Diagrams, formally

Three definitions

- A **diagram** in a category $\mathscr{C}$ is a functor $D\colon \mathscr{J} \to \mathscr{C}$ where the **shape** $\mathscr{J}$ is a small category
  - for our purposes, $\mathscr{C}$ is usually some category of “geometric sheaves” on a space, e.g. wedge products of the (co)tangent bundle of a manifold (so sections are differential forms and vector fields)
- Given a category $\mathscr{C}$ and an object $S \in \mathscr{C}$, the category $\mathrm{El}_S(\mathscr{C})$ of **generalised elements of shape** S is the coslice category $\mathrm{El}_S(\mathscr{C}) = S/\mathscr{C}$
- A **lift** of a diagram $D\colon \mathscr{J} \to \mathscr{C}$ through a functor $\pi\colon \mathscr{E} \to \mathscr{C}$ is a functor $\overline{D}\colon \mathscr{J} \to \mathscr{E}$ such that $\pi \circ \overline{D} = D$
  - we generally take π to be a discrete opfibration

$$\begin{tikzcd} & \mathsf{E} \arrow[d, "\pi"] \\ \mathsf{J} \arrow[ur, dashed, "\overline{D}"] \arrow[r, "D"'] & \mathsf{C} \end{tikzcd}$$

Solutions

Not limits, but lifts

$$C : \Omega^0_t \overset{\partial_t}{\underset{k\Delta}{\rightrightarrows}} \dot{C} : \Omega^0_t$$

The heat equation: $\dfrac{\partial C}{\partial t} = k \nabla C$

- Note that the diagram presenting an equation does ***not*** commute
  - if it did, then this would say that ***everything*** is a solution to the equation!
- Maybe surprisingly, we do ***not*** just take the limit (e.g. equaliser) of a diagram
  - this would give us ***all*** solutions — in (P)DEs, we're often just interested in single solutions! (we would also have to leave the categories of classical differential geometry in order for the relevant limits to exist)