Truncation Errors and Tailors Series

Chapra: Chapter-4

Truncation Error

- *Truncation errors* are those that result from using an approximation in place of an exact mathematical procedure.
- *Example:*

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

- *How much error is introduced with this approximation?*
- We can use *Tailors Series* to estimate truncation error.

Tailors Series

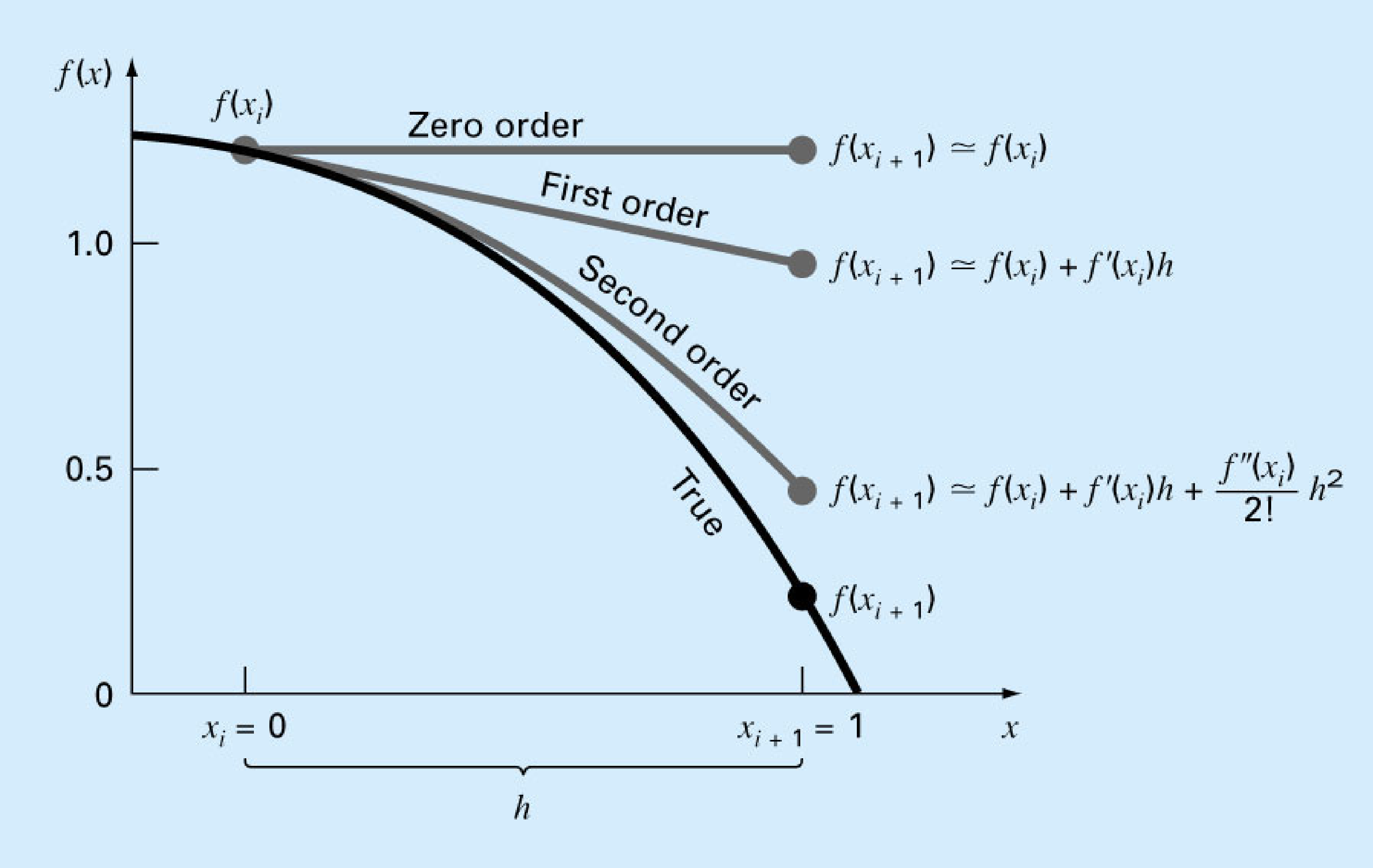
- *n^{th} order* approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''}{2!}(x_{i+1} - x_i)^2 + \ldots$$
$$+ \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

- $(x_{i+1}-x_i)= h$ *step size* (define first)

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)} \qquad x_i \le \varepsilon \le x_{i+1}$$

Reminder term, R_n, accounts for all terms from (n+1) to infinity.



$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

The approximation of *f(x)* at *x=1* by zero-order, fist-order and second-order Tailor series expansion

Insight

- Each additional term contributes to the approximation
- n^{th}-order Tailor Series gives exact value of n^{th}-order polynomial
- Inclusion of a few terms gives an approximation that is good enough for practical purpose.
- The Remainder:
  - ε is not exactly known.
  - Need to determine $f^{n+1}(x_{i+1})$, which require the determination of the $(n+1)$th derivative of $f(x)$. If we know $f(x)$ then we do not need to use Tailors series!
  - Yet, $R_n=O(h^{n+1})$ gives insight into error. E.g., if error is $O(h)$ then halving step size will halve the error. If error is $O(h^2)$ then halving the step size will quarter the error, and so on.

Effect of Step Size

Estimate

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i)$$

Actual

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

m=	68.1
c=	12.5
9=	9.81
Δt=	2

t	Actual	Estimate	Error%
0	0	0	
2	16.42172	19.62	19.47591
4	27.79763	32.03736	15.25213
6	35.67812	39.89621	11.82263
8	41.13722	44.87003	9.074043
10	44.91893	48.01792	6.899074
12	47.53865	50.01019	5.199019
14	49.35343	51.27109	3.885579
16	50.61058	52.06911	2.881848

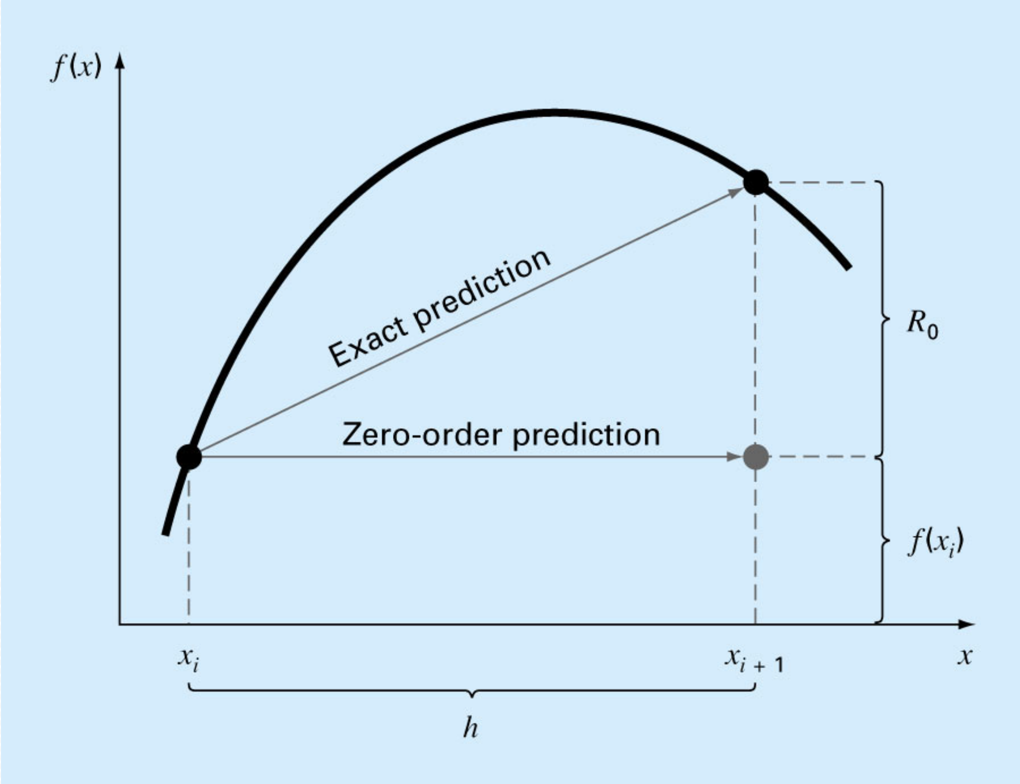
t	Actual	Estimate	Error%
0	0	0	
1	8.962318	9.81	9.458288
2	16.42172	17.81934	8.510793
3	22.63024	24.35854	7.637128
4	27.79763	29.69744	6.834436
5	32.09849	34.05637	6.099607
6	35.67812	37.6152	5.429315
7	38.65748	40.52079	4.820066
8	41.13722	42.89306	4.268249
9	43.20112	44.82988	3.770176
10	44.91893	46.4112	3.322138
11	46.34867	47.70225	2.920443
12	47.53865	48.75633	2.561458
13	48.52908	49.61693	2.241647
14	49.35343	50.31957	1.957596
15	50.03953	50.89323	1.706043
16	50.61058	51.36159	1.483897

Insight : R_n

$$f(x_{i+1}) \cong f(x_i)$$

$$R_0 = f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!} + \cdots\cdots$$

$$R_0 \cong f'(x_i)h$$

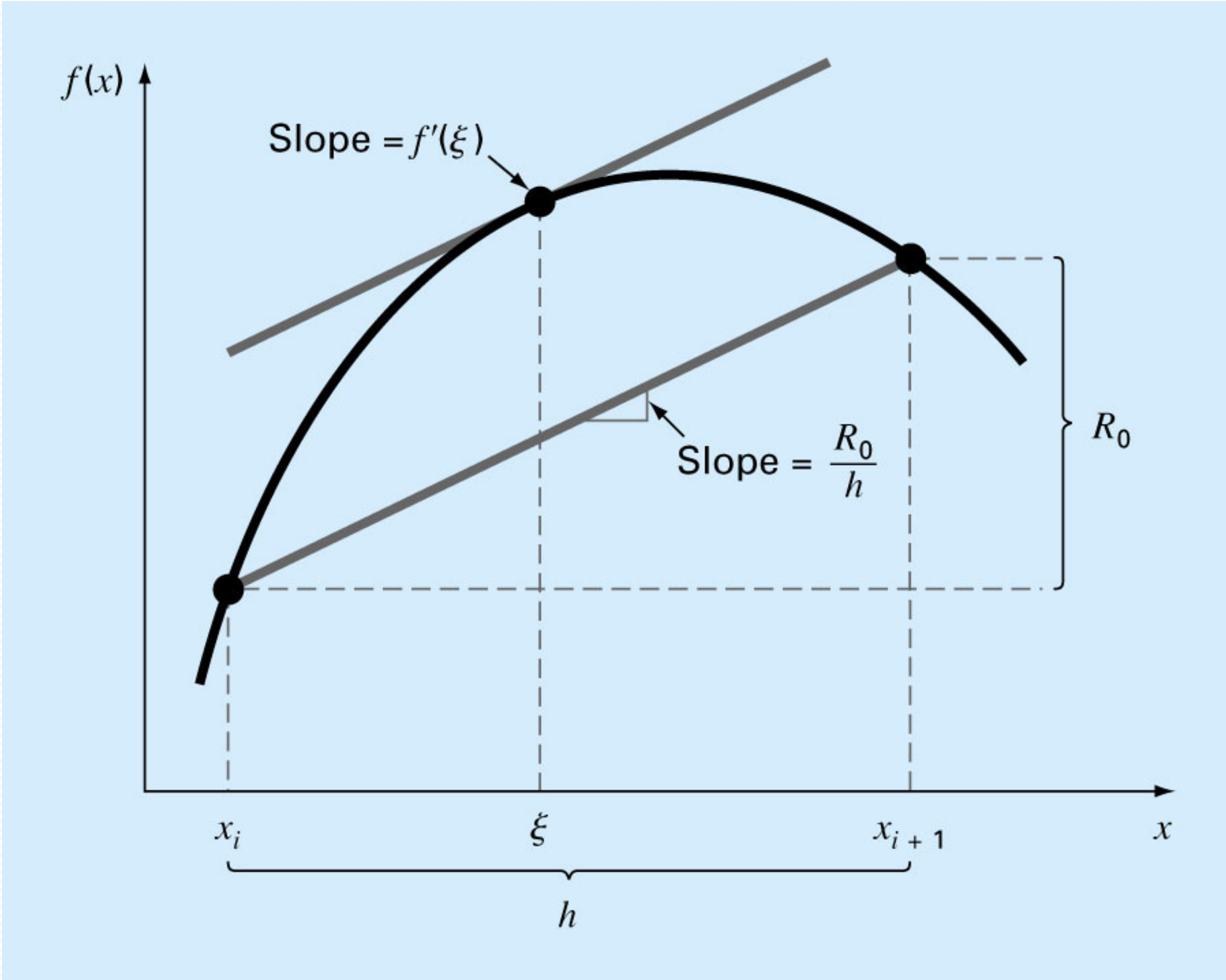


Insight : R_n

$$R_0 \cong f'(x_i)h$$

$$R_0 = f'(\varepsilon)h$$

$$R_1 = \frac{f''(\varepsilon)}{2!}h^2$$



How to get derivatives?

- We will be given value of unknown $f(x)$ for some value of x

- We can estimate $f^{n}(a)$, i.e. the n^{th} order derivate of $f(x)$ at $x=a$ numerically without knowing $f(x)$