X = max ...

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Assignment 12

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② Given

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\dot{x}(t) = A x(t) + B \cdot u(t)$$

(a). $x(t) = M \cdot q(t)$

$$\frac{d}{dt}(Mq(t)) = AMq(t) + Bu(t)$$

$$M\dot{q}(t) = AMq(t) + Bu(t)$$

$$\dot{q}(t) = \underbrace{M^{-1}AM}_{A_1} q(t) + \underbrace{M^{-1}B}_{B_1} u(t)$$

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} = |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda+2 & 2 & 0 \\ 0 & \lambda & -1 \\ 0 & 3 & \lambda+4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+2)(\lambda^2+4\lambda+3) = 0$$

$$\lambda = -2, \quad \lambda^2+3\lambda+\lambda+3 = 0$$

$$(\lambda)(\lambda+3)+1$$

$$\lambda = -1, -3, -2$$

Check whether $M^{-1}AM$ is digonalisable.

$N(A-\lambda I) \Rightarrow$ when $\lambda = -1$

$$N(A+I) \Rightarrow \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$N(A+2I) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = -3$

$$N(A+3I) \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix}, \quad M^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -3 & -1 \\ 2 & 8 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\dot{q}(t) = \Lambda(A).q(t) + B_1 u(t)$$

$$B_1(t) = M^{-1}B(t) \Rightarrow \underset{3\times3}{\frac{1}{2}\begin{bmatrix} 0 & -3 & -1 \\ 2 & 8 & 4 \\ 0 & -1 & -1 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & -4 \\ 6 & 12 \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \underset{3\times3}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix}} + \underset{3\times2}{\frac{1}{2}\begin{bmatrix} -1 & -4 \\ 6 & 12 \\ -1 & -2 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} = \begin{bmatrix} -q_1(t) \\ -2q_2(t) \\ -3q_3(t) \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -u_1(t) - 4u_2(t) \\ 6u_1(t) + 12u_2(t) \\ -u_1(t) - 2u_2(t) \end{bmatrix}$$

$$\dot{q}_1(t) = -q_1(t) - u_2(t) - \frac{1}{4}u_1(t)$$

$$\dot{q}_2(t) = -2q_2(t) + \frac{6}{4}u_1(t) + 3u_2(t)$$

$$\dot{q}_3(t) = -3q_3(t) - \frac{1}{4}u_1(t) - \frac{1}{2}u_2(t)$$

(b) $$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Here $x(t_0 = 0) = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix}$

$\Rightarrow$ Free response :-

$$e^{A(t)} = e^{\begin{bmatrix} -2t & -2t & 0 \\ 0 & 0 & t \\ 0 & -3t & -4t \end{bmatrix}} \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \cdot e^{At} \\ 5e^{At} \\ 2 \cdot e^{At} \end{bmatrix}$$

$$\Rightarrow \int_0^t e^{A(t)} \cdot B \cdot u(t)$$

$$Bu(t) \Rightarrow \underset{3\times 2}{\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}} \underset{2\times 1}{\begin{bmatrix}\overset{t}{u_1(t)}\\ \overset{1}{u_2(t)}\end{bmatrix}} = \begin{bmatrix}u_1(t)\\ u_2(t)\\ u_1(t)+u_2(t)\end{bmatrix}$$

$$= \int_0^t \begin{bmatrix} e^{At}\,\cancel{u_1(t)}\;t \\ e^{At}\,\cancel{u_2(t)} \\ e^{At}(\cancel{u_1(t)+u_2(t)})\;t+1 \end{bmatrix} dt$$

$$\Rightarrow \int_0^t e^{At}\cdot t\,dt = te^{At} - \frac{e^{At}}{A^2}$$

$$\int_0^t e^{At}\,dt = \frac{e^{At}}{A}$$

$$\int_0^t e^{At}(t+1)\,dt = e^{At}(t+1) - \frac{e^{At}}{A^2}$$

$$= \begin{bmatrix} \frac{e^{At}}{A}\,t \\ A\,\frac{e^{At}}{A}u_2(t) \\ \frac{e^{At}}{A}(u_1(t)+u_2(t)) \end{bmatrix}$$

$$x(t) = 10e^{At} + \frac{\cancel{10+u_1(t)}}{A} \quad \frac{1}{A}\begin{bmatrix} e^{At}u_1(t) \\ e^{At}u_2(t) \\ e \end{bmatrix}$$

$$\Rightarrow x(t) = \begin{bmatrix} 10e^{At} + te^{At} - \frac{e^{At}}{A^2} \\ 5e^{At} + \frac{e^{At}}{A} \\ e^{At}\cdot t + e^{At} - \frac{e^{At}}{A^2} - 2e^{At} \end{bmatrix}$$

③ Given :

$$\ddot{y}_1(t) + 3\dot{y}_1(t) + 2y_2(t) = 0 \rightarrow (a)$$

$$\ddot{y}_2(t) + 4\dot{y}_1(t) + 3y_2(t) = 0 \rightarrow (b)$$

state variables solving ① & ②

$x_1 = y_2(t)$; y

$$3\ddot{y}_1(t) + 9\dot{y}_1(t) + \cancel{6y_2(t)} = 0$$

$$2\ddot{y}_2(t) + 8\dot{y}_1(t) + \cancel{6}y_2(t) = 0$$

$$3\ddot{y}_1(t) - 2\ddot{y}_2(t) + \dot{y}_1(t) = 0 \rightarrow (c)$$

$x_1 = \dot{y}_1(t)$; x

$x_2 = \frac{dy_1}{dt}$; $x_2 = \dot{x}_1$ 4

$x_3 = \frac{d^2y_1^2}{dt^2}$; $\dot{x}_3 = \dot{x}_2$

ⓒ in ⓑ

$$2\ddot{y}_2(t) = -3\ddot{y}_1(t) - \dot{y}_1(t)$$

$$\Rightarrow \frac{-3\ddot{y}_1(t) - \dot{y}_1(t)}{2} + 4y_1(t) = -3y_2$$

$$\Rightarrow -3y_2 = -\frac{3}{2}\ddot{y}_1(t) - \frac{1}{2}\dot{y}_1(t) + 4y_1(t)$$

$$y_2 = +\frac{3}{2}\ddot{y}_1(t) + \frac{1}{6}\dot{y}_1(t) - \frac{4}{3}y_1(t)$$

$$y_2 = \frac{3}{2}\dot{x}_2 + \frac{1}{6}\dot{x}_1 - \frac{4}{3}x_1$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Big[$$

By lyapunov Inequality

$A^TP + PA = Q$

$\Rightarrow Q = -I$

4

④ Given

ⓐ
$$\frac{Y(s)}{U(s)} = \frac{1}{s^3+10s^2+27s+18}$$

$$s^3Y(s)+10s^2Y(s)+27sY(s)+18Y(s) = U(s)$$

$$\dddot{y}(t)+10\ddot{y}(t)+27\dot{y}(t)+18y(t) = u(t)$$

ⓑ Finding a state variable model

$$\left[\begin{array}{l} x_1 = y \\ x_2 = \frac{dy}{dt} \\ x_3 = \frac{d^2y}{dt^2} \\ \cancel{x_4 = \frac{d^3y}{dt^3}} \end{array}\right] \longrightarrow \left[\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -10x_3(t)+27x_2(t)+u(t)+18x_1(t) \end{array}\right.$$

$$\cancel{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 27 & 0 & \\ 0 & 0 & 10 & \\ 0 & 29 & -10 & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & 27 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[u(t)$$

ⓒ. In order to find the stability by Lyapunov Inequality

$$A^TP+PA = -Q \quad \text{let } Q = -I$$

$$A^T \otimes P + P \otimes A = -I$$

$$\begin{bmatrix} 0 & 0 & 18 \\ 1 & 0 & 27 \\ 0 & 1 & -10 \end{bmatrix} \otimes \begin{bmatrix} P_1 & P_2 & P_3 \\ P_2 & P_4 & P_6 \\ P_3 & P_6 & P_5 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 & P_3 \\ P_2 & P_4 & P_6 \\ P_3 & P_6 & P_5 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & 27 & -10 \end{bmatrix} = -I$$

(5) $A \in \mathbb{C}^{n\times n}$

$K \in N$

$(A \otimes I)^K = A^K \otimes I$

For a 2×2 A, I matrix

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^K$$

$$\left(\begin{bmatrix} a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}\right)^K = \left[\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & a & 0 & b \\ \hline c & 0 & d & 0 \\ 0 & c & 0 & d \end{array}\right]^K$$

$$= \left[\begin{array}{cc|cc} a^k & 0 & b^k & 0 \\ 0 & a^k & 0 & b^k \\ \hline c^k & 0 & d^k & 0 \\ 0 & c^k & 0 & d^k \end{array}\right]$$

$$= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^K$$

By Mathematic Induction it is true for all other matrices

$$(I \otimes A)^K = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^K = \left[\begin{array}{cc|c} a & b & 0 \\ c & d & \\ \hline 0 & & \begin{matrix} a & b \\ c & d \end{matrix} \end{array}\right]^K = \left[\begin{array}{c|c} \begin{matrix} a^k & b^k \\ c^k & d^k \end{matrix} & 0 \\ \hline 0 & \begin{matrix} a^k & b^k \\ c^k & d^k \end{matrix} \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} a^k & b^k \\ c^k & d^k \end{bmatrix} = I \otimes A^K$$

(6)

(b). $\exp(A \otimes I_n)$

$e^{(A \otimes I_n)^1}$

$e^{((A) \otimes I_n)}$

$(A \otimes I)^k = A^k \otimes I$

$(I \otimes A)^k = I \otimes A^k$

From $(A \otimes I)^k = A^k \otimes I$

$= \cancel{e}$

$e^A \otimes I_n$

$\exp(I_n \otimes A) = \cancel{\exp} \cdot I_n \otimes \exp A$

(6) (a). Given

$A, B, X \in \mathbb{C}^{n \times n}$

(a).

$\frac{d}{dt} X(t) = AX(t) + X(t)B$

$\dot{X}(t) = (A+B)X(t)$

Applying vectorisation on b.s

$\mathrm{Vec}(\dot{X}(t)) = \mathrm{Vec}((A+B)X(t))$

$= (I_m \otimes (A+B)) \mathrm{Vec}(X)$

$\boxed{\mathrm{Vec}(AB) = (I_{n \times n} \otimes \mathrm{Vec}(B)}$

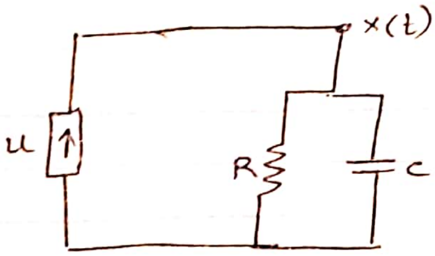
(b) We know from above results,

$\Rightarrow \exp(A \otimes I_n) = \exp(A) \otimes I_n$

$\exp(I_n \otimes A) = I_n \otimes \exp(A)$

$\Rightarrow$ Applying the above equations in (A)

(1)



$$u(t) = e^{-t/RC}\,\frac{10 - e^{-t_f/RC}\,x_0}{R\sinh(t_f/RC)}$$

$$R\,x(t) + \frac{1}{C}\int_{-\infty}^{t} x(t)\,dt = u(t)$$

$$\frac{d}{dt}u(t) = \frac{d}{dt}\left(e^{-t/RC}\cdot\underbrace{\left(\frac{10 - e^{-t_f/RC}\cdot x_0}{R\sinh(t_f/RC)}\right)}_{a}\right)$$

$$= e^{-t/RC}\cdot\frac{d}{dt}\left(\frac{10 - e^{-t_f/RC}\,x_0}{R\sinh(t_f/RC)}\right) + (a)\left(\frac{d}{dt}\left(e^{-t/RC}\right)\right)$$

$$= e^{-t/RC}\left[\frac{R\sinh(t_f/RC)\cdot(-e^{-t/RC}\cdot x_0)(-1) - (10 - e^{-t/RC}x_0)(R)(\cosh(t_f/RC))}{R^2\sinh^2(t_f/RC)}\right]$$

$$+ (a)\cdot\frac{-e^{-t}}{RC}$$

$$= e^{-t/RC}\left[\frac{+R\cdot\sinh(t/RC)\,(e^{-t/RC}x_0) - (10 - e^{-t/RC}x_0)(R)\left(\frac{e^{t/RC} + e^{-t/RC}}{2}\right)}{R^2\left(\frac{e^{t/RC} + e^{-t/RC}}{2}\right)^2}\right]$$

$$\Rightarrow \quad R\,X(t) + \frac{1}{C}\int_{-\infty}^{t} x(t)\,dt = u(t)$$