Double categories and structured categories

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Ehresmann, C. “Catégories doubles et catégories structurées”. *C. R. Acad. Sc.* **256** (1963), 1198–1201.

Abstract

Definition of structured categories; the particular case of double categories, which admit a **TO-DO** as a quotient category.

1 Double categories

p. 1

Definition. We define a *double category* to be a class $\mathscr{C}$ endowed with two composition laws, denoted $\bullet$ and $\perp$, satisfying the following conditions:

1. $(\mathscr{C},\bullet)$ is a category, denoted $\mathscr{C}^\bullet$; the right and left **TO-DO** of $f\in\mathscr{C}$ will be denoted by $\alpha^\bullet(f)$ and $\beta^\bullet(f)$ respectively, and the class of **TO-DO** by $\mathscr{C}_0^\bullet$;
2. $(\mathscr{C},\perp)$ is a category, denoted $\mathscr{C}^\perp$; the **TO-DO** of $f\in\mathscr{C}^\perp$ will be denoted by $\alpha^\perp(f)$ and $\beta^\perp(f)$ respectively, and the class of **TO-DO** by $\mathscr{C}_0^\perp$;
3. The maps $\alpha^\bullet$ and $\beta^\bullet$ (resp. $\alpha^\perp$ and $\beta^\perp$) are functors from $\mathscr{C}^\perp$ to $\mathscr{C}^\perp$ (resp. from $\mathscr{C}^\bullet$ to $\mathscr{C}^\bullet$);
4. *Axiom of permutability.* If the composites $k\bullet h$, $g\bullet f$, $k\perp g$, and $h\perp f$ are defined, then
$$(k\bullet h)\perp(g\bullet f) = (k\perp g)\bullet(h\perp f).$$

Let $\mathscr{C}$ be a class endowed with two composition laws $\bullet$ and $\perp$ satisfying axioms 1 and 2; consider the following axioms:

3'. $\mathscr{C}_0^\bullet$ (resp. $\mathscr{C}_0^\perp$) is stable with respect to $\perp$ (resp. to $\bullet$);

4'. If the composites $k\bullet h$, $g\bullet f$, $k\perp g$, and $h\perp f$ are defined, then both $(k\bullet h)\perp(g\bullet f)$ and $(k\perp g)\bullet(h\perp f)$ are defined and are equal to one another.

5. For all $f\in\mathscr{C}$, we have
$$\alpha^\bullet(\alpha^\perp(f)) = \alpha^\perp(\alpha^\bullet(f)), \qquad \beta^\bullet(\beta^\perp(f)) = \beta^\perp(\beta^\bullet(f));$$
$$\alpha^\bullet(\beta^\perp(f)) = \beta^\perp(\alpha^\bullet(f)), \qquad \alpha^\perp(\beta^\bullet(f)) = \beta^\bullet(\alpha^\perp(f)).$$

Proposition. *For $(\mathscr{C}, \bullet, \perp)$ to be a double category, it is necessary and sufficient that conditions 1, 2, 3′, 4′, and 5 be satisfied. In this case, $\mathscr{C}_0^\bullet$ (resp. $\mathscr{C}_0^\perp$) is a subcategory of $\mathscr{C}^\bullet$ (resp. $\mathscr{C}^\perp$).*

A *double subcategory* of a double category $\mathscr{C}$ is a subclass $\mathscr{C}'$ of $\mathscr{C}$ that is a subcategory of $\mathscr{C}^\bullet$ and of $\mathscr{C}^\perp$; then $\mathscr{C}'$ is a double category for the composition laws induced by $\bullet$ and $\perp$.

Definition. Let