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Master Thesis

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Provability Logic

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1 Introduction:

1.1 Notation

We denote the non-negative integers with ω; i.e $\omega = \{0, 1, 2, 3, \ldots\}$, and $\omega \times \omega = \omega^2$ and so on. Subsets of ω will be denoted by $A, B, C \ldots$ and arbitrary $\Gamma, \Phi, \Theta \ldots$. Lower case Latin letter a, b, c and x, y, z will denote integers. Further recursive functions will be denoted by $F, G, H, \ldots$, and other functions that are not recursive will be denoted by $f, g, h, \ldots$. $F(x) \downarrow$ will denote that $F(x)$ is defined and $F(x) = y$ denotes that $F(x)$ is defined and has value y. $F(x) \uparrow$ denotes that $F(x)$ is undefined. $\mathrm{dom}F$ and $\mathrm{im}F$ deontes the domain and image of $F(x)$. A few special functions have their own symbols: S for the successor function, C for the constant function and P for the projection. Relations will be denoted by R and $<, >, \leq, \geq$ will be used in the usual sense on integers.

We will use the standard symbols of propositional logic $\wedge, \vee, \rightarrow, \neg$. Further we will sometimes use $\exists$ and $\forall$ in the metalanguage of modal logic, and use them in the language of first-order arithmetics. The differences of these two uses should be clear from the context. We will argument the propositional logic with the modal unary connective $\Box$ and its dual $\Diamond := \neg\Box\neg$. We will denote modal formulas by the greek letters $\varphi, \psi, \theta, \ldots$. The symbol $\mathcal{H}$ will denote a Hintikka frame which is a tuple $\langle W, R \rangle$ where W is a set of "nodes" and R is a relation on $W$and $\mathcal{K}$ will denote a Kripke model, which is a Hintikka frame with a valuation ϕ on.

1.2 Historical Introduction

Both the studies of arithmetics and modal logic can be traced back to the accent Greeks.[1]

The first time that someone made a connection between these two different areas of mathematics was Kurt Gödel in his paper: {[a]sdf} To actually understand how this connection was made, a bit of the back story is needed.

A deeper look into the development of modal logic can be found in [reffff].W

[1]The study of arithmetics can be traced even further back

2 Preliminary

In this section I will state a few results and definitions from my two project I have written. These will be given without proof

3 Recursion Theory

We will begin this section with define two classes of functions; the *Partial Recursive Functions* and the *Turing Computable functions*. These two classes of functions gives rise to the same class of functions The idea behind these two classes of functions is to define what we intuitively mean with a computable function. The proofs in this section will be rather informal. The goal of this section will be to state and prove Kleene's recursion theorem, which will play a crucial role in our proof of the Solovay's completeness theorems. Further there will also be a few results and comments on the so called arithmetical hierarchy

3.1 Partial Recursive Functions and Recursive Functions

The partial recursive functions is an enlargement of the primitive recursive functions. The primitive recursive functions is defined in the following way:

Definition 3.1 The class of primitive recursive functions is the smallest class closed under the following schemata:

I. $S(x) = x + 1$ is primitive recursive.

II. $C(x) = k$ is primitive recursive.

III. $P_i^n(x_1, \dots, x_n) = x_i$ is primitive recursive.

IV. If $g, h_1, \dots h_m$ are primitive recursive then so is

$$f(x_1, \dots, x_n) = g(h_1(x_1, \dots, x_n), \dots, h_m(x_1, \dots, x_n)$$

V. If g and h are primitive recursive and $n \geq 1$ then f is also primitive recursive where:

$$f(0, x_1, \dots, x_n) = g(x_1, \dots, x_n)$$
$$f(x_1 + 1, x_2, \dots, x_n) = h(x_1, f(x_1, \dots, x_n), x_2, \dots, x_n)$$

⊣

In the appendix there is a collection of functions that are primitive recursive.

The primitive recursive functions do not include all computable functions, so we have to expand them in the following way:

Definition 3.2 The class of *partial recursive* (from now on some times called (p.r) functions is the least class closed under schemata I through V and the following VI schema.

VI. If $\theta(x_1, \dots, x_n, y)$ is a partial recursive function of $n+1$ variables and

$$\psi(x_1, \dots, x_n) = \mu y[\theta(x_1, \dots, x_n, y) \downarrow= 0 \\ \wedge \forall z \leq y[\theta(x_1, \dots, x_n, z) \downarrow]$$

Then ψ is a partial recursive function of n variables

A partial recursive function that is total is called a total recursive function; abbreviated to recursive function. ⊣

We will end this section with the following definition:

Definition 3.3 A relation $R \subset \omega^n$ where $n \geq 1$, is recursive (primtive recursive) if its characteristic function χ_R is recursive (primitive recursive). The case where $n = 1$ is the case where R is a set $A \subset \omega$ so we also have the definition of a set being recursive. ⊣

3.2 Turing Computable Functions

Another way to describe the intuitively computable functions is via a Turing machine.

Definition 3.4 A *Turing machine* M consists of a two-way infinite tape that is dived into different cells and a finite set of internal states $Q = \{q_0, \dots, g_n\}$, $n \geq 1$. Each is either blank: B or has a 1 written on them. The following three things can happen in a single step:

1. Change form one state to another.
2. Change the scanned symbol s to another symbol $s' \in S = \{1, B\}$
3. Move the reading head one cell to the right R or the left L.

The operation of M is controlled by a partial map:

$$\delta : Q \times S \to Q \times S \times \{R, L\}$$

Which may not be defined for all arguments. ⊣

The way to understand this definition is the following: if $(q, s, q', s', X) \in \delta$ it means that the machine M is in stage q where it scans symbol s then changes to state q' and replaces s by s'. Lastly it moves to the right if $X = R$ and the left if $X = L$. The map δ is called a Turing program if it can be views as a finite set of quintuples. If the input integer is x then it will be represented by a string of $x + 1$ consecutive $1's$, where all other cells are blank.

Further the machine M start in the state q_1 scanning the left-mst cell that contains a 1. The machine stops if it reaches the halting state q_0, and it will then output the number y which is the total number of 1's on the tape in the this state. If M with input x halts and outputs y we say that M computes the partial function $\psi(x) = y$

The conditi [Indfør noget mere tekst her]on a is determined by:

1. The current state q_i
2. The symbol s_0 that is being scanned.
3. The symbols on the tape to the right of s_0 up to the last 1. Denote this sequence by $s_1, s_2, \ldots s_n$
4. The symbols on the tape to the left of s_0 op to the first 1. Denote this sequence by $s_{-1}, s_{-2}, \ldots s_{-m}$

This is called the configuration of the machine and we can write it as follows:

$$s_{-m} \cdots s_{-1} q_0 s_0 s_1 \cdots c_n$$

Definition 3.5 A Turing computation according to to the Turing program P with input x is a sequence of configurations $c_0, c_1, \ldots, c_n$ such that c_0 represent the machine in the halting state q_0, and the transition $c_i \to c_{i+1}$ for all $i < n$ is giving by the Turing program P. ⊣

Proposition 3.1 Each Turing program P_e can be assigned a Gödel number e.

Proof. We will use the fact that each $x \in \omega$ has a unique prime decomposition:

$$x = p_0^{x_0} \cdots p_n^{x_n} \cdots$$

We can assign a number to each quintuple $(q_{i,j}, q_k, s_l, r_m)$ in a Turing program P in the following way[Omformuler denne sætning]:

$$p_0^{1+i} p_1^{1+j} p_2^{1+k} p_3^{1+l} p_4^{1+m}$$

Where we have that $r_0 = R$ and $r_1 = L$. Since the prime decomposition is unique, each different state of the program has a unique code. Each Turing is a sequence of different states and we can thus for an arbitrary Turing program P_e we let $e_0, \dots e_n$ denote the Gödel number of each different state and set:

$$e = p_0^{e_0} \cdots p_n^{e_n}$$

Thus each Turing program P_e has a unique Gödel number e. ⊣

Since each Turing program has a unique code we can list them and be able to find any program P_e by its code e. This gives the following definition:

Definition 3.6 The P_e be the Turing program with Gödel number e in the list and let $a_e^{(n)}$ be the function of n variables computed by P_e. Further let a_e abbreviate $a_e^{(1)}$. ⊣

3.3 Auxiliary Results

It can be proven that the two classes of functions; partial recursive and Turing computable functions gives rise to the same class of partial functions. This can be seen as evidence for *Church's Thesis* which states that this class of functions coincide with the function that we see as intuitively computable. In the rest of this project we will assume that the Church's Thesis is true.

We will being by proving the padding lemma, which states that each partial function φ_x has an infitine amount of indices.

Lemma 3.1 Each partial recursive function φ_i has $\aleph_0$ indicies, and for each x we can *effectively* find an infinite set A_x of indices for the same partial function.

Proof. For any program P_x that have internal states: $\{q_0, \dots, q_n\}$ we can add extra instructions $q_{n+1}B\ R, q_{n+2}B\ R, \dots$ such that we get a new program for the same computation. ⊣

The following theorem will show that each Turing computable function is in fact partial recursive. Latter we will prove that the converse also holds.

Theorem 3.1 There exist a predicate $T(e,x,y)$ and a function $U(y)$ that are primitive recursive such that:

$$\varphi_e(x) = U(\mu y T(e,x,y))$$

Proof. We showing that the predicate $T(e,x,y)$ exists and is primitive recursive. This predicate informally states that y is the code of Turing program P_e with input x. For each possible configuration c, we can assign a code:

$$\#(c) = 2^{1+i}3^{1+\#(s_0)}5^r7^l$$

Where $\#(s) = 0$ if $s = B$ and is equal to 1 otherwise, $r = \prod_{j\geq 1} p_j^{\#(s_j)}$ and $l = \prod_{j\leq -1} p_j^{\#(s_j)}$ We can now define the code of a Turing computation $c_0, c_1, \ldots c_n$ according to P_e to be:

$$y = 2^e \prod_{i\leq n} p_{i+1}^{\#(s_i)}$$

We can now define $T(e,x,y)$ to be [Læs i Kleene] Having defined the predicate T we can check if it holds. By proposition 2.1 we can "recover" the program P_e form e. Then we can recover the computation $c_0, c_1, \ldots c_n$ form y if y codes such a thing. We can now check if $c_0, c_1, \ldots, c_n$ is a computation according to P_e with x as the input in the first configuration c_0. If this is true, then $U(y)$ just outputs the number of 1's in the final configuration c_n. [Læs mere i Klenne] ⊣

This theorem also gives us that each partial recursive function can be created by two primitive recursive functions, with a single application of the μ-operator.

We will need the following notation in the next proof:

Definition 3.7 Set $\langle x,y\rangle$ to be the image of (x,y) under the injective recursive pairing function:

$$\frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$$

This function is from $\omega \times \omega$ onto ω.[Måske let mere?] ⊣

Theorem 3.2 For every $m,n \geq 1$ there exists an injective recursive function s_n^m of $m+1$ variable such that for all $x, y_1, \ldots, y_m$

$$\varphi^{(n)}_{s_n^m(x,y_1,\ldots,y_m)} = \lambda z_1, \ldots z_n[\varphi_x^{(m+n)}(y_1,\ldots,y_m,z_1,\ldots,z_n)]$$

Proof. I will follow Soare and only proof the case where $m = n = 1$. I.e the case where we have to proof:

$$a_{s_1^1(x,y)}(z) = [a_x^2(y,z)]$$

Let x and y på given. Then $s_1^1(x,y)$ can be described as follows:

1. Let P_x the Turing program with code x.
2. Change P_x into another Turing program $P_{x'}$ such that: $P_{x'}$ writes $y + 1$ "1" left of the input, such that there is a B between these 1 and the other input. Further i places the head to the left of the new input and proceeds to *run* P_x.
3. outputs x'

it is clear that $P_{x'}$ on input z compute the same as p_e would on input (x, y); i.e $\varphi_{x'} = \varphi_x^{(2)}(y, z)$. Further we have that $x' = s_s^1(x, y)$ By Church's Thesis the function $s = s_1^1$ is recursive, since it can be computed effectively. If it is not injective it can be replaced by a injective recursive function s' such that $\varphi_{s(x,y)} = \varphi_{s'(x,y)}$ by using the padding lemma and by defining $s'(x, y)$ in increasing order of $\langle x, y\rangle$. $\dashv$

3.4 Recursively Enumerable sets and the Graph of a function

In this section we will introduce the two concepts *recursive enumerable* sets and the *graph* of a function. We will further show that there is connection between these two concepts.

Definition 3.8 A set A is recursively enumerable (r.e.) if A is the domain of some primitive recursive function. Further we define the following two sets:

1. We let the eth r.e set be denoted by:

$$W_e = \text{dom } f_e = \{x : f(x) \downarrow\} = \{x : \exists y T(e, x, y)\}$$

2. $W_{e,s} = f_{e,s}$

$\dashv$

We asd???

Definition 3.9 A set A is the projection of some relation $R \subseteq \omega \times \omega$ if $A = \{x : \exists y : R(x, y)\}$. We further say that a set A is in Σ_1 form, if A is the projection of some recursive relation $R \subseteq \omega \times \omega$. $\dashv$

We can now show the following theorem:

Theorem 3.3 A set A is r.e iff A is Σ_1.

Proof. ($\Rightarrow$) Since A is r.e we have that $A = W_e = \text{dom} f_e$ for some e. This means that:

$$x \in W_e \Leftrightarrow \exists s(x \in W_{e,s}) \Leftrightarrow \exists s(T(e, x, s))$$

Since the relation T is primitive recursive and have that the set A is the projection a recursive relation.

($\Leftarrow$) Let $A = \{x : \exists y(Rx, y)\}$ where R is recursive. We then have that $A = \text{dom} f$ where $f(x) = \mu y(R(x, y))$ and thus A is r.e $\dashv$

We can say?

Theorem 3.4 If there is a recursive relation $R \subseteq \omega^{n+1}$ and if we have the following set:

$$A = \{x | \exists y_1 \ldots \exists y_n R(x, y_1, \ldots, y_n)\}$$

Then the set A is Σ_1

Proof. We will start of by defining the relation $S \subseteq \omega^2$ as follows:

$$S(x, z) \Leftrightarrow R(x, (z)_1, \ldots, (z)_n)$$

Where we have the following prime decomposition of z:

$$z = p_1^{(z)_1} \cdots p_k^{(z)_k}$$

Then the following equivalences holds:

$$\begin{aligned} \exists z S(x, z) &\Leftrightarrow \exists z R(x, (z)_1, \ldots (z)_n) \\ &\Leftrightarrow \exists y_1 \cdots \exists y_n R(x, y_1, \ldots, y_n) \end{aligned}$$

And thus the set A is clearly Σ_1. $\dashv$

From this theorem we can easily get the following corollary:

Corollary 3.1 The projection of an r.e relation is r.e

The next definition will also play a role in our proof of Solovay's completeness theorems.

Definition 3.10 The graph of a function f is the relation:

$$(x, y) \in \text{graph} f \Leftrightarrow f(x) = y$$

$\dashv$

Text

Theorem 3.5 If $R \subseteq \omega^2$ is an r.e relation, then there is a p.r function sel called the selector function for R such that:

$$\text{sel}(x) \text{ is defined } \Leftrightarrow \exists y(R(x,y))$$

and if this is the case we have that $(x, f(x)) \in R$

Proof. Since R is r.e it is Σ_1. This means that there is a recursive relation S such that $R(x,y)$ holds iff $\exists z(S(x,y,z))$. Thus we can define the following primitive recursive function:

$$g(x) = \mu u(S(x,(u)_1,(u)_2))$$

And now we put $f(x) = (g(x))_1$ ⊣

It will be the following theorem we will use in our proof later on.

Theorem 3.6 A partial function f is partial recursive iff its graph is recursive enumerable.

Proof. ($\Rightarrow$) The graph of f_e is r.e by theorem ??? and the definition of a graph.

($\Leftarrow$) Since the graph of f is assumed to be r.e we can conclude that f is its own prmitive recursive selector function. This is that $R = \text{graph} f$ can only have f as its selector function. [Whyyy?] ⊣

3.5 The Recursion Theorem

In this section we will state and prove the recursion theorem. It will be crucial in the next [chapter?], since we will need it to define a function.

Theorem 3.7 For every recursive function f there exists a fixed point n such that $\varphi_n = \varphi_{f(x)}$

Proof. We will start of be defining the following *diagonal* function $d(u)$ as:

$$\varphi_{d(u)}(z) = \begin{cases} \varphi_{\varphi_u(u)}(z) & \text{if } \varphi_u(u) \text{ converges} \\ \text{undefined} & \text{else} \end{cases} \tag{3.1}$$

By the $s-m-n$ theorem we have that the function d is injective and total. Further it is clearly seen that d is independent of f.

Given an arbitrary f we will choose an index v such that:

$$\varphi_v = f \circ d \tag{3.2}$$

Now set $n = d(v)$. We will show that this is a fixed point for the function f. Since f is total we also have that $f \circ d$ is total. This means that $\varphi_v(v)$ converges and that $\varphi_{d(v)} = \varphi_{\varphi_v(v)}$. Thus we have:

$$\varphi_n = \varphi_{d(v)} = \varphi_{\varphi_v(v)} = \varphi_{f \circ d(v)} = \varphi_{f(n)}$$

The second equality sign follows from (2.1) and the third follows from (2.2). $\dashv$

Following [Owens, find ref], the argument in the proof can be seen as a digitalization argument that fails. Commonly when we apply a digitalization argument, we have a class of sequences, with terms from an set A, that we arranges as the rows in a square matrix. We then have a map $h : A \to A$ that induces a operation h^* on the set of sequences such that if $\langle s(i), i \in I\rangle$ is a sequences in our matrix then

$$h^*(\langle s(i), i \in I\rangle) = \langle h(s(i)), i \in I\rangle$$

After having defined this map we will use it on the sequences that consists of the elements of the diagonal of the matrix and show that the resulting sequences is not one of the original sequences.

The digitalization argument "fails" in our case, since the sequences of the diagonal is already already one of the rows and thus the image h^* of this sequences will also be one of the rows; i.e the h has a fixed point.

The start of the proof can be seen as the following lemma:

Lemma 3.2 There is a diaognal function $d(u)$ such that:

$$\varphi_{d(u)}(z) = \begin{cases} \varphi_{\varphi_u(u)}(z) & \text{if } \varphi_u(u) \text{ converges} \\ \text{undefined} & \text{else} \end{cases} \tag{3.3}$$

Most of the times where one uses the Recursion Theorem, one actually uses this lemma to construct the given function.

3.5.1 Application of the Recursion Theorem

The recursion theorem is a "powerful" tool. It enables us to define a partial recursive function, which uses its own index as part of its definition. This The recursion theorem overrides this "self-reference" because we are using the $s-m-n$ theorem to define a function $f(x)$ and $\varphi_{f(x)}(z) = (\ldots, x \ldots)$ and then taking a fixed point: $\varphi_n = \varphi_{f(n)}$. When we are making constructions like this the only thing we cannot do is use specific properties of the function φ_n. We will use the theorem in this way in our proof of Solovay's Completeness Theorems to define a function with help of the functions own Gödel number; and the recursion theorem makes this a viable tactic.

The following examples will show a few uses of this theorem.

Example 3.1 We will show that there is a n such that:

$$W_n = \{n\}$$

We start of by using the $s - m - n$ theorem to define $W_{f(x)} = \{x\}$ then by the recursion theorem we can choose n such that we have:

$$W_n = W_{f(n)} = \{n\}$$

$\dashv$

3.6 The Arithmetical Hierarchy

Noget klort her :w

Definition 3.11 We define the sets Σ_n and Π_n in the following way:

1. A set A is in Σ_0 (Π_0) if and only if A is recursive.
2. For $n \geq 1$ the set A is in Σ_n if there is a recursive relation $R(x, y_1, \ldots, y_n)$ such that:
$$x \in A \text{ iff } \exists y_1 \forall y_2 \exists y_3 \cdots Q y_n R(x, y_1, \ldots, y_n)$$
Here Q is $\exists$ if n is odd and Q is $\forall$ if n is even. We define A being in Π_n likewise. A is in Π_n if:
$$x \in A \text{ iff } \forall y_1 \exists y_2 \forall y_3 \cdots Q y_n R(x, y_1, \ldots, y_n)$$
3. A is in Δ_n if $A \in \Sigma_n \cap \Pi_n$

We further say that a formula φ is Σ_n (Π_n) if it is Σ_n (Π_n) as a relation of the variables that are free in it. $\dashv$

In the rest of this project we will mostly look at formulas that are either Σ_n or Π_n and not sets, that have this property.

We can show a few properties of these sets.

Proposition 3.2

1. $A \in \Sigma_n \Leftrightarrow \overline{A} \in \Pi_n$
2. $A \in \Sigma_n(\Pi_n) \Rightarrow (\forall m > n)(A \in \Sigma_m \cap \Pi_m)$
3. $A, B \in \Sigma_n(\Pi_n) \Rightarrow A \cup B, A \cap B \in \Sigma_n(\Pi_n)$

4. $(R \in \Sigma_n \wedge n > 0 \wedge A = \{x : \exists y R(x, y)\}) \Rightarrow A \in \Sigma_n$

5. $(B \leq_m A \wedge A \in \Sigma_n) \Rightarrow B \in \Sigma_n$

6. If $R \in \Sigma_n(\Pi_n)$ and A and B are defined by:

$$\langle x, y \rangle \in A \Leftrightarrow \forall z < y R(x, y, z)$$

and

$$\langle x, y \rangle \in B \Leftrightarrow \exists z < y R(x, y, z)$$

Then we have $A, B \in \Sigma_n(\Pi_n)$

Proof. 1. If we have that:

$$A = \{x : \exists y_1 \forall y_2 \cdots R(x, y_1, \ldots)\}$$

Then we have:

$$\overline{A} = \{x : \forall y_1 \exists y_2 \cdots \neg R(x, y_1, \ldots)\}$$

Which is clearly Π_n.

2. If for example $A = \{x : \exists y_1 \forall y_2 R(x, y_1, y_2)\}$, then we can make the following reformulation of A:

$$A = \{x : \exists y_1 \forall y_2 \exists y_3 (Rx, y_1, y_2) \wedge y_3 = y_3)\}$$

This kind of reformulation can be done for any set in Σ_n (Π_n)

3. Let the following to sets be defined:

$$A = \{x : \exists y_1 \forall y_2 \cdots R(x, y_1, y_2, \ldots)\}$$

$$B = \{x : \exists z_1 \forall z_2 \cdots S(x, z_1, z_2, \ldots)\}$$

Then we have:

$$\begin{aligned} x \in A \cup B &\Leftrightarrow \exists y_1 \forall y_2 \cdots R(x, y_1, y_2, \ldots) \vee \exists z_1 \forall y_2 \cdots S(x, z_1, z_2 \ldots) \\ &\Leftrightarrow \exists y_1 \exists z_1 \forall y_2 \forall z_2 \cdots (R(x, y_1, y_2, \ldots) \vee S(x, z_1, z_2, \ldots)) \\ &\Leftrightarrow \exists u_1 \forall u_2 \cdots (R, (u_1)_0, (u_2)_0, \ldots) \vee S(x, (u_1)_1, (u_2)_1, \ldots)) \end{aligned}$$

Which is clearly Σ_n. The same argument can be made for $A \cap B$ and for Π_n sets.

4. This follows by quantifier contraction, in the same way as (3)

5. Let

$$A = \{x : \exists y_1 \forall y_2 \cdots R(x, y_1, y_2, \ldots)\}$$

And let $B \leq_m A$ via the function f. Then we have:

$$B = \{x : \exists y_1 \forall y_2 \cdots R(f(x), y_1, y_2, \ldots)\}$$

6. We will prove this by induction on n.

Base case: Let $n = 0$. Then A and B are clearly recursive.

Induction step: Now assume that $n > 0$ and suppose that $R \in \Sigma_n$. Our induction hypothesis says that (6) is true for all $m < n$. Then by (4) we have that $B \in \Sigma_n$. Further we have $S \in \Pi_{n+1}$ such that the following holds:

$$\begin{aligned}\langle x, y\rangle \in A &\Leftrightarrow (\forall z < y) R(x, y, z)\\ &\Leftrightarrow (\forall z < y) \exists u S(x, y, z, u)\\ &\Leftrightarrow \exists \sigma (\forall z < y) S(x, y, z, \sigma(z))\end{aligned}$$

WE have that σ range is in $\omega^{<\omega}$. By the indction hypothesis we have that $(\forall z < y) S \Pi_{n+1}$ but by the above deduction we must then have that $A \in \Sigma_n$. $\dashv$

3.7 Sigma completeness, put ind hvor dette passer i overstående

We can make a generalization of $D3$ by using [Lemma fra fagprojekt]. We use these genieralization in some of our proofs later on.

Theorem 3.8 Let $f v_0 \ldots v_{n-1}$ be a Σ_1 formula with free variables. Then:

$$\mathbf{PRA} \vdash f v_0 \ldots v_{n-1} \rightarrow \mathrm{Pr}(\ulcorner f v_0 \ldots v_{n-1} \urcorner)$$

Proof. By [ja, fra hvad?] we have that there is a Σ_1 formula $\exists v(g v v_0 \ldots v_{n-1} = \overline{0}$ such that we have:

$$\mathbf{PRA} \vdash f v_0 \ldots v_{n-1} \leftrightarrow \exists v(g v v_0 \ldots v_{n-1} = \overline{0}) \tag{3.4}$$

and by $D1$ we have:

$$\mathbf{PRA} \vdash \mathrm{Pr}(\ulcorner f v_0 \ldots v_{n-1} \leftrightarrow \exists v(g v v_0 \ldots v_{n-1} = \overline{0}) \urcorner) \tag{3.5}$$

We can now make the following deductions:

$$
\begin{aligned}
\mathbf{PRA} \vdash fv_0 \dots v_{n-1} &\to \exists v(gvv_0 \dots v_{n-1} = \overline{0}) && \text{By 3.4} \\
&\to \exists v \mathrm{Pr}(\ulcorner hvv_0 \dots v_{n-1} = \overline{0}\urcorner && \text{By theorem fagprojekt?)} \\
&\to \mathrm{Pr}(\ulcorner \exists v(hvv_0 \dots v_{n-1} = \overline{0})\urcorner) && \text{By D1 and D2} \\
&\to \mathrm{Pr}(\ulcorner fv_0 \dots v_{n-1}\urcorner) && \text{By 3.5}
\end{aligned}
$$

$\dashv$

4 Fragments of Arithmetics

In this section it will be shown how much induction there is in **PRA**and other arithmetical theories. We will look at different axioatizable subtheories of first order arithmetic called fragments. These fragments can be categories in the following three categories:

Strong fragments: The fragments that can prove the arithmetized cut elimination theorem.

Weak fragments: Those fragments that can not prove the arithmetized cut-elimination theorem.

Very weak fragments: The fragments which do not contain any induction axioms

We will start of by defining a very weak fragment called Robinson's theory. This fragments was first considered in [Ref]. We will denote this theory with Q This theory have the following axioms:

$$\begin{aligned}
&\forall x(\neg Sx \neq 0) \\
&\forall x \forall y(Sx = Sy \rightarrow x = y) \\
&\forall x(x \neq 0 \rightarrow \exists y(Sy = x) \\
&\forall x(x + 0 = x) \\
&\forall x \forall y(x + Sy = S(x + y)) \\
&\forall x(x \cdot 0 = 0) \\
&\forall (x \cdot Sy = x \cdot y + x)
\end{aligned}$$

This theory dose not have the inequality symbol. We will extend Q with the following axiom:

$$x \leq y \leftrightarrow \exists z(x + z = y)$$

This extension of Q is denoted by $Q_{\leq}$

Definition 4.1 Giving a class Γ of either Σ_n or Π_n formulas we define Γ-*induction* (Γ-*ind*) to be the following schema:

$$\varphi(0) \wedge \forall v(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall v \varphi(v)$$

for $\varphi \in \Gamma$. Further we define Γ-*Least Number Principle* (Γ-*MIN*) to be the following schema:

$$\exists x\varphi(x) \to \exists(\varphi(x) \land \neg\exists y(y < x \land \varphi(y)))$$

For $\varphi \in \Gamma$. Lastly we will define the replacement axioms for Γ, (Γ-REPL) as the following formulas:

$$(\forall x \leq t)\exists y\varphi(x,y) \to \exists z(\forall x \leq t)(\exists y \leq z)\varphi(x,y)$$

? $\dashv$

From the above axioms we can create a hierarchy of different strong fragments of arithmetics. We will define the following theories:

Definition 4.2 We define the following fragments:

1. The theory $I\Sigma_n$ is the theory that is axiomatized by the axioms of $Q_{\leq}$ and the Σ_n-IND axioms.
2. The theory $I\Delta_0$ is $Q_{\leq}$ plus the Δ_0-IND axioms.
3. The theory $L\Sigma_n$ is defined by the theory $I\Delta_0$ plus the Σ_n-MIN axioms
4. The theory $B\Sigma_n$ is defined by the theory $I\Delta_0$ plus the Σ_n-REPL axioms.
5. Lastly the theory Peano arithmetics is defined as the theory Q plus induction for all first order formulas.

$\dashv$

From the definition it is clear that the theory $I\Delta_0$ plays a crucial role. It can be shown that in Q a lot of the basic facts about arithmetic can be shown. These facts will not be shown here, but a list of them can be found in [Buss].

Definition 4.3 A predicate symbol $\dashv$

Definition 4.4 A function symbol. $\dashv$

4.1 PRA

Proposition 4.1 **PRA** is the thoery?

We will further define $\mathbf{PRA}^-$ as being the sub-theory of **PRA** where we restrict us self to only having induction for p.r formulas. Then we get the following result:

Proposition 4.2 Over $\mathbf{PRA}^-$ the following schemata are equivalent:

1. Σ_n-*Ind*
2. Π_n-*Ind*
3. Σ_n-*LNP*
4. Π_n-*LNP*

Proof. $\dashv$

5 General results on GL

In this section we will prove a lot of different properties about **GL**, that will be used in the following chapters. Further in the end we will introduce the system **GLS**.

But first we will look at a more general modal logic called **K4**.

5.0.1 The modal logic K4

We will start of by defining the modal logic **K**.

Definition 5.1 The modal logic **K** is the modal logic that has the following axioms and rules of inferences:

Axioms $\quad A1:$ All propositional tautologies.
$A2:$

⊣

We can now define the modal logic **K4**.

Definition 5.2 **K4** is **K** with the following axiom added:

$$\Box\varphi \to \Box\Box\varphi$$

⊣

We will start of by defining the system **GL**.

Definition 5.3 **GL** is the modal system with the following axioms and rules of inferences:

A1 All propositional tautologies.

A2 $\Box\varphi \land \Box(\varphi \to \psi) \to \Box\psi$

A3 $\Box(\Box\varphi \to \varphi) \to \Box\varphi$

R1 $\varphi, \varphi \to \psi \vdash \psi$

R2 $\varphi \vdash \Box\varphi$

$\dashv$

Further a Kripke model $\mathcal{K}$ of **GL**is a tuple such that $\mathcal{K} = \langle W, R, \phi \rangle$ where $W = \{0, \ldots, n\}$ and where R is a conversely well-founded transitive relation on W. Further the valuation function ϕ is defined in the usual way. :

5.1 Tress and GL

In [Tsss] it was shown that **GL** was complete with resect to conversely well-founded transitive frames. We will first start of by defining the notion of a tree.

Definition 5.4 A tree is a tuple $(W, <, w_0)$ where $(W, <)$ where:

1. $<$ is transitive and asymmetric.
2. w_0 is the minimal element of $<$. I.e $w_0 < w$ for all $w \in W$.
3. The set of predecessors of any element is finite and linearly ordered byh $<$.

$\dashv$

It is clear that finite trees are conversely well-founded frames. This fact is crucial in the proof of the following theorem:

Theorem 5.1 Let φ be a modal sentence. Then the followig are equivalent:

1. $\vdash_{\mathbf{GL}} \varphi$
2. φ is true in all models on finite tress
3. φ is valid in all models on finite tress

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follows by the completeness theorem. We also have that (2) $\Leftrightarrow$ (3) is true [Kom med et kort argument]. Thus we will just have to show (2) $\Rightarrow$ (1)

We will show this by contraposition. Assume that $\vdash_{GL} \varphi$ and let $\mathcal{K}$ be the following model $\mathcal{K} = \langle W, R, w_0, \phi \rangle$ be a counter model i.e $\nvDash_{w_0}^{K'} \varphi$. The goal is now to define a finite tree model: $\mathcal{K}_T = \langle W_T, <_T, \phi_T \rangle$. We will do this be letting W_T consists of finite R-increasing sequences form K.

Stage 0: Let the sequence w_0 be a part of K_T.

Stage $n+1$: For each sequence $(w_0, \ldots w_n) \in W_T$ we will look at $\Gamma = \{\Box\psi \in S(\varphi) : \vDash_{w_n} \Box\psi\}$. If $\Gamma = \emptyset$ then we do not extend the sequence $(w_0, \ldots w_n)$.

Otherwise we will for each $\Box\psi \in \Gamma$ choose a node $v \in W$ such that $w_0 Rv$ and such that we have:

$$\models_v \Box\psi,\ \not\models_v \psi$$

We can do this because of axiom 3. We will then add the sequence $(w_0, \dots w_n, v)$ to the set W_T

The model $\mathcal{K}_T$ then further consists of $<_T$ is the strict ordering by extension of finite sequences. ϕ_T is the defined in the following way:

$$(w_0, \dots, w_n) \in \phi_T(p) \Leftrightarrow w_n \in \phi(p)$$

This leads way to the following notation that we will use in the rest of the proof:

$$\models^{\mathcal{K}_T}_{(w_0,\dots,w_n)} p \Leftrightarrow \models^{\mathcal{K}}_{w_n} p$$

We will now prove two claims, by which the theorem will follow

Claim 1: $(W_T, <_T, w_0)$ is a finite tree with origin (w_0). It is clear that that is a finite tree with origin (w_0). **Udfyld og forstå resten**

Claim 2: For all $\psi \in S(\varphi)$ and for all $(w_0, \dots w_n) \in W_T$ we have:

$$\models^{\mathcal{K}_T}_{(w_0,\dots,w_n)} \psi \Leftrightarrow \models^{\mathcal{K}}_{w_n} \psi$$

This proof is done by induction on the complexity of ψ. We will only look at the case $\psi = \Box\theta$. So we have by the induction hypothesis:

$$\begin{aligned}
\models_{w_0} \Box\theta &\Rightarrow \forall v(w_0 Rv \Rightarrow \models_v \theta)\\
&\Rightarrow \forall v((w_0, \dots w_n, v) \in W_T \Rightarrow \models_v \theta)\\
&\Rightarrow \forall v((w_0, \dots w_n, v) \in W_T \Rightarrow \models^{\mathcal{K}_T}_{(w_0,\dots w_n,v)} \theta)
\end{aligned}$$

And the last line is the same as $\models_{(w_0,\dots,w_n)} \Box\theta$ For the other way, we will use contraposition:

$$\begin{aligned}
\not\models_{w_0} \Box\theta &\Rightarrow \exists v(w_0 Rv \;\&\; \not\models_v \theta)\\
&\Rightarrow \exists v((w_0, \dots w_n, v) \in W_T \;\&\; \not\models_v \theta)\\
&\Rightarrow \exists v((w_0, \dots w_n, v) \in W_T \;\&\; \not\models^{\mathcal{K}_T}_{(w_0,\dots,w_n,v)} \theta)
\end{aligned}$$

And the last line is the same as $\not\models_{(w_0,\dots,w_n)} \Box\theta$.

The theorem now follows since we have:

$$\not\models_{w_0} \varphi \Rightarrow \not\models^{\mathcal{K}_T}_{[w_0)} \varphi$$

$\dashv$

5.2 The system GLS

In this section we will define the modal logic **GLS**. Before we can do this in a smart way, we will define the notion of a $R2$ free axiomation; i.e a system that is not closed under $R2$.

5.2.1 Avoiding *R2*

Metatekst

We will need the following notation to define this. Given a set Γ:

1. $\Gamma \vdash \varphi$ if φ is derivable form Γ by only using $R1$
2. $\Gamma \vdash_2 \varphi$ if φ is derivable form Γ by using $R1$ and $R2$.

We will further define the following set for any set Γ:

$$\Gamma^2 = \Gamma \cup \{\Box\psi : \psi \in \Gamma\}$$

This set allows us to avoid $R2$. Why? We can now state and prove the following lemma:

Lemma 5.1 Let Γ include all the instanceses of $A1 - A3$. THen:

$$\Gamma \vdash_2 \varphi \Leftrightarrow \Gamma^2 \vdash \varphi$$

Proof. The way $\Leftarrow$ is trivial. So we will only show $\Rightarrow$. To prove this, it is enough to show that the set of theorems of Γ^2 is closed under $R2$; i.e to show that:

$$\Gamma^2 \vdash \varphi \Rightarrow \Gamma \vdash \Box\varphi$$

Let $\Gamma^2 \vdash \varphi$. Then we have a sequence $\varphi_1, \dots, \varphi_n = \varphi$, where for each $1 \leq i \leq n$ we have that φ_i is one of the following three:

1. We have that it is an axiom $\varphi_i \in \Gamma$
2. It is the necessitation $\Box\psi_1$ of an axiom $\psi_1 \in \Gamma$
3. It is a consequence of $R1$ of $\varphi_j, \varphi_k = \varphi_j \to \varphi_i$ where $j, k < i$.

We will show by induction on the lenght i of a 'intial' segment $\varphi_1, \dots, \varphi_i$ that $\Gamma^2 \vdash \Box\varphi_i$.

1. If we have that $\varphi_i \in \Gamma$. Then by definition we have that $\Box\varphi_i \in \Gamma^2$ and thus $\Gamma^2 \vdash \Box\varphi_i$.

2. If $\varphi_i = \Box\psi_i$ for some $\psi_i \in \Gamma$, then we can make the following deduction:

$$\begin{array}{lr} \Gamma^2 \vdash \Box\psi_i & \text{Since } \Box\psi_i \in \Gamma^2 \\ \quad \vdash \Box\psi_i \to \Box\Box\psi_i & \text{By } A3 \\ \quad \vdash \Box\Box\psi_i \; (= \Box\varphi_i) & \text{By } R1 \end{array}$$

   Which shows the result.

3. If φ_i follows form $\varphi_j, \varphi_k = \varphi_j \to \varphi_i$ by $R1$ we have by the induction hypothesis that $\Gamma^2 \vdash \Box\varphi_j$ and $\Gamma^2 \vdash \Box(\varphi_j \to \varphi_i)$. By $A2$ we also have that $\Gamma^2 \vdash \Box\varphi_j \wedge \Box(\varphi_j \to \varphi_i) \to \Box\varphi_i$ and by propositional logic we get that: $\Gamma^2 \vdash \Box\varphi_i$.

This ends the induction and the lemma will follow. ⊣

By this lemma we can get a $R2$ free definition of **GL**that we will call $\mathbf{GL}^2$. We can further by the deduction theorem from propositional logic get the following:

Lemma 5.2 $\Gamma \vdash \varphi$ iff there is a finite set $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq \Gamma$ such that: $K \vdash \wedge\theta_i \to \varphi$

5.2.2 Definition of GLS

Definition 5.5 The system of modal logic **GLS** is defined as the system of propositional modal logic that have all the theorems of **GL** and all the sentences $\Box\varphi \to \varphi$ as its axioms and which sole rule of inference is *modus ponens*. ⊣

It should be noted that this modal logic is not a *normal* one. The main theorem about this modal logic is the following:

Theorem 5.2 Let φ be a modal sentences. Define the following set: $S_\Box(\varphi) = \{\Box\psi : \Box\psi \text{ is a subformula of } \varphi\}$ the set of subformulas of φ that is boxed. Then the following to statements are equivalent:

1. $\vdash_{\mathbf{GLS}} \varphi$

2. $\vdash_{\mathbf{GL}} \bigwedge_{\Box\psi \in S_\Box(\varphi)} (\Box\psi \to \psi) \to \varphi$

Decidudud???

Definition 5.6 A model $\mathcal{K} = \langle W, R, \phi \rangle$ with start node w_0 is φ-sound if for every $\Box\psi \in S_\Box(\varphi)$ we have: $\vDash_{w_0} \Box\psi \to \psi$ ⊣

We can reduce theorem 5.2 to the following:

Theorem 5.3 Let φ be a modal sentences. Then:

1. $\vdash_{\mathbf{GLS}} \varphi$ iff φ is true in all φ-sound models of **GL**
2. $\vdash_{\mathbf{GL}} \varphi$ iff φ is valid in all φ-sound models of **GL**

Proof that 5.3 ⇒ 5.2: Assume 5.3.1. We hen have that $\vdash_{\mathbf{GLS}} \varphi$ iff φ is true in all φ-sound models of **GL**; which is the same all models of $\mathbf{GL} + \bigwedge_{\Box\psi \in S_{\Box}(\varphi)}(\Box\psi \to \psi) = \mathbf{GL}^s$ and hence:

$$\vdash_{\mathbf{GLS}} \varphi \Leftrightarrow \vdash_{\mathbf{GL}^s} \varphi$$

$\dashv$

For the proof of 5.3 we will make use of the following usefull construction:

Definition 5.7 Let $\mathcal{K} = \langle W, R, \phi\rangle$ be a Kripke Model. The derived model $\mathcal{K}'$ is defined as follows: $W' = W \cup \{w'\}$, where $w' \notin W$. We further have that R' is defined as $wR'v$ iff WRv for $w, v \in W$; w' is the new minimum element. Lastly we have that $w \in \phi'(p)$ iff $w \in \phi(p)$ for $w \in K$, where p is any atomic formula. Further we have that $w' \in \phi'(p)$ iff $w_0 \in \phi(p)$. $\dashv$

We can further define the notion of a sequences of successive derived models: $\mathcal{K}^{(1)}, \mathcal{K}^{(2)}, \ldots$ that have $w_1, w_2, \ldots$ as their minima. We define $\mathcal{K}^{(1)} = \mathcal{K}$ and $\mathcal{K}^{(n+1)} = (\mathcal{K}^{(n)})'$ and let w_n denote the minimum of $\mathcal{K}^{(n)}$. This can graphically been seen as the following:

Indfør tegninger

It should be noted that for $w \in \mathcal{K}$ and for any sentence θ that we have:

$$\vDash_w \theta \text{ in } \mathcal{K} \Leftrightarrow \vDash^{(n)}_w \theta \text{ in } \mathcal{K}^{(n)}$$

For the rest of this project we will omit the annoying superscripts.

The following lemma will explain a bit deeper how these derived models works:

Lemma 5.3 Let $\mathcal{K}$ be a φ-sound model, $S(\varphi)$ the set of subformulas of φ and $\mathcal{K}^{(n)}$ be the n-derived model. Then for all $\psi \in S(\varphi)$ we have:

$$\vDash_{w_0} \psi \Leftrightarrow \vDash_{w_n} \psi$$

Proof. We will prove this by induction on n. The lemma will follow from the case $n = 1$ [Why?], and this will be proven by induction om complexity of ψ. The only non trivial case is the case where $\psi = \Box\theta$, so this is the only case that will be shown:

Let $\psi = \Box\theta \in S(\varphi)$.

(⇒) The following is clear for $\Box\theta \in S(\varphi)$:

$$\vDash_{w_0} \Box\theta \Rightarrow \vDash_{w_0} \theta$$

This follows since our model is φ-sound. Therefore we have: $\forall v > w_1(\vDash_{w_1} \theta)$ and thus $\vDash_{w_1} \Box\theta$

(⇐) Here we can make the following deduction:

$$\begin{aligned} \vDash_{w_1} \Box\theta &\Rightarrow \forall v > w_1(\vDash_v \theta) \\ &\Rightarrow \forall v > w_0(\vDash_v \theta) \\ &\Rightarrow \vDash_{w_0} \Box\theta \end{aligned}$$

⊣

We can now return to our proof of the main theorem of this section by proving 5.3

Proof of 5.3. We will start of by proving (1). One way is easy:

(⇐) If φ is true in all φ-sound models $\mathcal{K}$ of **GL** then:

$$\vdash_{\mathbf{GL}^s} \varphi$$

And therefore we have $\vdash_{\mathbf{GLS}} \varphi$.

(⇒) This way takes a bit more work; it will be proven by contraposition. So suppose that φ is false in φ-sound model $\mathcal{K} = \langle W, R, \phi \rangle$. Then we have:

$$\vDash_w \bigwedge_{\Box\psi \in S_\Box(\varphi)} (\Box\psi \to \psi)$$

But still also have: $\nvDash_{w_0} \varphi$. We will show that $\nvdash_{\mathbf{GLS}} \varphi$ by showing that for any finite set Γ that for the system $\mathbf{GL}^\Gamma = \mathbf{GL} + \bigwedge_{\theta\in\Gamma}(\Box\theta \to \theta)$ that we have:

$$\nvdash_{\mathbf{GL}^\Gamma} \varphi$$

By lemma 5.3 we have that every model in the sequence $\mathcal{K}^{(0)}, \mathcal{K}^{(1)}, \dots$ is an φ-sound model, wherein φ is false. We will just have to show that for any finite set Γ that is a number m such that:

$$\vDash_{w_m} \bigwedge_{\theta\in\Gamma} (\Box\theta \to \theta)$$

This can be shown by showing that there is a m such that for each $n > m$ that the following holds:

$$\vDash_{w_n} \Box\theta \rightarrow \theta$$

But we either have that $\vDash_{w_m} \Box\theta$ for all n, thus $\vDash_{w_n} \theta$ for all n; or we have $\nvDash_{w_m} \Box\theta$ for some m and thus $\nvDash_{w_n} \Box\theta$ and $\vDash_{w_n} \Box\theta \rightarrow \theta$ for al $n > m$. This shows (1). We will now prove (2).**Bevis dette i hånden**

The implication $\Rightarrow$ is trivial. $\dashv$

6 Fixed point lemma

In this section we will prove the so called *fixed point theorem* for **GL**. Before stating this theorem, we will have to use the following two definitions:

Definition 6.1 We abbreviate $\Box\varphi \wedge \varphi$ as $\boxdot\varphi$ for every φ in our langauge. $\dashv$

Remark 6.1 By our semantic of modal logic, this we have that $\vDash_w \boxdot\varphi$ is true iff $\vDash_v \varphi$ for all $v \in \{w\} \cup \mathrm{acc}(w), where \mathrm{acc(w)}$ are $\mathrm{acc}(w)$ is the collection of "states" that can be "seen" from w, i.e: $\mathrm{acc} = \{v \in W : wRv\}$

Lemma 6.1 Måske Olsons lemma 3.

Another useful definition that we can make, since we for every Kripke model $\mathcal{K} = \langle W, R, \phi\rangle$ have that W is a finite set is the following:

Definition 6.2 Let $\mathcal{K} = \langle W, R, \phi\rangle$ be a Kripke model. The $\mathcal{K}$-rank of the notes $w \in W$ is defined in the following way: $\mathcal{K}-\mathrm{rank}(w) = 0$ iff there is no world v such that wRv. Otherwise we have that $\mathcal{K}-\mathrm{rank}(w) = 1+\max\{\mathcal{K}-\mathrm{rank}(v) : wRv\}$. $\dashv$

Since W is finite and that R is irreflexive we have that the for each $w \in W$ the $\mathcal{K}$-rank of w is unique.

Definition 6.3 A sentence φ is called modalized in p if every occurence of p in φ is under the scope of $\Box$. $\dashv$

We will also need the following definition:

Definition 6.4 A sentence φ is said to be n-decomposable iff for some sequence $q_1, \ldots, q_n$ consisting of distinct senctence letters that do not occur in φ we have some sentence $\psi(q_1, \ldots, q_n)$ that do not contain p and another sequence of distinct sentences $\theta_1(p), \ldots, \theta_n(p)$, which each contains p that we have

$$\varphi = \psi(\theta_1(p), \ldots, \theta_n(p))$$

. $\dashv$

It should be noted that if φ is modalized in p that we then have that φ is n-decomposable for some n. We can now state the following theorem:

Theorem 6.1 For very sentence φ modalized in p, there exists a sentence σ that only contains sentence letters that are contained in φ, but where p is not a sentence letter in σ, and such that:

$$\mathbf{GL} \vdash \boxdot(p \leftrightarrow \varphi) \rightarrow p \leftrightarrow \sigma)$$

Every such sentence σ is called a fixed point of φ.

We can specify this theorem. Since the fixed point σ dose not contain a p. So if we write $\varphi(p)$ instead of φ we get that:

$$\mathbf{GL} \vdash \boxdot(\sigma \leftrightarrow \varphi(p)) \leftrightarrow \boxdot(\sigma \leftrightarrow \sigma)$$

by substituting p with σ in the theorem, and this finally gives us:

$$\mathbf{GL} \vdash \sigma \leftrightarrow \varphi(\psi)$$

The first proofs of this theorem was found independently by Dick de Jongh [Ref] and Giovanni Sambin [Ref]. The proof I will follow here is due to Lisa Reidhaar-Olsona [Ref]. This proof is a semantical one and makes use of the Kripke model of **GL**.

We will need the following three lemmas in our proof of the Fixed Point Theorem:

Lemma 6.2 Given any Kripke model $\mathcal{K})\langle W, R, \phi\rangle$, $w \in W$ and sentence φ, we have that if $\vDash_w \Box\varphi$ then $\vDash_v \Box\varphi$ for any $v \in \mathrm{acc}(w)$. Further we have that $\vDash_v \boxdot\varphi$ for all $x \in \mathrm{acc}(w)$.

Proof. Assume that we have $\vDash_w \Box\varphi$, w is connected to v and v is connected v'. Since R is transitive we have that w is also connected to v' and thus we have that $\vDash_{v'} \varphi$. Since v' was chosen at random we have that $\vDash_v \Box\varphi$. We also have that $\vDash_v \varphi$ and thus we have $\vDash_v \boxdot\varphi$. ⊣

Lemma 6.3 Given any Kripke model $\mathcal{K} = \langle W, R, \phi\rangle$, $w \in W$ and sentence φ, if $\nvDash_w \Box\varphi$ then there is "notes" v connected to w such that $\vDash_v \Box\varphi$ and $\nvDash_v \varphi$.

Proof. Assume that $\nvDash_w \Box\varphi$ then there is a notes v connected to w such that $\nvDash_v \varphi$. Let v be the notes with the least rank with this property and suppose that vRv'. Since v' is of less rank than v we have that $\vDash_{v'} \varphi$. Now since v' was chosen arbitrarily we have that $\vDash_v \Box\varphi$ and the lemma is proven. ⊣

Lemma 6.4 For any sentences φ, ψ and θ we have that the following formula is valid:

$$\boxdot(\psi \leftrightarrow \theta) \rightarrow (\varphi(\psi) \leftrightarrow \varphi(\theta))$$

bevis as fixed punkt

Proof. We will prove this by showing that if φ is n-decomposable then it has a fixed point. We will show this by induction on n.

Base case: Suppose that φ is 0-decomposable. Then we have that p dose not occur in φ and it can thus itself be the sentence ψ.

Induction step: Assume that every sentence that is n-decomposable has a fixed point. We now have to show that every sentence that is $(n+1)$-decomposable also has a fixed point. To show this we will assume the following:

$$\varphi(P) = \psi(\Box\theta_1(p), \dots, \Box_{n+1}(p))$$

Further for each i let:

$$\varphi_i(p) = \psi(\Box\theta_1(p), \dots, \Box\theta_{i-1}, \top, \Box\theta_{i+1}, \dots, \Box\theta_{n+1}(p))$$

Thus we have that for each i that $\varphi(p)$ is n-decomposable, so it has a fixed point, that we call σ_i. Lastly we define:

$$\sigma = \psi(\Box\theta_1(\sigma_1), \dots, \Box\theta_{n+1}(\sigma_{n+1}))$$

Our goal is to show that σ is a fixed point of φ.

Lemma 6.5 For each i we have that:

$$\vdash_{\mathbf{GL}} \boxdot(p \leftrightarrow \varphi) \rightarrow \boxdot\Box\theta_i(p) \leftrightarrow \theta_i(\sigma_i))$$

Proof. Since we have that **GL** is complete, we just ave to show that for any model $\mathcal{K} = \langle W, R, \phi\rangle$ and any $w \in W$ that:

$$\mathcal{K} \vDash \boxdot(p \leftrightarrow \varphi) \rightarrow \boxdot(\Box\theta_i(p) \leftrightarrow \Box\theta_i(\sigma_i)) \tag{6.1}$$

So we will start of by fixing i, $\mathcal{K}$ and $w \in W$. We will show 6.1 by assuming $\vDash_w \boxdot(p \leftrightarrow \varphi)$ and then deduce: $\vDash_w \boxdot(\Box\theta_i(p) \leftrightarrow \Box\theta_i(\sigma_i))$; this is equivalent to $\vDash_v \Box\theta_i(p) \leftrightarrow \Box\theta_i(\sigma_i)$ for all $v \in \{w\} \sup \mathrm{acc}(w)$ by remark 6.1. So let $v \in \{w\} \cup \mathrm{acc}(w)$ and assume that $\vDash \Box\theta_i(p)$, i.e $\vDash_v \Box\theta_i(p) \leftrightarrow \top$. By lemma 6.2 we have that for any $v' \in \mathrm{acc}(v)$ that $\vDash_{v'} \Box\theta_i(p)$ and thus $\vDash_{v'} (\Box\theta_i(p) \leftrightarrow \top$. This means that we have:

$$\vDash_v \boxdot(\Box\theta_i(p) \leftrightarrow \top)$$

And thus by lemma 6.4 we get that $\vDash_v \varphi_i \leftrightarrow \varphi$ and since our v was chosen arbitrarily we have that $\vDash_w \Box(\varphi_i \leftrightarrow \varphi)$ and thus by lemma 6.2 we get that: $\vDash_v \boxdot(\varphi_i \leftrightarrow \varphi)$. Since we have assumed that $\vDash_w \boxdot(p \leftrightarrow \varphi)$ we again have by lemma 6.2 that $\vDash_v \boxdot(p \leftrightarrow \varphi)$, and hence we have $\vDash_v \boxdot(p \leftrightarrow \varphi_i)$. Since our logic is complete and we have assumed by the induction hypothesis that φ_i has a fixed point σ_i we have that $\vDash_v (p \leftrightarrow \theta_i)$, and thus, since v was chosen arbitrarily we have that $\vDash_w \Box(p \leftrightarrow \theta_i)$. So by using 6.2 again we get that $\vDash_v \boxdot(p \leftrightarrow \theta_i)$. We will now use lemma 6.4 again and get that:

$$\vDash_v \theta_i(p) \leftrightarrow \theta_i(\sigma_i) \tag{6.2}$$

and

$$\vDash_v \Box\theta_i(p) \leftrightarrow \Box\theta_i(\sigma_i) \tag{6.3}$$

Notice that these two holds for any $v \in \{w\} \cup \text{acc}(w)$ such that $\vDash_v \Box\theta_i(p)$. Further by 6.3 we can deduce $\vDash_v \Box\theta_i(p) \to \Box\theta_i(\sigma_i)$

For the next step of the prove of this lemma we will assume that $\nvDash_v \Box\theta_i(p)$. This means by lemma 6.3 that there is some world v' where $v' \in \text{acc}(v)$, such that $\nvDash_{v'} \theta_i(p)$ and $\vDash_{v'} \Box\theta_i(p)$. 6.2 holds for v' since $v' \in \{w\} \cup \text{acc}(w)$ and thus we have $\vDash_{v'} \theta_i(p) \leftrightarrow \theta_i(\sigma_i)$. This gives that $\nvDash_{v'} \theta_i(\sigma_i)$ and thus since vRv' we have that $\nvDash_v \Box\theta_i(\sigma_i)$. By contraposition we then get: $\vDash_v \Box\theta_i(\sigma_i) \to \Box\theta_i(p)$, and thus we have shown that:

$$\vDash_v \Box\theta_i(p) \leftrightarrow \Box\theta_i(\sigma_i)$$

We have now shown the lemma. $\dashv$

We now go back and finish our proof of the fixed point theorem. Suppose that $\mathcal{K}$ is a model and that $w \in W$ such that $\vDash_w \boxdot (p \leftrightarrow \varphi)$. By lemma 6.5 and completeness we get $\vDash_w \boxdot(\Box\theta_i(p) \leftrightarrow \Box\theta_i(\sigma_i))$. By using lemma 6.4 $(n+1)$ times we can deduce that:

$$\vDash_w \psi(\Box\theta_i(p), \ldots, \Box\theta_{n+1}(p)) \leftrightarrow \psi(\Box\theta_1(\sigma_1), \ldots, \Box\theta_{n+1}(\sigma_{n+1}))$$

i.e $\vDash_w \varphi \leftrightarrow \sigma$.

Since we have $\vDash_w p \leftrightarrow \varphi$ we get $\vDash_w p \leftrightarrow \sigma$, we can obtain $\vDash_w \boxdot(p \leftrightarrow \varphi) \to (p \leftrightarrow \sigma)$. Since our $\mathcal{K}$ and w was chosen at random we have that $\boxdot(p \leftrightarrow \varphi) \to (p \leftrightarrow \sigma)$ is valid. By completeness we then have: $\vdash_{\mathbf{GL}} \boxdot(p \leftrightarrow \varphi) \to (p \leftrightarrow \sigma)$ $\dashv$

7 Solovays completeness theorems

In this section we will prove Solovay's completeness theorems. The proof of these theorems follows a technique invented by Robert Solovay, which today is known as *Solovay construction*. This technique is a way of embedding Kripke models into arithmetic

7.1 The First Theorem

Lemma 7.1 For all modal sentences φ we have that :

$$GL \vdash \varphi \Rightarrow \forall^*(\mathbf{PRA} \vdash \varphi^*)$$

Proof. The proof will be done as an induction proof. $\dashv$

Theorem 7.1 (Solovay's first completeness theorem) For all modal sentences φ we have that:

$$\forall^*(\mathbf{PRA} \vdash \varphi^*) \Rightarrow \mathbf{GL} \vdash \varphi$$

This theorem will be shown by contraposition: If $\mathbf{GL} \nvdash \varphi$ then we have one * such that $\mathbf{PRA} \vdash \varphi^*$. This will be a rather complex construction, and will take the rest of this subsection to prove. So for the rest of this subsection fix a sentence φ such that $GL \vdash \varphi$ and let $\overline{K} = (W, R, \phi)$ be a Kripke model such that $w_0 \vdash \varphi$.

We will assume that $K = \{0, \ldots, n\}$ for some finite n and that $w_0 = 1$. R can be extended by setting $0Ri$ for each i in W. Is shall be noted that 0 is not part of our Kripke model.

Further we will first intuitively define a function $F\omega \to \{0, \ldots, n\}$ in the following way: Set $F(0) = 0$. Further we define $F(x+1)$ in the following way: If $x+1$ is the code of a proof that $\lim_{k\to\infty}(k) \neq z$ for some z accessible to $F(x)$ we set $F(x+1) = z$ otherwise we have that $F(x+1) = F(x)$.

The way this function works can be explained intuitively by the following quote:

> Imagine a refugee who is admitted from one country to another only if he/she provides a proof not to stay there forever. If the refugee is also never allowed to go to one of the previously visited countries, he/she must eventually stop somewhere. So, an honest refugee will never be

able to leave his/her country of origin. [Beklemeishev og artemov, find biktex reference]

Before we can give a formal definition of the function F, we will have to do some work, and introduce some notation. First of we will let G be a partial recursive function with the following Σ_1 graph: $\psi v_0 v_1$. From this graph we can obtain the following Σ_2 formula:

$$\exists v_0 \forall v_1 > v_0 : \psi v_1 v$$

Which says that G has limit v. We will abbreviate this as $L = v$. We will use this notation to define F in the following way:

$$F(v_0) = v_1 \leftrightarrow \begin{cases} (v_0 = \overline{0} \wedge v_1 = \overline{0}) \vee \\ (v_0 > \overline{0} \wedge Prov(v_0, \ulcorner L \neq \overline{0} \urcorner) \wedge F(v_0 - \overline{1})\overline{R}v_2) \vee \\ (v_0 > \overline{0} \wedge \forall v_2 \leq v_0 \neg (Prov(v_0, \ulcorner \neq v_2 \urcorner) \wedge \overline{F}(v_0 - \overline{1})\overline{R}v_2 \wedge \\ v_1 = \overline{F}(v_0 - \overline{1})) \end{cases}$$

Mere formelt defi:

Thus we have that F have the following properties:

1. $F(0) = 0$
2. If $x + 1$ proves that $L \neq \overline{z}$ and we have that $F(x)Rz$, then $F(x+1) = z$
3. Else we have that $F(x+1) = F(x)$

By the Recursion theorem this function exists. Further it is a total function since it is defined by recursion.

Further we will expand the langauge with a new function constant (This can be done, since ψ is the graph of a total function) $\overline{F}$ with the following defining axiom:

$$\overline{F}(v_0) = v_1 \leftrightarrow \psi(v_0) = v_1$$

We will now prove and state a few lemmas before proving the main theorem.

Lemma 7.2 The following three statements holds:

1. **PRA** $\vdash \forall v_0 (\overline{F} v_0 \leq \overline{n})$
2. For all $x \in \omega$ we have that:

$$\textbf{PRA} \vdash \forall v_o (\overline{F} v_0 = \overline{x} \rightarrow \forall v_1 > v_0 (\overline{F} v_1 = \overline{x} \vee \overline{x} \overline{RF} v_1))$$

3. **PRA** $\vdash \exists v_0 v_1 \forall v_2 > v_0 (\overline{F} v_2 = v_1)$.

Where (3) just means that **PRA** $\vdash \exists v_1(L = v_1)$.

Proof. **Dette bevis er ikke helt "færdigt". Færdiggør det selv senere**
We will prove each part separately

1. We will prove this part by induction.

   Base case: $F(0) = 0 \leq n$ is is clearly true.

   Induction step: Assume that $F(x) \leq n$ is true.Then we have that $F(x+1)$ is in the range of F and this $\leq n$ or we have that $F(x+1) \leq n$ so the induction is complete.

2. We will start of by write the formula we have to prove as the following equivalent formula:

$$\forall v_1 \forall v_0 (\overline{F} v_0 = \overline{x} \rightarrow F(v_0 + v_1 + \overline{1}) = \overline{x} \vee \overline{x} \overline{RF}(v_0 + v_1 + 1))$$

   To prove this we will use induction on v_1. This is an induction on a Π_1 formula; which is a possible induction for **PRA** by [Ref]

   Base case if $v_1 = 0$ then we have that

   Induction step bla

   Kom tilabe til dette sted

3. Here we will first prove the following:

$$\forall v_0 (\exists v_1 (\overline{F} v_1 = v_0) \rightarrow \exists v_1 (L = v_1))$$

   This is clearly true for $v_0 \leq n$. **Kig på dette senere**.

$\dashv$

Kommenter på Inductionen brugt ovenover.

Corollary 7.1 **PRA** $\vdash L \leq \overline{n}$, i.e **PRA** $\vdash \bigvee_{x \leq n} L = \overline{x}$

Proof. By lemma 7.2 1) and 3) and the following implication:

$$\mathbf{PRA} \vdash v \leq \overline{n} \rightarrow \bigvee_{x \leq n} v = \overline{x}$$

The corollary follows. $\dashv$

Lemma 7.3 For all $x, y \leq n$ we have that:

1. $L = \overline{x} \wedge \overline{x} \overline{R} \overline{y} \rightarrow \mathrm{Con}_{\mathbf{PRA}+L=\overline{y}}$

2. $\mathbf{PRA} \vdash L = \overline{x} \wedge \overline{x} \neq \overline{y} \wedge \neq \overline{x}\overline{R}\overline{y} \to \neg \mathrm{Con}_{\mathbf{PRA}+L=\overline{y}}$

3. $\mathbf{PRA} \vdash L = \overline{x} \wedge \overline{x} > \overline{0} \to \Pr(\ulcorner L = \neg x \urcorner)$

Proof. We will prove each statement separately:

1) Let xRy and assume for contradiction that $L = \overline{x} \wedge \Pr(\ulcorner L \neq \overline{y} \urcorner)$. Since we have that $L = \overline{x}$ we can chose a v_0 such that $\forall v_2(v_2 > v_0 \to \overline{F}v_2 = \overline{x})$ and we can also chose $v_1 + \overline{1} > v_0$ such that $\mathrm{Prov}(v_1 + \overline{1}, \ulcorner L \neq \overline{y} \urcorner)$ But we also have the following:

$$\mathbf{PRA} \vdash \mathrm{Prov}(v_1 + \overline{1}, \ulcorner L \neq y \urcorner) \wedge \overline{F}v_1 = \overline{x_1, \ldots, x_n} \wedge \overline{x}\overline{R}\overline{y} \to \overline{F}(v_1 + \overline{1}) = \overline{y}$$

This contradicts with $\forall v_2 > v_0(\overline{F}v_0 = \overline{x})$ which came from the assumption that $L = \overline{x}$ so we con conclude:

$$\mathbf{PRA} \vdash L = \overline{x} \wedge \overline{x}\overline{R}\overline{y} \to \neg \Pr(\ulcorner L \neq \overline{y} \urcorner)$$

Where we have that $\neg \Pr(L \neq \overline{y})$ is equivalent to $\mathrm{Con}_{\mathbf{PRA}+L=\overline{y}}$ and 1) follows.

2) By [ref???] We have the following deduction:

$$\begin{aligned} \mathbf{PRA} \vdash L = \overline{x} &\to \exists v_0(\overline{F}v_0 = \overline{x}) && (7.1) \\ &\to \Pr(\ulcorner \exists v_0(\overline{F}v_0 = \overline{x}) \urcorner) && (7.2) \end{aligned}$$

Further we have from lemma 7.2.2 and [D2, lob?] that:

$$\mathbf{PRA} \vdash \forall v_0(\overline{F}v_0 = \overline{x} \to (L = \overline{x} \vee \overline{x}\overline{R}L)) \quad (7.3)$$

$$\mathbf{PRA} \vdash (\ulcorner \forall v_0(\overline{F}v_0 = \overline{x} \to (L = \overline{x} \vee \overline{x}\overline{R}L)) \urcorner) \quad (7.4)$$

From 7.2 and 7.4 we have the following:

$$\mathbf{PRA} \vdash L = \overline{x} \to \Pr(\ulcorner L = \overline{x} \vee \overline{x}\overline{R}L \urcorner) \quad (7.5)$$

asd?

Which gives us [Why?]

$$\mathbf{PRA} \vdash \overline{x} \neq \overline{y} \wedge \neg(\overline{x}\overline{R}\overline{y}) \to \Pr(\ulcorner \overline{x} \neq y \wedge \neg(\overline{x}\overline{R}\overline{y}) \urcorner)$$

This with 7.5 gives us:

$$\mathbf{PRA} \vdash L = \overline{x} \wedge \overline{x} \neq \overline{y} \wedge \neg \overline{x}\overline{R}\overline{y} \to \Pr(\ulcorner L = \overline{x} \vee \overline{x}\overline{R}L \urcorner) \wedge \Pr(\ulcorner \overline{x} \neq \overline{y} \wedge \neg \overline{x}\overline{R}\overline{y} \urcorner)$$

From which we can deduce:

$$\mathbf{PRA} \vdash L\overline{x} \wedge \overline{x} \neq \overline{y} \wedge \neg \overline{x} \overline{R} \overline{y} \rightarrow \Pr(\ulcorner L \neq \overline{y} \urcorner)$$

Which gives us 2)

3) From the least number principible we have that:

$$\mathbf{PRA} \vdash L = \overline{x} \wedge \overline{x} > \overline{0} \rightarrow \exists v(\overline{F}(v + \overline{1}) = \overline{x} \wedge \overline{F}v \neq \overline{x})$$

By the definition of F we have for such a v the following:

$$\mathbf{PRA} \vdash \overline{F}(v + \overline{1}) = \overline{x}\overline{F}v \neq \overline{x} \rightarrow \mathrm{Prov}(v + \overline{1}, \ulcorner L \neq \overline{x} \urcorner)$$

And thus we have the following:

$$\mathbf{PRA} \vdash L = \overline{x} \wedge \overline{x} > \overline{0} \rightarrow \Pr(\ulcorner L \neq \overline{x} \urcorner)$$

$\dashv$

Lemma 7.2 and 7.3 gives us the most of the basic fact we need about L and F; at least the facts about them that we can prove in **PRA**. We will also need the following result, which cannot be proven i **PRA**.

Lemma 7.4 The following two statements are true, but they cannot be proven in **PRA**.

1. $L = \overline{0}$
2. For $0 \leq x \leq n$ we have that $\mathbf{PRA} + L = \overline{x}$ is consistent.

Proof. **1)** By lemma 7.2.3 the limit L exists. If $x > 0$ we have by lemma 7.4 that:

$$\begin{aligned} L = \overline{x} &\Rightarrow \mathbf{PRA} \vdash L \neq \overline{x} \\ &\Rightarrow L \neq \overline{x} \end{aligned}$$

Since **PRA** is sound. But this is a contradiction and we must conclude that $L = 0$.

2) Since we have that $L = \overline{0}$ is true and we have that **PRA** is sound, we have that $\mathbf{PRA} + L = \overline{0}$ is consistent. For $x > 0$ we will apply lemma 7.4.1 and get:

$$\mathbf{PRA} \vdash L = \overline{0} \wedge \overline{0}\overline{R}\overline{x} \rightarrow \mathrm{Con}_{\mathbf{PRA}+L=\overline{x}}$$

We have that the antecedent is true and hence that $\mathrm{Con}_{PRA+L=\overline{x}}$ is true, which proves this part of the lemma. $\dashv$

We have now shown all the important basic properties that F and L holds. In the next part of the proof we will simulate the Kripke model $\mathcal{K} = \langle W, R, \phi \rangle$, where $\mathcal{K}_1 \not\vdash \varphi$. For this end we will let $L = \overline{x}$ assume the nodes of $W = \{1, \dots n\}$. We will start of by defining the interpretation *.We will start of by defining the interpretation *. So for any p let:

$$p^* = \bigvee \{L = \overline{x} : 1 \leq x \leq n \text{ and } x \in \phi(p)\}$$

If this disjunction is empty, we will set it to be $\overline{0} = \overline{1}$. Further we are only interested the sentence φ; i.e the set:

$$S(\varphi) = \{\psi : \ \psi \text{ is a subformula of } \ \varphi\}$$

We will not look at $p \notin S(\varphi)$

The following lemma is the crucial result about this interpretation, and the theorem will follow easily from this result:

Lemma 7.5 Let $1 \leq x \leq n$. For any ψ and * as defined just above we have that:

1. $x \vdash \psi \Rightarrow \mathbf{PRA} \vdash L = \overline{x} \rightarrow \psi^*$
2. $x \not\vdash \psi \Rightarrow \mathbf{PRA} \vdash L = \overline{x} \rightarrow \neg\psi^*$

Proof. We will use induction on complexity of ψ. If $\psi = p$, then since p is a disjunct of p^* we get:

$$\vdash_x \varphi \Rightarrow \mathbf{PRA} \vdash L = \overline{x} \rightarrow p^*$$

Which proves 1, for ψ atomic. For 2 observe that if $\not\vdash_x p$, then $L = \overline{x}$ contradicts over disjunct of p^* and thus:

$$\not\vdash_x p \rightarrow \mathbf{PRA} \vdash L = \overline{x} \rightarrow \neg p^*$$

The cases where ψ is $\neg\theta, \theta \wedge \sigma, \theta \vee \sigma$ and $\theta \rightarrow \sigma$ are trivial. So we will just look at the case where $\psi = \Box\theta$. Here we make the following deductions:

$$\begin{aligned}
\vdash_x \Box\theta &\Rightarrow \forall y(xRy \Rightarrow \vdash_y \theta) \\
&\Rightarrow \forall (xR \Rightarrow \mathbf{PRA} \vdash L = \overline{y} \rightarrow \theta^*) \\
&\Rightarrow \bigwedge_{xRy} (\mathbf{PRA} \vdash L = \overline{y} \rightarrow \theta^*) \\
&\Rightarrow \mathbf{PRA} \vdash \bigvee_{xRy} L = \overline{y} \rightarrow \theta^* \\
&\Rightarrow \mathbf{PRA} \vdash \mathrm{Pr}(\ulcorner \textstyle\bigvee_{xRy} L = \overline{y} \urcorner) \rightarrow \mathrm{Pr}(\ulcorner \theta^* \urcorner)
\end{aligned}$$

And by the last line we can get the following by using the axioms of **GL**

$$\vdash_x \Box\theta \Rightarrow \mathbf{PRA} \vdash \mathrm{Pr}(\ulcorner \textstyle\bigvee_{xRy} L = \overline{y} \urcorner) \to \mathrm{Pr}(\ulcorner \theta^* \urcorner) \tag{7.6}$$

We can now invoke lemma 7.4 2 and 3 and get:

$$\mathbf{PRA} \vdash L = \overline{x} \to \bigwedge_{\neg(xRz)} \mathrm{Pr}(\ulcorner L \neq z \urcorner) \tag{7.7}$$

and therfore [Hvorfor?]

$$\mathbf{PRA} \vdash L = \overline{x} \to \mathrm{Pr}(\ulcorner \textstyle\bigvee_{xRy} L = \overline{y} \urcorner) \tag{7.8}$$

Now by 7.6 and 7.8 we get that:

$$\begin{aligned} \vdash_x \Box\theta &\Rightarrow \mathbf{PRA} \vdash L = \overline{x} \to \mathrm{Pr}(\ulcorner\ \theta^* \urcorner) \\ &\Rightarrow \mathbf{PRA} \vdash L = \overline{x} \to (\Box\theta)^* \end{aligned}$$

I.e we have proven part 1 of the lemma. Similarly we can do the following deduction:

$$\begin{aligned} \nvdash_x \Box\theta &\Rightarrow \exists y (xRy \wedge \nvdash_y \theta) \\ &\Rightarrow \exists y (xRy \wedge \mathbf{PRA} \vdash L = \overline{y} \to \neg\theta^*) \\ &\Rightarrow \exists y (xRy \wedge \mathbf{PRA} \vdash \theta^* \to L \neq \overline{y}) \\ &\Rightarrow \exists y (xRy \wedge \mathbf{PRA} \vdash \mathrm{Pr}(\ulcorner \theta^* \urcorner) \to \mathrm{Pr}(\ulcorner L \neq \overline{y} \urcorner) \end{aligned}$$

But by lemma 7.4 we get that if xRy:

$$\mathbf{PRA} \vdash L = \overline{x} \to \neg\mathrm{Pr}(\ulcorner L \neq \overline{y} \urcorner)$$

all in all this gives us:

$$\mathbf{PRA} \vdash L = \overline{x} \to \neg\mathrm{Pr}(\ulcorner \theta^* \urcorner)$$

Which just is the following we where trying to show:

$$\mathbf{PRA} \vdash L = \overline{x} \to \neg(\Box\theta)^*$$

$\dashv$

We can now finally prove Solovay's first completeness theorem:

Proof of Solovay's first completeness theorem. By lemma 7.5 we have that

$$\vdash_1 \varphi \Rightarrow \mathbf{PRA} \vdash L = \overline{1} \to \varphi^*$$

But by lemma [Hvad?] we have that **PRA** $+ L = 1$ is consistent, from which it follows that **PRA** $+ \neg\varphi^*$ is consistent; so φ^* is not a theorem af **PRA** and thus we have: **PRA** $\nvdash \varphi^*$ and the theorem follows by contraposition. $\dashv$

7.2 The Second Theorem

Solovay's Second Completeness Theorem is a strengthening of the first one, [How?]

Theorem 7.2 For all modal sentences φ, the following is equivalent:

1. **GLS** $\vdash \varphi$
2. **GL** $\vdash \bigwedge_{\Box\psi \in S(\varphi)} (\Box\psi \to \psi)\varphi$
3. φ is true in all φ-sound Kripke models
4. $\forall^*(\varphi^*$ is true$)$

Some parts of this theorem has already been proven. We have (1) $\Leftrightarrow$ (2) by theorem [asd?] and we have (2) $\Leftrightarrow$ (3) by theorem [asd?]. **Udbyd dette senere**

For proving (4) $\Rightarrow$ (3) we will again make use of contraposition. Let $\overline{\mathcal{K}} = ((1, \ldots, n), R, \phi)$ be given and let 1 be the root. Further let φ be such that $\nvdash_1 \varphi$. We will assume that $\overline{\mathcal{K}}$ is φ-sound. This means that we have: $\vdash_1 \Box\psi \to \psi$ for all $\psi \in S(\varphi)$

We will set $0Rw$ for all $w \in W$. We will now create a new model $\mathcal{K}'$ where we have added 0. We will create this new model in the following way:

$W' = \{0, 1 \ldots, n\}$
R' extends R by assuming that $0R'x$ for all $x \in W$
$\alpha_0 = 0$
ϕ' extends ϕ by putting $0 \in \phi(p)$ iff $1 \in \phi(p)$ for all $p \in S(\varphi)$

We will abuse notation and let R denote R' and ϕ denote ϕ'.

Lemma 7.6 For all $\psi \in S(\varphi)$ we have that:

$$0 \vdash \psi \text{ iff } 1 \vdash \psi$$

Proof. $\dashv$

Lemma 7.7 Let $0 \leq x \leq n$. For any $\psi \in S(\varphi)$ and any * as defined we have that:

1. $x \vdash \psi \Rightarrow$ **PRA** $\vdash L = \overline{x} \to \psi^*$

2. $x \nVdash \varphi \Rightarrow \mathbf{PRA} \vdash \overline{x} \to \neg\psi^*$

Proof. ⊣

We will now define a function F in the same way as before.

$$F(0) = 0$$
$$F(x+1) = \begin{cases} y & \text{Prov}(\overline{x}+\overline{1}, \ulcorner L \neq \overline{y} \urcorner) \wedge xRy \\ F(x) & \text{else} \end{cases}$$

All the Lemmas about F and L still holds. But the behavior ϕ has changed, since we must added the node 0 and we can only use sub formulas of φ. So we define:

$$p^* = \bigvee \{L = \overline{x} : 0 \leq x \leq n \wedge x \in \phi(p)\}$$

For $p \in S(\varphi)$ and let p^* be random for all p's that is not a subformula of φ. We now prove and state the following lemma (that reminds of what?):

Lemma 7.8 Let $0 \leq x \leq n$. For any $\psi \in S(\varphi)$ and * as defined above we have:

1. $\vdash_x \psi \Rightarrow \mathbf{PRA} \vdash L = \overline{x} \to \psi^*$

2. $\nVdash_x \psi \Rightarrow \mathbf{PRA} \nVdash L = \overline{x} \to \psi^*$

Proof. For $0 < x$ the proof is identical to the proof of [ref?]. So we will only prove the case where $x = 0$. The is again an induction on the complexity of ψ. We will only prove the cases where $\psi = \Box\theta$.

Let $\psi = \Box\theta$. We then have:

$$\begin{aligned} \vdash_0 \Box\theta &\Rightarrow \forall x(1 \leq x \leq n \Rightarrow \vdash_x \theta) \\ &\Rightarrow \forall x(1 \leq x \leq n \Rightarrow \mathbf{PRA} \vdash L = \overline{x} \to \theta^*) \end{aligned}$$

Since $x > 1$ and this case of the lemma has been proven. We can also make the following deduction by the induction hypothesis:

$$\begin{aligned} \vdash_0 \Box\theta &\Rightarrow \vdash_1 \theta \\ &\Rightarrow \vdash_0 \theta \\ &\Rightarrow \mathbf{PRA} \vdash L = \overline{0} \to \theta^* \end{aligned}$$

By combing these two we get:

$$\begin{aligned}\vdash_0 \Box\theta &\Rightarrow \bigwedge_{x\le n}(\mathbf{PRA} \vdash L = \overline{x} \to \theta^*)\\ &\Rightarrow \mathbf{PRA} \vdash (\bigvee_{x\le n} L = \overline{x}) \to \theta^*\\ &= \mathbf{PRA} \vdash \mathrm{Pr}(\ulcorner \bigvee L = \overline{x} \urcorner) \to \mathrm{Pr}(\ulcorner \theta^* \urcorner)\end{aligned}$$

But by corollary [<ref?] we have that $\mathbf{PRA} \vdash \bigvee L = \overline{x}$ and thus $\mathbf{PRA} \vdash \mathrm{Pr}(\ulcorner \bigvee L = \overline{x} \urcorner)$ so all in all we have:

$$\vdash_0 \Box\theta \Rightarrow \mathbf{PRA} \vdash \mathrm{Pr}(\ulcorner \theta^* \urcorner) \qquad \Rightarrow \mathbf{PRA} \vdash L = \overline{0} \to \mathrm{Pr}(\ulcorner \theta^* \urcorner)$$

This proves (1). The proof of (2) is a bit easier. We have:

$$\begin{aligned}\nvdash_0 \Box\theta &\Rightarrow \exists x(1 \le x \le n \wedge \nvdash_x \theta)\\ &\Rightarrow \exists x(1 \le x \le n \wedge \mathbf{PRA} \vdash L = \overline{x} \to \neg\theta^*)\\ &\Rightarrow \exists x(1 \le x \le n \wedge \mathbf{PRA} \vdash \theta^* \to L \neq \overline{x})\\ &\Rightarrow \mathbf{PRA} \vdash L = \overline{0} \to \neg\mathrm{Pr}(\ulcorner \theta^* \urcorner)\end{aligned}$$

And thus by Lemma [Ref?] we have that $\mathbf{PRA} \vdash L = \overline{0} \to \neg\mathrm{Pr}(\ulcorner L \neq \overline{x} \urcorner)$ for $x > 0$ $\dashv$

We thus have that $L = \overline{0}$ is true, and we can now finally prove the second completeness theorem:

Proof of the Second Completeness theorem. Assume that φ is false in $\overline{\mathcal{K}}$ i.e $\nvdash_1 \varphi$. Then by lemma [asd] we get $\nvdash_0 \varphi$ and the just proven lemma gives:

$$\mathbf{PRA} \vdash L = \overline{0} \to \neg\varphi^*$$

Since we have that $L = \overline{0}$ is true we then get $\neg\varphi^*$ which just means that φ^* is false. $\dashv$

7.2.1 Using the Second Completeness Theorem

We can use the second completeness theorem to prove the following theorem named after Rosser:

Theorem 7.3 There is a Σ_1-sentences φ such that:

1. $\mathbf{PRA} \nvdash \varphi$

2. **PRA** $\nvdash \not\varphi$
3. **PRA** $\vdash \mathrm{Con} \to \neg\,\mathrm{Pr}(\ulcorner \varphi \urcorner)$
4. **PRA** $\vdash \mathrm{Con} \to \neg PPr(\ulcorner \neg\varphi \urcorner)$

7.3 Further Completeness Results

8 Further results in Provability logic

asd

9 Bibliography

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