INTRODUCTION TO QUANTUM INFORMATION SCIENCE

Exercise Set 1
HT 2020

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1.1. **Omnipresent Wolfgang Pauli and his ubiquitous matrices.** The three Pauli matrices $\sigma_1 \equiv \sigma_x \equiv X$, $\sigma_2 \equiv \sigma_y \equiv Y$, and $\sigma_3 \equiv \sigma_z \equiv Z$, here supplemented by the identity matrix $\sigma_0 \equiv \mathbb{1}$, are written in the standard basis $\{|0\rangle, |1\rangle\}$ as

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

BIT FLIP   PHASE FLIP

The Pauli matrices are unitary as well as Hermitian. They square to the identity

$$X^2 = Y^2 = Z^2 = \mathbb{1}.$$

They anticommute

$$XY + YX = 0,$$
$$XZ + ZX = 0,$$
$$YZ + ZY = 0,$$

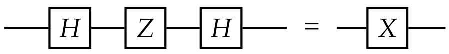
and satisfy

$$XY = iZ$$

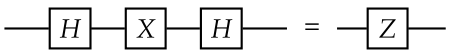
(and cyclic permutations). Their trace is zero and their determinant is -1.

(a) Find eigenvalues and eigenvectors of the three Pauli matrices.

(b) The two Pauli gates, X and Z, are often referred to as the bit flip and the phase flip respectively; we will use this terminology later on, when we discuss quantum error correction. Show that the Hadamard gate $H = \frac{1}{\sqrt{2}}(X + Z)$ turns phase flips into bit flips, $HZH = X$,



and bit flips into phase flips $HXH = Z$,



The set of complex $N \times N$ matrices form a Hilbert space with the inner product $(A|B) = \frac{1}{N} \mathrm{Tr}\, A^\dagger B$. This inner product is often called the *Hilbert-Schmidt product*.

(c) Show that the identity and the three Pauli matrices form an orthonormal basis, with respect to the Hilbert-Schmidt product, in the space of complex 2×2 matrices. Given that any 2×2 matrix A can be written in this basis as,

Here vector $\vec{a}$ has components a_x, a_y, a_z and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

$$A = a_0 \mathbb{1} + \vec{a} \cdot \vec{\sigma} \equiv a_0 \mathbb{1} + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z,$$

show that the coefficients a_k are given by the inner products $a_k = (\sigma_k|A) = \frac{1}{2} \mathrm{Tr}\, \sigma_k A$. If A is Hermitian then these coefficients are real numbers. Why?

1.2. **Mutually unbiased bases.** Mutually unbiased bases in a Hilbert space of dimension d are two orthonormal bases $\{|e_1\rangle, \ldots, |e_d\rangle\}$ and $\{|f_1\rangle, \ldots, |f_d\rangle\}$ such that

$$|\langle e_j | f_k \rangle|^2 = \frac{1}{d} \quad \text{for any basis states } |e_j\rangle \text{ and } |f_k\rangle$$

How many mutually unbiased bases one can find in a Hilbert space of an *arbitrary* dimension d? If you can answer this question for any d don't bother with handing in this classwork, just write it down and sent it to Nature or Physical Review Letter. Eternal fame awaits you for right now nobody knows the answer.

These bases are unbiased in the following sense: if a system is prepared in a state belonging to one of the bases, then all outcomes of the measurement with respect to the other basis will occur with equal probabilities. Show that eigenvectors of the three Pauli matrices give three mutually unbiased bases in a Hilbert space of dimension 2.

1.3. **How to ascertain the values of σ_x and σ_y of a qubit.** Alice prepares a qubit in any state of her choosing and gives it to Bob who secretly measures either σ_x or σ_y. The outcome of the measurement is seen only by Bob. Alice has no clue which measurement was chosen by Bob but right after his measurement she gets her qubit back and she can measure it as well. Some time later Bob tells Alice which of the two measurements was chosen, i.e. whether he measured σ_x or σ_y. Alice then tells him the outcome he obtained in his measurement. Bob is surprised for the two measurements have mutually unbiased bases and yet Alice always gets it right. How does she do it?

This is a simplified version of a beautiful quantum puzzle proposed in 1987 by Lev Vaidman, Yakir Aharonov, and David Z. Albert in a paper with the somewhat provocative title, "How to ascertain the values of σ_x, σ_y, and σ_z of a spin-$\frac{1}{2}$ particle." For the original see Phys. Rev. Lett. vol. 58, 1385 (1987).

1.4. **Tensor products of Pauli operators.** Show that the tensor product matrices $X \otimes X$ and $Z \otimes Z$ commute. What are the common eigenvectors of the two matrices?

An n-qubit Pauli operator is a tensor product of n Pauli operators $(X, Y, Z, \mathbb{1})$ with pre-factor $+1$ or -1. Show that n-qubit Pauli operators have the following properties

(a) They are both unitary and Hermitian.
(b) Any two operators either commute or anticommute.
(c) They have eigenvalues ± 1.
(d) They are self-inverse.

Consider the following two 10-qubit Pauli operators

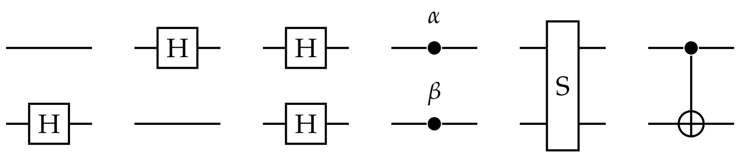
$$X \otimes X \otimes \mathbb{1} \otimes Z \otimes \mathbb{1} \otimes Y \otimes Z \otimes \mathbb{1} \otimes Z \otimes Z$$
$$X \otimes Y \otimes X \otimes \mathbb{1} \otimes \mathbb{1} \otimes Y \otimes X \otimes \mathbb{1} \otimes Z \otimes X$$

Do they commute or anticommute? There is a simple rule that allows to answer this question immediately, without any algebra. Can you see it?

1.5. **Two qubit operations.** The circuits below show six unitary operations on two qubits,

$$P(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

The square root of SWAP matrix has something in common with the square root of NOT. Start with writing the SWAP matrix.



The first four are described, respectively, by 4×4 unitary matrices which are tensor products $\mathbb{1} \otimes H$, $H \otimes \mathbb{1}$, $H \otimes H$ and $P(\alpha) \otimes P(\beta)$. The matrices of the two remaining gates, known as the square root of SWAP and controlled-NOT, stand out as they do not admit a tensor product decomposition in terms of single-qubit operations. Use the standard tensor product basis, $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, and write down unitary matrices for each of the six gates.

1.6. **Unitary evolution vs Hamiltonian for independent subsystems.** If subsystem $\mathcal{S}_1$ undergoes a unitary transformation U_1 and subsystem $\mathcal{S}_2$ undergoes a transformation U_2, then the overall unitary evolution is described by the operator $U_1 \otimes U_2$. Now, suppose that both subsystems evolve continuously in time and are characterised by the Hamiltonians H_1 and H_2. What is the overall Hamiltonian?

Hint: Over an infinitely small time interval dt, the subsystems evolve by $U_1 = \mathbb{1} - iH_1 dt$ and $U_2 = \mathbb{1} - iH_2 dt$, respectively.

1.7. **Basic entanglement.** Prove that the state of two qubits $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ is entangled iff $ad - bc \neq 0$. Deduce that the state $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + (-1)^k|11\rangle)$ is entangled for $k = 1$ and unentangled for $k = 0$. Express the latter case explicitly as a product state.

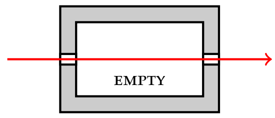
1.8. **Partial measurement.** Consider the three qubit state

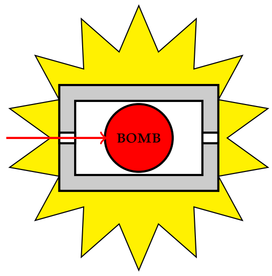
$$|\psi\rangle = \frac{i}{2}|000\rangle + \frac{12+5i}{26}|001\rangle - \frac{1}{2}|101\rangle + \frac{3}{10}|110\rangle - \frac{2i}{5}|111\rangle .$$

We make the standard measurement on the first qubit. What is the probability of seeing outcome 1 and what is the state of the three qubits right after the measurement?

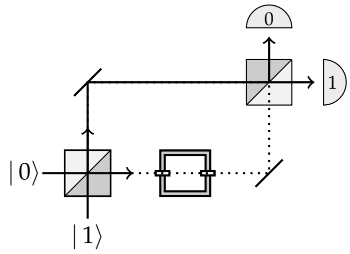
1.9. **The Quantum Bomb Tester.** You have been drafted by the government to help in the demining effort in a former war-zone. In particular, retreating forces have left very sensitive bombs in some of the sealed rooms. The bombs are configured such that if even one photon of light is absorbed by the fuse (i.e. if someone looks into the room), the bomb will go off. Each room has an input and output port which can be hooked up to external devices. An empty room will let light go from the input to the output ports unaffected, whilst a room with a bomb will explode if light is shone into the input port and the bomb absorbs even just one photon. Your task is to find a way of determining whether a room has a bomb in it without blowing it up, so that specialised (limited and expensive) equipment can be devoted to defusing that particular room. You would like to know with certainty whether a particular room had a bomb in it.

This is a slightly modified version of a bomb testing problem described by Avshalom Elitzur and Lev Vaidman in *Quantum-mechanical interaction-free measurement*, Found. Phys. **47**, 987-997 (1993).





Hint: Consider the setup where the input and output ports are hooked up in one of the arms of a Mach-Zehnder interferometer.



(1) To start with, consider the setup (see the margin) where the input and output ports are hooked up in the lower arm of a Mach-Zehnder interferometer (with symmetric beam splitters).
   (i) Assume an empty room. Send a photon to input port $|0\rangle$. Which detector, at the output port, will register the photon?
   (ii) Now assume that the room does contain a bomb. Again, send a photon to input port $|0\rangle$. Which detector will register the photon and with which probability?
   (iii) Design a scheme that allows you – at least part of the time – to decide whether a room has a bomb in it without blowing it up. If you iterate the procedure, what is its overall success rate for the detection of a bomb without blowing it up?

(2) Assume that the two beam splitters in the interferometer are different. Say the first beamsplitter reflects incoming light with probability r and transmits with probability $t = 1 - r$ and the second one transmits with probability r and reflects with probability t. Would the new setup improve the overall success rate of the detection of a bomb without blowing it up?

(3) There exists a scheme, involving many beamsplitters and something called "quantum Zeno effect", such that the success rate for detecting a bomb without blowing it up approaches 100%. Try to work it out or find a solution on internet.