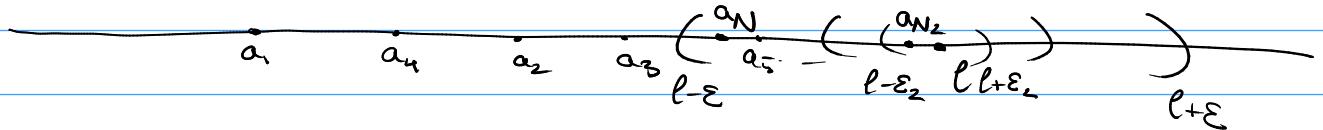


$$a_n = \frac{1}{n} \in \mathbb{R}$$

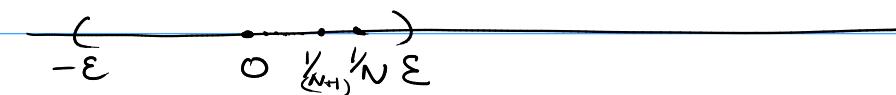
Онп: $(a_n) \in \mathbb{R}$, $l \in \mathbb{R}$ \Leftrightarrow нреженом $(a_n) \Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N}:$

$$n \geq N \Rightarrow |a_n - l| < \varepsilon$$

$$a_n \in (l - \varepsilon, l + \varepsilon)$$



Пример $a_n = \frac{1}{n}$, $l = 0$ — нрежен $(\frac{1}{n})$ ($\frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$)



$$\varepsilon = \frac{1}{1000}$$

$$\Rightarrow \varepsilon > 0, \quad a_N = \frac{1}{N} < \varepsilon \Leftrightarrow N > \frac{1}{\varepsilon} \quad \frac{1}{\varepsilon} = 1000 \quad \lceil \frac{1}{\varepsilon} \rceil = 1000$$

$$N = \lceil \frac{1}{\varepsilon} \rceil + 1 > \lceil \frac{1}{\varepsilon} \rceil \geq \frac{1}{\varepsilon}$$

$$N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$$

$$(1) \quad \forall n \geq N \quad \frac{1}{n} < \varepsilon$$

$$n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon$$

$$\Rightarrow \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$$

Примеры:

$$1) \quad a_n = 1 - \frac{1}{2^n} \xrightarrow{n \rightarrow \infty} 1 \quad |a_n - 1| = \left| 1 - \frac{1}{2^n} - 1 \right| = \left| -\frac{1}{2^n} \right| =$$

$$\underset{n \rightarrow \infty}{\text{означает что}} \underset{\text{сделано умножение на } 2^n}{\text{стремится к 0}} \quad \frac{1}{2^n} \xrightarrow{0}$$

$$\forall \varepsilon > 0 \quad \exists N: \quad n \geq N \Rightarrow \frac{1}{2^n} < \varepsilon$$

$$2) \quad a_n = 5 + \frac{2 - \frac{1}{n^3}}{6 + \frac{1}{n}} \xrightarrow{n \rightarrow \infty} \frac{16}{3} \quad \underset{\log_2 \frac{1}{\varepsilon}}{\lceil \log_2 \frac{1}{\varepsilon} \rceil + 1}$$

$$N > \lceil \log_2 \frac{1}{\varepsilon} \rceil \geq \log_2 \frac{1}{\varepsilon}$$

$$2^N > \frac{1}{\varepsilon} \quad n \geq N \Rightarrow \frac{1}{2^n} \leq \frac{1}{2^N} < \varepsilon$$

$$\frac{1}{2^N} < \varepsilon \quad a_n \rightarrow 1.$$

Утб: $a_n \rightarrow l \in \mathbb{R}$, $b_n \rightarrow k \in \mathbb{R}$. Тогда:

$$1) (a_n + b_n) \rightarrow (l+k)$$

$$2) (a_n - b_n) \rightarrow (l-k)$$

$$3) (a_n b_n) \rightarrow lk$$

$$4) \frac{a_n}{b_n} \rightarrow \frac{l}{k} \quad (\text{если } b_n \neq 0, \text{ то } k \neq 0)$$

Д-бо: 1) $\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n \geq N |(a_n + b_n) - (l+k)| < \varepsilon$ — это значит

$$\Rightarrow \varepsilon > 0 \exists N_a \in \mathbb{N} : \forall n \geq N_a |a_n - l| < \varepsilon_a = \frac{\varepsilon}{2}$$

$$\Rightarrow \varepsilon_a = \frac{\varepsilon}{2} \exists N_a \in \mathbb{N} : \forall n \geq N_a |a_n - l| < \varepsilon_a = \frac{\varepsilon}{2}$$

$$b_n \rightarrow k$$

$$\Rightarrow \varepsilon_b = \frac{\varepsilon}{2} \exists N_b \in \mathbb{N} : \forall n \geq N_b |b_n - k| < \varepsilon_b = \frac{\varepsilon}{2}$$

$$N = \max\{N_a, N_b\}$$

$$\Rightarrow n \geq N \Rightarrow n \geq N_a, n \geq N_b \Rightarrow$$

\downarrow

$$|b_n - k| < \frac{\varepsilon}{2}$$

$$|a_n - l| < \frac{\varepsilon}{2}$$

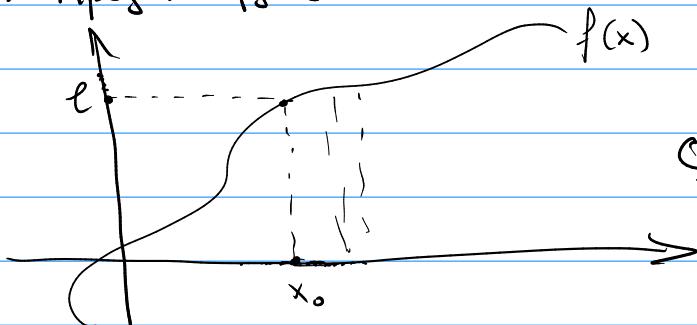
$$|(a_n + b_n) - (l+k)| = |(a_n - l) + (b_n - k)| \leq |a_n - l| + |b_n - k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\wedge \frac{\varepsilon}{2} \quad \wedge \frac{\varepsilon}{2} \quad || \varepsilon$

Тогда $(a_n + b_n) \rightarrow (l+k)$



2. Предел функции

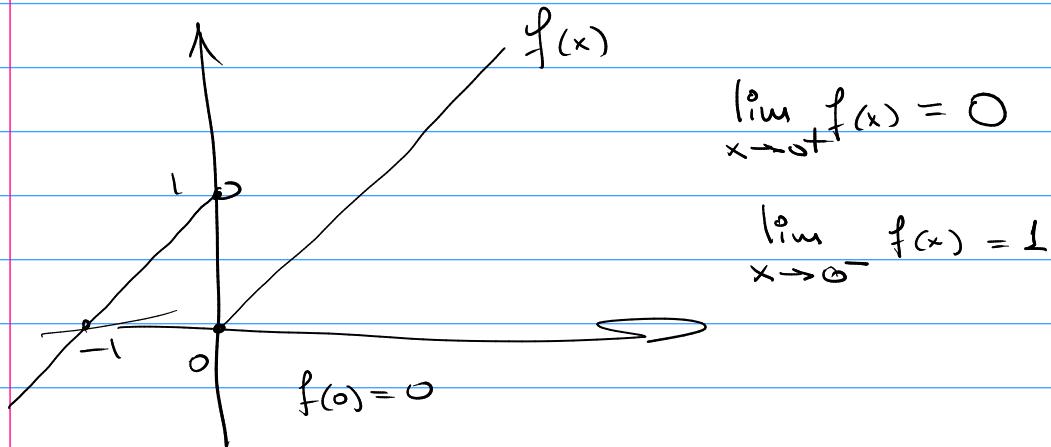
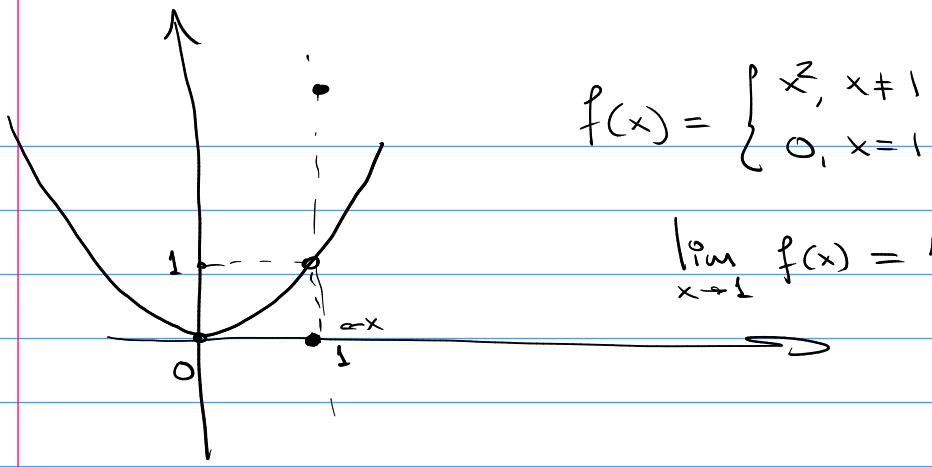


$$\lim_{x \rightarrow x_0} f(x)$$

Онт: $l \in \mathbb{R}$ @ непрерывной $f(x)$

если $x \rightarrow x_0$

$$\forall a_n \rightarrow x_0, f(a_n) \rightarrow l.$$



Teorema: $f, g: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) \xrightarrow{x \rightarrow x_0} l$, $g(x) \xrightarrow{x \rightarrow x_0} k$ —
то га

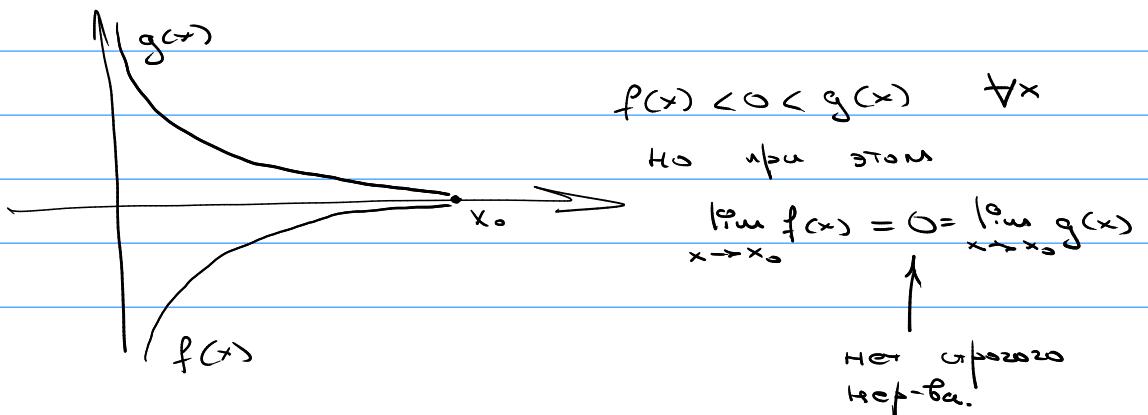
- 1) $(f(x) + g(x)) \xrightarrow{x \rightarrow x_0} l+k$
- и- $f(x)g(x)$
- и- $\frac{f(x)}{g(x)}$ (если $k \neq 0$)

Teorema $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) \xrightarrow{x \rightarrow x_0} l$, $h(x) \xrightarrow{x \rightarrow x_0} l$,
 $\exists \varepsilon > 0: \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon) \quad f(x) \leq g(x) \leq h(x)$

то га $g(x) \xrightarrow{x \rightarrow x_0} l$ (теорема о граничных неравенствах)

Teorema: $f, g: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) \xrightarrow{x \rightarrow x_0} l$, $g(x) \xrightarrow{x \rightarrow x_0} k$
 $\exists \varepsilon > 0: \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon) \quad f(x) \leq g(x)$.

то га $l \leq k$.



Задача

$$\frac{x+1}{x^2-1} \quad (x \rightarrow -1)$$

$\frac{0}{0} ?$

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1} = \frac{1}{-2} = -\frac{1}{2}$$

Теорема: $f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \dots + b_m x^m}$ — равноточная формула
(частные граничные случаи.)

Тогда:

1) $n=m$. Тогда $\lim_{x \rightarrow +\infty} f(x) = \frac{a_n}{b_n} = \frac{a_n}{b_n}$

2) $n > m$ Тогда $\lim_{x \rightarrow +\infty} f(x) = +\infty \cdot \operatorname{sign} \frac{a_n}{b_n}$

3) $n < m$ Тогда $\lim_{x \rightarrow +\infty} f(x) = 0$.

$$\begin{cases} -1, & \frac{a_n}{b_n} < 0 \\ 1, & \frac{a_n}{b_n} > 0 \end{cases}$$

1) $n=m$

$$f(x) = \frac{a_0 + a_1 x + \dots + a_n x^n}{b_0 + b_1 x + \dots + b_n x^n} = \frac{\overbrace{a_0/x^n + a_1/x^{n-1} + \dots + a_{n-1}/x}^0 + a_n}{\underbrace{b_0/x^n + \dots + b_n/x^n}_{\rightarrow 0} + b_n} \rightarrow \frac{a_n}{b_n}$$

$x \rightarrow +\infty \Rightarrow \frac{1}{x} \rightarrow 0 \Rightarrow \frac{1}{x^k} \rightarrow 0 \quad (\forall k \in \mathbb{N})$

$\Rightarrow \frac{a_{n-k}}{x^k} \rightarrow 0 \quad (\forall k > 0)$

2) $n > m$

$$f(x) = \frac{a_0 + a_1 x + \dots + a_n x^n}{b_0 + b_1 x + \dots + b_m x^m} = \frac{\overbrace{a_0/x^m + a_1/x^{m-1} + \dots + a_{m-1}/x}^0 + a_m}{\underbrace{b_0/x^m + b_1/x^{m-1} + \dots + b_{m-1}/x}^0 + b_m} \rightarrow \frac{a_m}{b_m}$$

$a_{m+1} x + a_{m+2} x^2 + \dots + a_n x^{n-m} =$

$$= x^{n-m} \left(\frac{a_{m+1}}{x^{n-m-1}} + \frac{a_{m+2}}{x^{n-m-2}} + \dots + \frac{a_{n-1}}{x} + a_n \right) \rightarrow$$

$\infty \quad (n > m)$

$\rightarrow a_m \cdot (+\infty) = \operatorname{sign} a_m \cdot (+\infty)$

$$f(x) \rightarrow \frac{\text{sign } a_n \cdot +\infty}{b_m} = \frac{\text{sign } a_n}{\text{sign } b_m} \cdot (+\infty) = \text{sign } \frac{a_n}{b_m} \cdot (+\infty)$$

$$\frac{-x^2 + 5x - 20000}{-x + 1000000} \rightarrow +\infty$$

3) $n < m$

$$f(x) = \frac{a_0 + a_1 x + \dots + a_n x^n}{b_0 + b_1 x + \dots + b_m x^m} = \frac{g(x)}{h(x)}$$

$g(x) \rightarrow \pm \infty$

$f(x) = \frac{1}{g(x)} \xrightarrow{x \rightarrow +\infty} 0$

$$\lim_{x \rightarrow +\infty} \frac{2x^2 - 3x - 5}{1 + x + 3x^2} \xrightarrow{(x \rightarrow +\infty)} \frac{2}{3}$$

$$2x^2 - 3x - 5 \rightarrow 2 \cdot (-1)^2 - 3(-1) - 5 = 2 + 3 - 5 = 0$$

$$x+1 \rightarrow ?$$

$$2x^2 - 3x - 5 = 2(x - x_1)(x - x_2)$$

$$\frac{2(x+1)(x - 5/2)}{x+1} \xrightarrow{=} 0$$

$$\begin{array}{r} 2x^2 - 3x - 5 \\ -2x^2 + 2x \\ \hline -5x - 5 \\ -5x - 5 \\ \hline 0 \end{array}$$

$$2(x+1)(x - 5/2)$$

$$2(-1 - 5/2)$$

//

$$-2 - 5 = -7$$

$$\cancel{x+2} \frac{8-2x^2}{x^2+4x-12} = \frac{2(4-x^2)}{x^2+4x-12} = \frac{2(2-x)(2+x)}{(x-2)(x+6)} = -\frac{2(2+x)}{x+6}$$

$$\begin{array}{r} x^2+4x-12 \\ -x^2-2x \\ \hline 6x-12 \\ \hline 6x-12 \\ \hline 0 \end{array} \quad \begin{array}{l} |x-2| \\ |x+6| \\ \downarrow \\ x \rightarrow 2 \end{array}$$

$$-\frac{8}{8} = -1$$

$$\frac{\sqrt{x+6} - \sqrt{10x-21}}{5(x-3)} = (x \rightarrow 3)$$

$$= \frac{(\sqrt{x+6} - \sqrt{10x-21})(\sqrt{x+6} + \sqrt{10x-21})}{5(x-3)(\sqrt{x+6} + \sqrt{10x-21})} =$$

$a^2 - b^2 = (a-b)(a+b)$
 xorum unum

$$= \frac{x+6 - (10x-21)}{5(x-3)(\sqrt{x+6} + \sqrt{10x-21})} = \frac{-9x + 27}{5(x-3)(\dots)} = -\frac{9}{5} \cdot \frac{x-3}{(x-3)(\dots)} =$$

$$= -\frac{9}{5} \cdot \frac{1}{(\dots)} \xrightarrow{x \rightarrow 3} -\frac{9}{5} \cdot \frac{1}{6} = \boxed{-\frac{3}{10}}$$

$$\frac{x^2+x-2}{\sqrt{x+6}-2} \quad (x \rightarrow -2)$$

$$\frac{\cancel{(x+2)(x-1)} \cdot \cancel{(x+6+2)}}{\cancel{x+6-4} \cancel{x+2}} =$$

$$\begin{array}{r} x^2+x-2 \\ -x^2-2x \\ \hline -x-2 \\ -x-2 \\ \hline 0 \end{array}$$

$$= (x-1) \cancel{(\sqrt{x+6}+2)} \xrightarrow{x \rightarrow -2} -12.$$

$$2x_{v_1} + x \cdot x_{v_2} = x_{v_3}$$

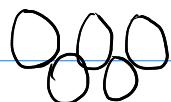
$$2y_{v_1} + x \cdot y_{v_2} = y_{v_3}$$

$$x \cdot v_2 = v_3 - 2v_1$$

$$v_1 \cdot v_2 = x_{v_1} \cdot x_{v_2} + y_{v_1} \cdot y_{v_2} \in \mathbb{R}$$

$$|v_1| \cdot |v_2| \cdot \cos \alpha \Rightarrow \cos \alpha = \frac{v_1 \cdot v_2}{|v_1| \cdot |v_2|}$$

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha}$$



1 форма: 95%

2 форма: 55%

$$P(\text{выиграл}) = 0.75$$

75% выиграл - выиграл.

x - вероятность выиграть

y - вероятность проиграть

$$0.35x + 0.45y = 0.75(x+y)$$

$$0.75x + 0.75y$$

$$0.2x = 0.2y$$

$$\frac{x}{y} = \frac{3}{2}$$

$$P(1 \text{ форма}) = \frac{3}{5}$$

$$\overline{abc}$$

$$\overline{a} \overline{b}$$

$$\overline{a} \overline{c}$$

$$(?) \quad \exists \quad \overline{abc} > 140:$$

$$\overline{abc} = \overline{a_1 b_1} \cdot \overline{a_2 b_2}$$

$$625 = 25 \cdot 25$$

$$143$$

$$\sqrt{143} < 12$$

$$440 - 500$$

$$440 \leq \overline{abc} \leq 500$$

$$\overline{a_1 b_1} \cdot \overline{a_2 b_2} \geq 40 \cdot 40 =$$

$$= 1600 > 500$$

$$\times$$

$$\begin{aligned} a &\geq 4 \\ b &\geq 4 \\ 0 &\leq c \leq 9 \end{aligned}$$

$$\begin{aligned} \overline{a_1 b_1} &\geq 40 \\ \overline{a_2 b_2} &\geq 40 \end{aligned}$$

$$\frac{e^{2x}}{x^{40}} \xrightarrow[x \rightarrow +\infty]{} 0$$

$$\frac{e^{\frac{x}{1000}}}{x} \xrightarrow[x \rightarrow +\infty]{} 0$$

$$\frac{x^7 + e^{2x}}{e^{3x}} \xrightarrow[x \rightarrow +\infty]{} 0$$

$$\frac{x^n}{e^{dx}} \xrightarrow[x \rightarrow +\infty]{} 0$$

$e \approx 2.718281828$

$\frac{x^7}{e^{3x}} + \frac{e^{2x}}{e^{3x}}$

$\frac{e^{2x} \cdot e^{-3x}}{e^{2x} \cdot e^{-3x}} = e^{-x} \xrightarrow[x \rightarrow +\infty]{} 0$

$\forall \alpha > 0$

$\forall n \in \mathbb{N}$

$$f(x) = \frac{\sqrt{e^{4x} + x^{10}}}{e^{2x}} \xrightarrow[x \rightarrow +\infty]{} ?$$

$$\frac{n+1}{n} \rightarrow 1$$

$$\frac{\sqrt{e^{4x} + x^{10}}}{e^{4x}} = \sqrt{1 + \left(\frac{x^{10}}{e^{4x}}\right)} \xrightarrow[x \rightarrow +\infty]{} 1$$

$$\frac{f(x)}{g(x)}$$

$$f(x) \geq g(x)$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} \geq 1 \quad (g(x) > 0)$$