Solutions for Problem Sheet 3

1 Partial traces and reduced density operators

1) Starting with $|\psi\rangle = \frac{1}{\sqrt{6}}\left(\sqrt{2}\,|10\rangle + |11\rangle + \sqrt{2}\,|00\rangle - |01\rangle\right)$, the density matrix $|\psi\rangle\langle\psi|$ can be written as

$$\begin{aligned}
&\frac{1}{6}\Big(|0\rangle\langle 0| \otimes (\sqrt{2}\,|0\rangle - |1\rangle)(\sqrt{2}\,\langle 0| - \langle 1|) + |1\rangle\langle 1| \otimes (\sqrt{2}\,|0\rangle + |1\rangle)(\sqrt{2}\,\langle 0| + \langle 1|)\Big) \\
+&\frac{1}{6}\Big(|0\rangle\langle 1| \otimes (\sqrt{2}\,|0\rangle - |1\rangle)(\sqrt{2}\,\langle 0| + \langle 1|) + |1\rangle\langle 0| \otimes (\sqrt{2}\,|0\rangle + |1\rangle)(\sqrt{2}\,\langle 0| - \langle 1|)\Big)
\end{aligned}$$

expanded out is

$$\begin{aligned}
& \tfrac{1}{3}(|10\rangle\langle 10| + |10\rangle\langle 00| + |00\rangle\langle 00| + |00\rangle\langle 10|) \\
+ \quad & \tfrac{\sqrt{2}}{6}(|10\rangle\langle 11| - |10\rangle\langle 01| + |00\rangle\langle 11| - |00\rangle\langle 01| + |11\rangle\langle 10| + |11\rangle\langle 00| - |01\rangle\langle 10| - |01\rangle\langle 00|) \\
+ \quad & \tfrac{1}{6}(|11\rangle\langle 11| - |11\rangle\langle 01| + |01\rangle\langle 01| - |01\rangle\langle 11|),
\end{aligned}$$

so in matrix form is

$$\begin{bmatrix}
\frac{1}{3} & -\frac{\sqrt{2}}{6} & \frac{1}{3} & \frac{\sqrt{2}}{6} \\
-\frac{\sqrt{2}}{6} & \frac{1}{6} & -\frac{\sqrt{2}}{6} & -\frac{1}{6} \\
\frac{1}{3} & -\frac{\sqrt{2}}{6} & \frac{1}{3} & \frac{\sqrt{2}}{6} \\
\frac{\sqrt{2}}{6} & -\frac{1}{6} & \frac{\sqrt{2}}{6} & \frac{1}{6}
\end{bmatrix}.$$

2) Taking the partial trace with respect to the first party we get

$$\begin{aligned}
\langle 0| \otimes \mathbb{I}\,(|\psi\rangle\langle\psi|)\,|0\rangle \otimes \mathbb{I} + \langle 1| \otimes \mathbb{I}\,(|\psi\rangle\langle\psi|)\,|1\rangle \otimes \mathbb{I} &= \frac{1}{6}\Big((\sqrt{2}\,|0\rangle - |1\rangle)(\sqrt{2}\,\langle 0| - \langle 1|) + (\sqrt{2}\,|0\rangle + |1\rangle)(\sqrt{2}\,\langle 0| + \langle 1|)\Big) \\
&= \frac{2}{3}\,|0\rangle\langle 0| + \frac{1}{3}\,|1\rangle\langle 1|,
\end{aligned}$$

which in matrix form is

$$\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

Taking the partial trace with respect to the second party we get

$$\begin{aligned}
\mathbb{I} \otimes \langle 0|\,(|\psi\rangle\langle\psi|)\,\mathbb{I} \otimes |0\rangle + \mathbb{I} \otimes \langle 1|\,(|\psi\rangle\langle\psi|)\,\mathbb{I} \otimes |1\rangle &= \frac{1}{3}\,(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&+ \frac{1}{6}\,(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&= \frac{2}{3}\,|+\rangle\langle +| + \frac{1}{3}\,|-\rangle\langle -|,
\end{aligned}$$

which in matrix form is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix}.$$

2 Trace distance

1) By the spectral decomposition, $A = \sum_i \lambda_i |i\rangle \langle i|$, thus $A^\dagger A = \sum_i |\lambda_i|^2 |i\rangle \langle i|$, and thus $\sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle \langle i|$. The trace of an operator is also the sum of the operator's eigenvalues, thus the trace norm of A is $\sum_i |\lambda_i|$. A density matrix only has positive eigenvalues, and it has trace 1, so the sum of the absolute value of the eigenvalues is 1.

2 updated) For pure states, we want to find the eigenvalues of $|\psi\rangle \langle\psi| - |\phi\rangle \langle\phi|$. To find the eigenvalues of this, we need only consider the support of this rank two operator, i.e. vectors in span$\{|\psi\rangle, |\phi\rangle = \alpha |\psi\rangle + \beta |\theta\rangle\}$ where $\langle\theta| \psi\rangle = 0$ and $\alpha = \langle\psi| \phi\rangle$. Therefore, we can reduce the operator $|\psi\rangle \langle\psi| - |\phi\rangle \langle\phi|$ to a 2-by-2 matrix:

$$\begin{bmatrix} 1-|\alpha|^2 = |\beta|^2 & -\beta^*\alpha \\ -\alpha^*\beta & -|\beta|^2 \end{bmatrix},$$

where the rows and columns are labelled by the vectors $|\psi\rangle$, $|\theta\rangle$. So we need to solve the following quadratic:

$$\lambda^2 = (|\alpha|^2 + |\beta|^2)|\beta|^2 = |\beta|^2 = 1 - |\langle\psi| \phi\rangle|^2, \tag{1}$$

which has the solutions

$$\lambda = \pm\sqrt{1 - |\langle\psi| \phi\rangle|^2}, \tag{2}$$

thus the trace distance is $\sqrt{1 - |\langle\psi| \phi\rangle|^2}$.

2) Clearly $d(\rho, \rho) = 0$ since it is the trace norm of the null operator. To prove that $d(\rho_1, \rho_2) = d(\rho_2, \rho_1)$, observe that $(\rho_i^\dagger - \rho_j^\dagger)(\rho_i - \rho_j) = (\rho_i - \rho_j)(\rho_i - \rho_j) = \rho_i^2 + \rho_j^2 - \rho_i\rho_j - \rho_j\rho_i$, which is symmetric under exchange of i and j. Thus $\|\rho_i - \rho_j\|_{tr} = \|\rho_j - \rho_i\|_{tr}$. To show that $d(\rho_1, \rho_3) \leq d(\rho_1, \rho_2) + d(\rho_2, \rho_3)$, observe that

$$d(\rho_1, \rho_3) = \|\rho_1 - \rho_3\|_{tr} = \frac{1}{2}\|\rho_1 - \rho_2 + \rho_2 - \rho_3\|_{tr} \leq \frac{1}{2}\|\rho_1 - \rho_2\|_{tr} + \frac{1}{2}\|\rho_2 - \rho_3\|_{tr} = d(\rho_1, \rho_2) + d(\rho_2, \rho_3),$$

since the trace norm will satisfy the triangle inequality. I think it's fine if students say this alone. However, for completeness, here's a proof of the triangle inequality $\|A + B\|_{tr} \leq \|A\|_{tr} + \|B\|_{tr}$. First observe that we can rewrite the operator A in terms of the singular value decomposition $U\Lambda V^\dagger$ where Λ is a diagonal matrix that has the absolute values of the eigenvalues of A on the diagonal, and U and V are unitaries. Now define $Q := UIV^\dagger$, where I is the identity matrix, thus Q has a singular value decomposition with singular values all being 1. Consider

$$\mathrm{Tr}(Q^\dagger A) = \mathrm{Tr}(VU^\dagger U\Lambda V^\dagger) = \mathrm{Tr}(\Lambda) = \|A\|_{tr} \leq \sup_{\sigma_1(P)\leq 1}\mathrm{Tr}(P^\dagger A), \tag{3}$$

where $\sigma_1(P)$ is the largest singular value of P. We can now prove that $\sup_{\sigma_1(P)\leq 1}\mathrm{Tr}(P^\dagger A) \leq \|A\|_{tr}$, and thus that they are equal:

$$\sup_{\sigma_1(P)\leq 1}\mathrm{Tr}(P^\dagger A) = \sup_{\sigma_1(P)\leq 1}\mathrm{Tr}(P^\dagger U\Lambda V^\dagger) = \sup_{\sigma_1(P)\leq 1}\mathrm{Tr}(V^\dagger P^\dagger U\Lambda) = \sum_i [V^\dagger P^\dagger U]_{ii}\Lambda_i \leq \sum_i \sigma_1(P)\Lambda_i = \sum_i \Lambda_i, \tag{4}$$

where Λ_i are the diagonal elements of Λ. Now this then implies that

$$\|A + B\|_{tr} = \sup_{\sigma_1(P)\leq 1}\mathrm{Tr}(P^\dagger(A + B)) \leq \sup_{\sigma_1(P)\leq 1}\mathrm{Tr}(P^\dagger A) + \sup_{\sigma_1(Q)\leq 1}\mathrm{Tr}(Q^\dagger B) = \|A\|_{tr} + \|B\|_{tr}. \tag{5}$$

A gold star to the students if they prove this!

3 Distinguishing two states

1) The states ρ_1 and ρ_2 commute and so $\rho_1 = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ and $\rho_2 = \sum_i q_i |\psi_i\rangle \langle\psi_i|$. So the trace norm of $\rho_1 - \rho_2$ is the sum of the absolute values $|p_i - q_i|$. The optimal measurement will just be to observe the system in the basis $\{|\psi_i\rangle\}$, and if $p_i > q_i$ then guess ρ_1, if $p_i < q_1$ guess ρ_2, and otherwise just randomly output a state. The probability of success when $\rho = \rho_1$ is p_i, and the probability of success when $\rho = \rho_2$ is q_i, with there being equal probability of getting either state. So the overall success is $\sum_i \frac{1}{2}\max(p_i, q_i)$, which can be rewritten as $\sum_i \frac{1}{4}(p_i + q_i + |p_i - q_i|) = \frac{1}{2}(1 + d(\rho_1, \rho_2))$, and is thus optimal.

2) We need to find the absolute values of the eigenvalues of $2/3(|0\rangle\langle 0| - |+\rangle\langle +|) + 1/3(|1\rangle\langle 1| - |-\rangle\langle -|)$, which is the operator

$$\frac{1}{6}(|0\rangle\langle 0| - |0\rangle\langle 1| - |0\rangle\langle 1| - |1\rangle\langle 1|), \tag{6}$$

and has eigenvalues $\{\pm\frac{1}{3\sqrt{2}}\}$, and so $d(\rho_1, \rho_2) = \frac{1}{3\sqrt{2}}$, and the probability is thus $\frac{1}{2} + \frac{1}{6\sqrt{2}}$.

4 Bloch vectors

1) Since the Paulis are traceless, the trace of this operator is $\text{Tr}\{\frac{1}{2}I\} = 1$. It is Hermitian, since it is the real linear combination of Hermitian matrices. Finally, we can show it's positive by considering its eigenvalues:

$$\frac{1}{2}(I + \vec{s}\cdot\vec{\sigma})|\psi\rangle = \frac{1}{2}(|\psi\rangle + \vec{s}\cdot\vec{\sigma}|\psi\rangle) = \lambda|\psi\rangle. \tag{7}$$

The eigenvalues of $\vec{s}\cdot\vec{\sigma}$ are obtained from explicit calculations as $\pm\sqrt{s_x^2+s_y^2+s_z^2}$, thus the eigenvalues of ρ are $\frac{1}{2}\left(1\pm\sqrt{s_x^2+s_y^2+s_z^2}\right)$. Since $\sqrt{s_x^2+s_y^2+s_z^2}\leq 1$, then ρ is definitely positive.

2) $d(\rho_1, \rho_2) = \frac{1}{4}\|I - I + \vec{s}_1\cdot\vec{\sigma} - \vec{s}_2\cdot\vec{\sigma}\|_{tr}$, resulting in the matrix

$$\frac{1}{4}\begin{bmatrix} s_1^z - s_2^z & (s_1^x - s_2^x) - i(s_1^y - s_2^y) \\ s_1^x - s_2^x + i(s_1^y - s_2^y) & s_2^z - s_1^z \end{bmatrix},$$

thus

$$\begin{aligned} \lambda^2 &= \frac{1}{16}\left[(s_1^z - s_2^z)^2 + (s_1^x - s_2^x + is_1^y - is_2^y)(s_1^x - s_2^x - is_1^y + is_2^y)\right] \\ &= \frac{1}{16}\left[(s_1^z - s_2^z)^2 + (s_1^y - s_2^y)^2 + (s_1^x - s_2^x)^2\right] \end{aligned}$$

So the trace distance is

$$\frac{1}{2}\sqrt{(s_1^z - s_2^z)^2 + (s_1^y - s_2^y)^2 + (s_1^x - s_2^x)^2} \tag{8}$$

5 Completely positive maps

1) To prove positivity, the density matrix ρ is positive, so it can be written as $\rho = M^\dagger M$ for a matrix M. Therefore, $A_k\rho A_k^\dagger = A_k M^\dagger M A_k^\dagger = N^\dagger N$ for $N = MA_k^\dagger$, and thus is positive. A sum of positive matrices is again positive, so the map is a positive map. To prove that it preserves trace, we have (by linearity and cyclicity of the trace) $\sum_k \text{Tr}(A_k\rho A_k^\dagger) = \sum_k \text{Tr}(A^\dagger A_k\rho) = \text{Tr}\rho = 1$. Two compose to completely positive maps, we have $\sum_{kl} B_l A_k \rho A_k^\dagger B_l^\dagger$ where $\sum_l B_l^\dagger B_l = I$, thus it defines a new CP map with matrices $C_{kl} := B_l A_k$, and notice that $\sum_{kl} C_{kl}^\dagger C_{kl} = \sum_k A_k^\dagger(\sum_l B_l^\dagger B_l)A_k = \sum_k A_k^\dagger A_k = I$.

2) Given the Bloch representation, we can consider how each Pauli components evolve under conjugation of Paulis. In particular, $\sigma_i\sigma_j\sigma_i = (-1)\sigma_j$ if $i \neq j$, and σ_i otherwise. This results in the new density matrix:

$$\frac{1}{2}(I - (1 - \frac{4p}{3})\left[s_x\sigma_x + s_y\sigma_y + s_z\sigma_z\right]), \tag{9}$$

where all components s_x, s_y and s_z are rescaled by $(1 - \frac{4p}{3})$.

6 Positive but not completely maps

1) We can straightforwardly see that $s_i \to -s_i$, i.e. a reflection with respect to every axis.

2) The density matrix after applying the map is still trace 1 since it only affects the Pauli terms in the Bloch representation. The Bloch vector after applying the map satisfies $\sqrt{(-s_x)^2 + (-s_y)^2 + (-s_z)^2} = \sqrt{(s_x)^2 + (s_y)^2 + (s_z)^2} =$

1, and thus results in another density matrix.

3) The density matrix after applying $\mathcal{N}$ to the first qubit is

$$\frac{1}{4}\left(I \otimes I - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right). \tag{10}$$

Now take the maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and apply the density matrix to this state. We get:

$$\frac{1}{4\sqrt{2}}\left(I \otimes I - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)(|00\rangle + |11\rangle) = \frac{1}{4}(\left|\Phi^+\right\rangle - \left|\Phi^+\right\rangle - \left|\Phi^+\right\rangle - \left|\Phi^+\right\rangle) = \frac{-1}{2}\left|\Phi^+\right\rangle, \tag{11}$$

which then implies the resulting density matrix is no longer positive, and so the map is not completely positive.

7 Approximate cloning

1) We start with two input states $|\phi\rangle$ and $|\psi\rangle$. Before the cloner the inner product between $|\phi, 0, R\rangle$ and $|\psi, 0, R\rangle$ is $\langle\phi\,|\psi\rangle$. Now assume without loss of generality that the cloner is a unitary (by enlarging the ancilla if necessary). After the cloner, the inner product between $|\phi, \phi, R'\rangle$ and $|\psi, \psi, R''\rangle$ is $\langle\phi\,|\psi\rangle^2 \langle R'\,|R''\rangle$. Since a unitary preserves the inner product between vectors, then $\langle\phi\,|\psi\rangle^2 \langle R'\,|R''\rangle = \langle\phi\,|\psi\rangle$ if $|\psi\rangle$ and $|\phi\rangle$ are at least identical or orthogonal. Therefore, for two arbitrary states we does not hold.

2) The reduced density matrices of the first and second qubits will be the same since the overall state is symmetric under exchange of the first two qubits.

3) The reduced state of the first two qubits is

$$\frac{2}{3}|\psi\psi\rangle\langle\psi\psi| + \frac{1}{6}\left(\left|\psi\psi^\perp\right\rangle + \left|\psi^\perp\psi\right\rangle\right)\left(\left\langle\psi\psi^\perp\right| + \left\langle\psi^\perp\psi\right|\right). \tag{12}$$

Now if we take the partial trace with respect to the first party we get

$$\frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{6}|\psi\rangle\langle\psi| + \frac{1}{6}\left|\psi^\perp\right\rangle\left\langle\psi^\perp\right|. \tag{13}$$

4) The test is the projective measurement in the basis $\{|\psi\rangle, \left|\psi^\perp\right\rangle\}$, so it will pass with probability $5/6$.

5) The Bloch vector of ρ is in the same direction as $|\psi\rangle\langle\psi|$ but shrunk by a factor of $2/3$. This can be seen by writing the matrix as $\frac{5}{12}(I + a\cdot\sigma) + \frac{1}{12}(I - a\cdot\sigma)$, which when expanded can be written as

$$\frac{1}{2}(I + \frac{2}{3}a\cdot\sigma), \tag{14}$$

and we thus see the shrinking of $\frac{2}{3}$.

8 Controlled unitaries revisited

1) The computation goes as $|0u\rangle \to |+u\rangle \to \frac{1}{\sqrt{2}}\left(|0u\rangle + e^{i\alpha}|1u\rangle\right) \to \frac{1}{2}\left((1+e^{i\alpha})|0u\rangle + (1-e^{i\alpha})|1u\rangle\right)$. Therefore, the probability of getting outcome 0 is $\frac{2+e^{i\alpha}+e^{-i\alpha}}{4} = \frac{1}{2}(1+\cos(\alpha))$.

2) If we now take the linear extension of the above for the state $|\psi\rangle$, at the end of the controlled unitary we have the state

$$\frac{1}{\sqrt{2}}\left(|0\psi\rangle + |1\rangle U|\psi\rangle\right). \tag{15}$$

The state after the final Hadamard is

$$\frac{1}{2}\left(|0\rangle\left(I+U\right)|\psi\rangle+|1\rangle\left(I-U\right)|\psi\rangle\right), \tag{16}$$

and the probability of getting outcome 0 is then $|\frac{1}{2}\langle\psi|\,(I+U^{\dagger})(I+U)\,|\psi\rangle\,|^{2} = \frac{1}{2}(1+\frac{\langle\psi|U+U^{\dagger}|\psi\rangle}{2})$. Note that $\langle\psi|\,U\,|\psi\rangle = \langle\psi|\,\phi\rangle = xe^{ip}$ and $\langle\psi|\,U^{\dagger}\,|\psi\rangle = \langle\phi|\,\psi\rangle = \langle\psi|\,\phi\rangle^{*} = xe^{-ip}$. Therefore the probability is

$$\frac{1}{2}(1+\frac{x(e^{-ip}+e^{ip})}{2}) = \frac{1}{2}(1+x(cos(p))), \tag{17}$$

therefore, $ve^{i\phi} = xe^{ip} = \langle\psi|\,U\,|\psi\rangle$.

3) Starting with $\text{Tr}(\rho U) = \sum_k \text{Tr}(p_k|u_k\rangle\langle u_k|U) = \sum_k \text{Tr}(e^{i\phi_k}p_k v_k|u_k\rangle\langle u_k|)$. Therefore, we have

$$\sum_k \text{Tr}(e^{i\phi_k}p_k v_k|u_k\rangle\langle u_k|) = \sum_k e^{i\phi_k}p_k v_k,$$

which is a complex number, so it can be written as $ve^{i\phi}$.

4) First observe that the expectation value of the Pauli-Z matrix on the first qubit is $\frac{1}{2}(1+v\cos\phi-(1-v\cos\phi)) = v\cos\phi = \text{Re}(ve^{i\phi})$. Therefore, given the circuit we can estimate the real part of $ve^{i\phi}$, which is the real part of $\text{Tr}(\rho U)$. To estimate $ve^{i\phi}$, we need to find the imaginary part of it. This can be achieved by replacing the final Hadamard and measurement in the circuit (together it evaluates a measurement in the Pauli-X basis) with a Pauli-Y measurement. After the controlled unitary we have the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\psi\rangle+|1\rangle\,U\,|\psi\rangle\right), \tag{18}$$

as before. The expectation of the Pauli-Y is then

$$\begin{aligned}
\frac{1}{2}\left(\langle 0u|+\langle 1u|\,U^{\dagger}\right)\sigma_y\left(|0u\rangle+U\,|1u\rangle\right) &= \frac{1}{2}\left(-i\,\langle\psi|\,U\,|\psi\rangle+i\,\langle\psi|\,U^{\dagger}\,|\psi\rangle\right)\\
&= \frac{1}{2}\left(-ive^{i\phi}+ive^{-i\phi}\right)\\
&= i^2 v\frac{\left(-e^{i\phi}+e^{-i\phi}\right)}{2i}\\
&= v\sin\phi,
\end{aligned}$$

which is the imaginary part of $ve^{i\phi}$. By the same calculation as before, we can generalise this to the case of density matrices, to get the imaginary part of $\text{Tr}(\rho U)$.

To estimate $\text{Tr}(U)$, we can take ρ to be the maximally mixed state. Then we just multiply the expectations from before by a constant d, which is the dimension of the Hilbert space.

9 Deutsch's algorithm and decoherence

After the query of the oracle, the state is

$$\frac{1}{2}\sum_x(-1)^{f(x)}\,|e_x,x,0\rangle-|e_x,x,1\rangle\,, \tag{19}$$

where e_x labels the state of the environment. Then applying the final Hadamard leads to

$$\begin{aligned}
\frac{1}{2}H\sum_x(-1)^{f(x)}\,|e_x,x\rangle\left(|0\rangle-|1\rangle\right) &= \frac{1}{2\sqrt{2}}\sum_x(-1)^{f(x)}\,|e_x\rangle\sum_y(-1)^{x.y}\,|y\rangle\left(|0\rangle-|1\rangle\right)\\
&= \frac{1}{2\sqrt{2}}\left(\sum_{x,y}(-1)^{f(x)+x.y}\,|e_x,y\rangle\right)\left(|0\rangle-|1\rangle\right).
\end{aligned}$$

The state of the first two qubits is

$$\frac{1}{2}\sum_{x,y}(-1)^{f(x)+x.y}\left|e_x,y\right\rangle . \tag{20}$$

For getting outcome $|0\rangle$ in the first qubit, we have the restriction of the state being

$$\frac{1}{2}\sum_{x}(-1)^{f(x)}\left|e_x,0\right\rangle . \tag{21}$$

So the probability of getting this outcome is

$$\frac{1}{4}\Big|\sum_{x,x'}(-1)^{f(x)+f(x')}\langle e_{x'}\left|e_x\right\rangle\Big|^2=\frac{1}{4}\left(2+2v(-1)^{f(0)+f(1)}\right), \tag{22}$$

which is equal to $(1/2)(1+v)$ if the function is constant, and $(1/2)(1-v)$ if it is balanced.