Homework-8

1. $A \in R^{n \times n}$ is symmetric and Idempotent matrix

$A^T . A = A$

$A^T = A$

$A^2 = A$

To prove :- $A^+ = A$

In order to prove pseudo-inverse, check whether it satisfies all the 4 laws of penrose

(1). $AGA = A$, G is pseudo inverse

(2) $GAG = G$

(3). $(AG)^T = AG$

(4) $(GA)^T = GA$

According to given problem:-

If $G = A$

$$AGA = A \cdot AA$$
$$= A^2 \cdot A$$
$$= A \cdot A$$
$$= A^2 = A$$

1st law satisfied

(2). $GAG = G$

$\Rightarrow A^{+}AA^{+} = A^{+}$

$\Rightarrow A \cdot A \cdot A = A$

$\Rightarrow A^{2} \cdot A = A$

$\Rightarrow A \cdot A = A$

$\Rightarrow A^{2} = A$

$\boxed{A = A}$

2nd law satisfied

(3) $(AG)^{T} = AG$

$(A \cdot A^{+})^{T} = A \cdot A$

$(A \cdot A)^{T} = A^{2}$

$(A^{2})^{T} = A$

$(A)^{T}$

A

3rd law satisfied

(4) $(GA)^{T} = GA$

$(A^{+}A)^{T} = A^{+} \cdot A$

$(A \cdot A)^{T} = AA$

$(A^{2})^{T} = A^{2}$

$(A)^{T} = A$

A^{T}

$\boxed{A = A}$

4th law satisfied

$\therefore A^{+} = A$ is the pseudoinverse for symmetric and idempotent matrix

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(7) Given :

$\min \|Ax-b\|_2$ using normal equations

$x = A^+ b$

(a) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

A is onto, i.e., it has independent rows.

$m = r = 2$

$\therefore A^+$ is the right inverse of A

Right inverse of A is calculated by

$$A^{-R} = A^T(AA^T)^{-1} = A^+$$

$$AA^T = \underset{2\times3}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 5 \end{bmatrix}}$$

$$= \underset{2\times2}{\begin{bmatrix} 21 & 30 \\ 30 & 45 \end{bmatrix}}$$

$1+4+16 \quad 2+8+20$

$2+8+20 \quad 4+16+25$

$$|AA^T| = (21)(45) - (30)(30)$$

$$= 945 - 900$$

$$= 45 \neq 0$$

45×21, 45, $90\times$, 5

$(AA^T)^{-1}$ is invertible

$$A^+ = \underset{3\times2}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 5 \end{bmatrix}} \underset{2\times2}{\left(\frac{1}{45} \begin{bmatrix} 45 & -30 \\ -30 & 21 \end{bmatrix} \right)} = \frac{1}{45} \underset{3\times2}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 5 \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} 45 & -30 \\ -30 & 21 \end{bmatrix}}$$

$$= \frac{1}{45} \begin{bmatrix} 45-60 & -30+42 \\ 90-120 & -60+84 \\ 180-150 & -120+105 \end{bmatrix}$$

$\frac{7 \times 4}{450}$, 180

$$A^{-R} = A^+ = \frac{1}{45} \begin{bmatrix} -15 & 12 \\ -30 & +24 \\ 30 & -15 \end{bmatrix}$$

$$x = A^+ b = \frac{1}{45} \underset{3\times2}{\begin{bmatrix} -15 & 12 \\ -30 & 24 \\ 30 & -15 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$= \frac{1}{45} \begin{bmatrix} -15+24 \\ -30+48 \\ 30-30 \end{bmatrix} \quad \frac{1}{45} \begin{bmatrix} 9 \\ 18 \\ 0 \end{bmatrix} = \begin{bmatrix} 9/45 \\ 18/45 \\ 0 \end{bmatrix}$$

(2) $A^+ = \lim_{\delta \to 0} (A^T A + \delta^2 I)^{-1} A^T = \lim_{\delta \to 0} A^T (AA^T + \delta^2 I)^{-1}$

$A_2 = \underset{1\times 2}{\begin{bmatrix} 3 & 4 \end{bmatrix}}$

(1). $A_2^T = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\Rightarrow \quad A \cdot A^T = \underset{1\times 2}{\begin{bmatrix} 3 & 4 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 9+16 \end{bmatrix} = 25$

$= \lim_{\delta \to 0} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 25 + \delta^2 I \end{bmatrix}^{-1}$

$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} \frac{1}{25} \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{25} \\ \frac{4}{25} \end{bmatrix}$

$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \lim_{\delta \to 0} (A^T \cdot A + \delta^2 I)^{-1} \cdot A^T$

$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$

$\lim_{\delta \to 0} \left(\begin{bmatrix} 5+\delta^2 & 10 \\ 10 & 20+\delta^2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\lim_{\delta \to 0} \left(\frac{1}{\delta^2 (25+\delta^2)} \right) \begin{bmatrix} 20+\delta^2-20 & 40+2\delta^2-40 \\ -10+10+\delta^2 & -20+\delta^2+20 \end{bmatrix}$

$= \lim_{\delta \to 0} \frac{\cancel{\delta^2}}{\cancel{\delta^2}(25+\delta^2)} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$= \frac{1}{25} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(5). $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, $A^+ = \begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}$

By Penrose properties

$AGA = A$

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

$$\begin{bmatrix} BB^+ & 0 \\ 0 & CC^+ \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

$$\begin{bmatrix} BB^+ & 0 \\ 0 & CC^+ \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \quad N(B) \subset N(C)$$

$$\begin{bmatrix} BB^+B & 0 \\ 0 & CC^+C \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

$GAG = G$

$$\begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}$$

$$\begin{bmatrix} B^+B & 0 \\ 0 & CC^+ \end{bmatrix}\begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}$$

$$\begin{bmatrix} B^+BB^+ & 0 \\ 0 & C^+CC^+ \end{bmatrix}$$

$N(B) = N(C)$

$(AG)^T = AG$

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^+ & 0 \\ C & C^+ \end{bmatrix}$$

$$\begin{bmatrix} BB^+ & 0 \\ 0 & CC^+ \end{bmatrix}^T$$

$$\begin{bmatrix} BB^+ & 0 \\ 0 & CC^+ \end{bmatrix} \Rightarrow$$

$AG = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}$

$$\begin{bmatrix} BB^+ & 0 \\ 0 & CC^+ \end{bmatrix}$$

$\therefore (AG)^T = AG$

$(GA)^T = GA$

$$\begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

$$\begin{bmatrix} B^+B & 0 \\ 0 & CC^+ \end{bmatrix}^T$$

$$\begin{matrix} B^+B & 0 \\ 0 & CC^+ \end{matrix}$$

GA

$$\begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

$$\begin{bmatrix} B^+B & 0 \\ 0 & C^+C \end{bmatrix}$$

$(GA)^T = GA$

$\therefore A^+ = \begin{bmatrix} B^+ & 0 \\ 0 & C^+ \end{bmatrix}$

(4). $A = \begin{bmatrix} B \\ C \end{bmatrix}$, $B \in R^{m \times n}$, $C \in R^{p \times n}$. $BC^T = 0$

$A^+ = [B^+ \ C^+]$

$\Rightarrow B C^T$

$\Rightarrow B_{m \times n} C^T_{n \times p} = 0_{m \times p}$

$C^T \in N(B)$.

so n-columns are dependent

B is not one-one

$\therefore R(A) \subseteq R(B)$.

$\Rightarrow$ [AGA = A $\quad \underset{2\times 1}{\begin{bmatrix} B \\ C \end{bmatrix}} \underset{1\times 2}{[B^+ \ C^+]} \begin{bmatrix} B \\ C \end{bmatrix}$

$= \underset{2\times 2}{\begin{bmatrix} BB^+ & BC^+ \\ CB^+ & CC^+ \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} B \\ C \end{bmatrix}}$

$= \begin{bmatrix} BB^+B + BC^+C \\ CB^+B + CC^+C \end{bmatrix}$

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$= \begin{pmatrix} B(BB^+ + C^+C) \\ C(B^+B + C^+C) \end{pmatrix} = \begin{bmatrix} B \\ C \end{bmatrix}$

$GAG = \underset{1\times 2}{[B^+ \ C^+]} \underset{2\times 1}{\begin{bmatrix} B \\ C \end{bmatrix}} (B^+ \ C^+]$

$=$

$[B^+B + C^+C][B^+ C^+]$

$[B^+BB^+ + C^+CB^+ \quad B^+BC^+ + C^+CC^+]$

$= B^+BB^+ = [B^+BB^+ + C^+CB^+ \quad C^+(B^+B + C^+C)]$

$= [B^+(BB^+ + CC^+) \quad C^+(B^+B + C^+C)]$

$G = (B^+ \quad C^+]$

$(AG)^T = \left(\underset{2\times 1}{\begin{bmatrix} B \\ C \end{bmatrix}} \underset{1\times 2}{[B^+ C^+]} \right)^T = \begin{bmatrix} BB^+ & BC^+ \\ CB^+ & CC^+ \end{bmatrix}^T = \begin{bmatrix} BB^+ & CB^+ \\ BC^+ & CC^+ \end{bmatrix}$

$= AG$

$(GA)^T = [B^+ \; C^+] \begin{bmatrix} B \\ C \end{bmatrix} = [B\overset{+}{B} + C^+C]^T$

$1\times 2 \qquad 2\times 1$

$= [B^+B + C^+C]$

$GA = [B^+ C^+]\begin{bmatrix} B \\ C \end{bmatrix} = [B^+B + C^+C]$

It satisfies **all** properties therefore it is A^+

③. $A \in R^{n\times m}$, $K \in R^{m\times m}_{m}$

K is an orthogonal matrix

$K\cdot K^T = I$

$\boxed{K^T = K^{-1}}$

$\underset{n\times m}{A} = \underset{n\times m}{B}\;\underset{m\times m}{K}$

then K is invertible

$AK^{-1} = B$

$\boxed{K^{-1} \in R(A)}$

$T: Y \to X$

$AY = T^{-1}x$

A

$BB^+ = A\cdot A^+$