Twisting cochains and twisted complexes

Simplicial methods in complex-analytic geometry

Tim Hosgood

24/07/19

Université d'Aix-Marseille
https://thosgood.github.io

Plan

History

First steps

- Edgar H Brown. “Twisted tensor products, I”. In: *Annals of Mathematics* 69.1 (1959), pp. 223–246.
- John C Moore. “Differential homological algebra”. In: *Actes du Congres International des Mathématiciens* 1 (1970), pp. 335–339.

Coherent sheaves

- Domingo Toledo and Yue Lin L Tong. "A parametrix for δ and Riemann-Roch in Čech theory". In: *Topology* 15.4 (1976), pp. 273–301.
- Domingo Toledo and Yue Lin L Tong. "Duality and Intersection Theory in Complex Manifolds. I". In: *Mathematische Annalen* 237 (1978), pp. 41–77.
- Nigel R O'Brian, Domingo Toledo, and Yue Lin L Tong. "The Trace Map and Characteristic Classes for Coherent Sheaves". In: *American Journal of Mathematics* 103.2 (1981), pp. 225–252.

- A I Bondal and M M Kapranov. “Enhanced Triangulated Categories”. In: *Math. USSR Sbornik* 70.1 (1991), pp. 1–15.
- Giovanni Faonte. *Simplicial nerve of an A-infinity category*. 2015. arXiv: 1312.2127 [math.AT].

Twisting cochains (OTT)

Nice spaces

Definition (Stein spaces)

A complex-analytic[1] manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

[1]analytic = $\mathcal{O}_Y$ is holomorphic functions, Y has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, Y has the Zariski topology.
[2]*Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

Nice spaces

Definition (Stein spaces)

A complex-analytic[1] manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

Motto

Stein things are nice.

[1] analytic = $\mathcal{O}_Y$ is holomorphic functions, Y has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, Y has the Zariski topology.
[2] *Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

Nice spaces

Definition (Stein spaces)

A complex-analytic[1] manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

Motto

Stein things are nice.

Throughout, X is a complex-analytic manifold with a nice[2] cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

[1] analytic = $\mathcal{O}_Y$ is holomorphic functions, Y has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, Y has the Zariski topology.
[2] *Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

Think of a bounded chain complex of vector bundles, but without the information of a differential.

Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded $\mathcal{O}_{U_\alpha}$-modules*:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

Think of a bounded chain complex of vector bundles, but without the information of a differential.

Definition (Endomorphisms)

The collection of *degree-q endomorphisms* $\mathrm{End}^q(V)$ *of* V is, over each $U_{\alpha_0 \ldots \alpha_p}$, given by

$$\mathrm{End}^q(V)|U_{\alpha_0 \ldots \alpha_p} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}\big(V_{\alpha_p}^i | U_{\alpha_0 \ldots \alpha_p}, V_{\alpha_0}^{i+q} | U_{\alpha_0 \ldots \alpha_p}\big).$$

Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q\in\mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

Think of a bounded chain complex of vector bundles, but without the information of a differential.

Definition (Endomorphisms)

The collection of *degree-q endomorphisms* $\mathrm{End}^q(V)$ *of* V is, over each $U_{\alpha_0\ldots\alpha_p}$, given by

$$\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}\left(V_{\alpha_p}^i|U_{\alpha_0\ldots\alpha_p}, V_{\alpha_0}^{i+q}|U_{\alpha_0\ldots\alpha_p}\right).$$

Warning

The maps are from the α_p part to the α_0 part.

The deleted Čech complex

Definition (Deleted Čech complex)

Define the chain complex $(\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathscr{C}}^p\big(\mathcal{U}, \mathrm{End}^q(V)\big) = \bigoplus_{(\alpha_0,\ldots,\alpha_p)} \mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p}$$

(where $\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = 0$ if $U_{\alpha_0\ldots\alpha_p} = \varnothing$) with the ***deleted*** *Čech differential*

$$\hat{\delta}\colon \hat{\mathscr{C}}^p\big(\mathcal{U}, \mathrm{End}^q(V)\big) \to \hat{\mathscr{C}}^{p+1}\big(\mathcal{U}, \mathrm{End}^q(V)\big)$$

$$(\hat{\delta}c)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=1}^{p} (-1)^i c_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}}.$$

A notational note

We use $\hat{\mathscr{C}}$ and $\hat{\delta}$ for the *deleted* Čech objects and $\check{\mathscr{C}}$ and $\check{\delta}$ for the ‘full’ Čech objects.

Further structure

- If V has a differential then this gives us a *bicomplex*.

Further structure

- If V has a differential then this gives us a *bicomplex*.
- There is a natural multiplication structure given by composition:

$$(c^{p,q} \cdot \tilde{c}^{\tilde{p},\tilde{q}})_{\alpha_0 \ldots \alpha_{p+\tilde{p}}} = (-1)^{q\tilde{p}} c^{p,q}_{\alpha_0 \ldots \alpha_p} \tilde{c}^{\tilde{p},\tilde{q}}_{\alpha_p \ldots \alpha_{p+\tilde{p}}}.$$

Further structure

- If V has a differential then this gives us a *bicomplex*.
- There is a natural multiplication structure given by composition:

$$(c^{p,q} \cdot \tilde{c}^{\tilde{p},\tilde{q}})_{\alpha_0 \ldots \alpha_{p+\tilde{p}}} = (-1)^{q\tilde{p}} c^{p,q}_{\alpha_0 \ldots \alpha_p} \tilde{c}^{\tilde{p},\tilde{q}}_{\alpha_p \ldots \alpha_{p+\tilde{p}}}.$$

- We could define the same complex for an arbitrary bounded graded vector bundle, i.e.

$$\hat{\mathscr{C}}^p(\mathcal{U}, V^q) = \bigoplus_{(\alpha_0,\ldots,\alpha_p)} V^q_{\alpha_0}$$

  but where the deleted Čech differential only omits the *first* index (but includes the $(p+1)$th).

A brief interlude on bundles

A holomorphic vector bundle E on X is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over U_β to over U_α.

A brief interlude on bundles

A holomorphic vector bundle E on X is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over U_β to over U_α.

These transition maps satisfy two conditions:

A brief interlude on bundles

A holomorphic vector bundle E on X is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over U_β to over U_α.

These transition maps satisfy two conditions:

1. $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\beta}$ (the *cocycle* condition); and

A brief interlude on bundles

A holomorphic vector bundle E on X is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over U_β to over U_α.

These transition maps satisfy two conditions:

1. $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\beta}$ (the *cocycle* condition); and
2. $g_{\alpha\alpha} = \mathrm{id}$ (the *invertibility* condition).

A brief interlude on bundles

A holomorphic vector bundle E on X is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over U_β to over U_α.

These transition maps satisfy two conditions:

1. $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\beta}$ (the *cocycle* condition); and
2. $g_{\alpha\alpha} = \mathrm{id}$ (the *invertibility* condition).

Note that these are maps from $E|U_{\alpha_p}$ to $E|U_{\alpha_0}$ in the specific case where $p = 1$.

Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\hat{\mathscr{C}}^1(\mathcal{U}, E)$, we see that

$$(\hat{\delta} g)_{\alpha\beta\gamma} = -g_{\alpha\gamma}$$
$$(g \cdot g)_{\alpha\beta\gamma} = g_{\alpha\beta} g_{\beta\gamma}.$$

Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\hat{\mathscr{C}}^1(\mathcal{U}, E)$, we see that

$$(\hat{\delta}g)_{\alpha\beta\gamma} = -g_{\alpha\gamma}$$
$$(g \cdot g)_{\alpha\beta\gamma} = g_{\alpha\beta}g_{\beta\gamma}.$$

This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

Definition (Twisting cochains)

A *(holomorphic) twisting cochain over V* is a formal sum

$$\mathrm{a} = \bigoplus_{k\in\mathbb{N}} \mathrm{a}^{k,1-k}$$

where $\mathrm{a}^{k,1-k} \in \hat{\mathscr{C}}^k(\mathcal{U}, \mathrm{End}^{1-k}(V))$ such that

1. $\hat{\delta}\mathrm{a} + \mathrm{a}\cdot\mathrm{a} = 0$; and
2. $\mathrm{a}^{1,0}_{\alpha\alpha} = \mathrm{id}$.

Twisting cochains

Definition (Twisting cochains)

A *(holomorphic) twisting cochain over V* is a formal sum

$$\mathrm{a} = \bigoplus_{k \in \mathbb{N}} \mathrm{a}^{k,1-k}$$

where $\mathrm{a}^{k,1-k} \in \hat{\mathscr{C}}^k(\mathcal{U}, \mathrm{End}^{1-k}(V))$ such that

1. $\hat{\delta}\mathrm{a} + \mathrm{a} \cdot \mathrm{a} = 0$; and
2. $\mathrm{a}_{\alpha\alpha}^{1,0} = \mathrm{id}$.

The invertibility condition "should" really be weakened by asking only that $\mathrm{a}_{\alpha\alpha}^{1,0}$ be *homotopic* to the identity.

Twisting cochains (cont.)

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

Twisting cochains (cont.)

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

- It might be the case that all but finitely many of the $\mathrm{a}^{k,1-k}$ are zero, but **never** $\mathrm{a}^{1,0}$, since it has to be the identity on $\alpha\alpha$.

Twisting cochains (cont.)

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

- It might be the case that all but finitely many of the $\mathrm{a}^{k,1-k}$ are zero, but **never** $\mathrm{a}^{1,0}$, since it has to be the identity on $\alpha\alpha$.
- If V has a differential then a is an element of total degree 1.

Twisting cochains (cont.)

Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

- It might be the case that all but finitely many of the $\mathrm{a}^{k,1-k}$ are zero, but **never** $\mathrm{a}^{1,0}$, since it has to be the identity on $\alpha\alpha$.
- If V has a differential then a is an element of total degree 1.
- We haven't said *when* twisting cochains exist, but under pretty mild assumptions they always do (by an inductive construction).

Unpacking the definition

$(k=0) \rightsquigarrow$ $\mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_\alpha^{0,1} = 0$, which tells us that $\mathrm{a}_\alpha^{0,1}$ is a *differential on* $V_\alpha^\bullet$.

Unpacking the definition

$(k=0) \rightsquigarrow$ $\mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_\alpha^{0,1} = 0$, which tells us that $\mathrm{a}_\alpha^{0,1}$ is a *differential on $V_\alpha^\bullet$.*

$(k=1) \rightsquigarrow$ $\mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_{\alpha\beta}^{1,0} = \mathrm{a}_{\alpha\beta}^{1,0} \cdot \mathrm{a}_\beta^{0,1}$, which tells us that we have a *chain map of chain complexes*

$$\mathrm{a}_{\alpha\beta}^{1,0} \colon \left(V_\beta^\bullet|U_{\alpha\beta}, \mathrm{a}_\beta^{0,1}\right) \to \left(V_\alpha^\bullet|U_{\alpha\beta}, \mathrm{a}_\alpha^{0,1}\right)$$

Unpacking the definition

$(k=0) \rightsquigarrow$ $\mathrm{a}^{0,1}_{\alpha} \cdot \mathrm{a}^{0,1}_{\alpha} = 0$, which tells us that $\mathrm{a}^{0,1}_{\alpha}$ is a *differential on $V^{\bullet}_{\alpha}$.*

$(k=1) \rightsquigarrow$ $\mathrm{a}^{0,1}_{\alpha} \cdot \mathrm{a}^{1,0}_{\alpha\beta} = \mathrm{a}^{1,0}_{\alpha\beta} \cdot \mathrm{a}^{0,1}_{\beta}$, which tells us that we have a *chain map of chain complexes*

$$\mathrm{a}^{1,0}_{\alpha\beta} \colon (V^{\bullet}_{\beta}|U_{\alpha\beta}, \mathrm{a}^{0,1}_{\beta}) \to (V^{\bullet}_{\alpha}|U_{\alpha\beta}, \mathrm{a}^{0,1}_{\alpha})$$

$(k=2) \rightsquigarrow$ $-\mathrm{a}^{1,0}_{\alpha\gamma} + \mathrm{a}^{1,0}_{\alpha\beta} \cdot \mathrm{a}^{1,0}_{\beta\gamma} = \mathrm{a}^{0,1}_{\alpha} \cdot \mathrm{a}^{2,-1}_{\alpha\beta\gamma} + \mathrm{a}^{2,-1}_{\alpha\beta\gamma} \cdot \mathrm{a}^{0,1}_{\gamma}$, which says that $\mathrm{a}^{2,-1}_{\alpha\beta\gamma}$ witnesses a *chain homotopy* between $\mathrm{a}^{1,0}_{\alpha\gamma}$ and $\mathrm{a}^{1,0}_{\alpha\beta} \cdot \mathrm{a}^{1,0}_{\beta\gamma}$. On $\alpha\beta\alpha$ and $\beta\alpha\beta$ this tells us that $\mathrm{a}^{1,0}_{\alpha\beta}$ and $\mathrm{a}^{1,0}_{\beta\alpha}$ are *chain homotopic inverses*, i.e. *quasi-isomorphism.*

Unpacking the definition (cont.)

$(k \geqslant 3) \rightsquigarrow$ some sort of 'higher homotopic gluings', whatever this might mean.

Unpacking the definition (cont.)

$(k \geqslant 3) \rightsquigarrow$ some sort of 'higher homotopic gluings', whatever this might mean.

This is one of the things that we want to formalise!

Unpacking the definition (cont.)

$(k \geqslant 3) \rightsquigarrow$ some sort of 'higher homotopic gluings', whatever this might mean.

This is one of the things that we want to formalise!

Extra-curricular

By taking (internal) homology we obtain something strict: a complex of *coherent sheaves* $\mathrm{H}^\bullet(\mathfrak{a})$. This is because quasi-isomorphisms become strict isomorphisms in homology.

We can use this fact to construct twisting cochains that resolve coherent sheaves by taking *local* resolutions by vector bundles.

The total differential

Lemma

For any $\mathrm{a} \in \mathrm{Tot}^1\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V))$*, the map*

$$\begin{aligned} \mathrm{D}_{\mathrm{a}} \colon \mathrm{Tot}^r\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ) &\to \mathrm{Tot}^{r+1}\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ) \\ c &\mapsto \hat{\delta}c + c \cdot \mathrm{a} \end{aligned}$$

defines a differential (i.e. squares to zero) if and only if a *is a twisting cochain.*

Proof.

(Tedious) definition chasing. □

We can actually define twisting cochains in a different way using this lemma (but we won't do so today).

The total differential (cont.)

We can actually define twisting cochains in a different way using this lemma (but we won't do so today).

But this approach lets us think of a twisting cochain as a *first-order perturbation of the deleted Čech differential*.

Example

Look at the most trivial example: let V be an *ungraded* vector bundle, and $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$, where $\mathrm{a}^{0,1}_{\alpha} = \mathrm{id}_{V_\alpha}$, and the $\mathrm{a}^{1,0}$ are the transition maps. Then

$$(\mathrm{D}_{\mathrm{a}} c)_{\alpha_0 \ldots \alpha_{p+1}} = \mathrm{a}^{1,0}_{\alpha_0\alpha_1} c_{\alpha_1 \ldots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha}_i \ldots \alpha_{p+1}}.$$

Example

Look at the most trivial example: let V be an *ungraded* vector bundle, and $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$, where $\mathrm{a}^{0,1}_\alpha = \mathrm{id}_{V_\alpha}$, and the $\mathrm{a}^{1,0}$ are the transition maps. Then

$$(\mathrm{D}_\mathrm{a} c)_{\alpha_0 \ldots \alpha_{p+1}} = \mathrm{a}^{1,0}_{\alpha_0\alpha_1} c_{\alpha_1 \ldots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha}_i \ldots \alpha_{p+1}}.$$

Note that we couldn't use the full Čech differential on $\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ)$ because everything has to lie over U_{α_0}, but this total differential solves that problem — recall that $\mathrm{a}^{1,0}_{\alpha_0\alpha_1}$ is a (quasi-)isomorphism.

Examples

Example

Look at the most trivial example: let V be an *ungraded* vector bundle, and $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$, where $\mathrm{a}^{0,1}_{\alpha} = \mathrm{id}_{V_\alpha}$, and the $\mathrm{a}^{1,0}$ are the transition maps. Then

$$(\mathrm{D}_{\mathrm{a}} c)_{\alpha_0 \ldots \alpha_{p+1}} = \mathrm{a}^{1,0}_{\alpha_0 \alpha_1} c_{\alpha_1 \ldots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha}_i \ldots \alpha_{p+1}}.$$

Note that we couldn't use the full Čech differential on $\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ)$ because everything has to lie over U_{α_0}, but this total differential solves that problem — recall that $\mathrm{a}^{1,0}_{\alpha_0\alpha_1}$ is a (quasi-)isomorphism.

A spectral-sequence argument shows that, in fact, D_{a} here really is 'the same as' the full Čech differential.

Examples (cont.)

Example

Now look at a slightly-less trivial example: let $V^\bullet$ consist of *complexes* $(V^\bullet_\alpha, \mathrm{d}_\alpha)$ of vector bundles, and $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$, where $\mathrm{a}^{0,1}_\alpha = \mathrm{d}_\alpha$, and the $\mathrm{a}^{1,0}$ are the transition maps. Then

$$\begin{aligned}(\mathrm{D}_\mathrm{a} c)_{\alpha_0 \ldots \alpha_{p+1}} = (-1)^p \mathrm{a}^{0,1}_{\alpha_0} c_{\alpha_0 \ldots \alpha_p} + \mathrm{a}^{1,0}_{\alpha_0 \alpha_1} c_{\alpha_1 \ldots \alpha_{p+1}} \\ + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_{p+1}}.\end{aligned}$$

Examples (cont.)

Example

Now look at a slightly-less trivial example: let $V^\bullet$ consist of *complexes* $(V^\bullet_\alpha, \mathrm{d}_\alpha)$ of vector bundles, and $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$, where $\mathrm{a}^{0,1}_\alpha = \mathrm{d}_\alpha$, and the $\mathrm{a}^{1,0}$ are the transition maps. Then

$$\begin{aligned}(\mathrm{D}_\mathrm{a}c)_{\alpha_0\ldots\alpha_{p+1}} = (-1)^p \mathrm{a}^{0,1}_{\alpha_0} c_{\alpha_0\ldots\alpha_p} + \mathrm{a}^{1,0}_{\alpha_0\alpha_1} c_{\alpha_1\ldots\alpha_{p+1}} \\ + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0\ldots\widehat{\alpha}_i\ldots\alpha_{p+1}}.\end{aligned}$$

Identifying the second and third terms with the full Čech differential, as above, gives the usual total differential of the Čech bicomplex:

$$\mathrm{D}_\mathrm{a} = \check{\delta} \pm \mathrm{d}_V.$$

Twisted complexes (BK)

Other fun things