

Simplicial Chern-Weil theory for coherent analytic sheaves, part I

Timothy Hosgood

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Abstract

In “Chern classes for coherent sheaves”, H.I. Green constructs Chern classes in de Rham cohomology of coherent analytic sheaves, but in a way that is directly amenable to being abstractified to give us a theory of *simplicial connections*, as well as a simplicial version of Chern-Weil theory. We construct here a formal $(\infty, 1)$ -categorical framework into which we can place Green’s work, as well as a more general idea as to what exactly a simplicial connection should be. The result will be the ability to work with generalised invariant polynomials (which will be introduced in the sequel to this paper) evaluated at the curvature of so-called *admissible* simplicial connections to get explicit Čech representatives in de Rham cohomology of characteristic classes of coherent analytic sheaves.

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1 Introduction

1.1 History and motivation

In 1980, H.I. Green, a student of O'Brian and Eells, wrote their thesis [Gre80] on the subject of Chern classes of coherent sheaves on complex-analytic manifolds. Although the thesis was never published, an exposition was given in [TT86], alongside a sketch of a proof of the Hirzebruch-Riemann-Roch formula for this construction of Chern classes. It combined the theory of twisting cochains, used with great success by Toledo, Tong, and O'Brian in multiple papers ([TT76; TT78; OTT81; OTT85]), with the fibre integration of Dupont ([Dup76]), to construct, from a coherent analytic sheaf, classes in $H^{2k}(X, \Omega_X^{\bullet \geq k})$ that coincide with those given by the classical construction of Chern classes in $H^{2k}(X, \mathbb{Z})$ by Atiyah-Hirzebruch ([AH62]). (It is of historical interest to mention that the approach of expressing characteristic classes in terms of transition functions is very much in line with ideas propounded by Bott; see e.g. the section entitled 'Concluding Remarks' in [BT82, §23].) This construction was considered by Grivaux in his thesis [Gri09], where he constructs unified Chern classes for coherent analytic sheaves (on *compact* analytic manifolds) in Deligne cohomology, and where he states an axiomatisation of Chern classes that ensures uniqueness in any sufficiently nice cohomology theory (of which de Rham cohomology is an example). Although he states that the Grothendieck-Riemann-Roch theorem for closed immersions is not known for Green's construction of Chern classes if X is non-Kähler, this turns out to not be a problem, since it follows from his other axioms by a purely formal, classical argument, involving deformation to the normal cone.

One reason that the study of Chern classes of coherent analytic sheaves is interesting is that it is notably less trivial than the algebraic version. In both the analytic and algebraic settings, Chern classes of locally free sheaves can be constructed by the splitting principle (as explained in e.g. [BT82, §21]) in the ‘most general’ cohomology theories (Deligne–Beilinson cohomology and Chow rings, respectively); but, although coherent algebraic sheaves admit global locally free resolutions, the same is *not* true of coherent analytic sheaves. In general, complex manifolds have very few holomorphic vector bundles, and there are whole classes of examples of coherent sheaves that do not admit a global locally free resolution ([Voi02, Corollary A.5]). One key insight of [Gre80], however, was that the holomorphic twisting resolutions of Toledo and Tong (whose existence was guaranteed by [TT78, Proposition 2.4]) could be used to construct a global resolution by ‘*simplicial* locally free sheaves’, or **locally free sheaves on the nerve**: objects that live over the Čech nerve $X_\bullet^\mathcal{U}$ of a cover \mathcal{U} of X . The existence of such a global resolution, glued together from local pieces, is mentioned in the introduction of [HS01] as a problem that should be amenable to the formal theory of descent. Indeed, these ‘simplicial sheaves’ can be constructed by taking the lax homotopy limit (in the sense of [Ber12, Definition 3.1]) of the diagram of model categories given by the pullback-pushforward Quillen adjunctions along the nerve of a cover of X . One very useful example of such an object is found by pulling back a global (i.e. classical) vector bundle to the nerve: given some $E \rightarrow X$, defining \mathcal{E}^\bullet by $\mathcal{E}^p = (X_p^\mathcal{U} \rightarrow X)^*E$. This actually satisfies a ‘strongly cartesian’ property: it is given by the ‘strict’ (i.e. not lax) homotopy limit of [Ber12]. The twisting cochains from which Green builds these resolutions are also interesting objects in their own right, having been studied extensively by Toledo, Tong, and O’Brian, as previously mentioned. In fact, they can be seen as specific examples of the twisted complexes of [BK91], which are used to pretriangulate arbitrary dg-categories. This gives a possible moral (yet entirely informal) reason to expect the existence of resolutions such as Green’s: twisted complexes give the ‘smallest’ way of introducing a stable structure on a dg-category, and perfect \mathcal{O}_X -modules can be defined as exactly the objects of the ‘smallest’ stable $(\infty, 1)$ -category that contains \mathcal{O}_X and is closed under retracts. Alternatively, one can appeal to [Wei16], which shows that, under certain restrictions on (X, \mathcal{O}_X) , twisting cochains constitute a dg-enrichment of the derived category of perfect complexes.

Another problem in trying to apply Chern–Weil theory to coherent analytic sheaves is that global holomorphic (Koszul) connections rarely exist: the Atiyah class (which coincides^[1] with the first Chern class in cohomology) measures the obstruction of the existence of such a connection. The other main result of Green’s thesis is the construction of ‘simplicial connections’, which are connections on ‘simplicial sheaves’ pulled back along the projection $X_\bullet^\mathcal{U} \times \Delta^\bullet \rightarrow X_\bullet^\mathcal{U}$. The idea behind this construction is powerfully simple: given local connections ∇_α (which always exist) on a locally-free sheaf

^[1]See [Huy05, Exercise 4.4.8], as well as the comments just below Exercise 4.4.11.

E (that is, ∇_α is a connection on $E|_{U_\alpha}$), on any intersection $U_{\alpha\beta}$ we consider the path $\gamma_{\alpha\beta}(t) = t\nabla_\beta + (1-t)\nabla_\alpha$ between the two local connections as some type of ‘connection’ on $X_1^{\mathcal{U}} \times \Delta^1$. More generally, on p -fold intersections $U_{\alpha_0 \dots \alpha_p}$, we can consider the ‘connection’ $\sum_{i=0}^p t_i \nabla_{\alpha_i}$ on $X_p^{\mathcal{U}} \times \Delta^p$. These objects then assemble to give us what might deserve to be called a **simplicial connection**. Green shows that we can take the curvature of such things, which consists of $\text{End}(E_p)$ -valued forms on $X_p^{\mathcal{U}} \times \Delta^p$; by certain technical properties of the sheaves in his resolution, Green shows that these forms satisfy the property necessary to define a simplicial differential form (the same property as found in the equivalence relation defining the fat geometric realisation of a simplicial space), which lets us apply Dupont’s fibre integration (after taking the trace, or evaluating under some other invariant polynomial) to recover (Čech representatives of) classes in de Rham cohomology. One thing that could be considered as missing from Green’s thesis is a formal study of simplicial connections, and so this forms one of the key parts of this paper. It is possible to define simplicial connections in a more general setting, and study conditions that ensure that Chern-Weil theory can be applied (these give the notions of **admissibility**, and being **generated in degree zero**). Green’s connections do indeed satisfy these formal conditions, and this provides a more rigorous reasoning for their usefulness.

Finally, as $(\infty, 1)$ -categories (presented by homotopical categories), modulo some subtleties in the definitions, complexes of sheaves with coherent cohomology are equivalent to the homotopy colimit of so-called **Green complexes** endowed with simplicial connections generated in degree zero. This means that applying Chern-Weil theory to Green complexes does indeed give us a working version of Chern-Weil theory for complexes of sheaves with coherent cohomology.

1.2 Purpose and overview

This diptych aims to construct an $(\infty, 1)$ -categorical framework in which we can formally understand Green’s construction, as well as showing that it agrees with other existing notions of Chern classes. In this first paper, we focus on the abstract definitions underlying the theory, and prove some technical results that will be used in the sequel; the next paper will focus on explicit calculations of simplicial connections (and their curvatures) in the case of pullbacks (to the nerve) of global vector bundles.

We briefly summarise Chern-Weil theory for those who are not familiar with it: it is a way of defining characteristic classes of bundles by looking at certain types of polynomials evaluated on the curvature of a connection on the bundle.

In Section 2 we define the necessary prerequisites (sheaves on simplicial spaces, simplicial differential forms, twisting cochains, and holomorphic twisting resolutions) and recall Green’s simplicial resolution [Gre80, §1.4], as well as exploring his example [Gre80, pp. 41–42] in more detail. We also define the fundamental notion of a **Green**

complex as any complex that behaves sufficiently like some complex coming from Green's resolution.

The purpose of Section 3 is to introduce the main object of study: **simplicial connections**. We give a geometric motivation for the definition of an **admissible** simplicial connection, but note that the real reason for the definition is that it is exactly the property needed in order to be able to apply a simplicial version of Chern-Weil theory (which we will do in part II). We then discuss an even finer property, namely that of **being generated in degree zero**, and show that Green complexes always admit such simplicial connections, and that this property does indeed imply admissibility. This (Theorem 3.5.5) is one of the two main results of this paper.

Finally, in Section 4, we define the relevant homotopical categories (and then $(\infty, 1)$ -categories) of the objects with which we have been working. The other main result of this paper (Corollary 4.2.14) is the fact that there is an equivalence of $(\infty, 1)$ -categories between complexes of sheaves with coherent cohomology and complexes coming from Green's resolution endowed with generated-in-degree-zero simplicial connections. When we develop the simplicial version of Chern-Weil theory, this equivalence will let us calculate Chern classes of coherent analytic sheaves by calculating the Chern classes of the Green complexes that resolve them.

2 Preliminaries

Throughout this entire paper, let (X, \mathcal{O}_X) be a paracompact complex-analytic manifold with its structure sheaf of holomorphic functions; let \mathcal{U} be a locally-finite Stein open cover of X such that finite intersections are again Stein.

2.1 Conventions

We write Δ to mean the abstract simplex category: its objects are the finite ordinals $[p] = [0, 1, \dots, p-1, p]$ for $p \in \mathbb{N}$; its morphisms are the order-preserving maps. For all $p \in \mathbb{N}$, we have, for $i \in \{0, \dots, p-1\}$, the **coface** maps, which are the injections $f_p^i: [p-1] \rightarrow [p]$ given by omitting i ; we also have, for $i \in \{0, \dots, p\}$, the **codegeneracy** maps, which are the surjections $s_i^p: [p+1] \rightarrow [p]$ that send both i and $i+1$ to i .

We write Δ^\bullet to mean the topological simplex category: its objects are the topological simplices $\Delta^p \subset \mathbb{R}^{p+1}$ for $p \in \mathbb{N}$, where Δ^p is the set of points

$$\Delta^p = \left\{ (t_0, \dots, t_p) \in \mathbb{R}^{p+1} \mid t_i \geq 0, \sum_{i=0}^p t_i = 1 \right\}$$

endowed with the topology induced by the inclusion $\Delta^p \hookrightarrow \mathbb{R}^{p+1}$. This is an example of a *cosimplicial space*: it has coface maps $\Delta^\bullet f_p^i: \Delta^{p-1} \rightarrow \Delta^p$ and codegeneracy maps

$\Delta^\bullet s_i^p: \Delta^{p+1} \rightarrow \Delta^p$. For this specific cosimplicial space, we simply write f_p^i (resp. s_i^p) to mean $\Delta^\bullet f_p^i$ (resp. $\Delta^\bullet s_i^p$).

Given a topological space Y with open cover \mathcal{V} , we define its **nerve** to be the simplicial space $Y_\bullet^\mathcal{V}$ given, in degree p , by

$$Y_p^\mathcal{V} = \coprod_{\substack{\beta_0 \dots \beta_p \\ V_{\beta_0 \dots \beta_p} \neq \emptyset}} V_{\beta_0 \dots \beta_p}$$

(where we write $V_{\beta_0 \dots \beta_p}$ to mean the intersection $V_{\beta_0} \cap \dots \cap V_{\beta_p}$) and where the **face** maps act by

$$Y_p^\mathcal{V} f_p^i: V_{\beta_0 \dots \beta_p} \mapsto V_{\beta_0 \dots \widehat{\beta_i} \dots \beta_p}$$

(where the hat denotes omission) and the **degeneracy** maps act by

$$Y_p^\mathcal{V} s_i^p: V_{\beta_0 \dots \beta_p} \mapsto V_{\beta_0 \dots \beta_i \beta_i \dots \beta_p}.$$

We denote the **Čech complex** (with respect to some cover \mathcal{U}) of an object A by either $\check{C}_\bullet^\mathcal{U}(A)$ or $\check{C}^\bullet(\mathcal{U}, A)$.

If an object has two gradings (for example, a cosimplicial object in the category of cochain complexes) then we denote one grading by \bullet and the other by \star (for example, $\mathcal{F}^{\bullet, \star}$). Generally, we try to keep the difference between gradings explicit in our notation, although this is sometimes at the cost of legibility when we have more than two gradings.

We tend to use the words ‘vector bundle’ and ‘locally free sheaf’ somewhat interchangeably.

As an abuse of notation, we write $U_{\alpha_0 \dots \alpha_p} \in \mathcal{U}$ to mean that each U_{α_i} is in \mathcal{U} .

2.2 Sheaves on simplicial spaces

Definition 2.2.1. Let $(Y_\bullet, \mathcal{O}_{Y_\bullet})$ be a simplicial ringed space, so that each space Y_p has structure sheaf \mathcal{O}_{Y_p} . Then a **sheaf of \mathcal{O}_{Y_\bullet} -modules on Y_\bullet** is a family \mathcal{E}^\bullet of sheaves, where \mathcal{E}^p is a sheaf of \mathcal{O}_{Y_p} -modules on Y_p , along with, for all $\varphi: [p] \rightarrow [q]$ in Δ , covariantly functorial^[2] morphisms

$$\mathcal{E}^\bullet(\varphi): (Y_\bullet \varphi)^* \mathcal{E}^p \rightarrow \mathcal{E}^q$$

^[2]That is, such that $\mathcal{E}^\bullet(\psi \circ \varphi) = \mathcal{E}^\bullet(\psi) \circ \mathcal{E}^\bullet(\varphi)$

of sheaves of \mathcal{O}_{Y_q} -modules, where we take the \mathcal{O} -linear pullback

$$(Y_\bullet \varphi)^* \mathcal{E}^p = (Y_\bullet \varphi)^{-1} \mathcal{E}^p \otimes_{(Y_\bullet \varphi)^{-1} \mathcal{O}_{Y_p}} \mathcal{O}_{Y_q}.$$

A **morphism** between two such sheaves \mathcal{E}^\bullet and \mathcal{F}^\bullet is a family φ^\bullet of morphisms of sheaves, where $\varphi^p: \mathcal{E}^p \rightarrow \mathcal{F}^p$, such that the square

$$\begin{array}{ccc} (Y_\bullet \varphi)^* \mathcal{E}^p & \xrightarrow{(Y_\bullet \varphi)^* \varphi^p} & (Y_\bullet \varphi)^* \mathcal{F}^p \\ \mathcal{E}^\bullet \varphi \downarrow & & \downarrow \mathcal{F}^\bullet \varphi \\ \mathcal{E}^q & \xrightarrow{\varphi^q} & \mathcal{F}^q \end{array}$$

commutes.

In the case where $Y_\bullet = X_\bullet^{\mathcal{U}}$ we often say, as a mild abuse of language, **vector bundle on the nerve** to mean ‘*locally free* sheaf of $\mathcal{O}_{X^{\mathcal{U}}}$ -modules’.

Remark 2.2.2. We refrain from calling such objects ‘simplicial sheaves’, because they are not simply simplicial objects in some category of sheaves (because, for example, each sheaf lives on a different space, not to mention the fact that these objects are cosimplicial). Rather, they are some sort of (co)lax limit objects.^[3]

We repeat the fact that these objects are *covariant* with respect to Δ (that is, *cosimplicial objects*), since they are contravariant objects (sheaves) on a contravariant space (a simplicial space).

Remark 2.2.3. There are no conditions on the ranks of a vector bundle on the nerve: it could be the case that \mathcal{E}^p is of rank r but \mathcal{E}^q is of rank s . By definition, however, the rank is constant over different open sets of the same simplicial degree: $\mathcal{E}^p|_{U_{\alpha_0 \dots \alpha_p}}$ is of the same rank as $\mathcal{E}^p|_{U_{\beta_0 \dots \beta_p}}$.

Definition 2.2.4. A sheaf \mathcal{E}^\bullet on a simplicial space is said to be **strongly cartesian** if the $\mathcal{E}^\bullet \varphi$ are isomorphisms for all $\varphi: [p] \rightarrow [q]$; a complex $\mathcal{E}^{\bullet, \star}$ of sheaves on a simplicial space is said to be **cartesian** if all the $\mathcal{E}^\bullet \varphi$ are quasi-isomorphisms, and **strongly cartesian** if they are all (strict) isomorphisms.

This former condition (of being cartesian) strengthens the ‘colax limit object’ description mentioned in Remark 2.2.2 to ‘strict limit object’.

Example 2.2.5. Given some locally free sheaf E on X , we can pull it back to the nerve: define $E^p = (X_p^{\mathcal{U}} \rightarrow X)^* E$. We call such sheaves **global vector bundles on the nerve**.

^[3]We can formalise this using the lax homotopy limit (of [Ber12, Definition 3.1]) of the diagram of model categories given by the pullback-pushforward Quillen adjunctions along the $X_\bullet^{\mathcal{U}}$, which, in the notation of [Ber12], means taking $F_{\alpha, \beta}^\theta = (X_\bullet^{\mathcal{U}} \theta)^*$ and $u_{\alpha, \beta}^\theta = \mathcal{E}^\bullet(\theta)$.

This gives a particularly well-behaved family of examples: all global vector bundles on the nerve have *constant rank* (across simplicial levels) and are *strongly cartesian* (in fact, all the $E^\bullet \varphi$ are identity maps, which is even stronger).

2.3 Simplicial differential forms

Definition 2.3.1. Let Y_\bullet be a simplicial complex manifold. Following [Dup76], we define a **simplicial differential r -form on Y_\bullet** to be a family ω_\bullet of forms, with ω_p a global section of the sheaf

$$\bigoplus_{i+j=r} \pi_{Y_p}^* \Omega_{Y_p}^i \otimes_{\mathcal{O}_{Y_p \times \Delta_{\text{extd}}^p}} \pi_{\Delta_{\text{extd}}^p}^* \Omega_{\Delta_{\text{extd}}^p}^j$$

(where Δ_{extd}^p is the affine subspace of \mathbb{R}^{p+1} given by the vanishing of $1 - \sum_{m=0}^p x_p$; and where Ω_{Y_p} is the sheaf of *holomorphic* forms, and $\Omega_{\Delta_{\text{extd}}^p}$ is the sheaf of *smooth* forms) such that, for all coface maps $f_p^i : [p-1] \rightarrow [p]$,

$$(Y_\bullet f_p^i \times \text{id})^* \omega_{p-1} = (\text{id} \times f_p^i)^* \omega_p \in \Omega^r(Y_p \times \Delta^{p-1}). \quad (2.3.1.1)$$

We write $\Omega^{r,\Delta}(Y_\bullet)$ to mean the algebra of all simplicial differential r -forms on Y_\bullet . We can describe each ω_p as a form **of type (i, j)** , by writing $\omega_p = \xi_p \otimes \tau_p$, where ξ_p is the Y_p -part of ω_p , and τ_p is the Δ^p -part of ω_p ; then $i = |\xi_p|$ and $j = |\tau_p|$. This lets us define a **differential**

$$d : \Omega^{r,\Delta}(Y_\bullet) \longrightarrow \Omega^{r+1,\Delta}(Y_\bullet)$$

which is given by the Koszul convention with respect to the type of the form:

$$\begin{aligned} d(\xi_p \otimes \tau_p) &= (d_{Y_\bullet} + (-1)^{|\xi_p|} d_{\Delta^\bullet})(\xi_p \otimes \tau_p) \\ &= d\xi_p \otimes \tau_p + (-1)^{|\xi_p|} \xi_p \otimes d\tau_p. \end{aligned}$$

Remark 2.3.2. The condition in (2.3.1.1) can be understood as asking that the forms descend to the fat geometric realisation, as explained in [Dup76], or via framings, as alluded to in [Hos20].

Lemma 2.3.3 (Dupont's fibre integration). *There is a quasi-isomorphism which, for each fixed degree r , consists of the map*

$$\int_{\Delta^\bullet} : \Omega^{r,\Delta}(Y_\bullet) \rightarrow \bigoplus_{p=0}^r \Omega^{r-p}(Y_p) \quad (2.3.3.1)$$

induced by **fibre integration**

$$\int_{\Delta^p} : \Omega^{r,\Delta}(Y_\bullet) \rightarrow \Omega^{r-p}(Y_p) \quad (2.3.3.2)$$

where the latter is given by integrating the type- $(r-p, p)$ part of a simplicial form over the geometric realisation of the p -simplex with its canonical orientation.

Proof. The proof in the *smooth* case is exactly [Dup76, Theorem 2.3] along with [Dup76, Remark 1, §2]; the proof in the holomorphic case works almost identically. More details (which are especially useful for explicit calculations) can be found in [Hos20]. \square

Remark 2.3.4. There is a possibility for confusion (and many sign errors) here: [Dup76] uses the convention of writing simplicial differential forms as forms on $\Delta^p \times Y_p$; [Gre80] does the opposite, writing $Y_p \times \Delta^p$. We opt for the latter.

However, this does not really concern us, given our present purposes: we do not perform any actual calculations with fibre integration in this paper; we mention it only to be able to state Example 2.3.5, which tells us that ‘having characteristic classes defined at the level of simplicial differential forms will let us recover classes defined at the level of de Rham cohomology’. This will form the basis of the sequel to this paper.

Example 2.3.5. Taking $Y_\bullet = X_\bullet^{\mathcal{U}}$ gives

$$\int_{\Delta^\bullet} : \Omega^{r,\Delta}(X_\bullet^{\mathcal{U}}) \rightarrow \bigoplus_{p=0}^r \Omega^{r-p}(X_p^{\mathcal{U}}) \cong \text{Tot}^r \check{C}_{\mathcal{U}}^\bullet(\Omega_X^\bullet)$$

where \check{C} denotes the Čech complex. It is interesting to note that the conditions (namely being locally finite and Stein) that we impose on \mathcal{U} are really (as far as fibre integration is concerned) only to ensure that this quasi-isomorphism will eventually calculate de-Rham cohomology; the *existence* of the quasi-isomorphism in Lemma 2.3.3 does *not* depend on the properties of the cover.

2.4 Green’s resolution

Definition 2.4.1. A (cochain) complex K^\bullet of sheaves on a locally-ringed space (X, \mathcal{O}_X) is said to be **perfect** if, locally, it is quasi-isomorphic to a bounded complex of locally free sheaves. That is, if, for every point $x \in X$, there exists some open neighbourhood U of x and some bounded complex E_U^\bullet of locally free sheaves on U such that $K^\bullet|_U$ is quasi-isomorphic to E_U^\bullet .

Definition 2.4.2. Suppose that, over each $U_\alpha \in \mathcal{U}$, we have a finite-length complex $(V_\alpha^\bullet, d_\alpha)$ of locally free \mathcal{O}_{U_α} -modules, the collection of which we refer to simply as V^\bullet . Define the collection $\text{End}^q(V)$ of **degree- q endomorphisms of V** over each $U_{\alpha_0 \dots \alpha_p}$ by

$$\text{End}^q(V) \big|_{U_{\alpha_0 \dots \alpha_p}} = \left\{ (f^i: V_{\alpha_p}^i \big|_{U_{\alpha_0 \dots \alpha_p}} \rightarrow V_{\alpha_0}^{i+q} \big|_{U_{\alpha_0 \dots \alpha_p}})_{i \in \mathbb{Z}} \bigg| d_{\alpha_p} \circ f^i = f^{i+1} \circ d_{\alpha_0} \right\}.$$

That is, an element of $\text{End}^q(V)$ is a ‘true’ (in that it commutes with the differentials) morphism^[4] of degree q of chain complexes $V_{\alpha_p}^\bullet \rightarrow V_{\alpha_0}^\bullet$.

Following [Gre80, 0.A], we define the **deleted Čech complex**

$$\hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) = \prod_{\substack{(\alpha_0 \dots \alpha_p) \\ \text{s.t. } U_{\alpha_0 \dots \alpha_p} \neq \emptyset}} \text{End}^q(V) \big|_{U_{\alpha_0 \dots \alpha_p}}$$

with **deleted Čech differential** $\hat{\delta}: \hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) \rightarrow \hat{\mathcal{C}}^{p+1}(\mathcal{U}, \text{End}^q(V))$ given by

$$(\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=1}^p (-1)^i c_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1}} \big|_{U_{\alpha_0 \dots \alpha_{p+1}}}$$

(note that the sum is only over $i \in \{1, \dots, p\}$, missing out both $i = 0$ and $i = p + 1$).

Definition 2.4.3. A **holomorphic twisting cochain for (\mathcal{U}, V^\bullet)** , where \mathcal{U} and V^\bullet are as in Definition 2.4.2, is an element

$$\mathfrak{a} = \sum_{k \geq 0} \mathfrak{a}^{k, 1-k} \in \text{Tot}^1 \hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\bullet(V))$$

such that

- (i) $\mathfrak{a}_{\alpha\alpha}^{1,0} = \text{id}$;
- (ii) $\mathfrak{a}_\alpha^{0,1} = d_\alpha$;
- (iii) \mathfrak{a} satisfies the Maurer-Cartan equation:

$$\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = 0.$$

Definition 2.4.4. Let F be a sheaf of \mathcal{O}_X -modules on X . Then a **holomorphic twisting resolution of F** is a triple $(\mathcal{U}, V^\bullet, \mathfrak{a})$ such that the following conditions are satisfied:

- (i) $\mathcal{U} = \{U_\alpha\}$ is a locally-finite Stein open cover of X ;

^[4]Alternatively, $\text{End}^q(V)$ can be thought of as Z^0 of the endomorphism module of V in the dg-category of chain complexes *but* with this modification of going from V_{α_p} to V_{α_0} .

- (ii) $V^\bullet = (V_\alpha^\bullet, d_\alpha)$ is a collection of local locally free resolutions of F over each U_α , of *globally*-bounded length^[5];
- (iii) \mathfrak{a} is a holomorphic twisting cochain for (\mathcal{U}, V^\bullet) over F — that is, \mathfrak{a} is a holomorphic twisting cochain for (\mathcal{U}, V^\bullet) such that we have the following commutative diagram:

$$\begin{array}{ccc} V^\bullet & \xrightarrow{\mathfrak{a}^{1,0}} & V^\bullet \\ & \searrow \quad \swarrow & \\ & F & \end{array}$$

- (iv) on degenerate simplices of the specific form $\alpha = (\alpha_0 \dots \alpha_p)$ with $\alpha_i = \alpha_{i+1}$ for some i , we have that $\mathfrak{a}^{k,1-k} = 0$ for $k \geq 2$.

Remark 2.4.5. There are a few existence criteria for holomorphic twisting resolutions. In particular, when F is coherent, [TT76, Lemma 8.13] and [TT78, Lemma 2.4] (with the latter being an applied version of the former) both show that a holomorphic twisting cochain exists (and the latter actually shows a stronger result using the Hilbert syzygy theorem, namely that we can ensure that our global-length bound B is no more than the dimension of X).

Definition 2.4.6. Given a ring R and some R -modules N_1, \dots, N_s , we say that a sequence $0 \rightarrow M_r \rightarrow \dots \rightarrow M_0 \rightarrow 0$ of R -modules is (N_1, \dots, N_s) -**elementary** if it is a direct sum of sequences of the form

$$(0 \rightarrow N_i \xrightarrow{\text{id}} N_i \rightarrow 0)[n]$$

for some $i \in \{1, \dots, s\}$ and $n \in \mathbb{Z}$. Given complexes $V_1^\bullet, \dots, V_t^\bullet$ of R -modules, we say that a sequence is $(V_1^\bullet, \dots, V_t^\bullet)$ -**elementary** if it is \mathcal{N} -elementary, where

$$\mathcal{N} = \left\{ V_i^j \mid 1 \leq i \leq t, 1 \leq j \leq s \right\}.$$

Theorem 2.4.7 (Green’s resolution). *Let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules on a paracompact complex-analytic manifold X with locally-finite Stein cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. Let $(\mathcal{U}, V^\bullet, \mathfrak{a})$ be a holomorphic twisted resolution of \mathcal{F} . Denote by \mathcal{F}^\bullet the pullback of \mathcal{F} to the nerve $X_\bullet^{\mathcal{U}}$. Then there exists a resolution^[6] $\mathcal{E}^{\bullet,\star}$ of \mathcal{F}^\bullet by vector bundles on the nerve:*

$$0 \rightarrow \mathcal{E}^{\bullet,n} \rightarrow \dots \rightarrow \mathcal{E}^{\bullet,0} \rightarrow \mathcal{F}^\bullet$$

where $n = \dim X$. Further, each $\mathcal{E}^{\bullet,j}$ satisfies the following properties:

^[5]That is, each V_α^\bullet is a resolution of $F|_{U_\alpha}$ by locally free \mathcal{O}_{U_α} -modules, and there exists some $B \in \mathbb{N}$ such that every V_α^\bullet is of length no greater than B .

^[6]That is, the morphism $\mathcal{E}^{p,\star} \rightarrow \mathcal{F}^p$ is a quasi-isomorphism in for every $p \in \mathbb{N}$.

(i) $\mathcal{E}^{0,\star} \mid U_\alpha \cong V_\alpha^\star$;

(ii) for all coface^[7] maps $f_p^i: [p-1] \rightarrow [p]$, the map

$$\mathcal{E}^{\bullet,\star} f_p^i: (X_\bullet^\mathcal{U} f_p^i)^* \mathcal{E}^{p-1,\star} \rightarrow \mathcal{E}^{p,\star}$$

of complexes of sheaves on $X_p^\mathcal{U}$ is injective, and $\text{Coker}(\mathcal{E}^{\bullet,\star} f_p^i)$ is an elementary sequence;^[8]

(iii) for all codegeneracy maps $s_i^p: [p+1] \rightarrow [p]$, the map

$$\mathcal{E}^{\bullet,\star} s_i^p: (X_\bullet^\mathcal{U} s_i^p)^* \mathcal{E}^{p+1,\star} \rightarrow \mathcal{E}^{p,\star}$$

of complexes of sheaves on $X_p^\mathcal{U}$ is surjective, and $\text{Ker}(\mathcal{E}^{\bullet,\star} s_i^p)$ is an elementary sequence.

These follow from the fact that, for all $\gamma \leq \beta \leq \alpha = (\alpha_0 \dots \alpha_p)$, writing \mathcal{E}_α to mean $\mathcal{E}^p \mid U_\alpha$, we have the following:

(iv) $\mathcal{E}_\alpha^\star \cong \mathcal{E}_\beta^\star \mid U_\alpha \oplus \mathcal{L}_{\alpha,\beta}^\star$ for some $(V_{\alpha_0}^\star, \dots, V_{\alpha_p}^\star)$ -elementary sequence $\mathcal{L}_{\alpha,\beta}^\star$;

(v) $\mathcal{L}_{\alpha,\gamma}^\star \cong \mathcal{L}_{\beta,\gamma}^\star \mid U_\alpha \oplus \mathcal{L}_{\alpha,\beta}^\star$;

(vi) over each U_α there is the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_\beta^\star & \longrightarrow & \mathcal{E}_\alpha^\star & \longrightarrow & \mathcal{L}_{\alpha,\beta}^\star \longrightarrow 0 \\ & & \uparrow \wr & & \uparrow \wr & & \text{id} \uparrow \\ 0 & \longrightarrow & \mathcal{E}_\gamma^\star \oplus \mathcal{L}_{\beta,\gamma}^\star & \longrightarrow & \mathcal{E}_\gamma^\star \oplus \mathcal{L}_{\alpha,\gamma}^\star & \longrightarrow & \mathcal{L}_{\alpha,\beta}^\star \longrightarrow 0 \end{array}$$

(omitting the restriction notation), where the bottom map is induced by the natural inclusion $\mathcal{L}_{\beta,\gamma}^\star \hookrightarrow \mathcal{L}_{\alpha,\gamma}^\star$ coming from $\mathcal{L}_{\alpha,\gamma}^\star \cong \mathcal{L}_{\beta,\gamma}^\star \oplus \mathcal{L}_{\alpha,\beta}^\star$.

Proof. This is [Gre80, §1.4]. □

Definition 2.4.8. We say that any complex $\mathcal{E}^{\bullet,\star}$ of vector bundles on the nerve satisfying conditions (iv), (v), and (vi) of Theorem 2.4.7 is **Green**. We say that a single vector bundle on the nerve \mathcal{E}^\bullet is Green if the complex $\mathcal{E}^\bullet[0]$ concentrated in degree zero is Green.

Note that we do not prove whether or not every Green complex actually comes from Green's resolution applied to a coherent sheaf, but this will not matter.

^[7]In fact, in [Gre80, §1.4], properties (ii) and (iii) are stated for arbitrary compositions of coface (resp. codegeneracy) maps instead of simply for single coface (resp. codegeneracy) maps.

^[8]Here the complex is indexed by \star from 0 to n , and $\mathcal{E}^{\bullet,\star} f_p^i$ is a map of complexes.

Remark 2.4.9. Both Definition 2.4.4 and Theorem 2.4.7 can be generalised to *complexes*^[9] of coherent sheaves: the fact that Green’s resolution still works follows exactly the same lines as the original proof (although we explain the details in [Hos20]); the fact that holomorphic twisting resolutions of complexes of coherent sheaves exist is explained in the proof of Lemma 4.2.7.

Corollary 2.4.10. *Green complexes are, in particular, cartesian complexes of locally free sheaves on the nerve.*

Proof. Since taking the direct sum with an elementary sequence is a quasi-isomorphism, this follows from properties (ii) and (iii) of Theorem 2.4.7. \square

Remark 2.4.11. The key point to make here is that Green’s simplicial resolution is **not** just the data of a resolution, but also the properties governing the (co)kernels. This is important because it will tell us, in particular, that we get *admissible* simplicial connections on each sheaf in the resolution, which is vital in defining characteristic classes. We use the coface-injectivity property when discussing admissibility of simplicial connections, but don’t seem to need the codegeneracy-surjectivity property anywhere. This might be because the simplicial condition for e.g. simplicial differential forms relies only on (co)face maps, and so we don’t care so much about (co)degeneracy maps in our formalism, and should really be talking about Δ - (or semi-simplicial-) sets.

Remark 2.4.12. Green’s resolution is *not* functorial, but this does not overly matter to us: we only use it to show essential surjectivity of a functor (in Lemma 4.2.7), and so only care about what it does to *objects*.

Example 2.4.13 (Green’s example). Let $X = \mathbb{P}_{\mathbb{C}}^1 \cong \mathbb{C} \cup \{\infty\}$ be the projective line over \mathbb{C} , with cover $\mathcal{U} = \{U_1, U_2\}$, where $U_1 = X \setminus \{\infty\}$ and $U_2 = X \setminus \{0\}$. Let \mathcal{F} be the coherent sheaf given by $\mathcal{O}_X/\mathcal{I}$, where $\mathcal{I} = \mathbb{I}(\{0\})$ is the sheaf of ideals corresponding to the subvariety $\{0\} \subset X$. We use α and β to denote indices in the indexing set $\{1, 2\}$ with the tacit assumption that, whenever we do so, $\alpha \neq \beta$.

The stalks of \mathcal{F} are easy enough to understand:

$$\mathcal{F}_z = \begin{cases} \mathbb{C} & \text{if } z = 0; \\ 0 & \text{if } z \neq 0. \end{cases}$$

This makes it easy to find local resolutions by \mathcal{O}_{U_α} -modules: over U_1 we have the resolution

$$(\xi_1^\bullet \rightarrow \mathcal{F} \mid U_1) = \underbrace{(0 \rightarrow \mathcal{O}_X \mid U_1 \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X \mid U_1)}_{\xi_1^\bullet} \rightarrow \mathcal{F} \mid U_1,$$

^[9]Since we are in the analytic setting, there is some subtlety surrounding the ‘good’ definition of the category of complexes of coherent sheaves. We discuss this further in Section 4.

and over U_2 we have the resolution

$$(\xi_2^\bullet \rightarrow \mathcal{F} \mid U_2) = \underbrace{(0 \rightarrow 0 \rightarrow 0)}_{\xi_2^\bullet} \rightarrow \mathcal{F} \mid U_2.$$

Since neither 0 nor ∞ are in U_{12} , the map $f \mapsto z \cdot f$ gives an automorphism of $\mathcal{O}_X \mid U_{12}$. This means that the homology of $\xi_1 \mid U_{12}$ is zero, and hence isomorphic to the homology of $\xi_2 \mid U_{12}$. The zero map thus gives a quasi-isomorphism $\xi_1 \mid U_{12} \xrightarrow{\sim} \xi_2 \mid U_{12}$, and higher homotopies can be constructed by an inductive method.

To construct a resolution by locally free sheaves on the nerve we need, in particular, a resolution of $\mathcal{F} \mid U_{12} = 0$ by locally free sheaves over U_{12} . Now we need to make a choice, since we have two different (but quasi-isomorphic) possibilities for such a complex:

$$\begin{aligned} (1) \xi_{12}^\bullet &= (0 \rightarrow \mathcal{O}_X \mid U_{12} \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X \mid U_{12}) \rightarrow \mathcal{F} \mid U_{12}, \\ (2) \xi_{12}^\bullet &= (0 \rightarrow 0 \rightarrow 0) \rightarrow \mathcal{F} \mid U_{12}. \end{aligned}$$

But we see that adding the elementary sequence

$$\begin{aligned} (1) \mathcal{L}^\bullet &= (0 \rightarrow \mathcal{O}_X \xrightarrow{\text{id}} \mathcal{O}_X \rightarrow 0) \\ (2) \mathcal{L}^\bullet &= (0 \rightarrow \mathcal{O}_X \xrightarrow{\text{id}} \mathcal{O}_X \rightarrow 0) \end{aligned}$$

to the latter gives us something isomorphic (since neither 0 nor ∞ are in $U_{\alpha\beta}$, so both $1/z$ and z are well defined and holomorphic) to the former:

$$\begin{array}{ccc} (1) \xi_{12}^\bullet & = & (0 \longrightarrow \mathcal{O}_X \mid U_{12} \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X \mid U_{12}) \\ & & \begin{array}{ccc} & f \mapsto z \cdot f \downarrow & \text{id} \downarrow \\ & & \end{array} \\ (2) \xi_{12}^\bullet \oplus (1) \mathcal{L}^\bullet & = & (0 \longrightarrow \mathcal{O}_X \mid U_{12} \xrightarrow{\text{id}} \mathcal{O}_X \mid U_{12}). \end{array}$$

We denote this isomorphism by

$$A_{12}^\bullet: (1) \xi_{12}^\bullet \xrightarrow{\sim} (2) \xi_{12}^\bullet \oplus (1) \mathcal{L}^\bullet$$

i.e. $A_{12}^0 = \text{id}$ and $A_{12}^1 = (z \cdot -)$. We don't actually need this second complex (the codomain of A_{12}^\bullet) to construct the resolution, but it will prove useful in Example 3.4.2.

At this point, the construction stabilises: since our cover consists of only two distinct opens sets, any p -intersection for $p \geq 3$ will be exactly some 2-intersection (either U_{11} , U_{12} , or U_{22}). We have thus constructed a complex of \mathcal{O}_{X^u} -modules on the nerve of X that give a resolution of \mathcal{F} (pulled back to the nerve):

$$0 \rightarrow \mathcal{E}^{\bullet,1} \rightarrow \mathcal{E}^{\bullet,0} \rightarrow \mathcal{F}^\bullet$$

where $\mathcal{E}^{\bullet,1}$ and $\mathcal{E}^{\bullet,0}$ are equal, and defined as

$$\begin{aligned}\mathcal{E}^{0,i} &= \mathcal{O}_X \big|_{U_1 \sqcup 0} \quad \text{over } U_1 \sqcup U_2 \\ \mathcal{E}^{1,i} &= \mathcal{O}_X \big|_{U_{12}} \quad \text{over } U_{12}\end{aligned}$$

and the map $\mathcal{E}^{\bullet,1} \rightarrow \mathcal{E}^{\bullet,0}$ is given by

$$\begin{aligned}(z \cdot -) \sqcup 0 &: \mathcal{E}^{0,1} \rightarrow \mathcal{E}^{0,0} \quad \text{over } U_1 \sqcup U_2 \\ (z \cdot -) &: \mathcal{E}^{1,1} \rightarrow \mathcal{E}^{1,0} \quad \text{over } U_{12}.\end{aligned}$$

3 Simplicial connections

Definition 3.0.1. We introduce the notation $\pi_{p,q}: X_p^{\mathcal{U}} \times \Delta^q \rightarrow X_p^{\mathcal{U}}$ for the projection map; we write π_p to mean $\pi_{p,p}$. Given some vector bundle \mathcal{E}^\bullet on the nerve, we sometimes write $\overline{\mathcal{E}^p}$ to mean $\pi_p^* \mathcal{E}^p$.

3.1 Connections and true morphisms

Definition 3.1.1. The **Atiyah exact sequence** (or **jet sequence**) of a holomorphic vector bundle E on X is the short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow E \otimes \Omega_X^1 \rightarrow J^1(E) \rightarrow E \rightarrow 0$$

where $J^1(E) = (E \otimes \Omega_X^1) \oplus E$ as a sheaf of \mathbb{C}_X -modules, but with an \mathcal{O}_X -action defined by

$$f(s \otimes \omega, t) = (fs \otimes \omega + t \otimes df, ft).$$

Definition 3.1.2. A **holomorphic (Koszul) connection** ∇ on a holomorphic vector bundle E on X is a (holomorphic) splitting of the Atiyah exact sequence of E . By enforcing the **Leibniz rule**

$$\nabla(s \otimes \omega) = \nabla s \wedge \omega + s \otimes d\omega$$

we can extend any connection $\nabla: E \rightarrow E \otimes \Omega_X^1$ to a map $\nabla: E \otimes \Omega_X^r \rightarrow E \otimes \Omega_X^{r+1}$. (Using the same symbol ∇ to denote the connection as well as any such extension is a common abuse of notation.) The **curvature** $\kappa(\nabla)$ of a connection ∇ is the \mathcal{O}_X -linear map

$$\kappa(\nabla) = \nabla \circ \nabla: E \rightarrow E \otimes \Omega_X^2.$$

Definition 3.1.3. A morphism $f: (A, \nabla_A) \rightarrow (B, \nabla_B)$ of vector bundles on X with connections is said to be a **true**^[10] **morphism** if

$$\nabla_B \circ f = (f \otimes \text{id}) \circ \nabla_A.$$

Note that, by the Leibniz rule, if such a morphism f is a true morphism, then the morphism $f: (A, \nabla'_A) \rightarrow (B, \nabla'_B)$ is also ‘true’, in some sense: it satisfies

$$\nabla'_B \circ f = (f \otimes \text{id}) \circ \nabla'_A.$$

In particular, if f is a true morphism, then $f: (A, \kappa(\nabla_A)) \rightarrow (B, \kappa(\nabla_B))$ is also a ‘true’ morphism (in this more general sense).

Definition 3.1.4. Given a vector bundle on the nerve \mathcal{E}^\bullet we define its *i*-th **comparison map**

$$\mathcal{C}_p^i(\mathcal{E}^\bullet): (X_\bullet^\mathcal{U} f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}} \longrightarrow (\text{id} \times f_p^i)^* \overline{\mathcal{E}^p},$$

for $i \in \{0, \dots, p-1\}$, to be the map

$$\pi_{p,p-1}^* (X_\bullet^\mathcal{U} f_p^i)^* \mathcal{E}^{p-1} \xrightarrow{\pi_{p,p-1}^*(\mathcal{E}^\bullet f_p^i)} \pi_{p,p-1}^* \mathcal{E}^p,$$

where we use the fact that

$$\begin{aligned} (X_\bullet^\mathcal{U} f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}} &= \pi_{p,p-1}^* (X_\bullet^\mathcal{U} f_p^i)^* \mathcal{E}^{p-1} \\ (\text{id} \times f_p^i)^* \overline{\mathcal{E}^p} &= \pi_{p,p-1}^* \mathcal{E}^p. \end{aligned}$$

If there is no chance of confusion, we usually omit the dependence on \mathcal{E}^\bullet from the notation. We often say ‘the’ comparison map \mathcal{C}_p when we really mean ‘all’ comparison maps \mathcal{C}_p^i for $i \in \{0, \dots, p-1\}$.

Corollary 3.1.5. *Let $\mathcal{E}^{\bullet,\star}$ be a Green complex. Then the comparison maps $\mathcal{C}_p^i(\mathcal{E}^{\bullet,\star})$ are injective.*

Proof. Pulling back along $\pi_{p,p-1}$ is an exact functor, and being Green tells us that the $\mathcal{E}^{\bullet,\star} f_p^i$ are injective. \square

^[10]These are sometimes called **flat** morphisms, but we opt for ‘true’ to avoid overloading the meaning of the word ‘flat’.

3.2 The motivating example

In an effort to motivate the definitions in this chapter, we start with a simplified example of what we wish to study: we replace vector bundles with vector spaces, and we replace curvatures of connections with endomorphisms.

Definition 3.2.1. Let C be the category whose objects are pairs (V, φ) of finite-dimensional vector spaces V and endomorphisms φ , and whose morphisms $f: (V, \varphi) \rightarrow (W, \psi)$ are the morphisms $f: V \rightarrow W$ of vector spaces such that $f \circ \varphi = \psi \circ f$.

Let $E: C \rightarrow C$ be the endofunctor that sends (V, φ) to $(V/\text{Ker } \varphi, \varphi)$. Write $L_E C$ to mean the localisation of C along all morphisms that become isomorphisms after applying E (the wide subcategory of which we denote by \mathcal{W}).

Recall that the **Grothendieck group** $K(C)$ of C is the group whose elements are isomorphism classes $[A]$ of objects $A \in C$, and where, for each short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in C , we introduce the relation $[A] - [B] + [C] = 0$ in $K(C)$, whence the group operation is given by

$$[(V, \varphi)] + [(W, \psi)] := (V \oplus W, \varphi \oplus \psi).$$

Definition 3.2.2. An object $(V, \varphi) \in C$ is **flat** if $\varphi = 0$. We define an equivalence relation \sim on $K(C)$ by saying that flat objects are equivalent to the zero object:

$$[(V, 0)] \sim [(0, 0)].$$

Definition 3.2.3. A morphism $f: (V, \varphi) \rightarrow (W, \psi)$ in C is **admissible** if there exist sub-bundles $V_1 \hookrightarrow V$ and $W_1 \hookrightarrow W$ such that

1. $V_1 \subseteq \text{Ker } \varphi$ and $W_1 \subseteq \text{Ker } \psi$;
2. f restricts to a morphism $V_1 \rightarrow W_1$;
3. f descends to an isomorphism $V/V_1 \xrightarrow{\sim} W/W_1$.

Lemma 3.2.4. A morphism $f: (V, \varphi) \rightarrow (W, \psi)$ in C is in \mathcal{W} if and only if it is admissible.

Proof. Let $f: (V, \varphi) \rightarrow (W, \psi)$ be a morphism in \mathcal{W} , so that $E(f)$ is an isomorphism. Then we take $V_1 := \text{Ker } \varphi$ and $W_1 := \text{Ker } \psi$.

Conversely, let $f: (V, \varphi) \rightarrow (W, \psi)$ be admissible. Then $V_1 \leq \text{Ker } \varphi$, and so, by the third isomorphism theorem, $(V/V_1)/(\text{Ker } \varphi/V_1) \cong V/\text{Ker } \varphi$. It remains then to show that $E(f)$ restricts to an isomorphism $\text{Ker } \varphi/V_1 \xrightarrow{\sim} \text{Ker } \psi/W_1$. But $f \circ \varphi = \psi \circ f$, and so $E(f) \circ E(\varphi) \circ E(f)^{-1} = E(\psi)$, whence

$$E(f): \text{Ker } \varphi/V_1 = \text{Ker } E(\varphi) \xrightarrow{\sim} \text{Ker } E(\psi) = \text{Ker } \psi/W_1. \quad \square$$

Lemma 3.2.5. There is a (canonical) isomorphism $K(L_E C) \xrightarrow{\sim} K(C)/\sim$.

Proof. It suffices to take the ‘identity’ map, as follows. Take two isomorphic objects $(V, \varphi) \cong (W, \psi) \in L_E C$. Then the isomorphism between them is given either by an isomorphism in C or by some morphism in \mathcal{W} . In the former case, we are done; in the latter case, Lemma 3.2.4 tells us that there is an admissible decomposition $f: V \cong V_1 \oplus V_2 \rightarrow W_1 \oplus W_2 \cong W$. But then both (V_1, φ) and (W_1, ψ) are equivalent to zero, whence $f: (V_2, \varphi) \xrightarrow{\sim} (W_2, \psi)$ gives us an isomorphism in $K(C)/\sim$.

Conversely, take two isomorphic objects $(V, \varphi) \cong (W, \psi) \in C$. Then they are also isomorphic in $L_E C$, since all isomorphisms are in \mathcal{W} . Further, if $[(V, \varphi)] \sim [(0, 0)]$ in $K(C)/\sim$, then $\varphi = 0$, whence $V/\text{Ker } \varphi = 0$, and so $[(V, \varphi)] = [(0, 0)] \in K(L_E C)$. \square

Now assume that we have some ‘invariant polynomial’^[11] P from $C^{\otimes n}$ to some (additive, say) abelian group G (such as \mathbb{C}). If P is additive then it will descend to a well-defined polynomial on $K(C)^{\otimes n}$. If P further sends an n -fold tensor product of flat objects to zero, then it also descends to a well-defined polynomial on $K(C)/\sim$. Thus, by Lemma 3.2.5, P is well defined on $K(L_E C)$; thus, by Lemma 3.2.4, the resulting characteristic classes (that is, the values of P) are invariant under admissible morphisms.

3.3 Admissibility

Definition 3.3.1. An **endomorphism-valued simplicial r -form** ω_\bullet on a vector bundle on the nerve \mathcal{E}^\bullet is a family of forms $\omega_\bullet = \{\omega_p\}_{p \in \mathbb{N}}$, where ω_p is a global section of the sheaf

$$\mathcal{E}nd(\overline{\mathcal{E}^p}) \otimes_{\mathcal{O}_{X_p^u}} \Omega_{X_p^u \times \Delta^p}^r$$

such that

$$\begin{aligned} & (\mathcal{C}_p^i \otimes \text{id}) \circ \left((X_\bullet^u f_p^i \times \text{id})^* \otimes (X_\bullet^u f_p^i \times \text{id})^* \right) \omega_{p-1} \\ &= \left((\text{id} \times f_p^i)^* \otimes (\text{id} \times f_p^i)^* \right) \omega_p \circ (\mathcal{C}_p^i \otimes \text{id}) \end{aligned}$$

as global sections of $\mathcal{H}om \left((X_\bullet^u f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}}, (\text{id} \times f_p^i)^* \overline{\mathcal{E}^p} \right) \otimes \Omega_{X_p^u \times \Delta^{p-1}}^r$.

Definition 3.3.2. We say that an endomorphism-valued simplicial r -form ω_\bullet is **admissible** if it is fibrewise \mathcal{C}_p -admissible: for all $p \in \mathbb{N}$, all $x \in X$, all $v_x \in \bigwedge^r T_x X$, the endomorphism

$$\overline{\mathcal{E}^p} \big|_{\{x\}} \xrightarrow{\omega_p(v_x)} \overline{\mathcal{E}^p} \big|_{\{x\}}$$

^[11]That is, invariant under a change of basis (the GL_n -action), but with the subtlety that we actually need a *sequence* of such polynomials, indexed by \mathbb{N} : one for each possible dimension of the vector space of an element of C . We describe these things in more detail when we need them: in part II.

is such that, for all $i \in \{0, \dots, p-1\}$, the comparison map $\mathcal{C}_p^i(\mathcal{E} \mid \{x\})$ is admissible (in the sense of Definition 3.2.3) with respect to the endomorphisms $\omega_p(v_x)$; in other words, if the induced map

$$(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \left((\overline{\mathcal{E}^{p-1}} \mid \{x\}) / \text{Ker } \omega_{p-1}(v_x) \right) \xrightarrow{\mathcal{C}_p^i} (\text{id} \times f_p^i)^* \left((\overline{\mathcal{E}^p} \mid \{x\}) / \text{Ker } \omega_p(v_x) \right)$$

is an isomorphism.

Lemma 3.3.3. *For an endomorphism-valued simplicial form ω_{\bullet} on \mathcal{E}_{\bullet} to be admissible it is sufficient to ask, for all $p \in \mathbb{N}$, for sub-bundles $L^p, M^p \hookrightarrow \overline{\mathcal{E}^p}$, lying in the kernel of the endomorphism part of the ω_p , such that the comparison maps \mathcal{C}_p^i restrict to isomorphisms*

$$\mathcal{C}_p^i : (X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* (\overline{\mathcal{E}^{p-1}} / L^{p-1}) \xrightarrow{\sim} (\text{id} \times f_p^i)^* (\overline{\mathcal{E}^p} / M^p).$$

Proof. If L^p lies in the kernel of (the endomorphism part of) ω_p then, in particular, $L^p \mid \{x\}$ lies in the kernel of $\omega_p(v_x)$ for any $v_x \in \bigwedge^r T_x X$ (and similarly for M^p). Then we appeal to Lemma 3.2.4. \square

3.4 Simplicial connections

Definition 3.4.1. Given an isomorphism $f : E \xrightarrow{\sim} F$ of vector bundles on X , along with a connection ∇_F on F , we define the **pullback connection** $f^* \nabla_F$ on E by

$$f^* \nabla_F = (f^{-1} \otimes f^*) \circ \nabla_F \circ f.$$

Locally, this takes the trivial connection d on F to the connection $d + f^{-1} df$ on E .

Note that this is *different* from the ‘other’ notion of the pullback of a connection, which normally means pulling back along some change of base $f : X \rightarrow Y$.

Example 3.4.2 (Green’s example (continued)). The prototypical example of what we define in this section is Green’s ‘barycentric connection’, which can be understood as a way of building a connection ∇_p on $\overline{\mathcal{E}^p}$ given local connections ∇_{α_i} on each \mathcal{E}^0 , by simply defining $\nabla_p = \sum_{i=0}^p t_i \nabla_{\alpha_i}$. We explain this in slightly more detail by continuing Example 2.4.13.

Assuming that we have a basis of local sections over U_1 , and another over U_2 , we can look at all the connections we have on the $\mathcal{E}^{p,i}$ given by using the isomorphism A_{12}^{\bullet} to pull back the local trivial connections. This data is given in Table 1.

bundle	p	p -intersection	local connection
$\mathcal{E}^{p,0}$	0	U_1	d
		U_2	d
	1	U_{12}	d (from U_1)
			$d + (A_{1,2}^0)^{-1} d A_{1,2}^0 = d$ (from U_2)
$\mathcal{E}^{p,1}$	0	U_1	d
		U_2	d
	1	U_{12}	d (from U_1)
			$d + (A_{1,2}^1)^{-1} d A_{1,2}^1 = d + \frac{dz}{z}$ (from U_2)

Table 1: All the local connections for this example.

Using these local connections, we can form the **barycentric connections** ∇_\bullet^i on $\mathcal{E}^{\bullet,i}$ as follows:

$$\begin{aligned} \nabla_\bullet^0 \text{ on } \mathcal{E}^{\bullet,0} \text{ is given by } & \begin{cases} \nabla_0^0 = t_0 d & = d \\ \nabla_1^0 = t_0 d + t_1 d & = d \end{cases} \\ \nabla_\bullet^1 \text{ on } \mathcal{E}^{\bullet,1} \text{ is given by } & \begin{cases} \nabla_0^1 = t_0 d & = d \\ \nabla_1^1 = t_0 d + t_1 \left(d + \frac{dz}{z}\right) & = d + t_1 \frac{dz}{z}. \end{cases} \end{aligned}$$

We could continue this example, using Example 2.3.5 to calculate an explicit Čech representative of the Chern class of \mathcal{F} in de Rham cohomology, but this is not within the goals of this current paper; see instead the sequel to this paper.

Definition 3.4.3. A **simplicial connection** ∇_\bullet on a vector bundle on the nerve \mathcal{E}^\bullet is a family $\nabla_\bullet = \{\nabla_p\}_{p \in \mathbb{N}}$ of connections, where ∇_p is a connection on $\overline{\mathcal{E}^p}$, such that the comparison maps

$$\left(X_\bullet^{\mathcal{U}} f_p^i \times \text{id}\right)^* \left(\overline{\mathcal{E}^{p-1}}, \nabla_{p-1}\right) \xrightarrow{\mathcal{C}_p^i(\mathcal{E}^\bullet)} \left(\text{id} \times f_p^i\right)^* \left(\overline{\mathcal{E}^p}, \nabla_p\right)$$

are true morphisms.^[12]

Remark 3.4.4. The individual connections ∇_p making up a simplicial connection ∇_\bullet on \mathcal{E}^\bullet are connections on $\overline{\mathcal{E}^p} = \pi_p^* \mathcal{E}^p$, *not* on \mathcal{E}^p itself.

^[12]The pullbacks act on both the vector bundle and the connection simultaneously: we do *not* mean e.g. ‘pull back the vector bundle on the nerve by $X_\bullet^{\mathcal{U}} f_p^i$ and the connection by id ’; we mean ‘pullback *both* the vector bundle on the nerve and the connection by $(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})$ ’.

Remark 3.4.5. We ask in Definition 3.4.3 that the morphisms between simplicial levels of the connection be true morphisms, but when it comes to defining the category of vector bundles on the nerve endowed with simplicial connections, a morphism $f: (\mathcal{E}^\bullet, \nabla_\bullet) \rightarrow (\mathcal{E}'^\bullet, \nabla'_\bullet)$ between such objects will *not* be asked to satisfy the corresponding requirement: it will simply be a morphism $f: \mathcal{E}^\bullet \rightarrow \mathcal{E}'^\bullet$ of vector bundles on the nerve.

Lemma 3.4.6. *The curvature (defined simplicial degree by simplicial degree) of a simplicial connection is an endomorphism-valued simplicial 2-form.*

Proof. This is a direct consequence of the definitions, as well as the aforementioned fact (in Definition 3.1.3) that a morphism that is true with respect to connections is also true with respect to their curvatures. \square

Definition 3.4.7. We say that a simplicial connection is **admissible** if its curvature is an *admissible* endomorphism-valued simplicial 2-form.

Lemma 3.4.8. *For a simplicial connection ∇_\bullet on \mathcal{E}^\bullet to be admissible, it is sufficient to ask for sub-bundles $A^p, B^p \hookrightarrow \overline{\mathcal{E}^p}$ such that*

- (i) A^p and B^p are ∇_p -flat;
- (ii) the comparison map

$$(X_\bullet^\mathcal{U} f_p^i \times \text{id})^* (\overline{\mathcal{E}^{p-1}}, \nabla_{p-1}) \xrightarrow{\mathbb{C}_p^i} (\text{id} \times f_p^i)^* (\overline{\mathcal{E}^p}, \nabla_p).$$

(which is already known to be a true morphism, since the connection is assumed to be simplicial) restricts to a morphism

$$(X_\bullet^\mathcal{U} f_p^i \times \text{id})^* (A^{p-1}, \nabla_{p-1}) \xrightarrow{\mathbb{C}_p^i} (\text{id} \times f_p^i)^* (B^p, \nabla_p);$$

- (iii) the above restriction of the comparison map induces an isomorphism

$$\mathbb{C}_p^i: (X_\bullet^\mathcal{U} f_p^i \times \text{id})^* (\overline{\mathcal{E}^{p-1}}/A^{p-1}) \xrightarrow{\sim} (\text{id} \times f_p^i)^* (\overline{\mathcal{E}^p}/B^p).$$

Proof. If A^p (resp. B^p) is ∇_p -flat then, in particular, it lies in the kernel of $\kappa(\nabla_p)$. Since $\kappa(\nabla_p)$ is simply $\nabla_p \circ \nabla_p$, the (again) aforementioned fact (that a morphism that is true with respect to some connections is also true with respect to their curvatures) tells us that the (true) morphism

$$(X_\bullet^\mathcal{U} f_p^i \times \text{id})^* (A^{p-1}, \nabla_{p-1}) \rightarrow (\text{id} \times f_p^i)^* (B^p, \nabla_p)$$

induces a *true* morphism

$$(X_\bullet^\mathcal{U} f_p^i \times \text{id})^* (A^{p-1}, \kappa(\nabla_{p-1})) \rightarrow (\text{id} \times f_p^i)^* (B^p, \kappa(\nabla_p)).$$

But then, by Lemma 3.3.3, we are done. \square

Definition 3.4.9. The difference $\nabla'_\bullet - \nabla_\bullet$ of two admissible simplicial connections has *no a priori reason* to be an admissible endomorphism-valued simplicial 1-form, which prompts the following definition: a set of admissible simplicial connections is said to be **compatible** if the difference of any two simplicial connections is indeed an admissible endomorphism-valued simplicial 1-form.

Remark 3.4.10. In summary of the definitions in this chapter so far:

- the *simplicial* condition (Definition 3.4.3) ensures that various forms (such as the curvature) defined by a connection satisfy the gluing condition needed to give a simplicial differential form;
- the *admissibility* condition (Definition 3.4.7) will ensure that we can evaluate ‘generalised invariant polynomials’ on the curvature and get something that is the same in *all* simplicial degrees;
- the *compatibility* condition (Definition 3.4.9) will ensure that characteristic classes will be independent of the choice of connection.

The last two points will be further explained and justified in the sequel to this paper.

3.5 Being generated in degree zero

Remark 3.5.1. For an arbitrary vector bundle on the nerve \mathcal{E}^\bullet , there is no reason for \mathcal{E}^0 and \mathcal{E}^p to be isomorphic. We might imagine, however, that there should be some sort of structural condition ensuring that \mathcal{E}_α^0 and $\mathcal{E}_{\alpha \dots \alpha}^p$ agree, so that the vector bundle is somehow ‘built up’ from its degree-zero part. Indeed, Green bundles can be thought of exactly in this way: the cokernel of any $\mathcal{E}^\bullet f_p^i$ is elementary, and these cokernels satisfy a cocycle condition. If we write $\zeta_p^i: [0] \rightarrow [p]$ to mean the morphism in Δ that sends 0 to i , then we have an *equality* of sheaves

$$(X_\bullet^{\mathcal{U}} \zeta_p^i)^* \mathcal{E}^0 \big|_{U_{\alpha_0 \dots \alpha_p}} = (\mathcal{E}^0 \big|_{U_{\alpha_i}}) \big|_{U_{\alpha_0 \dots \alpha_p}} \quad (3.5.1.1)$$

as well as an isomorphism (Theorem 2.4.7 (iv))

$$\mathcal{E}^p \big|_{U_{\alpha_0 \dots \alpha_p}} \cong ((X_\bullet^{\mathcal{U}} \zeta_p^i)^* \mathcal{E}^0 \big|_{U_{\alpha_0 \dots \alpha_p}}) \oplus \mathcal{K}_{0,i,p} \quad (3.5.1.2)$$

where $\mathcal{K}_{0,i,p} = \text{Coker}(\mathcal{E}^\bullet \zeta_p^i)$ is elementary in the $\mathcal{E}_{\alpha_i}^0$. This splitting lets us think of \mathcal{E}^p as being built up from \mathcal{E}^0 by adding an elementary sequence, and the cocycle condition (Theorem 2.4.7 (v)) satisfied by the $\mathcal{K}_{0,i,p}$ tells us that building \mathcal{E}^p from \mathcal{E}^0 and then \mathcal{E}^q from \mathcal{E}^p (for $q > p$) is the same as building \mathcal{E}^q from \mathcal{E}^0 directly.

Remark 3.5.2. Applying π_p^* (which is exact) to (3.5.1.2), we get a splitting

$$\pi_p^* \mathcal{E}^p \big|_{U_{\alpha_0 \dots \alpha_p}} \cong \pi_p^* \left((X_{\bullet}^{\mathcal{U}} \zeta_p^i)^* \mathcal{E}^0 \big|_{U_{\alpha_0 \dots \alpha_p}} \right) \oplus \pi_p^* \mathcal{K}_{0,i,p}. \quad (3.5.2.1)$$

So if we have local connections $\widetilde{\nabla}_{\alpha_i}$ on $\mathcal{E}^0 \big|_{U_{\alpha_i}}$ for $i = 0, \dots, p$, then (3.5.1.1) tells us that $\pi_p^* \widetilde{\nabla}_{\alpha_i}$ defines a connection on the first summand. But we can also construct a connection on the elementary summand by taking a direct sum of the $\widetilde{\nabla}_{\alpha_i}$. Together then, this lets us build a connection ∇_p on each $\pi_p^* \mathcal{E}^p$.

Importantly, this direct sum of connections on the elementary summand gives a **compatible**^[13] **sequence of connections**: the connections commute with the differentials. This is because being an elementary sequence means that all differentials are identity maps, so if we place the same connection in the two non-zero degrees of each elementary component (i.e. on both copies of M in $0 \rightarrow M \rightarrow M[1] \rightarrow 0$) then commutativity is trivial. But it is known, by applying [BB72, Lemma 4.22] to the exact sequence $0 \rightarrow M \rightarrow (M \rightarrow M[1]) \rightarrow M[1] \rightarrow 0$, that such a sequence of connections gives a trivial characteristic class, and so extending by such a thing will ensure that we do not change the characteristic class of our bundle, thanks to additivity.

Definition 3.5.3. A simplicial connection ∇_{\bullet} on a *Green* vector bundle on the nerve \mathcal{E}^{\bullet} is said to be **generated in degree zero** if, for all $U_{\alpha_0 \dots \alpha_p} \in \mathcal{U}$, it is of the form

$$\nabla_p \big|_{U_{\alpha_0 \dots \alpha_p}} = \sum_{i=0}^p t_i \pi_p^* \widetilde{\nabla}_{\alpha_i}$$

where each $\widetilde{\nabla}_{\alpha_i}$ is a connection on $\mathcal{E}^0 \big|_{U_{\alpha_i}}$. That is, if it is built as in Remark 3.5.2.

Remark 3.5.4. In [Hos20], we try to use the terminology ‘barycentric’ only to describe connections generated in degree zero on pullbacks (to the nerve) of global vector bundles; the phrase ‘generated in degree zero’ applies to arbitrary vector bundles on the nerve. A mild abuse of language is permitted, however, by the fact that, if the bundle \mathcal{E}^{\bullet} is strongly cartesian (which is the case for pullbacks of global bundles), then the inclusion maps $\mathcal{E}^{\bullet} \zeta_p^i$ are isomorphisms. This means, in particular, that the definition of ‘being generated in degree zero’ agrees with that of the barycentric connection on pullbacks of global vector bundles.

Theorem 3.5.5. *Let $\mathcal{E}^{\bullet, \star}$ be a Green complex. Then we can endow each $\mathcal{E}^{\bullet, j}$ with a simplicial connection generated in degree zero. Further, these simplicial connections are admissible.*

Proof. We split the proof into three steps: *defining* connections that are *generated in degree zero*; showing that they are *simplicial*; and then showing that they are *admissible*. For ease of notation, we write \mathcal{E}^{\bullet} instead of $\mathcal{E}^{\bullet, j}$.

^[13]Not to be confused with a compatible family of simplicial connections.

1. Take *arbitrary* local connections $\widetilde{\nabla}_\alpha$ on $\mathcal{E}^0|_{U_\alpha}$ for all α . Since $\mathcal{E}^{\bullet,\bullet}$ is Green, we can use Remarks 3.5.1 and 3.5.2 to define connections ∇_\bullet on $\overline{\mathcal{E}^\bullet}$ that are *generated in degree zero*.
2. To show that each ∇_\bullet defined above is a *simplicial* connection, we need to show that

$$\begin{array}{ccc}
(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \pi_{p-1}^* \mathcal{E}^{p-1} & \xhookrightarrow{\mathcal{C}_p^i(\mathcal{E}^\bullet)} & (\text{id} \times f_p^i)^* \pi_p^* \mathcal{E}^p \\
\downarrow (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \nabla_{p-1} & & \downarrow (\text{id} \times f_p^i)^* \nabla_p \\
(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \pi_{p-1}^* \mathcal{E}^{p-1} & \xhookrightarrow{\mathcal{C}_p^i(\mathcal{E}^\bullet) \otimes \text{id}} & (\text{id} \times f_p^i)^* \pi_p^* \mathcal{E}^p \\
\otimes \Omega_{X_p^{\mathcal{U}} \times \Delta^p}^1 & & \otimes \Omega_{X_p^{\mathcal{U}} \times \Delta^p}^1
\end{array} \tag{3.5.5.1}$$

commutes (where we use Corollary 3.1.5, which tells us that the comparison maps are injective, and we also use the fact that tensoring preserves splittings to see that the bottom horizontal arrow is also an injection). We start by making some simplifications.

First of all, all of the sheaves in (3.5.5.1) lie over $X_p^{\mathcal{U}} \times \Delta^{p-1}$, but we make the identification

$$\begin{aligned}
X_p^{\mathcal{U}} \times \Delta^{p-1} &\simeq X_p^{\mathcal{U}} \times f_p^i(\Delta^{p-1}) \\
&\subset X_p^{\mathcal{U}} \times \Delta^p
\end{aligned}$$

so that we can label *both* (i.e. on the nerve and on the simplex) simplicial parts with the same indices: the nerve being labelled with $\{0, 1, \dots, p\}$; the simplex being labelled with $\{0, 1, \dots, \hat{i}, \dots, p\}$ (so that i is now fixed).

Next, we can use the commutativity of the square

$$\begin{array}{ccc}
X_p^{\mathcal{U}} & \xrightarrow{X_\bullet^{\mathcal{U}} f_p^i} & X_{p-1}^{\mathcal{U}} \\
\uparrow & & \uparrow \\
U_{\alpha_0 \dots \alpha_p} & \hookrightarrow & U_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_p}
\end{array} \tag{3.5.5.2}$$

to see that

$$\begin{aligned} & \left((X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}} \right) \big| \left(U_{\alpha_0 \dots \alpha_p} \times f_p^i(\Delta^{p-1}) \right) \\ &= \left(\overline{\mathcal{E}^{p-1}} \big| \left(U_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_p} \times \Delta^{p-1} \right) \right) \big| \left(U_{\alpha_0 \dots \alpha_p} \times f_p^i(\Delta^{p-1}) \right) \end{aligned}$$

as sheaves over $U_{\alpha_0 \dots \alpha_p} \times f_p^i(\Delta^{p-1})$.

So, to prove the commutativity of (3.5.5.1), we start by calculating how the two pullbacks of the connections (that is, the vertical arrows in the square) act (after restricting all sheaves to $U_{\alpha_0 \dots \alpha_p}$). The first pullback is

$$(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \nabla_{p-1} = (X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \sum_{\substack{j=0 \\ j \neq i}}^p t_j \pi_{p-1}^* \widetilde{\nabla}_{\alpha_j} = \sum_{\substack{j=0 \\ j \neq i}}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j}$$

where the first equality is simply the definition, and the second equality is the tricky one. We are really writing $\widetilde{\nabla}_{\alpha_j}$ to mean two different things: on the left-hand side, it means the connection $\pi_{p-1}^* \widetilde{\nabla}_{\alpha_j}$ extended by the trivial connection^[14] on $\mathcal{K}_{0,j,p-1}$; on the right-hand side, it means the connection $\pi_p^* \widetilde{\nabla}_{\alpha_j}$ extended by the trivial connection on $\mathcal{K}_{0,j,p}$ (all of this using the notation and properties of Remarks 3.5.1 and 3.5.2). So this second equality will really follow from (3.5.5.4), which tells us that extending trivially on $\mathcal{K}_{0,j,p-1}$ and then again to the rest of $\mathcal{K}_{0,j,p}$ (which we do for $\pi_{p-1}^* \widetilde{\nabla}_{\alpha_j}$) is the same as simply extending trivially on $\mathcal{K}_{0,j,p}$ (which we do for $\pi_p^* \widetilde{\nabla}_{\alpha_j}$).

The second pullback is much simpler:

$$(\text{id} \times f_p^i)^* \nabla_p = (\text{id} \times f_p^i)^* \sum_{j=0}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j} = \sum_{\substack{j=0 \\ j \neq i}}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j}$$

which is ‘exactly the same’ as the first pullback — the scare quotes being important, because these connections are on *different sheaves*. But, since the horizontal arrows in (3.5.5.1) are injections, it means that these two connections really are the same when we just follow how they act on the top-left sheaf in the square; i.e., the square commutes.

Looking ahead to (3.5.5.3), since we have extended by something *compatible* on $\mathcal{K}_{p-1,i,p}$, the characteristic class of $\mathcal{K}_{p-1,i,p}$ will be 1, as mentioned above. This means, by additivity of characteristic classes, that the classes of $\mathcal{E}^{p-1} \big| U_{\alpha_0 \dots \alpha_p}$

^[14]That is, we extend it in a compatible way, as explained in Remark 3.5.2.

and $\mathcal{E}^p \mid U_{\alpha_0 \dots \alpha_p}$ will agree. This is the content of half of the proof of [Gre80, Lemma 2.2].

3. To show that each ∇_\bullet is an *admissible* simplicial connection, it suffices to show that the conditions in Lemma 3.4.8 are satisfied.

Before proceeding with the proof, however, we take some time to look at how the splittings that we have been using respect the simplicial structure. The Green assumption implies that

$$\mathcal{E}^p \mid U_{\alpha_0 \dots \alpha_p} \cong \left(\mathcal{E}^{p-1} \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \quad (3.5.5.3)$$

which, combined with (3.5.2.1), says that (for $i \neq j$)^[15]

$$\begin{aligned} \mathcal{E}^p \mid U_{\alpha_0 \dots \alpha_p} &\cong \left(\left(X^\mathcal{U} \zeta_{p-1}^j \right)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \oplus \mathcal{K}_{0, j, p-1} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \\ &\cong \left(\left(X^\mathcal{U} \zeta_{p-1}^j \right)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \left(\mathcal{K}_{0, j, p-1} \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \right) \end{aligned}$$

and (again by the Green assumption) these \mathcal{K} satisfy some cocycle condition:

$$\mathcal{K}_{0, i, p} \cong \mathcal{K}_{0, j, p-1} \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \quad (3.5.5.4)$$

whence

$$\mathcal{E}^p \mid U_{\alpha_0 \dots \alpha_p} \cong \left(\left(X^\mathcal{U} \zeta_{p-1}^j \right)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{0, i, p}. \quad (3.5.5.5)$$

Now, for each comparison map \mathfrak{C}_p^i , we set

$$\begin{aligned} A^{p-1} &= \pi_p^* \mathcal{K}_{0, j, p-1} \\ B^p &= \pi_p^* \mathcal{K}_{0, i, p} \end{aligned}$$

for an arbitrary $j \neq i$.

- (i) A^p is ∇_p flat by definition: we extended the connection $\pi_p^* \widetilde{\nabla}_{\alpha_i}$ by the trivial connection on this direct summand.
- (ii) The comparison map \mathfrak{C}_p^i is simply the pullback (along $\pi_{p, p-1}$) of $\mathcal{E}^\bullet f_p^i$, and so respects the splitting (3.5.2.1) almost by definition, because these splittings come from the Green assumption.

^[15]And we somehow don't really care about when $i = j$, since we will always be working on the embedding of Δ^{p-1} into Δ^p by f_p^i , which is equivalent to working with the labelling $(\alpha_0, \dots, \widehat{\alpha}_i, \dots, \alpha_p)$ on Δ^{p-1} .

- (iii) We want the comparison map to induce an isomorphism when we take the quotients. To avoid getting lost in a mire of notation, instead of using (3.5.2.1), we use the language of Definition 2.4.8. Set

$$\begin{aligned}\alpha &= (\alpha_0, \dots, \alpha_p) \\ \beta &= (\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p) \\ \gamma &= (\alpha_j)\end{aligned}$$

and note that $A^{p-1} = \pi_{p-1}^* \mathcal{L}_{\beta, \gamma}$ and $B^p = \pi_p^* \mathcal{L}_{\alpha, \gamma}$. Then

$$\begin{aligned}\mathcal{E}_\alpha^\bullet &\cong \mathcal{E}_\gamma^\bullet | U_\beta | U_\alpha \oplus \mathcal{L}_{\alpha, \gamma} \\ \mathcal{E}_\beta^\bullet &\cong \mathcal{E}_\gamma^\bullet | U_\beta \oplus \mathcal{L}_{\beta, \gamma}\end{aligned}$$

and we are interested in

$$(X_\bullet^\mathcal{U} f_p^i \times \text{id})^* \pi_{p-1}^* (\mathcal{E}_\gamma^\bullet | U_\beta) \xrightarrow{\pi_{p,p-1}^* (\mathcal{E}^\bullet f_p^i)} (\text{id} \times f_p^i)^* \pi_p^* (\mathcal{E}_\gamma^\bullet | U_\beta | U_\alpha).$$

Consider first the source of this map: by Definition 3.1.4, we know that

$$(X_\bullet^\mathcal{U} f_p^i \times \text{id})^* \pi_{p-1}^* (\mathcal{E}_\gamma^\bullet | U_\beta) \cong \pi_{p,p-1}^* (X_\bullet^\mathcal{U} f_p^i)^* (\mathcal{E}_\gamma^\bullet | U_\beta)$$

but, as in (3.5.5.2), we know that

$$(X_\bullet^\mathcal{U} f_p^i)^* (\mathcal{E}_\gamma^\bullet | U_\beta) | U_\alpha = (\mathcal{E}_\gamma^\bullet | U_\beta) | U_\beta | U_\alpha = \mathcal{E}_\gamma^\bullet | U_\beta | U_\alpha$$

The target is much simpler, since, as explained in Definition 3.1.4, the pull-back along f_p^i on the simplex part changes nothing:

$$(\text{id} \times f_p^i)^* \pi_p^* (\mathcal{E}_\gamma^\bullet | U_\beta | U_\alpha) \cong \pi_{p,p-1}^* (\mathcal{E}_\gamma^\bullet | U_\beta | U_\alpha).$$

But then we see that the source and target, *when we restrict to U_α* , are identical, so it would suffice to show that the comparison map descends to an injection when we take these quotients. But these quotients are exactly the sheaf with which we started: we added the cokernels and then quotiented them out; and we know, from Corollary 3.1.5, that the comparison map is injective here.

□

Remark 3.5.6. We can, in particular, think of Theorem 3.5.5 as a proof that, *when working with Green complexes*, being generated in degree zero implies admissibility.

Lemma 3.5.7. *Let \mathcal{E}^\bullet be a Green vector bundle on the nerve, and let ∇_\bullet and ∇'_\bullet be two simplicial connections on \mathcal{E}^\bullet that are generated in degree zero. Then the difference $\nabla'_\bullet - \nabla_\bullet$ is an admissible endomorphism-valued simplicial form. In other words, the set of generated-in-degree-zero connections on a Green vector bundle on the nerve is a compatible family.*

Proof. The fact that the difference of two arbitrary simplicial connections on an arbitrary vector bundle on the nerve is an endomorphism-valued simplicial 1-form follows ‘immediately’ from the definitions, without any extra hypotheses. The content of this lemma is that *generated-in-degree-zero* connections on a *Green* vector bundle on the nerve have an *admissible* difference.

Write $\nabla_\bullet^{(1)}$ and $\nabla_\bullet^{(2)}$ to mean the two connections, so that $\nabla_\bullet^{(i)}$ is constructed as in Remark 3.5.1, but with a different choice of local connections $\widetilde{\nabla}_\alpha^{(i)}$ on each $\mathcal{E}_0|_{U_\alpha}$. This means that, for each $p \in \mathbb{N}$, we can write

$$\nabla_p^{(i)} = \sum_{j=0}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j}^{(i)}$$

which means that the difference is of the form

$$\nabla_p^{(2)} - \nabla_p^{(1)} = \sum_{j=0}^p t_j \pi_p^* \left(\widetilde{\nabla}_{\alpha_j}^{(2)} - \widetilde{\nabla}_{\alpha_j}^{(1)} \right) = \sum_{j=0}^p t_j \pi_p^* \eta_{\alpha_j} = \sum_{j=0}^p t_j \overline{\eta_{\alpha_j}}$$

where η_{α_j} is an endomorphism-valued simplicial 1-form on U_{α_j} . The claim is then that $\sum_{j=0}^p t_j \overline{\eta_{\alpha_j}}$ is an *admissible* endomorphism-valued simplicial 1-form, and this follows almost exactly as in the proof of Theorem 3.5.5, because the difference $\sum t_j \overline{\eta_{\alpha_j}}$ is somehow also ‘generated in degree zero’, in that it is given by trivially extending on each $(X_\bullet^{\mathcal{U}} f_p^i)^* \mathcal{E}_{p-1} \hookrightarrow \mathcal{E}_p$. \square

4 Coherent sheaves

4.1 Homotopical categories

Definition 4.1.1. In this chapter we are interested in **relative categories**: pairs (C, \mathcal{W}) where C is a category and \mathcal{W} (whose morphisms we call *weak equivalences*) is a wide subcategory of C . A relative category is said to be a **homotopical category** if its weak equivalences satisfy the **2-out-of-6 property**: if

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

is a sequence of composable morphisms such that the compositions gf and hg are weak equivalences, then f , g , h , and hgf are all weak equivalences too. We often write

a relative (or homotopical) category (C, \mathcal{W}) simply as C , omitting the weak equivalences from our notation.

Using the formalism of [Rez00] along with the results of [BK13], we can think of a homotopical category (C, \mathcal{W}) as presenting the $(\infty, 1)$ -category LC , which is the complete Segal space given by taking a Reedy fibrant replacement of the Rezk/simplicial nerve $N(C, \mathcal{W})$. In particular, [BK13, §1.2 (ii)] tells us that any homotopically full relative subcategory of a partial model category is again a partial model category, and all of the categories that we study here are such subcategories of either the category of complexes of sheaves on X or of the category of complexes of sheaves on $X_\bullet^{\mathcal{U}}$, both of which^[16] are model categories, and thus partial model categories.

Remark 4.1.2. If a relative category has weak equivalences defined as being exactly the morphisms that become isomorphisms under some functor into another (say, abelian) category, then it is a homotopical category, because isomorphisms satisfy the 2-out-of-6 property.^[17]

Definition 4.1.3. We begin by formally defining some relative categories (all of which are actually *homotopical* categories, by Remark 4.1.2). All complexes are *cochain* complexes, concentrated in non-negative degrees and bounded above; if we don't mention what the morphisms are then they are simply morphisms of cochain complexes (i.e. degree-wise morphisms that commute with the differentials); and if we don't mention what the weak equivalences are then they are simply quasi-isomorphisms of complexes.

- $\text{Coh}(X)$ is the category of complexes of sheaves of \mathcal{O}_X -modules that are quasi-isomorphic to a complex of coherent sheaves.
- $\text{Coh}_{\mathcal{U}}(X)$ is the category of complexes of sheaves of \mathcal{O}_X -modules whose restriction to any $U \in \mathcal{U}$ is quasi-isomorphic to a complex of coherent sheaves on U .
- $\text{CCoh}(X)$ is the category of complexes of sheaves of \mathcal{O}_X -modules that have coherent (internal) cohomology.
- $\text{Sh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$ is the category of *cartesian* complexes of sheaves on the nerve. Note that morphisms between two such complexes are maps such that, in every *internal* degree (i.e. in every degree of the complex), we have a *morphism* of sheaves

^[16]The former having, for example, the projective model structure from [Hov01]; the latter having the model structure coming from its construction as a lax homotopy limit via the formalism of [Ber12], as described in the footnote in Remark 2.2.2 of this current paper.

^[17]If gf is an isomorphism then f must be a monomorphism, and g an epimorphism; if hg is an isomorphism then g must be a monomorphism, and h an epimorphism. But then g is both a monomorphism and an epimorphism, and thus an isomorphism. Then we can use the fact that isomorphisms satisfy 2-out-of-3, for example.

on the nerve (and such that these commute with the differentials of the complexes); weak equivalences are given by morphisms of complexes such that, in each *simplicial* degree, we have a quasi-isomorphism of complexes.

- $\text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ is the full subcategory of $\text{Sh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ consisting of complexes that are *locally (with respect to \mathcal{U}) quasi-isomorphic* to a (cartesian) complex of *locally free* sheaves on the nerve. That is, for $\mathcal{F}^{\bullet, \star} \in \text{Sh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ to be in $\text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$, there must exist, for all $U_{\alpha_0 \dots \alpha_p} \in \mathcal{U}$, a complex $\mathcal{G}_{U_{\alpha_0 \dots \alpha_p}}^{\star}$ of locally free sheaves on $U_{\alpha_0 \dots \alpha_p}$ such that

$$\mathcal{F}_{\alpha_0 \dots \alpha_p}^{p, \star} \simeq \mathcal{G}_{U_{\alpha_0 \dots \alpha_p}}^{\star}.$$

Similarly, $\text{Coh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ is the full subcategory of $\text{Sh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ consisting of complexes that are *locally (with respect to \mathcal{U}) quasi-isomorphic* to a (cartesian) complex of coherent sheaves on the nerve.

- $\text{Green}(X_{\bullet}^{\mathcal{U}})$ is the full subcategory of $\text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ spanned by objects that are locally (with respect to \mathcal{U}) quasi-isomorphic to some Green complex (Definition 2.4.8). The fact that this actually is a subcategory is justified by Corollary 2.4.10. By definition, every object of $\text{Green}(X_{\bullet}^{\mathcal{U}})$ is a Green complex.

Note that all of these categories that depend on \mathcal{U} are natural in the choice of cover: taking a refinement $\mathcal{V} \supset \mathcal{U}$ induces a functor e.g. $\text{Green}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{Green}(X_{\bullet}^{\mathcal{V}})$. This lets us take homotopy colimits (where $\text{hocolim}: [\mathcal{D}^{\text{op}}, (\infty, 1)\text{-Cat}] \rightarrow (\infty, 1)\text{-Cat}$, with \mathcal{D} small, is defined as the $(\infty, 1)$ -categorical left adjoint to the constant-diagram functor) over refinements of covers, e.g. $\text{hocolim}_{\mathcal{U}} \text{Green}(X_{\bullet}^{\mathcal{U}})$.

Remark 4.1.4. Our categories are defined as having ‘objects *locally quasi-isomorphic* to a certain class of objects’, and *not* ‘objects in the class of objects’. For example, an object $\mathcal{F}^{\bullet} \in \text{Coh}_{\mathcal{U}}(X)$ is such that, for all $U \in \mathcal{U}$, there exists some complex of coherent sheaves \mathcal{G}_U^{\bullet} such that $\mathcal{G}_U^{\bullet} \simeq \mathcal{F}^{\bullet}|_U$. This does *not* a priori imply the existence of some global complex of coherent sheaves \mathcal{G}^{\bullet} such that $\mathcal{G}^{\bullet} \simeq \mathcal{F}^{\bullet}$. This subtlety makes the proofs in this section more technical than they morally are.

Remark 4.1.5. Given a full embedding $(C, \mathcal{W}') \hookrightarrow (\mathcal{D}, \mathcal{W})$ of one homotopical category into another such that $\mathcal{W}' = C \cap \mathcal{W}$, there are *two* $(\infty, 1)$ -categories defined by C that, in general, do *not* agree:

1. $L_{\mathcal{W}'} C$, given by localising C along \mathcal{W}' ;
2. $L_{\mathcal{D}} C$, given by taking the full sub- $(\infty, 1)$ -category of \mathcal{D} spanned by C .

As mentioned in Remark 4.1.4, we define all our subcategories in the same way: given some $(\mathcal{D}, \mathcal{W})$, we construct some (C, \mathcal{W}') by taking the full subcategory of \mathcal{D} of objects that are (locally) connected via \mathcal{W} to objects satisfying some specific property. This means that, when we write $L C$, we implicitly mean $L_{\mathcal{D}} C$.

Remark 4.1.6. In the algebraic setting, there is an equivalence between the category of complexes of coherent sheaves and the category of complexes of sheaves with coherent (internal) cohomology. In the analytic case, things are more subtle, and so we have to take a slightly longer route to prove our desired result. Indeed, to the best of our knowledge, the question of whether or not $\text{Coh}(X)$ and $\text{CCoh}(X)$ are equivalent is still open, except in low dimensions, where it is known to be true (see [Yu13, §2.2.2]).

Note that, given a refinement of our cover $\mathcal{V} \supset \mathcal{U}$, we have full embeddings

$$\text{Coh}(X) \hookrightarrow \text{Coh}_{\mathcal{U}}(X) \hookrightarrow \text{Coh}_{\mathcal{V}}(X) \hookrightarrow \text{CCoh}(X)$$

that preserve and reflect quasi-isomorphisms.

Definition 4.1.7. We define the category $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ via the Grothendieck construction applied to the functor $F: \text{Green}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{Set}$ given, on an object $\mathcal{E}^{\bullet,*}$, by

$$F(\mathcal{E}^{\bullet,j}) = \{\text{generated-in-degree-zero simplicial connections on } \mathcal{E}^{\bullet,j}\},$$

where Theorem 3.5.5 tells us that this set is non-empty. So an object of $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ is a pair $(\mathcal{E}^{\bullet,*}, \nabla_{\bullet}^*)$, where $\mathcal{E}^{\bullet,*}$ is an object of $\text{Green}(X_{\bullet}^{\mathcal{U}})$, and ∇_{\bullet}^j is a simplicial connection generated in degree zero (and thus admissible) on $\mathcal{E}^{\bullet,j}$; the morphisms of $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ are exactly those of $\text{Green}(X_{\bullet}^{\mathcal{U}})$. In particular, $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ is a homotopical category with the same weak equivalences as $\text{Green}(X_{\bullet}^{\mathcal{U}})$.

Remark 4.1.8. By construction, the forgetful functor

$$\textcircled{4}: \text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{Green}(X_{\bullet}^{\mathcal{U}})$$

(that forgets about the connections) is fully faithful and essentially surjective, and thus induces an equivalence of categories. More importantly, though, it also restricts to give an equivalence of the corresponding subcategories of weak equivalences, which will be useful later on.

In summary, we have the following diagram (of 1-categories), where we write \succrightarrow to denote a fully-faithful functor, and \twoheadrightarrow to denote an essentially-surjective one:

$$\begin{array}{ccccccc} \text{Green}(X_{\bullet}^{\mathcal{U}}) & \xrightarrow{\textcircled{1}} & \text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}}) & \xrightarrow{\textcircled{2}} & \text{Coh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}}) & \overset{\textcircled{3}}{\dashrightarrow} & \text{Coh}_{\mathcal{U}}(X) \\ \textcircled{4} \uparrow & & & & & & \\ & & & & & & \\ \text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) & & & & & & \end{array} \quad (4.1.8.1)$$

where we will define $\textcircled{3}$ later, and, even then, only at the level of $(\infty, 1)$ -categories. Note that $\textcircled{1}$ and $\textcircled{2}$ are fully-faithful by definition (as full subcategories), and the forgetful functor $\textcircled{4}$ is an equivalence by definition (as explained in Remark 4.1.8).

Our goal for the rest of this section is to prove that, when we localise all the categories in (4.1.8.1), all the functors become equivalences of $(\infty, 1)$ -categories.

This is the content of Theorem 4.2.9 and Corollary 4.2.14.

Remark 4.1.9. We omit the localisation notation from our functors: we will write ① to mean both the functor between homotopical categories and the induced functor between their localisations.

We delay our study of ③ for a little while, and consider first the rest of (4.1.8.1). Our first goal is to show that this diagram descends to a diagram at the level of localisations. For this we need to know that all our functors ① really are functors of relative categories, in that they preserve weak equivalences.

Remark 4.1.10. Note that ① and ② are inclusions of full subcategories, and so trivially preserve weak equivalences; similarly, ④ is constructed in such a way that it automatically preserves weak equivalences, as explained in Definition 4.1.7.

In fact, by Remark 4.1.8, ④ actually directly induces an equivalence at the level of localisations:

$$\textcircled{4}: \mathrm{LGreen}_{\nabla, 0}(X_{\bullet}^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{LGreen}(X_{\bullet}^{\mathcal{U}}).$$

This is because an equivalence of relative categories that restricts to an equivalence of the wide subcategories of weak equivalences induces an equivalence of the localisations, as can be shown by using [BK12, Lemma 5.4], as well as the fact that an equivalence of categories induces an equivalence of their nerves.

As a side note, even though ④ gives an equivalence at the level of localisations, it there no longer looks like a Grothendieck construction: there is no reason for two weakly equivalent Green complexes (or even, more simply, quasi-isomorphic complexes of vector bundles) to admit local connections that are in bijective correspondence with one another. This isn't something that poses a problem, but is interesting to note.

Now we wish to study this mysterious functor ③. We aim to build an adjunction of $(\infty, 1)$ -categories, step by step.

Writing $i: X_{\bullet}^{\mathcal{U}} \rightarrow X$ to mean the map given by inclusion of each open subset into X , we have an adjunction^[18]

$$\mathrm{Sh}(X_{\bullet}^{\mathcal{U}}) : (i_{*} \dashv i^{*}) : \mathrm{Sh}(X).$$

Then, recalling that the limit functor can be defined as being the right adjoint to the constant diagram functor, and writing $\mathrm{Sh}(X_{\bullet}^{\mathcal{U}})$ to mean the category of complexes of

^[18]Note that, to eventually agree with the orientation of our diagram, we write our adjunctions as $(R \dashv L)$ instead of as $(L \dashv R)$.

sheaves of $\mathcal{O}_{X_X^{\mathcal{U}}}$ -modules, thought of as a category of $X_{\bullet}^{\mathcal{U}}$ -diagrams, we can compose this adjunction with the above to get a Quillen adjunction

$$\mathrm{Sh}(X_{\bullet}^{\mathcal{U}}) : (\lim \circ i_* \vdash \mathrm{cst} \circ i^*) : \mathrm{Sh}(X)$$

where being Quillen follows from the fact that the pullback/pushforward adjunction and the limit/constant adjunction are both Quillen.

So we are now in the following situation: we have a diagram

$$\begin{array}{ccc} \mathrm{Sh}(X_{\bullet}^{\mathcal{U}}) & \xrightleftharpoons[\mathrm{cst} \circ i^*]{\lim \circ i_*} & \mathrm{Sh}(X) \\ \uparrow & & \uparrow \\ \mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) & & \mathrm{Coh}_{\mathcal{U}}(X) \end{array}$$

where the adjunction is Quillen. Deriving the functors (and localising the categories) then gives us the diagram

$$\begin{array}{ccc} \mathrm{LSh}(X_{\bullet}^{\mathcal{U}}) & \xrightleftharpoons[\mathbb{L}(\mathrm{cst} \circ i^*)]{\mathbb{R}(\lim \circ i_*)} & \mathrm{LSh}(X) \\ \uparrow & & \uparrow \\ \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) & & \mathrm{LCoh}_{\mathcal{U}}(X) \end{array}$$

and we wish to know if the adjunction restricts to give an adjunction of the subcategories. Following [TV08, Lemma 2.2.2.13], we write \mathbb{f} to mean the total right derived functor $\mathbb{R}(\lim \circ i_*)$; we write \mathbb{l}^* to mean the total left derived functor $\mathbb{L}(\mathrm{cst} \circ i^*)$. So we want to show that

$$\begin{aligned} \mathbb{f} : \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) &\rightarrow \mathrm{LCoh}_{\mathcal{U}}(X) \\ \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) &\leftarrow \mathrm{LCoh}_{\mathcal{U}}(X) : \mathbb{l}^* \end{aligned}$$

(where we omit the restriction of the functors from our notation).

Lemma 4.1.11. *The image of $\mathbb{l}^* : \mathrm{LCoh}_{\mathcal{U}}(X) \rightarrow \mathrm{LSh}(X_{\bullet}^{\mathcal{U}})$ is contained in $\mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$.*

Proof. Let $\mathcal{F}^{\star} \in \mathrm{LCoh}_{\mathcal{U}}(X)$. The pullback functor i^* is exact since it is given by the topological pullback tensored with the structure sheaf, and the topological pullback is exact, and tensoring along a disjoint union of open immersions is also exact. Then, since the constant diagram functor is (trivially) also exact, we see that \mathbb{l}^* is just the pullback to the nerve (as with global vector bundles in Example 2.2.5). But, as mentioned there, the simplicial maps are then simply identity maps, which means that the resulting object is indeed cartesian; and being coherent is a local property, so it suffices to check it on each $U_{\alpha_0 \dots \alpha_p}$ in $X_p^{\mathcal{U}}$, but, over such an open set, $\mathbb{l}^* \mathcal{F}^j$ is simply $\mathcal{F}^j|_{U_{\alpha_0 \dots \alpha_p}}$, which is coherent by definition. \square

Remark 4.1.12. It is a good idea to fully understand the cosimplicial structure of $\iota_* \mathcal{F}^{\bullet, \star}$ before proceeding, so we spell out all the details here.

Recall that $\iota_* = \mathbb{R}i_* = i_*$, where $i: X_{\bullet}^{\mathcal{U}} \rightarrow X$. We want to describe what $(\iota_* \mathcal{F}^{\bullet, \star})^p$ is for each $p \in \mathbb{N}$, as well as how these ‘fit together’ to give a cosimplicial object. This can be explained by just improving our notation: write $i_p: X_p^{\mathcal{U}} \rightarrow X$ to mean the map given by inclusion of each $U_{\alpha_0 \dots \alpha_p}$ in $X_p^{\mathcal{U}}$ into X , so that i is exactly the data of $(i_p)_{p \in \mathbb{N}}$. Then define^[19]

$$(\iota_* \mathcal{F}^{\bullet, \star})^p = (i_p)_* \mathcal{F}^{p, \star} = \bigoplus_{(\alpha_0 \dots \alpha_p)} (i_p)_* \mathcal{F}_{\alpha_0 \dots \alpha_p}^{p, \star}.$$

Then, given some $\varphi: [p] \rightarrow [q]$ in Δ , we want to know how to define

$$(i_p)_* \mathcal{F}^{p, \star} \xrightarrow{(\iota_* \mathcal{F}^{\bullet, \star})^\diamond(\varphi)} (i_q)_* \mathcal{F}^{q, \star},$$

but, using the pullback/pushforward adjunction, this is the same as asking for a map

$$(i_q)^*(i_p)_* \mathcal{F}^{p, \star} \longrightarrow \mathcal{F}^{q, \star}.$$

Firstly, we claim, for $p < q$ (dealing with the other case shortly), that there is a natural map $(i_q)^*(i_p)_* \mathcal{F}^{p, \star} \rightarrow (X_{\bullet}^{\mathcal{U}} \varphi)^* \mathcal{F}^{p, \star}$; secondly, we claim that this gives us the map that we want. To see the first claim, we appeal to the geometric nature of pushforwards and pullbacks: since the U_{α} do not necessarily have trivial intersection with one another, it is not necessarily true that $(i^* i_* \mathcal{F})|_U = \mathcal{F}|_U$; what *is* true, however, is that the right-hand side is a direct summand of the left:

$$(i^* i_* \mathcal{F})|_U \cong \mathcal{F}|_U \oplus \mathcal{G}.$$

This gives us the first claim (for $p < q$; we deal with the other case shortly), since restriction is the same as pulling back along $X_{\bullet}^{\mathcal{U}} \varphi$. For the second claim, by definition of what it means to be an element of $\text{Coh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$, we have maps $(X_{\bullet}^{\mathcal{U}} \varphi)^* \mathcal{F}^{p, \star} \rightarrow \mathcal{F}^{q, \star}$ for every $\varphi: [p] \rightarrow [q]$. Combining all of the above then gives us the desired maps, and thus the cosimplicial structure.

Finally then, when $p > q$ we can do something similar. Here the map $X_{\bullet}^{\mathcal{U}} \varphi$ is given by inserting degenerate intersections,^[20] and so it again isn’t necessarily the case that $(i_q)^*(i_p)_* \mathcal{F}^{p, \star} = (X_{\bullet}^{\mathcal{U}} \varphi)^* \mathcal{F}^{p, \star}$, since we have intersections $U_{\alpha_0 \dots \alpha_p}$ that can be strictly smaller than any $U_{\beta_0 \dots \beta_q}$. But we can still construct some $(i_q)^*(i_p)_* \mathcal{F}^{p, \star} \rightarrow \mathcal{F}^{q, \star}$ by precomposing the $(X_{\bullet}^{\mathcal{U}} \varphi)^* \mathcal{F}^{p, \star} \rightarrow \mathcal{F}^{q, \star}$ with the projection maps

$$(i_q)^*(i_p)_* \mathcal{F}^{p, \star} = (i_p)_* \mathcal{F}^{p, \star} |_{U_{\beta_0 \dots \beta_q}} = \bigoplus_{(\alpha_0 \dots \alpha_p)} (i_p)_* \mathcal{F}_{\alpha_0 \dots \alpha_p}^{p, \star} |_{U_{\beta_0 \dots \beta_q}} \twoheadrightarrow (i_p)_* \mathcal{F}_{\beta_0 \dots \beta_q \beta_q \dots \beta_q}^{q, \star}$$

^[19]It would probably suffice to simply write $\iota_* \mathcal{F}^{\diamond, \star}$ instead of $(\iota_* \mathcal{F}^{\bullet, \star})^\diamond$, since the cosimplicial structure is described by \bullet , but then we should really write $(\iota_{\bullet})_* \mathcal{F}^{\bullet, \star}$, which is equally unsightly.

^[20]Obtaining things that look like $U_{\alpha_0 \dots \alpha_i \alpha_i \dots \alpha_p}$.

where the first ‘equality’ really means that we work locally over each $U_{\beta_0 \dots \beta_p}$, and we write $(\beta_0 \dots \beta_q \beta_q \dots \beta_q)$ to mean some degenerate embedding of $(\beta_0 \dots \beta_q)$ into $X_p^{\mathcal{U}}$.

Lemma 4.1.13. *The image of $\int: \mathrm{L}\mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) \rightarrow \mathrm{L}\mathrm{Sh}(X)$ is contained in $\mathrm{L}\mathrm{Coh}_{\mathcal{U}}(X)$.*

Proof. Before giving the proof, we give a short summary of how it will proceed, to save anybody familiar with such arguments the arduous task of following the notation. In particular, this proof is incredibly similar to that of Lemma 4.2.3, which is really an analytic version of [TV08, Lemma 2.2.2.13] in that it follows the same line of argument. We can argue everything locally on some $U \in \mathcal{U}$; by definition, weak equivalences are quasi-isomorphisms; we can use the total complex construction to calculate the homotopy limit in the definition of \int ; the fact that our complexes are cartesian gives us a weak equivalence between this total complex and the (total complex of the) Čech complex of the simplicial-degree-zero part of our original complex; the latter is weakly equivalent to the simplicial-degree-zero part of our original complex (since all covers can be taken to be Stein); a commuting triangle then tells us that the desired quasi-isomorphism is indeed a quasi-isomorphism.

Let $\mathcal{F}^{\bullet, \star} \in \mathrm{L}\mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$ and $U \in \mathcal{U}$. Then $\mathcal{F}^{0, \star}|_U$ is a complex of coherent sheaves on U , and so it would suffice to show that there is a quasi-isomorphism

$$\mathcal{F}^{0, \star}|_U \xrightarrow{\sim} \left(\int \mathcal{F}^{\bullet, \star} \right)|_U.$$

Firstly, note that there exists a good candidate morphism: the cosimplicial structure of \mathcal{F} means that we have a morphism

$$i_p^* \mathcal{F}^{0, \star} \rightarrow \mathcal{F}^{p, \star}$$

of sheaves over $X_p^{\mathcal{U}}$ for all $p \in \mathbb{N}$, where $i_p: X_p^{\mathcal{U}} \rightarrow X$; by the pull/push adjunction, this gives us a morphism

$$\mathcal{F}^{0, \star} \rightarrow (i_p)_* \mathcal{F}^{p, \star}$$

and so, by the universal property of the homotopy limit (since $\mathbb{R}\mathrm{lim} = \mathrm{holim}$), we get a morphism

$$\mathcal{F}^{0, \star} \rightarrow \mathrm{holim}_p (i_p)_* \mathcal{F}^{p, \star} = \int \mathcal{F}^{\bullet, \star}$$

which, denoting restriction to U by a subscript U , induces

$$\mathcal{F}_U^{0, \star} \rightarrow \left(\int \mathcal{F}^{\bullet, \star} \right)_U. \tag{4.1.13.1}$$

Now we wish to show that (4.1.13.1) is indeed a quasi-isomorphism. We can (justified by Remark 4.1.15) calculate the homotopy limit with the **total construction**: writing $F^{\bullet,\star}$ to mean the cosimplicial object $((i_{\bullet})_* \mathcal{F}^{\bullet,\star})$, we define

$$\begin{aligned} \mathrm{Tot}(F)^n &= \bigoplus_{\ell \in \mathbb{N}} F^{\ell, n-\ell} \\ d_{\mathrm{Tot}(F)} &= d_F + (-1)^n \delta \end{aligned}$$

where d_F is the differential $d_{\mathcal{F}^{\bullet,\star}}$ coming from the \star -grading of $\mathcal{F}^{\bullet,\star}$, and where

$$\delta^n = \sum_{i=0}^{n+1} (-1)^i F^{\bullet,\star}(f_{n+1}^i)$$

is the alternating sum of coface maps, whose action is given by the cosimplicial structure of $F^{\bullet,\star}$. To see that this makes sense in terms of degrees, note that

$$\begin{aligned} \delta^m &: F^{\ell, m} \rightarrow F^{\ell+1, m} \\ d_F^\ell &: F^{\ell, m} \rightarrow F^{\ell, m+1}. \end{aligned}$$

So (4.1.13.1) becomes

$$\mathcal{F}_U^{0,\star} \rightarrow \mathrm{Tot}(F)_U^\star, \quad (4.1.13.2)$$

and to show that this is a quasi-isomorphism, it is enough to show that

$$\mathcal{F}_U^{0,\star}(V) \rightarrow \mathrm{Tot}(F)_U^\star(V) \quad (4.1.13.3)$$

is a quasi-isomorphism for all $V \in \mathcal{V}$, where \mathcal{V} is some cover of U .

The cartesian condition gives us quasi-isomorphisms

$$\mathcal{F}_{\beta_0 \dots \beta_p}^{0,\star} \xrightarrow{\sim} \mathcal{F}_{\beta_0 \dots \beta_p}^{p,\star}$$

of complexes of (coherent) sheaves over any $V_{\beta_0 \dots \beta_p} \in \mathcal{V}$, and so we can refine \mathcal{V} to a cover (which we can always take to be Stein) $\mathcal{W} = \{W_\gamma\}$ of $V_{\beta_0 \dots \beta_p}$ such that we have quasi-isomorphisms

$$\mathcal{F}_{\beta_0 \dots \beta_p}^{0,\star}(W_{\gamma_0 \dots \gamma_p}) \xrightarrow{\sim} \mathcal{F}_{\beta_0 \dots \beta_p}^{p,\star}(W_{\gamma_0 \dots \gamma_p}) \quad (4.1.13.4)$$

of complexes of abelian groups.

But, as complexes of sheaves over $V_{\beta_0 \dots \beta_p}$, we trivially have that

$$\mathcal{F}_{\beta_0 \dots \beta_p}^{p,\star} = (i_p)_* \mathcal{F}_{\beta_0 \dots \beta_p}^{p,\star} \Big|_{W_{\beta_0 \dots \beta_p}}$$

and so the right-hand side of (4.1.13.4) is exactly $F_U^{\bullet,\star}(V_{\beta_0 \dots \beta_p})$; further, the left-hand side is exactly $\check{C}_{\mathcal{W}}^{\bullet,\star}(\mathcal{F}_U^{0,\star})$. Together, then, this tells us that (4.1.13.4) gives a morphism of bicomplexes

$$\check{C}_{\mathcal{W}}^{\bullet,\star}(\mathcal{F}_U^{0,\star}) \rightarrow F_U^{\bullet,\star}(V_{\beta_0 \dots \beta_p})$$

which is a quasi-isomorphism on each row. By a classical spectral sequence argument^[21] we can show that such a morphism of bicomplexes induces a quasi-isomorphism on the respective total complexes:

$$\text{Tot } \check{C}_{\mathcal{W}}^{\bullet,\star}(\mathcal{F}_U^{0,\star}) \xrightarrow{\sim} \text{Tot } F_U^{\bullet,\star}(V_{\beta_0 \dots \beta_p}).$$

We can see that the triangle

$$\begin{array}{ccc} \mathcal{F}_U^{0,\star}(V_{\beta_0 \dots \beta_p}) & \longrightarrow & \text{Tot } F_U^{\bullet,\star}(V_{\beta_0 \dots \beta_p}) \\ \uparrow & \nearrow & \\ \text{Tot } \check{C}_{\mathcal{W}}^{\bullet,\star}(\mathcal{F}_U^{0,\star}) & & \end{array}$$

commutes, where the horizontal arrow is exactly the morphism (4.1.13.3). But the vertical arrow is a quasi isomorphism (because taking the Čech complex with respect to a Stein cover gives a resolution), and we have already shown that the diagonal arrow is a quasi-isomorphism (by the cartesian condition), and so the horizontal arrow must also be a quasi-isomorphism. \square

In summary then, we have an adjunction

$$\text{L Coh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}}) : (\int \vdash \iota^*) : \text{L Coh}_{\mathcal{U}}(X) \quad (4.1.13.5)$$

Definition 4.1.14. $\textcircled{3} = \int$.

Remark 4.1.15. The projective model structure on non-negatively-graded cochain complexes gives us a simplicial model category (by the dual of Dold-Kan), and so, if we can show that $F^{\diamond,\star}$ is Reedy fibrant, then we can apply [Hir03, Theorem 19.8.7], which tells us that $\text{holim } F^{\diamond,\star} \simeq \text{Tot}(F^{\diamond,\star})$, for some abstract definition of Tot. The fact that the totalization (in the sense of Hirschhorn) agrees with the totalization of a bicomplex (in the usual homological algebra sense), and that the Bousfield-Kan spectral sequence and the spectral sequence(s) associated to a bicomplex coincide, can be found^[22] in [Fre17, III.1.1.13].

^[21]Taking the mapping cone, applying the spectral sequences associated to a bicomplex, and using induction on the number of non-zero rows, combined with the fact that the direct limit functor for complexes of abelian groups is exact.

^[22]We thank Maximilien Péroux for suggesting this reference.

To show Reedy fibrancy of some cosimplicial object X^\bullet , we need to show that the maps $X^n \rightarrow M_n(X^\bullet)$ are fibrant for all $n \geq 0$, where $M_n(X^\bullet)$ is the **matching object** given by

$$M_n(X^\bullet) = \lim_{\substack{\varphi: [n] \rightarrow [i] \\ i \neq n}} X^i.$$

We can write this down more explicitly as

$$\begin{aligned} M_0(X^\bullet) &= \{*\} \\ M_1(X^\bullet) &= X^0 \\ M_2(X^\bullet) &= X^1 \times_{X^0} X^1 \end{aligned}$$

and so on.

It is a purely formal consequence of the simplicial identities that any *simplicial* set is Reedy *cofibrant* in the *injective* model structure; formally dual to this is the fact that any *cosimplicial* set is Reedy *fibrant* in the *projective* model structure. For example, the fact that $X^1 \rightarrow M_1(X^\bullet) = X^0$ is fibrant (i.e. surjective) is due to the fact that it admits a right inverse (namely either of the two face maps $X^0 \rightarrow X^1$), thanks to the (co)simplicial identities (namely $s_0^0 f_1^0 = s_0^0 f_0^0 = \text{id}_{[0]}$).

4.2 Equivalences

Lemma 4.2.1. *The left adjoint ι^* of the adjunction (4.1.13.5) is conservative.*

Proof. Let $f: \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ be a morphism in $\text{LCCoh}(X)$ such that $\iota^* f$ is an isomorphism in $\text{L Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$. By definition of the weak equivalences in $\text{Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$, and the calculation of ι^* in Definition 4.1.14, over each $U_{\alpha_0 \dots \alpha_p} \in X_p^{\mathcal{U}}$, this says that the map

$$\iota^* f = f \mid U_{\alpha_0 \dots \alpha_p}: \mathcal{F}^\bullet \mid U_{\alpha_0 \dots \alpha_p} \rightarrow \mathcal{G}^\bullet \mid U_{\alpha_0 \dots \alpha_p}$$

is a quasi-isomorphism. But this is just saying that every restriction of f to some open subset of X is a quasi-isomorphism, which implies that f is a quasi-isomorphism, i.e. a weak equivalence in $\text{CCoh}(X)$. \square

Lemma 4.2.2. *Let $\varphi^\bullet: \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ be a morphism^[23] in $\text{Sh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$. Then φ^\bullet is a weak equivalence if and only if $\varphi^0: \mathcal{F}^0 \rightarrow \mathcal{G}^0$ is a weak equivalence.*

Proof. If φ^\bullet is a weak equivalence then, by definition, each φ^p is a weak equivalence, and so it remains only to show ‘if’ part of the claim. Recalling that pulling back along

^[23]Where we omit from our notation the internal grading \star of the complexes, writing e.g. \mathcal{F}^\bullet instead of $\mathcal{F}^{\bullet, \star}$.

the inclusion of an open subset is the same as restriction to that same open subset, the cartesian condition on \mathcal{F}^\bullet tells us that, for all $U_{\alpha_0 \dots \alpha_p} \in \mathcal{U}$,

$$\mathcal{F}^0|_{U_{\alpha_0 \dots \alpha_p}} \xrightarrow{\sim} \mathcal{F}^p|_{U_{\alpha_0 \dots \alpha_p}}$$

as (complexes of) sheaves over $U_{\alpha_0 \dots \alpha_p}$, and similarly for \mathcal{G}^\bullet . Combining this with the commutative square that follows from the definition of what it means to be a morphism of sheaves on the nerve, we get the commutative square

$$\begin{array}{ccc} \mathcal{F}^0 & \xrightarrow{\sim} & \mathcal{F}^p \\ \varphi^0 \downarrow & & \downarrow \varphi^p \\ \mathcal{G}^0 & \xrightarrow{\sim} & \mathcal{G}^p \end{array}$$

from which it follows that, if $\varphi^0: \mathcal{F}^0 \rightarrow \mathcal{G}^0$ is a weak equivalence, then so too is φ^p for all $p \in \mathbb{N}$, and hence also φ^\bullet . \square

Lemma 4.2.3. *The counit*

$$\iota^* \circ \int \implies \text{id}_{\text{L Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})}$$

of the adjunction (4.1.13.5) is a weak equivalence.

Proof. Let $\mathcal{F}^{\bullet, \star} \in \text{Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$. First of all, by Lemma 4.2.2, we know that it suffices to show that the counit is a weak equivalence in simplicial degree zero. But we can further simplify things: recalling the definition of \int , and using the fact that ι^* is simply the pullback along i , it suffices to show that the induced morphism

$$(i^* \text{holim}_{p \in \mathbb{N}} (\iota_* \mathcal{F}^{\bullet, \star})^p)|_{U_\alpha} \longrightarrow \mathcal{F}^{0, \star}|_{U_\alpha} \tag{4.2.3.1}$$

in $\text{Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$ is a weak equivalence for all $U_\alpha \in \mathcal{U}$. But since the composite $U_\alpha \hookrightarrow \mathcal{U} \rightarrow X$ is exactly $U_\alpha \hookrightarrow X$, we see that pulling back along i and then restricting to U_α is the same as restricting directly to U_α . Finally, just to simplify notation, we write $F^{\bullet, \star}$ to mean the cosimplicial object $(\iota_* \mathcal{F}^{\bullet, \star})^\bullet$, we write d_F to mean the differential $d_{\mathcal{F}^{\bullet, \star}}$ coming from the \star -grading of $\mathcal{F}^{\bullet, \star}$, and we write a subscript α to denote restriction to U_α . All together then, (4.2.3.1) becomes

$$(\text{holim}_{p \in \mathbb{N}} F^{p, \star})|_{U_\alpha} \longrightarrow \mathcal{F}_\alpha^{0, \star} \tag{4.2.3.2}$$

But now we can proceed almost exactly as in the proof of Lemma 4.1.13: we use the total construction, and construct the same commuting triangle but with the horizontal arrow going in the other direction. \square

Remark 4.2.4. The counit of an adjunction being a weak equivalence (or an isomorphism, in the 1-categorical case) is equivalent to the right adjoint being fully faithful.

Lemma 4.2.5. *Let $C : (L \dashv R) : \mathcal{D}$ be an adjunction with L conservative and R fully faithful. Then $(L \dashv R)$ gives an equivalence $C \simeq \mathcal{D}$.*

Proof. It suffices to show that R is essentially surjective, so let $c \in C$, and define $d = L(c)$. Then $LR(d) \rightarrow d$ is an equivalence (because R being fully faithful is equivalent to the counit of the adjunction being an equivalence). But $LR(d) \rightarrow d$ is, by definition, $LRL(c) \rightarrow L(c)$, and since this is an equivalence and L is conservative, we see that $RL(c) \rightarrow c$ is an equivalence. That is, $R(d) \simeq c$. \square

Corollary 4.2.6. *The adjunction (4.1.13.5) gives an equivalence of $(\infty, 1)$ -categories*

$$\mathrm{L} \mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) \simeq \mathrm{L} \mathrm{Coh}_{\mathcal{U}}(X)$$

and thus an equivalence

$$\mathrm{hocolim}_{\mathcal{U}} \mathrm{L} \mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) \simeq \mathrm{hocolim}_{\mathcal{U}} \mathrm{L} \mathrm{Coh}_{\mathcal{U}}(X).$$

Lemma 4.2.7. *The composite functor*

$$\mathrm{hocolim}_{\mathcal{U}} \mathrm{L} \mathrm{Green}(X_{\bullet}^{\mathcal{U}}) \xrightarrow{\textcircled{3}\textcircled{2}\textcircled{1}} \mathrm{hocolim}_{\mathcal{U}} \mathrm{L} \mathrm{Coh}_{\mathcal{U}}(X)$$

is essentially surjective.

Proof. Since $\textcircled{3}$ is an equivalence (Corollary 4.2.6), it suffices to show that

$$\textcircled{2}\textcircled{1} : \mathrm{hocolim}_{\mathcal{U}} \mathrm{L} \mathrm{Green}(X_{\bullet}^{\mathcal{U}}) \rightarrow \mathrm{hocolim}_{\mathcal{U}} \mathrm{L} \mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$$

is essentially surjective. So let $\mathcal{F}^{\bullet, \star} \in \mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$.

By definition, for all $U_{\alpha} \in \mathcal{U}$, there exists some complex $\mathcal{G}_{\alpha}^{\star}$ of coherent sheaves on U_{α} such that $\mathcal{F}_{\alpha}^{0, \star} \simeq \mathcal{G}_{\alpha}^{\star}$. We know that we can always locally resolve $\mathcal{G}_{\alpha}^{\star}$ by locally free sheaves, and so, by possibly taking a refinement $\mathcal{V} \supset \mathcal{U}$ (and using α, β, \dots to now label the open sets $V_{\alpha}, V_{\beta}, \dots$ of the refinement), we can obtain some (bounded) complex $\mathcal{H}_{\alpha}^{\star}$ of free sheaves (of finite rank) on V_{α} such that $\mathcal{F}_{\alpha}^{0, \star} \simeq \mathcal{H}_{\alpha}^{\star}$.

But this is simply saying that $\mathcal{F}_{\alpha}^{0, \star}$ is perfect, and so [OTT85, Proposition 1.2.3] (or [Wei16, Proposition 3.20]), tells us that, after possibly taking another refinement of our cover, there exists some holomorphic twisting resolution of $\mathcal{F}^{0, \star}$. Applying the construction of Green's resolution then gives us some $\mathcal{E}^{\bullet, \star} \xrightarrow{\sim} j^*(\mathcal{F}^{0, \star})^{\bullet}$, where $\mathcal{E}^{\bullet, \star}$ is a complex of locally free sheaves on the Čech nerve of $X_0^{\mathcal{U}} = \coprod_{\alpha} V_{\alpha}$, and j is the map from the Čech nerve of $X_0^{\mathcal{U}}$ to $X_0^{\mathcal{U}}$ itself. Note, however, that the Čech nerve of $X_0^{\mathcal{U}}$ is identical to the Čech nerve of X , and so it suffices to prove that $j^*(\mathcal{F}^{0, \star})^{\bullet} \simeq \mathcal{F}^{\bullet, \star}$ as sheaves on $X_{\bullet}^{\mathcal{U}}$, since $\textcircled{1}$ and $\textcircled{2}$ are both simply inclusions of full subcategories.

By Lemma 4.2.2, it suffices to show that we have a weak equivalence in simplicial degree zero, but $j^*(\mathcal{F}^{0, \star})^0 \xrightarrow{\sim} \mathcal{F}^{0, \star}$ is simply the identity map. \square

Lemma 4.2.8. *Let C be a partial model category, and $(\mathcal{D}_\lambda)_{\lambda \in \mathcal{P}}$ be a diagram of full subcategories of C indexed by some filtered poset^[24] \mathcal{P} . Assume further that each \mathcal{D}_λ is stable under weak equivalences. Then*

$$\mathrm{hocolim}_{\lambda \in \mathcal{P}} L\mathcal{D}_\lambda \simeq \bigcup_{\lambda \in \mathcal{P}} L\mathcal{D}_\lambda$$

where $\bigcup_{\lambda \in \mathcal{P}} L\mathcal{D}_\lambda$ is the full sub- $(\infty, 1)$ -category of LC spanned by the union of the objects of all of the $L\mathcal{D}_\lambda$.

In particular, the induced map

$$\mathrm{hocolim}_{\lambda \in \mathcal{P}} L\mathcal{D}_\lambda \rightarrow LC$$

is fully faithful.

Proof. Since partial model categories present $(\infty, 1)$ -categories, it suffices to prove the corresponding claim for $\mathrm{hocolim} \mathcal{D}_\lambda$ instead of $\mathrm{hocolim} L\mathcal{D}_\lambda$. Write Y_\bullet^λ to mean the complete Segal space $L\mathcal{D}_\lambda$, and X_\bullet to mean the complete Segal space LC . But since each \mathcal{D}_λ is a full subcategory of C stable under weak equivalences, each Y_n^λ is a union of connected components of X_n , corresponding to the span of the objects of \mathcal{D}_λ . This means that, in particular, for each $\mathcal{D}_\lambda \hookrightarrow \mathcal{D}_\mu$, the maps $Y_n^\lambda \hookrightarrow Y_n^\mu$ are all closed embeddings, and thus cofibrations. Hence

$$\mathrm{hocolim}_\lambda Y_\bullet^\lambda \cong \mathrm{colim}_\lambda Y_\bullet^\lambda.$$

Similarly, each $Y_n^\lambda \hookrightarrow X_n$ is a closed embedding, and thus a cofibration.

We claim that

$$\mathrm{colim}_\lambda Y_\bullet^\lambda \cong \bigcup_\lambda Y_\bullet^\lambda$$

where $\bigcup_\lambda Y_\bullet^\lambda$ is the subspace of X_\bullet spanned by the connected components of all the Y_\bullet^λ . Note that this is the complete Segal space $\bigcup_\lambda L\mathcal{D}_\lambda$, and so proving the above claim will finish the proof.

By the universal property of the colimit, we have a commutative diagram

$$\begin{array}{ccc} (Y_\bullet^\lambda)_{\lambda \in \mathcal{P}} & & \\ (f_\lambda)_{\lambda \in \mathcal{P}} \downarrow & \searrow \iota & \\ \mathrm{colim}_\lambda Y_\bullet^\lambda & \xrightarrow{f} & \bigcup_\lambda Y_\bullet^\lambda \end{array}$$

[24]The poset assumption is not necessarily necessary

and we wish to show that f is an isomorphism. Since ι is simply the inclusion of each Y_\bullet^λ into the union, it is surjective. Thus, given any $y \in \bigcup_\lambda Y_\bullet^\lambda$, there exists some λ such that $y \in Y_\bullet^\lambda$, and then, by commutativity, $f(f_\lambda(y)) = y$, which shows surjectivity. To show injectivity, let $y, z \in \operatorname{colim}_\lambda Y_\bullet^\lambda$ be such that $f(y) = f(z)$. Since \mathcal{P} is filtered, there exists some λ such that $f(y), f(z) \in Y_\bullet^\lambda$. Now $f(y) = f f_\lambda f(y)$ and $f(z) = f f_\lambda f(z)$, whence $f f_\lambda f(y) = f f_\lambda f(z)$. But f is surjective, and so $f f_\lambda(y) = f f_\lambda(z)$, but $f f_\lambda = \iota_\lambda$ is simply the inclusion of Y_\bullet^λ into the union, whence $y = z$. \square

Theorem 4.2.9. *There is an equivalence of $(\infty, 1)$ -categories*

$$\operatorname{hocolim}_{\mathcal{U}} \operatorname{L Green}_{\nabla, 0}(X_\bullet^{\mathcal{U}}) \simeq \operatorname{hocolim}_{\mathcal{U}} \operatorname{L Coh}_{\mathcal{U}}(X).$$

Proof. Note that, in the following, all the ① are the functors at the level of localisations. By Lemma 4.2.3 and Remark 4.2.4, ③ is fully faithful; since ① and ② are inclusions of full subcategories, they remain fully faithful at the level of localisations, and we can use Lemma 4.2.8 to see that they remain fully faithful after taking homotopy colimits; this means that the composite functor ③②① is also fully faithful. Lemma 4.2.7 tells us that ③②① is essentially surjective. All together then, this tells us that the composite functor ③②① is an equivalence, and since all of the composite functors are fully faithful, each one must itself also be an equivalence. By Remark 4.1.10, we know that ④ is an equivalence. \square

Lemma 4.2.10. *The functor*

$$\operatorname{hocolim}_{\mathcal{U}} \operatorname{L Coh}_{\mathcal{U}}(X) \rightarrow \operatorname{L CCoh}(X)$$

induced by the full embedding $\operatorname{Coh}_{\mathcal{U}}(X) \hookrightarrow \operatorname{CCoh}(X)$ is essentially surjective.

Proof. We need to show that, given any \mathcal{K}^\bullet with coherent cohomology, there exists some cover \mathcal{U} such that, on each $U_\alpha \in \mathcal{U}$, there exists some bounded complex of coherent sheaves quasi-isomorphic to $\mathcal{K}^\bullet|_{U_\alpha}$. Following the proof of [KS90, Proposition 1.7.11], we see that it suffices to show that, for any surjective morphism $u: \mathcal{G} \rightarrow \mathcal{H}$ of analytic sheaves with \mathcal{H} coherent, and any point $x \in X$, there exists a neighbourhood U_x of x , a coherent sheaf \mathcal{F} on U_x , and a morphism $t: \mathcal{F} \rightarrow \mathcal{G}$ such that the composite morphism $ut: \mathcal{F} \rightarrow \mathcal{H}$ is surjective.

To see this, let $x \in X$. Since \mathcal{H} is coherent, \mathcal{H}_x is of finite type over the local ring $\mathcal{O}_{X, x}$, and so can be generated by a finite number of sections s_1, \dots, s_r . Since the map $\mathcal{G}_x \rightarrow \mathcal{H}_x$ is surjective, we can lift these sections to sections t_1, \dots, t_r of \mathcal{G}_x defined on some neighbourhood U_x of x . Define the sheaf \mathcal{F} on U_x to be the free sheaf $(\mathcal{O}_{U_x})^r$, and define the morphism $t: \mathcal{F} \rightarrow \mathcal{G}$ by the sections t_1, \dots, t_r . Consider the cokernel \mathcal{C} of $ut: \mathcal{F} \rightarrow \mathcal{H}$, which is coherent, since both \mathcal{F} and \mathcal{H} are. Since \mathcal{C} is coherent, its support is an analytic set, and thus closed. By construction, the map $(ut)_x: \mathcal{F}_x \rightarrow \mathcal{H}_x$ is surjective. Since the complement of $\operatorname{supp}(\mathcal{C})$ is open, and not equal to $\{x\}$, there exists

a neighbourhood V_x of x that does not intersect with $\text{supp}(\mathcal{C})$. Thus $ut: \mathcal{F} \rightarrow \mathcal{H}$ is surjective on V_x . \square

Lemma 4.2.11. *The functor*

$$\text{hocolim}_{\mathcal{U}} \text{L Coh}_{\mathcal{U}}(X) \rightarrow \text{L CCoh}(X)$$

induced by the full embedding $\text{Coh}_{\mathcal{U}}(X) \hookrightarrow \text{CCoh}(X)$ is fully faithful.

Proof. If we identify each $\text{L Coh}_{\mathcal{U}}(X)$ with its ‘weakly-essential’ image in $\text{L CCoh}(X)$ (i.e. the subcategory spanned by objects *weakly equivalent* to those in the image of the inclusion), then we can simply apply Lemma 4.2.8. \square

Definition 4.2.12. Let $\underline{\text{Vect}}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ be the subcategory of $\text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ spanned by complexes that actually *are* cartesian complexes of locally free sheaves on the nerve; let $\underline{\text{Green}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ be the subcategory of $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ spanned by complexes that actually *are* Green complexes.

Lemma 4.2.13. $\text{hocolim}_{\mathcal{U}} \text{L Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) \simeq \text{hocolim}_{\mathcal{U}} \text{L } \underline{\text{Green}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$.

Proof. The homotopy colimit over refinements of all covers is equivalent to the homotopy colimit over some truncation below of refinements over all covers, i.e. we can always assume that our covers are as fine as we wish when computing the homotopy colimit. But, by taking a fine enough cover \mathcal{U} , every object of $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ restricted to each U_{α} is free, and so, as in the proof of Lemma 4.2.7, we can invert quasi-isomorphisms whose target is in $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$. This means that, given some morphism in the localisation $\text{L Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$, expressed as a chain of roofs with the left-legs all quasi-isomorphisms, we can invert the quasi-isomorphisms and compose the resulting morphisms to obtain a single morphism in $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ which is equal to that in the localisation with which we started. This means that $\text{L Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ is equivalent to $\text{L } \underline{\text{Green}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$, and so their homotopy colimits agree. \square

Corollary 4.2.14. *There is an equivalence of $(\infty, 1)$ -categories*

$$\text{hocolim}_{\mathcal{U}} \text{L } \underline{\text{Green}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) \simeq \text{L CCoh}(X).$$

Proof. This is a consequence of Theorem 4.2.9 and Lemmas 4.2.10, 4.2.11, and 4.2.13. \square

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