EXERCISES

1.1) $\forall a,b \int_a^b f(x)\,dx = 0 \quad \Rightarrow f(x) = 0$

PBC: $\forall a,b \int_a^b f(x)\,dx = 0 \quad \Rightarrow f(x) \neq 0$

- $\exists x_0 \mid f(x_0) \neq 0$

  Because of continuous function:

  $$\exists \varepsilon \mid f(x_0+\varepsilon) \neq 0 \wedge \forall x \in [x_0, x_0+\varepsilon]: f(x) \neq 0$$

  $$\Rightarrow \int_{x_0}^{x_0+\varepsilon} f(x)\,dx \neq 0 \quad \lightning$$

  $$\Rightarrow \forall a,b \int_a^b f(x)\,dx = 0 \quad \Rightarrow f(x) = 0$$

1.2) $\Theta(t,x) = T(t',x')$

$x' = \frac{x}{L} \qquad t' = \frac{t}{\tau}$

$\Theta(t,x) = T\left(\frac{t}{\tau}, \frac{x}{L}\right)$

$\partial_t \Theta(t,x) = \underbrace{\partial_t T\left(\frac{t}{\tau}, \frac{x}{L}\right)}_{\text{Ableitung nach } t \; *}$

$= T'\left(\frac{t}{\tau}, \frac{x}{L}\right) \cdot \frac{1}{\tau}$

$= \partial_1 T\left(\frac{t}{\tau}, \frac{x}{L}\right) \cdot \frac{1}{\tau}$

$= \partial_1 T\left(t', \frac{x}{L}\right) \cdot \frac{1}{\tau}$

$= \partial_{t'} T(t', x') \cdot \frac{1}{\tau}$

$= \partial_t T\left(\frac{t}{\tau}, x'\right) \cdot \frac{1}{\tau}$

$= \partial_{x'x'} T(t', x') \cdot \frac{1}{\tau}$

$= \partial_{x'x'} \Theta(t' \cdot \tau, x' \cdot L) \cdot \frac{1}{\tau}$

$= \partial_{x'} \Theta'(t' \cdot \tau, x' \cdot L) \cdot L \cdot \frac{1}{\tau}$

$= \Theta''(t' \cdot \tau, x' \cdot L) \cdot L^2 \cdot \frac{1}{\tau}$

$= \Theta''(t, x) \cdot L^2 \cdot \frac{1}{\tau}$

$= \partial_{xx} \Theta(t,x) \cdot \frac{L^2}{\tau}$

$= \alpha \, \partial_{xx} \Theta(t,x)$

$\partial_t \Theta\left(\frac{t}{\tau}, \frac{x}{L}\right)$

Kettenregel:

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$T \quad \frac{t}{\tau}$

$T'\left(\frac{t}{\tau}, \frac{x}{L}\right) \cdot \frac{1}{\tau}$

1.3.) $q = -k \nabla T$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -k \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} T$$

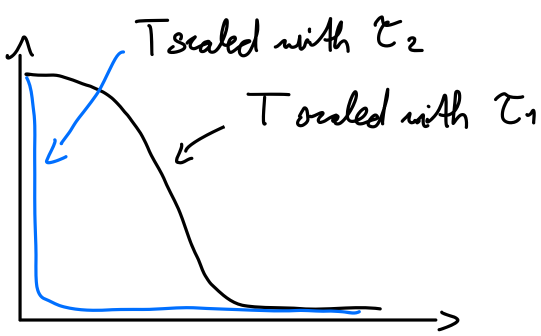
y, z independent

$$q_2 = -k \, \partial_y T = -k \cdot 0 = 0$$
$$q_3 = -k \, \partial_z T = -k \cdot 0 = 0$$

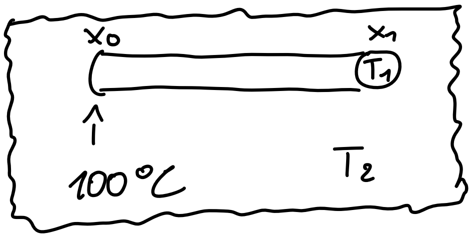
In our case the rods length has much more impact than the rods diameter. Therefore the time for temperature change along the diameter of the rod is neglectable.

τ reference time $\rightarrow$ $\tau_1 = \frac{L^2}{\alpha}$ $\quad \tau_2 = \frac{\varepsilon^2}{\alpha}$

„How fast the dynamic happens"



1.4) $q = h(T_2 - T_1)$



T_2 can be seen as constant, if the surrounding fluid can transfer the received heat from the solid part efficiently. This can for example be achieved by exchanging or cooling down the fluid.

$T(x_0) = 100$

$$q = -k\,\partial_x T = h(20 - \underbrace{T(x_1)}_{T_1})$$

$$\Rightarrow \partial_x T(x_1) - \frac{h}{k} T(x_1) = \frac{h}{k} \cdot 20$$

Boundary condition modelling!

1.8) Implicit Euler Scheme:

$$T_i^{n+1} = T_i^n + \frac{\Delta t}{\Delta x^2}\left(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}\right)$$

Explicit Euler Scheme:

$$T_i^{n+1} = T_i^n + \frac{\Delta t}{\Delta x^2}\left(T_{i+1}^{n} + 2T_i^{n} + T_{i-1}^{n}\right)$$

T_i^n as a Fourier series:

$$T_i^n = \sum_k \hat{T}_k^n e^{ikx_i}$$

Implicit Euler rewritten:

$$\sum_k \left(\hat{T}_k^{n+1} - \hat{T}_k^n - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\hat{T}_k^{n+1}\right)e^{ikx_i} = 0$$

Explicit Euler rewritten:

$$\sum_k \left(\hat{T}_k^{n+1} - \hat{T}_k^n - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\hat{T}_k^{n}\right)e^{ikx_i} = 0$$

From Implicit Euler Scheme we get the following:

$$\hat{T}_k^{n+1} - \hat{T}_k^n - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\hat{T}_k^{n+1} = 0$$

$$G_k = \frac{\hat{T}_k^{n+1}}{\hat{T}_k^n}$$

$$\hat{T}_k^n = \hat{T}_k^{n+1} - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\hat{T}_k^{n+1}$$

$$G_k = \frac{\hat{T}_k^{n+1}}{\hat{T}_k^{n+1} - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\hat{T}_k^{n+1}}$$

$$G_k = \frac{\hat{T}_k^{n+1}}{\hat{T}_k^{n+1}\left(1 - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right)}$$

$$G_k = \frac{1}{1 - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)}$$

$$|G_k| \leq 1$$

$$\frac{1}{\left|1 - \frac{\Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right|} \leq 1$$

$$\frac{1}{\left|1 - \frac{2\Delta t}{\Delta x^2}\left(\cos(k\Delta x) - 1\right)\right|} \leq 1$$

$$1 \leq 1 - \frac{2\Delta t}{\Delta x^2}\left(\cos(k\Delta x) - 1\right)$$

$$0 \leq -\frac{2\Delta t}{\Delta x^2}\left(\cos(k\Delta x) - 1\right)$$

$$0 \geq \underbrace{\frac{2\Delta t}{\Delta x^2}}_{\substack{\Delta t \text{ positive} \\ \Delta x \text{ positive}}} \underbrace{\left(\underbrace{\cos(k\Delta x)}_{[-1;\,1]} - 1\right)}_{[-2;\,0]}$$

$\square$