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Assignment-7.

1. $L:(V,F) \to (W,F)$

$$L(v) = e^v$$

a) Prove L is linear:

we need to show
$L(\alpha V_1 + \beta V_2) = \alpha L(V_1) + \beta L(V_2)$

Let
$V_1, V_2 \in V$ & $\alpha, \beta \in R$

$$L(\alpha V_1 + \beta V_2) = e^{(\alpha V_1 + \beta V_2)}$$

$$= e^{\alpha V_1} \cdot e^{\beta V_2}$$

$$= (e^{V_1})^{\alpha} \cdot (e^{V_2})^{\beta}$$

$$= \alpha \odot_W L(V_1) \oplus_W \beta \odot_W L(V_2)$$

$\therefore L(\alpha \odot_V V_1 \oplus_V \beta \odot_V V_2) = \alpha \odot_W L(V_1) \oplus_W \beta \odot_W L(V_2)$
L is linear.

b). L is invertible.

It is both one-one and onto. It is one-one because $e^{x_1} = e^{x_2}$ iff $x_1 = x_2$ $\forall x_1, x_2 \in V$. & It is on-to as the range of L is the image of V in W.

$$f(x) = e^x$$

$$y = e^x.$$

$$\ln y = x.$$

$$\Rightarrow f^{-1}(x) = \ln x.$$

2) $A_{35\times 41}$ rank = 17

$$\begin{array}{c} c_1\ c_2\ \dots\ c_{41} \\ \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{35\times 41} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{41} \end{bmatrix}_{41\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{41\times 1}$$

$\Rightarrow c_1x_1 + c_2x_2 + \dots + c_{41}x_{41} = 0.$

Since rank = 17, there are 17 independent columns.

Therefore, we ~~can~~ have 17 pivots & $(41-17)$ free variables in the solution.

Therefore, The dimension of Null space is 24.

$\Rightarrow$ 24 linearly Independent Solutions.

3) $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $[A|b]$ $m\times(n+1)$

Given, ~~if~~ $b \in R(A)$.

$Ax = b$

$$\begin{array}{c} c_1\ c_2\ \dots\ c_n \\ \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{m\times n} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

if $b \in R(A)$, then

$b = c_1x_1 + c_2x_2 + \dots + c_nx_n$, where c_i is the i^{th} column in A.

$$[A|b] = \begin{bmatrix} c_1 & c_2 & \dots & c_n & b \\ | & | & & | & | \\ | & | & & | & | \\ | & | & & | & | \end{bmatrix}$$

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Since, b is linearly combination of columns of A, adding column to A as [A|b] does not increase the rank of A.

$\therefore$ rank(A) = rank(A|b)

if rank(A) = rank(A|b)

Since, adding b does not change the rank of A, we can say b is linear combination of columns of A.

That means b is in column space of A.
This implies $b \in R(A)$.

4) $L: R^3 \to R^3$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ -x+2y-z \\ z-y \end{bmatrix}$$

$$L^{-1}\begin{bmatrix} 2x-y \\ -x+2y-z \\ z-y \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$2x-y=r$
$-x+2y-z=s$
$z-y=t$

$z=t+y$
$x=\frac{r+y}{2}$

$\Rightarrow -\frac{r+y}{2}+2y-t-y=s$
$-r-y+4y-2t-2y=2s$
$y=2s+r+2t$

$\Rightarrow x=r+s+t$
$z=2s+r+3t$
$y=2s+r+2t$

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b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ -x + 2y - z \\ z - y \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

c) A^{-1}

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -2 & 4 & -2 & & & \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$

$R_2 \leftarrow 2R_2 + R_1$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & -3 & 3 & 0 & 0 & 3 \end{array}\right] \xrightarrow{R_3 \leftarrow 3R_3 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ & & 2 & 2 & 4 & 6 \end{array}\right] \xrightarrow{R_2 \leftarrow 2R_3 + R_2} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 3 & 6 & 6 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right]$$

$R_1 \leftarrow R_2 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Checking with part (a).

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} r+s+t \\ r+2s+2t \\ r+2s+3t \end{bmatrix}.$$

It is same as part (a).

d)

The matrix repesentation with changed basis

$$Z = P^{-1} A P$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$Z = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

$\det(z) \neq 0.$

$\therefore$ Invertible

5) (a) $L_1([x \ \ y]^T) = \alpha_1 x + \beta_1 y$

$L_2([x \ \ y]^T) = \alpha_2 x + \beta_2 y$.

(i) $\langle L_1, L_1 \rangle = \alpha_1^2 + \beta_1^2$.

$\therefore \langle L_1, L_1 \rangle \geq 0$. & $\langle L_1, L_1 \rangle = 0$ iff $\alpha_1 = 0, \beta_1 = 0$.

(ii) $\langle L_1, L_2 \rangle = \alpha_1 \alpha_2 + \beta_1 \beta_2 = \langle L_2, L_1 \rangle$.

(iii) $\langle L_1, \alpha L_2 + \beta L_3 \rangle = \cancel{\alpha_1 \alpha_2 \alpha_3 \alpha \beta + \beta_1 \beta_2 \beta_3 \alpha \beta}$

$\alpha_1 x + \beta_1 y \ (\alpha \alpha_2 x + \alpha \beta_2 y + \beta \alpha_3 x + \beta \beta_3 x$.

$\alpha \ \alpha_1 \alpha_2 + \beta_1 \beta_2 + \beta$ $= \alpha \langle L_1, L_2 \rangle + \beta \langle L_1, L_3 \rangle$.

$\therefore$ It is an inner product space.

(b) $L([x \ \ y]^T) = \alpha x + \beta y$.

$L \rightarrow$ ~~2x~~ 1x2 mat.

~~A~~ L' : 1x1

$AL = L'$

$A \Rightarrow 1 \times \cancel{2}1$ matrix.

~~$L = [\ \ ]$~~ $L = [\alpha_1 \ \ \beta]$

6) $Hom_R(R^n, R^m)$ is isomorphic to R^{mn} ?

Sol: $Hom_R(R^n, R^m)$ is a space of all linear transformations from R^n to R^m. Therefore the transformation matrix $A_{m \times n}$ is of order $m \times n$.
All $m \times n$ matrices form a $Hom_R(R^n, R^m)$. The basis set is of dimension mn.

To show that $Hom_R(R^n, R^m)$ is isomorphic to R^{mn}, we show that the cardinality of basis set of $Hom_R(R^n, R^m)$ is equal to cardinality of basis of R^{mn}.

cardinality of basis set of $Hom_R(R^n, R^m) = mn$
cardinality of basis set of $R^{mn} = mn$.

Since, there cardinalities are equal, there exists a bijection b/t the basis sets & therefore spaces.

$\Rightarrow$ $Hom_R(R^n, R^m)$ is isomorphic to R^{mn}

7) a) $Hom_C(C^{m \times n}, C^{p \times q})$

A Transformation matrix dimension is $p \times m$ & n should be equal to q.

$\therefore$ Dimension is pm.

b) $Hom_R(P_n^R, R^m)$

dimension $= nm$.

18) ~~Given~~ $L_1 : V \to W$, $L_2 : V \to W$ & $L_1^* \& L_2^*$ are adjoints.

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(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 0 & 0 & b_2 \end{array}\right] \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 1 & 0 & 0 & b_2/2 \end{array}\right].$$

Solutions exists only $b_2 = 0$

~~Infinit~~

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow 2R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

~~Yes~~, Infinitely many solutions exist.

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 2 & 4 & 6 & 20 \end{array}\right] \xrightarrow{R_2 \leftarrow 2R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$

Yes, Infinitely many solution exist.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow 2R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

~~This~~ No Solutions.

c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 1 & 0 & 0 & b_2 \end{array}\right]$ $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $b \in R^2$

No, It has infinite solutions.

d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$\det(A) = -1(2-6) = 4.$

$\therefore$ Inverse exists.

$\therefore$ $x = A^{-1}b.$ $\forall\, b \in R^3.$

~~So~~ It has unique solution.

11) Given

$$\|Px\|_2 = \|x\|_2$$

Let $x = m+n$, where $m \in R(P)$, $n \in R(P)$, so that $m \perp n$

$$\|x_2\|^2 = \|m+n\|_2^2$$

$$= \|m\|_2^2 + \|n\|_2^2$$

Then

$$\|x\|_2^2 = \|Px\|_2^2 = \|P(m+n)\|_2^2$$

$$= \|Pm\|_2^2 = \|m\|^2$$

Therefore, $n=0$ & thus $x = m \in R(P)$

10) $\underset{5\times 2}{A: U} \to \underset{3\times 1}{V}$; $A^*: V \to U$.

$u \in U$

$v \in V$

$$\langle v, A(u)\rangle_Q = v^H Q A u.$$

$$\langle u, A^*(v)\rangle = u^H R A^* v$$

$$\Rightarrow v^H Q A u = u^H R A^* v.$$

$$\Rightarrow QA = (R \cdot A^*)^H.$$

$$QA = R^H \cdot (A^*)^H \cdot R^H.$$

$$(A^*)^H = QA(R^H)^{-1}.$$

$$(A^*)^H = \begin{bmatrix} 1 & j & 0 \\ -j & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{3\times3} \begin{bmatrix} 3 & j \\ 2j & 0 \\ -j & 0 \end{bmatrix}_{3\times2} \begin{bmatrix} 0 & j \\ -j & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3-2 & j \\ -3j+4j-j & 1 \\ 2j & 0 \end{bmatrix} \begin{bmatrix} 1 & -j \\ j & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & j \\ 0 & 1 \\ 2j & 0 \end{bmatrix} \begin{bmatrix} 1 & -j \\ j & 0 \end{bmatrix}$$

$$(A^*)^H = \begin{bmatrix} 0 & -1 \\ j & 0 \\ 2j & -2 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 0 & -j & -2j \\ -1 & 0 & -2 \end{bmatrix}_{2\times3}.$$

Adjoint of A.

9) Given,

P_1, P_2 is an orthogonal projection

(i) We need to show $(P_2-P_1)^T = P_2^T - P_1^T$ & $(P_2-P_1)^2 = (P_2-P_1)$ for it to be a orthogonal projection

(i) $(P_2-P_1)^2 = P_2^2 + P_1^2 - 2P_2P_1$

Since P_1 & P_2 are orthogonal projections

$P_1^2 = P_1$, $P_2^2 = P_2$ & $P_1P_2 = P_2P_1 = P$.

$= P_2 + P_1 - 2P_1 = P_2 - P_1$.

$\Rightarrow (P_2-P_1)^2 = P_2 - P_1$.

(ii) $(P_2 - P_1)^T = P_2^T - P_1^T = P_2 - P_1$.

$\therefore$ It is orthogonal projection.

8) Given,

$L_1 : V \to W$, $L_2 : V \to W$, are Linear Transformation.

a) $(L_1 + L_2)^* = L_1^* + L_2^*$

$\langle (L_1+L_2)V, W \rangle = \langle V, (L_1+L_2)^* W \rangle$ [because adjoint].

$= \langle L_1 V, W \rangle + \langle L_2(V), W \rangle$

$= \langle V, L_1(W) \rangle + \langle V, L_2^*(W) \rangle$

$= \langle V, L_1^*(W) + L_2^*(W) \rangle$.

$= \langle V, L_1^* + L_2^* (W) \rangle$.

$\Rightarrow (L_1+L_2)^* = L_1^* + L_2^*$

b) $(\alpha L_1)^* = \alpha^* L_1^*$

$\langle (\alpha L) V, W \rangle = \langle V, (\alpha L)^* W \rangle$

$= \langle \alpha L(V), W \rangle$.

$= \alpha \langle L(V), W \rangle$

$= \alpha \langle V, L^*(W) \rangle$

$= \langle V, \bar{\alpha} L^*(W) \rangle$. $\Rightarrow$ ~~(αL_1)~~

d) $(L_1^*)^* = L_1$

$\langle (L_1^*)^*(V), W \rangle = \langle V, (L_1^*)^*(W) \rangle$

$= \langle V, L^*(W) \rangle$

$= \langle L(V), W \rangle$

$= \langle W, L(V) \rangle$

$\Rightarrow (L_1^*)^* = L$