Double categories and structured categories

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Abstract

Definition of structured categories; the particular case of double categories, which admit a category of squares as a quotient category.

TO-DO fix ◫ and ⊟; fix •

1 Double categories

p. 1

Definition. We define a *double category* to be a class $\mathscr{C}$ endowed with two composition laws, denoted $\bullet$ and $\perp$, satisfying the following conditions:

1. $(\mathscr{C},\bullet)$ is a category, denoted $\mathscr{C}^\bullet$; the right and left **TO-DO** of $f \in \mathscr{C}$ will be denoted by $\alpha^\bullet(f)$ and $\beta^\bullet(f)$ respectively, and the class of **TO-DO** by $\mathscr{C}_0^\bullet$;
2. $(\mathscr{C},\perp)$ is a category, denoted $\mathscr{C}^\perp$; the **TO-DO** of $f \in \mathscr{C}^\perp$ will be denoted by $\alpha^\perp(f)$ and $\beta^\perp(f)$ respectively, and the class of **TO-DO** by $\mathscr{C}_0^\perp$;
3. The maps $\alpha^\bullet$ and $\beta^\bullet$ (resp. $\alpha^\perp$ and $\beta^\perp$) are functors from $\mathscr{C}^\perp$ to $\mathscr{C}^\perp$ (resp. from $\mathscr{C}^\bullet$ to $\mathscr{C}^\bullet$);
4. *Axiom of permutability.* If the composites $k\bullet h$, $g\bullet f$, $k\perp g$, and $h\perp f$ are defined, then

$$(k\bullet h)\perp(g\bullet f) = (k\perp g)\bullet(h\perp f).$$

Let $\mathscr{C}$ be a class endowed with two composition laws $\bullet$ and $\perp$ satisfying axioms 1 and 2; consider the following axioms:

3'. $\mathscr{C}_0^\bullet$ (resp. $\mathscr{C}_0^\perp$) is stable with respect to $\perp$ (resp. to $\bullet$);

4'. If the composites $k\bullet h$, $g\bullet f$, $k\perp g$, and $h\perp f$ are defined, then both $(k\bullet h)\perp(g\bullet f)$ and $(k\perp g)\bullet(h\perp f)$ are defined and are equal to one another.

5. For all $f \in \mathscr{C}$, we have

$$\alpha^\bullet(\alpha^\perp(f)) = \alpha^\perp(\alpha^\bullet(f)), \qquad \beta^\bullet(\beta^\perp(f)) = \beta^\perp(\beta^\bullet(f));$$
$$\alpha^\bullet(\beta^\perp(f)) = \beta^\perp(\alpha^\bullet(f)), \qquad \alpha^\perp(\beta^\bullet(f)) = \beta^\bullet(\alpha^\perp(f)).$$

Proposition. *For* $(\mathscr{C}, \bullet, \perp)$ *to be a double category, it is necessary and sufficient that conditions 1, 2, 3′, 4′, and 5 be satisfied. In this case,* $\mathscr{C}_0^\bullet$ *(resp.* $\mathscr{C}_0^\perp$*) is a subcategory of* $\mathscr{C}^\bullet$ *(resp.* $\mathscr{C}^\perp$*).*

A *double subcategory* of a double category $\mathscr{C}$ is a subclass $\mathscr{C}'$ of $\mathscr{C}$ that is a subcategory of $\mathscr{C}^\bullet$ and of $\mathscr{C}^\perp$; then $\mathscr{C}'$ is a double category for the composition laws induced by $\bullet$ and $\perp$.

Definition. Let $\mathscr{C}$ be a double category; we define a *left ideal* (resp. *right ideal*) of $\mathscr{C}^\perp$ to be a subcategory $I^\perp$ of $\mathscr{C}^\perp$ such that $\mathscr{C} \bullet I^\perp$ (resp. $I^\perp \bullet \mathscr{C} = I^\perp$), where $\mathscr{C} \bullet I^\perp$ (resp. $I^\perp \bullet \mathscr{C}$) is the class of composites $f \bullet g$ (resp. $g \bullet f$) for $g \in I^\perp$ and $f \in \mathscr{C}$. We similarly define an *ideal* of $\mathscr{C}^\bullet$.

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Proposition. *Let* $\mathscr{C}$ *be a double category; a left ideal* $I^\perp$ *of* $\mathscr{C}^\perp$ *is a* **TO-DO**[1] *over* $\mathscr{C}^\bullet$ *for the composition law* $(f, g) \mapsto f \bullet g$ *if and only if* $f \bullet g$ *is defined, where* $f \in \mathscr{C}$ *and* $g \in I^\perp$*. The corresponding category* $\mathscr{E}(I^\perp)$ ***is this right?*** *of hypermorphisms*[1] *is a double category for the composition laws* ***is this right?***

$$(f', g') \bullet (f, g) = (f' \bullet f, g)$$

if and only if $g' = f \bullet g$*; further* ***is this right? or is it two joined iffs***

$$(f', g') \perp (f, g) = (f' \perp f, g' \perp g)$$

if and only if $f' \perp f$ *and* $g' \perp g$ *are defined.*

2 Double categories of squares

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two categories with the same class of **TO-DO** . Let $\square(\mathscr{C}_2, \mathscr{C}_1)$ be the set of quadruples (g_2, g_1, f_1, f_2), with $f_i, g_i \in \mathscr{C}_i$ for $i = 1, 2$, such that

$$\begin{aligned} \alpha(f_1) &= \alpha(f_2), & \alpha(g_1) &= \beta(f_2); \\ \beta(f_1) &= \alpha(g_2), & \beta(g_1) &= \beta(g_2). \end{aligned}$$

We define two composition laws on $\square(\mathscr{C}_2, \mathscr{C}_1)$:

- *Longitudinal* multiplication

  $$(g_2', g_1', f_1', f_2') \boxbar (g_2, g_1, f_1, f_2) = (g_2', g_1' g_1, f_1' f_1, f_2)$$

  if and only if $f_2' = g_2$ **?**;

- *Lateral* multiplication

  $$(g_2', g_1', f_1', f_2') \boxminus (g_2, g_1, f_1, f_2) = (g_2' g_2, g_1', f_1, f_2' f_2)$$

  if and only if $f_1' = g_1$ **?**.

Proposition. $\square(\mathscr{C}_2, \mathscr{C}_1)$ *is a double category for longitudinal and lateral multiplication.*

Suppose that $\mathscr{C} = \mathscr{C}_1 = \mathscr{C}_2$; recall[2] that a *square* in $\mathscr{C}$ is an element $(g_2, g_1, f_1, f_2) \in \square(\mathscr{C}, \mathscr{C})$ such that $g_1 f_2 = g_2 f_1$.

Corollary. *The class* $\square\mathscr{C}$ *of squares in* $\mathscr{C}$ *is a double subcategory of* $\square(\mathscr{C}, \mathscr{C})$*.*

Corollary. *Let* $\mathscr{C}$ *be a double category; then* $\mathscr{C}^\bullet$ *admits the longitudinal category* $\boxbar(\mathscr{C}_0^\bullet, \mathscr{C}_0^\perp)$ *as a quotient category*[1]*, where* $\mathscr{C}_0^\bullet$ *(resp.* $\mathscr{C}_0^\perp$*) is endowed with its structure as a subcategory of* $\mathscr{C}^\perp$ *(resp. of* $\mathscr{C}^\bullet$*).*

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[1] *Espèces de structures locales; élargissements de catégories*, Séminaire Top. et Géo Diff. (Ehresmann), **III**, Paris, 1961; Jahres. Deutsch. Math. Ver., **60**, 1957, p. 49.

3 Functors into a double category

Let Γ be a category and $\mathscr{C}$ a double category; let $\mathscr{F}(\mathscr{C}^\bullet,\Gamma)$ be the class of functors from Γ to $\mathscr{C}^\bullet$.

Proposition. *$\mathscr{F}(\mathscr{C}^\bullet,\Gamma)$ is a category for the composition law $(\Phi',\Phi)\mapsto\Phi'\perp\Phi$, where $(\Phi'\perp\Phi)(f)=\Phi'(f)\perp\Phi(f)$, if and only if $\Phi'(f)\perp\Phi(f)$ is defined for all $f\in\mathscr{C}$.*

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Definition. Let $\mathscr{C}$ and $\mathscr{C}_1$ be two double categories; we define a *double functor* from $\mathscr{C}$ to $\mathscr{C}_1$ to be a map Φ from $\mathscr{C}$ to $\mathscr{C}_1$ such that Φ is a functor from $\mathscr{C}^\bullet$ to $\mathscr{C}_1^\bullet$ and a functor from $\mathscr{C}^\perp$ to $\mathscr{C}_1^\perp$. The class of double functors from $\mathscr{C}$ to $\mathscr{C}_1$ is denoted $\mathscr{F}(\mathscr{C}_1,\mathscr{C})$.

Proposition. *$\mathscr{F}(\mathscr{C}_1,\mathscr{C})$ is a subcategory of $\mathscr{F}(\mathscr{C}_1^\bullet,\mathscr{C}^\bullet)$ and of $\mathscr{F}(\mathscr{C}_1^\perp,\mathscr{C}^\perp)$; endowed with the two induced composition laws, $\mathscr{F}(\mathscr{C}_1,\mathscr{C})$ is a double category.*

Proposition. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories; the longitudinal category $\mathfrak{N}(\mathscr{C}',\mathscr{C})$ of natural transformations[2] between functors from $\mathscr{C}$ to $\mathscr{C}'$ can be identified with the category $\mathscr{F}(\boxminus\mathscr{C}',\mathscr{C})$, by identifying the natural transformation (φ',τ,ϕ) with the functor Φ such that*

$$\Phi(f)=\big(\varphi'(f),\,\tau(\beta(f)),\,\tau(\alpha(f)),\,\varphi(f)\big)$$

for all $f\in\mathscr{C}$.

Consequently, if $(\mathscr{C}^\bullet,\mathscr{C}^\perp)$ is a double category, then a functor Φ from a category Γ into $\mathscr{C}^\bullet$ can be considered as a generalised natural transformation from $\alpha^\perp\Phi$ to $\beta^\perp\Phi$. We will see another generalisation of natural transformations (the double category of quintets) in a following publication.

4 Structured categories

Let

[2] *Catégorie des foncteurs types*, Rev. Un. Mat. Argentina, **20**, 1960, p. 194.