Twisting cochains and twisted complexes

Simplicial methods in complex-analytic geometry

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Plan

History

- Edgar H Brown. "Twisted tensor products, I". In: *Annals of Mathematics* 69.1 (1959), pp. 223–246.
- John C Moore. "Differential homological algebra". In: *Actes du Congrès International des Mathématiciens* 1 (1970), pp. 335–339.

Coherent sheaves

- Domingo Toledo and Yue Lin L Tong. "A parametrix for δ and Riemann-Roch in Čech theory". In: *Topology* 15.4 (1976), pp. 273–301.
- Domingo Toledo and Yue Lin L Tong. "Duality and Intersection Theory in Complex Manifolds. I". In: *Mathematische Annalen* 237 (1978), pp. 41–77.
- Nigel R O'Brian, Domingo Toledo, and Yue Lin L Tong. "The Trace Map and Characteristic Classes for Coherent Sheaves". In: *American Journal of Mathematics* 103.2 (1981), pp. 225–252.

- A I Bondal and M M Kapranov. "Enhanced Triangulated Categories". In: *Math. USSR Sbornik* 70.1 (1991), pp. 1–15.
- Giovanni Faonte. *Simplicial nerve of an A-infinity category*. 2015. arXiv: 1312.2127 [math.AT].

Lemma

Let $x \in \{twisting, twisted\}$ and $y \in \{cochain, complex\}$.

Then there exists somebody who uses "$x\,y$" to denote what you call "$x^c\,y^c$".

Corollary

Sometimes it can be hard to figure out what people mean.

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Nomenclature

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But that's very possibly just me.

Twisting cochains (OTT)

Nice spaces

Definition (Stein spaces)

A complex-analytic[1] manifold *Y* is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

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[1]analytic = $\mathcal{O}_Y$ is holomorphic functions, *Y* has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, *Y* has the Zariski topology.

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Throughout, *X* is a complex-analytic manifold with a nice[2] cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

[1] analytic = $\mathcal{O}_Y$ is holomorphic functions, *Y* has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, *Y* has the Zariski topology.
[2] *Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded $\mathcal{O}_{U_\alpha}$-modules*:

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Definition (Endomorphisms)

The collection of *degree-q endomorphisms $\mathrm{End}^q(V)$ of V* is, over each $U_{\alpha_0 \ldots \alpha_p}$, given by

$$\mathrm{End}^q(V)|_{U_{\alpha_0 \ldots \alpha_p}} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}\big(V_{\alpha_p}^i|_{U_{\alpha_0 \ldots \alpha_p}}, V_{\alpha_0}^{i+q}|_{U_{\alpha_0 \ldots \alpha_p}}\big).$$

Warning

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We discuss this later.

Definition (Deleted Čech complex)

Define the chain complex $(\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathscr{C}}^p(\mathcal{U}, \mathrm{End}^q(V)) = \bigoplus_{(\alpha_0, \ldots, \alpha_p)} \mathrm{End}^q(V)|U_{\alpha_0 \ldots \alpha_p}$$

(where $\mathrm{End}^q(V)|U_{\alpha_0 \ldots \alpha_p} = 0$ if $U_{\alpha_0 \ldots \alpha_p} = \varnothing$) with the **deleted Čech differential**

$$\hat{\delta} \colon \hat{\mathscr{C}}^p(\mathcal{U}, \mathrm{End}^q(V)) \to \hat{\mathscr{C}}^{p+1}(\mathcal{U}, \mathrm{End}^q(V))$$

$$(\hat{\delta} c)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=1}^{p} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_{p+1}}.$$

We use $\hat{\mathscr{C}}$ and $\hat{\delta}$ for the *deleted* Čech objects and $\check{\mathscr{C}}$ and $\check{\delta}$ for the 'full' Čech objects.

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- We could define the same complex for an arbitrary bounded graded vector bundle, i.e.

$$\hat{\mathscr{C}}^p(\mathcal{U}, V^q) = \bigoplus_{(\alpha_0, \ldots, \alpha_p)} V^q_{\alpha_0}$$

but where the deleted Čech differential only omits the *first* index (but includes the $(p+1)$th).

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Note that these are maps from $E|U_{\alpha_p}$ to $E|U_{\alpha_0}$ in the specific case where $p = 1$.

Thinking of $g_{\alpha\beta}$ as an element of $\hat{\mathscr{C}}^1(\mathcal{U}, E)$, we see that

$$(\hat{\delta}g)_{\alpha\beta\gamma} = -g_{\alpha\gamma}$$
$$(g \cdot g)_{\alpha\beta\gamma} = g_{\alpha\beta}g_{\beta\gamma}.$$

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This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

Twisting cochains

Definition (Twisting cochains)

A *(holomorphic) twisting cochain over V* is a formal sum

$$a = \bigoplus_{k \in \mathbb{N}} a^{k,1-k}$$

where $a^{k,1-k} \in \hat{\mathscr{C}}^k(\mathcal{U}, \mathrm{End}^{1-k}(V))$ such that

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1. $\hat{\delta}a + a \cdot a = 0$; and
2. $a^{1,0}_{\alpha\alpha} = \mathrm{id}$.

The invertibility condition "should" really be weakened by asking only that $a^{1,0}_{\alpha\alpha}$ be *homotopic* to the identity.

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(a \cdot b)^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} a^{q,t} \cdot b^{q',t'}.$$

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Twisting cochains (cont.)

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- If V has a differential then a is an element of total degree 1.
- We haven't said *when* twisting cochains exist, but under pretty mild assumptions they always do (by an inductive construction).

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$$a_{\alpha\beta}^{1,0} \colon \left(V_\beta^\bullet | U_{\alpha\beta}, a_\beta^{0,1} \right) \to \left(V_\alpha^\bullet | U_{\alpha\beta}, a_\alpha^{0,1} \right)$$

Unpacking the definition

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$(k = 2) \rightsquigarrow -\mathrm{a}_{\alpha\gamma}^{1,0} + \mathrm{a}_{\alpha\beta}^{1,0} \cdot \mathrm{a}_{\beta\gamma}^{1,0} = \mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_{\alpha\beta\gamma}^{2,-1} + \mathrm{a}_{\alpha\beta\gamma}^{2,-1} \cdot \mathrm{a}_\gamma^{0,1}$, which says that $\mathrm{a}_{\alpha\beta\gamma}^{2,-1}$ witnesses a *chain homotopy* between $\mathrm{a}_{\alpha\gamma}^{1,0}$ and $\mathrm{a}_{\alpha\beta}^{1,0} \cdot \mathrm{a}_{\beta\gamma}^{1,0}$. On $\alpha\beta\alpha$ and $\beta\alpha\beta$ this tells us that $\mathrm{a}_{\alpha\beta}^{1,0}$ and $\mathrm{a}_{\beta\alpha}^{1,0}$ are *chain homotopic inverses*, i.e. *quasi-isomorphism*.

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Extra-curricular

By taking (internal) homology we obtain something strict: a complex of *coherent sheaves* $H^\bullet(a)$. This is because quasi-isomorphisms become strict isomorphisms in homology.

We can use this fact to construct twisting cochains that resolve coherent sheaves by taking *local* resolutions by vector bundles.

The total differential

Lemma

For any $\mathrm{a} \in \mathrm{Tot}^1 \hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V))$, the map

$$\mathrm{D}_\mathrm{a} \colon \mathrm{Tot}^r \hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ) \to \mathrm{Tot}^{r+1} \hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ)$$
$$c \mapsto \hat{\bar{\delta}} c + c \cdot \mathrm{a}$$

defines a differential (i.e. squares to zero) if and only if a *is a twisting cochain.*

Proof.

(Tedious) definition chasing. $\qquad \square$

We can actually define twisting cochains in a different way using this lemma (but we won't do so today).

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But this approach lets us think of a twisting cochain as a *first-order perturbation of the deleted Čech differential.*

Examples

Example

Look at the most trivial example: let *V* be an *ungraded* vector bundle, and $a = a^{0,1} + a^{1,0}$, where $a_\alpha^{0,1} = \mathrm{id}_{V_\alpha}$, and the $a^{1,0}$ are the transition maps. Then

$$(D_a c)_{\alpha_0 \ldots \alpha_{p+1}} = a^{1,0}_{\alpha_0 \alpha_1} c_{\alpha_1 \ldots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_{p+1}}.$$

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Note that we couldn't use the full Čech differential on $\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ)$ because everything has to lie over U_{α_0}, but this total differential solves that problem — recall that $a_{\alpha_0 \alpha_1}^{1,0}$ is a (quasi-)isomorphism.

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A spectral-sequence argument shows that, in fact, D_a here really is 'the same as' the full Čech differential.

Examples (cont.)

Example

Now look at a slightly-less trivial example: let $V^\bullet$ consist of *complexes* $(V_\alpha^\bullet, d_\alpha)$ of vector bundles, and $a = a^{0,1} + a^{1,0}$, where $a_\alpha^{0,1} = d_\alpha$, and the $a^{1,0}$ are the transition maps. Then

$$(D_a c)_{\alpha_0 \ldots \alpha_{p+1}} = (-1)^p a_{\alpha_0}^{0,1} c_{\alpha_0 \ldots \alpha_p} + a_{\alpha_0 \alpha_1}^{1,0} c_{\alpha_1 \ldots \alpha_{p+1}}$$
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$$(D_a c)_{\alpha_0 \dots \alpha_{p+1}} = (-1)^p a^{0,1}_{\alpha_0} c_{\alpha_0 \dots \alpha_p} + a^{1,0}_{\alpha_0 \alpha_1} c_{\alpha_1 \dots \alpha_{p+1}}$$
$$+ \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1}}.$$

Identifying the second and third terms with the full Čech
differential, as above, gives the usual total differential of the
Čech bicomplex:

$$D_a = \check{\delta} \pm d_V.$$

- Transition maps naturally go from α_p to α_0.

Why this emphasis on the first index?

- Transition maps naturally go from α_p to α_0.
- We want to be able to compare local things, and we need to pull everything back to lie over the same open set in order to do so.

The Maurer-Cartan equation in other fields

Subject	Equation	Interpretation
Differential geometry	$F_\nabla = \mathrm{d}A + A \cdot A$	curvature of a Koszul connection[3]
Gauge theory	$\Omega = \mathrm{d}A + \frac{1}{2}[A \wedge A]$	curvature of a principal connection
Deformation theory	$\partial a + \frac{1}{2}[a, a]$	deformations of f.d. associative k-algebras with unit[4]

---

[3] Here be Christoffel symbols.

[4] There is also the beautiful fact (that we won't explain at all) that $\mathrm{MC}(A \otimes \mathfrak{g}) \simeq \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), A)$.

Motto

Solutions to (i.e. zeros of) the Maurer-Cartan equation are always (in some sense) *flat objects*.

Twisted complexes (BK)

Motto

Pretriangulated dg-categories are those whose homotopy category is *triangulated*, where *triangulated* denotes the structure left over from taking the homotopy category of a *stable* $(\infty, 1)$-*category*.[5]

---

[5]Loop spaces and suspensions form an equivalence.

Problem

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Solution

Twisted complexes à la Bondal and Kapranov.

dg-categories

Definition

A *dg-category* is a category enriched over chain complexes. That is, the hom-sets are hom-*complexes*.

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Definition

Let $\mathcal{A}$ be a dg-category. Then a *twisted complex* $\mathfrak{C}$ *over* $\mathcal{A}$ is a collection

$$\mathfrak{C} = \{E_i \in \mathcal{A}, q_{ij} \colon E_i \to E_j \mid i, j \in \mathbb{Z}\}$$

such that

- all but finitely many of the E_i are zero;
- the q_{ij} are of degree $i - j + 1$; and
- $\mathrm{d}q_{ij} + \sum_{s \in \mathbb{Z}} q_{sj} q_{is} = 0$.

The main result

Theorem

Given a dg-category, the smallest dg-category containing it in which we can define shifts and functorial cones is exactly its category of twisted complexes.

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Given a dg-category, the smallest dg-category containing it in which we can define shifts and functorial cones is exactly its category of twisted complexes.

Further, if the original dg-category is pretriangulated then this embedding is a quasi-equivalence (which lets us pull back the shift and the cones, which descend exactly to a triangulated structure on the homotopy category).

Warning

It is **not** the case that, by picking the 'right' dg-category $\mathcal{A}$, we can recover the definition of holomorphic twisting cochains from that of twisted complexes.

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It is **not** the case that, by picking the 'right' dg-category $\mathcal{A}$, we can recover the definition of holomorphic twisting cochains from that of twisted complexes.

Rather, twisting cochains are a *specific* case of twisted complexes.

Constructing a twisted complex from a twisting cochain

- Let $\mathcal{A} = \mathrm{dgMod}_B$ where $B = \hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathcal{O}_X)$ with differential $\hat{\delta}$.

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- We need a degree-1 B-linear endomorphism $\mathrm{a} = q_{00}$ of E_0 such that $\hat{\delta}\mathrm{a} + \mathrm{aa} = 0$, but we can show that the dg-algebra $\mathrm{End}_B(E_0)$ is exactly $\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V))$.
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Motto

Twisting cochains are twisted complexes that are *concentrated in degree zero* and *'projective/free'*.

Other fun things