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Project outside the course scope

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Introduction to Modal logic

A study of possible worlds and provability logic

Date: November 8, 2021

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1 Introduction

This project will be about *modal logic*. *modal logic* is a study in the cross-field between philosophy, mathematics and computer science. In analytical philosophy it is an important tool in discussions about such diverse topics as: metaphysics (what is necessarily and what is possible?), deontic ethics (What am i obliegted to do?) and epistemology (What is knowledge and what is belief?). In mathematics is has proven a useful tool in understanding provability theory and in computer sciences it has been used to understand how a computer process or could processes different things. It has even found use in linguistics.

So what is modal logic and why can it be used in such diverse areas? Shortly started: *modal logic is the study of relational logical structures*, and since each of these areas deal or can deal with relational structures, it is clear that the tools modal logic provides can be used so wide variety off different cases. In this project it will be shown how modal logic can be a tool in both philosophy and mathematics.

In this project we will only look at propositional modal logic, i.e modal logic were we ”expand” our well-know propositional logic with a ’modal’ symbol.

Lastly it shall be noted that this project will mostly follow Lemmon [1977] and Blackburn [2002]. Some part of the section on provability logic will follow Liu [2013].

We will start of the project with a short historical introduction to the subject.

2 Historical introduction

Modalities in logic has been a theme at least since the time of Aristoteles. To begging with philosophers looked at the modalities *necessarily* and was is merely *possible*. Leibniz was the first to give a somewhat clear definition of when a proposition was necessarily true: a proposition is necessarily true if it is true in every possible world.

But this definition was still to vague too use for a ”orrrrd” study of modalities. It was first in the late fifties with the work of Jaakoo Hintikka og Saul Kripke[1] that there was developed a precise semantic for modal logics that made it possible to prove completeness theorems. The main goal of this project will be to show different completeness theorems for different modal logics, and this is only possible by the work done by Hintikka and Kripke.[2]

3 The Modal language, frames and models

3.1 Modal language

As noted in the introduction, the study of this project will be about proportional modal logic; i.e modal logic without quantifiers. So we will start with a set called Φ which consists of propositional letter $p, q, \ldots$. We will start of by stating our *primitive symbols* of our modal language.

[1]Kripke wrote his first and extremely influential paper on the subject when he was still in high school

[2]A deeper look at the history of modal logic can be found in the introduction of Lemmon [1977] and in Goldblatt [2003]

Definition 1. We have the following primitive symbols in our language:

i Every propositional letter from our set Φ

ii The logical constants $\perp$ (zero-ary) and $\rightarrow$ (binary)

iii The model operator $\Box$ (unary)

$\dashv$

Now we can define the well formed formulas (*wff*) of this language. In the rest of the project we will call the well formed formulas of our language for *modal formulas* or just formulas.

Definition 2. We define a *modal formulas*recursively as follows:

i A proportional letter p is a *modal formula*

ii $\perp$ is a *modal formula*

iii If φ and ψ are *modal formulas* then $\varphi \rightarrow \psi$ is a *modal formula*

iv If φ is a *modal formula* then $\Box\varphi$ is a *modal formula*

v Nothing is a formula except as prescribed by (i)-(iv)

$\dashv$

Here $\Box\varphi$ should be read as φ is necessarily true. From this point on we will use the Greek lowercase letters: $\varphi,\ \psi,\ \vartheta, \ldots$ as meta-logical variable. In particular we will use them to denoted modal formulas. The uppercase Greek letters Γ, Θ and so on will denoted sets of modal formulas Further we define $\wedge,\ \neg,\ \vee$ and $\leftrightarrow$ in the following way:

$$
\begin{aligned}
\neg\varphi &:= \varphi \rightarrow \perp \\
\varphi \vee \psi &:= \neg\varphi \rightarrow \psi \\
\varphi \wedge \psi &:= \neg(\neg\varphi \vee \neg\psi) \\
\varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)
\end{aligned}
$$

These definition should be know from propositional logic. We also define the following symbols which are specific to modal logic:

- $\Diamond\varphi = \neg\Box\neg\varphi$.
- $\Box^n\varphi = \underbrace{\Box\cdots\Box}_{n}\varphi$
- $\Diamond^n\varphi = \underbrace{\Diamond\cdots\Diamond}_{n}\varphi$

Here $\Diamond\varphi$ should be read as φ is possible. And for example $\Box^n\varphi$ is that it is necessarily that it φ is necessarily true.

3.2 Hintikka frames and Kripke models

We have now defined our *modal language*. It is now possible for us to define the notion of a Hintikka Frame and a Kripke Model.[3] When we have defined these two, we can finally define the notion of truth

Definition 3 (Hintikka frame)**.** A Hintikka frame is a tuple $\mathcal{H} = \langle W, R\rangle$ where W is a non-empty set (we will call its members for worlds) and where $R \subseteq W \times W$ is a relation that we will call the accessibility relation. We will use the notation wRv to say that w is related to v, or that v is accessible from w. We do not put any conditions on our relation R at this point. $\dashv$

Having defined the notion of a frame we will now define a Kripke model.

Definition 4 (Kripke model)**.** A Kripke model is a tuple $\mathcal{K} = \langle \mathcal{H}, \phi\rangle$, where $\mathcal{H}$ is a Hintikka frame and ϕ is function that for each proportional letter $p \in \Phi$ assign a subset $\phi(p)$ of W. Formally

$$\phi : \Phi \to \mathcal{P}(W)$$

$\dashv$

We can now define the what it means for a modal formula φ to be true at a world w in a Kripke model $\mathcal{K} = \langle \mathcal{H}, \phi\rangle$; we abbreviate this to:

$$\models^{\mathcal{K}}_{w} \varphi$$

like our definition of our language this definition will also be done recursively.

Definition 5 (Truth definition)**.** We define the notion of truth as follows:

(A) if φ is a propositional letter p then:

$$\models^{\mathcal{K}}_{w} \varphi \text{ iff } w \in \phi(p)$$

(B) If φ is $\perp$ then:

$$\models^{\mathcal{K}}_{w} \varphi \text{ iff never}$$

(C) If φ is $\psi \to \vartheta$ then:

$$\models^{\mathcal{K}}_{w} \varphi \text{ iff if } \models^{\mathcal{K}}_{w} \psi \text{ then } \models^{\mathcal{K}}_{w} \vartheta$$

(D) if φ is $\Box\psi$ then:

$$\models^{\mathcal{K}}_{w} \varphi \text{ iff for all } v \text{ such that } wRv \ \models^{\mathcal{K}}_{v} \psi$$

$\dashv$

The last bullet-point exactly describes our intuition about necessity; a formula $\Box\varphi$ is necessarily true in world w if φ is true in all worlds v that are accessible from w. In the case (B) we will also say that not $\models^{\mathcal{K}}_{w} \varphi$ and use the following notation if this is the case: $\not\models^{\mathcal{K}}_{w} \varphi$. It is now also clear that we can say that if φ has the form $\Diamond\psi$ then:

[3]The usual name for a Hintikka Frame is actually a Kripke Frame, but it was Hintikka that was the first to use the notion of this entity in modal logic

(E) $\models_w^{\mathcal{K}} \varphi$ iff there exists v such that wRv and $\models_v^{\mathcal{K}} \psi$

Which states that a formula $\Diamond\varphi$ is true in world w if there is a world v that is accessible from w where φ is true. We can also generalize our truth definition to $\Box^n$ and $\Diamond^n$. Here we define R^n as the relative product of R with itself n times, so we get that:

$$wR^0v \text{ iff } w = v$$
$$wR^{n+1}v \text{ iff } \exists v'(wRv' \wedge v'R^nv)$$

With this definition we get the following:

Proposition 1. *For a modal formula φ we have the following:*

(F) $\models_w^{\mathcal{K}} \Box^n\varphi$ *iff* $\forall v(wR^nv \rightarrow \models_v^{\mathcal{K}} \varphi)$

(G) $\models_w^{\mathcal{K}} \Diamond^n\varphi$ *iff* $\exists v(wR^nv \wedge \models_v^{\mathcal{K}} \varphi)$

Here it should be noted that we onwards will use propositional and first order predicate logic notation *informally* in our metalanguage. We do this to shorten and clarify our statements. It should be clear when we use propositional logic in our metalanguage and not in our modal language. Since we have not defined quantifiers in our language it will be clear that those symbols will only be used in our metalanguage.

Proof. We can prove (F) by induction on n.
Base case
If $n = 0$ we get that

$$\models_w^{\mathcal{K}} \varphi \text{ iff } \forall v(w = v \rightarrow \models_v^{\mathcal{K}} \varphi)$$

And this is trivially true.
Induction step Now assume that it is true for n. Then by (D) we have that $\models_w^{\mathcal{K}} \Box^{n+1}\varphi$ iff $\forall v(wRv \rightarrow \models_v^{\mathcal{K}} \Box^n\varphi)$ and this is true if and only if

$$\forall v(wRv \rightarrow \forall v'(vR^nv' \rightarrow \models_{v'}^{\mathcal{K}} \varphi))$$

by our induction hypothesis. This statement can be change to the following by quantifier reasoning:

$$\forall v'\exists v((wRv \wedge vR^nv') \rightarrow \models_{v'}^{\mathcal{K}} \varphi)$$

And by our definition of R^{n+1} this is:

$$\forall v(wR^{n+1}v \rightarrow \models_v^{\mathcal{K}} \varphi)$$

So we have now shown (F), and (G) follows from (F). The proof of the last part should be clear and is omitted here. ⊣

Having defined the notion of truth in a given Kripke Model, we will now define the concepts *valid* and *satisfiable*.

Definition 6. For a given Hintikka frame $\mathcal{H} = \langle W, R\rangle$ we say that φ is valid in $\mathcal{H}$ (In this case we write: $\models^{\mathcal{H}} \varphi$) if and only if $\models_w^{\mathcal{K}} \varphi$ for all Kripke Models on our frame $\mathcal{H}$ and all worlds $w \in W$. φ is satisfiable in $\mathcal{H}$ if and only if $\models_w^{\mathcal{K}} \varphi$ for some Kripke Model $\mathcal{K}$ on Hintikka frame $\mathcal{H}$ and some world $w \in W$.
Furthermore: φ is valid if and only if φ is valid in all frames $\mathcal{H}$ and satisfiable if and only if φ is satisfiable on all frames $\mathcal{H}$. If φ is valid we will simply just write: $\models \varphi$. ⊣

We can also define the notion of validity for a Kripke Model. We will also use this notion in the project, since it often convenient.

Definition 7. We say that φ is valid in a Kripke model $\mathcal{K} = \langle W, R, \phi \rangle$ if and only if $\vDash_w^{\mathcal{K}} \varphi$ for all worlds $w \in W$. In this case we will write: $\vDash^{\mathcal{K}} \varphi$. $\dashv$

3.3 Modal logics

We will end this chapter with the definition of a modal logic and some general notions about these:

Definition 8. A modal logic Λ is a set of modal formulas that contains all propositional tautologies[4] and is closed under *modus ponens* (MP) and uniform substitution: i.e if $\varphi \in \Lambda$, and if ψ is obtained by replacing proposition letters in φ with arbitrary formulas then ψ is in Λ.. If $\varphi \in \Lambda$ we say that φ is a theorem of Λ and write $\vdash_\Lambda \varphi$, else we write $\nvdash_\Lambda$.

If Λ_1 and Λ_2 are modal logics such that $\Lambda_1 \subseteq \Lambda_2$ we say that Λ_2 is an extension of Λ_1. In the rest of the text we will drop the word "modal" in the most cases, and simply talk of "logics".

Lastly we also define Λ_{S} to be $\{\varphi | \vDash^{\mathfrak{S}} \varphi$, for all structures $\mathfrak{S} \in \mathrm{S}\}$. Here S is any class of frames. So Λ_{S} is the set of all formulas valid in a given class of frames. $\dashv$

We will end this section with the definition of when a given formula is deducible in a modal logic:

Definition 9. Let $\psi_1, \ldots, \psi_n, \varphi$ be modal formulas. We say that φ is deducible in propositional calculus from $\psi_1, \ldots, \psi_n$ if $(\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$ is a tautology.

If $\Gamma \cup \{\varphi\}$ is a set of modal formulas then φ is deducible in Λ from Γ if $\vdash_\Lambda \varphi$ or if there are formulas $\psi_1, \ldots, \psi_n \in \Gamma$ such that

$$\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$$

In this case we write $\Gamma \vdash_\Lambda \varphi$ else we write $\Gamma \nvdash_\Lambda \varphi$. We say that Γ is Λ-consistent if $\Gamma \nvdash_\Lambda \bot$, and if not we say that it is Λ-inconsistent. $\dashv$

Another way to say that a modal logic Λ is inconsistent is to say that If there there is a formula φ such that $\Gamma \vdash_\Lambda \varphi \wedge \neg\varphi$. Further we have that a set Γ is Λ-consistent if and only if every finite subset of Γ is, i.e our notion of deductibility has the compactness property.

4 Completeness of the logic K

In this section we will show that a modal logic which we will call **K**[5] is complete and sound; i.e that a sentence φ is a theorem of **K** if and only if it is a valid sentence. In later sections we will look at other systems and show completeness results for them. Before we define our **K** we will follow Lemmon and show some *preliminary* results:

[4]In Lemmons definition of this concept he only needs a few of the tautologies, but in Blackburn and acc they pick all of them. Here I follow Blackburn

[5]After Saul Kripke

Theorem 1. *For any Kripke model $\mathcal{K}$ we have if $\models^{\mathcal{K}} \varphi$ then $\models^{\mathcal{K}} \Box\varphi$*

Proof. Let $\mathcal{K} = \langle W, R, \phi\rangle$ be given and assume $\models^{\mathcal{K}} \varphi$. Then $\forall w \in W$ we have that $\models^{\mathcal{K}}_w \varphi$; if we pick an arbitrary $w \in W$ and any $v \in W$ such that wRv we then have that $\models^{\mathcal{K}}_v \varphi$. But then by the truth definition 4.(D) we have that $\models^{\mathcal{K}}_w \Box\varphi$ and since w was picked arbitrary we have that $\models^{\mathcal{K}} \Box\varphi$. ⊣

Since this holds for all Kripke models, the two following corollaries also follows:

Corollary 1. *For any Hintikka frame $\mathcal{H}$: If $\models^{\mathcal{H}} \varphi$ then $\models^{\mathcal{H}} \Box\varphi$.*

Corollary 2. *If $\models \varphi$ then $\models \Box\varphi$.*

We will now show the following theorem

Theorem 2. $\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Proof. We start of by letting $\mathcal{H} = \langle W, R\rangle$ be any Hintikka frame, and $\mathcal{K}$ be a Kripke model on $\mathcal{H}$. We then pick a $w \in W$ and assume that $\models^{\mathcal{K}}_w \Box(\varphi \to \psi)$ and $\models^{\mathcal{K}}_w \Box\varphi$ Holds.
We 'just' have to show that $\models^{\mathcal{K}}_w \Box\varphi$. So assume that we have a v such that wRv and hence we have that $\models^{\mathcal{K}}_v \varphi \to \psi$ and $\models^{\mathcal{K}}_v \varphi$. By definition 4.(B) we then have that $\models^{\mathcal{K}}_v \psi$. Since v was picked arbitrarily we then have that $\models^{\mathcal{K}}_w \Box\psi$. ⊣

4.1 Normal modal logics and the logic K

Having proved those prelimanry result we can define the concept a normal modal logic:

Definition 10. A modal logic Λ is called normal if it contains the following formula:

$$\text{(K)} \quad \Box(p \to q) \to (\Box p \to \Box q)$$

And if for any formula if $\vdash_\Lambda \varphi$ then $\vdash_\Lambda \Box\varphi$ (The rule of necessitation, RN)
If Γ is a set of modal formulas, we call the smallest normal modal logic containing Γ the normal modal logic axiomatized by Γ. . We call the normal modal logic generate by the empty set **K**, and this is the smallest normal modal logic. ⊣

Normal modal logics have some nice properties. Some of them will be shown now:

Proposition 2. *If Λ is a normal modal logic then:*

1. *if $\vdash_\Lambda \varphi \to \psi$ the $\vdash_\Lambda \Box\varphi \to \Box\psi$*
2. $\vdash_\Lambda \Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$
3. $\vdash_\Lambda \Box(\varphi_1 \wedge \cdots \wedge \varphi_n) \leftrightarrow \Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n$ *for $n \geqslant 2$*
4. *if $\vdash_\Lambda \psi_1 \wedge \cdots \wedge \psi_n \to \varphi$ then $\vdash_\Lambda \Box\psi_1 \wedge \cdots \wedge \Box\psi_n \to \Box\varphi$ for $n \geqslant 0$*

Proof. We show the results one at a time:

1. Suppose that $\vdash_\Lambda \varphi \to \psi$. Since Λ is normal we can use (RN) and get $\vdash_\Lambda \Box(\varphi \to \psi)$ and then use (K) and MP and get $\vdash_\Lambda \Box\varphi \to \Box\psi$

2. ($\rightarrow$) We have that $\varphi \wedge \psi \rightarrow \varphi$ and $\varphi \wedge \psi \rightarrow \psi$ are tautologies of propositional logic, and we have from 1 of this proposition that $\vdash_\Lambda \Box(\varphi \wedge \psi) \rightarrow \Box\varphi$ and $\vdash_\Lambda \Box(\varphi \wedge \psi) \rightarrow \Box\psi$. Again by predicate logic this gives us: $\vdash_\Lambda \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
($\leftarrow$) We have that $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$ is clearly a tautology of propositional logic, so from (1) we get that $\vdash_\Lambda \Box\varphi \rightarrow \Box(\psi \rightarrow \varphi \wedge \psi)$. So from (K) we get $\vdash_\Lambda \Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))$ and this in turn gives us $\vdash_\Lambda \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

3. We use induction on n. If $n = 2$ it is just (2). Assume it holds for n and look at: $\Box(\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \varphi_{n+1})$ and set $\vartheta = \varphi_n \wedge \varphi_{n+1}$. Then we have that since it holds for n conjutcs that:
$$\vdash_\Lambda \Box(\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \wedge \vartheta) \leftrightarrow \Box\varphi_1 \wedge \cdots \wedge \Box\varphi_{n-1} \wedge \Box\vartheta$$
The result follows if we exchange ϑ with $\varphi_n \wedge \varphi_{n+1}$ and see that $\Box(\varphi_n \wedge \varphi_{n+1})$ is equivalent too $\Box\varphi_n \wedge \Box\varphi_{n+1}$ by (2).

4. For $n = 0$ this just RN. For $n = 1$ it is just (1). If $n \geqslant 2$ then it holds by using (1) to get $\vdash_\Lambda \Box(\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \Box\varphi$. Then use (2) to get $\vdash_\Lambda \Box\psi_1 \wedge \cdots \wedge \Box\psi_n \rightarrow \Box\varphi$.

$\dashv$

4.2 Different kinds of normal modal logics:

K is not the only normal modal logic that we can define. There is a lot of others and some of the traditional ones can be generated by adding the following axioms[6] to **K**:

T	$\Box\varphi \rightarrow \varphi$	D	$\Box\varphi \rightarrow \Diamond\varphi$
4	$\Box\varphi \rightarrow \Box\Box\varphi$	G	$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
4^n	$\Box^n\varphi \rightarrow \Box^{n+1}\varphi$	G'	$\Diamond^m\Box^n\varphi \rightarrow \Box^p\Diamond^q\varphi$
B	$\Diamond\Box\varphi \rightarrow \varphi$		
E	$\Diamond\Box\varphi \rightarrow \Box\varphi$	L	$\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$

In the following there will be a few comments of these axioms. If **A** is an axioms then **KA** will denote the normal modal logic generated by **A**.[7]

We have **G'** as the generalization of the most of these axioms. It can be seen that **T** is **G'** with $m, p, q = 0$ and $n = 1$ etc. We will use this fact in our completeness theorems, where we will show a completeness result for **G'** and then look at the more "local" cases after.[8] The only axiom of these that **G'** do not generalize is the axiom **L**, which is called *Löb's formula*. The modal logic **KL** plays a crucial role in provability logic. In this logic the $\Box\varphi$ reads as *it is provable that* φ.
The history behind the names of these axioms can be found in most books about modal logic, so it will be omitted it in this project.

[6]The axioms have a different form in Blackburn, since they have choosen $\Diamond$ as the fundamental modal operator. Here I follow Lemmon

[7]There is a lot of different names for the different modal logics. But I will follow this principle.

[8]It should be noted that there are a more general scheme than **G'** that can be shown to be complete; the Sahlqvist fragment. But it is outside the scope of this project to show that these are complete

4.3 Completeness theorem for the modal logic K

Before showing the next theorems, we will need a definition about *soundness* and *completeness*, which are the some of the crucial definitions in this project.

Definition 11. Let S be a class of frames or models.

1. A normal modal logic is sound with respect to S if $\Lambda \subseteq \Lambda_{\mathrm{S}}$. I.e if $\vdash_\Lambda \varphi$ implies $\models^{\mathcal{S}} \varphi$ for all structures in S.

2. We say that Λ is strongly complete with respect to S if for any set of formulas $\Gamma \cup \{\varphi\}$, if $\Gamma \models^{\mathrm{S}} \varphi$ then $\Gamma \vdash_\Lambda \varphi$ A logic Λ is weakly complete with respect to S if for any formula φ, if $\models^{\mathrm{S}} \varphi$ then $\vdash_\Lambda \varphi$

$\dashv$

Here it should be noted that weak completeness is the special case of strong completeness where Γ is the empty set. So any logic that is strongly complete is also weakly complete. It should also be noted that we can formulate our definition of weak completeness in a similar was we did with soundness: Λ is weakly complete with respect to S if $\Lambda_{\mathrm{S}} \subseteq \Lambda$. So if Λ is both sound and weakly complete with respect to S we have that $\Lambda_{\mathrm{S}} = \Lambda$.
We can show a short lemma that makes it easier for us to show that a given logic is strongly complete with respect to a class of structures.

Lemma 1. *Let Λ be a modal logic. If every Λ-consistent set of formulas is satisfiable on some $\mathcal{S} \in \mathrm{S}$ then Λ is strongly complete with respect to this class of structures.*

Proof. We will argue by contraposition. So assume that Λ is not strongly complete with respect to S. This means there is a set of formulas $\Gamma \cup \{\varphi\}$ such that $\Gamma \models^{\mathrm{S}} \varphi$ but $\Gamma \nvdash_\Lambda \varphi$. Hence we have shown that $\Gamma \cup \{\neg\varphi\}$ is Λ-consistent but is not satisfiable on any structure in S. By contraposition the lemma follows. $\dashv$

We will also need the definitions of a maximal Λ consistent set (sometimes denoted Λ-MCS) and a canonical model:

Definition 12. A logic Λ is complete (or maximal) if and only if for all formulas φ either $\varphi \in \Lambda$ or $\neg\varphi \in \Lambda$.
A set of formulas Γ is maximal Λ consistent if Γ is Λ consistent and complete $\dashv$

The following proporties of maximal consistent sets will be given here without proof:

Proposition 3. *If Λ is a logic and Γ is a maximal Λ consistent set then:*

1. *Γ is closed under modus ponens.*

2. *$\Lambda \subseteq \Gamma$.*

3. *For all formulas φ and ψ we have that: $\varphi \vee \psi \in \Gamma$ if and only if either $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

The following result called Lidenbaum's Lemma show that any consistent set, can be extended to a maximal consistent set, which is important in our definition of a canonical model.

Lemma 2 (Lindenbaum's Lemma). *Every Λ-consistent set Γ of formulas has a maximal consistent Λ-extension; i.e there is a maximal consistent set Γ^+ such that $\Gamma \subseteq \Gamma^+$*

Proof. Let Γ be a Λ-consistent set of formulas and let $\varphi_0, \varphi_1, \ldots$ be an enumeration of all formulas in our language. We define a series of Γ extensions as follows:

$$\Gamma_0 = \Gamma$$
$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if this is } \Lambda\text{-consistent} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{else} \end{cases}$$
$$\Gamma^+ = \bigcup_{n \geqslant 0} \Gamma_n$$

It is clear that $\Gamma \subseteq \Gamma^+$. It is also clear that φ_n or $\neg\varphi_n$ is in Γ, since each formula appears at least once. We will now show that each Γ_n is consistent. This will be done by induction. The case where $n = 0$ is trivial true. Assume that Γ_n is consistent. Then if $\Gamma_n \cup \{\varphi_n\}$ is Λ-consistent Γ_{n+1} is consistent. If this is not the case then we would have $\Gamma_n \nvdash \varphi_n$ and since Γ_n is consistent we would have that $\Gamma_n \cup \{\neg\varphi\}$ is consistent, and hence our induction is done.
Γ^+ is consistent since it is a union of consistent sets. It is maximal since for any formula φ in our language we have that φ is somewhere in our list of formulas φ_i, and hence either $\varphi \in \Gamma_{n+1}$ or $\neg\varphi \in \Gamma_{n+1}$. Since $\Gamma_{n+1} \subseteq \Gamma^+$ we have that $\varphi \in \Gamma^+$ or $\neg\varphi \in \Gamma^+$ and thus Γ^+ is complete and consistent and hence maximal consistent. $\dashv$

Before defining canonical models we will first show another theorem:

Theorem 3. *For any set of formulas Γ we have that $\Gamma \vdash_\Lambda \varphi$ if and only if φ belongs to all Λ-MCS's extension of Γ. Specially this gives: $\vdash_\Lambda \varphi$ if and only if φ belongs to all Λ-MCS's*

Proof. The last part of the theorem follows from the first if we pick $\Gamma = \varnothing$. We will prove the first part.
$\Rightarrow$
if $\Gamma \vdash_\Lambda \varphi$ then it is clear that φ belongs to all Λ-MCS's of Γ; else there would be an inconsistent Λ-MCS of Γ that would exists inconsistent.
$\Leftarrow$
Assume that $\Gamma \nvdash_\Lambda \varphi$ and look at $\Gamma' = \Gamma \cup \{\neg\varphi\}$. Then Γ' is Λ-consistent, since else we would have that $\Gamma' \vdash_\Lambda \bot$ which means that we would have $\Gamma \vdash \neg\varphi \to \bot$ and hence $\Gamma \vdash \varphi$ which is a contradiction. Then by Lindenbaum's Lemma we have that Γ' has an Λ-MCS Λ^+ and this is also a Λ-MCS of Γ since we have $\Gamma \subseteq \Gamma'$. But we have that $\neg\varphi \in \Gamma^+$ and hence $\varphi \notin \Gamma^+$ and the proof is done. $\dashv$

We can now define the notion of a canonical model, which is crucial in showing strong completeness results.[9]

[9]The name *canonical model* is not used by Lemmon, but is a name this concept got after his text. The definition of this concept is also a bit different in Blackburn [2002], but here I will follow Lemmon [1977]

Definition 13. The canonical model $\mathcal{K}_\Lambda$ for a normal modal logic, is the tuple $\langle W_\Lambda, R_\Lambda, \phi_\Lambda \rangle$, where:

1. W_Λ is the set set of all Λ-MCS's (This set is non-empty by Lindenbaum's Lemma)
2. For $w, v \in W_\Lambda$ we define R_Λ as follows: $wR_\Lambda v$ iff $\{\varphi : \Box\varphi \in w\} \subseteq v$
3. $\phi_\Lambda(p) = \{w \in W_\Lambda | p \in w\}$

⊣

With this definition we will show two lemmas. The first one will tell us something about the relation in the canonical model and the second will tell us something about the *truth* in the model.

Lemma 3. *For all normal modal logic Λ we have for any $w \in W_\Lambda$ that $\Box\varphi \in w$ iff for all v such that $wR_\Lambda v$ that $\varphi \in v$*

Proof. If $\Box\varphi \in w$ and $wR_\Lambda v$ then it follows by the definition of R_Λ that $\varphi \in v$. The other way is a bit more tricky. Suppose that $\varphi \in v$ for all v such that $\Theta = \{\varphi | \Box\varphi \in u\} \subseteq v$. φ then belongs to every maximal consistent extension of Θ. By theorem (3) we get that $\Theta \vdash_\Lambda \varphi$. Then have a set of formulas $\psi_1, \ldots, \psi_n$ such that $\Box\psi_1, \ldots, \Box\psi_n \in w$ and:

$$\vdash_\Lambda \psi_1 \wedge \cdots \wedge \psi_n \to \varphi$$

Since Λ is normal we can use proposition 2.(4) and get:

$$\vdash_\Lambda \Box\psi_1 \wedge \cdots \Box\psi_n \to \Box\varphi$$

But this means that $\Box\varphi \in w$ and we are done. ⊣

With this lemma we can show that a formula φ is in world w if and only if it is true in the model $\mathcal{K}_\Lambda$ at this world. So membership of a world is the same as "truth" at the world in our canonical model.

Lemma 4. *For any normal modal logic Λ and any formula φ we have that:*

$$\models_w^{\mathcal{K}_\Lambda} \varphi \textit{ iff } \varphi \in w$$

There follows a corollary to this lemma that we state here without proof:

Corollary 3. *For any consistent normal modal logic Λ we have that $\vdash_\Lambda \varphi$ if and only if for all $w \in W_\Lambda$ that $\models_w^{\mathcal{K}_\Lambda} \varphi$.*

Proof. We will show this result by complexity on φ. This technique is the same as used in first order predicate logic, but the last part we will switch $\forall\varphi$ with $\Box\varphi$.If φ is a propositional letter p then $\models_w^{\mathcal{K}_\Lambda}$ if and only if $w \in \phi_\Lambda(p)$ and by the definition of ϕ_Λ this means exactly that $\varphi \in w$. We will now show the (A), (B), (C) criteria:

(A) If φ is $\bot$ then $\not\models_w^{\mathcal{K}_\Lambda}$ by our truth definition and $\bot \notin w$ since w is a consistent set. So the claim is true in this case.

(B) If φ is $\psi \to \theta$ then we have that $\vDash_w^{\mathcal{K}_\Lambda} \varphi$ iff if $\vDash_w^{\mathcal{K}_\Lambda} \psi$ then $\vDash_w^{\mathcal{K}_\Lambda} \theta$ by our truth definition. Our induction hypothesis then gives us that this is equivalent to: if $\psi \in w$ then $\theta \in v$ and since w is maximal this is equivalent to $\psi \to \theta \in w$; i.e $\varphi \in w$.

(C) If φ is $\Box\psi$. By truth definition of "$\Box$" we have that $\vDash_w^{\mathcal{K}_\Lambda} \Box\psi$ iff $\forall v(wR_\Lambda v \to \vDash_v^{\mathcal{K}_\Lambda} \psi)$. By our induction hypothesis this is $\forall v(wR_\Lambda v \to \psi \in v)$. This gives us exactly that $\Box\varphi \in w$ by lemma 3

$\dashv$

With this lemma we can show our first main theorem of this project: *The canonical model theorem.* This theorem will be used in this and the next section to show some completeness results concerning different normal modal logics.

Theorem 4. *Any normal modal logic is strongly complete with respect to its canonical model.*

Proof. Suppose that Γ is a consistent set of the normal modal logic Λ, hence by Lindenbaum's lemma there is a Λ maximal consistent set Γ^+ that extend Γ, and hence $\Gamma^+ \in W_\Lambda$ so it is a world in our model. The lemma 4 then gives that $\vDash_{\Gamma^+}^{\mathcal{K}_\Lambda} \Gamma$. $\dashv$

We will in the following use this theorem to show that **K** is sound and strongly complete with respect to the class of all frames. We will start with soundness:

Theorem 5. *if $\vdash_K \varphi$ then $\models \varphi$; i.e any theorem of K is valid.*

This theorem says exactly that **K** is sound to the class of all frames. Further; to show this we just have to show that the axioms are valid and that the rules of inference preserve validity on the frames we want to show that it is sound with respect to.

Proof. We will prove this by induction on φ's proofs length in **K**. If $\varphi \in PC$ then it is known that $\vDash \varphi$. Further we know that *modus ponens* is a valid rule of inference and by corollary 4.1.2 that the rule of necessitation is a valid rule of inference. We have also shown in Themorem ??? that the scheme (K) is a true for all worlds in all frames. So our axioms are sound and our rules of inference are valid, we can conclude that every theorem of **K** is valid in all frames, and this was what we should prove. $\dashv$

Theorem 6. ***K** is strongly complete with the class of all frames.*

Proof. We just have to shown that for any **K**-consistent set Γ there is a model $\mathcal{K}$ and a world in w this model such that $\vDash_w^{\mathcal{K}} \Gamma$. But we can just chose the canonical model $\mathcal{K}_{\mathbf{K}} = \langle \mathcal{H}_{\mathbf{K}}, \phi_{\mathbf{K}} \rangle$ and let Γ^+ be a **K**-maximal consistent set extending Γ. By canonical model theorem we then have that $\vDash_{\Gamma^+}^{\mathcal{K}_{\mathbf{K}}} \Gamma$ $\dashv$

So all in all we have shown that, since strong completeness implies weak completes:

$$\vdash_{\mathbf{K}} \varphi \text{ iff } \vDash \varphi$$

5 Completeness results for the axiom G'

Recall that we have the following axiom:

Definition 14. We define the scheme G' as follows:

$$G' :\ \Diamond^m \Box^n \varphi \to \Box^p \Diamond^q \varphi$$

Where $m, n, p, q \in \omega$ ⊣

We will want to show that for the systems KG' (The system K with G' added as an axiom), that theoremhood is equivalent to validity in the class frames that satisfies the following condition:

$$(g') :\ \forall w, v, v' (wR^m v \wedge wR^p v' \to \exists v_0 (tR^n v_0 \wedge v' R^q v_0))$$

5.1 Soundness

We will start of by showing soundness and then later prove completeness

Theorem 7. *$\vDash^{\mathcal{H}} G'$ for any $\mathcal{H}$ satisfying g'*

Proof. Let $\mathcal{H} = \langle W, R \rangle$ be a frame satisfying g'. Our goal is to show that $\vDash^{\mathcal{H}} \Diamond^m \Box^n \varphi \to \Box^p \Diamond^q \varphi$ for any φ. So let $\mathcal{K} = \langle \mathcal{H}, \phi \rangle$, select a $w \in W$ and assume that

$$\vDash^{\mathcal{K}}_w \Diamond^m \Box^n \varphi$$

By the truth definition (F) we have that there exists some t such that $uR^m t$ and also $\vDash^{\mathcal{K}}_t \Box^n \varphi$. We will now assume that $uR^p t'$ and then since $\mathcal{H}$ satisfies g' we can conclude that there exists t_0 such that $tR^n t_0$ and $t' R^q t_0$. Now we have that $tR^n t_0$ and that $\vDash^{\mathcal{K}}_t \Box^N \varphi$ so we can conclude that $\vDash^{\mathcal{K}}_{t_0} \varphi$. Since $t' R^q t_0$ it then follows that $\vDash^{\mathcal{K}}_{t'} \Diamond^q$. Finally we can conclude

$$\vDash^{\mathcal{K}}_w \Box^p \Diamond^q \varphi$$

This completes our proof. ⊣

It is easily seen that we also have shown:

Corollary 4. *If $\vdash_{KG'} \varphi$ then $\vDash^{\mathcal{H}} \varphi$ for all $\mathcal{H}$ satisfying g'.*

If we can show the converse of this corollary we have showed completeness. That way is a bit more complicated, so we need some preliminary results and thoughts first.

5.2 Completeness

We will start of by showing a result that holds for any normal modal logic.

Proposition 4. *If Λ is a normal modal logic then:*

1. $\vdash_\Lambda \Box^n \varphi_1 \vee \cdots \Box^n \varphi_k \to \Box^n (\varphi_1 \vee \cdots \varphi_k)$
2. *if* $\vdash_\Lambda \varphi \wedge \Diamond^n \psi_1 \wedge \cdots \wedge \Diamond^n \psi_k \to \bot$ *then* $\vdash_\Lambda \Box \varphi \to \Box^{n+1} (\neg \psi_1 \vee \cdots \vee \neg \psi_k)$ *for* $k \geqslant 1$ *and* $n \geqslant 0$.

Proof. (1):We have that $\varphi_i \to \varphi_1 \vee \cdots \vee \varphi_k$ for $1 \leqslant i \leqslant k$ is a tautology so by using prop 2 part 1 n times we get that $\vdash_\Lambda \Box^n\varphi_i \to \Box(\varphi_1 \vee \cdots \vee \varphi_k)$ **Tænkt lige over dette**
(2): Assume that $\vdash_\Lambda \varphi \wedge \Diamond^n\psi_1 \wedge \cdots \wedge \Diamond^n\psi_k \to \bot$. This gives by definition of $\Diamond^n$ and use of a tautology that $\vdash_\Lambda \varphi \to \Box^n\neg\psi_1 \vee \cdots \vee \Box^n\neg\psi_k$ and hence by part (1) we get that $\vdash_\Lambda \varphi \to \Box^n(\neg\psi_1 \vee \cdots \vee \psi_k)$. The result noew follows from propostion 2 part 1. $\dashv$

Next we will show the following proposition about the canonical model. We will use this proposition in our completeness proof

Proposition 5. *Let Λ be a normal modal logic. For any $w, v \in W_\Lambda$ we have that:*

$$wR^m_\Lambda v \text{ iff } \{\varphi | \Box^n\varphi \in w\} \subseteq v$$

Proof. We will use induction on n to show this result.
Base case:
If $n = 0$ we have that show that $w = v$ if and only if $\{\varphi|\varphi \in w\} \subseteq v$. The last part here is the same as stating that $w \subseteq v$. It is clear that if $w = v$ then $w \subseteq v$. So assume that $w \subseteq v$ and $\varphi \in v$ and suppose for contradiction that $\varphi \notin w$. Since w is complete by definition we have that $\neg\varphi \in w$ and hence $\neg\varphi \in v$, which contradicts that v is consistent, and hence the basis for the induction holds
The induction step:
Assume the result holds for n. We will show the two ways separately
($\Rightarrow$)
Assume that $wR^{n+1}_\Lambda v$, which means that $\exists v'(wR_\Lambda v' \wedge v'R^n_\Lambda v)$. Now pick $\Box^{n+1}\varphi \in w$. Then by the definition of R_Λ this gives us that $\Box^n\varphi \in v'$ and hence by our induction hypothesis and that $v'R^n_\Lambda v$ we get that $\varphi \in v$, which was what we wanted to show.
($\Leftarrow$)
Assume that $\{\varphi|\Box^{n+1}\varphi \in w\} \subseteq v$ and consider the following set:

$$\Gamma = \{\varphi|\Box\varphi \in w\} \cup \{\Diamond^n\varphi|\varphi \in v\}$$

We will show that this set is consistent and then use Lindenbaum's Lemma. So assume for contradiction that it is not consistent. So we have that $\Gamma \vdash_\Lambda \bot$ i.e we have formulas $\psi_1, \ldots \psi_l$ with $\Box\psi_1, \ldots, \Box\psi_l \in w$ and formulas $\vartheta_1, \ldots, \vartheta_k \in v$ such that:

$$\vdash_\Lambda \psi_1 \wedge \cdots \wedge \psi_l \wedge \Diamond^n\vartheta_1 \wedge \cdots \wedge \Diamond^n\vartheta_k \to \bot$$

By proposition 3.(2) and proposition 2(3) we get that:

$$\vdash_\Lambda \Box\psi_1 \wedge \cdots \wedge \Box\psi_l \to \Box^{n+1}(\neg\vartheta_1 \vee \cdots \neg\vartheta_k)$$

Now since w is an Λ-extension we have that $\Box^{n+1}(\neg\vartheta_1 \vee \cdots \vee \vartheta_k) \in w$ and hence we can conclude that $\neg\vartheta_1 \vee \cdots \neg\vartheta_k \in v$. Since v is a maximal consistent set we have that $\neg\vartheta_i \in v$ for some i with $1 \leqslant i \leqslant k$, and this contradicts with v being consistent. So Γ is Λ-consistent. Thus we can by Lindenbaum's Lemma find maximal consistent Λ-extension of Γ that we will call v'. Now since $\{\varphi|\Box\varphi\} \subseteq \Gamma$ we have that $wR_\Lambda v'$ by definition of R_Λ. Furthermore since $\{\Diamond^n\varphi|\varphi \in v\} \subseteq \Gamma$ we also have that $\{\Diamond^n\varphi|\varphi \in v\} \subseteq v'$, and thus $\{\varphi|\Box^n\varphi \in v'\} \subseteq v$. But then by the induction hypothesis we get that $v'R^n_\Lambda v$ and since $wR\Lambda v'$ we get that $wR^{n+1}_\Lambda v$ and we are done. $\dashv$

We are now close to show completeness of our system **KG'**. To show this it is actually enough to show that $\mathcal{H}_{\mathbf{KG'}} = \langle W_{\mathbf{KG'}}, R_{\mathbf{KG'}} \rangle$ satisfies g'. By theorem 7 we have that the logics **KG'** are consistent it is also clear that there exists frames $\mathcal{H}$ satisfying g' for any m, n, p, q, and these frames are clearly by definition normal. Thus we can use corollary 3 to say that if $\nvdash_{\mathbf{KG'}} \varphi$ we can find a $w \in W_{\mathbf{KG'}}$ such that $\nvDash_w^{\{} \mathcal{K}_{\mathbf{KG'}} \varphi$. So all in all we just need to show that $\mathcal{H}_{\mathbf{KG'}}$ satisfies g'.

Theorem 8. *For any consistent normal K-system S, if S contains G' then $\mathcal{H}_S$ satisfies g'.*

Proof. Let Λ be a normal modal logic such that $G' \subseteq \Lambda$. We will show that:

$$\forall w, v, v'(wR_\Lambda^m v \wedge wR_\Lambda^p v' \rightarrow \exists v_0(vR_\Lambda^n v_0 \wedge v' R_\Lambda^q v_0)$$

And hence then have shown that $\mathcal{H}_\Lambda$ satisfies g'. So for $w, v, v' \in W_\Lambda$ assume that $wR_\Lambda^m v$ and $wR_\Lambda^p v'$. By the just shown porpostion we then have that $\{\varphi | \Box^m \varphi \in w\} \subseteq v$ and $\{\varphi | \Box^p \varphi \in w\} \subseteq v'$. We will now define the following set and show that it is consistent:

$$\Gamma = \{\varphi | \Box^n \varphi \in v\} \cup \{\varphi | \Box^q \varphi \in v'\}$$

So assume for contradiction that this set is inconsistent. So there are formulas: $\psi_1, \ldots, \psi_j$ and $\vartheta_1, \ldots, \vartheta_k$ with $\Box^n \psi_i \in v$ for $1 \leqslant i \leqslant j$ and $\Box^q \vartheta_i \in v$ for $i \leqslant i \leqslant k$ such that $\vdash_\lambda \neg(\psi \wedge \vartheta)$ where we have set $\psi = \psi_1 \wedge \cdots \wedge \psi_n$ and have defined ϑ in the same way. By proposition 2 part 3 we then get $\Box^n \psi \in v$ and $\Box^q \vartheta \in v'$. Further since we have $\vdash_\Lambda \neg(\psi \wedge \vartheta)$ we have $\vdash_\Lambda \psi \rightarrow \neg\vartheta$ and hence $\vdash_\Lambda \Box^n \psi \rightarrow \Box^n \neg\vartheta$.
We have that v is an exteinsion(???) of Λ so we have $\Box^n \neg\vartheta \in v$. But we have that $\{\varphi | \Box^m \varphi \in w\} \subseteq v$ so specially we must have that $\Diamond^m \Box^n \neg\vartheta \in w$ and since we have that $G' \subseteq \Lambda \subseteq w$ we get that $\Box^p \Diamond^q \neg\vartheta \in w$. Since $\{\varphi | \Box^p \varphi \in w\} \subseteq v'$ we then get that $\Diamond^q \neg\vartheta \in v'$, by duality this gives us: $\neg\Box^q \vartheta \in v'$. But this contradicts that v' is consistent and hence Γ is a Λ-consistent set and by Lindenbaum's Lemma it has a maximal consistent Λ extension. We will call this extension v_0. But we then have that $\{\varphi | \Box^n \varphi \in v\} \subseteq v_0$ and that $\{\varphi | \Box^q \varphi \in v'\} \subseteq v_0$ and then by propostion shown just before this theorem we have that $vR_\Lambda^n v_0$ and $v' R_\Lambda^q v_0$ and hence we have shown that $\mathcal{K}_\Lambda$ satisfies g'. $\dashv$

From this theorem, another main results of this project follows:

Theorem 9. ***KG'*** *is strongly complete with respect to the class of all frames* H *that satisfies g'. Specially we have that: $\vdash_{KG'} \varphi$ if and only if $\vDash^{\mathcal{H}} \varphi$ for all $\mathcal{H}$ satisfying g'*

Proof. By lemma it is sufficient 1 we have to find for a given set of **KG'**-consistent formulas Γ a model $\mathcal{K} = \langle \mathcal{H}, \phi \rangle$ that satisfies g' and a world w form this model such that $\vDash_w^{\mathcal{K}} \Gamma$. We can just pick the canonical model for **KG'** i.e the model $\mathcal{K}_{KG'} = \langle W_{KG'}, R_{KG'}, \phi_{KG'} \rangle$. This models satisfies g' by the theorem we have just shown. Further let Γ^+ be any **KG'** maximal consistent set that extends Γ. By our truth lemma we get that $\vDash_{\Gamma^+}^{\mathcal{K}_{KG'}} \Gamma$ $\dashv$

5.3 Using the compltenes theorem

In this section we will use the completeness theorem for **G'** to show that different modal logics are strongly complete and sound with respect to different classes of frames. If we set $m, p, q = 0$ and $n = 1$ we get the axiom T. g' then becomes (remember that R^0 is just identity):

$$\forall w, v, v'(w = v \wedge w = v' \rightarrow \exists v_0(vRv_0 \wedge v' = v_0))$$

But this is clearly just $\forall u(uRu)$, and this is exactly that R is reflexive. We can now conclude the following:

Theorem 10. *$\vdash_T \varphi$ if and only if $\models^{\mathcal{H}} \varphi$ for all reflexive frames $\mathcal{H}$*

If now we put $m = q = 0$ and set $p = n + 1$ We get axiom 4^n and get that g' is:

$$\forall w, v, v'(w = t \wedge wR^{n+1}v' \rightarrow \exists v_0(tR^n v_0 \wedge v' = v_0))$$

And this simplifies to:

$$\forall w, v'(wR^{n+1}v' \rightarrow wR^n v')$$

I.e R is a relation that is n-transtive. If $n = 1$ we get that:

$$\forall w, v, v'(wRv \wedge vRv' \rightarrow wRv')$$

I.e that R is transitive. So we have shown the following:

Theorem 11. *$\vdash_{K4^n} \varphi$ iff $\models^{\mathcal{H}} \varphi$ for all n-transitve $\mathcal{H}$. If $n = 1$ it is for all transitives frames.*

This process can be done for a all our axioms (except axiom **L**) in our list from section 4.2. But too not bore the reader we will turn to something a bit more interesting.[10]

6 Provability logic

In the last section of this project we will look at the logic we get by adding Löb's formula to **K**. We will denoted this logic: **KL**. This logic is usually called provability logic and as noted earlier we will now read $\Box\varphi$ as there is a proof of φ. This logic

6.1 Soundness

We will start off with defining the *term* frame definability:

Definition 15. Let φ be a modal formula and H a class of Hintikka frames. We say that φ defines H if for all frames $\mathcal{H}$, $\mathcal{H}$ is in H if and only iff $\models^{\mathcal{H}} \varphi$. Further if Γ is a set modal formulas of this type, we say that Γ defines H if $\mathcal{H}$ is in H if and only if $\models^{\mathcal{H}} \Gamma$ $\dashv$

[10] A lot of different completeness results can be found in Lemmon [1977]

If we have a formula that defines a class of frames, then we also have that the modal logic where we add the formula as an axiom is sound with respect to that class of frames. With this definition we can show that Löbs formula defines the class of frames where R is transitive and conversely well-founded, and hence **KL** is sound with respect to that class. We will start off by defining what it means for a relation to be conversely well-founded and then show the result:

Definition 16. A relation R on a set Γ is called well-founded if every nonempty subset $\Theta \subseteq \Gamma$ has a minimal element with respect to R. In other words R is well-founded if there is no infinite sequence such that $\ldots Rw_2Rw_1Rw_0$.
A relation R on a set Γ is called conversely well-founded if the converse R^{-1} of R is well-founded, i.e if there is no infinite sequence such that $w_0Rw_1Rw_2R\ldots$. $\dashv$

Proposition 6. *L defines the class of Hintikka frames $\langle W, R\rangle$ such that R is transitive and R's converse is well founded. This means that the logic KL is sound with respect to this class of frames.*

Proof. We will first show that if $\mathcal{H} = \langle W, R\rangle$ is a frame where R is transitive and conversely well founded, then L is valid in this frame.
For the sake of contradiction we will assume that L is not valid in $\mathcal{H}$. So we have a valuation ϕ (and hence a model $\mathcal{K}$ with this valuation) and a state w such that $\not\models_w^{\mathcal{K}} \Box(\Box p \to p) \to \Box p$; i.e that $\models_w^{\mathcal{K}} \Box(\Box p \to p) \to \Box p$ but $\not\models_w^{\mathcal{K}} \Box p$. So we have a world w_1 such that wRw_1 and $w_1 \notin \phi(p)$, but as $\Box p \to p$ holds at all successors of w we must have that $\not\models_{w_1}^{\mathcal{K}} \Box p$. So we have a w_2 such that w_1Rw_2 where p is false, and since R is transitive we have that wRw_2. We can repeat this argument *ad infinitum*, so w_2 has a world w_3 with w_2Rw_3 where p is false at w_3 and so on. So we have found an infinite path $wRw_1Rw_2Rw_3R\ldots$, and this contradicts that R is conversely well-founded.
We will show that if L is valid in a frame $\mathcal{H} = \langle W, R\rangle$, then R is transitive and conversely well-founded. We will use contraposition So we will assume that either R is not transitive or conversely well-founded, and then find valuation ϕ and a world w such that $\not\models_w^{(\mathcal{H},\phi)} L$
R is not transitive:
We will first show that L is equivalent to $\Diamond p \to \Diamond(p \wedge \neg\Diamond p)$:

$$\Box(\Box p \to p) \to \Box p \text{ iff } \Diamond\neg p \to \Diamond\neg(\Box p \to p) \text{ iff } \Diamond\neg p \to \Diamond(\neg\Diamond\neg p \wedge \neg p)$$

If we substitute p with a $\neg q$ we then have $\Diamond q \to \Diamond(q \wedge \neg\Diamond q)$ hence **L** is equivalent to this formula. Assume that R is conversely well-founded but not transitive. Since R is not transitive we will have three worlds: w_1, w_2, w_3, such that w_1Rw_2, w_2Rw_3 but not w_1Rw_3, and since R is conversely well-founded we have that there do not exist a world v such that: w_3Rv. We will now choose a valuation: $\phi(p) = \{w_2, w_3\}$ and put $\mathcal{K} = \langle \mathcal{H}, \phi\rangle$. With this model we will claim that $\not\models_{w_1}^{\mathcal{K}} \Diamond p \to \Diamond(p \wedge \neg\Diamond p)$. This claim is equivalent to $\models_{w_1}^{\mathcal{K}} \neg(\Diamond p \to \Diamond(p \wedge \neg\Diamond p)$, which in turn is equivalent to $\models_{w_1}^{\mathcal{K}} \Diamond p \wedge \Box(\neg p \vee \Diamond p)$. The first conjunct is true since we have a world w_2 such that w_1Rw_2 where p is true. The second conjunct is true since there is only one world there is a successor of w_1 which is w_2. So we have to show that $\models_{w_2} \neg p \vee \Diamond p$. But this is true since p is true in w_3 and we have that w_2Rw_3.
R is not conversely well-founded

We then have a infinite sequence $w_0Rw_1Rw_2R\ldots$, and we will use this to define our valuation:

$$\phi(p) = W \backslash \{v \in W |\text{ there is an infinite path starting from } v\}$$

we define $\mathcal{K} = \langle \mathcal{H}, \phi \rangle$ It is clear that under this valuation that $\Box p \to p$ is true in every world. Since if $\vDash_v^{\mathcal{K}} \Box p$ in a arbitrary world v means that for every v' such that vRv' we would have that $v' \in \phi(p)$, and this means exactly that non of these worlds v' have an infinite chain of connected worlds, and thus neither do v has such a chain. But we then have that $\vDash_{w_0}^{\mathcal{K}} \Box(\Box p \to p)$; the claim follows from the fact that we have defined our $\phi(p)$ such that we have: $\nvDash_{w_0}^{\mathcal{K}} \Box p$ and a *modus tollens* argument by using Löb's formula. $\dashv$

6.2 KL is not strongly complete

In this section it will be prove that **KL** is not strongly complete and sound with respect to any class of frames. So even though it is sound with respect to the class of all frames such that the relation R is transitive and conversely well-founded, it is not strongly complete with respect to this class of frames.

Theorem 12. ***KL*** *is not sound and strongly complete with respect to any class of frames.*

Proof. We will start of by defining the following set:

$$\Gamma = \{\Diamond q_1\} \cup \{\Box(q_i \to q_{i+1} | 1 \leqslant i \in \omega\}$$

We will show that this set is consistent in **KL** and that no model based on a **KL**-frame can satisfy all the formulas in Γ. When this has been shown the theorem will follow.

We will show that Γ is consistent by showing that any subset $\Psi \subset \Gamma$ is consistent (remember our logics all have the compactness property). If Ψ is such a subset, then for some $n \in \omega$ there is a finite set Θ of the form $\{\Diamond q_1\} \cup \{\Box(q_i \to \Diamond q_{i+1} | 1 \leqslant ii < n\}$ where $\Psi \subseteq \Theta \subset \Gamma$. If we show that Θ is consistent we would also have shown that Ψ is consistent.

Let $\widehat{\Theta}$ be the conjunction of all the formulas in Θ. We want to see that this set is consistent and for this we will show that it can be satisfied in a $\mathcal{K} = \langle \mathcal{H}, \phi \rangle$ where $\mathcal{H}$ is transitive and conversely well-founded; i.e the formula is valid on the frame that **KL** defines. Let $\mathcal{H}$ be the frame with $W = \{0, \ldots, n\}$ in the usual order. This frame is transitive and conversely well-founded. Now let $\mathcal{K}$ be a model based on this frame where for all $1 \leqslant i \leqslant n$ we define $\phi(q_i) = \{i\}$. For this model it is easily seen that we have that $\vDash_0^{\mathcal{K}} \widehat{\Theta}$ and hence $\widehat{\Theta}$ is **KL**-consistent. So Θ is **KL**-consistent and hence so is Ψ and since this subset was picked arbitrarily we can conclude that Γ is a **KL** consistent set.

We can now show that **KL** is not sound and strongly complete with respect to any class of frames. So for the sake of contradiction we assume that **KL** is sound and strongly complete with some class of frames called H. Since **KL** is not inconsistent we have that H is non-empty, so any **KL**-consistent set of formulas can be satisfied at some world in a model based on a frame from H. Since Γ is such a set of formulas there is a model $\mathcal{K}$ based on a frame $\mathcal{H}$ from H and a world $w \in \mathcal{K}$ such that $\vDash_w^{\mathcal{K}} \Gamma$. We can then define a path from w such that $wR_v1Rv_2\dots$ that will continue *ad infinitum*, but this contradicts that R is a conversely well-founded relation and hence **KL** is not sound and strongly complete with respect to any class of frames. $\dashv$

6.3 Weak completeness

We have just shown that **KL** is not strongly complete with respect to any class of frames. But is it weakly complete with respect to any class for frames? In this section it will be shown that it is indeed complete with respect to the class of frames that are transitive and conversely well-founded. Hence the last result of this project will be the following:[11]

Theorem 13. ***KL*** *is weakly complete with respect to the class of frames where $\mathcal{H}$ is transitive and conversely well founded. I.e for any modal formula φ we have the following:*

$$\models^{\mathcal{H}} \varphi \ \Rightarrow \vdash_{\boldsymbol{KL}} \varphi$$

So when we have shown this theorem we have shown that $\mathbf{KL} = \Lambda_{\mathrm{H}}$ where H is the class of all frames that are transitive and conversely well-founded. In the rest of this section H will denote this class of frames.

We will show the contrapositive of this theorem. So let $\mathcal{K} = \langle W, R, \phi\rangle$ be a model based on H, and let γ be a modal formula that is not a theorem of **KL**, so we have: $\nvdash_{\mathbf{KL}} \gamma$. Our goal is now to construct a model $\mathcal{K}$ based on $\mathcal{H} \in \mathrm{H}$ such that $\not\models^{\mathcal{K}} \gamma$. For this we need the following definitions and lemma:

Definition 17. Let φ be a modal formula. The length of φ is a number $l(\varphi)$ defined recursively in the following way:

1. for $p \in \Phi$ we set $l(p) = 1$
2. $l(\bot) = 1$
3. $l(\psi \to \vartheta) = l(\psi) + l(\vartheta) + 1$
4. $l(\Box\psi) = l(\psi) + 1$

$\dashv$

From this definition it is seen that any modal formula φ does not have more than $2^{l(\varphi)}$ subformulas. We will now define two other concepts, that are similar too our concept of consistent and maximal consistent:

Definition 18. We call a set Γ of subformulas of γ for γ-consistent in **KL** if $\nvdash_{\mathbf{KL}} \neg\widehat{\Theta}$ for all $\Theta \subseteq \Gamma$ and here $\widehat{\Theta}$ is the conjunction of all the members of the set Θ.
Further we say that Γ is maximal γ-consistent if for any subformula ψ of γ either $\psi \in \Gamma$ or $\neg\psi \in \Gamma$. $\dashv$

Since there is not more than $2^{l(\gamma)}$ subformulas of γ we also have that there are at most $2^{l(\gamma)}$ γ-consistent set Γ. So we can define the following finite[12] Kripke Model $\mathcal{K} = \langle W, R, \phi\rangle$, where :

[11] Here I will follow Yang Lius' text

[12] It is called finite since W is finite set

1. W contains all the maximal γ-consistent sets, i.e

$$W = \{w \mid w \text{ is maximal } \gamma\text{-consistent}\}$$

2. For any $w, v \in W$ we have:

$$wRv \text{ iff } \begin{cases} (1) \text{ for all } \Box\varphi & \text{if } \Box\varphi \in w \text{ then } \Box\varphi, \varphi \in v \\ (2) \text{ for some } \Box\psi & \text{if } \Box\psi \in v \text{ then } \neg\Box\psi \in w \end{cases}$$

3. We define $\phi(p)$ in the usually way, i.e $\vDash_w^{\mathcal{K}} p$ iff $w \in \phi(p)$

If we can show that γ is not valid in this model, we have shown the weak completeness theorem that we wanted. Note that both (1) and (2) shall hold in the definition of the relation. In the rest of this section $\mathcal{K}$ will denoted this model. For this we need two lemmas, both of which resembles some of our lemmas for canonical models.

Lemma 5. *We have that:*

1. *for any subformula of the form $\Box\varphi$ of γ and any $w \in W$ we have:*

$$\Box\varphi \in w \textit{ iff for any } v,\ wRv \textit{ implies } \varphi \in v$$

2. *R is transitive and conversely well-founded*

Proof. We will start by showing 1.
$(\Rightarrow)$
This way follows immediately from (1) from our definition of our relation R in our model.
$(\Leftarrow)$
For this way we will show the contraposition: if $\Box\varphi \notin w$ then we have a v such that wRv and $\varphi \notin v$. For this we define:

$$\Gamma = \{\neg\varphi, \Box\varphi\} \cup \{\psi, \Box\psi \mid \Box\psi \in w\}$$

If Γ is inconsistent we can pick $\psi_1, \ldots \psi_n, \Box\psi_1 \ldots, \Box\psi_n$ to be an enumeration of formulas of the kind on the right side of $\cup$. This can be done since w is a finite set. Since Γ is inconsistent we can now do the following deduction:

$$\vdash_{\mathbf{KL}} \neg(\neg\varphi \wedge \Box\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n \wedge \Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \tag{1}$$
$$\vdash_{\mathbf{KL}} (\psi_1 \wedge \cdots \wedge \psi_n \wedge \Box\psi_1 \wedge \cdots \wedge \Box\psi_n \wedge \Box\varphi) \to \varphi \tag{2}$$
$$\vdash_{\mathbf{KL}} (\psi_1 \wedge \cdots \wedge \psi_n \wedge \Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \to (\Box\varphi \to \varphi) \tag{3}$$
$$\vdash_{\mathbf{KL}} (\Box\psi_1 \wedge \Box^2\psi_1 \wedge \cdots \Box\psi_n \Box^2\psi_n) \to \Box(\Box\varphi \to \varphi) \tag{4}$$
$$\vdash_{\mathbf{KL}} (\Box\psi_1 \wedge \cdots \Box\psi_n) \to \Box\varphi \tag{5}$$

(1) is true since Γ is inconsistent. (2) is true since $\wedge$ is commutative and by our definition of $\wedge$. Line (3) follows form (2) by the deduction theorem. Line (4) follows by line (3) since **KL** is normal and by proposition 2.(3). The last line follows by axiom **4** and **L**. Line (5) implies that $\Box\varphi \in w$ given that all the $\Box\psi_i$ are in w since $\Box\varphi$ is a

subformula of γ. So if $\Box\varphi \notin w$ then we can say that Γ is consistent. If X is consistent we have that $\Gamma \subseteq v$ for some v. Further since $\Box\varphi \notin w$ we have that $\neg\Box\varphi \in w$ and by (2) from our definition of R we have that wRv since $\Box\varphi \in \Gamma \subseteq v$. Lastly we have that $\neg\varphi \in \Gamma$ and $X \subseteq v$ we have that $\varphi \notin v$, so we have shown 1.
We will now show 2. Transitivity follow from (1) from our definition of R.
We will now show that R is conversely well-founded, and here we will use the fact that W is finite set. First we will show that in this case that R is conversely well-founded if and only if it is irreflexive. Every relation that is conversely well-founded is clearly irreflexive. Since if R was reflexive then for each $w \in W$ there would be an infinite chain $wRwR\dots$. We will show the other way with contraposition. So assume that R is a transitive relation on a finite set W. Since W is a finite set of worlds, we can write it as: $W = \{w_1, \dots, w_n\}$. Now assume that R is not conversely well-founded; i.e there is a infinite chain such that: $w_1Rw_2R\dots R_m\dots$. In this chain there must be a world, say w_k such that it appears at least two times in the chain. Since R is transitive we must then have that w_kRw_k. Since there is at least one of such worlds the relation R is not irreflexive. So we have shown that if a transitive relation R on a finite set is not conversely well-founded it is not irreflexive and by contraposition we have that irreflexivity implies conversely well-founded in this case.
We will use this to show that the relation R is conversely well-founded. So assume for contradiction that R is not conversely well-founded. So R is not irreflexive and hence wRw for all $w \in W$. By (2) of our definition of R we have that for some $\Box\psi$ that both $\Box\psi \in w$ and $\neg\Box\psi \in w$ and hence w is inconsistent, and since each w is maximal γ-consistent this is a contradiction and hence R is conversely well-founded. $\dashv$

We will need one more lemma before we can prove our completeness theorem. This lemma looks a lot like our truth lemma for the canonical model and it plays a similarly role.

Lemma 6. *For every subformula φ of γ and any $w \in W$ we have:*

$$\varphi \in w \textit{ iff } \vDash^{\mathcal{K}}_{w} \varphi$$

The proof will be omitted since it is almost identical to the truth lemma. The only difference is that we use lemma 5 in the part with $\Box$. We can now show the wanted weak completeness theorem.

proof of theorem 13. We will use contraposition and assume that $\nvdash_{\mathbf{KL}} \gamma$. This means that $\{\neg\gamma\}$ is consistent in **KL** and this implies that $\neg\gamma \in w'$ where w' is a maximal γ-consistent set. Since this set is consistent we have that $\gamma \notin w'$ and hence by the lemma above that $\nvDash^{\mathcal{K}}_{w'} \gamma$ and hence by contraposition the theorem is proven. $\dashv$

So even though we in the previous section proved that **KL** was not sound and strongly complete to any class of frames, we have just shown that it is sound and weakly complete to the class of frames that are transitive and conversely well-founded, i.e any valid formula have a proof and any theorem of **KL** is valid.

7 Perspective to proof theory

In the last section we proved that *provability logic* was weakly complete and sound with respect to the class of frames that were transitive and conversely well-founded. Robert Solovay used this result in Solovay [1976] and the model constructed in the proof to show that the logic **KL** catches everything that Peano Arithmetic can say about in modal terms about its own provability predicate.[13] In other words we can say that **KL** is arithmetical sound. This result makes it possible to use provability logic to construct arithmetical sentences that are not provable in Peano Arithmetics. Further it can be shown that **KL** is decidable and hence a part of the undecidable Peano Arithmetics can be studied with a decidable modal logic. So Solovay's theorem is of the places where the use of modal logic has played a part in widening over knowledge in another part of mathematics.[14]

So even though modal logic started out as something purely used in philosophy, and its early development was done by philosophers and logicians; the framework developed by Kripke and Hintikka has proven so general effective that it has found use in a lot of different areas.

[13] The proof and the setup for these theorem is technical and way beyond the scope of this project.

[14] There is some other places where modal logic has been used in mathematics. An overview can be found in chapter 16 in Blackburn et al. [2007]

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