Twisting cochains and twisted complexes

Simplicial methods in complex-analytic algebraic geometry

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Plan

History

Twisting cochains (OTT)

- Bicomplexes
- Maurer-Cartan

Twisted complexes (BK)

- Pretriangulated vs. triangulated
- Generalisation of twisting cochains

Other fun things

History

First steps

- Edgar H Brown. “Twisted tensor products, I”. In: *Annals of Mathematics* 69.1 (1959), pp. 223–246.
- John C Moore. “Differential homological algebra”. In: *Actes du Congres International des Mathématiciens* 1 (1970), pp. 335–339.

Coherent sheaves

- Domingo Toledo and Yue Lin L Tong. “A parametrix for $\bar{\partial}$ and Riemann-Roch in Čech theory”. In: *Topology* 15.4 (1976), pp. 273–301.
- Domingo Toledo and Yue Lin L Tong. “Duality and Intersection Theory in Complex Manifolds. I”. In: *Mathematische Annalen* 237 (1978), pp. 41–77.
- Nigel R O’Brian, Domingo Toledo, and Yue Lin L Tong. “The Trace Map and Characteristic Classes for Coherent Sheaves”. In: *American Journal of Mathematics* 103.2 (1981), pp. 225–252.

- A I Bondal and M M Kapranov. “Enhanced Triangulated Categories”. In: *Math. USSR Sbornik* 70.1 (1991), pp. 1–15.
- Giovanni Faonte. *Simplicial nerve of an A-infinity category*. 2015. arXiv: 1312.2127 [math.AT].

Twisting cochains (OTT)

Definition (Stein spaces)

A complex-analytic[1] manifold Y is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

[1]analytic = $\mathcal{O}_Y$ is holomorphic functions, Y has the $\mathbb{C}^n$-induced topology; algebraic = $\mathcal{O}_Y$ is algebraic functions, Y has the Zariski topology.

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Throughout, X is a complex-analytic manifold with a nice[2] cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

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Endomorphisms of bounded-graded modules

Let $V = \{V_\alpha^\bullet\}$ be a collection of *bounded-graded* $\mathcal{O}_{U_\alpha}$*-modules*:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

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$$\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}\big(V_{\alpha_p}^i|U_{\alpha_0\ldots\alpha_p}, V_{\alpha_0}^{i+q}|U_{\alpha_0\ldots\alpha_p}\big).$$

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Warning

The maps are from the α_p part to the α_0 part.

Definition (Deleted Čech complex)

Define the chain complex $(\hat{\mathscr{C}}^\bullet(\mathcal{U}, \mathrm{End}^\circ(V)), \hat{\delta})$ by

$$\hat{\mathscr{C}}^p(\mathcal{U}, \mathrm{End}^q(V)) = \bigoplus_{(\alpha_0,\ldots,\alpha_p)} \mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p}$$

(where $\mathrm{End}^q(V)|U_{\alpha_0\ldots\alpha_p} = 0$ if $U_{\alpha_0\ldots\alpha_p} = \varnothing$) with the ***deleted Čech differential***

$$\hat{\delta}\colon \hat{\mathscr{C}}^p(\mathcal{U}, \mathrm{End}^q(V)) \to \hat{\mathscr{C}}^{p+1}(\mathcal{U}, \mathrm{End}^q(V))$$

$$(\hat{\delta}c)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=1}^{p} (-1)^i c_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{p+1}}.$$

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- We could define the same complex for an arbitrary bounded graded vector bundle, i.e. $\hat{\mathscr{C}}^\bullet(\mathcal{U}, V^\circ)$, but where the deleted Čech differential only omits the *first* index (but includes the $(p+1)$th).

A brief interlude on bundles

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Note that these are maps from $E|U_{\alpha_p}$ to $E|U_{\alpha_0}$ in the specific case where $p = 1$.

Rewriting the cocycle condition

Thinking of $g_{\alpha\beta}$ as an element of $\hat{\mathscr{C}}^1(\mathcal{U}, E)$, we see that

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This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

Twisting cochains

Definition (Twisting cochains)

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Twisted complexes (BK)

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