Hamiltonian Cycle is NP-Complete

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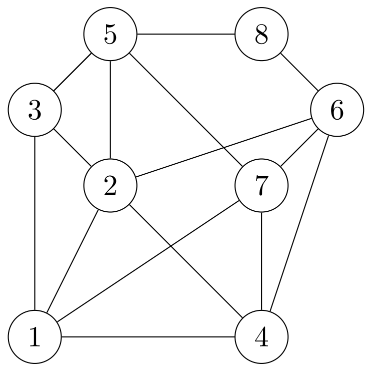
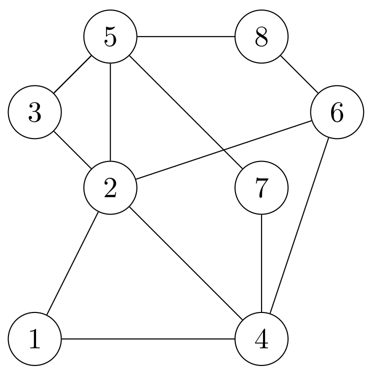
Definition 1. *A* Hamiltonian cycle *on a graph $G = (V, E)$ is a cycle that visits every vertex.*



$1 - 2 - 3 - 5 - 8 - 6 - 7 - 4 - 1$
is a Hamiltonian cycle



This graph does not have
a Hamiltonian cycle

Definition 2. *Let $G = (V, E)$ be a graph. The* Hamiltonian cycle *feasibility problem is to determine whether there is a Hamiltonian cycle in $G = (V, E)$.*

Theorem 1. *The Hamiltonian cycle problem is $\mathcal{N}P$-complete.*

First show the problem is in $\mathcal{N}P$:
Our certificate of feasibility consists of a list of the edges in the Hamiltonian cycle. We can check quickly that this is a cycle that visits every vertex.

We show that **3-SAT** can be polynomially transformed to the Hamitonian cycle problem. This requires the construction of two sets of gadgets, and specification of how to link them together.

We have an instance of 3-SAT with n boolean variables $z_1, \ldots, z_n$ and m clauses $C_1, \ldots, C_m$, with each clause containing exactly 3 literals.

The first type of gadget is to force satisfaction of a clause. Each clause is mapped into a gadget with 13 vertices, shown in Figure 1. There are multiple ways to traverse the gadget, each corresponding to different ways to satisfy the clause. Any Hamiltonian cycle that enters

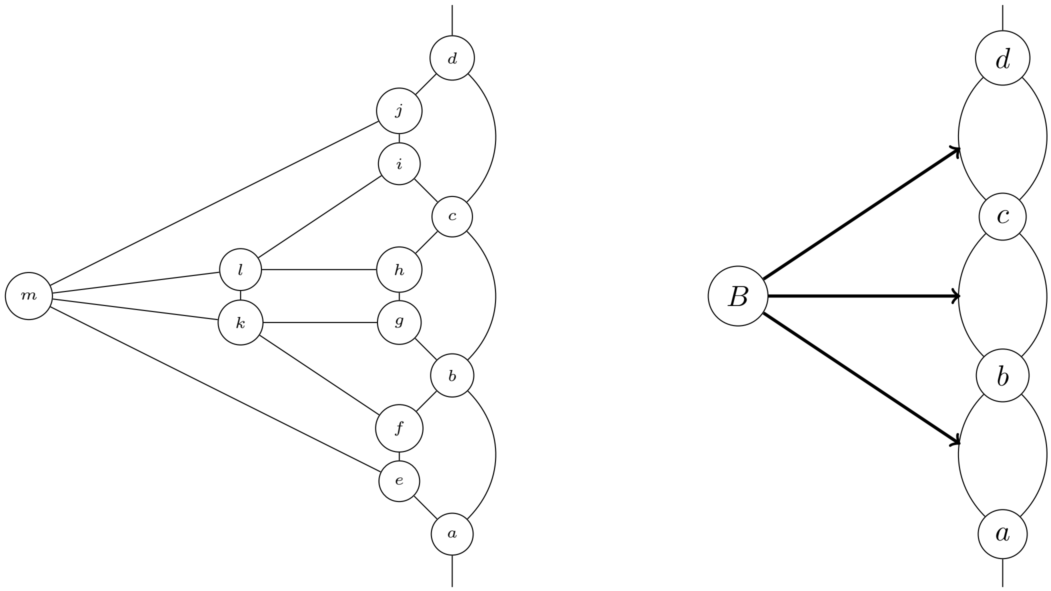


Figure 1: Gadget B, to ensure each clause is satisfied

at vertex a and leaves at vertex d must visit all the other vertices of the gadget in between. There are multiple ways to traverse the gadget.

Each of the edges (a, b), (b, c), (c, d) corresponds to a literal in the clause. **Not traversing** the edge corresponds to the literal taking the value TRUE. It is **not possible to traverse all three of the edges** (a, b)**,** (b, c)**,** (c, d), but any subset of the edges can be traversed. Hence, any Hamiltonian cycle will correspond to each clause being satisfied. Note also that if any of the edges (a, b), (b, c), (c, d) is traversed, then it must be traversed from bottom to top. A shorthand representation of the gadget is given on the right.

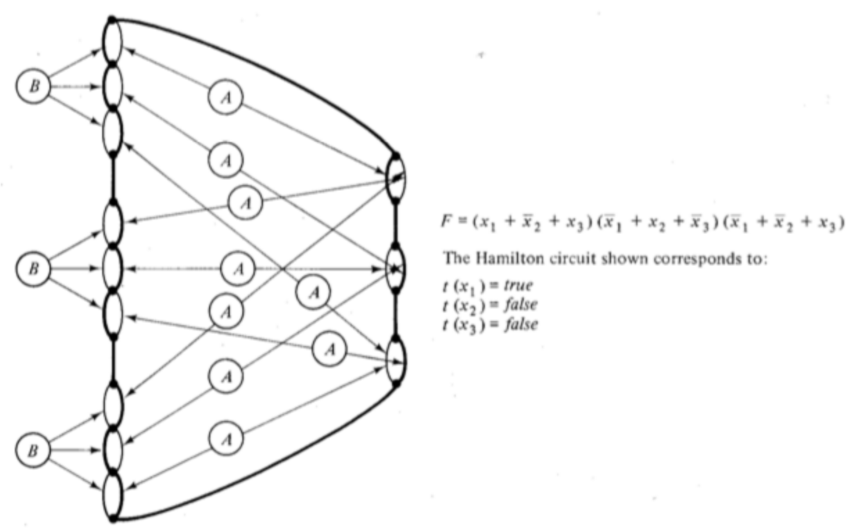


Figure 15–10

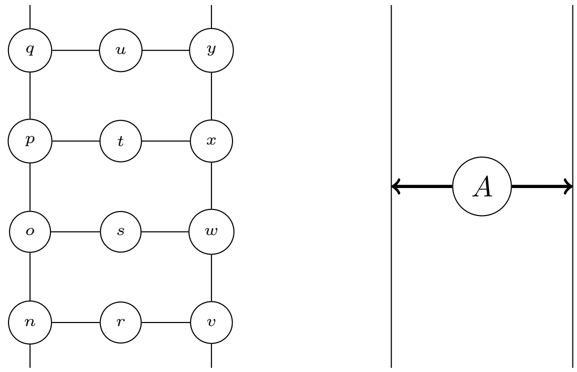


Figure 2: Gadget A to ensure consistency

The gadgets B are placed in series, one for each clause. A pair of edges is constructed for each variable, as shown in Figure 15-10 taken from the text by Papadimitriou and Steiglitz. Traversing the left hand edge of the two corresponds to setting the boolean literal to TRUE, and traversing the right corresponds to setting it to FALSE. As long as at least one of the right hand edges is not traversed, the clause is satisfied.

Consistency can be ensured by constructing another gadget, pictured in Figure 2. If we traverse an edge corresponding to a literal taking the value FALSE, we need to ensure that the literal takes the value FALSE in every other clause where it appears.

We can traverse the gadget in two ways: (i) enter below n and leave above q, or (ii) enter above y and leave below v. A copy of this gadget will be used for every literal. The vertices n through q are placed on the right hand edges of the B gadgets. If this right hand edge corresponds to the unnegated literal z_j then the vertices y through v are placed on the TRUE side for variable z_j in the pair of edges on the right of the graph. Conversely, if the right hand edge of the B gadget corresponds to the literal $\bar{z}_j$ then the vertices y through v are placed on the FALSE side for variable z_j in the pair of edges on the right of the graph. The use of the A gadgets is illustrated in Figure 15-10 from Papadimitriou and Steiglitz.

Definition 3. *An instance of the* traveling salesman problem (TSP) with upper bound *is defined by a complete graph K_n with edge weights w_e and an upper bound B. The answer to the feasibility problem is* YES *if there exists a Hamiltonian tour in the graph of length no greater than B; otherwise, the answer is* NO*.*

Theorem 2. *TSP with upper bound is NP-complete.*

Proof. Any tour is a certificate of feasibility, so the problem is in $\mathcal{N}P$.

We reduce Hamiltonian cycle to TSP with upper bound. Given an instance of Hamiltonian cycle on graph $G = (V, E)$, we construct a TSP instance by using the same vertices and giving the edges of the complete graph the following weights:

$$w_e = \begin{cases} 1 & \text{if } e \in E \\ 2 & \text{if } e \notin E \end{cases}$$

The upper bound is chosen as $B = |V|$. There exists a Hamiltonian cycle in the original graph if and only if there is a TSP tour which only uses edges with $w_e = 1$, so if and only if there is a TSP tour of length no greater than $|V|$. □

Note that the edge lengths in the constructed TSP instance satisfy the *triangle inequality*:

$$w_{ab} + w_{bc} \geq w_{ac} \quad \text{for any vertices } a, b, c.$$

Hence, we've proved the restricted version of the TSP where the edge lengths must satisfy the triangle inequality is also $\mathcal{NP}$-complete.

Definition 4. *A* Hamiltonian path *on a graph $G = (V, E)$ is a path that visits every vertex.*

Definition 5. *Let $G = (V, E)$ be a graph. The* Hamiltonian path *feasibility problem is to determine whether there is a Hamiltonian path in $G = (V, E)$.*

Theorem 3. *The Hamiltonian path problem is $\mathcal{NP}$-complete.*

Proof. Any Hamiltonian path is a certificate of feasibility, so the problem is in $\mathcal{NP}$.

We reduce an instance of Hamiltonian cycle on a graph $G = (V, E)$ to Hamiltonian path in two different ways. (There are many ways to do this.)

- *First reduction:* Pick any vertex $v \in V$, split it into two vertices v_1 and v_2. If $(v, u) \in E$ then edges (v_1, u) and (v_2, u) are included in the new graph. Introduce two new nodes s and t and two new edges (s, v_1) and (v_2, t). The nodes s and t are leaves in the new graph, so they must be the endpoints of any Hamiltonian path in the new graph. This Hamiltonian path directly returns a Hamiltonian cycle in the original graph.

- *Second reduction:* The second reduction requires more work. For each edge $e = (u, v)$ in E, perform the following steps:

  1. Remove edge e from E.
  2. Add two new nodes s and t and two new edges (s, u) and (v, t).
  3. Search for a Hamiltonian path in the modified graph. If one exists, it must have endpoints s and t, so it must correspond to a Hamiltonian cycle in the original graph.

  The number of calls to the Hamiltonian path algorithm is equal to the number of edges in the original graph with the second reduction.

Hence the $\mathcal{NP}$-complete problem Hamiltonian cycle can be reduced to Hamiltonian path, so Hamiltonian path is itself $\mathcal{NP}$-complete. □