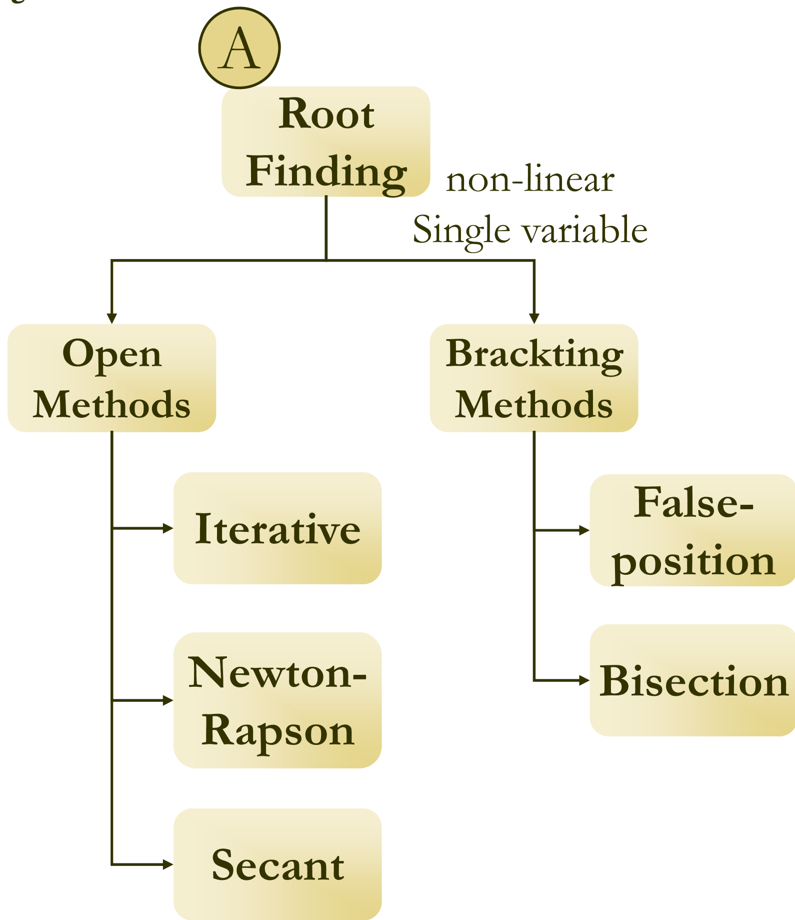
A.3

Newton-Raphson Method

Equation Solving

- Given an approximate location (initial value)
- find a single real root



Method

- We want to solve $f(x)=0$ near x_r

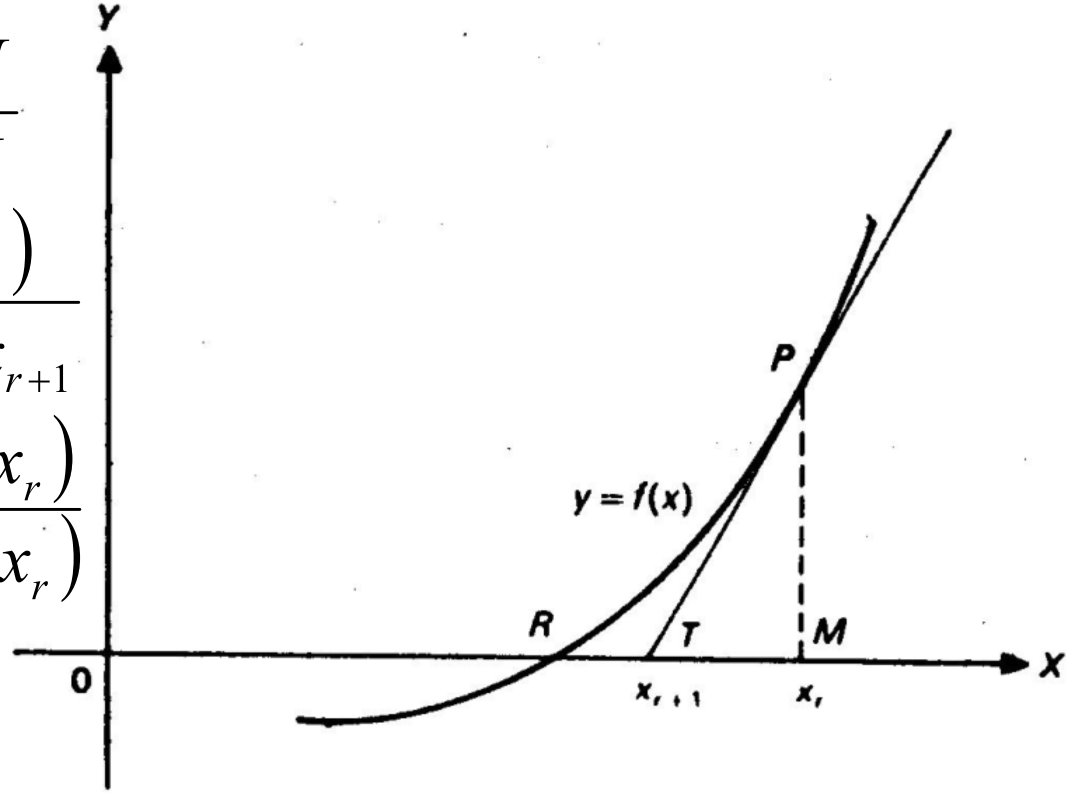
x_r = approximate root

x_{r+1} = intersection of $f'(x_r)$ and x-axis

$$\tan\angle PTM = \frac{PM}{TM}$$

$$\Rightarrow f'(x_r) = \frac{f(x_r)}{x_r - x_{r+1}}$$

$$\therefore \quad x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$



Algorithm

$x_{r+1} \leftarrow$ initial guess value

do {

$$x_r \leftarrow x_{r+1}$$

$$x_{r+1} \leftarrow x_r - \frac{f(x_r)}{f'(x_r)}$$

} while ($| x_{r+1} - x_r | >$ Q)

We can compute the derivative numerically as follows:

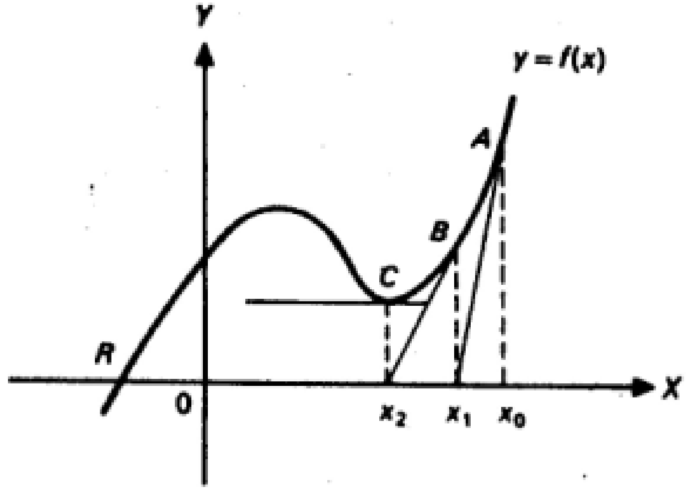
$$f'(x_r) = \frac{f(x_r + h) - f(x_r - h)}{2h}$$

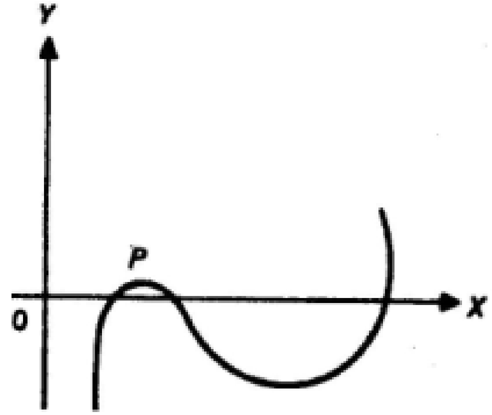
Limitations

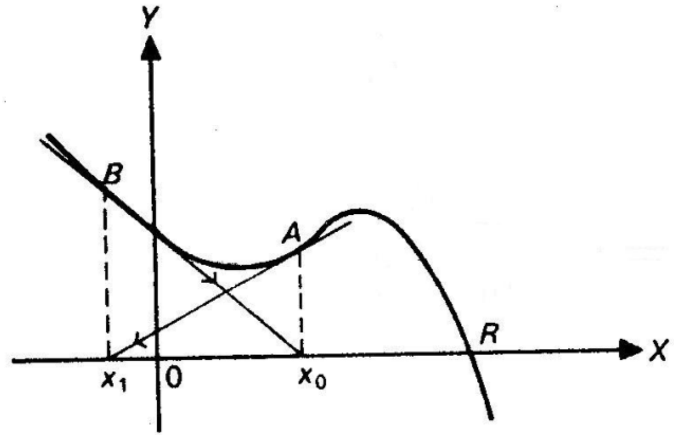
1. $f'(x_{r+1})=0$, local minima.

2. $f'(x_{r+1})\approx 0$, occurs when two roots are very close.

3. x_r and x_{r+1} recurs







Convergence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= a - \varepsilon_n - \frac{f(a-\varepsilon_n)}{f'(a-\varepsilon_n)} \qquad [\because x_n = a - \varepsilon_n]$$

$$= a - \varepsilon_n - \frac{f(a) - \varepsilon_n f'(a) + \frac{1}{2}\varepsilon_n^2 f''(a) - \cdots}{f'(a) - \varepsilon_n f''(a) + \frac{1}{2}\varepsilon_n^2 f'''(a) - \cdots}$$

$$\approx a - \varepsilon_n + \varepsilon_n \left[1 - \frac{1}{2}\varepsilon_n \frac{f''(a)}{f'(a)} - \cdots\right]\left[1 - \varepsilon_n \frac{f''(a)}{f'(a)}\right]^{-1} \qquad [\because f(a) = 0]$$

$$\approx a + \frac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)}$$

$$\therefore \varepsilon_{n+1} = -\frac{1}{2}\varepsilon_n^2 \frac{f''(a)}{f'(a)}$$

Second order convergence

Example

---

x^2-5x+4=0
0 Xr=5.000 Xr+1=4.200
1 Xr=4.200 Xr+1=4.012
2 Xr=4.012 Xr+1=4.000
3 Xr=4.000 Xr+1=4.000
3 Xr=4.000 Xr+1=4.000

---

e^(-x)-x=0
0 Xr=1.000 Xr+1=0.538
1 Xr=0.538 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567

---

xsin(x*x)-2cos(x)
0 Xr=1.000 Xr+1=0.538
1 Xr=0.538 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567
2 Xr=0.567 Xr+1=0.567

A.4

Secant Method

April 5, 2009

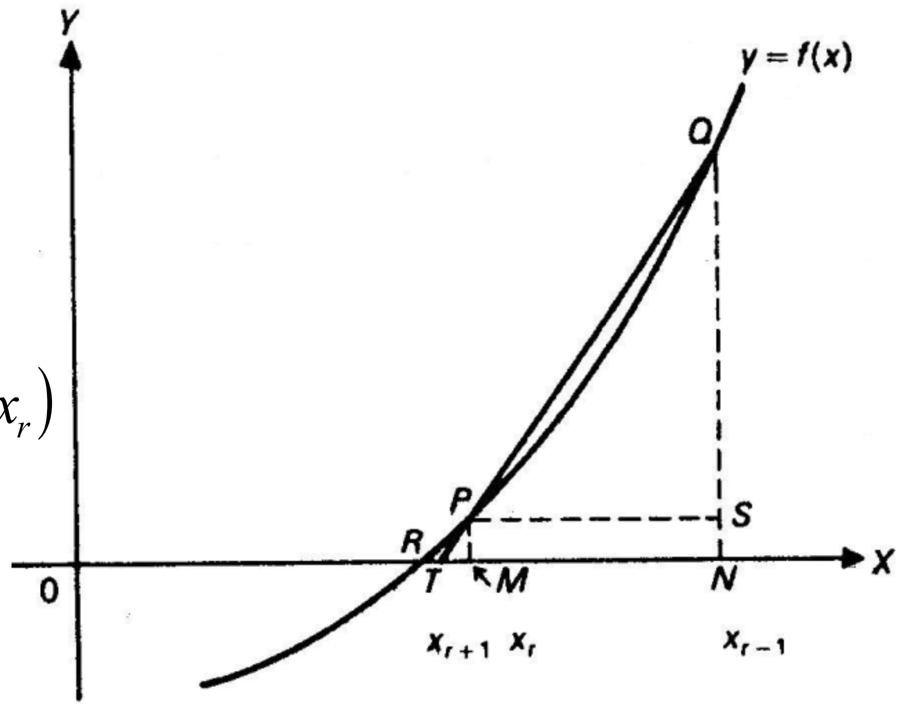
Concept

- Newton-Raphson method needs to compute *f′(x)*
  - It may be analytically complicated, or
  - Numerical evaluation may be time consuming

$$\frac{TM}{PM} = \frac{PS}{QS}$$

$$\Rightarrow \frac{x_r - x_{r+1}}{f(x_r)} = \frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)}$$

$$\Rightarrow x_{r+1} = x_r - \left[\frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)}\right] f(x_r)$$



- Tangent is replaced with chord ⇒ lower convergence rate

Algorithm

$x_r \leftarrow$ initial guess value

$x_{r+1} \leftarrow$ initial guess value

do {

$$x_{r-1} \leftarrow x_r$$

$$x_r \leftarrow x_{r+1}$$

$$x_{r+1} = x_r - \left[\frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)}\right] f(x_r)$$

} while ($| x_{r+1} - x_r | >$ Q)

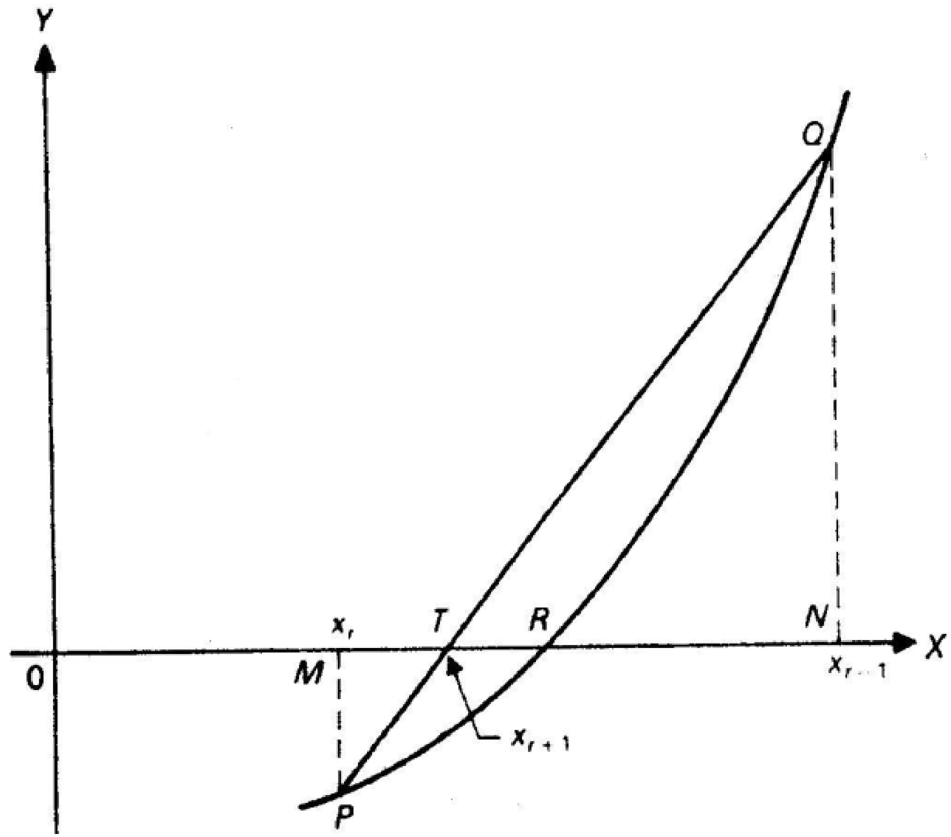
A.4

False Position Method

April 5, 2009

Concept

- Same as Secant method, except that the root is always between $\boldsymbol{x_r}$ and $\boldsymbol{x_{r-1}}$ i.e., $\boldsymbol{f(x_r)*f(x_{r-1}) < 0}$



- Convergence is lower than Newton-Raphson method
- Guarantees success

Algorithm

Choose x_r and x_{r-1} such that $f(x_r)*f(x_{r-1}) < 0$

do {

$$x_{r+1} \leftarrow x_r - \left[\frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)}\right] f(x_r)$$

$$if \;\; f(x_{r+1}) \times f(x_{r-1}) < 0 \;\; then \; x_r \leftarrow x_{r-1}$$

$$x_{r-1} \leftarrow x_r$$

$$x_r \leftarrow x_{r+1}$$

} *while* ($| \, x_{r+1} - x_r \, | >$ Q)