# Twisting cochains and twisted complexes

## Simplicial methods in complex-analytic algebraic geometry

Tim Hosgood

24/07/19

Université d'Aix-Marseille

# Plan

- History
- Twisting cochains (OTT)
  - Bicomplexes
  - Maurer-Cartan
- Twisted complexes (BK)
  - Pretriangulated vs. triangulated
  - Generalisation of twisting cochains
- Other fun things

# History

# First steps


- Edgar H Brown. "Twisted tensor products, I". In: *Annals of Mathematics* 69.1 (1959), pp. 223–246.
- John C Moore. "Differential homological algebra". In: *Actes du Congres International des Mathématiciens* 1 (1970), pp. 335–339.

- Domingo Toledo and Yue Lin L Tong. “A parametrix for $\bar\partial$ and Riemann-Roch in Čech theory”. In: *Topology* 15.4 (1976), pp. 273–301.
- Domingo Toledo and Yue Lin L Tong. “Duality and Intersection Theory in Complex Manifolds. I”. In: *Mathematische Annalen* 237 (1978), pp. 41–77.
- Nigel R O’Brian, Domingo Toledo, and Yue Lin L Tong. “The Trace Map and Characteristic Classes for Coherent Sheaves”. In: *American Journal of Mathematics* 103.2 (1981), pp. 225–252.

- A I Bondal and M M Kapranov. "Enhanced Triangulated Categories". In: *Math. USSR Sbornik* 70.1 (1991), pp. 1–15.
- Giovanni Faonte. *Simplicial nerve of an A-infinity category*. 2015. arXiv: 1312.2127 [math.AT].

# Twisting cochains (OTT)

# Nice spaces

## Definition (Stein spaces)

A complex-analytic[1] manifold $Y$ is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

---

[1] analytic = $\mathcal{O}_Y$ is holomorphic functions, $Y$ has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, $Y$ has the Zariski topology.

[2] *Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

## Definition (Stein spaces)

A complex-analytic[1] manifold $Y$ is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

## Motto

Stein things are nice.

[1] analytic = $\mathcal{O}_Y$ is holomorphic functions, $Y$ has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, $Y$ has the Zariski topology.
[2] *Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

**Definition (Stein spaces)**

A complex-analytic[1] manifold $Y$ is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

**Motto**

Stein things are nice.

Throughout, $X$ is a complex-analytic manifold with a nice[2] cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

[1]analytic = $\mathcal{O}_Y$ is holomorphic functions, $Y$ has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, $Y$ has the Zariski topology.
[2]*Locally finite*, *Stein*, and *trivialising* (for the bundles in question).

# Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

## Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

Think of a bounded chain complex of vector bundles, but without the information of a differential.

# Endomorphisms of bounded-graded modules

Let $V = \{V^\bullet_\alpha\}$ be a collection of *bounded-graded $\mathcal{O}_{U_\alpha}$-modules*:

$$V^\bullet_\alpha = \bigoplus_{q\in\mathbb{N}} V^q_\alpha \quad \text{such that } V^q_\alpha \text{ is zero for all but finitely many } q.$$

Think of a bounded chain complex of vector bundles, but without the information of a differential.

## Definition (Endomorphisms)

The collection of *degree-q endomorphisms* $\mathrm{End}^q(V)$ *of* $V$ is, over each $U_{\alpha_0\ldots\alpha_p}$, given by

$$\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}\big(V^i_{\alpha_p}|U_{\alpha_0\ldots\alpha_p}, V^{i+q}_{\alpha_0}|U_{\alpha_0\ldots\alpha_p}\big).$$

# Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q\in\mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

Think of a bounded chain complex of vector bundles, but without the information of a differential.

**Definition (Endomorphisms)**

The collection of *degree-q endomorphisms* $\mathrm{End}^q(V)$ *of* $V$ is, over each $U_{\alpha_0\ldots\alpha_p}$, given by

$$\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}\big(V_{\alpha_p}^i|U_{\alpha_0\ldots\alpha_p}, V_{\alpha_0}^{i+q}|U_{\alpha_0\ldots\alpha_p}\big).$$

**Warning**

The maps are from the $\alpha_p$ part to the $\alpha_0$ part.

# The deleted Čech complex

## Definition (Deleted Čech complex)

Define the chain complex $(\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathscr{C}}^p\big(\mathcal{U}, \mathrm{End}^q(V)\big) = \bigoplus_{(\alpha_0,\ldots,\alpha_p)} \mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p}$$

(where $\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = 0$ if $U_{\alpha_0\ldots\alpha_p} = \varnothing$) with the ***deleted*** *Čech differential*

$$\hat{\delta}\colon \hat{\mathscr{C}}^p\big(\mathcal{U}, \mathrm{End}^q(V)\big) \to \hat{\mathscr{C}}^{p+1}\big(\mathcal{U}, \mathrm{End}^q(V)\big)$$

$$(\hat{\delta}c)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=1}^{p}(-1)^i c_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}}.$$

# Further structure

- If $V$ has a differential then this gives us a *bicomplex*.

## Further structure

- If $V$ has a differential then this gives us a *bicomplex*.
- There is a natural multiplication structure given by composition:

$$(c^{p,q} \cdot \tilde{c}^{\tilde{p},\tilde{q}})_{\alpha_0 \dots \alpha_{p+\tilde{p}}} = (-1)^{q\tilde{p}} c^{p,q}_{\alpha_0 \dots \alpha_p} \tilde{c}^{\tilde{p},\tilde{q}}_{\alpha_p \dots \alpha_{p+\tilde{p}}}.$$

# Further structure

- If $V$ has a differential then this gives us a *bicomplex*.
- There is a natural multiplication structure given by composition:

$$(c^{p,q} \cdot \tilde{c}^{\tilde{p},\tilde{q}})_{\alpha_0 \ldots \alpha_{p+\tilde{p}}} = (-1)^{q\tilde{p}} c^{p,q}_{\alpha_0 \ldots \alpha_p} \tilde{c}^{\tilde{p},\tilde{q}}_{\alpha_p \ldots \alpha_{p+\tilde{p}}}.$$

- We could define the same complex for an arbitrary bounded graded vector bundle, i.e. $\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ)$, but where the deleted Čech differential only omits the *first* index (but includes the $(p+1)$th).

# A brief interlude on bundles

A holomorphic vector bundle $E$ on $X$ is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over $U_\beta$ to over $U_\alpha$.

# A brief interlude on bundles

A holomorphic vector bundle $E$ on $X$ is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over $U_\beta$ to over $U_\alpha$.

These transition maps satisfy two conditions:

# A brief interlude on bundles

A holomorphic vector bundle $E$ on $X$ is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over $U_\beta$ to over $U_\alpha$.

These transition maps satisfy two conditions:

1. $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\beta}$ (the *cocycle* condition); and

# A brief interlude on bundles

A holomorphic vector bundle $E$ on $X$ is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over $U_\beta$ to over $U_\alpha$.

These transition maps satisfy two conditions:

1. $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\beta}$ (the *cocycle* condition); and
2. $g_{\alpha\alpha} = \mathrm{id}$ (the *invertibility* condition).

# A brief interlude on bundles

A holomorphic vector bundle $E$ on $X$ is described exactly by its *transition maps* $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$, which describe the change in trivialisation from over $U_\beta$ to over $U_\alpha$.

These transition maps satisfy two conditions:

1. $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\beta}$ (the *cocycle* condition); and
2. $g_{\alpha\alpha} = \mathrm{id}$ (the *invertibility* condition).

Note that these are maps from $E|U_{\alpha_p}$ to $E|U_{\alpha_0}$ in the specific case where $p = 1$.

# Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\hat{\mathscr{C}}^1(\mathcal{U}, E)$, we see that

$$
\begin{aligned}
(\hat{\delta} g)_{\alpha\beta\gamma} &= -g_{\alpha\gamma} \\
(g \cdot g)_{\alpha\beta\gamma} &= g_{\alpha\beta} g_{\beta\gamma}.
\end{aligned}
$$

# Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\hat{\mathscr{C}}^1(\mathcal{U}, E)$, we see that

$$(\hat{\delta}g)_{\alpha\beta\gamma} = -g_{\alpha\gamma}$$
$$(g \cdot g)_{\alpha\beta\gamma} = g_{\alpha\beta}g_{\beta\gamma}.$$

This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

**Definition (Twisting cochains)**

A *(holomorphic) twisting cochain over V* is a formal sum

$$\mathrm{a} = \bigoplus_{k \in \mathbb{N}} \mathrm{a}^{k,1-k}$$

where $\mathrm{a}^{k,1-k} \in \hat{\mathscr{C}}^k(\mathcal{U}, \mathrm{End}^{1-k}(V))$ such that

1. $\hat{\delta}\mathrm{a} + \mathrm{a} \cdot \mathrm{a} = 0$; and
2. $\mathrm{a}^{1,0}_{\alpha\alpha} = \mathrm{id}$.

# Twisting cochains (cont.)

## Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

# Twisting cochains (cont.)

**Warning**

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

- It might be the case that all but finitely many of the $\mathrm{a}^{k,1-k}$ are zero, but **never** $\mathrm{a}^{1,0}$, since it has to be the identity on $(\alpha\alpha)$.

# Twisting cochains (cont.)

## Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

- It might be the case that all but finitely many of the $\mathrm{a}^{k,1-k}$ are zero, but **never** $\mathrm{a}^{1,0}$, since it has to be the identity on $(\alpha\alpha)$.
- If $V$ has a differential then $\mathrm{a}$ is an element of total degree 1.

# Twisting cochains (cont.)

## Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(\mathrm{a} \cdot \mathrm{b})^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} \mathrm{a}^{q,t} \cdot \mathrm{b}^{q',t'}.$$

- It might be the case that all but finitely many of the $\mathrm{a}^{k,1-k}$ are zero, but **never** $\mathrm{a}^{1,0}$, since it has to be the identity on $(\alpha\alpha)$.
- If $V$ has a differential then $\mathrm{a}$ is an element of total degree 1.
- We haven't said *when* twisting cochains exist, but under pretty mild assumptions they always do (by an inductive construction).

# Unpacking the definition

$(k = 0) \rightsquigarrow$ $\mathrm{a}_\alpha^{0,1} \cdot \mathrm{a}_\alpha^{0,1} = 0$, which tells us that $\mathrm{a}_\alpha^{0,1}$ is a *differential on* $V_\alpha^\bullet$.

# Unpacking the definition

$(k=0) \rightsquigarrow$ $\mathrm{a}_{\alpha}^{0,1} \cdot \mathrm{a}_{\alpha}^{0,1} = 0$, which tells us that $\mathrm{a}_{\alpha}^{0,1}$ is a *differential on $V_{\alpha}^{\bullet}$.*

$(k=1) \rightsquigarrow$ $\mathrm{a}_{\alpha}^{0,1} \cdot \mathrm{a}_{\alpha\beta}^{1,0} = \mathrm{a}_{\alpha\beta}^{1,0} \cdot \mathrm{a}_{\beta}^{0,1}$, which tells us that we have a *chain map of chain complexes*

$$\mathrm{a}_{\alpha\beta}^{1,0} \colon \left(V_{\beta}^{\bullet}|U_{\alpha\beta}, \mathrm{a}_{\beta}^{0,1}\right) \to \left(V_{\alpha}^{\bullet}|U_{\alpha\beta}, \mathrm{a}_{\alpha}^{0,1}\right)$$

# Twisted complexes (BK)

# Other fun things