Twisting cochains and twisted complexes

Simplicial methods in complex-analytic geometry

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Plan

- History
- Twisting cochains (OTT)
  - The bicomplex
  - The total complex
  - Maurer-Cartan
- Twisted complexes (BK)
  - Pretriangulated, triangulated, and stable
  - Generalisation of twisting cochains
- Other fun things
  - The A-infinity Yoneda embedding
  - The bar construction

History

First steps

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- John C Moore. “Differential homological algebra”. In: *Actes du Congres International des Mathématiciens* 1 (1970), pp. 335–339.

Coherent sheaves

- Domingo Toledo and Yue Lin L Tong. "A parametrix for $\bar{\partial}$ and Riemann-Roch in Čech theory". In: *Topology* 15.4 (1976), pp. 273–301.
- Domingo Toledo and Yue Lin L Tong. "Duality and Intersection Theory in Complex Manifolds. I". In: *Mathematische Annalen* 237 (1978), pp. 41–77.
- Nigel R O'Brian, Domingo Toledo, and Yue Lin L Tong. "The Trace Map and Characteristic Classes for Coherent Sheaves". In: *American Journal of Mathematics* 103.2 (1981), pp. 225–252.

Triangulation and stability

- A I Bondal and M M Kapranov. "Enhanced Triangulated Categories". In: *Math. USSR Sbornik* 70.1 (1991), pp. 1–15.
- Giovanni Faonte. *Simplicial nerve of an A-infinity category*. 2015. arXiv: 1312.2127v2 [math.AT].

Nomenclature

Lemma

Let $x \in \{twisting, twisted\}$ and $y \in \{cochain, complex\}$.

Then there exists somebody who uses "$x\,y$" to denote what you call "$x^c\,y^c$".

Corollary

Sometimes it can be hard to figure out what people mean.

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But that's very possibly just me.

Twisting cochains (OTT)

Nice spaces

Definition (Stein spaces)

A complex-analytic[1] manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

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[1]analytic = $\mathcal{O}_Y$ is holomorphic functions, Y has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, Y has the Zariski topology.

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Throughout, X is a complex-analytic manifold with a nice[2] cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

[1]analytic = $\mathcal{O}_Y$ is holomorphic functions, Y has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, Y has the Zariski topology.
[2]*Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

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Definition (Endomorphisms)

The collection of *degree-q endomorphisms* $\mathrm{End}^q(V)$ *of* V is, over each $U_{\alpha_0 \ldots \alpha_p}$, given by

$$\mathrm{End}^q(V)|U_{\alpha_0 \ldots \alpha_p} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}\big(V_{\alpha_p}^i | U_{\alpha_0 \ldots \alpha_p}, V_{\alpha_0}^{i+q} | U_{\alpha_0 \ldots \alpha_p}\big).$$

Source and target

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We discuss this later.

Definition (Deleted Čech complex)

Define the chain complex $(\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathscr{C}}^p\big(\mathcal{U}, \mathrm{End}^q(V)\big) = \bigoplus_{(\alpha_0,\ldots,\alpha_p)} \mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p}$$

(where $\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = 0$ if $U_{\alpha_0\ldots\alpha_p} = \varnothing$) with the ***deleted Čech differential***

$$\hat{\delta}\colon \hat{\mathscr{C}}^p\big(\mathcal{U}, \mathrm{End}^q(V)\big) \to \hat{\mathscr{C}}^{p+1}\big(\mathcal{U}, \mathrm{End}^q(V)\big)$$

$$(\hat{\delta}c)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=1}^{p} (-1)^i c_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}}.$$

A notational note

We use $\hat{\mathscr{C}}$ and $\hat{\delta}$ for the *deleted* Čech objects and $\check{\mathscr{C}}$ and $\check{\delta}$ for the ‘full’ Čech objects.

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- We could define the same complex for an arbitrary bounded graded vector bundle, i.e.

$$\hat{\mathscr{C}}^p(\mathcal{U}, V^q) = \bigoplus_{(\alpha_0, \ldots, \alpha_p)} V^q_{\alpha_0}$$

  but where the deleted Čech differential only omits the *first* index (but includes the $(p+1)$th).

A brief interlude on bundles

A holomorphic vector bundle E on X is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over U_β to over U_α.

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Note that these are maps from $E|U_{\alpha_p}$ to $E|U_{\alpha_0}$ in the specific case where $p = 1$.

Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\hat{\mathscr{C}}^1(\mathcal{U}, E)$, we see that

$$(\hat{\delta}g)_{\alpha\beta\gamma} = -g_{\alpha\gamma}$$
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This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

Twisting cochains

Definition (Twisting cochains)

A *(holomorphic) twisting cochain over V* is a formal sum

$$\mathrm{a} = \bigoplus_{k \in \mathbb{N}} \mathrm{a}^{k,1-k}$$

where $\mathrm{a}^{k,1-k} \in \hat{\mathscr{C}}^k(\mathcal{U}, \mathrm{End}^{1-k}(V))$ such that

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The invertibility condition "should" really be weakened by asking only that $\mathrm{a}^{1,0}_{\alpha\alpha}$ be *homotopic* to the identity.

Twisting cochains (cont.)

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

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- If V has a differential then a is an element of total degree 1.
- We haven't said *when* twisting cochains exist, but under pretty mild assumptions they always do (by an inductive construction).

Unpacking the definition

$(k=0) \rightsquigarrow$ $\mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_\alpha^{0,1} = 0$, which tells us that $\mathrm{a}_\alpha^{0,1}$ is a *differential on $V_\alpha^\bullet$*.

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$(k = 1) \rightsquigarrow$ $\mathrm{a}^{0,1}_{\alpha} \cdot \mathrm{a}^{1,0}_{\alpha\beta} = \mathrm{a}^{1,0}_{\alpha\beta} \cdot \mathrm{a}^{0,1}_{\beta}$, which tells us that we have a *chain map of chain complexes*

$$\mathrm{a}^{1,0}_{\alpha\beta} \colon \left(V^{\bullet}_{\beta}|U_{\alpha\beta}, \mathrm{a}^{0,1}_{\beta}\right) \to \left(V^{\bullet}_{\alpha}|U_{\alpha\beta}, \mathrm{a}^{0,1}_{\alpha}\right)$$

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$(k=1) \rightsquigarrow$ $\mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_{\alpha\beta}^{1,0} = \mathrm{a}_{\alpha\beta}^{1,0} \cdot \mathrm{a}_\beta^{0,1}$, which tells us that we have a *chain map of chain complexes*

$$\mathrm{a}_{\alpha\beta}^{1,0} \colon (V_\beta^\bullet|U_{\alpha\beta}, \mathrm{a}_\beta^{0,1}) \to (V_\alpha^\bullet|U_{\alpha\beta}, \mathrm{a}_\alpha^{0,1})$$

$(k=2) \rightsquigarrow$ $-\mathrm{a}_{\alpha\gamma}^{1,0} + \mathrm{a}_{\alpha\beta}^{1,0} \cdot \mathrm{a}_{\beta\gamma}^{1,0} = \mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_{\alpha\beta\gamma}^{2,-1} + \mathrm{a}_{\alpha\beta\gamma}^{2,-1} \cdot \mathrm{a}_\gamma^{0,1}$, which says that $\mathrm{a}_{\alpha\beta\gamma}^{2,-1}$ witnesses a *chain homotopy* between $\mathrm{a}_{\alpha\gamma}^{1,0}$ and $\mathrm{a}_{\alpha\beta}^{1,0} \cdot \mathrm{a}_{\beta\gamma}^{1,0}$. On $\alpha\beta\alpha$ and $\beta\alpha\beta$ this tells us that $\mathrm{a}_{\alpha\beta}^{1,0}$ and $\mathrm{a}_{\beta\alpha}^{1,0}$ are *chain homotopic inverses*, i.e. *quasi-isomorphism*.

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Extra-curricular

By taking (internal) homology we obtain something strict: a complex of *coherent sheaves* $\mathrm{H}^\bullet(\mathfrak{a})$. This is because quasi-isomorphisms become strict isomorphisms in homology.

We can use this fact to construct twisting cochains that resolve coherent sheaves by taking *local* resolutions by vector bundles.

The total differential

Lemma

For any $\mathrm{a} \in \mathrm{Tot}^1 \hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V))$*, the map*

$$\begin{aligned} \mathrm{D}_\mathrm{a} \colon \mathrm{Tot}^r \hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ) &\to \mathrm{Tot}^{r+1} \hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ) \\ c &\mapsto \hat{\delta} c + c \cdot \mathrm{a} \end{aligned}$$

defines a differential (i.e. squares to zero) if and only if a *is a twisting cochain.*

Proof.

(Tedious) definition chasing. □

The total differential (cont.)

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But this approach lets us think of a twisting cochain as a *first-order perturbation of the deleted Čech differential*.

Examples

Example

Look at the most trivial example: let V be an *ungraded* vector bundle, and $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$, where $\mathrm{a}^{0,1}_{\alpha} = \mathrm{id}_{V_\alpha}$, and the $\mathrm{a}^{1,0}$ are the transition maps. Then

$$(\mathrm{D}_{\mathrm{a}} c)_{\alpha_0 \ldots \alpha_{p+1}} = \mathrm{a}^{1,0}_{\alpha_0 \alpha_1} c_{\alpha_1 \ldots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha}_i \ldots \alpha_{p+1}}.$$

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We can't use the full Čech differential on $\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ)$ because everything has to lie over U_{α_0}, but this total differential solves that problem — $\mathrm{a}^{1,0}_{\alpha_0\alpha_1}$ is a (quasi-)isomorphism.

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A spectral-sequence argument shows that, in fact, D_a here really is 'the same as' the full Čech differential.

Examples (cont.)

Example

Now look at a slightly-less trivial example: let $V^\bullet$ consist of *complexes* $(V_\alpha^\bullet, \mathrm{d}_\alpha)$ of vector bundles, and $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$, where $\mathrm{a}_\alpha^{0,1} = \mathrm{d}_\alpha$, and the $\mathrm{a}^{1,0}$ are the transition maps. Then

$$(\mathrm{D_a}c)_{\alpha_0\ldots\alpha_{p+1}} = (-1)^p \mathrm{a}_{\alpha_0}^{0,1} c_{\alpha_0\ldots\alpha_p} + \mathrm{a}_{\alpha_0\alpha_1}^{1,0} c_{\alpha_1\ldots\alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0\ldots\widehat{\alpha}_i\ldots\alpha_{p+1}}.$$

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$$\begin{aligned}(\mathrm{D}_\mathrm{a} c)_{\alpha_0 \ldots \alpha_{p+1}} = (-1)^p \mathrm{a}^{0,1}_{\alpha_0} c_{\alpha_0 \ldots \alpha_p} + \mathrm{a}^{1,0}_{\alpha_0 \alpha_1} c_{\alpha_1 \ldots \alpha_{p+1}} \\ + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha}_i \ldots \alpha_{p+1}}.\end{aligned}$$

Identifying the second and third terms with the full Čech differential, as above, gives the usual total differential of the Čech bicomplex:

$$\mathrm{D}_\mathrm{a} = \check{\delta} \pm \mathrm{d}_V.$$

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- Transition maps naturally go from α_p to α_0.
- We want to be able to compare local things, and we need to pull everything back to lie over the same open set in order to do so.

The Maurer-Cartan equation in other fields

Subject	Equation	Interpretation
Differential geometry	$F_\nabla = \mathrm{d}A + A \cdot A$	curvature of a Koszul connection[3]
Gauge theory	$\Omega = \mathrm{d}A + \frac{1}{2}[A \wedge A]$	curvature of a principal connection
Deformation theory	$\partial a + \frac{1}{2}[a, a]$	deformations of f.d. associative k-algebras with unit[4]

[3] Here be Christoffel symbols.

[4] There is also the beautiful fact (that we won't explain at all) that $\mathrm{MC}(A \otimes \mathfrak{g}) \simeq \mathrm{Hom}_{\mathsf{dgAlg}}(\mathrm{CE}(\mathfrak{g}), A)$.

Flatness

Motto

Solutions to (i.e. zeros of) the Maurer-Cartan equation are always (in some sense) *flat objects*.

Twisted complexes (BK)

Motto

Pretriangulated dg-categories are those whose homotopy category is *triangulated*, where *triangulated* denotes the structure left over from taking the homotopy category of a *stable* $(\infty, 1)$-*category*.[5]

[5] Loop spaces and suspensions form an equivalence.

Stability (cont.)

Problem

There is no reason for an arbitrary dg-category to be pretriangulated, which means that homotopy theorists might be unhappy. This is bad.

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Solution

Twisted complexes à la Bondal and Kapranov.

dg-categories

Definition

A *dg-category* is a category enriched over chain complexes. That is, the hom-sets are hom-*complexes*.

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Definition

Let $\mathcal{A}$ be a dg-category. Then a *twisted complex* $\mathfrak{C}$ *over* $\mathcal{A}$ is a collection

$$\mathfrak{C} = \{E_i \in \mathcal{A}, q_{ij} \colon E_i \to E_j \mid i, j \in \mathbb{Z}\}$$

such that

- all but finitely many of the E_i are zero;
- the q_{ij} are of degree $i - j + 1$; and
- $\mathrm{d}q_{ij} + \sum_{s \in \mathbb{Z}} q_{sj} q_{is} = 0$.

Theorem

Given a dg-category, the smallest dg-category containing it in which we can define shifts and functorial cones is exactly its category of twisted complexes.

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Further, if the original dg-category is pretriangulated then this embedding is a quasi-equivalence (which lets us pull back the shift and the cones, which descend exactly to a triangulated structure on the homotopy category).

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Further, if the original dg-category is pretriangulated then this embedding is a quasi-equivalence (which lets us pull back the shift and the cones, which descend exactly to a triangulated structure on the homotopy category).

We can also use the simplicial nerve *to construct stable* $(infty, 1)$*-categories.*

Twisting cochains as twisted complexes

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It is **not** the case that, by picking the 'right' dg-category $\mathcal{A}$, we can recover the definition of holomorphic twisting cochains from that of twisted complexes.

Rather, twisting cochains are a *specific* case of twisted complexes.

Constructing a twisted complex from a twisting cochain

- Let $\mathcal{A} = \mathrm{dgMod}_B$ where $B = \hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathcal{O}_X)$ with differential $\hat{\delta}$.

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- We need a degree-1 B-linear endomorphism $\mathrm{a} = q_{00}$ of E_0 such that $\hat{\delta}\mathrm{a} + \mathrm{a}\mathrm{a} = 0$, but we can show that the dg-algebra $\mathrm{End}_B(E_0)$ is exactly $\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V))$.
- Decomposing a into $\mathrm{a}^{k,1-k}$ we can see that the equation that must be satisfied is exactly what we want.

Constructing a twisted complex from a twisting cochain

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Motto

Twisting cochains are twisted complexes that are *concentrated in degree zero* and '*projective/free*'.

Other fun things

The reference

Bernhard Keller. *Introduction to A-infinity algebras and modules*. 2001. arXiv: 9910179v2 [math.RA].

Definition

- An A_∞-*algebra* is 'like a loop space' — it is a k-algebra with a graded derivation, but where associativity of multiplication holds only up to homotopy, which hold only up to homotopy, which…;
- $\mathcal{C}_\infty\mathcal{A}$ is the *category of* A_∞*-modules* over an A_∞-algebra $\mathcal{A}$;
- $\mathcal{D}_\infty\mathcal{A}$ is the *homotopy category of* $\mathcal{C}_\infty\mathcal{A}$.

Factorisation of the Yoneda functor

Theorem

Let $\mathcal{A}$ be an A_∞-category (with strict identities). Then the Yoneda functor よ factors through the A_∞-category $\mathrm{tw}\mathcal{A}$ of twisting cochains

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\text{よ}_1"] \arrow[dr, swap, "\text{よ}"] & \mathrm{tw}\mathcal{A} \arrow[d, "\text{よ}_2"] \\
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\end{tikzcd}$$

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Further, $よ_1$ is (strictly) fully faithful, and $よ_2$ induces an equivalence

$$\mathrm{H}^0\mathrm{tw}\mathcal{A} \xrightarrow{\sim} \mathrm{tria}\mathcal{A}.$$

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$$\mathrm{d}_A \circ \tau + \tau \circ \mathrm{d}_C + \mu \circ (\tau \otimes \tau) \circ \Delta = 0.$$

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Note that this *is* Maurer-Cartan, since the last term is the product $\tau \star \tau$ in the *convolution algebra*.

The cobar-bar adjunction

The *bar-cobar adjunction* is $(\Omega \dashv \mathrm{B})$ between *cocomplete* dg-coalgebras $\mathsf{dgCog}_{\mathrm{cc}}$ and dg-algebras dgAlg.

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We can do something similar to then turn τ_f into a chain map (again, by Maurer-Cartan) $f'\colon \Omega C \to A$.

Koszul duality and twisted tensor products — full circle

Given some twisting cochain $\tau\colon C \to A[1]$ we can define the *twisted tensor products* $- \otimes_\tau A$ and $- \otimes_\tau C$ on the level of (co)derived categories of (co)modules.

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These two functors form an equivalence if and only if

$$A \otimes_\tau C \otimes_\tau A \to A$$

is a quasi-isomorphism.

If this is the case then

$$\mathrm{H}_\bullet C = \mathrm{Tor}^A_\bullet(k, k)$$
$$\mathrm{H}^\bullet A = \mathrm{Ext}^\bullet_C(k, k).$$