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Linear Algebra

1) $(1,1)$, $(2,1)$, $(3,4)$

a) $y = \alpha x + \beta$

$1 = \alpha + \beta$

$1 = 2\alpha + \beta$

$4 = 3\alpha + \beta$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$A \quad X \quad = \quad b$

$$X = (A^T A)^{-1} A^T b$$

$$= \left[\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}_{3\times2} \right]^{-1} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 1+2+3 \\ 1+2+3 & 1+1+1 \end{bmatrix}^{-1} \begin{bmatrix} 1+2+12 \\ 1+1+4 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 45-36 \\ -90+84 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$$\boxed{\alpha = \frac{3}{2}, \quad \beta = 1}$$

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b) $x = \alpha y + \beta$

$1 = \alpha + \beta$

$2 = \alpha + \beta$

$3 = 4\alpha + \beta$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$$X = (A^T A)^{-1} A^T \cdot b$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \left[\underset{2\times3}{\begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{bmatrix}} \right]^{-1} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+16 & 1+1+4 \\ 1+1+4 & 1+1+1 \end{bmatrix}^{-1} \begin{bmatrix} 1+2+12 \\ 1+2+3 \end{bmatrix}$$

$$= \frac{1}{54-36} \begin{bmatrix} 3 & -6 \\ -6 & 18 \end{bmatrix} \begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

$$= \frac{1}{18} \begin{bmatrix} 9 \\ 18 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

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2) $A^H = -A$, $A \in \mathbb{C}^{n \times n}$

$Ax = \lambda x$; λ is eigen value of A; for vector x

$\cancel{y^H A x = \lambda y^H x}$. $x^H A^H = \lambda^* x^H$

$\cancel{x^H A^H y = \lambda^* x^H y}$.

$\cancel{-x^H A y = \lambda^* x^H y}$.

$-x^H \lambda x$

$x^H A x = \lambda x^H x$.

$x^H A^H x = \lambda^* x^H x$.

$-x^H A x = \lambda^* x^H x$.

$+x^H \lambda x = (-\lambda^*) x^H x$.

$\Rightarrow \quad x^H x (\lambda + \lambda^*) = 0$

Since $x^H x \neq 0$, $\quad \lambda + \lambda^* = 0$

$\Rightarrow \quad \lambda = -\lambda^*$

This is only possible if λ is imaginary

$\therefore$ Eigen values of A are Imaginary

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3) $A = \begin{bmatrix} 1 & 1+j \\ 2-2j & 4 \end{bmatrix}$

$|A - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 1+j \\ 2-2j & 4-\lambda \end{bmatrix} \right| \Rightarrow \lambda^2 - 5\lambda = 0.$

$\Rightarrow \lambda = 0, \lambda = 5$

$(A - 0)x = 0.$

$\Rightarrow Ax = 0.$

$\begin{bmatrix} 1 & 1+j \\ 2-2j & 4-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} -(1+j) \\ 1 \end{bmatrix}$

$(A - 5I)x = 5x.$

$\begin{bmatrix} -4 & 1+j \\ 2-2j & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2-2j \end{bmatrix}$

$\Rightarrow V = \begin{bmatrix} -(1+j) & 1 \\ 1 & (2-2j) \end{bmatrix}$

$\Rightarrow V^{-1} = \frac{-1}{5} \begin{bmatrix} (2-2j) & -1 \\ -1 & -(1+j) \end{bmatrix}$

$V^{-1}AV = \frac{-1}{5} \begin{bmatrix} 2+2j & -1 \\ -1 & -(1+j) \end{bmatrix} \begin{bmatrix} 1 & 1+j \\ 2-2j & 4 \end{bmatrix} \begin{bmatrix} -(1+j) & 1 \\ 1 & (2-2j) \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

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(b) Left Eigen Vector:

$$\cancel{y^T A}\quad x^H A = \lambda \cdot x^H$$

$$\Rightarrow \det\left|(A^H - \lambda^* I)\right| = 0$$

$$\det(A^H - \lambda^* I) = \begin{bmatrix} 1-\lambda^* & 1-j \\ 2+2j & 4-\lambda^* \end{bmatrix} \quad \left|\begin{bmatrix} 1-\lambda & 2+2j \\ 1-j & 4-\lambda \end{bmatrix}\right|$$

$$\lambda = 0, 5$$

$\lambda = 0$

$$\begin{bmatrix} 1 & 2+2j \\ 1-j & 4 \end{bmatrix} x = 0$$

$$x = \alpha \begin{bmatrix} -(2+2j) \\ 1 \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix} -4 & 2+2j \\ 1-j & -1 \end{bmatrix} x = 0$$

$$x = \alpha \begin{bmatrix} \frac{1}{4}(2+2j) \\ 1 \end{bmatrix}$$

$$W = \begin{bmatrix} -(2+2j) & \frac{1}{4}(2+2j) \\ 1 & 1 \end{bmatrix}$$

$$W^H A V = \begin{bmatrix} -(2-2j) & 1 \\ \frac{1}{4}(2-2j) & 1 \end{bmatrix} \begin{bmatrix} 1 & 1+j \\ 2-2j & 4 \end{bmatrix} \begin{bmatrix} -(1+j) & 1 \\ 1 & (2-2j) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

6) A, $n \times n$ strictly upper or lower

$$A_{n\times n} = \begin{bmatrix} 0_{n-1\times 1} & A' \\ 0_1 & 0_{1\times n-1} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0_{n-1\times 1} & A' \\ 0_1 & 0_{1\times n-1} \end{bmatrix} \begin{bmatrix} 0_{n-1\times 1} & A' \\ 0_1 & 0_{1\times n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ A strictly lower or upper is always zero.

if nilpotent

$A^r = 0$ for $r \in Z$

$Ax = \lambda x$ $\quad x \neq 0$

$A^2 x = \lambda A x$

$\Rightarrow Ax = 0(x)$

$\therefore$ Eigen values zero

if $\lambda = 0$

$Ax = 0$

$\Rightarrow A^2 x = 0$

$\Rightarrow A^r x = 0$

Since $x \neq 0$, $A^r = 0$

7)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det(A - \lambda I) =$

$(1-\lambda)(-3-\lambda)(1-\lambda)$

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$\phi(A) = (1-A)(-3-A)(1-A) = 0$

$\Rightarrow A^3 + A^2 - 5A + 3I = 0$

$\Rightarrow A(A^2 + A - 5I) = -3I.$

$\frac{-1}{3}(A^2 + A - 5I) = A^{-1}$

$$A^2 = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 9 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 + A - 5I = \begin{bmatrix} 2 & -2 & 5 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{3}\begin{bmatrix} -3 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

4)

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 1 & 1 & & & \\ & 2-\lambda & 2 & & & \\ & & 3-\lambda & & & \\ & & & 1-\lambda & 2 & 3 \\ & & & & 4-\lambda & 4 \\ & & & & & 5-\lambda \end{bmatrix}$$

8) $\det(A-\lambda I) =$
$$(1-\lambda)^2\,(2-\lambda)\,(3-\lambda)\,(4-\lambda)\,(5-\lambda)$$

$\lambda = 1,\ m=2,$
$\lambda=2,\ \lambda=3,\ \lambda=4,\ \lambda=5,\quad m=1$

If $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 1 & & & \\ & 1 & 2 & & & \\ & & 2 & & & \\ & & & 0 & 2 & 3 \\ & & & & 3 & 4 \\ & & & & & 4 \end{bmatrix}$$

$\text{rank}(A-\lambda I) = 4.$
$\text{nullity}(A-\lambda I) = 2.$; $\therefore$ 2 independent vectors.

So

$$\text{Jordan form of } A = \begin{bmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & 3 & & \\ & 0 & & 4 & \\ & & & & 5 \end{bmatrix}$$

(B)

$$B - \lambda I = \begin{bmatrix} 2-\lambda & 3 & 1 \\ 0 & 2-\lambda & -2 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(B - \lambda I) = (2-\lambda)^2 (1-\lambda)$$

$\lambda = 2$, $m = 2$

$\lambda = 1$

If $\lambda = 2$

$$(B - \lambda I) = \begin{bmatrix} 0 & 3 & 1 \\ & 0 & -2 \\ & & -1 \end{bmatrix}$$

$\text{rank}(B - 2I) = 2.$

$\Rightarrow$ Nullity $= 1.$

It needs two But multiplicity is 2.

J_2 is of the form

$$J_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$J(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

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5) Given

$A = uv^H$, rank $= 1$.

u, v column vectors.

$\det(A - \lambda I) = 0$

$\Rightarrow$ $\lambda = 0$ is an eigen value.

$$Au = (uv^H)u$$
$$= u(v^H u)$$
$$= (v^H u)u.$$

$\therefore (v^H u)$ is eigenvalue of A.

Assuming, $V \neq 0$, The orthogonal complement of linear subspace generated by v is $(n-1)$ dimensional.

Let $\phi_1, \ldots \phi_{n-1}$ be basis for their space.

Then they are linearly independent. & $u^T v \phi_i = (v\phi_i)u = 0$

$\therefore$ Eigen values '0' has multiplicity $n-1$.

Eigen vector $\lambda = 0$,

Nullity $(uv^H) = n - 1$, $n-1$ Independent vectors

$$J = \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & \ddots & 0 \\ & & & v^H u \end{bmatrix}$$