Final Honour School of Mathematics Part C

C7.4 Introduction to Quantum Information
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Model Solutions

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1. The CHSH inequality involves four observables: A_1, A_2, pertaining to qubit $\mathcal{A}$, and B_1, B_2, pertaining to qubit $\mathcal{B}$. Each observable takes values ± 1.

   (a) [4 marks] Consider the local hidden-variable scenario, that is, assume that values ± 1 can be assigned simultaneously to all four observables. The CHSH quantity S is defined as

   $$S = A_1(B_1 - B_2) + A_2(B_1 + B_2).$$

   Explain why any statistical average of S must satisfy $-2 \leqslant \langle S \rangle \leqslant 2$, so that classical correlations are bounded by the CHSH inequality,

   $$|\langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle| \leqslant 2.$$

In quantum theory the observables A_1, A_2, B_1, B_2 become 2×2 Hermitian matrices with two eigenvalues ± 1, and $\langle S \rangle$ becomes the expectation value of the 4×4 CHSH matrix

$$S = A_1 \otimes (B_1 - B_2) + A_2 \otimes (B_1 + B_2).$$

   (b) [3 marks] Explain why for any state $|\psi\rangle$ the expectation value $\langle S \rangle = \langle \psi | S | \psi \rangle$ cannot exceed the largest eigenvalue of S.

The largest eigenvalue (in absolute value) of a Hermitian matrix M, denoted by $||M||$, is a matrix norm and has the following properties (which you may use):

$$||M \otimes N|| = ||M|| \, ||N||, \quad ||MN|| \leqslant ||M|| \, ||N||, \quad ||M + N|| \leqslant ||M|| + ||N||.$$

   (c) [2 marks] Explain why $||A_k|| = 1$ and $||B_l|| = 1$ $(k, l = 1, 2)$, and show that $||S|| \leqslant 4$.

One can, however, derive a tighter bound.

   (d) [5 marks] Show that

   $$S^2 = 4 \, \mathbb{1} \otimes \mathbb{1} + [A_1, A_2] \otimes [B_1, B_2].$$

   (e) [3 marks] Explain why the norm of each of the commutators, $||[A_1, A_2]||$ and $||[B_1, B_2]||$, cannot exceed 2 and why $||S^2|| = ||S||^2$.

   (f) [4 marks] Show that quantum correlations are bounded by the (Tsirelson) inequality

   $$||S|| \leqslant 2\sqrt{2}, \quad \text{which implies} \quad |\langle S \rangle| \leqslant 2\sqrt{2}.$$

   (g) [4 marks] How is this inequality modified when the operators A_1, A_2, B_1, B_2 commute with each other? Provide physical interpretation.

SOLUTION TO QUESTION 1

(a) [4 marks]
$$S = A_1(B_1 - B_2) + A_2(B_1 + B_2).$$
Given that B_1 and B_2 have definite values, one of the terms $(B_1 + B_2)$ or $(B_1 - B_2)$ must be equal to 0 and the other one ± 2. Thus S can only take two values: $S = \pm 2$.

(b) [3 marks] S, as a Hermition operator has spectral decomposition,
$$S = \sum_k \lambda_k |s_k\rangle\langle s_k| ,$$
where λ_k are real eigenvalues with corresponding eigenvector $|s_k\rangle$. For the expectation value,
$$\langle S\rangle := \langle\psi|S|\psi\rangle = \sum_k \lambda_k |\langle\psi|s_k\rangle|^2 = \sum_k \lambda_k p_k,$$
we find
$$\lambda_{\min} \leqslant \langle S\rangle \leqslant \lambda_{\max},$$
where $\lambda_{\min}, \lambda_{\max}$ denotes the smallest and largest eigenvalue of S, respectively. The bounds are due to the fact that for the expectation value the eigenvalues of S are added with statistical weights ($p_k \geqslant 0$, $\sum_k p_k = 1$).

(c) [2 marks] The A_k and B_l operators have two eigenvalues ± 1, so $||A_k|| = ||B_l|| = 1$.
$$\begin{aligned} ||S|| &\leqslant ||A_1 \otimes (B_1 - B_2)|| + ||A_2 \otimes (B_1 + B_2)|| = ||A_1||\ ||B_1 - B_2|| + ||A_2||\ ||B_1 + B_2|| \\ &\leqslant ||B_1|| + ||B_2|| + ||B_1|| + ||B_2|| = 4. \end{aligned}$$

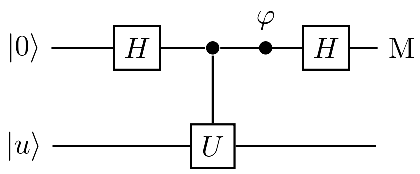
(d) [5 marks]
$$\begin{aligned} S^2 &= (A_1 \otimes (B_1 - B_2) + A_2 \otimes (B_1 + B_2))^2 = \\ &= A_1^2 \otimes (B_1 - B_2)^2 + A_2^2 \otimes (B_1 + B_2)^2 + A_1A_2 \otimes (B_1 - B_2)(B_1 + B_2) + \\ &+ A_2A_1 \otimes (B_1 + B_2)(B_1 - B_2) = \\ &= \mathbb{I} \otimes (B_1 - B_2)^2 + \mathbb{I} \otimes (B_1 + B_2)^2 + A_1A_2 \otimes (B_1^2 - B_2^2 + B_1B_2 - B_2B_1) + \\ &+ A_2A_1 \otimes (B_1^2 - B_2^2 - B_1B_2 + B_2B_1) = \\ &= \mathbb{I} \otimes (B_1^2 + B_2^2 - B_1B_2 - B_2B_1) + \mathbb{I} \otimes (B_1^2 + B_2^2 + B_1B_2 + B_2B_1) + \\ &+ A_1A_2 \otimes (B_1B_2 - B_2B_1) - A_2A_1 \otimes (B_1B_2 - B_2B_1) = \\ &= 4\mathbb{I} \otimes \mathbb{I} + [A_1, A_2] \otimes [B_1, B_2]. \end{aligned}$$

(e) [3 marks]
$$||A_1A_2 - A_2A_1|| \leqslant ||A_1A_2|| + ||A_2A_1|| \leqslant ||A_1||\ ||A_2|| + ||A_2||\ ||A_1|| = 1 + 1 = 2.$$
For every Hermitian operator the norm of S is $|\lambda_{\max}|$ and the norm of S^2 is $|\lambda_{\max}|^2$.

(f) [4 marks]
$$||S^2|| = ||S||^2 \leqslant ||4\mathbb{I} \otimes \mathbb{I}|| + ||[A_1, A_2] \otimes [B_1, B_2]|| = 4 + 2 \cdot 2 = 8.$$
Therefore:
$$||S|| \leqslant \sqrt{8} = 2\sqrt{2}.$$

(g) [4 marks] When the operators commute, $S^2 = 4\mathbb{I} \otimes \mathbb{I}$ and $||S|| = 2$. Therefore $|\langle S\rangle| \leqslant 2$. When A_1, A_2, B_1, B_2 commute, we can assign numerical values to all of them and therefore we have the classical scenario.

2. Consider the following quantum network composed of the two Hadamard gates, one phase gate φ , one controlled-U operation and the measurement M in the standard basis,



The top horizontal line represents a qubit and the bottom one an auxiliary physical system.

(a) [4 marks] Suppose $|u\rangle$ is an eigenvector of U, such that $U|u\rangle = e^{i\alpha}|u\rangle$. Step through the execution of this network, writing down quantum states of the qubit and the auxiliary system after each computational step. In measurement M outcome 0 is registered with probability $\Pr(0)$. What is this probability?

(b) [5 marks] Show that for any pure state $|u\rangle$ the probability $\Pr(0)$ can be expressed as

$$\Pr(0) = \frac{1}{2}\left[1 + \mathrm{Re}\left(\, e^{i\varphi}\langle u|\, U\, |u\rangle\right)\right].$$

(c) [6 marks] Suppose the auxiliary system is prepared in a mixed state described by the density operator ρ,

$$\rho = p_1\, |u_1\rangle\langle u_1| + p_2\, |u_2\rangle\langle u_2| + ... + p_n\, |u_n\rangle\langle u_n|\,,$$

where vectors $|u_k\rangle$ form an orthonormal basis, $p_k \geqslant 0$ and $\sum_{k=1}^{n} p_k = 1$. Show that

$$\Pr(0) = \frac{1}{2}\left[1 + \mathrm{Re}\left(\, e^{i\varphi}\,\mathrm{Tr}(U\rho)\right)\right]. \tag{1}$$

(d) [5 marks] Suppose you have control over the phase gate φ and you can prepare any state of the auxiliary system ρ, and run the circuit as many times as you wish. How would you estimate the trace of U?

(e) [5 marks] Assume the auxiliary system is maximally entangled with some other system that does not participate in the quantum evolution induced by the circuit. The joint state of the two systems is

$$|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} |u_i\rangle\, |v_i\rangle\,,$$

where vectors $|u_i\rangle$ and $|v_i\rangle$ form orthonormal bases in their respective Hilbert spaces. What is the probability $\Pr(0)$ is this case? How is it related to the probability given in Eq.(1)?

SOLUTION TO QUESTION 2

(a) [4 marks] In the following it is shown how the initial state changes after each applied quantum gate:

$$
\begin{aligned}
|0\rangle|u\rangle &\overset{\text{first } H}{\to} \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)|u\rangle \\
&\overset{\text{C}-U}{\to} \frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle+|1\rangle U|u\rangle\right)=\frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle+e^{i\alpha}|1\rangle|u\rangle\right) \\
&\overset{\varphi}{\to} \frac{1}{\sqrt{2}}\Big(|0\rangle|u\rangle+e^{i(\varphi+\alpha)}|1\rangle|u\rangle\Big) \\
&\overset{\text{second } H}{\to} \frac{1}{2}\Big((|0\rangle+|1\rangle)|u\rangle+e^{i(\varphi+\alpha)}(|0\rangle-|1\rangle)|u\rangle\Big)=:|\psi_f\rangle\,.
\end{aligned}
$$

The probability, $\mathrm{Pr}(0)$, that 0 is the outcome of the measurement M can be calculated via

$$
\begin{aligned}
\mathrm{Pr}(0) &= \langle\psi_f|(|0\rangle\langle 0|\otimes\mathbb{1})|\psi_f\rangle \\
&= \left|\frac{1}{2}\left(1+\mathrm{e}^{\mathrm{i}(\varphi+\alpha)}\right)\right|^2 \\
&= \left|\frac{1}{2}\mathrm{e}^{\mathrm{i}(\varphi+\alpha)/2}\left(\mathrm{e}^{-\mathrm{i}(\varphi+\alpha)/2}+\mathrm{e}^{\mathrm{i}(\varphi+\alpha)/2}\right)\right|^2 \\
&= \cos^2[(\varphi+\alpha)/2] \\
&= \frac{1}{2}\left(1+\cos\left(\varphi+\alpha\right)\right).
\end{aligned}
$$

(b) [5 marks] We do the same again for an arbitrary pure state $|u\rangle$:

$$
\begin{aligned}
|0\rangle|u\rangle &\overset{\text{first } H}{\to} \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)|u\rangle \\
&\overset{\text{C}-U}{\to} \frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle+|1\rangle U|u\rangle\right) \\
&\overset{\varphi}{\to} \frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle+e^{i\varphi}|1\rangle U|u\rangle\right) \\
&\overset{\text{second } H}{\to} \frac{1}{2}\left((|0\rangle+|1\rangle)|u\rangle+e^{i\varphi}(|0\rangle-|1\rangle)U|u\rangle\right):=|\psi_f\rangle\,.
\end{aligned}
$$

Evaluation of $\mathrm{Pr}(0)=\langle\psi_f|(|0\rangle\langle 0|\otimes\mathbb{1})|\psi_f\rangle$ yields

$$
\mathrm{Pr}(0)=\frac{1}{2}(1+\mathrm{Re}\{e^{i\varphi}\langle u|U|u\rangle\}).
$$

(c) [6 marks] Simply take linear combinations of the pure states that form the orthonormal basis of the density operator:

$$
\begin{aligned}
\mathrm{Pr}(0) &= \sum_k p_k\frac{1}{2}(1+\mathrm{Re}\{e^{i\varphi}\langle u_k|U|u_k\rangle\})=\frac{1}{2}\{\sum_k p_k+\sum_k p_k\mathrm{Re}\{e^{i\varphi}\langle u_k|U|u_k\rangle\}\} \\
&= \frac{1}{2}\{1+\sum_k\mathrm{Re}\{e^{i\varphi}\langle u_k|p_kU|u_k\rangle\}\}=\frac{1}{2}\{1+\mathrm{Re}\{e^{i\varphi}\sum_k\mathrm{Tr}\;p_kU|u_k\rangle\langle u_k|\}\} \\
&= \frac{1}{2}\{1+\mathrm{Re}\{e^{i\varphi}\mathrm{Tr}\;U\sum_k p_k|u_k\rangle\langle u_k|\}\}=\frac{1}{2}\{1+\mathrm{Re}\{e^{i\varphi}\mathrm{Tr}(U\rho)\}\}.
\end{aligned}
$$

(d) [5 marks] Prepare $\rho = \frac{1}{n}\mathbb{1}$ and run the circuit many times with $\varphi = 0$ to get the real part, $\mathrm{Re}\{(\mathrm{Tr}\ U)\}$, and with $\varphi = \frac{\pi}{2}$ to get the imaginary part, $\mathrm{Im}\{(\mathrm{Tr}\ U)\}$.

(e) [5 marks] By taking partial trace it is clear that the reduced density operator for the auxiliary system of the quantum network is a maximally mixed state to begin with. Therefore, it is equivalent to setting $\rho \propto \mathbb{1}$ in the answer to part (c).

3. Any linear map A acting on density matrices of a qubit can be completely characterised by its action on the four basis matrices $|i\rangle\langle j|$, where $i, j = 0, 1$, and represented as a 4×4 matrix, known as the Choi matrix,

$$\tilde{A} = \frac{1}{2}\left[\begin{array}{c|c} A(|0\rangle\langle 0|) & A(|0\rangle\langle 1|) \\ \hline A(|1\rangle\langle 0|) & A(|1\rangle\langle 1|) \end{array}\right].$$

Depending on the map A, the Choi matrix may or may not be a density matrix.

(a) [3 marks] Explain why a matrix must be both positive semi-definite and have trace 1 in order to be considered a density matrix.

For a physically admissible map A we require that both A and its extension to any other physical systems, written as $\mathbb{1} \otimes A$, map density operators into density operators. Such maps are called completely positive trace-preserving (CPTP) maps.

(b) [6 marks] Show that the Choi matrix can be expressed as $(\mathbb{1} \otimes A)\,|\psi\rangle\langle\psi|$, where

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

What does this tell you about the Choi matrix associated with completely positive maps?

(c) [5 marks] The transpose operation T is defined as $|i\rangle\langle j| \to |j\rangle\langle i|$. Is the transpose of a density matrix also a density matrix? Using the Choi matrix, or otherwise, show that transpose is not a physically admissible operation.

A quantum state of two qubits described by the density matrix ϱ is called separable if ϱ is of the form

$$\varrho = \sum_k p_k\, \rho_k \otimes \upsilon_k,$$

where $p_k \geqslant 0$ and $\sum_{k=1} p_k = 1$. Otherwise ϱ is called entangled.

(d) [5 marks] Show that partial transpose, $\mathbb{1} \otimes T$, maps separable states into separable states.

(e) [6 marks] Consider a quantum state of two qubits described by the density matrix

$$\rho = p\,|\psi\rangle\langle\psi| + (1-p)\frac{1}{4}\mathbb{1} \otimes \mathbb{1}, \quad p \in [0, 1].$$

Apply partial transpose $\mathbb{1} \otimes T$ to this state and check if the resulting matrix is a density matrix. For which values of p the density matrix ρ represents an entangled state?

SOLUTION TO QUESTION 3

(a) [3 marks] The diagonal elements of a density matrix in any basis are interpreted as probabilities, hence they all must be non-negative (Condition I) and must add up to 1 (Condition II).

(Condition I) The non-negativity is ensured by requiring the matrix to be positive semi-definite, for which it is ensured that all the eigenvalues are non-negative. Let us denote the kth non-negative eigenvalue of a positive semi-definite matrix ϱ with λ_k and the corresponding eigenstate with $|\lambda_k\rangle$. From the spectral decomposition we have $\varrho = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ which ensures that for any state $|\alpha\rangle$ we have

$$\langle\alpha|\varrho|\alpha\rangle = \sum_k \underbrace{\lambda_k}_{\geqslant 0} \underbrace{|\langle\alpha|\lambda_k\rangle|^2}_{\geqslant 0} \geqslant 0,$$

and thus the diagonal elements of a positive semi-definite matrix are in all bases non-negative.

(Condition II) The trace of a matrix is independent of its basis representation and a trace of 1 ensures that the non-negative diagonal elements of a positive semi-definite matrix add up to 1.

Hence (Condition I) and (Condition II) for a matrix ensure that its diagonal elements can be interpreted as probabilities, the necessary condition that a matrix is a density matrix.

(b) [6 marks] We first write explicitly

$$\begin{aligned}|\psi\rangle\langle\psi| = \frac{1}{2}\Big(& |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ & + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\ & + |1\rangle\langle 0| \otimes |1\rangle\langle 0| \\ & + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big),\end{aligned}$$

which gives

$$\begin{aligned}(\mathbb{1} \otimes A)\, |\psi\rangle\langle\psi| = \frac{1}{2}\Big(& |0\rangle\langle 0| \otimes A(|0\rangle\langle 0|) \\ & + |0\rangle\langle 1| \otimes A(|0\rangle\langle 1|) \\ & + |1\rangle\langle 0| \otimes A(|1\rangle\langle 0|) \\ & + |1\rangle\langle 1| \otimes A(|1\rangle\langle 1|)\Big).\end{aligned}$$

We now use for the first qubit the explicit representation $|0\rangle \to \begin{pmatrix}1 & 0\end{pmatrix}^T$ and $|1\rangle \to \begin{pmatrix}0 & 1\end{pmatrix}^T$, which gives

$$\begin{aligned}(\mathbb{1} \otimes A)\, |\psi\rangle\langle\psi| &= \frac{1}{2}\Bigg[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes A(|0\rangle\langle 0|) \\ &\quad + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \otimes A(|0\rangle\langle 1|) \\ &\quad + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \otimes A(|1\rangle\langle 0|) \\ &\quad + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \otimes A(|1\rangle\langle 1|)\Bigg] \\ &= \frac{1}{2}\begin{pmatrix}A(|0\rangle\langle 0|) & A(|0\rangle\langle 1|)\\ A(|1\rangle\langle 0|) & A(|1\rangle\langle 1|)\end{pmatrix} = \tilde{A},\end{aligned}$$

where in the last line we have used the **Kronecker product** multiplication rule.

(c) [5 marks] Let us denote with ϱ a density matrix. To check whether ϱ^T is density matrix we have to check that $\mathrm{Tr}\{\varrho^T\} = 1$ (Condition I) and that ϱ^T is positive semi-definite (Condition II).

(Condition I) The trace condition holds trivially since the transpose operation let the diagonal elements invariant.

(Condition II) To check whether ϱ^T is positive semi-definite we first note that ϱ^T is Hermitian. Let us denote the kth eigenvalue of ϱ with λ_k and the corresponding eigenvector with $|\lambda_k\rangle$. We apply the transpose operation to the eigenequation $\varrho\,|\lambda_k\rangle = \lambda_k\,|\lambda_k\rangle$:

$$\overline{\langle\lambda_k|}\varrho^T = \lambda_k\overline{\langle\lambda_k|},$$

and due to ϱ^T being Hermitian, we find the eigenequation for the transposed density matrix

$$\varrho^T\overline{|\lambda_k\rangle} = \lambda_k\overline{|\lambda_k\rangle},$$

where $\overline{|\lambda_k\rangle}$ is the state $|\lambda_k\rangle$ with complex conjugated entries. Hence, ϱ^T has the same eigenvalues as ϱ from which directly follows that ϱ^T is positive semi-definite.

Condition I and Condition II hold for ϱ^T so that it is a density matrix.

Next we show that the transpose operation T is not a physically admissible map. For this we have to check whether $\mathbb{1}\otimes T$ is CPTP. We demonstrate this on the Choi matrix:

$$\begin{aligned}\tilde{T} = (\mathbb{1}\otimes T)\,|\psi\rangle\langle\psi| &= \frac{1}{2}\begin{pmatrix} T(|0\rangle\langle 0|) & T(|0\rangle\langle 1|) \\ T(|1\rangle\langle 0|) & T(|1\rangle\langle 1|)\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (|0\rangle\langle 0|) & (|1\rangle\langle 0|) \\ (|0\rangle\langle 1|) & (|1\rangle\langle 1|)\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

The first and the last row of the Choi matrix are decoupled and yield an eigenvalue of $\lambda_+ = 1/2$, respectively. The second and the third row are coupled, so we have to determine the eigenvalues of

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},$$

which yields the eigenvalues $\lambda_\pm = \pm 1/2$. The Choi matrix $\tilde{T}$ has a negative eigenvalue and is thus no density matrix. Therefore, $\mathbb{1}\otimes T$ is not CPTP and hence T is not a physically admissible map.

(d) [5 marks] We have to show that $\mathbb{1}\otimes T$ maps separable states to separable states. Using the general form of a separable state, $\varrho = \sum_k p_k\rho_k\otimes v_k$, where $p_k \geqslant 0$, and $\{\rho_k\}_k$, $\{v_k\}_k$ are sets of mixed states of subsystem 1 and 2, respectively. Applying to this expression the partial transpose yields

$$(\mathbb{1}\otimes T)\,\varrho = \sum_k p_k\rho_k\otimes Tv_k,$$

and $Tv_k = v_k^T$ is the transpose of v_k, which is again a density operator as we showed in (c). Hence the partial transpose preserves the decomposition structure of separable states and therefore maps separable into separable states.

(e) [6 marks] Applying the partial transpose, $\mathbb{1} \otimes T$ to the density matrix $\rho = p\,|\psi\rangle\langle\psi| + [(1-p)/4]\mathbb{1} \otimes \mathbb{1}$ yields $(\mathbb{1} \otimes T)\,\rho = p\tilde{T} + [(1-p)/4]\mathbb{1} \otimes \mathbb{1}$, where $\tilde{T}$ is the Choi matrix of T, for which we derived the explicit representation in (c). Thus, for the representation of the partial transpose of ρ we find

$$(\mathbb{1} \otimes T)\,\rho = \begin{pmatrix} p/2 + (1-p)/4 & 0 & 0 & 0 \\ 0 & (1-p)/4 & p/2 & 0 \\ 0 & p/2 & (1-p)/4 & 0 \\ 0 & 0 & 0 & p/2 + (1-p)/4 \end{pmatrix}.$$

To check whether the resulting state is a density matrix we have to check whether the trace is 1 and all eigenvalues are non-negative. The trace is in fact 1. We now determine the eigenvalues: The first and the last row are again decoupled and have the same diagonal entry, which yields for both subspaces the eigenvale $\lambda_1 = (1+p)/4 > 0$. The second and third row are coupled and we have to determine the eigenvalues of the matrix

$$B := \begin{pmatrix} (1-p)/4 & p/2 \\ p/2 & (1-p)/4 \end{pmatrix},$$

which yields the characteristic polynomial $\det\{B - \lambda\mathbb{1}\} = ([(1-p)/4] - \lambda)^2 - p^2/4$ which has to match zero for the eigenvalues. We find with this the eigenvalues $\lambda_1 = (1+p)/4$ and $\lambda_2 = (1-3p)/4$. Only λ_2 can get negative which happens for $p > 1/3$. Therefore, only for $p \leqslant 1/3$ is the partial transpose of ρ a density matrix. To check for which values of ρ the density matrix is entangled we can use the **Peres-Horodecki criterion**, which states that ρ is entangled when $(\mathbb{1} \otimes T)\,\rho$ has a negative eigenvalue. We already showed that this is the case for $p > 1/3$, for which therefore ρ is entangled. Note that the **Peres-Horodecki criterion** is sufficient here since we consider a two-qubit system.