Solutions

Not limits, but lifts

$$C : \Omega^0_t \overset{\partial_t}{\underset{k\Delta}{\rightrightarrows}} \dot{C} : \Omega^0_t$$

The heat equation: $\dfrac{\partial C}{\partial t} = k \nabla C$

- Instead, *solutions correspond to lifts of the diagram* D *through the codomain functor* $\mathrm{cod}\colon \mathsf{El}_S(\mathscr{C}) \to \mathscr{C}$
  - when working with (P)DEs, we usually take $S = \underline{\mathbb{R}}$ (the constant sheaf of $\mathbb{R}$ on M), since morphisms $\underline{\mathbb{R}} \to \mathscr{F}$ of sheaves (of real vector spaces on M) are in bijection with global sections of $\mathscr{F}$
- Note that we can recover limits from lifts, since a lift through cod is exactly a cone over D with apex S

Morphisms of diagrams

Two constructions

- Given a category $\mathscr{C}$, the **backwards category of diagrams in** $\mathscr{C}$ has objects $(\mathscr{J}, D\colon \mathscr{J} \to \mathscr{C})$ and morphisms $(R\colon \mathscr{J}' \to \mathscr{J}, \rho\colon D \circ R \Rightarrow D')$; the **forwards** category has morphisms $(R\colon \mathscr{J} \to \mathscr{J}', \rho\colon D \Rightarrow D' \circ R)$

$$
\begin{tikzcd}
\mathbf{J} \arrow[dr, "D"'] & & \mathbf{J}' \arrow[ll, "R"'] \arrow[dl, "D'"] \\
& \mathbf{C} \arrow[u, phantom, "\overset{\rho}{\Rightarrow}"] &
\end{tikzcd}
$$

Forwards

$$
\begin{tikzcd}
\mathbf{J} \arrow[rr, "R"] \arrow[dr, "D"'] & & \mathbf{J}' \arrow[dl, "D'"] \\
& \mathbf{C} \arrow[u, phantom, "\overset{\rho}{\Rightarrow}"] &
\end{tikzcd}
$$

Backwards

- If ρ is the identity, then we recover (the opposite of) the slice category $\mathsf{Cat}/\mathscr{C}$

- Morphisms in the *backwards* category send solutions to solutions

  - thus these tend to go in the direction of *increasing generality*, e.g. from static Maxwell–Faraday *with* potentials to static Maxwell–Faraday *without* potentials