Function defined on a measure space:-

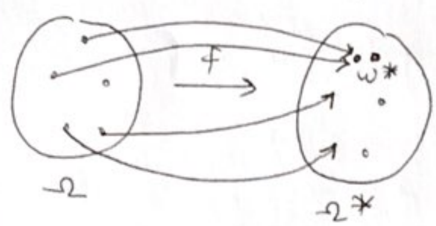
Up till now, we have studied measures, which are set functions. Set function are not easy to handle with since the basic arithmetic operations can not be performed on sets. So we define function on ~~measurable~~ mble space.

Let $(\Omega, \mathcal{A})$ be a mble space.

Let $f : \Omega \to \Omega^*$

(if $\Omega^* = \mathbb{R}$, then f is a real valued function).

$f : \Omega \to \Omega^*$ means for $\omega \in \Omega$, $f(\omega) \in \Omega^*$.



$$\{\omega \mid f(\omega) = \omega^*\} \subset \Omega$$

__Def__: $f^{-1}(\omega^*) = \{\omega \in \Omega \mid f(\omega) = \omega^*\}$

__Def__: for $A^* \subset \Omega^*$

$$f^{-1}(A^*) = \{\omega \mid f(\omega) \in A^*\} \subset \Omega$$
$$= \bigcup_{\omega^* \in A^*} f^{-1}(\omega^*)$$

Let $\{A_1^*, A_2^*\}$ be a collection of sets of Ω^*.

__Def__. then $\{f^{-1}(A_1^*), f^{-1}(A_2^*)\} = f^{-1}\{A_1^*, A_2^*\}$

In general, For a class $\mathcal{C}^*$ of subsets of Ω^*,

$$f^{-1}(\mathcal{C}^*) = \{A \subset \Omega \mid A = f^{-1}(A^*), \text{ for some } A^* \in \mathcal{C}^*\}.$$

$$f^{-1}(A^* \cup B^*) = \{\omega \mid f(\omega) \in A^* \cup B^*\}$$
$$= \{\omega \mid f(\omega) \in A^*\} \cup \{\omega \mid f(\omega) \in B^*\}$$
$$= f^{-1}(A^*) \cup f^{-1}(B^*)$$

In general

Let T be an indexing set (may be finite, infinite, countable, uncountable)

then $f^{-1}\left(\bigcup_{t\in T} A_t^*\right) = \bigcup_{t\in T} f^{-1}(A_t^*)$

Similarly,

$f^{-1}(A^* \cap B^*) = f^{-1}(A^*) \cap f^{-1}(B^*)$

& hence $f^{-1}\left(\bigcap_{t\in T} A_t^*\right) = \bigcap_{t\in T} f^{-1}(A_t^*)$.

—x—

Thm : $f^{-1}(A^*)' = [f^{-1}(A^*)]'$.

Proof :- Consider

$$f^{-1}(A^*)' = \{\omega \mid f(\omega) \in (A^*)'\}$$

$$= \{\omega \mid f(\omega) \notin A^*\}$$

$$= \{\omega \mid f(\omega) \in A^*\}'$$

$$= [f^{-1}(A^*)]'$$

—x—

Thm 1 : Inverse image of a σ-field is a σ-field.

Let $f : \Omega \to \Omega^*$.

let $\mathcal{A}^*$ be a σ-field of subsets of Ω^*.

~~i.e. $\mathcal{A}^* = \{A^* \mid A^* \subset \Omega^*\}$ be a σ-field.~~

i.e. $\mathcal{A}^* = \{A^* \mid A^* \subset \Omega^*\}$ be a σ-field.

then $\mathcal{A} = f^{-1}(\mathcal{A}^*)$ is also a σ-field.

Proof : clearly $\mathcal{A}$ is non empty because

$\Omega = f^{-1}(\Omega^*) \in f^{-1}(\mathcal{A}^*) = \mathcal{A}$.

Let $A \in \mathcal{A}$ to prove $A' \in \mathcal{A}$.

Now $A \in \mathcal{A} \Rightarrow A \in f^{-1}(\mathcal{A}^*)$

$\Rightarrow A = f^{-1}(A^*)$ for some $A^* \in \mathcal{A}^*$

but $\mathcal{A}^*$ is a σ-field $\Rightarrow (A^*)' \in \mathcal{A}^*$

(16) Now $A = f^{-1}(A^*)$

$\therefore A' = [f^{-1}(A^*)]'$

$= f^{-1}(A^*)'$ where $(A^*)' \in \mathcal{A}^*$

$\Rightarrow A' \in f^{-1}(\mathcal{A}^*)$ i.e $A' \in \mathcal{A}$

$\Rightarrow \mathcal{A}$ is closed under compliments.

Further let $A_n \in \mathcal{A}$, $n = 1, 2, 3, \dots$

i.e $A_n \in f^{-1}(\mathcal{A}^*)$ $\forall n$

i.e $A_n = f^{-1}(A_n^*)$ for some $A_n^* \in \mathcal{A}^*$

$$\therefore \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} f^{-1}(A_n^*)$$

$$= f^{-1}\left(\bigcup_{n=1}^{\infty} A_n^*\right)$$

but $\mathcal{A}^*$ is a σ-field $\Rightarrow \bigcup_{n=1}^{\infty} A_n^* \in \mathcal{A}^*$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in f^{-1}(\mathcal{A}^*) = \mathcal{A}$$

$\Rightarrow \mathcal{A}$ is closed under countable unions.

$\Rightarrow \mathcal{A}$ is a σ-field.

—x—

Thm: Let $\mathcal{A}$ be a σ-field of subsets of Ω.

Then $\mathcal{C}^* = \{A^* \mid f^{-1}(A^*) \in \mathcal{A}\}$ is a σ-field of subsets of Ω^*.

Proof: Note that $\mathcal{A}$ is a σ-field

$\Rightarrow \Omega \in \mathcal{A}$

but $\Omega = f^{-1}(\Omega^*)$

$\Rightarrow \Omega^* \in \mathcal{C}^*$

So that $\mathcal{C}^*$ is non-empty.

Let $A^* \in \mathcal{C}^*$, to prove $(A^*)' \in \mathcal{C}^*$.

Now $A^* \in \mathcal{C}^*$

$\Rightarrow f^{-1}(A^*) \in \mathcal{A}$

$\Rightarrow [f^{-1}(A^*)]' \in \mathcal{A}$

$\Rightarrow f^{-1}[(A^*)'] \in \mathcal{A}$

$\Rightarrow (A^*)' \in \mathcal{C}^*$

$\Rightarrow \mathcal{C}^*$ is closed under compliments.

(47) Finally, let $A_n^* \in \mathcal{C}^*$, $n=1,2\ldots$ $\longrightarrow f^{-1}(A_n^*) \in \mathcal{A}\ \forall n$

to prove $\bigcup_n A_n^* \in \mathcal{C}^*$.

Consider $f^{-1}(\bigcup_n A_n^*) = \bigcup_{n=1}^{\infty} f^{-1}(A_n^*) \in \mathcal{A}$, $\because \mathcal{A}$ is a σ-field.

$$\Rightarrow \bigcup_n A_n^* \in \mathcal{C}^*$$

$\Rightarrow \mathcal{C}^*$ is closed under countable unions.

& hence $\mathcal{C}^*$ is also a σ-field.

—x—

Remark: If $A^* \subset B^*$

$\Rightarrow f^{-1}(A^*) \subset f^{-1}(B^*)$

Similarly, if $\mathcal{C}_1^*$ & $\mathcal{C}_2^*$ are two collection of subsets of Ω^*.

then $\mathcal{C}_1^* \subset \mathcal{C}_2^* \Rightarrow f^{-1}(\mathcal{C}_1^*) \subset f^{-1}(\mathcal{C}_2^*)$.

—x—

Thm: $\sigma[f^{-1}(\mathcal{C}^*)] = f^{-1}[\sigma(\mathcal{C}^*)]$

where $\mathcal{C}^*$ is a collection of subsets of Ω^*.

Proof: Recall $\mathcal{C}^* \subset \sigma(\mathcal{C}^*)$

$\therefore f^{-1}(\mathcal{C}^*) \subset f^{-1}[\sigma(\mathcal{C}^*)]$

$\therefore \sigma[f^{-1}(\mathcal{C}^*)] \subset f^{-1}[\sigma(\mathcal{C}^*)]$ — (1)

because $f^{-1}\sigma(\mathcal{C}^*)$ is a σ-field, by thm 1.

Next to prove $f^{-1}[\sigma(\mathcal{C}^*)] \subset \sigma[f^{-1}(\mathcal{C}^*)]$

Consider the class of sets

$\mathcal{D} = \{B \subset \Omega^* \mid f^{-1}(B) \in \sigma[f^{-1}(\mathcal{C}^*)]\}$

then, by thm 2, $\mathcal{D}$ is a σ-field.

~~Let~~ Suppose a set $E \in \mathcal{C}^*$

then $f^{-1}(E) \in f^{-1}(\mathcal{C}^*) \subset \sigma(f^{-1}(\mathcal{C}^*))$

$\Rightarrow E \in \mathcal{D}$

$\Rightarrow \mathcal{C}^* \subset \mathcal{D}$

(48) $\because \sigma(\mathcal{C}^*) \subset \mathcal{D}$ ($\because \mathcal{D}$ is a σ-field)

$\Rightarrow f^{-1}(\sigma(\mathcal{C}^*) \subset f^{-1}(\mathcal{D})$ —— (2)

Now
$$\mathcal{D} = \{ B \mid f^{-1}(B) \in \sigma(f^{-1}(\mathcal{C}^*))\}$$

$$\Rightarrow f^{-1}(\mathcal{D}) = \sigma(f^{-1}(\mathcal{C}^*) \quad \text{—(3)}$$

using (3) in (2), we have

$$f^{-1}(\sigma(\mathcal{C}^*)) \subset \sigma(f^{-1}(\mathcal{C}^*)) \quad \text{—— (4)}$$

From (1) & (4), we have

$$f^{-1}(\sigma(\mathcal{C}^*)) = \sigma(f^{-1}(\mathcal{C}^*))$$

—— x ——