$L(\alpha v + \beta v) = \alpha L(v) + \beta L(v)$

$\boxed{\text{Assignment - 7}}$

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(2) $AX = 0$

$A_{35 \times 41} \times {}_{41 \times 1} = 0$

Dim(Soln. space) + Rank A = no. of columns of A

$\downarrow$

35 - rows
41 - columns

41 - rows
1 - columns

$\downarrow$

Rank - 17 - 17 - independent rows

$m = 17$, $n = 41$

$\downarrow$ No. of equations

$\downarrow$ no. of unknowns

Therefore we have $(n-m)$ independent solutions

$(41-17)$

$\underline{24}$

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(4) $L: R^3 \to R^3$

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times3} = \begin{bmatrix} 2x-y \\ -x+2y-z \\ z-y \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2x-y \\ -x+2y-z \\ z-y \end{bmatrix}$$

$$\Rightarrow (M_L) \to \begin{bmatrix} -2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow A$$

$$M^{-1}\begin{pmatrix} -2x-y \\ -x+2y-z \\ z-y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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$L^{-1} \circ L \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ -

$-2x - y = 0$

$-2x = y$

$x = -y/2$

$x = 0$

$\cancel{-}\frac{+y}{2} + \frac{2y}{1} - z = 0$

$\frac{y + 4y}{2} = z$

$\frac{5y}{2}$

$\boxed{z = 0}$

$z - y = 0$

$\frac{5y}{2} - \frac{y}{1} = 0$

$y = 0$

$N(L) = 0$ $\therefore$ it is one-one

$-2x - y = a$

$-x + 2y - z = b$

$z - y = c$

$\Rightarrow a + b + c = -2x - y - x + 2y - z + z - y$

$= -3x$

$a + b + c = -3x$

$x = \frac{-1}{3}(a + b + c) = \frac{1}{3}(-a - b - c)$

$-2\left(\frac{-1}{3}(a + b + c)\right) - y = a$

$\Rightarrow + y = -a + \frac{2}{3}(a + b + c)$

$\Rightarrow \frac{-a}{1} + \frac{2a}{3} + \frac{2}{3}(b + c)$

$= \frac{-3a + 2a}{3} + \frac{2}{3}(b + c)$

$y = \frac{-a}{3} + \frac{2b}{3} + \frac{2c}{3} \Rightarrow \frac{1}{3}(-a + 2b + 2c)$

$z = c + y$

$= \frac{-a}{3} + \frac{2b}{3} + \frac{2c}{3} + \frac{c}{1}$

$+ \frac{2c + 3c}{3}$

$= \frac{-a}{3} + \frac{2b}{3} + \frac{5c}{3}$ $\frac{1}{3}(-a + 2b + 5c)$

$\frac{1}{3}\cancel{(-a - 2b - 5c)}$

$= \frac{-1}{3}\cancel{(a + 2b + 5c)}$

$L^{-1} = \frac{1}{3} \begin{pmatrix} -1 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & +2 & 5 \end{pmatrix} \begin{pmatrix} 2x - y \\ -x + 2y - z \\ z - y \end{pmatrix} =$

$\Rightarrow -x$

$-2x + \cancel{y} + x - \cancel{2y} + \cancel{z} - \cancel{z} + \cancel{y}$

ⓒ $A^{-1} = +\frac{1}{3}\begin{pmatrix} -1 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$

Here L^{-1} and A^{-1} are same.

ⓓ Matrix Representation of L and L^{-1} w.r.t

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow L(S) = L\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$-1+2$

$-1-1$

$$L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} ; \quad L\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$$

$$L(S) = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -2 \\ -1 & -1 & 0 \end{pmatrix}$$

$$L(S) = A \cdot S$$

$$A^{-1} \cdot L(S) = S$$

$$A^{-1} = S \cdot L^{-1}(S).$$

$$(x)\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = ($$

$$x \cdot S = A$$

$$x = A \cdot S^{-1}$$

$$S^{-1} = \begin{pmatrix} -\frac{5}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{5}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{3}{8} & -\frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -2 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1+1+0 & 0-1+0 & -1-1+2 \\ 1-2+0 & 0+2+0 & -1+2-2 \\ -1+1+0 & 0-1+0 & 1-1+0 \end{pmatrix}$$

$$\boxed{L^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & -1 \\ -\frac{1}{2} & -1 & -\frac{3}{2} \end{pmatrix}}$$

ⓔ

$$\boxed{L = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}}$$

(11)

If $x \in R(P)$

$Px = x$

suppose $\|Px\|_2 = \|x\|_2$

let $x = m + n$ where $m \in R(P)$, $n \in R(P)$

Sothat $m \perp n$

$$\|x_2\|^2 = \|m+n\|_2^2$$

$$= \|m\|_2^2 + \|n\|_2^2$$

then

$$\|x\|_2^2 = \|Px\|_2^2 = \|P(m+n)\|_2^2$$

$$= \|Pm\|_2^2 = \|m\|_2^2$$

Therefore $n = 0$ and thus

$x = m \in R(P)$

③ Given

$A \in R^{m\times n}$

$b \in R^{m}$

If $\text{Rank}(A) = \text{Rank}(A|b)$

$[A|b] = m\times(n+1)$ matrix with that additional column b

~~R(A)~~ $R(A) = \{Ax \mid x \in R^{m\times n}\}$

is the subset of R^m

Here the system is consistent for any $b \in R^n$

$\therefore \boxed{b \in R(A).}$

$Ax = b$

$$\underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{m\times n}\underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{m\times 1}$$

This means that b is obtained from the linear combination of A.

For ex:- ~~2X2 matrix can be obtained by~~

$$\cancel{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}}$$

$\therefore$ Rank remains the same and we know $\text{Rank}(A|b)$ is a linear combination of $A \Rightarrow$ Rank is equal to the columns of A

$\therefore b \in R(A)$

and vice versa

6. ~~Sol~~ Given

$Hom_R(R^n, R^m)$ is isomorphic to R^{mn}.

$Hom_R(R^n, R^m)$

Set of all possible linear mapping from $R^n \to R^m$

Set of All linear mapping here in the above case corresponds to all possible $m \times n$ matrices.

$$\underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{n \times 1}$$

$A_{m \times n}$ $\quad V_{n \times 1}$ $\quad W_{m \times 1}$

$\qquad\qquad R^n \qquad R^m$

Here A is a vector space which consists of all the linear operators to ~~poss~~ that map $R^n \to R^m$

Dimension of vector space $A = m * n$
$= mn$

$\Rightarrow$ Here in-order to be isomorphic it has to be both injective and surjective.

In order to be injective, $N(A) = 0$

It is observable that $N(A) = 0$

$R(A)$ is the column space of A

$\therefore$ It is isomorphic to R^{mn}

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(a). Given :-

Dimension of $Hom_C(C^{m \times n}, C^{p \times q})$

Here it means the ~~~~ set of all linear mapping of

$(A \times C^{m \times n}) \to (C^{p \times q})$

A is a vector space which consists of all linear mappings that perform the mappings of $C^{m \times n} \to C^{p \times q}$

$$(A)_{p \times m} (C)_{m \times n} = C_{p \times q}$$

Here the mapping is possible only when $(n = q)$ ✓

The In-order to find the dimension of A, we should find the ~~number~~ basis of (A)✓

The basis of Vectorspace A is given by set of pm - $p \times m$ matrices

$\therefore$ The dimension here is $\underline{pm}$

(b) $Hom_R(P_n^R, R^m)$

$P_n^R = (a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n)$

It is the set of all linear mapping from $P_n^R \to R^m$.

$$(A)_{m \times 1} (P_n^R)_{1 \times 1} = (R^m)_{m \times 1}$$

Dimension of Hom is given by dimension of A

To Find the dimension of vector space A

we need to find the basis of A

Basis of A is given by set of $m(1)$ $m \times 1$ matrices

There the dimension here is $\underline{m}$

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(8) $L_1 : V \to W$

$L_2 : V \to W$ are linear transformations on inner product spaces $(V, \mathbb{C})$ and $(W, \mathbb{C})$

L_1^* & L_2^* are adjoints w.r.t complex inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$

$(L_1+L_2)^*$ $\qquad L_1^* : W \to V$

$L_2^* : W \to V$

(a) $(L_1+L_2)^* = L_1^* + L_2^*$

$\langle (L_1+L_2)V, W \rangle = \langle V, (L_1+L_2)^* W \rangle$

$\langle (L_1+L_2)V, W \rangle = \langle L_1 V_1 + L_2 V, W \rangle$

$= \langle V, L_1^*(W) \rangle + \langle V, L_2^*(W) \rangle$

$= \langle V, L_1^* + L_2^*(W) \rangle$

$\therefore (L_1+L_2)^* = L_1^* + L_2^*$

(b). $(\alpha L_1)^* = \alpha^* L_1^*$

$\langle \alpha L, V, W \rangle = \langle V, (\alpha L)^* W \rangle$

$= \langle \alpha L(V), W \rangle$

$= \alpha \langle L(V), W \rangle$

$= \alpha \langle V, L^*(W) \rangle$

$= V, \bar{\alpha} L^*(W) \rangle$

$(\alpha L_1^*) = \alpha^* L_1^*$

(d). $(L_1)^* = L_1$

$\langle (L_1^*)^*(V), W \rangle = \langle V, (L_1^*)^*(W) \rangle$

$\langle L_1^*(W), V \rangle = \overline{\langle V, L^*(W) \rangle}$

$= \langle L(V), W \rangle$

$= \langle W, L(V) \rangle$

$(L_1^*)^* = L_1$

(1.2) Linear equation, $Ax = b$

(a) Does for all $b \in R^2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix}$$

$A \in R^{2\times 3}$, $b \in R^2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\Rightarrow R(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$$

since $b \in R(A)$, therefore it has a solution infinitely many solutions

(b).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$$

$\Rightarrow$ $R_2 \to 2R_1 - R_2$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$m = 2$

(1) ✓

(1) $R(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, Dim $R(A) = 1$

for $b_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Here $\underline{m \leq n}$

There e

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It has infinitely many solutions

(b_2). $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 2 & 4 & 6 & 20 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 1 & 2 & 3 & 10 \end{array}\right]$

$R_2 \to \frac{R_2}{2}$

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

It has infinitely many solutions

$b_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A|b = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1/2 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & -1/2 \end{array}\right]$

It is not a consistent system

(c). $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \end{bmatrix}_{m\times n} \Rightarrow 2\times 3$

At most one solution,

if columns are independent

$n(A) = 0$

$m \geq n$

$2 \geq 3$ ✗

No. it does not contain atmost one solutio

(d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow A$ is non-singular?

$(1)(0) - 2(1) + 3(2)$

$\Rightarrow -2+6$

$= -4 \neq 0$; A is non-singular

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$

$x_1 + 2x_2 + 3x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$x_3 = -2x_2$

$2x_2 = -3x_3$

$x_1 = 0$

$x_2 = \frac{-3x_3}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -3/2 \\ 1 \end{bmatrix} \Rightarrow$$

$N(A) \neq 0$

It does not have a unique solution

①

$L : (V, F) \rightarrow (W, F)$

$F = R, \; V = R$

$L(v) = e^{v}$

$(v_1 \oplus_v v_2) = v_1 + v_2$

$\alpha \odot_v v = \alpha v \qquad \alpha \in F, \; W = R^{+}$

~~$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$~~

$L(\alpha \odot_v (v_1) \oplus \beta \odot_v v_2) =$

$L(\alpha v_1 + \beta v_2)$

$e^{(\alpha v_1 + \beta v_2)} =$ ~~$w^{(\alpha+\beta)}$~~

$= e^{(\alpha \odot w_1 \oplus_w \cdot \beta \odot_w w_2)}$

$= e^{w_1^{\alpha} \cdot w^{\beta}}$

$= e^{w^{(\alpha+\beta)}}$

$\Rightarrow \alpha L(v_1) + \beta L(v_2)$

$= \alpha \odot_w w_1 \oplus_w \beta \odot_w w_2$

$\Rightarrow e^{w^{\alpha} \oplus_w w^{\beta}}$

$= e^{(w^{\alpha} * w^{\beta})}$

$= e^{w^{(\alpha+\beta)}}$

$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$

$\therefore$ it is linear.

(b). To prove L is invertible, it has to be injective and surjective

To prove one-one, $N(A) = 0$

$\Rightarrow$ Here $N(A) \neq 0$.

$\therefore$ it cannot be invertible

(9) Let P_1, P_2 be the orthogonal projections on a plane with unit normal vector x

$P_1 \cancel{P_2} = x - \langle n, x \rangle$

$P_1 = (x - n^T x) n$

$P_2 = (x - n^T x) n$ $\Rightarrow \boxed{P_1 + P_2 = I}$

is Identity

$P_1 P_2 = P_2 P_1 = P_2$

$(P_1 - P_2)^2 = P_1^2 + P_2^2 - P_1 P_2 - P_2 P_1$

$= P_1 - P_2$

Here $P_1 - P_2$ is idempotent

Ad: $(P_1 - P_2)^* = P_1^* - P_2^*$

$= \underline{P_1 - P_2}$

is a orthogonal matrix

(5). $L_1 \begin{bmatrix} x \\ y \end{bmatrix}^{\cancel{T}} = \alpha_1 x + \beta_1 y$

$L_2 \begin{bmatrix} x \\ y \end{bmatrix} = \alpha_2 x + \beta_2 y$

$\langle L_1, L_2 \rangle = \langle \alpha_1 x + \beta_1 y, \alpha_2 x + \beta_2 y \rangle$

$\Rightarrow \alpha_1 \alpha_2 + \beta_1 \beta_2$

(b). $L\begin{bmatrix} x \\ y \end{bmatrix} = \alpha x + \beta y$

$A(L) = L'$

$$\underset{1\times 2}{L'}\underset{2\times 1}{\begin{pmatrix} x \\ y \end{pmatrix}} = (\alpha - \beta) y$$

$$= \alpha\begin{bmatrix} x \\ 0 \end{bmatrix} + \beta\begin{bmatrix} y \\ 0 \end{bmatrix}$$

$A(L) = \underline{L'}$

$L \Rightarrow [\alpha \ \beta]$, $\quad L' \Rightarrow [0 \ \ \alpha - \beta]$

$$\underset{1\times 1}{A}\underset{1\times 2}{[\alpha \ \ \beta]} = \underset{1\times 2}{[0 \ \ \ \alpha - \beta]}$$

$\Rightarrow A \Rightarrow$

(c). A is one-one

and not onto

(d). $L_1\begin{bmatrix} x \\ y \end{bmatrix} = [x]$, $L_2\begin{bmatrix} x \\ y \end{bmatrix} = y$

$L_1 = [1 \quad 0] \qquad L_2 = [0 \quad 1]$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(10)

$$A = \begin{bmatrix} 3 & j \\ 2j & 0 \\ -j & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & j & 0 \\ -j & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & j \\ -j & 1 \end{bmatrix}$$

$$\langle x y \rangle_Q = x^H Q y$$

$$\Rightarrow \langle x, y, A^* \rangle$$

$$\langle u, v \rangle_R = u^H R V$$

$$\Rightarrow \begin{bmatrix} 1 & j & 0 \\ -j & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3} \begin{bmatrix} 3 & j \\ 2j & 0 \\ -j & 0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 3+2j^2 & j \\ -3j+4j-j & -j^2 \\ 2j & 0 \end{bmatrix}$$

$$= \left\langle \begin{bmatrix} 1 & j \\ 0 & 1 \\ 2j & 0 \end{bmatrix} \right\rangle$$

$\Rightarrow$