

Twisting cochains and twisted complexes

Simplicial methods in complex-analytic geometry

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History

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Nomenclature

Lemma

Let $x \in \{\text{twisting, twisted}\}$ and $y \in \{\text{cochain, complex}\}$.

Then there exists somebody who uses “ xy ” to denote what you call “ $x^c y^c$ ”.

Corollary

Sometimes it can be hard to figure out what people mean.

I like “twisting cochain” for the topological Čech definition, which is a special case of a “twisted complex”, defined in the dg-categorical setting.

But that’s very possibly just me.

Twisting cochains (OTT)

Nice spaces

Definition (Stein spaces)

A complex-analytic¹ manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

Motto

Stein things are nice.

Throughout, X is a complex-analytic manifold with a nice² cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

¹analytic = \mathcal{O}_Y is holomorphic functions, Y has the \mathbb{C}^n -induced topology;
algebraic = \mathcal{O}_Y is algebraic functions, Y has the Zariski topology.

²*Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* \mathcal{O}_{U_α} -modules:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

Think of a bounded chain complex of vector bundles, but without the information of a differential.

Definition (Endomorphisms)

The collection of *degree- q endomorphisms* $\text{End}^q(V)$ of V is, over each $U_{\alpha_0 \dots \alpha_p}$, given by

$$\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_{\alpha_p}^i|_{U_{\alpha_0 \dots \alpha_p}}, V_{\alpha_0}^{i+q}|_{U_{\alpha_0 \dots \alpha_p}}).$$

Warning

The maps are from the α_p part to the α_0 part.

We discuss this later.

The deleted Čech complex

Definition (Deleted Čech complex)

Define the chain complex $(\hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) = \bigoplus_{(\alpha_0, \dots, \alpha_p)} \text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}}$$

(where $\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = 0$ if $U_{\alpha_0 \dots \alpha_p} = \emptyset$) with the **deleted Čech differential**

$$\hat{\delta}: \hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) \rightarrow \hat{\mathcal{C}}^{p+1}(\mathcal{U}, \text{End}^q(V))$$

$$(\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=1}^p (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

A notational note

We use $\hat{\mathcal{C}}$ and $\hat{\delta}$ for the *deleted* Čech objects and \mathcal{C} and δ for the ‘full’ Čech objects.

Further structure

- If V has a differential then this gives us a *bicomplex*.
- There is a natural multiplication structure given by composition:

$$(c^{p,q} \cdot \tilde{c}^{\tilde{p},\tilde{q}})_{\alpha_0 \dots \alpha_{p+\tilde{p}}} = (-1)^{q\tilde{p}} c_{\alpha_0 \dots \alpha_p}^{p,q} \tilde{c}_{\alpha_p \dots \alpha_{p+\tilde{p}}}^{\tilde{p},\tilde{q}}.$$

- We could define the same complex for an arbitrary bounded graded vector bundle, i.e.

$$\hat{\mathcal{C}}^p(\mathcal{U}, V^q) = \bigoplus_{(\alpha_0, \dots, \alpha_p)} V_{\alpha_0}^q$$

but where the deleted Čech differential only omits the *first* index (but includes the $(p+1)$ th).

A brief interlude on bundles

A holomorphic vector bundle E on X is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over U_β to over U_α .

These transition maps satisfy two conditions:

1. $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ (the *cocycle condition*); and
2. $g_{\alpha\alpha} = \mathrm{id}$ (the *invertibility condition*).

Note that these are maps from $E|_{U_{\alpha p}}$ to $E|_{U_{\alpha 0}}$ in the specific case where $p = 1$.

Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\mathcal{C}^1(\mathcal{U}, E)$, we see that

$$\begin{aligned}(\hat{\delta}g)_{\alpha\beta\gamma} &= -g_{\alpha\gamma} \\ (g \cdot g)_{\alpha\beta\gamma} &= g_{\alpha\beta}g_{\beta\gamma}.\end{aligned}$$

This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

Twisting cochains

Definition (Twisting cochains)

A (holomorphic) twisting cochain over V is a formal sum

$$a = \bigoplus_{k \in \mathbb{N}} a^{k, 1-k}$$

where $a^{k, 1-k} \in \hat{\mathcal{C}}^k(\mathcal{U}, \text{End}^{1-k}(V))$ such that

1. $\hat{\delta}a + a \cdot a = 0$; and
2. $a_{\alpha\alpha}^{1,0} = \text{id}$.

The invertibility condition “should” really be weakened by asking only that $a_{\alpha\alpha}^{1,0}$ be *homotopic* to the identity.

Twisting cochains (cont.)

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(a \cdot b)^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} a^{q,t} \cdot b^{q',t'}.$$

- It might be the case that all but finitely many of the $a^{k,1-k}$ are zero, but **never** $a^{1,0}$, since it has to be the identity on $\alpha\alpha$.
- If V has a differential then a is an element of total degree 1.
- We haven't said *when* twisting cochains exist, but under pretty mild assumptions they always do (by an inductive construction).

Unpacking the definition

$(k = 0) \rightsquigarrow a_{\alpha}^{0,1} \cdot a_{\alpha}^{0,1} = 0$, which tells us that $a_{\alpha}^{0,1}$ is a *differential on V_{α}^{\bullet}* .

$(k = 1) \rightsquigarrow a_{\alpha}^{0,1} \cdot a_{\alpha\beta}^{1,0} = a_{\alpha\beta}^{1,0} \cdot a_{\beta}^{0,1}$, which tells us that we have a *chain map of chain complexes*

$$a_{\alpha\beta}^{1,0}: (V_{\beta}^{\bullet} | U_{\alpha\beta}, a_{\beta}^{0,1}) \rightarrow (V_{\alpha}^{\bullet} | U_{\alpha\beta}, a_{\alpha}^{0,1})$$

$(k = 2) \rightsquigarrow -a_{\alpha\gamma}^{1,0} + a_{\alpha\beta}^{1,0} \cdot a_{\beta\gamma}^{1,0} = a_{\alpha}^{0,1} \cdot a_{\alpha\beta\gamma}^{2,-1} + a_{\alpha\beta\gamma}^{2,-1} \cdot a_{\gamma}^{0,1}$, which says that $a_{\alpha\beta\gamma}^{2,-1}$ witnesses a *chain homotopy* between $a_{\alpha\gamma}^{1,0}$ and $a_{\alpha\beta}^{1,0} \cdot a_{\beta\gamma}^{1,0}$. On $\alpha\beta\alpha$ and $\beta\alpha\beta$ this tells us that $a_{\alpha\beta}^{1,0}$ and $a_{\beta\alpha}^{1,0}$ are *chain homotopic inverses*, i.e. *quasi-isomorphism*.

Unpacking the definition (cont.)

$(k \geq 3) \rightsquigarrow$ some sort of ‘higher homotopic gluings’, whatever this might mean.

This is one of the things that we want to formalise!

Extra-curricular

By taking (internal) homology we obtain something strict: a complex of *coherent sheaves* $H^\bullet(a)$. This is because quasi-isomorphisms become strict isomorphisms in homology.

We can use this fact to construct twisting cochains that resolve coherent sheaves by taking *local* resolutions by vector bundles.

The total differential

Lemma

For any $a \in \text{Tot}^1 \hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V))$, the map

$$\begin{aligned} D_a : \text{Tot}^r \hat{\mathcal{C}}^\bullet(\mathcal{U}, V^\circ) &\rightarrow \text{Tot}^{r+1} \hat{\mathcal{C}}^\bullet(\mathcal{U}, V^\circ) \\ c &\mapsto \hat{\delta}c + c \cdot a \end{aligned}$$

defines a differential (i.e. squares to zero) if and only if a is a twisting cochain.

Proof.

(Tedious) definition chasing. □

The total differential (cont.)

We can actually define twisting cochains in a different way using this lemma (but we won't do so today).

But this approach lets us think of a twisting cochain as a *first-order perturbation of the deleted Čech differential*.

Examples

Example

Look at the most trivial example: let V be an *ungraded* vector bundle, and $a = a^{0,1} + a^{1,0}$, where $a_{\alpha}^{0,1} = \text{id}_{V_{\alpha}}$, and the $a^{1,0}$ are the transition maps. Then

$$(D_a c)_{\alpha_0 \dots \alpha_{p+1}} = a_{\alpha_0 \alpha_1}^{1,0} c_{\alpha_1 \dots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

We can't use the full Čech differential on $\mathcal{C}^{\bullet}(\mathcal{U}, V^{\circ})$ because everything has to lie over U_{α_0} , but this total differential solves that problem — $a_{\alpha_0 \alpha_1}^{1,0}$ is a (quasi-)isomorphism.

A spectral-sequence argument shows that, in fact, D_a here really is 'the same as' the full Čech differential.

Examples (cont.)

Example

Now look at a slightly-less trivial example: let V^\bullet consist of complexes $(V_\alpha^\bullet, d_\alpha)$ of vector bundles, and $a = a^{0,1} + a^{1,0}$, where $a_\alpha^{0,1} = d_\alpha$, and the $a^{1,0}$ are the transition maps. Then

$$\begin{aligned}(D_a c)_{\alpha_0 \dots \alpha_{p+1}} &= (-1)^p a_{\alpha_0}^{0,1} c_{\alpha_0 \dots \alpha_p} + a_{\alpha_0 \alpha_1}^{1,0} c_{\alpha_1 \dots \alpha_{p+1}} \\ &\quad + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.\end{aligned}$$

Identifying the second and third terms with the full Čech differential, as above, gives the usual total differential of the Čech bicomplex:

$$D_a = \check{\delta} \pm d_V.$$

Why this emphasis on the first index?

- Transition maps naturally go from α_p to α_0 .
- We want to be able to compare local things, and we need to pull everything back to lie over the same open set in order to do so.

The Maurer-Cartan equation in other fields

Subject	Equation	Interpretation
Differential geometry	$F_{\nabla} = dA + A \cdot A$	curvature of a Koszul connection ³
Gauge theory	$\Omega = dA + \frac{1}{2}[A \wedge A]$	curvature of a principal connection
Deformation theory	$\partial a + \frac{1}{2}[a, a]$	deformations of f.d. associative k -algebras with unit ⁴

³Here be Christoffel symbols.

⁴There is also the beautiful fact (that we won't explain at all) that $\mathrm{MC}(A \otimes \mathfrak{g}) \simeq \mathrm{Hom}_{\mathrm{dAlg}}(\mathrm{CE}(\mathfrak{g}), A)$.

Motto

Solutions to (i.e. zeros of) the Maurer-Cartan equation are always (in some sense) *flat objects*.

Twisted complexes (BK)

Motto

Pretriangulated dg-categories are those whose homotopy category is *triangulated*, where *triangulated* denotes the structure left over from taking the homotopy category of a *stable* $(\infty, 1)$ -category.⁵

⁵Loop spaces and suspensions form an equivalence.

Problem

There is no reason for an arbitrary dg-category to be pretriangulated, which means that homotopy theorists might be unhappy. This is bad.

Solution

Twisted complexes à la Bondal and Kapranov.

Definition

A *dg-category* is a category enriched over chain complexes. That is, the hom-sets are hom-complexes.

Definition

Let \mathcal{A} be a dg-category. Then a *twisted complex* \mathfrak{C} over \mathcal{A} is a collection

$$\mathfrak{C} = \{E_i \in \mathcal{A}, q_{ij}: E_i \rightarrow E_j \mid i, j \in \mathbb{Z}\}$$

such that

- all but finitely many of the E_i are zero;
- the q_{ij} are of degree $i - j + 1$; and
- $dq_{ij} + \sum_{s \in \mathbb{Z}} q_{sj}q_{is} = 0$.

The main result

Theorem

Given a dg-category, the smallest dg-category containing it in which we can define shifts and functorial cones is exactly its category of twisted complexes.

Further, if the original dg-category is pretriangulated then this embedding is a quasi-equivalence (which lets us pull back the shift and the cones, which descend exactly to a triangulated structure on the homotopy category).

We can also use the simplicial nerve to construct stable $(\text{infty}, 1)$ -categories.

Twisting cochains as twisted complexes

Warning

It is **not** the case that, by picking the ‘right’ dg-category \mathcal{A} , we can recover the definition of holomorphic twisting cochains from that of twisted complexes.

Rather, twisting cochains are a *specific* case of twisted complexes.

Constructing a twisted complex from a twisting cochain

- Let $\mathcal{A} = \text{dgMod}_B$ where $B = \hat{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{O}_X)$ with differential $\hat{\delta}$.
- Let $E_0 = \hat{\mathcal{C}}^\bullet(\mathcal{U}, V)$ (with **trivial** differential) and $E_i = 0$ for all $i \neq 0$.
- We need a degree-1 B -linear endomorphism $a = q_{00}$ of E_0 such that $\hat{\delta}a + aa = 0$, but we can show that the dg-algebra $\text{End}_B(E_0)$ is exactly $\hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V))$.
- Decomposing a into $a^{k, 1-k}$ we can see that the equation that must be satisfied is exactly what we want.

Motto

Twisting cochains are twisted complexes that are *concentrated in degree zero* and *'projective/free'*.

Other fun things

Bernhard Keller. *Introduction to A-infinity algebras and modules*. 2001. arXiv: 9910179v2 [math.RA].

Definition

- An A_∞ -algebra is ‘like a loop space’ — it is a k -algebra with a graded derivation, but where associativity of multiplication holds only up to homotopy, which hold only up to homotopy, which...;
- $\mathcal{C}_\infty \mathcal{A}$ is the *category of A_∞ -modules* over an A_∞ -algebra \mathcal{A} ;
- $\mathcal{D}_\infty \mathcal{A}$ is the *homotopy category of $\mathcal{C}_\infty \mathcal{A}$* .

Factorisation of the Yoneda functor

Theorem

Let \mathcal{A} be an A_∞ -category (with strict identities). Then the Yoneda functor \mathfrak{y} factors through the A_∞ -category $\mathrm{tw}\mathcal{A}$ of twisting cochains

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathfrak{y}_1} & \mathrm{tw}\mathcal{A} \\ & \searrow \mathfrak{y} & \downarrow \mathfrak{y}_2 \\ & & \mathcal{C}_\infty\mathcal{A} \end{array}$$

Further, \mathfrak{y}_1 is (strictly) fully faithful, and \mathfrak{y}_2 induces an equivalence

$$H^0\mathrm{tw}\mathcal{A} \xrightarrow{\sim} \mathrm{tria}\mathcal{A}.$$

The set-up

- dg-coalgebra (C, d_C) with comultiplication Δ ;
- dg-algebra (A, d_A) with multiplication μ .

Then a *twisting cochain* is a morphism $\tau: C \rightarrow A[1]$ such that

$$d_A \circ \tau + \tau \circ d_C + \mu \circ (\tau \otimes \tau) \circ \Delta = 0.$$

Note that this is Maurer-Cartan, since the last term is the product $\tau \star \tau$ in the *convolution algebra*.

The cobar-bar adjunction

The *bar-cobar adjunction* is $(\Omega \dashv B)$ between *cocomplete* dg-coalgebras $\mathrm{dgCog}_{\mathrm{cc}}$ and dg-algebras dgAlg .

Given $f \in \mathrm{dgCog}_{\mathrm{cc}}(C, BA)$ we can define $\tau_f = f \circ \rho$, where $\rho: BA \rightarrow A[1]$ is the natural projection. Then $\tau_f: C \rightarrow A[1]$.

Maurer-Cartan then reduces to asking that f is a *chain map* (which is true, by definition).

We can do something similar to then turn τ_f into a chain map (again, by Maurer-Cartan) $f^\sharp: \Omega C \rightarrow A$.

Koszul duality and twisted tensor products — full circle

Given some twisting cochain $\tau: C \rightarrow A[1]$ we can define the *twisted tensor products* — $\otimes_{\tau} A$ and $- \otimes_{\tau} C$ on the level of (co)derived categories of (co)modules.

These two functors form an equivalence if and only if

$$A \otimes_{\tau} C \otimes_{\tau} A \rightarrow A$$

is a quasi-isomorphism.

If this is the case then

$$H_{\bullet}C = \mathrm{Tor}_{\bullet}^A(k, k)$$

$$H^{\bullet}A = \mathrm{Ext}_{\tilde{C}}^{\bullet}(k, k).$$