

Twisting cochains and twisted complexes

Simplicial methods in complex-analytic geometry

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Plan

History

Twisting cochains (OTT)

- The bicomplex

- The total complex

- Maurer-Cartan

Twisted complexes (BK)

- Pretriangulated vs. triangulated

- Generalisation of twisting cochains

Other fun things

History

- Edgar H Brown. “Twisted tensor products, I”. In: *Annals of Mathematics* 69.1 (1959), pp. 223–246.
- John C Moore. “Differential homological algebra”. In: *Actes du Congres International des Mathématiciens* 1 (1970), pp. 335–339.

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- Domingo Toledo and Yue Lin L Tong. “Duality and Intersection Theory in Complex Manifolds. I”. In: *Mathematische Annalen* 237 (1978), pp. 41–77.
- Nigel R O’Brian, Domingo Toledo, and Yue Lin L Tong. “The Trace Map and Characteristic Classes for Coherent Sheaves”. In: *American Journal of Mathematics* 103.2 (1981), pp. 225–252.

- A I Bondal and M M Kapranov. “Enhanced Triangulated Categories”. In: *Math. USSR Sbornik* 70.1 (1991), pp. 1–15.
- Giovanni Faonte. *Simplicial nerve of an A-infinity category*. 2015. arXiv: 1312.2127 [math.AT].

Twisting cochains (OTT)

Definition (Stein spaces)

A complex-analytic¹ manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

¹analytic = \mathcal{O}_Y is holomorphic functions, Y has the \mathbb{C}^n -induced topology;
algebraic = \mathcal{O}_Y is algebraic functions, Y has the Zariski topology.

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Throughout, X is a complex-analytic manifold with a nice² cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

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Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* \mathcal{O}_{U_α} -modules:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

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The collection of *degree- q endomorphisms* $\text{End}^q(V)$ of V is, over each $U_{\alpha_0 \dots \alpha_p}$, given by

$$\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_{\alpha_p}^i|_{U_{\alpha_0 \dots \alpha_p}}, V_{\alpha_0}^{i+q}|_{U_{\alpha_0 \dots \alpha_p}}).$$

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Warning

The maps are from the α_p part to the α_0 part.

The deleted Čech complex

Definition (Deleted Čech complex)

Define the chain complex $(\hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) = \bigoplus_{(\alpha_0, \dots, \alpha_p)} \text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}}$$

(where $\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = 0$ if $U_{\alpha_0 \dots \alpha_p} = \emptyset$) with the **deleted Čech differential**

$$\hat{\delta}: \hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) \rightarrow \hat{\mathcal{C}}^{p+1}(\mathcal{U}, \text{End}^q(V))$$

$$(\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=1}^p (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

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- There is a natural multiplication structure given by composition:

$$(c^{p,q} \cdot \tilde{c}^{\tilde{p},\tilde{q}})_{\alpha_0 \dots \alpha_{p+\tilde{p}}} = (-1)^{q\tilde{p}} c^{p,q}_{\alpha_0 \dots \alpha_p} \tilde{c}^{\tilde{p},\tilde{q}}_{\alpha_p \dots \alpha_{p+\tilde{p}}}.$$

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- We could define the same complex for an arbitrary bounded graded vector bundle, i.e. $\hat{\mathcal{C}}^\bullet(\mathcal{U}, V^\circ)$, but where the deleted Čech differential only omits the *first* index (but includes the $(p+1)$ th).

A brief interlude on bundles

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Note that these are maps from $E|_{U_{\alpha p}}$ to $E|_{U_{\alpha 0}}$ in the specific case where $p = 1$.

Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\mathcal{C}^1(\mathcal{U}, E)$, we see that

$$\begin{aligned}(\hat{\delta}g)_{\alpha\beta\gamma} &= -g_{\alpha\gamma} \\ (g \cdot g)_{\alpha\beta\gamma} &= g_{\alpha\beta}g_{\beta\gamma}.\end{aligned}$$

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This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

Twisting cochains

Definition (Twisting cochains)

A (holomorphic) twisting cochain over V is a formal sum

$$a = \bigoplus_{k \in \mathbb{N}} a^{k, 1-k}$$

where $a^{k, 1-k} \in \hat{\mathcal{C}}^k(\mathcal{U}, \text{End}^{1-k}(V))$ such that

1. $\hat{\delta}a + a \cdot a = 0$; and
2. $a_{\alpha\alpha}^{1,0} = \text{id}$.

Twisting cochains (cont.)

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(a \cdot b)^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} a^{q,t} \cdot b^{q',t'}.$$

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- If V has a differential then a is an element of total degree 1.
- We haven't said *when* twisting cochains exist, but under pretty mild assumptions they always do (by an inductive construction).

Unpacking the definition

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$(k = 2) \rightsquigarrow -a_{\alpha\gamma}^{1,0} + a_{\alpha\beta}^{1,0} \cdot a_{\beta\gamma}^{1,0} = a_{\alpha}^{0,1} \cdot a_{\alpha\beta\gamma}^{2,-1} + a_{\alpha\beta\gamma}^{2,-1} \cdot a_{\gamma}^{0,1}$, which says that $a_{\alpha\beta\gamma}^{2,-1}$ witnesses a *chain homotopy* between $a_{\alpha\gamma}^{1,0}$ and $a_{\alpha\beta}^{1,0} \cdot a_{\beta\gamma}^{1,0}$. On $\alpha\beta\alpha$ and $\beta\alpha\beta$ this tells us that $a_{\alpha\beta}^{1,0}$ and $a_{\beta\alpha}^{1,0}$ are *chain homotopic inverses*, i.e. *quasi-isomorphism*.

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Extra-curricular

By taking (internal) homology we obtain something strict: a complex of *coherent sheaves* $H^\bullet(\mathfrak{a})$. This is because quasi-isomorphisms become strict isomorphisms in homology.

We can use this fact to construct twisting cochains that resolve coherent sheaves by taking *local* resolutions by vector bundles.

The total differential

Lemma

For any $a \in \text{Tot}^1 \hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V))$, the map

$$\begin{aligned} D_a : \text{Tot}^r \hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V)) &\rightarrow \text{Tot}^{r+1} \hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V)) \\ c &\mapsto \hat{\delta}c + c \cdot a \end{aligned}$$

defines a differential (i.e. squares to zero) if and only if a is a twisting cochain.

Proof.

(Tedious) definition chasing. □

The total differential (cont.)

We can actually define twisting cochains in a different way using this lemma (but we won't do so today).

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This approach lets us think of a twisting cochain as a *first-order perturbation of the deleted Čech differential*.

MENTION HOW THIS IS A WAY OF VIEWING EVERYTHING OVER
 U_α

Twisted complexes (BK)

Other fun things
