

Twisting cochains and twisted complexes

Simplicial methods in complex-analytic algebraic geometry

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Plan

History

Twisting cochains (OTT)

Bicomplexes

Maurer-Cartan

Twisted complexes (BK)

Pretriangulated vs. triangulated

Generalisation of twisting cochains

Other fun things

History

- Edgar H Brown. “Twisted tensor products, I”. In: *Annals of Mathematics* 69.1 (1959), pp. 223–246.
- John C Moore. “Differential homological algebra”. In: *Actes du Congres International des Mathématiciens* 1 (1970), pp. 335–339.

Coherent sheaves

- Domingo Toledo and Yue Lin L Tong. “A parametrix for δ and Riemann-Roch in Čech theory”. In: *Topology* 15.4 (1976), pp. 273–301.
- Domingo Toledo and Yue Lin L Tong. “Duality and Intersection Theory in Complex Manifolds. I”. In: *Mathematische Annalen* 237 (1978), pp. 41–77.
- Nigel R O’Brian, Domingo Toledo, and Yue Lin L Tong. “The Trace Map and Characteristic Classes for Coherent Sheaves”. In: *American Journal of Mathematics* 103.2 (1981), pp. 225–252.

- A I Bondal and M M Kapranov. “Enhanced Triangulated Categories”. In: *Math. USSR Sbornik* 70.1 (1991), pp. 1–15.
- Giovanni Faonte. *Simplicial nerve of an A-infinity category*. 2015. arXiv: 1312.2127 [math.AT].

Twisting cochains (OTT)

Definition (Stein spaces)

A complex-analytic¹ manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

¹analytic = \mathcal{O}_Y is holomorphic functions, Y has the \mathbb{C}^n -induced topology;
algebraic = \mathcal{O}_Y is algebraic functions, Y has the Zariski topology.

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Throughout, X is a complex-analytic manifold with a nice² cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

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Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* \mathcal{O}_{U_α} -modules:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

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Definition (Endomorphisms)

The collection of *degree- q endomorphisms* $\text{End}^q(V)$ of V is, over each $U_{\alpha_0 \dots \alpha_p}$, given by

$$\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_{\alpha_p}^i|_{U_{\alpha_0 \dots \alpha_p}}, V_{\alpha_0}^{i+q}|_{U_{\alpha_0 \dots \alpha_p}}).$$

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Warning

The maps are from the α_p part to the α_0 part.

The deleted Čech complex

Definition (Deleted Čech complex)

Define the complex $(\hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) = \bigoplus_{(\alpha_0, \dots, \alpha_p)} \text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}}$$

(where $\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = 0$ if $U_{\alpha_0 \dots \alpha_p} = \emptyset$) with the **deleted Čech differential**

$$\hat{\delta}: \hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) \rightarrow \hat{\mathcal{C}}^{p+1}(\mathcal{U}, \text{End}^q(V))$$

$$(\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=1}^p (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

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- There is a natural multiplication structure given by composition:

$$(c^{p,q} \cdot \tilde{c}^{\tilde{p},\tilde{q}})_{\alpha_0 \dots \alpha_{p+\tilde{p}}} = (-1)^{q\tilde{p}} c_{\alpha_0 \dots \alpha_p}^{p,q} \tilde{c}_{\alpha_p \dots \alpha_{p+\tilde{p}}}^{\tilde{p},\tilde{q}}.$$

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