Exercise Set 4
HT 2021

INTRODUCTION TO QUANTUM INFORMATION SCIENCE

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4.1. **CP maps revisited.** Any linear transformation (superoperator) T acting on density matrices of a qubit can be completely characterised by its action on the four basis matrices $|a\rangle\langle b|$, where $a, b = 0, 1$, and can be represented as a 4×4 matrix,

$$\tilde{T} = \left[\begin{array}{c|c} T(|0\rangle\langle 0|) & T(|0\rangle\langle 1|) \\ \hline T(|1\rangle\langle 0|) & T(|1\rangle\langle 1|) \end{array}\right].$$

Write down $\tilde{T}$ for:

(1) transposition, $\varrho \mapsto \varrho^T$,
(2) depolarising channel, $\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x\varrho\sigma_x + \sigma_y\varrho\sigma_y + \sigma_z\varrho\sigma_z\right)$, for some $0 \le p \le 1$.

Show that for completely positive maps T matrix $\tilde{T}$ must be positive semidefinite.

4.2. **Quantum error correction.**

(1) Draw a quantum network (circuit) that encodes a single qubit state $\alpha|0\rangle + \beta|1\rangle$ into the state $\alpha|00\rangle + \beta|11\rangle$ of two qubits. Here and in the following α and β are some unknown generic complex coefficients.
(2) Two qubits were prepared in state $\alpha|00\rangle + \beta|11\rangle$, exposed to bit flip-errors, and then measured with an ancillary qubit, as shown in Fig. 1. The result of the measurement is x. Can you infer the absence of errors when $x = 0$? Can you infer the presence of errors when $x = 1$? Can you correct any detected errors?

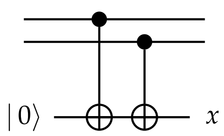


Fig. 1

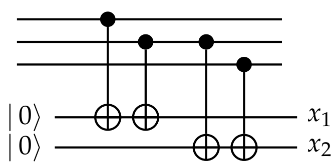
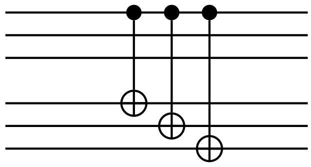


Fig. 2

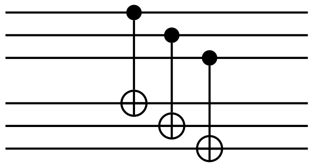
Three qubits were prepared in state $\alpha|000\rangle + \beta|111\rangle$ and then, by mistake, someone applied the Hadamard gate to one of them, but nobody remembers which one. Your task is to recover the original state of the three qubits.

(3) Express the Hadamard gate as the sum of two Pauli matrices. Pick up one of the three qubits and apply the Hadamard gate. How is the state $\alpha|000\rangle + \beta|111\rangle$ modified? Interpret this in terms of bit-flip and phase-flip errors.
(4) You perform the error syndrome measurement shown in Fig. 2. Suppose the outcome of the measurement is $x_1 = 0, x_2 = 1$. How would you recover the original state? Describe the recovery procedure when $x_1 = 0, x_2 = 0$.

The figure below shows two implementations of a controlled-NOT gate acting on the encoded states of the three qubit code.



Implementation A



Implementation B

(5) Assume that the only sources of errors are individual controlled-NOT gates which produce bit-flip errors in their outputs. These errors are independent and occur with a small probability p. For each of the two implementations find the probability of generating unrecoverable errors at the output. Which of the two implementations is fault-tolerant?

4.3. **Stabilisers define vectors and subspaces.**

(1) We say that a unitary S stabilises $|\psi\rangle$ if $S|\psi\rangle = |\psi\rangle$. Show that the set of stabilisers of $|\psi\rangle$ forms a group (known as the stabiliser group).

(2) The n-qubit Pauli group is defined as

$$\mathcal{P}_n = \{\mathbb{1}, X, Y, Z\}^{\otimes n} \otimes \{\pm 1, \pm i\}$$

where X, Y, Z are the Pauli matrices. Each element of $\mathcal{P}_n$ is, up to an overall phase ± 1, $\pm i$, a tensor product of Pauli matrices and identity matrices acting on the n qubits. Show that the elements of the Pauli group either commute or anticommute.

(3) We shall restrict our attention to stabilisers which form Abelian subgroups of $\mathcal{P}_n$ and do not contain the element $-\mathbb{1}$. Explain why all such stabilisers (except the trivial one, i.e. the tensor product of the identities) have trace zero and square to $\mathbb{1}$.

(4) Show that each stabiliser S has the same number of eigenvectors with eigenvalues $+1$ and -1, and hence "splits" the 2^{2n} dimensional Hilbert space in half. How would you describe the action of the two operators $\frac{1}{2}(\mathbb{1} \pm S)$?

(5) Consider two stabiliser generators, S_1 and S_2. Show that eigenvalue $+1$ subspace of S_1 is split again in half by S_2. That is, in that subspace exactly half of the S_2 eigenvectors have eigenvalue $+1$ and the other half -1.

Hint: Show that Tr $\frac{1}{2}(\mathbb{1} + S_1)S_2 = 0$.

(6) If a stabiliser group in the Hilbert space of dimension 2^n has a minimal number of generators, $S_1, \ldots, S_r$, what is dimension of the stabiliser subspace?

(7) State $|0\rangle$ is stabilised by Z and state $|1\rangle$ is stabilised by $-Z$. What are stabiliser generators for the standard basis of two qubits, i.e. for the states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$? What are stabiliser generators for each of the four Bell states?

(8) Construct stabiliser generators for an $n = 3$, $k = 1$ (n physical qubits encoding k logical qubits) code that can correct a single bit flip, i.e. ensure that recovery is possible for any of the errors in the set $\mathcal{E} = \{\mathbb{1}\mathbb{1}\mathbb{1}, X\mathbb{1}\mathbb{1}, \mathbb{1}X\mathbb{1}, \mathbb{1}\mathbb{1}X\}$. Find an orthonormal basis for the two-dimensional code subspace.

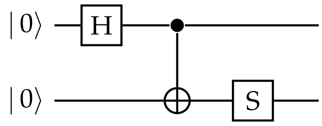
We often drop the tensor product symbol, e.g. $\mathbb{1}X\mathbb{1} \equiv \mathbb{1} \otimes X \otimes \mathbb{1}$

(9) Describe the subspace fixed by the stabiliser generators $X \otimes X \otimes \mathbb{1}$ and $\mathbb{1} \otimes X \otimes X$ and its relevance for quantum error correction.

(10) Let S_1 and S_2 be stabiliser generators for a two qubit state $|\psi\rangle$. The state is modified by a unitary operation U. What are the stabiliser generators for $U|\psi\rangle$?

(11) Step through the circuit

Here S is a phase gate: $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto i|1\rangle$.



writing down quantum states of the two qubits after each unitary operation and their respective stabiliser generators. How would you describe the action of the three gates, H, S and controlled-NOT, in the stabiliser language?

4.4. **Shor's 9-qubit code.** Use 8 stabiliser generators for Shor's 9-qubit code and explain why this code can correct an arbitrary single-qubit error. In fact, it can also correct some multiple-qubit errors. Which of the following errors can be corrected by the nine-qubit code: X_1X_3, X_2X_7 , X_5Z_6 , Z_5Z_6 , Y_2Z_8?

X_i, Y_i, or Z_i represents X, Y, or Z applied to the i-th qubit.