Final Honour School of Mathematics Part C

C7.4 Introduction to Quantum Information
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Model Solutions

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Do not turn this page until you are told that you may do so

1. (a) [3 marks] Consider a Boolean function $f : \{0,1\}^n \to \{0,1\}$. Explain how the quantum evaluation of f,

$$|\, x \,\rangle |\, y \,\rangle \mapsto |\, x \,\rangle |\, y \oplus f(x) \,\rangle,$$

where $x \in \{0,1\}^n$ and $y \in \{0,1\}$ are the binary strings placed in the first and the second register respectively, can be reduced to the unitary operation U_f on the first register,

$$U_f |\, x \,\rangle = (-1)^{f(x)} |\, x \,\rangle.$$

What is the geometric interpretation of this transformation?

(b) [2 marks] The reflection in the subspace that is orthogonal to $|\, a \,\rangle$ can be written as

$$V_a = \mathbb{1} - 2|\, a \,\rangle\langle\, a \,|.$$

Provide the geometric interpretation of $U V_a U^\dagger$, where U is a unitary operator.

(c) [5 marks] Let $f(x) = 1$ for $x = s$ and $f(x) = 0$ otherwise. We denote the Hadamard transform on n qubits by H_n and the reflection in the subspace that is orthogonal to $|\, 0 \,\rangle$ by V_0. (The vector $|\, 0 \,\rangle$ represents the binary string of n zeros, $|\, 0 \ldots 0 \,\rangle$.) Describe the action of the Grover iteration operator

$$G = -H_n V_0 H_n U_f,$$

in the plane spanned by $H_n |\, 0 \,\rangle$ and the unknown $|\, s \,\rangle$?

A quantum algorithm $\mathcal{A}$, which solves a certain problem in the complexity class NP, can be viewed as a unitary operation A on n qubits. The result of $A|\, 0 \,\rangle$ is the state $|\, \psi \,\rangle$, which is a superposition of binary strings representing possible, not necessarily correct, outputs. It is known that a subsequent measurement in the computational basis provides a correct answer to the problem with the probability $p = \sin^2 \theta \ll 1$ and that $|\, \psi \,\rangle$ can be written as

$$|\, \psi \,\rangle = \sin\theta \,|\, \psi_g \,\rangle + \cos\theta \,|\, \psi_b \,\rangle,$$

where $|\, \psi_g \,\rangle$ and $|\, \psi_b \,\rangle$ are normalised projections of $|\, \psi \,\rangle$ on the subspace spanned by the binary strings corresponding to good and bad answers, respectively. Let U_f corresponds to the Boolean function $f : \{0,1\}^n \to \{0,1\}$ that verifies the outputs of $\mathcal{A}$, that is, $f(x) = 1$ if x is the correct output and $f(x) = 0$ otherwise. You can assume that A, $A^\dagger$ and U_f can be efficiently implemented.

(d) [8 marks] Show that the subspace spanned by $|\, \psi_g \,\rangle$ and $|\, \psi_b \,\rangle$ is invariant under the action of the modified Grover iteration operator Q,

$$Q = -A V_0 A^\dagger U_f,$$

and express $Q|\, \psi_g \,\rangle$ and $Q|\, \psi_b \,\rangle$ as linear superposition of $|\, \psi_g \,\rangle$ and $|\, \psi_b \,\rangle$.

(e) [3 marks] Show that after r applications of Q to the state $|\, \psi \,\rangle$ we obtain

$$Q^r |\, \psi \,\rangle = \sin((2r+1)\theta) |\, \psi_g \,\rangle + \cos((2r+1)\theta) |\, \psi_b \,\rangle.$$

How many applications of Q are required before you can perform a measurement and obtain a correct answer with probability at least $1 - p$?

(f) [4 marks] Provide an informal description of the complexity class NP. Does it matter here that $\mathcal{A}$ solves a problem which is in NP?

SOLUTION TO QUESTION 1

(a) [3 marks] [Bookwork] Prepare the qubit in the second register in the state $\frac{1}{\sqrt{2}}(|\,0\,\rangle - |\,1\,\rangle)$ and evaluate f.

$$|\,x\,\rangle\frac{|\,0\,\rangle - |\,1\,\rangle}{\sqrt{2}} \rightarrow |\,x\,\rangle\frac{|\,f(x)\,\rangle - |\,f(x)\oplus 1\,\rangle}{\sqrt{2}}$$

For $f(x)=0$ the RHS is $|\,x\,\rangle\frac{1}{\sqrt{2}}(|\,0\,\rangle - |\,1\,\rangle)$ whereas for $f(x)=1$ it is $-|\,x\,\rangle\frac{1}{\sqrt{2}}(|\,0\,\rangle - |\,1\,\rangle)$ which is the transformation

$$|\,x\,\rangle \rightarrow (-1)^{f(x)}|\,x\,\rangle$$

on the first register. This can be interpreted as a reflection in the subspace orthogonal to each $|\,x\,\rangle$ such that $f(x)=1$.

(b) [2 marks] [Similar] Since,

$$UV_aU^\dagger = \mathbb{1} - 2U|\,a\,\rangle\langle\,a\,|U^\dagger$$

this is a reflection in the subspace orthogonal to the new state $U|\,a\,\rangle$.

(c) [5 marks] [Bookwork] From the previous two questions, $-H_nV_0H_n$ is a reflection in the subspace orthogonal to the state $H_n|\,0\,\rangle$ and U_f is a reflection in the subspace orthogonal to the state $|\,s\,\rangle$. Overall, this is a rotation in the subspace spanned by $H_n|\,0\,\rangle$ and $|\,s\,\rangle$.

(d) [8 marks] [Similar] Since $Q = -AV_0A^\dagger U_f$, then from the previous question this is a rotation in the subspace spanned by $A|\,0\,\rangle = |\,\psi\,\rangle = \sin\theta\,|\,\psi_g\,\rangle + \cos\theta\,|\,\psi_b\,\rangle$ and $|\,\psi_g\,\rangle$ which is the subspace spanned by $|\,\psi_g\,\rangle$ and $|\,\psi_b\,\rangle$. Thus $\text{span}\{|\,\psi_g\,\rangle, |\,\psi_b\,\rangle\} = \text{span}\{Q|\,\psi_g\,\rangle, Q|\,\psi_b\,\rangle\}$.

$$\begin{aligned}
Q|\,\psi_g\,\rangle &= (\mathbb{1} - 2|\,\psi\,\rangle\langle\,\psi\,|)|\,\psi_g\,\rangle \\
&= |\,\psi_g\,\rangle - 2\langle\psi|\psi_g\rangle|\,\psi\,\rangle \\
&= |\,\psi_g\,\rangle - 2\sin\theta\,(\sin\theta|\,\psi_g\,\rangle + \cos\theta|\,\psi_b\,\rangle) \\
&= (1-2\sin^2\theta)|\,\psi_g\,\rangle - 2\sin\theta\cos\theta|\,\psi_b\,\rangle \\
&= \cos 2\theta|\,\psi_g\,\rangle - \sin 2\theta|\,\psi_b\,\rangle
\end{aligned}$$

Similarly,

$$\begin{aligned}
Q|\,\psi_b\,\rangle &= |\,\psi_b\,\rangle - 2\langle\psi|\psi_b\rangle|\,\psi\,\rangle \\
&= |\,\psi_b\,\rangle - 2\cos\theta\,(\sin\theta|\,\psi_g\,\rangle + \cos\theta|\,\psi_b\,\rangle) \\
&= (1-2\cos^2\theta)|\,\psi_b\,\rangle - 2\sin\theta\cos\theta|\,\psi_g\,\rangle \\
&= \sin 2\theta|\,\psi_g\,\rangle + \cos 2\theta|\,\psi_b\,\rangle
\end{aligned}$$

(e) [3 marks] [Similar] In the previous question we showed that Q was a rotation in the $|\,\psi_g\,\rangle$,$|\,\psi_b\,\rangle$ subspace by an angle of 2θ. Hence repeated application will be a rotation by multiples of 2θ, with r such applications upon ψ leading to

$$Q^r|\,\psi\,\rangle = \sin((2r+1)\theta)|\,\psi_g\,\rangle + \cos((2r+1)\theta)|\,\psi_b\,\rangle.$$

For the state $|\,\psi\,\rangle$ the success probability was $p = \sin^2\theta$ thus in order to achieve a probability of at least $1-p$ we require the $|\,\psi_g\,\rangle$ coefficient of the output state to be $\cos\theta = \sin(\theta+\pi/2)$. Thus the minimum number is given by solving the equation $2r\theta = \pi/2$ which gives $r = \left\lfloor\frac{\pi}{4\theta}\right\rfloor$. For $\theta << 1$ we have $\sqrt{p} = \sin\theta \approx \theta$ and thus $r \approx \left\lfloor\frac{\pi}{4\sqrt{p}}\right\rfloor$

(f) [4 marks] [New] NP describes the class of problems that are difficult to solve but easy to verify. It does matter here that the problem is in NP for it can be efficiently verified by U_f.

2. (a) [3 marks] Two qubits are prepared in one of the four Bell states

$$| \Phi_\pm \rangle = \tfrac{1}{\sqrt{2}}(| 00 \rangle \pm | 11 \rangle), \qquad | \Psi_\pm \rangle = \tfrac{1}{\sqrt{2}}(| 01 \rangle \pm | 10 \rangle).$$

Show that the Bell states form an orthonormal basis in the Hilbert space associated with two qubits. What does it mean that the Bell states are stabilised by $\pm Z \otimes Z$ and $\pm X \otimes X$? Specify stabiliser generators for each of the four Bell states.

(b) [2 marks] Let S_1 and S_2 be stabiliser generators for a two qubit state $| \psi \rangle$. The state is modified by a unitary operation U. What are the stabiliser generators for $U| \psi \rangle$?

(c) [3 marks] Recall that the n-qubit Pauli group is defined as

$$\mathcal{P}_n = \{\mathbb{1}, X, Y, Z\}^{\otimes n} \otimes \{\pm 1, \pm i\}$$

where X, Y, Z are the Pauli matrices. Each element of $\mathcal{P}_n$ is, up to an overall phase ± 1, $\pm i$, a tensor product of Pauli matrices and identity matrices acting on the n qubits. Elements of the Pauli group either commute or anticommute. Show, using stabiliser generators or otherwise, that, up to an overall phase, the elements of $\mathcal{P}_2$ map the Bell states into the Bell states.

(d) [4 marks] Charlie prepares three qubits in the state

$$\frac{1}{\sqrt{2}} (| 000 \rangle + | 111 \rangle) . \tag{1}$$

He gives one qubit to Alice and one to Bob, and keeps the third one for himself. Trace over the third qubit and show that Alice and Bob share a bipartite state described by the density operator

$$\varrho = \frac{1}{2}| \Phi_+ \rangle\langle \Phi_+ | + \frac{1}{2}| \Phi_- \rangle\langle \Phi_- |.$$

Is this an entangled state?

(e) [4 marks] Given the bipartite state ϱ, Alice applies one of the four unitary operations $\{\mathbb{1}, X, Y, Z\}$ to her qubit and sends it to Bob. Can Bob, who performs the measurement in the Bell basis, tell which operation was chosen by Alice? How many bits of information can Alice communicate to Bob?

(f) [5 marks] Charlie, after preparing the state (1) and giving the two qubits to Alice and Bob, applies the Hadamard gate to his qubit and then measures it in the standard basis. He communicates the outcome of the measurement to Bob. Can Bob now tell which of the four operations was chosen by Alice, and if so, how? Does it matter whether Charlie performs his measurement before or after Bob's measurement?

(g) [4 marks] Suppose a third party, who may or may not know the outcome of Bob's measurement, intercepts the qubit that Alice sent to Bob. Explain why there is no measurement that the third party can perform to determine which message Alice transmits?

SOLUTION TO QUESTION 2

(a) [3 marks] [Bookwork] The orthonormality can be verified directly $\langle \Phi_\pm | \Psi_\pm \rangle = 0$, $\langle \Phi_\pm | \Phi_\pm \rangle = 1$ etc. [1 mark]
The Bell states are stabilised by $\pm Z \otimes Z$ and $\pm X \otimes X$, which means they are the eigenstates of $\pm Z \otimes Z$ and $\pm X \otimes X$ with eigenvalue $+1$. [1 mark]
The four Bell states and the corresponding stabiliser generators are

State	Stabiliser generators
Φ_+	$X \otimes X, \quad Z \otimes Z$
Φ_-	$-X \otimes X, \quad Z \otimes Z$
Ψ_+	$X \otimes X, \quad -Z \otimes Z$
Ψ_-	$-X \otimes X, \quad -Z \otimes Z$

(b) [2 marks] [Similar] If $S_k | \psi \rangle = | \psi \rangle$, then $US_kU^\dagger U | \psi \rangle = U | \psi \rangle$. Therefore, $US_kU^\dagger$ for $k = 1, 2$ are the new stabiliser generators.

(c) [3 marks] [New] Any two elements of the Pauli group either commute or anti-commute, therefore they can change at most the sign of the stabiliser generators $\pm X \otimes X, \pm Z \otimes Z$ e.g. for $X \otimes Z$

$$(X \otimes Z)(X \otimes X)(X \otimes Z) = -(X \otimes X)$$
$$(X \otimes Z)(Z \otimes Z)(X \otimes Z) = -(Z \otimes Z).$$

Thus $X \otimes Z$ maps $| \Phi_+ \rangle$ into $| \Psi_- \rangle$, and similarly for other combinations.

(d) [4 marks] [Similar] The density matrix corresponding to the state $\frac{1}{\sqrt{2}} (| 000 \rangle + | 111 \rangle)$ is

$$\varrho_{ABC} = \frac{1}{2} (| 000 \rangle\langle 000 | + | 111 \rangle\langle 111 | + | 000 \rangle\langle 111 | + | 111 \rangle\langle 000 |) .$$

After tracing over the third qubit, we get

$$\varrho = \frac{1}{2} (| 00 \rangle\langle 00 | + | 11 \rangle\langle 11 |) ,$$

Similarly $\frac{1}{2} | \Phi_+ \rangle\langle \Phi_+ | + \frac{1}{2} | \Phi_- \rangle\langle \Phi_- | = \frac{1}{2} (| 00 \rangle\langle 00 | + | 11 \rangle\langle 11 |)$ after expanding the states in the computational basis. This is a separable state.

(e) [4 marks] [Similar] Alice will map the Bell states in the mixture $| \Phi_+ \rangle$ and $| \Phi_- \rangle$ into some other two Bell states. Therefore, Bob can only differentiate between pairs of Bell states and Alice can send at most 1 bit. This is summarised in the following table:

Alice operation	State	Bob's dectection
$\mathbb{1}$	$\frac{1}{2} | \Phi_+ \rangle\langle \Phi_+ | + \frac{1}{2} | \Phi_- \rangle\langle \Phi_- |$	$\mathbb{1}$ or Z
X	$\frac{1}{2} | \Psi_+ \rangle\langle \Psi_+ | + \frac{1}{2} | \Psi_- \rangle\langle \Psi_- |$	X or Y
Z	$\frac{1}{2} | \Phi_- \rangle\langle \Phi_- | + \frac{1}{2} | \Phi_+ \rangle\langle \Phi_+ |$	$\mathbb{1}$ or Z
Y	$\frac{1}{2} | \Psi_- \rangle\langle \Psi_- | + \frac{1}{2} | \Psi_+ \rangle\langle \Psi_+ |$	X or Y

from which it is clear the Bob can only distinguish between the pairs $(\mathbb{1}, Z)$ and (X, Y).

(f) [5 marks] [New] Applying Hadamard to the third qubit results in

$$\frac{1}{\sqrt{2}} (| 000 \rangle + | 111 \rangle) \mapsto \frac{1}{2} (| 00 \rangle(| 0 \rangle + | 1 \rangle) + | 11 \rangle(| 0 \rangle - | 1 \rangle))$$
$$= \frac{1}{2} ((| 00 \rangle + | 11 \rangle)| 0 \rangle + (| 00 \rangle - | 11 \rangle)| 1 \rangle) .$$

Thus Bob learns whether the state he shared with Alice was $|\Phi_+\rangle$ or $|\Phi_-\rangle$. If he knows that it was $|\Phi_+\rangle$ he can determine Alice's operation according to the following:

Alice's operation	Bob's state
$\mathbb{1}$	$|\Phi_+\rangle$
X	$|\Psi_+\rangle$
Z	$|\Phi_-\rangle$
Y	$|\Psi_-\rangle$

Otherwise, if the initial state was $|\Phi_-\rangle$ he finds

Alice's operation	Bob's state
$\mathbb{1}$	$|\Phi_-\rangle$
X	$|\Psi_-\rangle$
Z	$|\Phi_+\rangle$
Y	$|\Psi_+\rangle$

The chronological order in which Bob and Charlie perform their respective measurements does not matter.

(g) [4 marks] [New] The reduced density matrix of the transmitted qubit is a maximally mixed state $\varrho_A = \mathrm{Tr}_B\,\varrho = \frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|)$. Therefore, no information can be retrieved from it.

3. Any density matrix of a single qubit can be parametrised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ and written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right),$$

where σ_x, σ_y and σ_z are the Pauli matrices, and $\vec{s}\cdot\vec{\sigma} = s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$.

(a) [3 marks] Check that such parametrised ϱ is a density matrix. Explain why the length of the Bloch vector cannot exceed 1.

(b) [5 marks] Any physically admissible operation on a qubit is described by a completely positive map which can always be written as

$$\varrho \mapsto \varrho' = \sum_k A_k \varrho A_k^\dagger \tag{2}$$

where matrices A_k must satisfy

$$\sum_k A_k^\dagger A_k = \mathbb{1}. \tag{3}$$

Show that this map preserves positivity and trace. Show that any composition of completely positive maps is also completely positive.

(c) [9 marks] Any linear transformation T acting on density matrices of a qubit can be completely characterised by its action on the four basis matrices $|\,a\,\rangle\langle\,b\,|$, where $a, b = 0, 1$,

$$T(|\,a\,\rangle\langle\,b\,|) = \sum_{\alpha,\beta=0,1} T_{(\alpha a)(\beta b)}|\,\alpha\,\rangle\langle\,\beta\,|.$$

Using conditions (2) and (3), or otherwise, show that for completely positive maps the 4×4 matrix $T_{(\alpha a)(\beta b)}$ must be positive semidefinite and must satisfy

$$\sum_\alpha T_{(\alpha a)(\alpha b)} = \delta_{ab}, \qquad T^\star_{(\alpha a)(\beta b)} = T_{(\beta b)(\alpha a)}.$$

(d) [8 marks] Let T be defined as,

$$T(\mathbb{1}) = \mathbb{1}, \quad T(\sigma_x) = x\sigma_x, \quad T(\sigma_y) = y\sigma_y, \quad T(\sigma_z) = z\sigma_z.$$

where x, y, z are some real numbers. What is the range of x, y, z for which the map T is positive? Using the matrix representation of T, or otherwise, determine the range for which it is completely positive.

[*The Pauli matrices $\sigma_x \equiv X$, $\sigma_y \equiv Y$, and $\sigma_z \equiv Z$ are*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They anticommute and square to the identity $X^2 = Y^2 = Z^2 = \mathbb{1}$.]

SOLUTION TO QUESTION 3

(a) [3 marks] [Bookwork] $\mathrm{Tr}\,\rho = \mathrm{Tr}\,½(\mathbb{1}+s_x\sigma_x+s_y\sigma_y+s_z\sigma_z) = \mathrm{Tr}\,½\mathbb{1} = 1$, because Pauli matrices have trace 0. The two eigenvalues of

$$\rho = \frac{1}{2}\begin{bmatrix} 1+s_z & s_x - i\,s_y \\ s_x + i\,s_y & 1-s_z \end{bmatrix}$$

are $½(1 \pm \sqrt{s_x^2+s_y^2+s_z^2}) = ½(1\pm|\vec{s}|)$. Hence, for $\rho \geqslant 0$ we must require that $|\vec{s}| \leqslant 1$.

(b) [5 marks] [Bookwork] For any vector $|\,v\,\rangle$ we can write $\langle\,v\,|\sum_k A_k\varrho A_k^\dagger|\,v\,\rangle$ as $\sum_k\langle\,v_k\,|\rho|\,v_k\,\rangle$ where $|\,v_k\,\rangle = A_k^\dagger|\,v\,\rangle$. We have $\langle\,v_k\,|\rho|\,v_k\,\rangle \geqslant 0$ for $\rho \geqslant 0$, hence $\langle\,v\,|\sum_k A_k\varrho A_k^\dagger|\,v\,\rangle \geqslant 0$ for any $|\,v\,\rangle$ which implies that $\rho' \geqslant 0$. [2 marks]

The trace is linear and cyclic thus $\mathrm{Tr}\left(\sum_k A_k\varrho A_k^\dagger\right) = \sum_k\left(\mathrm{Tr}\,A_k^\dagger A_k\varrho\right) = \mathrm{Tr}\left[(\sum_k A_k^\dagger A_k)\varrho\right] = \mathrm{Tr}(\mathbb{1}\varrho) = 1$. [1 mark]

Consider two completely positive maps described by the Kraus operators A_k and B_l. The composition $\sum_{k,l} B_l(A_k\varrho A_k^\dagger)B_l^\dagger = \sum_{k,l} B_lA_k\varrho(B_lA_k)^\dagger$ is a completely positive map with Kraus operators B_lA_k because $\sum_{k,l}(B_lA_k)^\dagger B_lA_k = \sum_k A_k^\dagger(\sum_l B_l^\dagger B_l)A_k = \mathbb{1}$. [2 marks]

(c) [9 marks] [New] We can write the action of a completely positive map T in terms of its Kraus operators: $T(|\,a\,\rangle\langle\,b\,|) = \sum_k A_k|\,a\,\rangle\langle\,b\,|A_k^\dagger$. Then the matrix $T_{(\alpha a)(\alpha b)}$ becomes

$$T_{(\alpha a)(\alpha b)} = \sum_k\langle\,\alpha\,|A_k|\,a\,\rangle\langle\,b\,|A_k^\dagger|\,\alpha\,\rangle. \qquad \text{[1 mark]} \tag{4}$$

Hence,

$$\begin{aligned}\sum_\alpha T_{(\alpha a)(\alpha b)} &= \sum_\alpha\sum_k\langle\,\alpha\,|A_k|\,a\,\rangle\langle\,b\,|A_k^\dagger|\,\alpha\,\rangle \\ &= \sum_k\sum_\alpha\langle\,b\,|A_k|\,\alpha\,\rangle\langle\,\alpha\,|A_k^\dagger|\,a\,\rangle \\ &= \sum_k\langle\,b\,|A_kA_k^\dagger|\,a\,\rangle = \langle\,b\,|\mathbb{1}|\,a\,\rangle = \delta_{ab}. \qquad \text{[3 marks]}\end{aligned}$$

Similarly,

$$\begin{aligned}T^\star_{(\alpha a)(\beta b)} &= \sum_k\langle\,\alpha\,|A_k|\,a\,\rangle^\star\langle\,b\,|A_k^\dagger|\,\beta\,\rangle^\star \\ &= \sum_k\langle\,\beta\,|A_k|\,b\,\rangle\langle\,a\,|A_k^\dagger|\,\alpha\,\rangle = T_{(\beta b)(\alpha a)}. \qquad \text{[2 marks]}\end{aligned}$$

In order to show that the matrix $T_{(\alpha a)(\alpha b)}$ is positive semi-definite notice that each term $\langle\,\alpha\,|A_k|\,a\,\rangle\langle\,b\,|A_k^\dagger|\,\beta\,\rangle$ in the sum (4) has the form $v_iv_j^\star$, where $i = (\alpha a)$ and $j = (\beta b)$. Now for any vector c_i it holds $\sum_{i,j} c_i^\star v_iv_j^\star c_j \geqslant 0$, which means that $T_{(\alpha a)(\alpha b)}$ is a sum of positive semi-definite matrices and therefore is positive semi-definite itself. [3 marks]

Alternative approach. Students should know that if T is a completely positive map then any extension of this map is a permissible physical operation (this was explained during the lectures). Thus they may notice that matrix $\frac{1}{2}T_{(\alpha a)(\alpha b)}$ is a density matrix pertaining to two qubits for it is the result of $\mathbb{1}\otimes T$ acting on the maximally entangled state $\frac{1}{\sqrt{2}}(|\,00\,\rangle + |\,11\,\rangle)$. More explicitly,

$$\mathbb{1}\otimes T\,\frac{1}{2}(|\,0\,\rangle\langle\,0\,|\otimes|\,0\,\rangle\langle\,0\,| + |\,0\,\rangle\langle\,1\,|\otimes|\,0\,\rangle\langle\,1\,| + |\,1\,\rangle\langle\,0\,|\otimes|\,1\,\rangle\langle\,0\,| + |\,1\,\rangle\langle\,1\,|\otimes|\,1\,\rangle\langle\,1\,|)$$

gives

$$\frac{1}{2}(|\,0\,\rangle\langle\,0\,| \otimes T(|\,0\,\rangle\langle\,0\,|) + |\,0\,\rangle\langle\,1\,| \otimes T(|\,0\,\rangle\langle\,1\,|) + |\,1\,\rangle\langle\,0\,| \otimes T(|\,1\,\rangle\langle\,0\,|) + |\,1\,\rangle\langle\,1\,| \otimes T(|\,1\,\rangle\langle\,1\,|)).$$

This expression can be also written as the block matrix form

$$\frac{1}{2}\left[\begin{array}{c|c} T(|\,0\,\rangle\langle\,0\,|) & T(|\,0\,\rangle\langle\,1\,|) \\ \hline T(|\,1\,\rangle\langle\,0\,|) & T(|\,1\,\rangle\langle\,1\,|) \end{array}\right]$$

which is indeed $\frac{1}{2}T_{(\alpha a)(\alpha b)}$. The requested properties of $T_{(\alpha a)(\alpha b)}$ follow from the properties of density matrices.

(d) [8 marks] [New] T is positive if for all $\varrho \geqslant 0$, $T(\varrho) \geqslant 0$. The eigenvalues of $\varrho' = T(\varrho)$ will be positive if the length of the Bloch vector $|s'| \leqslant 1$ which implies $|x|, |y|, |z| \leqslant 1$. [2 marks]
For the complete positivity, we first find the matrix elements $T_{(\alpha a)(\alpha b)}$

$$\begin{aligned}
T(|\,0\,\rangle\langle\,0\,|) &= T\left(\frac{\mathbb{1}+\sigma_z}{2}\right) = \frac{1+z}{2}|\,0\,\rangle\langle\,0\,| + \frac{1-z}{2}|\,1\,\rangle\langle\,1\,|,\\
T(|\,0\,\rangle\langle\,1\,|) &= T\left(\frac{\sigma_x+i\sigma_y}{2}\right) = \frac{x+y}{2}|\,0\,\rangle\langle\,1\,| + \frac{x-y}{2}|\,1\,\rangle\langle 0,\,|\\
T(|\,1\,\rangle\langle\,0\,|) &= T\left(\frac{\sigma_x-i\sigma_y}{2}\right) = \frac{x-y}{2}|\,0\,\rangle\langle\,1\,| + \frac{x+y}{2}|\,1\,\rangle\langle\,0\,|,\\
T(|\,1\,\rangle\langle\,1\,|) &= T\left(\frac{\mathbb{1}-\sigma_z}{2}\right) = \frac{1-z}{2}|\,0\,\rangle\langle\,0\,| + \frac{1+z}{2}|\,1\,\rangle\langle\,1\,|. \qquad \text{[3 marks]}
\end{aligned}$$

We can write explicitly the resulting matrix

$$\frac{1}{2}\begin{bmatrix} 1+z & 0 & 0 & x+y \\ 0 & 1-z & x-y & 0 \\ 0 & x-y & 1-z & 0 \\ x+y & 0 & 0 & 1+z \end{bmatrix}$$

The first two condition for $T_{(\alpha a)(\alpha b)}$ are clearly satisfied. Using the block structure of the matrix we can find its eigenvalues and therefore the range of x, y, z for which it is positive semi-definite

$$\begin{aligned}
1 &\geqslant -x-y-z,\\
1 &\geqslant -x+y+z,\\
1 &\geqslant x-y+z,\\
1 &\geqslant x+y-z. \qquad \text{[3 marks]}
\end{aligned}$$