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SUBJECT: Measure theory

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UNIT-1

→ Let Ω be some abstract space.
Let $A \subset \Omega$, $B \subset \Omega$. Then we know how to define $A \cup B$, $A \cap B$, A^c, B^c etc (set theory)

→ Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of set i.e $A_n \subset \Omega$, $\forall n = 1, 2$
and the sequence is $\{A_1, A_2, A_3, \ldots\}$

Recall :- The concept of $\lim a_n$ where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real number.

→ First we will define limit superior and limit inferior
So let $\{A_n\}$ be a sequence of sets of Ω i.e
$A_n \subset \Omega$ $\forall n = 1, 2, \ldots$
then

2018 2 (a)

$$\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \quad \text{Def}^n$$

and

$$\lim\inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$$

Note that $\lim\sup A_n$ and $\lim\inf A_n$, both are sets and subsets of Ω.

Result 1: $\lim\inf A_n \subset \lim\sup A_n$

Proof Let $\omega \in \lim\inf A_n$.
to prove $\omega \in \lim\sup A_n$

So let $\omega \in \lim\inf A_n$
i.e. $\omega \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

let $\bigcap_{n=k}^{\infty} A_n = C_k$

then $\omega \in \bigcup_{k=1}^{\infty} C_k$

$\Leftrightarrow \omega \in C_k$ for some k

$\Leftrightarrow \omega \in \bigcap_{n=k}^{\infty} A_n$ for some k

$\Leftrightarrow \omega \in A_n \quad \forall n \geq k$, for some +ve integer k — (1)

Now to prove $\omega \in \lim\sup A_n$

i.e $\omega \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcap_{r=1} \bigcup_{n \geq r} A_n$

let $\bigcup_{n=r}^{\infty} A_n$ ~~A_n~~ $= B_r$ (say)

Hence $\Longleftrightarrow$ to prove $\omega \in \bigcap_{r=1}^{\infty} B_r$

i.e $\Longleftrightarrow$ to prove $\omega \in B_r \ \forall r$

i.e $\Leftrightarrow$ to prove $\omega \in \bigcup_{n=r}^{\infty} A_n \quad \forall r \geq 1$ — (2)

let r be any fixed positive integer

to prove $\omega \in \bigcup_{n=r}^{\infty} A_n$

Suppose $r < k$ $\qquad r < r+1 < \ldots < k < k+1 < \ldots$

from (1) we know that $\qquad k \longrightarrow$

$\omega \in A_n \quad \forall n \geq k$

$\Rightarrow \omega \in \bigcup_{n=k}^{\infty} A_n \subset \bigcup_{n=r}^{\infty} A_n$

$\Rightarrow \omega \in \bigcup_{n=r}^{\infty} A_n$

If $k \leq r$, again from (1)

$\omega \in A_n \quad \forall n \geq k$

$\Rightarrow \omega \in A_r, A_{r+1}, \ldots$

$\Rightarrow \omega \in \bigcup_{n=r}^{\infty} A_n$

Thus in either case, $\omega \in \bigcup_{n=r}^{\infty} A_n$. Since r is a arbitrary, $\omega \in \bigcup_{n=r}^{\infty} A_n \quad \forall r \geq 1$ Thus (2) holds

Hence the proof

Theorem: $\limsup A_n = \{\omega \in \Omega \mid \omega \in A_n$ for an infinite no. of values of $n\}$

$= \{\omega \in \Omega \mid \omega \in A_n$ infinitely often (i.o)$\}$

Proof We know that $\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

Let $\omega \in \limsup A_n$

$\Leftrightarrow \omega \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

$\Leftrightarrow \omega \in \bigcup_{n=k}^{\infty} A_n, \ \forall k \geq 1$

$\Leftrightarrow \omega \in A_n$ for some $n \geq k$, $\forall k \geq 1$

Thus $\forall k \geq 1$, $\exists$ an integer $n \geq k$, such that $\omega \in A_n$.

$\Leftrightarrow \omega \in A_n$ infinitely often

Theorem: $\liminf A_n = \{\omega \in \Omega \mid \omega \in$ all A_n except possibly a finite no. of them$\}$

Proof We know that

$$\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$$

Let $\omega \in \liminf A_n$

$\Leftrightarrow \omega \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

$\Leftrightarrow \omega \in \bigcap_{n=k}^{\infty} A_n$ for some $k \geq 1$

$\Leftrightarrow \omega \in$ all A_n $\forall n \geq k$

where k is some +ve integer

$\Leftrightarrow \omega \in$ all A_n, except possibly a finite no. of them

Definition :-

If for a sequence of set $\{A_n\}$ $\lim\inf A_n = \lim\sup A_n$, we say that $\lim A_n$ exists and $\lim A_n = \lim\inf A_n = \lim\sup A_n$

— x —

Result :- $(\lim\inf A_n)^c = \lim\sup A_n^c$

$(\lim\inf A_n)' = \lim\sup A_n'$

Proof

$$(\lim\inf A_n)' = \left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n\right)'$$

$$= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n'$$

$$= \lim\sup A_n'$$

Similarly $(\lim\sup A_n)' = (\lim\inf A_n')$

Examples

(1) Suppose A & $B \subset \Omega$

Define $A_n = \begin{cases} A & \text{if } n \text{ is even} \\ B & \text{if } n \text{ is odd} \end{cases}$

check whether $\lim A_n$ exists or not?

Here $\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

$$= \bigcap_{k=1}^{\infty} (A \cup B)$$

$$= A \cup B$$

and $\lim\inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

$$= \bigcup_{k=1}^{\infty} (A \cap B)$$

$$= A \cap B$$

In general $\lim\inf A_n \neq \lim\sup A_n$
If $A = B$, then only $\lim\inf A_n = \lim\sup A_n$
Thus $\lim A_n$ exists only when $A = B$

⇒ Let $\{A_n\}$ be a sequence of disjoint sets. Does $\lim A_n$ exists?

Clearly there is now ω which belongs to an infinite no. of A_n's

$$\Rightarrow \lim\sup A_n = \phi$$
but $\lim\inf A_n \subset \lim\sup A_n$
$$\Rightarrow \lim\inf A_n = \phi$$
Thus $\lim A_n = \lim\inf A_n = \lim\sup A_n = \phi$

☆ **Definition :- Monotone Sequence**

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets of Ω.
$\{A_n\}$ is said to be a monotone increasing sequence if $A_n \subset A_{n+1}$ $\forall n \geq 1$ i.e
$$A_1 \subset A_2 \subset A_3 \subset \ldots \quad \text{Notation: } A_n \uparrow$$

Similarly, $\{A_n\}$ is said to be a monotone decreasing sequence if $A_n \supset A_{n+1}$ $\forall n \geq 1$ i.e
$$A_1 \supset A_2 \supset A_3 \supset \ldots \quad \text{Notation: } A_n \downarrow$$

Result If sequence $\{A_n\}$ is such that $A_n \uparrow$, then $\lim A_n = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \downarrow$, then $\lim A_n = \bigcap_{n=1}^{\infty} A_n$

Proof Suppose $A_n \uparrow$ i.e $A_n \subset A_{n+1}$, $\forall n \geq 1$
to prove that $\lim A_n = \bigcup_{n=1}^{\infty} A_n$

Consider $\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

Consider

$$\begin{aligned}\bigcup_{n=1}^{\infty} A_n &= A_1 \cup A_2 \cup A_3 \cup \cdots \\ &= A_2 \cup A_3 \cup \cdots \quad (\because A_1 \subset A_2) \\ &= A_3 \cup A_4 \cup \cdots \quad (\because A_2 \subset A_3) \\ &= \bigcup_{n=k}^{\infty} A_n\end{aligned}$$

Thus $\bigcup_{n=k}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n \quad \forall\ k \geq 1$

Hence $\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

$$= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n \quad \text{—} (1)$$

Next Consider

$$\lim\inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$$

$$= \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n \quad \text{—} (2)$$

From (1) and (2) $\lim\inf A_n = \lim\sup A_n$ and
Hence $\lim A_n$ exists and $\lim A_n = \bigcup_{n=1}^{\infty} A_n$

Now Suppose $A_n \downarrow$ to prove $\lim A_n = \bigcap_{n=1}^{\infty} A_n$

Consider $\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$.

Now $A_n \downarrow \Rightarrow A_1 \supset A_2 \supset A_3 \supset \dots$

$\Rightarrow \bigcup_{n=k}^{\infty} A_n = A_k$

and Hence $\lim\sup A_n = \bigcap_{k=1}^{\infty} A_k$ — (3)

Further $\lim\inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

now consider $\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \dots$

but $A_1 \supset A_2 \supset A_3 \supset \dots$

$\Rightarrow \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n = \bigcap_{n=3}^{\infty} A_3 = \dots \bigcap_{n=k}^{\infty} A_n$

and

Hence $\lim\inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$ — (4)

from (3) and (4), we have

$\lim A_n = \lim\inf A_n = \lim\sup A_n = \bigcap_{n=1}^{\infty} A_n$

——x——

Indirect proof using first part of the result

Suppose $A_n \downarrow$ i.e $A_1 \supset A_2 \supset A_3 \supset \dots$

to prove $\lim A_n = \bigcap_{n=1}^{\infty} A_n$

Since $A_1 \supset A_2 \supset A_3 \supset \dots \Rightarrow A_1^c \subset A_2^c \subset A_3^c \subset \dots$

i.e $A_1' \subset A_2' \subset A_3^c \subset \dots$

Thus $\{A_n\} \uparrow$ then using the first part of the result $\lim\inf A_n' = \lim\sup A_n' = \bigcup_{n=1}^{\infty} A_n'$

Taking Compliment we have.

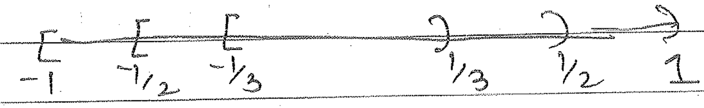
$(\lim\inf A_n')' = (\lim\sup A_n')' = \left(\bigcup_{n=1}^{\infty} A_n'\right)'$

$\Rightarrow \lim\sup A_n = \lim\inf A_n = \bigcap_{n=1}^{\infty} A_n = \lim A_n$

Example

① let $A_n = [-\frac{1}{n}, \frac{1}{n})$ $n \geq 1$
check whether $\lim A_n$ exists or not?

$A_1 = [-1, 1)$
$A_2 = [-1/2, 1/2)$
$A_3 = [-1/3, 1/3)$
$\vdots$

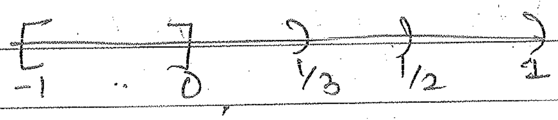


Note that $A_n \downarrow$, hence $\lim A_n = \bigcap_{n=1}^{\infty} A_n = \{0\}$

2019 - 2(a)

② for each of the following seq^n of set, check whether $\lim A_n$ exists or not?

$A_n = [-1, 1/n)$ $\forall n \geq 1$
$A_1 = [-1, 1)$
$A_2 = [-1, 1/2)$
$A_3 = [-1, 1/3)$

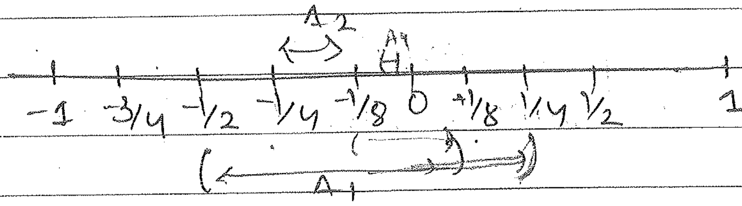


Note that $A_n \downarrow$ and $\lim A_n = \bigcap_{n=1}^{\infty} A_n = [-1, 0]$

③ $A_n = (-2^{-n}, (-1/2)^{n+1})$
$= \left(-\frac{1}{2^n}, \left(\frac{-1}{2}\right)^{n+1}\right)$, $n \geq 1$

$A_1 = \left(\frac{-1}{2}, \frac{1}{4}\right)$

$A_2 = \left(\frac{-1}{4}, \frac{-1}{8}\right)$



$A_3 = \left(\frac{-1}{8}, \frac{+1}{16}\right)$

$A_{2n+1} \to \{0\}$
and A_{2n} are disjoint set

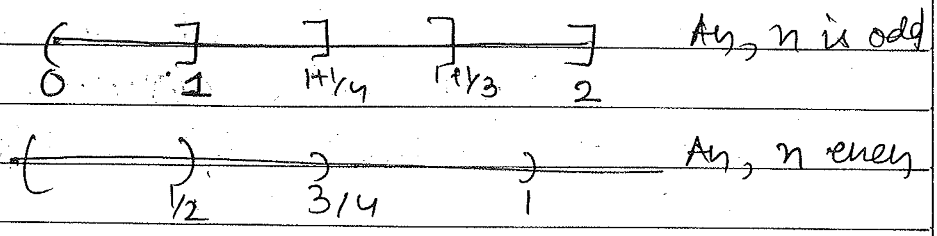
$A_4 = \left(\frac{-1}{16}, \frac{-1}{32}\right)$

we observe that $\lim\sup A_n = \{\omega \in \Omega \mid \omega \in A_n \text{ i.o}\}$
$= \{0\}$

and $\lim\inf A_n = \phi$, Hence $\lim A_n$ doesn't exist

(4) $A_n = \begin{cases} (0, 1-1/n), & \text{if } n \text{ is even} \\ (0, 1+1/n], & \text{if } n \text{ is odd} \end{cases}$

$A_1 = (0, 2]$
$A_2 = (0, 1/2)$
$A_3 = (0, 4/3]$
$A_4 = (0, 3/4)$



We note that as n increases
$A_n \to (0,1)$ for n even.
and $A_n \to (0,1]$ for n odd
i.e $A_{2n} \to (0,1)$
$A_{2n+1} \to (0,1]$

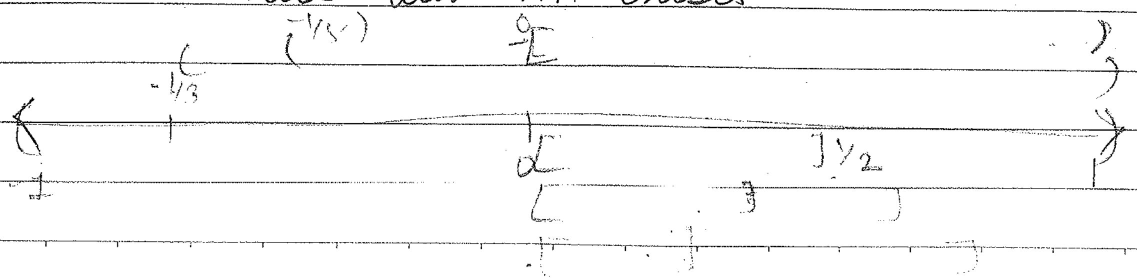
$\Rightarrow$ lim inf $A_n = (0,1)$ and lim sup $A_n = (0,1]$
Hence limit does not exist

(5) $A_{2n+1} = \left(\frac{-1}{2n+1}, 1\right)$, $A_{2n} = \left[0, 1-\frac{1}{2n}\right]$

$A_1 = (-1, 1)$ $\quad A_2 = \left[0, \frac{1}{2}\right]$
$A_3 = \left(\frac{-1}{3}, 1\right)$
$\quad A_4 = [0, 3/4]$
$A_5 = \left(\frac{-1}{5}, 1\right)$ $\quad A_6 = [0, 5/6]$

We note that $A_{2n+1} \to [0,1)$
$A_{2n} \to [0,1)$

Thus lim inf A_n = lim sup A_n = $[0,1)$ = lim A_n
Thus lim A_n exists



Field and σ - Fields (Algebra and σ - Algebra)

let Ω be an abstract Space.
let $A, B, C \ldots$ be subsets of Ω.
let $\mathcal{C}$ denote some collection of subsets of Ω
eg. $\mathcal{C} = \{A, B, C\}$, $\mathcal{C} = \{\phi, \Omega\}$, $\mathcal{C} = \{\phi, A, B\}$ etc

$\mathcal{C}$ can be even empty collection
or $\mathcal{C} = \{\phi\}$ consists of one set ϕ, so this is a non-empty collection

Definition - Field :-

A non-empty collection $\mathcal{C}$ of subsets of Ω is known as a field or Algebra if it satisfies the following conditions
(i) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$ i.e $A' \in \mathcal{C}$
(ii) $A, B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C}$

In other words
A non-empty collection $\mathcal{C}$ of subsets of Ω is known as a field if it closed under compliments and finite unions

—x—

Remark : A field is a non-empty collection
So let $A \in \mathcal{C}$ then $A' \in \mathcal{C}$
Also $\Rightarrow A \cup A' \in \mathcal{C}$
i.e $\Omega \in \mathcal{C}$
also $\Omega' \in \mathcal{C}$ i.e $\phi \in \mathcal{C}$

Thus
(i) Every field always contains ϕ and Ω
(ii) $\mathcal{C} = \{\phi, \Omega\}$ is the smallest field
(Also known as trivial field)

(iii) If $A \in \mathcal{C} \Rightarrow A' \in \mathcal{C}$
Thus $\{\phi, \Omega, A, A'\}$ is also a field.

(iv) Suppose $\mathcal{C}$ is a field and suppose $A, B \in \mathcal{C}$
then $A' \in \mathcal{C}$, $B' \in \mathcal{C}$

$\Rightarrow A \cup B,\ A \cup B',\ A' \cup B,\ A' \cup B' \in \mathcal{C}$
but $A' \cup B' = (A \cap B)' \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ and so on

$\mathcal{C} = \{\phi, \Omega, A, B, A', B', A \cup B', A \cup B', A' \cup B, A' \cup B', A \cap B \ldots\}$
All possible union, intersection and compliments have to be in $\mathcal{C}$.

—x—

Alternate Def^n:- A non-empty collection $\mathcal{C}$ of subsets of Ω is a field if it is closed under compliments and finite intersection i.e

$$A \in \mathcal{C} \Rightarrow A' \in \mathcal{C} \text{ and}$$
$$\text{if } A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$$

—x—

* Largest field :- Power of set of $\Omega = \mathcal{P}(\Omega) = \{A \mid A \subset \Omega\}$

—x—

Notation :- "$\mathcal{F}$" for a field

—x—

σ - field :- A non-empty collection of subset of Ω ($\mathcal{A}$) is called a σ - field (or σ - algebra) if

(i) $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$
(ii) $A_n \in \mathcal{A}$, $\forall n = 1, 2, \ldots$
then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

i.e closed under compliments and countable unions

Alternative def :- A non-empty collection $\mathcal{A}$ of subsets of Ω is called a σ-field if it closed under compliments and countable intersections.

* **Properties of σ-field :-**

let $\mathcal{A}$ be a σ-field. let $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$.
Choose A_n s.t $A_1 = A'$ & $A_2 = A_3 = \cdots = A$
then
$$\bigcup_{n=1}^{\infty} A_n = A' \cup A = \Omega \in \mathcal{A}$$
Hence $\Omega' = \phi \in \mathcal{A}$
Thus every ~~fi~~ σ-field must contain ϕ and Ω.
In fact $\{\phi, \Omega\}$ is the smallest σ-field and $P(\Omega)$ is the largest σ-field

—x—

① → let $\mathcal{A}$ be a σ-field
let $\{A_n\} \in \mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$
also $A_n \in \mathcal{A} \Rightarrow A_n^c \in \mathcal{A} \Rightarrow \cup A_n^c \in \mathcal{A}$
$\Rightarrow (\cap A_n)^c \in \mathcal{A}$
$\Rightarrow \cap A_n \in \mathcal{A}$

Thus $\mathcal{A}$ is closed under countable intersection

—x—

② → lim sup A_n, lim inf $A_n \in \mathcal{A}$
let $A_n \in \mathcal{A}$ $\forall n = 1, 2, \ldots$

then $\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

let $B_k = \bigcup_{n=k}^{\infty} A_n$ then $B_k \in \mathcal{A}$ $\forall k$

$\therefore \bigcap_{k=1}^{\infty} B_k \in \mathcal{A}$

$\Leftrightarrow \limsup A_n \in \mathcal{A}$

Similarly, $\lim \inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n \in \mathcal{A}$

Further, if $\lim A_n$ exists, then $\lim A_n \in \mathcal{A}$

—x—

(2016) 1(b)
(3) → Let $\mathcal{A}$ be a σ-field. Then $\mathcal{A}$ is also a field
(2016) → It is sufficient to check that $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

(2016) 1(b) Take $A_1 = A$ & $A_2 = A_3 = A_4 = \dots = B$, then

$$A_n \in \mathcal{A} \quad \forall n \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = A \cup B \in \mathcal{A}$$

$$\Rightarrow \mathcal{A} \text{ is a field}$$

Thus every σ-field is also a field

But the converse is not true.
The following example will prove that every field is not necessarily a σ-field

Ex Let $\Omega = \{1, 2, 3, \dots\}$, Define
$\mathcal{F} = \{A \subset \Omega \,/\, \text{either } A \text{ or } A^c \text{ is a finite set}\}$

Claim: 1) $\mathcal{F}$ is non-empty
Let $A \in \mathcal{F} \Rightarrow A^c$ is finite if A is not finite
and A^c is not finite if A is finite
Thus $A \in \mathcal{F}$, $A^c \in \mathcal{F}$
Thus $\mathcal{F}$ is closed under compliments.

Further let $A, B \in \mathcal{F}$. To prove $A \cup B \in \mathcal{F}$
When A & $B \in \mathcal{F}$, then one of the following holds
(i) A & B both finite $\Rightarrow A \cup B$ is finite
(ii) A & B' are finite $\Rightarrow A' \cap B'$ are finite
$\Rightarrow A \cup B \in \mathcal{F}$

(iii) A' & B' are finite. By similar argument $A \cup B \in \mathcal{F}$

(iv) A' and B' are finite $\Rightarrow A' \cap B'$ are finite
$\Rightarrow (A \cup B)'$ is finite
$\Rightarrow A \cup B \in \mathcal{F}$

Thus $\mathcal{F}$ is also closed under finite unions.
Thus $\mathcal{F}$ is a field

$\rightarrow$ Now define A_n as $A_n = \{n\}$, if n even
& $A_n = \{\phi\}$, if n odd

Then $A_n \in \mathcal{F} \ \forall n$
$\cup A_n = \{2, 4, 6, 8, \dots\}$ not finite
$(\cup A_n)' = \{1, 3, 5, 7, \dots\}$ not finite

Thus $\cup A_n \notin \mathcal{F}$
$\Rightarrow \mathcal{F}$ is not closed under countable unions
$\Rightarrow \mathcal{F}$ is not a σ-field

Theorem A finite field is a σ-field

Proof Let $\mathcal{F}$ be a finite field
i.e $\mathcal{F}$ is a field containing finite no. of sets

So let $\mathcal{F} = \{C_1, \dots C_N\}$ where N is finite
Let $A_n \in \mathcal{F} \ \forall n$, then A_n are some or all of sets $C_1, \dots C_N$ and hence

$\bigcup_{n=1}^{\infty} A_n$ is union of some or all sets $C_1 \dots C_N$

i.e $\bigcup_{n=1}^{\infty} A_n$ is a finite union

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ ($\because \mathcal{F}$ is a field)

Hence $\mathcal{F}$ is also a σ-field

2019 3b

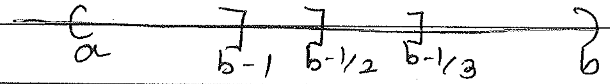
Def^n:- A non-empty collection $\mathcal{M}$ of subset of Ω is called a monotone class is $\mathcal{M}$ is closed under limits of monotone sequence.
i.e $A_n \in \mathcal{M}$, $n = 1, 2, \ldots$ & $\{A_n\}$ is monotone then $\lim A_n \in \mathcal{M}$.

Result:- Is a σ-field a monotone class?
Yes, Every σ-field is a monotone class because σ-field is closed under limits of any type of sequence, if it exists.

The converse is not always true
i.e a monotone class is not always a σ-field

→ Let $\mathcal{C} = \{I \mid I \text{ is an interval in } \mathbb{R}\}$
i.e any type of the interval (a,b), $(a,b]$, $[a,b)$, $[a,b]$, $(-\infty,b)$, $(-\infty,b]$, (a,∞), $[a,\infty)$
then $\mathcal{C}$ is a monotone class.

eg. Let $A_n = (a, b-1/n]$, $n \geq 1$
then $A_n \uparrow$ and $\lim A_n = (a,b) \in \mathcal{C}$



Here we note that $\mathcal{C}$ is closed under limits of monotone sequence. Hence $\mathcal{C}$ is a monotone class. But if $(a,b) \in \mathcal{C}$ $(a,b)' \notin \mathcal{C}$
$\Rightarrow$ $\mathcal{C}$ is not closed under compliments
$\Rightarrow$ $\mathcal{C}$ is not a σ-field

(i) $A_n = \left[a - \frac{1}{n}, b\right]$, $\forall n \geq 1$

then $\lim_{n \to \infty} A_n = [a, b]$

(ii) let $B_n = \left[a, b + \frac{1}{n}\right)$

then $\lim_{n \to \infty} B_n = [a, b]$

(iii) let $C_n = \left(a - \frac{1}{n}, b\right]$

then $\lim_{n \to \infty} C_n = [a, b]$

(iv) $D_n = \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$

then $\lim_{n \to \infty} D_n = [a, b]$

2019 - 3(b) (Montone defn)

Thm A monotone field is a σ-field
i.e a field which is a monotone class or a monotone class ~~or a monotone class~~ which is a field is also a σ-field

Proof Let $\mathcal{C}$ be a monotone field. Being a field, it is always closed under compliments
Let $A_n \in \mathcal{C}$ $\forall n = 1, 2, \ldots$
&

To prove $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$

Define $B_k = \bigcup_{n=1}^{k} A_n$

then $B_k \in \mathcal{C}$ $\forall k$ and $B_k \uparrow \Rightarrow \lim_{k \to \infty} B_k \in \mathcal{C}$

But $\lim_{k \to \infty} B_k = \bigcup_k B_k = \bigcup_{n=1}^{\infty} A_n \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$

$\Rightarrow \mathcal{C}$ is a closed under ~~com~~ countable unions
$\Rightarrow \mathcal{C}$ is a σ-field.

Thm A field is a σ-field iff it is a monotone class, Also

Thm A monotone class is a σ-field iff it is a field

— x —

→ Let $\Omega = R$, let $A_x = (x, \infty)$, $x \in R$,
Define $\mathcal{A}_x = \{\Omega, \phi, A_x, A_x'\}$ then $\mathcal{A}_x$ is σ-field $\forall x \in R$
Thus on a given space, we can define infite no. of σ-fields
So let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two σ-fields of subsets of Ω

clearly $\phi, \Omega \in (\mathcal{A}_1 \cap \mathcal{A}_2)$
Thus $\mathcal{A}_1 \cap \mathcal{A}_2$ is non-empty

let $A \in \mathcal{A}_1 \cap \mathcal{A}_2$
$\Rightarrow A \in \mathcal{A}_1$ and $A \in \mathcal{A}_2$
$\Rightarrow A' \in \mathcal{A}_1$ and $A' \in \mathcal{A}_2$
$\Rightarrow A' \in \mathcal{A}_1 \cap \mathcal{A}_2$
$\Rightarrow \mathcal{A}_1 \cap \mathcal{A}_2$ is closed under complimentation

let $\{A_n\}_{n=1}^{\infty} \in \mathcal{A}_1 \cap \mathcal{A}_2$
$\Rightarrow A_n \in \mathcal{A}_1 \ \forall n$ & $A_n \in \mathcal{A}_2 \ \forall n$
$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_1$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_2$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow \mathcal{A}_1 \cap \mathcal{A}_2$ is closed under countable unions
Hence $\mathcal{A}_1 \cap \mathcal{A}_2$ is also σ-field.

— x —

This idia can be extended to intersection of any number of σ-fields.

let $\{\mathcal{A}_t, t \in T\}$ be a collection of σ-field of subsets of Ω.
(T can be finite, infinite, countable or uncountable)
Then $\bigcap_{t \in T} \mathcal{A}_t$ is also a σ-field

— x —

$\rightarrow$ Is Union of two σ-field also a σ-field?
No, Not necessarily

eg Let $\Omega = \mathbb{R}$, $A = [0,\infty)$, $B = (0,\infty)$
then let
$$\mathcal{A}_1 = \{\phi, \Omega, [0,\infty), (-\infty,0)\}$$
$$\mathcal{A}_2 = \{\phi, \Omega, (0,\infty), (-\infty,0]\}$$
then
$\mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, \Omega, [0,\infty), (-\infty,0), (0,\infty), (-\infty,0]\}$
Now
$$(-\infty,0] \cap [0,\infty) = \{0\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$$

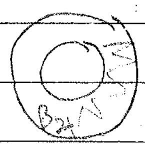
$\Rightarrow$ $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-field.

—x—

Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-fields. Further Suppose $\mathcal{A}_1 \cup \mathcal{A}_2$ is a σ-field, then what is the relation between $\mathcal{A}_1$ & $\mathcal{A}_2$?
The relation is either $\mathcal{A}_1 \subset \mathcal{A}_2$ or $\mathcal{A}_2 \subset \mathcal{A}_1$

Proof If possible Suppose $\mathcal{A}_1 \not\subset \mathcal{A}_2$ or $\mathcal{A}_2 \not\subset \mathcal{A}_1$

$\Rightarrow \exists$ set $B_1 \in \mathcal{A}_1$ but $B_1 \notin \mathcal{A}_2$
and $\exists$ set $B_2 \in \mathcal{A}_2$ but $B_2 \notin \cancel{\mathcal{A}_2} (\mathcal{A}_1)$
Now
$$B_1 \in \mathcal{A}_1 \Rightarrow B_1 \in \mathcal{A}_1 \cup \mathcal{A}_2$$
$$B_2 \in \mathcal{A}_2 \Rightarrow B_2 \in \mathcal{A}_1 \cup \mathcal{A}_2$$



Now
$$B_1 \cup B_2 \in \mathcal{A}_1 \quad \because B_2 \notin \mathcal{A}_1$$
$$B_1 \cup B_2 \in \mathcal{A}_2 \quad \because B_1 \notin \mathcal{A}_2$$
$$\Rightarrow B_1 \cup B_2 \notin \mathcal{A}_1 \cup \mathcal{A}_2$$

$\Rightarrow$ $\mathcal{A}_1 \cup \mathcal{A}_2$ is not closed under finite union
$\Rightarrow$ $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a field
$\Rightarrow$ $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-field which is a contradiction, Thus our assumption must be wrong

$\Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is a σ-field iff either $\mathcal{A}_1 \subset \mathcal{A}_2$ or $\mathcal{A}_2 \subset \mathcal{A}_1$.

—x—

→ Given a class $\mathcal{C}$ of subset of Ω, find the smallest σ-field containing $\mathcal{C}$.

Note that power set of Ω always contains $\mathcal{C}$ and $P(\Omega)$ ~~po~~ is a σ-field.

Now to find smallest σ-field containing $\mathcal{C}$. Let us consider the collection of all σ-field that contain $\mathcal{C}$.

i.e let $\{\mathcal{A}_t, t \in T\}$ be such that $\mathcal{A}_t \supset \mathcal{C}, \forall t \in T$

Let $\mathcal{A} = \bigcap_{t \in T} \mathcal{A}_t \supset \mathcal{C}$

We know that $\mathcal{A}$ is also a σ-field and
$\mathcal{A} \subset \mathcal{A}_t$ ~~$\not\supset$~~ $\forall t \in T$

Thus, $\mathcal{A}$ is the smallest σ-field containing $\mathcal{C}$.
$\sigma(\mathcal{C})$

—x—

Defn:- σ-field generated by class $\mathcal{C}$.
Let $\mathcal{C}$ be a non-empty collection of subset of Ω. Then the smallest σ-field containing the class $\mathcal{C}$ is called the σ-field generated by class $\mathcal{C}$ and is denoted by $\sigma(\mathcal{C})$.

[Remark:- If $\mathcal{C}$ itself is a σ-field then $\sigma(\mathcal{C}) = \mathcal{C}$

But if $\mathcal{C}$ is simply a non-empty collection to reach up to σ-field containing $\mathcal{C}$, we need to add all posibly required sets, such that it becomes a σ-field]

eg. let $A \subset \Omega$ and $\mathcal{C} = \{A\}$ then $\sigma(\mathcal{C}) = \{A, \phi, \Omega, A'\}$
Thus $\mathcal{C} \subset \sigma(\mathcal{C})$

let $\mathcal{C} = \{A, B\}$, $A, B \subset \Omega$
then $\sigma(\mathcal{C}) = \{\phi, \Omega, A, B, A', B', A\cup B, A\cap B, A\cup B', A'\cup B, A'\cup B', A\cap B', A'\cap B, A'\cap B', \ldots\}$

— x —

Borel σ-field :-

let $\Omega = R$. Then the σ-field defined over real line is known as Borel σ-field and is denoted by $\mathcal{B}$.

How Borel σ-field is generated?
Define
$\mathcal{H} = \{A \mid A$ is finite union of sets of the type $(a, b], (-\infty, \alpha], (\beta, \infty)\}$
$a \le b,\ a, b, \alpha, \beta \in R$

Then $\mathcal{H}$ is a field. Why?
eg. $(-\infty, 5] \cup [5, \infty) = R \in \mathcal{H}$
Let $a = 5$, $b = 5$ then $(5, 5] = \phi \in \mathcal{H}$
Thus $\phi, R \in \mathcal{H}$

further $\mathcal{H}$ is closed under finite unions
$\Rightarrow \mathcal{H}$ is a field.
Then the σ field generated by $\mathcal{H}$ i.e $\sigma(\mathcal{H})$ is known as the Borel σ-field.

Thus $\mathcal{B} = \sigma(\mathcal{H})$

Borel -σ- field can also be generated as follows:
Define the following 4 classes of subsets of $\mathbb{R}$

$\mathcal{C}_1 = \{ I \mid I = (a,b) , -\infty < a \leq b < \infty \}$
$\mathcal{C}_2 = \{ I \mid I = [a,b] , -\infty \leq a \leq b < \infty \}$
$\mathcal{C}_3 = \{ I \mid I = (a,b] , -\infty < a \leq b < \infty \}$
$\mathcal{C}_4 = \{ I , I = [a,b) , -\infty < a \leq b < \infty \}$

<u>Remark</u>:- Any member of any class can be written as member of other class.

eg let $A_n = (a, b - 1/n]$, $n = 1, 2$
then $A_n \in \mathcal{C}_3$. Note that $A_n \uparrow (a,b)$
i.e $\lim A_n = (a,b) \in \mathcal{C}_1$
i.e set in $\mathcal{C}_1$ can be written as limit of sets in $\mathcal{C}_3$

Similarly $\underbrace{(\alpha, \beta]}_{\in \mathcal{C}_3} = \lim_{n \to \infty} \underbrace{(\alpha, \beta + \frac{1}{n})}_{\mathcal{C}_1}$

OR

$$\underbrace{[a,b]}_{\in \mathcal{C}_2} = \lim_{n \to \infty} \underbrace{(a - \frac{1}{n}, b + \frac{1}{n})}_{\in \mathcal{C}_1}$$

Thus it doesn't matter, among $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \& \mathcal{C}_4$ which one we choose
Then

$$\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2) = \sigma(\mathcal{C}_3) = \sigma(\mathcal{C}_4) = \mathcal{B}$$

[Smallest σ- field containing $\mathcal{C}_1$ OR σ- field generated by $\mathcal{C}_1$]

This is how Borel σ- field is generated

— x —

Extended Real line :-

Real line $(-\infty, \infty) = R$

Extended Real line $[-\infty, \infty] = \overline{R}$

σ - field over R : Borel σ - field $(\mathcal{B})$

σ - field over $\overline{R}$: extended Borel σ - field $(\overline{\mathcal{B}})$

— x —

Def :- set $B \in \mathcal{B}$ are called Borel sets

Examples of Borel set :- Open sets, closed set, semi-open or semiclosed sets, singleton sets, set of all rationals, set of all irrationals etc are all Borel sets.

* Remark :- Note that Borel sets are always subset of R, but every subset of R need not be a Borel set.

* Remark :- To describe a Borel set is highly impossible we can simply give a number of example of Borel sets.

— x —

Let Ω be some abstract space and let $\mathcal{A}$ be a fixed σ-field of subset of Ω.

Then $(\Omega, \mathcal{A})$ is known as a Measurable Space

Similarly, $(R, \mathcal{B})$ is also a measurable space.

Let $(\Omega, \mathcal{A})$ be a measurable space. Sets belonging to $\mathcal{A}$ are known as measurable sets (mble set)

eg let $A, B \subset \Omega$

let $\mathcal{A}_1 = \{\phi, \Omega, A, A'\}$

$\mathcal{A}_2 = \{\phi, \Omega, B, B'\}$

then A is measurable w.r.t $\mathcal{A}_1$ but not measurable w.r.t $\mathcal{A}_2$

Thus whenever a measurable space is defined, the σ-field has to be kept fixed. Then the mble sets are also fixed

→ Now the question is how to measure a mble set?
For this, we have the concept of set function which assign every mble set some value over R or $\bar{R}$

Set function

A function $\varphi : \mathcal{A} \to \bar{R}$ is known as a set function
eg. Let $(R, \mathcal{B})$ be a mble space
Let $B = \{5, 7\}$ $C = \{2, 4, 8\}$ etc.
(1) define $\varphi(B) =$ no. of elements in B
$\Rightarrow \varphi(B) = 2$
$\Rightarrow \varphi(C) = 8$ etc.

(2) Or let $E = (a, b)$, $F = [a, b)$
and define $\varphi(E) =$ length of interval E
so that $\varphi(E) = b - a$, $\varphi(E) = b - a$

(3) Or say $\varphi(E) = \begin{cases} 1 & \text{if } E \text{ is non-empty} \\ 0 & \text{if } E \text{ is empty} \end{cases}$

Thus we can define a variety of set function on $\mathcal{A}$.

All sets function are of not much use. We have some special properties associated with set function and we prefer those set functions which have these properties.

* **Finitely additive set function** (finitly additive)

A set function Ψ is said to be finitely additive if $\Psi(A_1 \cup A_2) = \Psi(A_1) + \Psi(A_2)$ where A_1 & $A_2 \in \mathcal{A}$ and $A_1 \cap A_2 = \phi$

Remark :- The above definition is for finite no. of set i.e. If $A_1, \dots A_k \in \mathcal{A}$ such that $A_i \cap A_j = \phi \quad \forall \; i \neq j$, then

$$\Psi\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} (\Psi(A_i))$$

Remark It is not necessary that the set function is to be defined on a σ-field. It can also be defined over some non-empty collection $\mathcal{C}$.

Remark : Since Ψ takes value in $\bar{R} = [-\infty, \infty]$, Ψ must not takes values in $\bar{R}$ both $\pm\infty$.
At the most, Ψ can be take $+\infty$ or $-\infty$. but not both

eg. the third set function defined above is not finitely additive.

eg. let E_1 & E_2 be two non-empty subsets and $E_1 \cap E_2 = \phi$
then $E_1 \cup E_2$ is also non-empty
$\therefore \Psi(E_1 \cup E_2) = 1, \quad \Psi(E_1) = 1, \quad \Psi(E_2) = 1$

$\therefore \psi(E_1) + \psi(E_2) = 2$ and
$\psi(E_1 \cup E_2) \neq \psi(E_1) + \psi(E_2)$

* Countablly additive set function (σ- additive)

A set function ψ is said to be countably additive if

$$\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$$

where $A_n \in \mathcal{A}$ $\forall n$ and $A_i \cap A_j = \phi$ $\forall i \neq j$
i.e $\{A_n\}$ is a sequence of disjoint sets.
Another name :- σ - additive set functions

Remark :- 1) $0 \cdot \infty \to$ undefined
2) $-\infty + (-\infty) = -\infty$
3) $\infty + \infty = \infty$
4) $\infty - \infty \Rightarrow$ undefined

Hence the set function can assume values only $+\infty$ or $-\infty$ but not both. Otherwise $\psi(A) + \psi(B)$ is not well defined

— x —

Finite set function

Defn :- A set function ψ is said to be a finite set function if $|\psi(A)| < \infty$ $\forall A \in \mathcal{A}$
Hencefourth, we consider only non -ve valued set function i.e $\psi(A) \in [0, \infty]$
So we can revise the above definition as follow.

A set function ψ is said to be a finite set fun^n if $\psi(A) < \infty$ $\forall A \in \mathcal{A}$
eg The set function in example 3 above

σ- finite set function

Def :- A non-ve set function ψ is said to be σ-finite if set $E \in \mathcal{A}$, $\exists$ a seqⁿ $\{A_n\}$, $A_n \in \mathcal{A}$ $\forall n$ such that $\bigcup_n A_n \supset E$ and $\psi(A_n) < \infty$ $\forall n$

Remark: A finite set function is σ-finite but a σ-finite set function need not be finite.

eg set fⁿ in ex 3 is finite & hence σ-finite.
The first two function are σ-finite but not finite

eg in (1) $\psi(E)$ = no. of elements in E
let $E = \{1, 2, 3, \ldots\}$ and let $A_n = \{n\}$
then $\psi(A_n) = 1$ $\forall n$.
and $E \subset \bigcup_{n=1}^{\infty} A_n$ or $\bigcup_{n=1}^{\infty} A_n \supset E$

Thus ψ is σ-finite, but ψ is not finite
$\because$ $\psi(E)$ is not finite

Continuity theorem for additive set functions

Def A set function ψ is said to be continuous from below if $A_n \uparrow A \Rightarrow \psi(A_n) \uparrow \psi(A)$.
Similarly,

Def A set function ψ is said to be continuous from above if $A_n \downarrow A \Rightarrow \psi(A_n) \downarrow \psi(A)$

Def A set function ψ is continuous at A if $A_n \to A$ $\Rightarrow \psi(A_n) \to \psi(A)$

Thm 1) A σ- additive set function ψ defined on a σ-field is finitely additive if $\psi(\phi)=0$
Conversely
If ψ is finitely additive and is continuous from below, then ψ is a σ-additive

Proof Suppose ψ is σ- additive and $\psi(\phi)=0$
Now ψ σ- additive

$$\Rightarrow \psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n),$$

where $A_i \cap A_j = \phi \quad \forall i = j$.

let $A_n = \phi \quad \forall n \geq 3$, then $A_i \cap A_j = \phi \quad \forall i \neq j$
and $\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \psi(A_1) + \psi(A_2) + \psi(\phi) + \psi(\phi) + \dots$

i.e $\psi(A_1 \cup A_2) = \psi(A_1) + \psi(A_2)$
$\Rightarrow \psi$ is finitely additive

Conversely, Suppose ψ is finitely additive and is continuous from below then to prove ψ is σ- additive
Now ψ is a finitely additive

$$\Rightarrow \psi\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{n} \psi(B_k), \quad B_i \cap B_j = \phi \quad \forall i \neq j.$$

Let $\{A_n\}$ be a sequence of disjoint sets
To prove that

$$\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$$

Define $D_k = \bigcup_{n=1}^{k} A_n \uparrow \bigcup_{n=1}^{\infty} A_n = A$ Say

Since ψ is continuous from below.

$\Rightarrow \psi(D_k) \uparrow \psi(A)$

i.e $\psi\left(\bigcup_{n=1}^{k} A_n\right) \uparrow \psi(A)$

i.e $\lim_{k\to\infty} \psi\left(\bigcup_{n=1}^{k} A_n\right) = \psi\left(\bigcup_{n=1}^{\infty} A_n\right)$

i.e $\lim_{k\to\infty} \sum_{n=1}^{k} \psi(A_n) = \psi\left(\bigcup_{n=1}^{\infty} A_n\right)$

i.e $\sum_{n=1}^{\infty} \psi(A_n) = \psi\left(\bigcup_{n=1}^{\infty} A_n\right)$

i.e $\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$

$\Rightarrow \psi$ is σ - additive

— x —

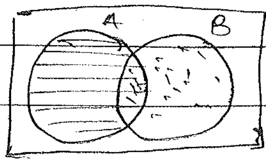
Thm If ψ is finite, finitely additive and ψ is conti. at ϕ, then ψ is σ - additive

To prove this thm, we will use following result

Result :- For any sequence $\{A_n\}_{n=1}^{\infty}$ $\bigcup_{n=1}^{\infty} A_n$ can always be written as union of disjoint sets.

Proof :- let us first consider the case of two sets

$A \cup B = A \cup (B \cap A')$



Where A & $B \cap A'$ are disjoint sets

Consider a Seq^n $\{A_n\}$ of sets

Define $B_1 = A_1$, $B_2 = A_2 \cap A_1'$, $B_3 = A_3 \cap A_1' \cap A_2'$

$\vdots$

$B_n = A_n \cap A_{n-1} \cap \dots \cap A_1$

then $\{B_n\}$ is a sequence of disjoint sets and

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Hence the result

— x —

Proof of theorem

Suppose ψ is finite i.e $\psi(A) < \infty \; \forall A \in \mathcal{A}$.
Also let ψ be finitely additive and ψ is continuous at ϕ.
To prove that ψ is σ- additive.

Let $\{A_n\}$ be a seq^n of disjoint sets
To prove

$$\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$$

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B_k = A - \bigcup_{n=1}^{k} A_n$

then $B_k \downarrow \phi$

Now $\psi(B_k) = \psi(A) - \psi\left(\bigcup_{n=1}^{k} A_n\right)$

$$= \psi(A) - \sum_{n=1}^{k} \psi(A_n)$$ ($\because$ A_n's are disjoint and ψ is finitely additive)

$\therefore \lim_{k\to\infty} \psi(B_k) = \psi(A) - \lim_{k\to\infty} \sum_{n=1}^{k} \psi(A_n)$

but $B_k \downarrow \phi \Rightarrow \lim_{k\to\infty} \psi(B_k) \downarrow \psi(\phi)$

Now $\psi(\phi)$ has to be equal to zero, because ψ is finitely additive.

$\therefore \Psi(A_1 \cup \phi) = \Psi(A_1) + \Psi(\phi)$

$\Rightarrow \Psi(A_1) = \Psi(A_1) + \Psi(\phi)$

$\Rightarrow \Psi(\phi) = 0$

Thus $\lim_{k\to\infty} \Psi(B_k) = \Psi(\phi) = 0$

$$\Rightarrow \Psi(A) = \lim_{k\to\infty} \sum_{n=1}^{k} \Psi(A_n)$$

$$\Rightarrow \Psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Psi(A_n)$$

$\Rightarrow \Psi$ is σ- additive

—x—

Now we define a set function with specific properties.

Measure

Let $(\Omega, \mathcal{A})$ be a measurable space.

2016 1(c) A set function μ is said to be a measure if

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \geq 0$

(iii) μ is σ-additive (or countablly additive)

—x—

Measure space

The triplet $(\Omega, \mathcal{A}, \mu)$ is known as 'Measure Space'

Properties of a Measure

① If $A \subset B$, $A, B \in \mathcal{A}$ then $\mu(A) \leq \mu(B)$

proof Since $A \subset B$, $B = A \cup (B \cap A')$
and $A, B \cap A'$ are disjoint sets

$\therefore \mu(B) = \mu(A) + \mu(B \cap A')$

Note that μ is non-negative

$\Rightarrow \mu(B \cap A') \geq 0$

$\Rightarrow \mu(B) \geq \mu(A)$ OR $\mu(A) \leq \mu(B)$

2019-2(b)

2. If $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$

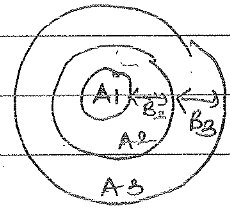
Proof: let $A_n \uparrow A$ i.e. $A_1 \subset A_2 \subset A_3 \subset \dots$

Define $B_1 = A_1$

$B_2 = A_2 \cap A_1'$

$\vdots$

$B_j = A_j \cap A_{j-1}' \cap \dots \cap A_1'$



then

$$A_n = B_1 \cup B_2 \cup \dots \cup B_n$$

and B_j's are disjoint sets

$$\therefore \mu(A_n) = \mu\left(\bigcup_{j=1}^{n} B_j\right) = \sum_{j=1}^{n} \mu(B_j) \quad \text{--- (1)}$$

Now consider

$$A = \bigcup_{n=1}^{\infty} A_n \quad (\because A_n \uparrow)$$

$$= \bigcup_{n=1}^{\infty} B_n \text{, where } B_n\text{'s are disjoint sets}$$

$$\therefore \mu(A) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

(Because μ is σ-additive)

$$\therefore \mu(A) = \lim_{k\to\infty} \sum_{n=1}^{k} \mu(B_n)$$

$$= \lim_{k\to\infty} \mu\left(\bigcup_{n=1}^{k} B_n\right)$$

$$= \lim_{k \to \infty} \mu(A_k)$$

$$= \lim_{n \to \infty} \mu(A_n)$$

Thus $\mu(A_n) \uparrow \mu(A)$ i.e. $[\mu(\lim A_n) = \lim \mu(A_n)]$ when $A_n \uparrow$

(remember if $A \subset B$, $\mu(A) \leq \mu(B)$)

—x—

Note:- It is sufficient if $\mu(A_k) < \infty$ for $k \geq 1$. Then in place of A, choose the set A_k and above proof hold

—x—

What will happen if μ is not finite

Example : let $\Omega = \mathbb{R}$, $\mathcal{A} = \mathcal{B}$

Consider the sequence $A_n = \{n, n+1, \dots\}$, $n \geq 1$

and define $\mu(A_n)$ = no. of elements in A_n

then $A_n \downarrow \phi$

$\mu(A_n) = \infty \quad \forall n$

$\Rightarrow \mu(A_n) \to +\infty$ but $\mu(\phi) = 0$

Thus $\mu(A_n) \not\downarrow 0$

Thus $\mu(A_k) < \infty$ for some $k \geq 1$ is a necessary condition for the above result.

Note:- Above properties are specially for monotone sequence. The following properties are for any general seq^n $\{A_n\}$

→ Let $\{A_n\}$ be a sequence of mble sets.
then we know that
$$\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = A \text{ (say) } \&$$
$$\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = B \text{ (say)}$$

2018-2(d) 2M

Thm (a) $\liminf \mu(A_n) \geq \mu(\liminf A_n)$
(b) $\limsup \mu(A_n) \leq \mu(\limsup A_n)$
if μ is a finite measure

2019 2(f)

Proof [Recall if $\{\alpha_n\}$ & $\{\beta_n\}$ are seq^n of real line nos & $\alpha_n \leq \beta_n$
$\Rightarrow \liminf \alpha_n \leq \liminf \beta_n$]

(a) Consider $\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} B_k$

where $B_k = \bigcap_{n=k}^{\infty} A_n$, then $B_k \uparrow$.

$$\therefore \mu(\liminf A_n) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right)$$
$$= \mu(\lim B_k)$$
$$= \lim \mu(B_k) \quad \text{(from earlier result)} \quad -(1)$$

Now $B_k \subset A_k$
$\Rightarrow \mu(B_k) \leq \mu(A_k)$
$\Rightarrow \liminf \mu(B_k) \leq \liminf \mu(A_k)$
i.e $\liminf \mu(A_k) \geq \liminf \mu(B_k)$
$= \lim \mu(B_k)$
$= \mu(\liminf A_n)$

Thus $\liminf \mu(A_n) \geq \mu(\liminf A_n)$

(b) We know that

$$\limsup A_n = (\liminf A_n')'$$
$$= \Omega - \liminf A_n'$$

$$\therefore \mu(\limsup A_n) = \mu(\Omega) - \mu(\liminf A_n')$$
$$\geq \mu(\Omega) - \liminf \mu(A_n')$$
(by part a)

[Recall : $\alpha - \liminf \alpha_n = \limsup(\alpha - \alpha_n)$]

Hence

$$\mu(\limsup A_n) \geq \limsup[\mu(\Omega) - \mu(A_n')]$$
$$= \limsup[\mu(A_n)]$$
$\because \mu$ is finite

Thus

$$\mu(\limsup A_n) \geq \limsup \mu(A_n)$$

—x—

Give an independent proof of (b) and hence prove (a)

Proof Consider $\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} B_k$

where $B_k = \bigcup_{n=k}^{\infty} A_n \downarrow$

$$\therefore \mu(\limsup A_n) = \mu(\bigcap_{k=1}^{\infty} B_k)$$
$$= \mu(\lim B_k)$$
$$= \lim \mu(B_k)$$
(using earlier results & μ is finite)

Now $A_k \subset B_k$

$\Rightarrow \mu(A_k) \leq \mu(B_k)$

$$\limsup \mu(A_k) \leq \limsup \mu(B_k)$$

Hence $\limsup \mu(A_k) \leq \limsup \mu(B_k)$
$= \lim \mu(B_k)$
$= \mu(\lim B_k)$
$= \mu(\limsup A_k)$

Hence $\limsup \mu(A_n) \leq \mu(\limsup A_n)$

Now $\liminf A_n = (\limsup A_n')'$
$= \Omega - \limsup A_n'$

$\therefore \mu(\liminf A_n) = \mu(\Omega) - \mu(\limsup A_n')$
$\leq \mu(\Omega) - \limsup \mu(A_n')$
$= \liminf \mu(\Omega - A_n')$
$= \liminf \mu(A_n)$

Hence the proof

$\rightarrow$ If $A_n \rightarrow A$ and μ is a finite measure then
$\mu(A) = \mu(\lim A_n) = \lim \mu(A_n)$

Proof Recall Since $A_n \rightarrow A$
$\Rightarrow \liminf A_n = \limsup A_n = \lim A_n = A$
Hence
$\mu(A) = \mu(\liminf A_n)$
$\leq \liminf \mu(A_n)$
$\leq \limsup \mu(A_n)$
$\leq \mu(\limsup(A_n)) = \mu(A)$

$\Rightarrow$ equality holds everywhere
$\Rightarrow \mu(A) = \liminf \mu(A_n) = \limsup \mu(A_n)$
$= \lim \mu(A_n)$

Thus $\mu(\lim A_n) = \lim \mu(A_n)$

2017 (c)

Def: Let Ω be some abstract space. If $\mathcal{C}_1 \subset \mathcal{C}_2$ and let μ_i be a measure defined on $\mathcal{C}_i$, $i = 1, 2$ such that

$$\mu_1(C_1) = \mu_2(C_1) \quad \forall \, C_1 \in \mathcal{C}_1$$

i.e μ_1 and μ_2 agrees on all sets in $\mathcal{C}_1$. Then μ_1 is known as a restriction of μ_2 on $\mathcal{C}_1$ & μ_2 is known as an extension of μ_1 on $\mathcal{C}_2$

Example: Let $\Omega = R$

Define $\mathcal{C}_1 = \{\phi, \Omega\}$, $\mathcal{C}_2 = \{\phi, \Omega, A, A'\}$.

$\mathcal{C}_3 = \{\phi, \Omega, B, B'\}$

$\mu_1 : \mathcal{C}_1 \to R$ s.t $\mu_1(\phi) = 0$ & $\mu_1(\Omega) = 1$

$\mu_2 : \mathcal{C}_2 \to R$ s.t $\mu_2(\phi) = 0$, $\mu_2(\Omega) = 1$

$\mu_2(A) = \mu_2(A') = 1/2$

$\mu_3 : \mathcal{C}_3 \to R$ s.t $\mu_3(\phi) = 0$, $\mu_3(\Omega) = 1$

$\mu_3(B) = 1/3$, $\mu_3(B') = 2/3$

Note that $\mathcal{C}_1 \subset \mathcal{C}_2$, $\mathcal{C}_1 \subset \mathcal{C}_3$

μ_1 agrees with μ_2 on $\mathcal{C}_1$

μ_1 agrees with μ_3 on $\mathcal{C}_1$

$\Rightarrow$ μ_1 is restriction of μ_2 on $\mathcal{C}_1$ and μ_1 is restriction of μ_3 on $\mathcal{C}_1$.

OR

μ_2 is extension of μ_1 on $\mathcal{C}_2$

μ_3 is extension of μ_1 on $\mathcal{C}_3$

Note that there can be more than one extension of a measure from $\mathcal{C}$ to $\mathcal{C}^*$, where $\mathcal{C} \subset \mathcal{C}^*$.

2017 (b)

Caratheodary Extension theorem:-
(Only statement)

Suppose μ is a σ-finite measure defined on a field. Then there exists a unique measure $\bar{\mu}$ on the σ-field $\sigma(\mathcal{F})$, which is also σ-finite.]

Lebesgue Measure

Consider the measurable space $(\mathbb{R}, \mathcal{B})$

We want to define a measure on this space

Now to define a measure on $\mathcal{B}$ is not practically possible, as we do not know that nature of $\mathcal{B}$.

We know simply some types of Borel sets but not all. Hence we first define a measure on a field and using above theorem extend it to $\mathcal{B}$.

Consider the field

$$\mathcal{F} = \{ I \mid I \text{ is a finite union of disjoint intervals of the type } (a,b], (-\infty, \alpha] \ \& \ (\beta, \infty) \}$$

We know that $\mathcal{F}$ is a field.

Define a set function on $\mathcal{F}$ as follows.

Let $A \in \mathcal{F}$, Then $A = \bigcup_{j=1}^{N} I_j$,

where I_j's are disjoint intervals of above type

Define

$$\mu(A) = \mu\left(\bigcup_{j=1}^{N} I_j\right) = \sum_{j=1}^{\infty} \mu(I_j)$$

where $\mu\{(a,b)\} = b-a$

$\mu[-\infty, \alpha] = \infty$

$\mu(\beta, \infty) = \infty$

Then μ is a measure, μ is not finite but μ is surely σ-finite.

→ Define $A_n = (n, n+1]$ then $\mu(A_n) = n+1-n = 1 < \infty$
and $R \subset \bigcup_{n=-\infty}^{\infty} A_n$

Thus μ is σ-finite

→ Then by Caratheodory extension theorem $\exists$ a unique measure λ on $\mathcal{B} = \sigma(\mathcal{H})$
such that μ is a restriction of λ on $\mathcal{H}$ OR
λ is extension of μ on $\mathcal{B}$.
and

$$\lambda\{(a, b]\} = \mu\{(a, b]\} = b - a.$$

This measure λ is known as the 'Lebesgue measure'

Sets which can be measured by λ are known as Lebesgue measurable sets.

→ Is every Borel set Lebesgue measurable?
Yes

Lebesgue measure of some Borel sets :-
(1) $B \in \mathcal{B}$ where B is a singleton set
Say $B = \{x\}$, $x \in R$
Now

$$\{x\} = \lim_{n \to \infty} \left[x - \frac{1}{n}, x\right]$$

$\therefore \lambda(B) = \lambda\{x\} = \lambda\left\{\lim_{n\to\infty}\left(x-\frac{1}{n}, x\right]\right\}$

$= \lim_{n\to\infty} \lambda\left(x-\frac{1}{n}, x\right]$

$= \lim_{n\to\infty} x - \left(x - \frac{1}{n}\right)$

$= \lim_{n\to\infty} \frac{1}{n}$

$= 0$

Thus lebesgue measure of every singleton set is zero.

(2) Let $B = [a,b] = \{a\} \cup (a,b]$

$\therefore \lambda(B) = \lambda[\{a\} \cup (a,b]]$

$= \lambda\{a\} + \lambda(a,b]$

$= 0 + b - a = b-a$

(3) Let $B = [a,b)$

$= \{a\} \cup (a,b)$

$= \lim_{n\to\infty}\left[\{a\} \cup \left(a, b-\frac{1}{n}\right]\right]$

$\therefore \lambda(B) = \lambda\{a\} + \lim_{n\to\infty} \lambda\left(a, b-\frac{1}{n}\right]$

$= 0 + \lim_{n\to\infty}\left(b - \frac{1}{n} - a\right)$

$= b-a$

(4) Let $B = (a,b)$

$= \lim_{n\to\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right]$

$$\therefore \lambda(B) = \lim_{n\to\infty} \lambda\left[a+\frac{1}{n}, b-\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left(b-\frac{1}{n}-a-\frac{1}{n}\right)$$

$$= b-a$$

Thus, whatever type of interval we have, its lebesgue measure is $b-a$.

In general

$$\lambda(A-B) = \lambda(A)-\lambda(B) \text{ if } B\subset A$$

provided both $\lambda(A)$ & $\lambda(B)$ are not $+\infty$

—x—

→ Let B = Set of rationals $= \{r_1, r_2, \dots\}$

then $B = \bigcup_k A_k$ where $A_k = \{r_k\}$

$$\therefore \lambda(B) = \lambda\left(\bigcup_k A_k\right) = \sum_k \lambda(A_k) = 0$$

* The lebesgue measure of the set of rational is zero

eg: Let B = Set of rationals between $(3,10]$ then $\lambda(B)=0$

Let B_1 = Set of irrationals between $(3,10]$

then

$$B_1 = (3,10] - \{\text{the set of rational bet}^n\ (3,10]\}$$

$$\therefore \lambda(B_1) = \lambda\{(3,10]\} - 0 = 7$$

let B_2 = set of irrational in $(-\infty,\infty)$

then $\lambda(B_2) = \infty$

Remark :- For whatever Borel set we want to find the lebesgue Measure, try to write that set in the form $(a,b]$

Lebesgue - Stiltjes Measure (L-S Measure)

Let F be a function on real line such that F is non-decreasing, right continuous
$F(-\infty) = 0$ and $F(+\infty) = 1$

(In probability theory, we call such a function as distribution function).

Let $\mathcal{H}$ be the field as defined above.
$\mathcal{H} = \{ I \mid I$ is finite union of disjoint intervals of the type $(a, b], (-\infty, \alpha], (\beta, \infty) \}$

Define a measure μ on $\mathcal{H}$ as follows.
$\mu\{(a, b]\} = F(b) - F(a) \geq 0$ ($\because F$ is non-decreasing)

$$\mu(\phi) = \lim_{b \downarrow a} F(b) - F(a)$$
$$= F(a^+) - F(a) = 0 \quad (\because F \text{ is right continuous})$$

$\mu(\beta, \infty) = \lim_{n \to \infty} (\beta, n]$ and for $I \in \mathcal{H}$.

$$\mu(I) = \mu\left(\bigcup_{j=1}^{k} I_j\right) = \sum_{j=1}^{k} \mu(I_j)$$

where I_j's are intervals of above type
$\mu(\mathbb{R}) = F(+\infty) - F(-\infty) = 1 - 0 = 1$

$\Rightarrow \mu$ is a finite measure & hence μ is σ-finite

Hence by Caratheodory Extension theorem $\exists$ a unique measure μ_F on $\mathcal{B}$, $\ni \mu_F\{(a, b]\} = F(b) - F(a)$

This measure μ_F is known as the 'Lebesgue - Stiltjes measure' or L-S measure corresponding to F.

Ex Let $F = \begin{cases} 0 & \text{if } x<0 \\ 1/2 & \text{if } 0 \le x < 1 \\ x/2 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

Note that $F(0^-) = 0$, $F(0) = F(0^+) = 1/2$

$\Rightarrow x=0$ is a point of discontinuity and all $x \neq 0$ $x \in R$ is a continuity point.

Let us find L.S measure of different sets

Let $B = \{x\}$, $x \in R$

Let $x=0$ i.e $B = \{0\}$

then $\mu_F(B) = \mu_F\{0\}$

$$= \lim_{n\to\infty} \mu_F\left(0-\frac{1}{n}, 0\right]$$

$$= \lim_{n\to\infty} F(0) - F\left(0-\frac{1}{n}\right)$$

$$= F(0) - F(0^-)$$

$$= 1/2 - 0 = 1/2$$

Let $x \neq 0$ say $x = 3/4$, $B = \{3/4\}$

$\mu_F(B) = \mu_F\{3/4\}$

$$= \lim_{n\to\infty} \mu_F\left(\frac{3}{4}-n, \frac{3}{4}\right]$$

$$= \lim_{n\to\infty}\left[F\left(\frac{3}{4}\right) - \cancel{\lim_{n\to\infty}} F\left(\frac{3}{4}-n\right)\right]$$

$$= \cancel{\lim} F\left(\frac{3}{4}\right) - \lim_{n\to\infty} F\left(\frac{3}{4}-n\right)$$

$\mu_F(B) = F\left(\frac{3}{4}\right) - F\left(\frac{3}{4}\right) = \frac{1}{2} - \frac{1}{2} = 0$

Similarly if $x=4$, $\mu_F\{4\} = 0$

Thus, the LS measure of every continuity pt of F is zero and for points of discontinuity

L-S measure = height of jump at that point

→ Now Suppose $B = [a,b] = \{a\} \cup (a,b]$

$\therefore \mu_F(B) = \mu_F\{a\} + \mu_F(a,b]$

$= \mu_F\{a\} + F(b) - F(a)$

let $B = (a,b) = (a,b] - \{b\}$

$\therefore \mu_F(B) = \mu_F(a,b] - \mu_F\{b\}$

let $B = [a,b) = (a,b] + \{a\} - \{b\}$

$\therefore \mu_F(B) = \mu_F(a,b] + \mu_F\{a\} - \mu_F\{b\}$

$= F(b) - F(a) + \mu_F\{a\} - \mu_F\{b\}$

So eg. let $B = [0,3] = \{0\} \cup (0,3]$

$\therefore \mu_F(B) = \mu_F\{0\} + F(3) - F(0))$

$= \frac{1}{2} + 1 - \frac{1}{2} = 1$

let $B = (0,3) = (0,3] - \{3\}$

$\therefore \mu_F(B) = \mu_F(0,3] - \mu_F\{3\}$

$= F(3) - F(0) - 0$

$= 1 - 1/2 - 0$

$= 1/2$

$B = \left(\frac{3}{2}, \frac{7}{2}\right]$

$\mu_F(B) = F\left(\frac{7}{2}\right) - F\left(\frac{3}{2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$

2018 - 3(b)
2019 - 3(a)

→ Let $f(x) = \begin{cases} 0 & x < -3 \\ 1/4 & -3 \le x < 0 \\ \frac{x+1}{2} & 0 \le x < 1/2 \\ 1 & x \ge 1/2 \end{cases}$

Find L.S measure of the following sets
$A = \{0, 1/2\}$ $B = [0, 1/2]$, $C = (-3, 0)$, $D = (0, 1/2)$
$E = [-3, 0] = \{-3\} \cup (-3, 0]$

Here we note that

x	$F(x^-)$	$F(x)$	$F(x^+)$
-3	0	$1/4$	$1/4$
0	$1/4$	$1/2$	$1/2$
$1/2$	$3/4$	1	1

Thus $x = -3, 0, 1/2$ are points of discontinuity and all $x \in R$ except $x = -3, 0, 1/2$ are continuity point of F.

$\therefore \mu_F\{-3\} = 1/4 - 0 = 1/4$
$\mu_F\{0\} = 1/2 - 1/4 = 1/4$
$\mu_F\{1/2\} = 1 - 3/4 = 1/4$

$\therefore \mu_F(A) = \mu_F\{0, 1/2\} = \mu_F\{0\} + \mu_F\{1/2\}$
$= 1/4 + 1/4 = 1/2$

$\therefore \mu_F(B) = \mu_F[0, 1/2]$
$= \mu_F\{0\} + \mu_F(0, 1/2]$
$= 1/4 + F(1/2) - F(0)$
$= 1/4 + 1 - 1/2 = 3/4$

$\mu_F(C) = \mu_F(-3, 0)$
$= \mu_F\{(-3, 0] - \{0\}\}$
$= \mu_F(-3, 0] - \mu_F\{0\}$
$= F(0) - F(-3) - 1/4$
$= 1/2 - 1/4 - 1/4 = 0$

$$\mu_F(D) = \mu_F(0, 1/2)$$
$$= \mu_F(0, 1/2] - \mu_F\{1/2\}$$
$$= F\left(\tfrac{1}{2}\right) - F(0) - \tfrac{1}{4}$$
$$= 1 - 1/2 - 1/4 = 1/4$$

$$\mu_F(E) = \mu_F\{[-3, 0]\}$$
$$= \mu_F\{-3\} + \mu_F(-3, 0]$$
$$= 1/4 + F(0) - F(-3)$$
$$= \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{2}$$

—x—

let $F(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ 1/2 & , \text{ if } 0 \le x < 1 \\ 1 - \frac{1}{2} e^{-x+1} & , \text{ if } x \ge 1 \end{cases}$

Find L-S measure of the sets $\{0\}, \{0, 1\}, [0, 1], (0, 1)$

Here $F(0^-) = 0$, $F(0) = F(0+) = 1/2$

and $x = 0$ is the only point of discontinuity

and $\forall x \neq 0$ is continuity point of F

$\therefore \mu_F\{0\} = 1/2 - 0 = 1/2$ and $\mu_F\{x\} = 0 \quad \forall x \neq 0$

$\therefore \mu_F\{0, 1\} = \tfrac{1}{2} + 0 = \tfrac{1}{2}$

$$\mu_F[0, 1] = \mu_F\{0\} + \mu_F(0, 1]$$
$$= 1/2 + F(1) - F(0)$$
$$= 1/2 + 1/2 - 1/2 = 1/2$$

$$\mu_F(0, 1) = \mu_F(0, 1] - \mu_F\{1\}$$
$$= F(1) - F(0) - 0$$
$$= 1/2 - 1/2$$
$$= 0$$

→ Let $\Omega = R$

Let $A = (-\infty, 0]$ and $\mathcal{C} = \{\Omega, \phi, A, A'\}$

then $\mathcal{C}$ is a field.

Define μ on $\mathcal{C}$ as $\mu(B) = \begin{cases} 1 & \text{if } 4 \in B \\ 0 & \text{if } 4 \notin B \end{cases}$

then $\mu(\Omega) = 1$, $\mu(\phi) = 0$, $\mu(A) = 0$ & $\mu(A') = 1$

Now consider a set $C = (-5, 2) \subset A$

but $\mu(A) = 0$

$\not\Rightarrow \mu(C) = 0$, $\because C \notin \mathcal{C}$ and μ is defined only for sets in $\mathcal{C}$ and not for any other sets

In fact, here $\mu(C)$ is not defined so to rectify this, define a σ-field $\mathcal{C}^*$ as

$$\mathcal{C}^* = \sigma[\mathcal{C} \cup \{\text{all subsets of } A\}]$$

[to get rid of the problem that subject of set of measure zero are not measurable, we need to define $\mathcal{C}^*$ as above]

Def: A measure μ is called a complete measure if $\mu(A) = 0$, $B \subset A \Rightarrow B$ is a measurable set and hence $\mu(B) = 0$

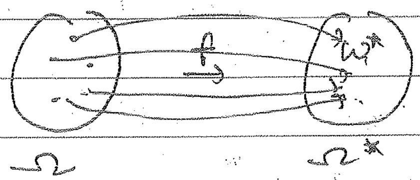
Thus $(\Omega, \mathcal{A}, \mu)$ is known as complete measure space, if for mble set $A \in \mathcal{A}$ of measure zero all subsets of A are measurable.

A set of measure zero i.e $\mu(A) = 0$ is known as a null set (or μ null set)

Remark :- Any measure space can be completed by the procedure of defining $\mathcal{C}^*$

* Functions defined on a Measure Space

Uptill now, we have studied measures, which are set function. Set function are not easy to handle with since the basic arithematic operation cannot be performed on sets.
So we defined function on mble space
Let $(\Omega, \mathcal{A})$ be a mble space
Let $f : \Omega \to \Omega^*$
(if $\Omega^* = R$, then f is a real valued functions)
$f : \Omega \to \Omega^*$ means for $w \in R$, $f(w) \in \Omega^*$



$$\{w \mid f(w) = w^*\} \subset \Omega$$

Def: $f^{-1}(w^*) = \{w \in \Omega \mid f(w) = w^*\}$

Def: for $(A^*) \subset \Omega^*$

$$f^{-1}(A^*) = \{w \mid f(w) \subset A^*\} \subset \Omega$$

$$= \bigcup_{w^* \in A^*} f^{-1}(w^*)$$

Let $\{A_1^*, A_2^*\}$ be a collection of sets of Ω^*
then

$$\{f^{-1}(A_1^*), f^{-1}(A_2^*)\} = f^{-1}\{A_1^*, A_2^*\}$$

Def: In general, for a class $\mathcal{C}^*$ of subsets of Ω^*,

$$f^{-1}(\mathcal{C}^*) = \{A \subset \Omega \mid A = f^{-1}(A^*), \text{ for some } A^* \in \mathcal{C}^*\}$$

$$f^{-1}(A^* \cup B^*) = \{\omega \mid f(\omega) \in A^* \cup B^*\}$$
$$= \{\omega \mid f(\omega) \in A^*\} \cup \{\omega \mid f(\omega) \in B^*\}$$
$$= f^{-1}(A^*) \cup f^{-1}(B^*)$$

In general
let T be an indexing set (may be finite, infinite Countable, uncountable)
then

$$f^{-1}\left(\bigcup_{t \in T} A_t^*\right) = \bigcup_{t \in T} f^{-1}(A_t^*)$$

Similarly

$$f^{-1}(A^* \cap B^*) = f^{-1}(A^*) \cap f^{-1}(B^*)$$

and

hence $f^{-1}\left(\bigcap_{t \in T} A_t^*\right) = \bigcap_{t \in T} f^{-1}(A_t^*)$

Thm $f^{-1}(A^*)' = [f^{-1}(A^*)]'$

Proof Consider

$$f^{-1}(A^*)' = \{\omega \mid f(\omega) \in (A^*)'\}$$

$$= \{\omega \mid f(\omega) \notin A^*\}$$

$$= \{\omega \mid f(\omega) \in A^*\}'$$

$$= [f^{-1}(A^*)]'$$

2019 - 3(b)

Thm 1: Inverse image of a σ-field is a σ-field
Let $f : \Omega \to \Omega^*$

2016 2(a) let $\mathcal{A}^*$ be a σ-field of subsets of Ω^*

i.e $\mathcal{A}^* = \{A^* / A^* \subset \Omega^*\}$ be a σ-field
then

$$\mathcal{A} = f^{-1}(\mathcal{A}^*) \text{ is also a } \sigma\text{-field}$$

Proof clearly A is non empty ~~became~~ because

$$\Omega = f^{-1}(\Omega^*) \in f^{-1}(\mathcal{A}^*) = \mathcal{A}$$

let $A \in \mathcal{A}$ to prove $A' \in \mathcal{A}$

Now

$$A \in \mathcal{A} \Rightarrow A \in f^{-1}(\mathcal{A}^*)$$

$\Rightarrow A = f^{-1}(A^*)$ for some $A^* \in \mathcal{A}^*$

but

$\mathcal{A}^*$ is a σ- field $\Rightarrow (A^*)' \in \mathcal{A}^*$

Now

$$A = f^{-1}(A^*)$$

$$\therefore A' = [f^{-1}(A^*)]'$$

$$= f^{-1}(A^*)' \quad \text{where } (A^*)' \in \mathcal{A}^*$$

$\Rightarrow A' \in f^{-1}(\mathcal{A}^*)$ i.e $A' \in \mathcal{A}$

$\Rightarrow \mathcal{A}$ is closed under compliments

Further let $A_n \in \mathcal{A}$, $n = 1, 2, 3.$

i.e

$$A_n \in f^{-1}(\mathcal{A}^*) \quad \forall n$$

i.e $A_n = f^{-1}(A_n^*)$ for some $A_n^* \in \mathcal{A}^*$

$$\therefore \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} f^{-1}(A_n^*)$$

$$= f^{-1}\left(\bigcup_{n=1}^{\infty} A_n^*\right)$$

but $\mathcal{A}^*$ is a σ- field $\Rightarrow \bigcup_{n=1}^{\infty} A_n^* \in \mathcal{A}^*$

$$\bigcup_{n=1}^{\infty} A_n \in f^{-1}(\mathcal{A}^*) = \mathcal{A}$$

$\Rightarrow \mathcal{A}$ is closed under countable union.
$\Rightarrow \mathcal{A}$ is a σ-field.

—x—

Thm (2018 3(b)(ii) (2.5)) Let $\mathcal{A}$ be a σ-field of subsets of Ω then

$$\mathcal{C}^* = \{A^* \mid f^{-1}(A^*) \in \mathcal{A}\}$$ is a σ-field of subsets of Ω^*.

Proof Note that $\mathcal{A}$ is a σ-field. $\Rightarrow \Omega \in \mathcal{A}$,
but $\Omega = f^{-1}(\Omega^*) \Rightarrow \Omega^* \in \mathcal{C}^*$

So that $\mathcal{C}^*$ is non-empty.
Let $A^* \in \mathcal{C}^*$ to prove $(A^*)' \in \mathcal{C}^*$

Now $A^* \in \mathcal{C}^* \Rightarrow f^{-1}(A^*) \in \mathcal{A}$
$\Rightarrow [f^{-1}(A^*)]' \in \mathcal{A}$
$\Rightarrow f^{-1}[(A^*)'] \in \mathcal{A}$
$\Rightarrow (A^*)' \in \mathcal{C}^*$

$\Rightarrow \mathcal{C}^*$ is closed under compliments

Finally, let $A_n^* \in \mathcal{C}^*$, $n = 1, 2, \ldots$ $(f^{-1}(A_n^*) \in \mathcal{A} \;\forall n)$
to prove

$$\bigcup_n A_n^* \in \mathcal{C}^*$$

Consider

$$f^{-1}\left(\bigcup_n A_n^*\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n^*) \in \mathcal{A}, \quad \because \mathcal{A} \text{ is a } \sigma\text{-field}$$

$$\Rightarrow \bigcup_n A_n^* \in \mathcal{C}^*$$

$\Rightarrow \mathcal{C}^*$ is closed under countable unions and hence $\mathcal{C}^*$ is also a σ-field

Remark If $A^* \subset B^* \Rightarrow f^{-1}(A^*) \subset f^{-1}(B^*)$

Similarly

if $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ are two collection of subsets of Ω^*

then

$$\mathcal{C}_1^* \subset \mathcal{C}_2^* \Rightarrow f^{-1}(\mathcal{C}_1^*) \subset f^{-1}(\mathcal{C}_2^*)$$

Thm 2018 (5b) $\sigma[f^{-1}(\mathcal{C}^*)] = f^{-1}[\sigma(\mathcal{C}^*)]$

where $\mathcal{C}^*$ is a collection of subsets of Ω^*

Proof Recall $\mathcal{C}^* \subset \sigma(\mathcal{C}^*)$

so

$$\therefore f^{-1}(\mathcal{C}^*) \subset f^{-1}[\sigma(\mathcal{C}^*)]$$

$$\therefore \sigma[f^{-1}(\mathcal{C}^*)] \subset f^{-1}[\sigma(\mathcal{C}^*)] \quad \text{— (1)}$$

because $f^{-1}\sigma(\mathcal{C}^*)$ is a σ-field by thm 1

Next to prove

$$f^{-1}[\sigma(\mathcal{C}^*)] \subset \sigma[f^{-1}(\mathcal{C}^*)]$$

Consider the class of sets

$$\mathcal{D} = \{B \subset \Omega^* \mid f^{-1}(B) \in \sigma[f^{-1}(\mathcal{C}^*)]\}$$

then by thm 2, $\mathcal{D}$ is a σ-field

Suppose a set $E \in \mathcal{C}^*$

Then $f^{-1}(E) \in f^{-1}(\mathcal{C}^*) \subset \sigma(f^{-1}(\mathcal{C}^*))$

$\Rightarrow E \in \mathcal{D}$

$\Rightarrow \mathcal{C}^* \subset \mathcal{D}$

$$\therefore \sigma(\mathcal{C}^*) \subset \mathcal{D} \quad (\because \mathcal{D} \text{ is a } \sigma\text{-field})$$

$$\Rightarrow f^{-1}(\sigma(\mathcal{C}^*)) \subset f^{-1}(\mathcal{D}) \quad \text{— (2)}$$

Now

$$\mathcal{D} = \{ B \mid f^{-1}(B) \in \sigma(f^{-1}(\mathcal{C}^*)) \}$$

$$\Rightarrow f^{-1}(\mathcal{D}) = \sigma(f^{-1}(\mathcal{C}^*)) \quad — (3)$$

using (3) in (2) we have

$$f^{-1}(\sigma(\mathcal{C}^*)) \subset \sigma(f^{-1}(\mathcal{C}^*)) \quad — (4)$$

From (1) and (4), we have

$$f^{-1}(\sigma(\mathcal{C}^*)) = \sigma[f^{-1}(\mathcal{C}^*)]$$

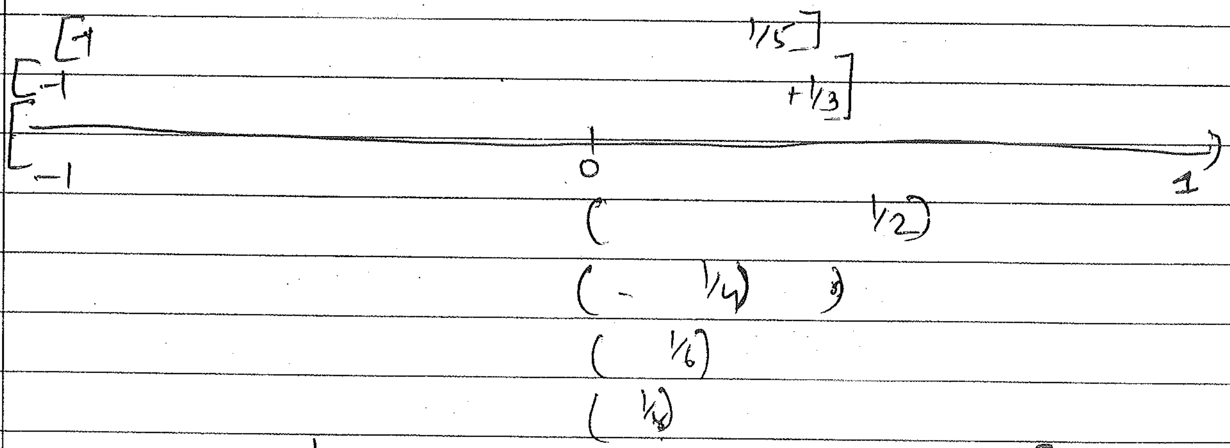
Example $A_{2n} = \left(0, \frac{1}{2n}\right)$, $A_{2n+1} = \left[-1, \frac{1}{2n+1}\right]$

$A_1 = \left[-1, \frac{+1}{1}\right]$ $\quad A_2 = (0, 1/2)$

$A_3 = [-1, 1/3]$ $\quad A_4 = (0, 1/4)$

$A_5 = [-1, +1/5]$ $\quad A_6 = (0, 1/6)$

$A_7 = [-1, 1/7]$ $\quad A_8 = (0, 1/8)$



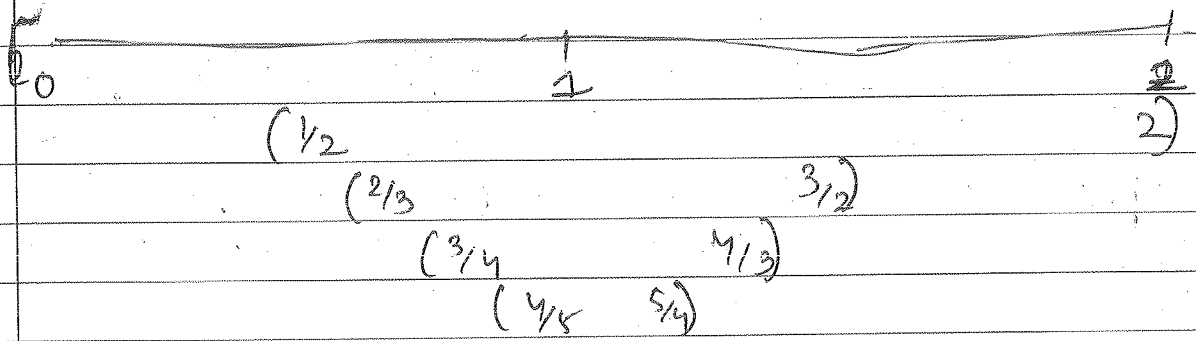
$\lim_{n\to\infty} A_{2n} = \phi$, $\quad \lim_{n\to\infty} A_{2n+1} = [-1, 0]$

(2) A_n = The set of rational integers

$$= \left(1 - \frac{1}{n+1},\ 1 + \frac{1}{n}\right)$$

$$A_1 = \left(\frac{1}{2}, 2\right), \quad A_2 = \left(\frac{2}{3}, \frac{3}{2}\right)$$

$$A_3 = \left(\frac{3}{4}, \frac{4}{3}\right), \quad A_4 = \left(\frac{4}{5}, \frac{5}{4}\right)$$



In fact $1 - \frac{1}{n+1} \to 1$ as $n \to \infty$

$1 - \frac{1}{n+1} < 1$ but $1 + \frac{1}{n} \to 1$ as $n \to \infty$

but $1 + 1/n > 1$

Note that A_n is decreasing sequence and

$$\lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n = \{1\}$$

(2) $A_n = \left(0,\ b + \frac{(-1)^n}{n}\right)$, $b \geq 1$ increasing

$A_1 = (0,\ b-1)$ $\quad$ $A_2 = (0,\ b+1/2)$

$A_3 = (0,\ b-1/3)$ $\quad$ $A_4 = (0,\ b+1/4)$

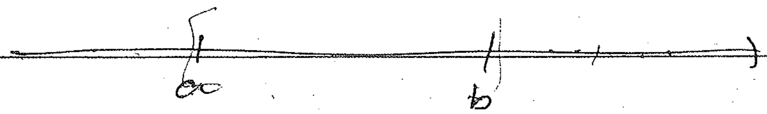
$A_{2n+1} = (0, b)$ $\quad$ $A_{2n} = \left(0,\ b+\frac{1}{2n}\right)$

$= (0, b]$

$\liminf A_n = (0, b)$
$\limsup A_n = (0, b]$ $\quad$ because $(0,b) \subset (0,b]$

$\left(a-\frac{1}{n},\ b+\frac{1}{n}\right) \to [a, b]$

$(a,\ b-1) \to (a, b]$



Measurable function (3 - defn)

Recall $R : (-\infty, \infty) \rightarrow \sigma$- field : $\mathcal{B}$
$\bar{R} : [-\infty, \infty]$: extended real line
Then $\bar{\mathcal{B}}$: σ- field on $\bar{R}$

$\bar{\mathcal{B}}$ is generated in the same way as $\mathcal{B}$.
Now as we discussed earlier, set function defined on σ- field are not convenient to work with, so we define function on Ω

So consider $f : \Omega \rightarrow \bar{R}$
We will study different types of function

Defn A function $f : \Omega \rightarrow \bar{R}$ is called an indicator function, if for some $A \subset \Omega$, $(A \in \mathcal{A}^*)$

$$f(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

i.e

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Simple function :- A function : $f : \Omega \rightarrow \bar{R}$ is called a simple function, if it takes a finite number of distinct values $C_1, C_2, \ldots C_n$ on sets $A_1, A_2, \ldots A_n$, where $A_1, \ldots A_n$ are subsets of Ω s.t

$$A_i \cap A_j = \phi \ \forall \ i \neq j \ \ \& \ \ \bigcup_{k=1}^{n} A_k = \Omega$$

Thus $f(\omega) = C_k$ if $\omega \in A_k$, $k = 1, \ldots n$

Another way of writing f is

$$f = \sum_{k=1}^{n} C_k \cdot I_{A_k}$$

or

$$f(\omega) = \sum_{k=1}^{n} C_k \cdot I_{A_k}(\omega)$$

$\Rightarrow f(\omega) = C_k$ if $\omega \in A_k$

Remark : 1) Indicator funⁿ is special case of simple function

2) In above definition, any of the C_k may be $+\infty$ or $-\infty$

Elementary function :- A function $f : \Omega \to \overline{R}$ is called an elementary function if it takes a countable no. of distinct values.

i.e

$$f(\omega) = C_k \quad \text{if} \quad \omega \in A_k \quad , k = 1, 2 \ldots$$

where $A_i \cap A_j = \phi \quad \forall\, i \neq j \quad \& \quad \bigcup_{k=1}^{\infty} A_k = \Omega$

Result :- If f and g are two simple function Then all arethematic operation on simple function result is simple function provided the resulting function is well defined. Thus $f+g$, $f-g$, $f \times g$, & f/g are all simple function, provided. We are not coming accross the terms like $\infty - \infty$, ∞/∞, $0/\infty$, $0 \times \infty$ etc

— x —

Thus we can define a no. of function we are interested in function having specific property & we call such function as measurable function

Measurable function :-

Def^n Let $(\Omega, \mathcal{A})$ be fixed.
A function $f: \Omega \to \overline{\mathbb{R}}$ is called measurable function if $f^{-1}(B) \in \mathcal{A}$ $\forall B \in \overline{\mathcal{B}}$
i.e inverse image under f of a Borel set is in $\mathcal{A}$ i.e a measurable set.

Remark :- This is a discriptine definition, since it only discribes the concept, but do not help to construct a mble function.

Def (D'): (Discriptine type)
A function $f: \Omega \to \overline{\mathbb{R}}$ is called a measurable function if $f^{-1}(C) \in \mathcal{A}$ $\forall C \in \mathcal{C}$ where $\sigma(\mathcal{C}) = \mathcal{B}$

eg $\mathcal{C} = \{ I \mid I = (a, b], -\infty \leq a \leq b \leq \infty \}$
or $\mathcal{C} = \{ I \mid I = (-\infty, \alpha], -\infty \leq \alpha \leq \infty \}$

Again this is a discriptine definition

* Properties of function measurable according to def D' :-

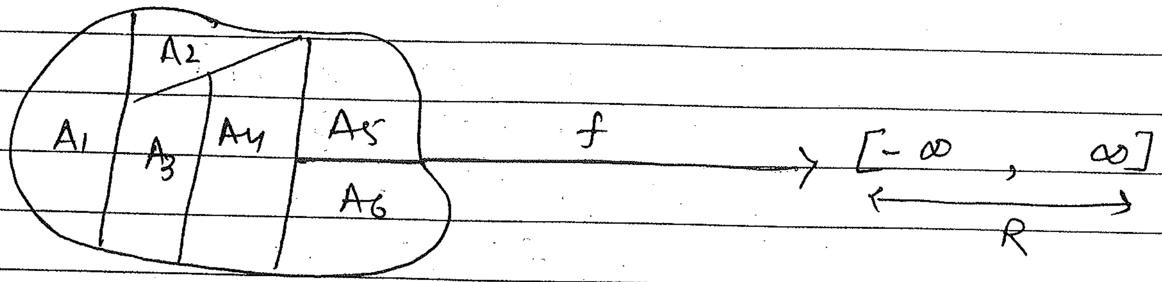
(1) Suppose f is a simple function. Then f is measurable (D').

Proof

Let $f = \sum_{j=1}^{n} c_j I_{A_j}$ be a simple function

where $A_1, \ldots A_n \in \mathcal{A}$, $A_i \cap A_j = \phi$, $\bigcup_{j=1}^{n} A_j = \Omega$

Now $\mathcal{A}$ is a σ-field. So all possible unions / intersection / compliments of $A_1, \ldots A_n$ are in $\mathcal{A}$



then if $\mathcal{C} = \{ I \mid I = (a,b],\ -\infty \le a \le b \le \infty \}$

and $f^{-1}\{(a,b]\} = \{ \omega \mid a < f(\omega) \le b \}$

$$= \bigcup_{\{j \mid c_j \in (a,b]\}} A_j \in \mathcal{A}$$

Thus

$$f^{-1}(C) \in \mathcal{A} \ \forall\, C \in \mathcal{C} \quad \text{s.t } \sigma(\mathcal{C}) = \bar{\mathcal{B}}$$

Hence f is measurable according to definition (D')

2) let $f_1, f_2, \ldots$ be function mble (D')
Then $g = \max_k f_k$ is mble (D')

let $\mathcal{C} = \{ I \mid I = [-\infty, \alpha] \}$
Then $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$

$$g^{-1}(I) = \{ \omega \mid \max f_k(\omega) \le \alpha \}$$
$$= \{ \omega \mid f_k(\omega) \le \alpha \ \forall k \}$$
$$= \bigcap_k \{ \omega \mid f_k(\omega) \le \alpha \}$$
$$= \bigcap_k f_k^{-1}(I) \in \mathcal{A}$$

$\therefore f_k$ is mble (D').

$\Rightarrow \max_k f_k$ is mble (D')

(3) If f is mble (D'), $-f$ is mble (D')
let $g = -f$. let
$\mathcal{C} = \{ I \mid I = [-\infty, \alpha],\ -\infty \le \alpha \le \infty \}$

Consider

$$g^{-1}(I) = \{\omega \mid g(\omega) \in [-\infty, \alpha]\}$$
$$= \{\omega \mid g(\omega) \leq \alpha\}$$
$$= \{\omega \mid -f(\omega) \leq \alpha\}$$
$$= \{\omega \mid f(\omega) \geq -\alpha\}$$
$$= f^{-1}[-\alpha, \infty] \in \mathcal{A} \quad (\because A \text{ is a } \sigma\text{-field})$$

$\Rightarrow g^{-1}(I) \in \mathcal{A} \;\; \forall I \in \mathcal{C}$ where $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$

$\Rightarrow g$ is mble (D')

— x —

On similar lines, we can prove that $\min f_k$, $\lim\sup f_n$, $\lim\inf f_n$ and $\lim f_n$ if it exists are all mble (D'), if each f_n is mble (D').

— x —

Constructive Definition

Def – A function $f : \Omega \to \overline{R}$ is called mble func^n if $\exists$ a sequence $\{f_n\}$ of simple functions s.t $f_n(\omega) \to f(\omega)$; Here each f^n, f_n is

$$F_n = \sum_{k=1}^{m_k} C_{n,k} \, I_{A_{n,k}}$$

(each f_n is a simple function)

2019-4(a)

Result : All these definitions are equivalent

Proof Suppose f is mble (D)

i.e $f^{-1}(B) \in \mathcal{A} \;\; \forall B \in \overline{\mathcal{B}}$

Since $\mathcal{C} \subseteq \overline{\mathcal{B}}$ and $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$

$\Rightarrow f^{-1}(C) \in \mathcal{A} \;\; \forall C \in \mathcal{C}$ with $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$

$\Rightarrow f$ is mble (D')

Conversely, suppose f is mble (D')

i.e $f^{-1}(C) \in \mathcal{A} \;\; \forall C \in \mathcal{C}$ where $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$

To prove that $f^{-1}(B) \in \mathcal{A} \quad \forall B \in \mathcal{B}$

i.e. to prove $f^{-1}(\mathcal{B}) \subseteq \mathcal{A}$

Now, we know that

$$\sigma[f^{-1}(\mathcal{C})] = f^{-1}(\sigma(\mathcal{C})) = f^{-1}(\mathcal{B})$$

From (1), we have

$$f^{-1}(C) \in \mathcal{A} \quad \forall C \in \mathcal{C}$$

$\Rightarrow \sigma(f^{-1}(\mathcal{C})) \subset \mathcal{A}$ ($\because \mathcal{A}$ is a σ-field)

$\Rightarrow f^{-1}(\mathcal{B}) \subset \mathcal{A}$

Thus f is mble (D)

Thus we proved that Def (D) $\equiv$ Def (D')

—x—

Now suppose that $\{f_n\}$ is a seq^n of simple functions s.t $f_n \to f$

(i.e f is mble according to def C)

(Recall if f_n is mble (D') ($\because$ it is a simple f^n)

so $\lim f_n = f$ is mble (D')

and f is mble (D) ($\because D' \equiv D$)

Thus C $\Rightarrow$ D' and D

Finally, let f be mble (D'/D)

To prove f is mble according 'C'

Define $f_n(\omega)$ as follows

$$f_n(\omega) = \begin{cases} \dfrac{k-1}{2^n} & \text{if } \dfrac{k-1}{2^n} \leq f(\omega) < \dfrac{k}{2^n} \\ & k = -n2^n+1, \dots n2^n \\ -n & \text{if } f(\omega) < -n \\ n & \text{if } f(\omega) \geq n \end{cases}$$

The function f_n takes $2n2^n+1$ values
let us write the f^n for say $n=1,2..$

where $n=1$ range for k is $-1, 0, 1, 2$
then $\frac{k-1}{2^n}$ will be $-1, -\frac{1}{2}, 0, \frac{1}{2}$

then

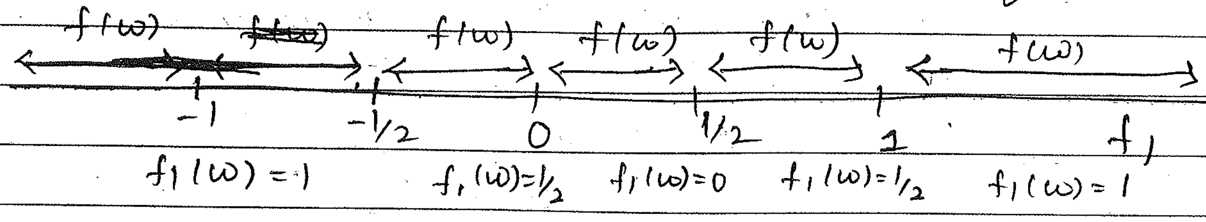
$$f_1(\omega) = \begin{cases} -1 & \text{if} \quad f(\omega) < -1 \\ -1 & \text{if} \quad -1 \leq f(\omega) < 1/2 \\ -1/2 & \text{if} \quad -1/2 \leq f(\omega) < 0 \\ 0 & \text{if} \quad 0 \leq f(\omega) < 1/2 \\ 1/2 & \text{if} \quad 1/2 \, f(\omega) < 1 \\ 1 & \text{if} \quad f(\omega) \geq 1 \end{cases}$$

Similarly, let us write $f_2(\omega)$
Here $n=2$. Hence
$k = -n2^n+1, \dots n2^n$ becomes
$k = -7, -6, -5, \dots -1, 0, 1, 2, \dots 8$
So that
$$\frac{k-1}{2^n} = -\frac{8}{4}, -\frac{7}{4}, -\frac{6}{4}, \dots -\frac{2}{4}, -\frac{1}{4}, 0, \frac{1}{4} \dots \frac{7}{4}$$

and hence $f_2(\omega)$ is

$$f_2(\omega) = \begin{cases} -2 & \text{if} \quad f(\omega) < -2 \\ -8/4 \; (=-2) & \text{if} \quad -2 \leq f(\omega) \leq -7/4 \\ -7/4 & \text{if} \quad -7/4 \leq f(\omega) < -6/4 \\ \vdots & \\ -1/4 & \text{if} \quad -1/4 \leq f(\omega) < 0 \\ 0 & \text{if} \quad 0 \leq f(\omega) < 1/4 \\ 1/4 & \text{if} \quad 1/4 \leq f(\omega) < 2/4 \\ \vdots & \\ 7/4 & \text{if} \quad 7/4 \leq f(\omega) < +8/4 = +2 \\ 2 & \text{if} \quad f(\omega) \geq 2 \end{cases}$$

In general $f_n(\omega)$ takes $2n2^n+1$ distinct values. So each f_n is a simple function



While defining $f_2(\omega)$ we make still smaller intervals betn $(-2, 2)$ and approximate $f(\omega)$ by $f_2(\omega)$

$\Rightarrow$ as n increases, the difference betn $f(\omega)$ & $f_n(\omega)$ becomes smaller and smaller

Thus being simple function, each f_n is a mble function

If $f(\omega) = +\infty$ then $f_n(\omega) = n \to \infty$ as $n \to \infty$

If $f(\omega) = -\infty$ then $f_n(\omega) = -n \to -\infty$ as $n \to \infty$

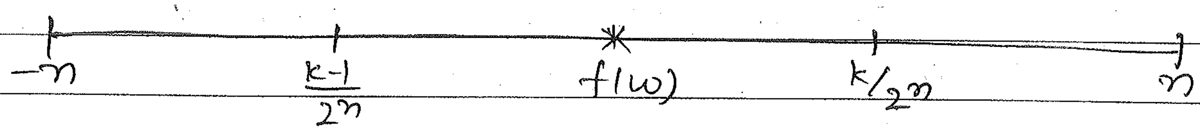
Thus in either case $f_n \to f$

If $-\infty < f(\omega) < \infty$, then $\exists$ n (large) s.t

$$-n \leq f(\omega) < n$$

then

$$\exists\ k \text{ s.t } \frac{k-1}{2^n} \leq f(\omega) < \frac{k}{2^n}$$



In that case

$$|f_n(\omega) - f(\omega)| < \frac{1}{2^n} \to 0 \text{ as } n \to \infty$$

Thus $f_n \to f$ in all cases.

i.e f is a limit of seqn of simple functions $\{f_n\}$. $\Rightarrow$ f is mble (c)

Thus all the three definitions of mble f^n are equivalent

Remark Let $f:(\Omega, \mathcal{A}_1) \to (\bar{R}, \bar{B})$ be a mble function. Let $\mathcal{A}_2$ be another σ-field. Then f may not be measurable w.r.t $(\Omega, \mathcal{A}_2)$.

eg. Let $\Omega = R$ and
$\mathcal{A}_1 = \{\Omega, \phi, (-\infty, 0], (0, \infty)\}$ and
$\mathcal{A}_2 = \{\Omega, \phi, (-\infty, 5], [5, \infty)\}$

Define $f(\omega) = \begin{cases} 1 & \text{if } \omega \le 0 \\ 2 & \text{if } \omega > 0 \end{cases}$

Then $f(\omega) = 1 \cdot I_{(-\infty, 0]} + 2 \cdot I_{(0, \infty)}$
$= 1 \cdot I_A + 2 I_{A^c}$

Note A & $A^c \in \mathcal{A}_1$ hence f is mble $(\Omega, \mathcal{A}_1)$
but A & $A^c \notin \mathcal{A}_2 \Rightarrow f$ is not mble $(\Omega, \mathcal{A}_2)$

for any σ-field not containing A & A^c, f will not be mble w.r.t that σ-field

Now suppose we have another σ-field say $\mathcal{A}$ s.t $\mathcal{A}_1 \subset \mathcal{A}$ then f is surely mble w.r.t $(\Omega, \mathcal{A})$. We know that $\mathcal{A}_1 \subset P(\Omega)$
$\Rightarrow f$ is mble w.r.t $(\Omega, P(\Omega))$

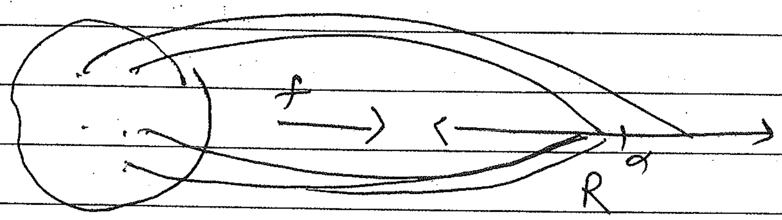
Thus f is mble w.r.t every σ-field containing $\mathcal{A}$. So we need to choose smallest one

Def^n The Smallest σ-field w.r.t which a given f^n is mble is called the σ-field included by f on Ω.

In the above example, where f takes only 2 distinct values A & A^c the σ-field included by f is $\{\phi, \Omega, A, A^c\}$

—x—

Let $\mathcal{A} = \{\Omega, \phi\}$: trivial σ-field
To find which function are mble w.r.t this σ-field
Define $f(\omega) = \alpha \ \forall \omega \in \Omega$ (constant function)



To check the condition $f^{-1}(C) \in \mathcal{A}$

$$f^{-1}(C) = \phi \quad \text{if } \alpha \notin C$$
$$\Omega \quad \text{if } \alpha \in C$$

Thus the smallest σ-field induced by f is $\{\Omega, \phi\}$

<u>Conclusion</u>: The only f^n's mble w.r.t trivial σ-field are constant function, which takes only one value.

—x—

Now $\{\phi, \Omega\} \subset$ every σ-field $\Rightarrow$ constant function are mble w.r.t every σ-field

—x—

The earlier theorem is for mble f^n f which takes values over R or $\overline{R}$
The following thm is specifically for non-ve mble function.

Th^m 2016 2(b) Given a non-ve mble f^n f, $\exists$ a non-decreasing sequence of non-ve simple fun^n f_n s.t $f_n \uparrow f$. i.e

$\exists \{f_n\}$, f_n simple $\forall n$ s.t $0 \leq f_n \uparrow f$

Proof Define $f_n(\omega) = \frac{k}{2^n}$ if $\frac{k}{2^n} \leq f(\omega) < \frac{k+1}{2^n}$

$k = 0, 1, 2, \dots n2^n$

$= n$ if $f(\omega) \geq n$

let us observe say $f_1(\omega)$, $f_2(\omega)$

for $n=1$, $k = 0, 1, 2$

$$\therefore f_1(\omega) = \begin{cases} 0 & \text{if } 0 \leq f(\omega) < 1/2 \\ 1/2 & \text{if } 1/2 \leq f(\omega) < 1 \\ 1 & \text{if } f(\omega) \geq 1 \end{cases}$$

for $n=2$, $k = 0, 1, 2, \dots 8$

and

$$f_2(\omega) = \begin{cases} 0 & \text{if } 0 \leq f(\omega) < 1/4 \\ 1/4 & \text{if } 1/4 \leq f(\omega) < 2/4 \\ 2/4 = 1/2 & \text{if } 2/4 \leq f(\omega) < 3/4 \\ 3/4 & \text{if } 3/4 \leq f(\omega) < 4/4 \\ 4/4 & \text{if } 4/4 \leq f(\omega) < 5/4 \\ 5/4 & \text{if } 5/4 \leq f(\omega) < 6/4 \\ 6/4 & \text{if } 6/4 \leq f(\omega) < 7/4 \\ 7/4 & \text{if } 7/4 \leq f(\omega) < 8/4 = 2 \\ 2 & \text{if } f(\omega) \geq 2 \end{cases}$$

Now observe that $f_1(\omega) \leq f_2(\omega)$ $\forall \omega$

further each f_n is a simple f^n $\forall n$

and $|f_n(\omega) - f(\omega)| \leq \frac{1}{2^n}$ when $f(\omega) < n$

and when $f(\omega) \geq n$, $f_n(\omega) = n \to \infty$, as $n \to \infty$
Thus in either case as $n \to \infty$ $f_n(\omega) \uparrow f(\omega)$
Hence the theorem.

— x —

Result: A mble function can also be obtained as a limit of a sequence of elementary function. Further this convergence is uniform if f is bounded

— x —

Basically we are interested in three types of measurable function

1. Non -ve simple function
(Takes non-ve, finitely many distinct values)
2. Non -ve mble function $(f \geq 0)$
3. Any mble function (mble f^n which takes values over $\bar{\mathbb{R}}$)

Let f be any measurable f^n. Then f can be written as

$$\underbrace{f}_{} = \underbrace{f^{+}}_{\text{+ve part of } f} - \underbrace{f^{-}}_{\text{-ve part of } f}$$

where $f^{+} = \max(f, 0) = \begin{cases} f & \text{if } f \geq 0 \\ 0 & \text{if } f < 0 \end{cases}$

$f^{-} = -\min(f, 0) = \begin{cases} 0 & \text{if } f \geq 0 \\ -f & \text{if } f < 0 \end{cases}$

Note that $f^{+} \geq 0$ and $f^{-} \geq 0$
Now since f is mble also '0' being constant function always mble $\Rightarrow f^{+} = \max(f, 0)$ is mble
Simlly $-f$ is also mble $\Rightarrow f^{-} = -\min(f, 0)$ is mble

The converse is non-necessarily true i.e If f^+ and f^- are mble, f may not be mble

— x —

Ex Suppose f_1 and f_2 be mble $f^n(\mathcal{A})$

$\Rightarrow \{g_n\} \to f_1$ g_n simple $\forall n$

and $\{h_n\} \to f_2$, h_n simple $\forall n$

$\Rightarrow \{g_n + h_n\}$ Simple $\forall n$

$\Rightarrow g_n + h_n \longrightarrow f_1 + f_2$

and hence $f_1 + f_2$ is mble $(\mathcal{A})$

$\alpha f_1 + \beta f_2$ is mble $(\mathcal{A})$, $\alpha, \beta \in R$

$f_1 \cdot f_2$ is mble $(\mathcal{A})$

$\max(f_1, f_2)$ mble $(\mathcal{A})$

$\min(f_1, f_2)$ mble $(\mathcal{A})$

All ordered function are mble $(\mathcal{A})$

— x —

Ex $\exists$ a f^n $f: \Omega \to R \ni f^2$ is mble $\mathcal{A}$ but f is not mble.

Let $\Omega = R$ and $\mathcal{A} = \{\phi, \Omega\}$

Define $f^2(\omega) = 1 \quad \forall \omega \in \Omega$

then $f(\omega) = \begin{cases} 1 & \text{if } \omega > 0 \\ -1 & \text{if } \omega \le 0 \end{cases}$

then f is not mble $(\mathcal{A})$

— x —

Ex $\exists$ a f^n $f: \Omega \to R \ni f^2$ is mble $\mathcal{A}$ but f is not mble

Let $\Omega = R$ and $\mathcal{A} = \{\phi, \Omega\}$

Define $f^2(\omega) = 1 \quad \forall \omega \in \Omega$

then $f(\omega) = \begin{cases} 1 & \text{if } \omega > 0 \\ -1 & \text{if } \omega \le 0 \end{cases}$

then f is not mble $(\mathcal{A})$

Ex Define $|X| = \begin{cases} X & \text{if } X \geq 0 \\ -X & \text{if } X < 0 \end{cases}$

i.e. $|X| = X^+ + X^-$.

Remark :- Instead of notation 'x' is used.

Suppose X is mble $\Rightarrow X^+$ & X^- are mble $\Rightarrow |X|$ is mble. Is the converse true? i.e If $|X|$ is mble, then is X mble?

No - not necessarily.

Thus the question of interest is
Is $g(x)$ mble if x is mble?
The answer is in the following theorem.

1) Let f be a mble f^n, $f : \Omega \to R$
Let $g : R \to R$ be a continuous f^n
Then $g(f) : \Omega \to R$ is also a mble function

2) Define a class of function $\mathcal{C}$ as.
$\mathcal{C} = \{ g \mid$ either g is continuous or is a limit of sequence of continuous functions $\}$
A function h is called a baire function if $h \in \mathcal{C}$.

Result: A baire function of a mble f^n is also mble i.e f mble, $h \in \mathcal{C} \Rightarrow h(f)$ is also mble

3) Borel function :-
A function $g : R \to R$ is called a borel fun^n if $g^{-1}(B)$ is a Borel set.

Result :- Borel function of a mble function is mble eg. if f is mble $\Rightarrow f^2$ is mble or say e^f is mble.

Convergence :-

Let $(\Omega, \mathcal{A}, \mu)$ be fixed measure space.
We consider mble function on this space.
We want to study the convergence pattern of f_n to f

We say $f_n(\omega) \to f(\omega)$ $\forall \omega \in \Omega$
(Remember, $f_n(\omega)$ $\forall n$ & $f(\omega)$ are real numbers)
So, by the concept of convergence of sequence of real numbers

we say $f_n(\omega) \to f(\omega)$ if $\forall \varepsilon > 0$, $\exists$ N large enough s.t $|f_n(\omega) - f(\omega)| < \varepsilon$
$\forall n \geq N(\varepsilon, \omega)$

(we assume both f_n and f are finite valued)

The above condition can also be written as

$$\bigcap_{\varepsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n>N}^{\infty} \{\omega \mid |f_n(\omega) - f(\omega)| < \varepsilon\}$$

$\forall \varepsilon > 0$ for some N large $\forall n > N$

and the above set is known as set of convergence

We will discuss two types of convergences
1) Convergence almost everywhere
2) Convergence in measure.

* Convergence almost everywhere :-

A set N is said to hold almost everywhere if $\mu(N') = 0$ i.e N' is a μ - null set.

In general, a concept is said to hold almost everywhere, if the measure of set on which concept doesnot hold is zero

eg :- 1) A function f is said to be a.e. finite valued if $\mu\{\omega \mid |f(\omega)| = \infty\} = 0$

2) Two function f and g are said to be equivalent a.e if $\mu\{\omega \mid f(\omega) \neq g(\omega)\} = 0$

Henceforth we consider $f : \Omega \to R$ which are finite a.e i.e even though the f^n f takes values $+\infty$ or $-\infty$, the μ measure of such sets is 0.

So let $\{f_n\}$ be a seq^n of a.e finite valued mble function

Def^n A seq^n of mble function $\{f_n\}$ converges to a measurable f^n f a.e if

$$\mu\{\omega \mid f_n(\omega) \not\to f(\omega)\} = 0$$

i.e if $\omega \in N$, $f_n \to f$
and if $\omega \in N'$, $f_n \not\to f$ (N' has measure zero)

Criteria for a.e convergence.

Result :- $f_n \to f$ a.e iff

$$\lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} \left[|f_n - f| \geq \varepsilon\right]\right) = 0 \quad \forall\, \varepsilon > 0$$

provided $\mu\left(\bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right) < \infty$ for some k

proof We know that $f_n \to f$ a.e

iff $\mu[\omega \mid f_n(\omega) \not\to f(\omega)] = 0$.

Now consider

$$\{\omega \mid f_n(\omega) \to f(\omega)\} = \bigcap_{\varepsilon>0} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{\omega \mid |f_n(\omega) - f(\omega)| < \varepsilon\}$$

This is the set of convergence (N) then.

$f_n \to f$ a.e if $\mu(N') = 0$

i.e

iff $\mu\left[\bigcup_{\varepsilon>0} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right] = 0$

i.e

iff $\mu\left[\bigcap_{k=1}^{\infty} \underbrace{\bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon}_{B_K}\right] = 0 \quad \forall\, \varepsilon > 0$

then $B_K \downarrow$ and $\lim B_K = \bigcap_{k=1}^{\infty} B_K$

i.e iff $\mu\left[\lim_{k\to\infty} B_K\right] = 0 \quad \forall\, \varepsilon > 0$

i.e iff $\mu\left[\lim_{k\to\infty} \bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right] = 0 \quad \forall\, \varepsilon > 0$

i.e iff $\lim_{k\to\infty} \mu\left[\bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right] = 0 \quad \forall\, \varepsilon > 0$

provided

$\mu\left[\bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right] < \infty$ for some k

This is known as a.e convergence criteria

Def^n $\{f_n\}$ converges mutually a.e iff
$$\mu\{\omega \mid |f_m(\omega) - f_n(\omega)| \not\to 0\} = 0 \text{ as } m, n \to \infty$$

Remark ① Proceeding as above, the criteria for convergence mutually a.e is
$$\mu\left[\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} |f_{n+k} - f_n| \geq \varepsilon\right] = 0$$

② $f_n \to f$ a.e $\Leftarrow\Rightarrow$ $\{f_n\}$ converges mutually a.e

* **Convergence in Measure** (2016 - 2(c)) 2019 - 3(c)

Def^n A sequence of mble function $\{f_n\}$ is said to converge in measure to a mble f^n f if $\forall \varepsilon > 0$
$$\mu[\omega \mid |f_n(\omega) - f(\omega)| > \varepsilon] \to 0 \text{ as } n \to \infty$$

[Note that nothing can be said about compliment because we don't know whether μ is finite or not also what is value of $\mu(\Omega)$.]

Notation $f_n \xrightarrow{\mu} f$

Def^n A sequence of mble function $\{f_n\}$ converges mutually in measure if $\forall \varepsilon > 0$
$$\mu[|f_m - f_n| \geq \varepsilon] \to 0 \text{ as } m, n \to \infty$$

Suppose $f_n \xrightarrow{\mu} f$, Now consider.

$$\mu[|f_m - f_n| \geq \varepsilon] = \mu[|f_m - f + f - f_n| \geq \varepsilon]$$
$$\leq \mu[|f_m - f| + |f_n - f| \geq \varepsilon]$$
$$\leq \mu[|f_m - f| \geq \tfrac{\varepsilon}{2} \text{ or } |f_n - f| \geq \varepsilon/2]$$
$$\leq \mu[|f_m - f| \geq \tfrac{\varepsilon}{2}] + \mu[|f_n - f| \geq \tfrac{\varepsilon}{2}]$$
$$\to 0 \text{ as } m, n \to \infty$$

$\Rightarrow$ $\{f_n\}$ converges mutually in measure. Thus if $f_n \xrightarrow{\mu} f$ $\Rightarrow$ $\{f_n\}$ converges mutually in measure.

Conversely,
if $\{f_n\}$ converges mutually in measure,
$\Rightarrow f_n \xrightarrow{\mu} f$ to some mble f^n f

— x —

To study relation between two type of convergence

Result 1 : a.e convergence $\Rightarrow$ convergence in measure provided the measure is finite.

proof: Suppose $\{f_n\}$ is a seq^n of mble function such that $f_n \xrightarrow{a.e} f$ and let μ be a finite measure

$\Rightarrow$ Criteria of a.s convergence holds.

$$\Rightarrow \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right) = 0 \quad \forall \varepsilon > 0$$

and $\mu\left(\bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right) < \infty$ $\because$ μ is finite

Now

$$\lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} |f_n - f| \geq \varepsilon\right) = 0$$

$\Rightarrow \mu(|f_n - f| \geq \varepsilon) \to 0$ as $n \to \infty$

$\Rightarrow f_n \xrightarrow{\mu} f$

—x—

If the measure is not finite, a.e convergence may or may not imply convergence in measure.

Ex.1 let $\Omega = [0, \infty)$

Define $f_n = \begin{cases} 1 & \text{if } \omega \in (0, 1/n) \\ 0 & \text{o.w.} \end{cases}$

Fix a $\omega \in \Omega$ i.e $\omega \geq 0$

$\forall\ \omega \in \Omega$, $f_n(\omega) = 0$ for n large

i.e $f_n \to 0$ a.e $\therefore \mu[\omega | f_n \not\to 0] = 0$

let $\varepsilon > 0$

Consider $\mu[|f_n - 0| \geq \varepsilon] = \mu[f_n = 1]$

$= \mu[0, 1/n)]$

let μ be the lebesgue measure λ & we know that this measure is not finite

$= \lambda[(0, 1/n)]$

$= \frac{1}{n} \to 0$ as $n \to \infty$

$\Rightarrow f_n \xrightarrow{\mu} 0$ Thus A.S. convergence

$\Rightarrow$ Convergence in measure and the measure is not finite.

Ex2 let $\Omega = [0, \infty)$

$f_n = \begin{cases} 1 & \text{if } \omega \in [n, n+1] \\ 0 & \text{o.w.} \end{cases}$

Fix a $\omega \in \Omega$

then $\forall\ \omega \in \Omega$, $f_n(\omega)=0$ for n large
Thus $f_n(\omega) \to 0$, Thus $f_n \to 0$ a.e

Now let $\varepsilon > 0$ consider
$\mu[|f_n - 0| \geq \varepsilon] = \mu[f_n = 1]$
$= \mu[[n, n+1]]$

let μ be the lebesgue measure λ then
$= \lambda[n, n+1]$
$= 1 \not\to 0$

Thus $X_n \overset{\mu}{\not\to} 0$
Thus a.s convergence $\not\Rightarrow$ convergence in measure where μ is not finite

Convergence in measure may not imply convergence a.e

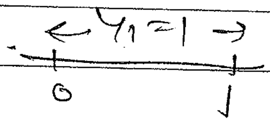
Ex let $\Omega = [0, \infty)$ Define
$$X_{nk} = \begin{cases} 1 & \text{if } \omega \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \\ 0 & \text{o.w} \end{cases}$$
let
$Y_1 = X_{11}$, $Y_2 = X_{21}$, $Y_3 = X_{22}$, $Y_4 = X_{31}$,
$Y_5 = X_{32}$, $Y_6 = X_{33}$, and so on
Thus
$\{Y_m\}$ is a countable seq^n of r.v.s
what happens to $\{Y_m\}$ as $m \to \infty$

[given an m, $\exists$ an n & k s.t $Y_m = X_{nk}$]
i.e for each $m \geq 1$, $\exists\ n_0, k_0$ s.t
$X_{n_0 k_0} = Y_m$ and hence

$$Y_m = \begin{cases} 1 & \text{if } \omega \in \left[\frac{k_0-1}{n_0}, \frac{k_0}{n_0}\right] \\ 0 & \text{o.w} \end{cases}$$

let $n=1$

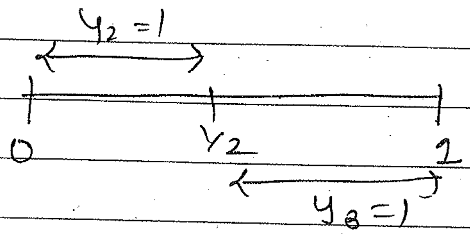
$$X_{11} = Y_1 = \begin{cases} 1 & \text{if } \omega \in [0,1] \\ 0 & \text{o.w} \end{cases}$$



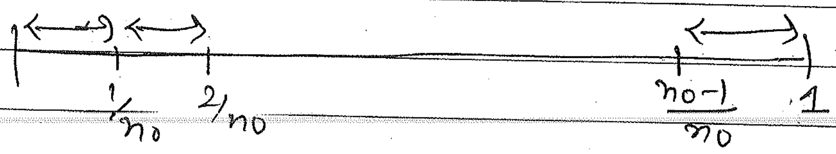
$n=2$

$X_{21} = Y_2$

$X_{22} = Y_3$



In general $n = n_0$



let $\omega \geq 0$ be fixed. $\{\Omega = [0, \infty)\}$

If $\omega > 1$, all $Y_m = 0$

Suppose $0 \leq \omega \leq 1$, then clearly for each n there is only one X_{nk} which takes value 1 at ω and all other X_{nj}'s are zero

Thus $\{Y_m(\omega)\}$ has a value 1 at several places and zero at a lot of other places.

Hence $X_{nk} \not\to$ to any no. at any $\omega \in [0,1]$

Thus X_{nk} converges nowhere.

Hence X_{nk} doesnot converge a.e

but $\{X_{nk}\}$ converges in measure

let μ be the lebsegue measure

$$\mu\left[|X_{nk} - 0| \geq \varepsilon\right] = \lambda\left[|X_{nk} = 1\right]$$

$$= \lambda\left[\frac{k-1}{n}, \frac{k}{n}\right]$$

$$= \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow X_{nk} \xrightarrow{\mu} 0$ but $X_n \overset{a.e}{\not\to} 0$

So under what condition convergence in measure implies convergence a.e

Thm let $\{f_n\}$ be a seq^n of finite valued mble fu^n which converges mutually in measure. Then $\exists$ a subsequence $\{n_k\}$ such that $\{f_{n_k}\}$ converges a.e to some finite valued function f.

Also $f_n \xrightarrow{\mu} F$

UNIT-3 INTEGRATION

2016 3(a) 2018 4(b)

Integration of a mble function w.r.t a measure μ

~~let~~

Let $f: \Omega \to \overline{R}$ be a measurable function

To define $\int_{\Omega} f\,d\mu$ where μ is a measure defined on $(\Omega, \mathcal{A})$

So let $(\Omega, \mathcal{A}, \mu)$ be a measure space

let $f: \Omega \to \overline{R}$.

We will define $\int_{\Omega} f\,d\mu$ when

(i) f is non-ve simple function

(ii) f is non-ve mble function

(iii) f is any mble function

(i) let f be a non-ve simple function

i.e $f = \sum_{j=1}^{n} a_j I_{A_j}$, $a_j \geq 0$ and distinct $A_i \cap A_j = \phi$ $\forall i \neq j$ &

$\bigcup_{j=1}^{n} A_j = \Omega$; $A_j \in \mathcal{A}$ $\forall j$

Defn Integral of a ~~ve~~ non-ve simple function

$f = \sum_{j=1}^{n} a_j I_{A_j}$ is defined as

$$\int_{\Omega} f\,d\mu = \sum_{j=1}^{n} a_j \mu(A_j)$$

(ii) let f be non-ve mble function we know that if $f \geq 0$, mble, then $\exists \{f_n\}$ such that $f_n \geq 0$, f_n simple and $0 \leq f_n \uparrow f$

then

Def^n $$\int_\Omega f\,du = \lim_{n\to\infty} \int f_n\,du$$

let f be any mble function then $\int f\,du$ is said to be exist, iff $\int f^+ du < \infty$ or $\int f^- du < \infty$ and whenever it exists

$$\int_\Omega f\,du = \int_\Omega f^+ du - \int_\Omega f^- du$$

[check for existance is necessary first]

[Note:- $\int_\Omega f\,du$ does not exist if $\int_\Omega f^\pm du = \infty$

eg let $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$

and
$$E(x) = \int x\, f(x)\,dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$$
$$= \frac{1}{\pi}\left[\int_{\infty}^{0} + \int_0^\infty\right]$$

Thus $\int x\, f(x)\,dx$ does not exist]

Def^n :- A mble function f is said to be integrable if both $\int f^+ du$ and $\int f^- du$ are finite and then $\int f\,du < \infty$

Remark:- It is necessary to show that the definition of integrals are unambiguous.

Properties of integrals:-

If $A \in \mathcal{A}$ and $A \subset \Omega$ then $\int_A f\,du = \int_\Omega f I_A\,du$

Let $\int f\,du$, $\int g\,du$, and $\int f\,du + \int g\,du$ exists

(i.e either both finite or both $+\infty$ or $-\infty$ or one of them finite)

☆ Then, we have the following properties of integral

(A) Linearity property:-

(i) $\int_\Omega (f+g)\,du = \int_\Omega f\,du + \int_\Omega g\,du$

(ii) $\int_{A \cup B} f\,du = \int_A f\,du + \int_B f\,du$, $A, B \in \mathcal{A}$, $A \cap B = \phi$

(iii) For a constant c, $\int cf\,du = c\int f\,du$

(B) Order preserving property

(i) $f \geq 0 \Rightarrow \int f\,du \geq 0$
(ii) $f \geq g \Rightarrow \int f\,du \geq \int g\,du$
(iii) $f = g$ a.e. $\Rightarrow \int f\,du = \int g\,du$

Step I Establish Def 1 is full proof

Step II Establish properties A & B for non-ve simple function

Step III Establish Def 2 is full proof

IV Establish properties A & B for non-ve mble function

V Establish Def 3 is full proof

VI Establish properties A & B for any mble function

—x—

Step I: let f be a non-ve simple function
Suppose we write

$$f = \sum_{i=1}^{m} a_i I_{A_i} \quad \text{and} \quad f = \sum_{j=1}^{n} b_j I_{B_j}$$

then

$$\int f \, d\mu = \sum_{i=1}^{m} a_i \mu(A_i) \text{ and also}$$

$$\int f \, d\mu = \sum_{j=1}^{n} b_j \mu(B_j)$$

To prove

$$\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{j=1}^{n} b_j \mu(B_j)$$

Proof Note $w \in A_i \cap B_j \Rightarrow a_i = b_j$

Consider
$$\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{i=1}^{m} a_i \mu(A_i \cap \Omega)$$

$$= \sum_{i=1}^{m} a_i \mu\left(A_i \cap \left(\bigcup_{j=1}^{n} B_j\right)\right)$$

$$= \sum_{i=1}^{m} a_i \mu\left[\bigcup_{j=1}^{n} (A_i \cap B_j)\right]$$

$$= \sum_{i=1}^{m} a_i \sum_{j=1}^{n} \mu(A_i \cap B_j) \quad \left(\because A_i \cap B_j \subset A_i \text{ are disjoint set}\right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \mu(A_i \cap B_j) \quad \text{—(1)}$$

Similarly,

$$\sum_{j=1}^{n} b_j \mu(B_j) = \sum_{j=1}^{n} \sum_{i=1}^{m} b_j \mu(A_i \cap B_j) \quad \text{— (2)}$$

but $a_i = b_j$ on $A_i \cap B_j$

$$\Rightarrow \int f \, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \mu(A_i \cap B_j) = \sum_{j=1}^{n} \sum_{i=1}^{n} b_j \mu(A_i \cap B_j)$$

$$\text{i.e } \sum_{i=1}^{m} a_i \mu(A_i) = \sum_{j=1}^{n} b_j \mu(B_j)$$

Thus the definition of integral of a non-ve simple function is full proof.

—x—

Step II : A (i)

To prove $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

let $f = \sum_{i=1}^{m} a_i I_{A_i}$ & let $g = \sum_{j=1}^{n} b_j I_{B_j}$

then

$$f + g = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i + b_j) I_{A_i \cap B_j}$$

$$\therefore \int (f+g) \, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i + b_j) \mu(A_i \cap B_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i \mu(A_i \cap B_j)) + \sum_{i=1}^{m} \sum_{j=1}^{n} b_j \mu(A_i \cap B_j)$$

$$= \sum_{i=1}^{m} a_i \mu(A_i \cap \bigcup_{j=1}^{n} B_j) + \sum_{j=1}^{n} b_j \mu\left(\bigcup_{i=1}^{m} A_i \cap B_j\right)$$

$$= \sum_{i=1}^{m} a_i \mu(A_i) + \sum_{j=1}^{n} b_j \mu(B_j)$$

$$= \int f \, d\mu + \int g \, d\mu$$

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A (ii) To prove $\int_{A \cup B} f\, du = \int_A f\, du + \int_B f\, du$

where $A \cap B = \phi$

proof $\int_{A\cup B} f\, du = \int f\, I_{A\cup B}\, du$

but $I_{A\cup B} = I_A + I_B$

$\because A \cap B = \phi$

$$\therefore \int_{A\cup B} f\, du = \int f(I_A + I_B)\, du$$

$$= \int f I_A\, du + \int f I_B\, du$$

$$= \int_A f\, du + \int_B f\, du$$

A (iii) let C be a constant

To prove

$$\int cf\, du = c \int f\, du$$

If $C = 0$ then $Cf = 0$ & hence both sides are zero

If $C > 0$ then $f = \sum_{i=1}^{m} a_i I_{A_i}$

$(0 < c < \infty)$ $\therefore cf = \sum_{i=1}^{m} (ca_i) I_{A_i}$

then $\int cf\, du = \sum_{i=1}^{m} (ca_i)\, \mu(A_i)$

$$= C \sum_{i=1}^{m} a_i\, \mu(A_i)$$

$$= C \int f\, du$$

(B) (i) To prove if $f \geq 0 \Rightarrow \int f d\mu \geq 0$

$F \geq 0 \Rightarrow$ all a_i's ≥ 0 also μ is non-ve

$\Rightarrow \int f d\mu = \sum_{i=1}^{m} a_i \mu_i (A_i) \geq 0$

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(ii) $f \geq g \Rightarrow \int f d\mu \geq \int g d\mu$

$f \geq g \Rightarrow f - g \geq 0$

Now

$$f = f - g + g$$

$$\therefore \int f d\mu = \int (f-g) d\mu + \int g d\mu$$

$$\geq \int g d\mu$$

(iii) If $f = g$ a.e then $\int f d\mu = \int g d\mu$

let $A = \{f = g\}$ then $f = g$ a.e $\Rightarrow \mu(A') = 0$

Now consider

$$\int (f-g) d\mu = \int_A (f-g) d\mu + \int_{A'} (f-g) d\mu$$

$$= 0 + \int (f-g) I_{A'} d\mu$$

$$= 0 + 0$$

$\because A_i \cap B_j \cap A' \subset A'$

and $\mu(A_i \cap B_j \cap A') = 0$

$\Rightarrow \int (f-g) d\mu = 0$

$$\therefore \int f \, d\mu = \int (f-g+g) \, d\mu$$
$$= \int (f-g) \, d\mu + \int g \, d\mu$$
$$= \int g \, d\mu$$ Hence proved.

— x —

Step III : TP Def (2) is full proof

We accept this result without proof

[Result: let $\{X_n\}$ be a non-decreasing sequence of non-ve simple function s.t $X_n \to X$ let $X \geq Y$ where Y is non-ve simple f^n then $\lim \int X_n \, d\mu \geq \int Y \, d\mu$]

Assuming the above result holds, we can prove Def (2) is full proof or unambiguous

Def 2: $0 \leq f_n \uparrow f$, f_n simple $\forall n$ then

$$\int f \, d\mu = \lim_n \int f_n \, d\mu$$

To prove this def^n is full proof, if possible Suppose $0 \leq f_n \uparrow f$, f_n simple
$0 \leq g_n \uparrow f$, g_n simple

To prove

$$\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu$$

Note that $\forall$ fixed $k \geq 1$

$f \geq g_k$, g_k simple

$\therefore$ by above result

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int g_k \, d\mu \quad \forall \, k \geq 1$$

Let $k \to \infty$ then

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \lim_{k\to\infty} \int g_k \, d\mu \quad -(1)$$

On similar lines, we can prove that

$$\lim_{n\to\infty} \int g_n \, d\mu \geq \lim_{n\to\infty} \int f_n \, d\mu \quad -(2)$$

From (1) and (2), we have

$$\lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu$$

Thus $\int f \, d\mu$ is unambiguously defined if $f \geq 0$

Step IV : To prove properties of integral of non-ne mble f^n :-

A (i) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

where f and $g \geq 0$, mble f^n

Recall $0 \leq f_n \uparrow f$, f_n simple $0 \leq g_n \uparrow g$, g_n simple
then $0 \leq f_n + g_n \uparrow f+g$, $f_n + g_n$ simple

$$\therefore \int (f+g)\, d\mu = \lim_{n\to\infty} \int (f_n + g_n)\, d\mu$$

$$= \lim_{n\to\infty} \left[\int f_n\, d\mu + \int g_n\, d\mu \right]$$

$$= \lim_{n\to\infty} \int f_n\, d\mu + \lim_{n\to\infty} \int g_n\, d\mu$$

$$= \int f\, d\mu + \int g\, d\mu$$

Hence proved.

A (ii) $\int_{A\cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu$, $A\cap B = \phi$

In fact, all properties A(ii), A(iii) & B(i), B(ii) B(iii) hold for non -ve mble f^n
(can be proved on similar lines as A(i))

Def 8: Integral of a mble f^n

Step V: Let f be a mble function
$\int f\, d\mu$ is said to exist if $\int f^+ d\mu < \infty$ or

$\int f^- d\mu < \infty$. Then $\int f\, d\mu = \int f^+ d\mu - \int f^- d\mu$

This definition is unambiguous by itself, because $\int f^+ d\mu$ and $\int f^- d\mu$ are unambiguously defined.

Properties :- Let f and g be any mble function such that $\int f d\mu$, $\int g\, d\mu$ & $\int f d\mu + \int g\, d\mu$ exists

$\int f\, d\mu$ can be $\pm\infty$, $\int g\, d\mu$ can be $\pm\infty$
but $\int f\, d\mu + \int g\, d\mu$ exists

$\Rightarrow$ $\int f\,d\mu$ and $\int g\,d\mu$ both can be $+\infty$
or both can be $-\infty$
but not one $+\infty$ & the other $-\infty$

Then A(i) $\int_\Omega (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$

proof Define
$A_1 = \{\omega \mid f \geq 0,\ g \geq 0\}$

$A_2 = \{\omega \mid f \geq 0,\ g < 0,\ f+g \geq 0\}$
$A_3 = \{\omega \mid f \geq 0,\ g < 0,\ f+g < 0\}$
$A_4 = \{\omega \mid f < 0,\ g < 0,\ f+g < 0\}$
$A_5 = \{\omega \mid f < 0,\ g \geq 0,\ f+g \geq 0\}$
$A_6 = \{\omega \mid f < 0,\ g \geq 0,\ f+g < 0\}$

then $A_i \cap A_j = \phi \ \forall i \neq j$ and $\bigcup_i A_i = \Omega$
we shall prove that

$$\int_{A_k} (f+g)\,d\mu = \int_{A_k} f\,d\mu + \int_{A_k} g\,d\mu, \quad k = 1, 2, \ldots 6$$

For set A_1 & A_4 i.e for $k=1$ & 4, the result is Obvious.
Suppose $k=2$
write $f+g = f-(-g)$
i.e $f = f+g+(-g)$ on A_2

Here f, $f+g$ & $-g$ all are non-ve mble $f^{\underline{n}}$ &
Hence
$$\int_{A_2} f\,d\mu = \int_{A_2} (f+g)\,d\mu + \int_{A_2} -g\,d\mu$$

$$= \int_{A_2} (f+g)\,d\mu - \int_{A_2} g\,d\mu$$

i.e $\int_{A_2} (f+g)\, du = \int_{A_2} f\, du + \int_{A_2} g\, du$

On similar lines, we can prove that

$$\int_{A_k} (f+g)\, du = \int_{A_k} f\, du + \int_{A_k} g\, du \quad \text{for } k = 3, 5, 6$$

$\therefore$ adding all such results, we have (and rearranging the term)

$$\sum_{k=1}^{6} \int_{A_k} (f+g)\, du = \sum_{k=1}^{6} \int_{A_k} f\, du + \sum_{k=1}^{6} g\, du$$

$$\Rightarrow \int_{\Omega} (f+g)\, du = \int_{\Omega} f\, du + \int_{\Omega} g\, du$$

On similar lines, we can prove the remaining properties.

—x—

* Integrability property :-

A mble f^n f is said to be integrable if

$$\int f^+ du < \infty \text{ and } \int f^- du < \infty$$

In that case $\int f\, du = \int f^+ du - \int f^- du < \infty$

Now $|f| = f^+ + f^-$

$$\therefore \int |f|\, du = \int f^{-1} du + \int f^- du < \infty$$

Note that in general $\int |f|\, d\mu$ always exists $\because$ it is either $< \infty$ or $= +\infty$

Thus

(i) f is integrable iff $\int |f|\, d\mu < \infty$

(ii) If $|f| \leq g$, g integrable then f is integrable

proof: $|f| \leq g \Rightarrow \int |f|\, d\mu \leq \int g\, d\mu < \infty$
$\Rightarrow \int |f|\, d\mu < \infty$
$\Rightarrow f$ is integrable

(iii) $f = g$ a.e $\Rightarrow \int f\, d\mu = \int g\, d\mu$

let

$N = \{\omega \mid f = g\}$ then $N' = \{\omega \mid f \neq g\}$ &
$\mu(N') = 0$

Now $\int_{\Omega} f\, d\mu = \int_{N} f\, d\mu + \int_{N'} f\, d\mu = \int_{N} f\, d\mu$

Similarly $\int_{\Omega} g\, d\mu = \int_{N} g\, d\mu + \int_{N'} g\, d\mu = \int_{N} g\, d\mu$

$\because \mu(N') = 0$

but on N, $f = g \Rightarrow \int_{N} f\, d\mu = \int_{N} g\, d\mu$

$\Rightarrow \int_{\Omega} f\, d\mu = \int_{\Omega} g\, d\mu$

<u>Remark</u>: The 2^{nd} definition uses sequences of non-ve simple function $\{f_n\}$ s.t $f_n \uparrow f$, then $\int f d\mu \to \int f d\mu$ where f is non-ve mble f^n.

What will happen if $\{f_n\}$ is a seq^n of non-ve mble function?

The answer is in the following famous thm Convergence thm.

Monotone Convergence Theorem (MCT) :-
(2017) 3(c) 2018 - 4(c) (5M)

Statement :- Suppose $0 \le f_n \uparrow f$, where f_n is a mble f^n $\forall n$ Then $0 \le \int f_n \, d\mu \uparrow \int f \, d\mu$

Proof Since $f_n \ge 0 \Rightarrow \int f_n \, d\mu \ge 0$ &

$$\int f_n \, d\mu \le \int f_{n+1} \, d\mu$$

Now for some k, $f_k \ge 0 \Rightarrow \exists$ a sequence $\{f_{kn}\}$ of non -ve simple function s.t $0 \le f_{kn} \uparrow f_k$ as $n \to \infty$

Thus

$$\begin{array}{llll}
f_{11} & f_{12} \cdots & f_{1n} \cdots & \longrightarrow f_1 \\
f_{21} & f_{22} \cdots & f_{2n} \cdots & \longrightarrow f_2 \\
\vdots & \vdots & \vdots & \\
f_{k1} & f_{k2} \cdots & f_{kn} \cdots & \longrightarrow f_k \\
\vdots & \vdots & & \\
f_{n1} & f_{n2} \cdots & f_{nn} \cdots & \longrightarrow f_n
\end{array}$$

Then $0 \le f_{kn} \le f_k \le f_n \quad \forall k \le n$

let $X_n = \max_{k \le n} f_{kn}$, which is a simple f^n $\forall n$

then $0 \le f_{kn} \le X_n \le f_n \quad \forall k \le n$ — ①

and

$$0 \le \int f_{kn} \, d\mu \le \int X_n \, d\mu \le \int f_n \, d\mu \quad \text{— ②}$$

let $n \to \infty$, then in ① & ②

$$0 \le f_k \le \lim_{n \to \infty} X_n \le f \quad \forall k \quad \text{— ③}$$

$\forall k$

and
$$0 \leq \int f_k \, d\mu \leq \lim_{n\to\infty} \int x_n \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu \quad \text{—— (4)}$$

Now let $k \to \infty$ in (3) & (4), we have

$$0 \leq f \leq \lim_{n\to\infty} x_n \leq f \quad \text{and} \quad \text{—— (5)}$$

and
$$0 \leq \lim_{k\to\infty} \int f_k \, d\mu \leq \lim_{n\to\infty} \int x_n \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu \quad \text{—— (6)}$$

From (5), we have $\lim x_n = f$ and from (6) we have
$$\lim_{n\to\infty} \int x_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

but
x_n's are simple f^n s.t $\lim_{n\to\infty} x_n = f$.

$$\therefore \lim_{n\to\infty} \int x_n \, d\mu = \int \lim_{n\to\infty} x_n \, d\mu = \int f \, d\mu$$

$$\Rightarrow \lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Hence the proof

✩ Application of MCT

Let $f_n \geq 0$ be mble $f^{n}s$ for every n then
$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \left(\sum_{n=1}^{\infty} f_n \right) d\mu$$

i.e indefinite integral is σ- additive

Proof let $g_k = \sum_{1}^{k} f_n$, then $0 \le g_k \uparrow \sum_{n=1}^{\infty} f_n$.

Each g_k is mble and so is $\sum_{n=1}^{\infty} f_n$

($\because$ limit of sequence of mble function is mble)

Then by MCT,

$$\lim_{k\to\infty} \int g_k \, du = \int \left(\lim_{k\to\infty} g_k\right) du$$

i-e

$$\lim_{k\to\infty} \int \left[\sum_{1}^{k} f_n\right] du = \int \left(\sum_{n=1}^{\infty} f_n\right) du$$

$$\Rightarrow \lim_{k\to\infty} \sum_{1}^{k} \int f_n \, du = \int \left(\sum_{n=1}^{\infty} f_n\right) du$$

$$\Rightarrow \sum_{n=1}^{\infty} \int f_n \, du = \int \left(\sum_{n=1}^{\infty} f_n\right) du$$

i-e countable sum can be taken inside integral.

— x —

Example (i) Compute $\lim_{n\to\infty} \int_{1}^{2} \frac{n}{1+nx^2} \, dx$.

Ans First we note that $f_n = \frac{n}{1+nx^2}$ is such that

if $n < m$ then $f(n) \le f(m)$ i-e each f_n is a mble f^n s.t $0 \le f_n \uparrow$

[eg. let $x = 1.5$,

n	$f_n = \frac{n}{1+n(1.5)^2}$
1	0.3076
2	0.3636
3	0.3570 & so on

]

$\therefore$ By MCT

$$\lim_{n\to\infty} \int_1^2 \frac{n}{1+nx^2}\,dx = \int_1^2 \lim_{n\to\infty} \frac{n}{1+nx^2}\,dx$$

$$= \int_1^2 \lim_{n\to\infty} \frac{1}{\frac{1}{n}+x^2}\,dx$$

$$= \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = 1-\frac{1}{2} = \frac{1}{2}$$

* **Fatou's lemma:-** 2016 - 3 (c) (with proof)
2017 - 3(c)

(a) Suppose $f_n \geq g$, g mble and f_n $\forall n$ are mble & g is integrable. Let $\int f_n d\mu$ exists $\forall n$, then

$$\int \liminf f_n\, d\mu \leq \liminf \int f_n\, d\mu$$

(b) Suppose $f_n \leq h$, f_n $\forall n$ & h are mble, h is integrable then

$$\int (\limsup f_n)\, d\mu \geq \limsup \int f_n\, d\mu$$

(c) Suppose $g \leq f_n \leq h$ & $\lim f_n = f$, where h and g are integrable then

$$\lim \int f_n\, d\mu = \int f\, d\mu$$

(Remark:- In MCT, we have an $\uparrow$ seq^n of mble function f_n. But what is the seq^n is not $\uparrow$. Then Fatou's lemma is applicable.)

(9) Assume first that $g \geq 0$

Recall $\liminf f_n = \lim_{k \to \infty} \inf_{n \geq k} f_n$

let $g_k = \inf_{n \geq k} f_n$

$\therefore \liminf f_n = \lim_{k \to \infty} g_k$, where $0 \leq g_k \uparrow$

Hence by MCT,

$$\lim_{k \to \infty} \int g_k \, du = \int \lim_{k \to \infty} g_k \, du$$

$$= \int \liminf f_n \, du$$

i.e $\int \liminf f_n \, du = \lim_{k \to \infty} \int g_k \, du$

$$= \lim_{k \to \infty} \int \left(\inf_{n \geq k} f_n \right) du \quad \text{—} (1)$$

but $\int \left(\inf_{n \geq k} f_n \right) du \leq \int f_k \, du \quad \forall k$

$\therefore \liminf \int \left(\inf_{n \geq k} f_n \right) du \leq \liminf \int f_k \, du \quad \text{—} (2)$

but from (1)

$\lim_{k \to \infty} \int \left(\inf_{n \geq k} f_n \right) du$ exists

$\Rightarrow \liminf \int \left(\inf_{n \geq k} f_n \right) du = \lim_{k \to \infty} \int \left(\inf_{n \geq k} f_n \right) du$

$$= \int \liminf f_n \, du \quad \text{—} (3)$$

using (3) in (2), we have

$\int (\lim\inf f_n)\, d\mu \leq \lim\inf \int f_n\, d\mu$ if $g \geq 0$

Suppose $g < 0$, then $-g > 0$ &
define $f_n^* = f_n - g \geq 0$

Hence applying previous argument to f_n^*, we have

$$\int \lim\inf f_n^*\, d\mu \leq \lim\inf \int f_n^*\, d\mu$$

i.e $\int \lim\inf (f_n - g)\, d\mu \leq \lim\inf \int (f_n - g)\, d\mu$

i.e $\int [\lim\inf (f_n - g)\, d\mu \leq \lim\inf [\int f_n\, d\mu - \int g\, d\mu]$

i.e $\int \lim\inf f_n\, d\mu - \int g\, d\mu \leq \lim\inf \int f_n\, d\mu - \int g\, d\mu$

and $\because g$ is integrable $\Rightarrow \int g\, d\mu < \infty$

$\Rightarrow \int \lim\inf f_n\, d\mu \leq \lim\inf \int f_n\, d\mu$

Thus (a) is proved

—x—

(b) Proof using part (a)

To prove $\int \lim\sup f_n\, d\mu \geq \lim\sup \int f_n\, d\mu$

if $f_n \leq h$, h integrable

Note that $-f_n \geq -h$ and $-h$ is integrable

By part (a)

$$\int \lim\inf (-f_n)\, d\mu \leq \lim\inf \int (-f_n)\, d\mu$$

i.e $\int -\lim\sup f_n \, d\mu \leq -\lim\sup \int f_n \, d\mu$

i.e $\int \lim\sup f_n \, d\mu \geq \lim\sup \int f_n \, d\mu$

Thus. (b) is proved

—x—

(c) Combining (a) and (b) we have

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu$$
$$\leq \lim\sup \int f_n \, d\mu$$
$$\leq \int \lim\sup f_n \, d\mu$$

But $\lim f_n = f$ exists

$\Rightarrow \liminf f_n = \lim\sup f_n = \lim f_n = f$

$$\Rightarrow \int \liminf f_n \, d\mu = \int \lim\sup f_n \, d\mu$$
$$= \int \lim f_n \, d\mu$$
$$= \int f \, d\mu$$

$\Rightarrow$ equality holds everywhere

$$\Rightarrow \liminf \int f_n \, d\mu = \lim\sup \int f_n \, d\mu$$
$$= \lim \int f_n \, d\mu$$
$$= \int \lim f_n \, d\mu = \int f \, d\mu$$

Thus $\lim \int f_n \, d\mu = \int \lim f_n \, d\mu$

—x—

<u>Note</u>:- 1) The result in (c) holds everywhere $f_n \to f$ a.e [i.e $\mu[f_n \not\to f] = 0$]

Note 2) If no convergence is specified, we consider it as pointwise convergence.

—x—

2017-4(a) – 2016 – 3(c) – only statement & Application (5M)
2019-4(b)
2018-4(d)

Lebesgue's Dominated Convergence Theorem:–

Statement:– let $\{f_n\}$ be a sequence of mble function such that $|f_n| \leq g$ a.e where g is integrable. let $f_n \to f$ a.e or in measure, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu \quad \text{— (1)}$$

Remark:

In fact $\int f_n \, d\mu \to \int f \, d\mu$

$\Rightarrow \left| \int f_n \, d\mu - \int f \, d\mu \right| \to 0$

but

$$\left| \int f_n \, d\mu - \int f \, d\mu \right| = \left| \int (f_n - f) \, d\mu \right| \leq \int |f_n - f| \, d\mu$$

Hence if

$$\int |f_n - f| \, d\mu \to 0 \quad \text{— (2)}$$

$\Rightarrow \left| \int f_n \, d\mu - \int f \, d\mu \right| \to 0$

i.e $\int f_n \, d\mu \to \int f \, d\mu$

i.e $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$ which is (1)

Thus (2) $\Rightarrow$ (1)

Hence it is enough to prove (2) holds

proof Write $|f_n - f| = g_n$

Since $|f_n| \le g \Rightarrow |g_n| \le 2g$

and

$g_n \to 0$ a.e or in measure

write $|f_n - f| = g_n$

Since $|f_n| \le g \Rightarrow |g_n| \le 2g$

and $g_n \to 0$ a.e or in measure

First consider the case of convergence a.e

If $f_n \to f$ a.e i.e $g_n \to 0$, a.e,
then by part (c) of fatou's lemma.

$$\int g_n \, du \to 0$$

i.e $\int |f_n - f| \, du \to 0$ i.e (2) holds

Thus we need to consider only the case $g_n \to 0$ in measure.

To prove $\int g_n \, du \to 0$ as $n \to \infty$

Now $g_n \ge 0 \Rightarrow \int g_n \, du \ge 0$

It is sufficient to prove that

$\limsup \int g_n \, du = 0$ — (3)

[because then $\liminf \int g_n \, du = 0$ and
Hence $\lim \int g_n \, du = 0$]

Let $\limsup \int g_n du = C > 0$ [if possible]
let n' be a sequence of ~~the~~ integers.

s.t

$$C_{n'} = \int g_{n'} \, d\mu \to c \quad \text{as } n \to \infty$$

Recall the following result

[If $h_n \xrightarrow{\mu} h$, then $\exists\, n_k$ s.t $h_{n_k} \to h$ a.e]

here $g_{n'} \to 0$ in measure $g_{n'} \xrightarrow{\mu} 0$

$\Rightarrow \exists$ a further subsequence

$\{n''\} \subset \{n'\}$ s.t

$g_{n''} \to 0$ a.e

$\therefore$ By Fatou's lemma,

$$C'' = \int g_{n''} \, d\mu \to \int 0 \, d\mu = 0 \quad \text{— } (*)$$

but

$\{C_{n''}\} \subset \{C_{n'}\}$ which converges to c

$$\Rightarrow C_{n''} \to c \quad \text{— } (**)$$

$(*)$ & $(**)$ together implies

$$c \equiv 0$$

$$\Rightarrow \limsup \int g_n \, d\mu = 0$$

$$\Rightarrow \lim \int g_n \, d\mu = 0$$

$$\Rightarrow \lim \int |f_n - f| \, d\mu = 0$$

(2) holds

Extension (Application) :-

$f_n : \Omega \to R$, If f is continuous.

$$f(x, \omega) \to f(x_0, \omega) \text{ as } x \to x_0$$

$f_n(\omega) \to f(\omega)$ is equivalent to

$f(x, \omega) \to f(x_0, \omega)$ as $x \to x_0$

(i.e $n \to \infty$ is replaced by $x \to x_0$ along some arbitrary set)

Thus the above result hold with f_n replaced by $f(x)$ and $n \to \infty$ replaced by $x \to x_0$

New form of DCT :-

(i) If $|f(x)| \le g$, g integrable and

$$f(x) \to f(x_0) \text{ as } x \to x_0$$

then $\int f(x) \to \int f(x_0)$

Application 1 :-

(ii) Suppose for ($x \in$ some arbitrary set in which $x \to x_0$),

$\frac{d}{dx} f(x)$ exists at x_0 and

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \le y, \quad y \text{ integrable}$$

then $$\left(\frac{d}{dx} \int f(x) \right)_{x_0} = \int \left(\frac{d}{dx} f(x) \right)_{x_0}$$

Proof

LHS $\left|\frac{d}{dx}\int f(x)\right|_{x_0} = \lim_{x \to x_0} \frac{\int F(x) - \int f(x_0)}{x - x_0}$

$= \lim_{x \to x_0} \int \left(\frac{f(x) - f(x_0)}{x - x_0} \right) du$

by LDCT

$= \int \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] du$

$= \int \left[\frac{d}{dx} f(x_0) \right]_{x = x_0} du$

$=$ RHS

In the above the result is written for at single point x_0, It is extended in the following way.

(iii) Let $F(x, \omega) : \Omega \to \mathbb{R}$, $x \in [a, b]$

Suppose on this finite interval

$\frac{d f(x)}{dx}$ exists and $\left|\frac{d f(x)}{dx}\right| \leq y$.

y integrable, then f on $[a, b]$

$$\frac{d}{dx}\int F(x)\, du = \int \frac{d}{dx} f(x)\, du \quad \forall x \in [a, b]$$

(iv) Let $f(x,w)$ be a continuous f^n of x for each $x \in [a,b]$ and

$|f(x)| \leq Y$, Y integrable, then $\forall x \in [a,b]$

$$\int_a^x \int_\Omega f(t,w)\, du\, dt = \int_\Omega \int_a^x f(t,w)\, dt\, du \quad \text{---} (*)$$

$\rightarrow$) $\int \dots dx$: Reymann - Integral

proof Let $G(x)$ = LHS of (*)
and $H(x)$ = RHS of (*)

Note :- $F(t) = \int_0^t f(x)\, dx$

If f is continuous and $F'(t) = f(t)$

Now

$$G(x) = \int_a^x \int_\Omega f(t,w)\, du\, dt$$

Since f is continuous,

$$\Rightarrow G'(x) = \int_\Omega f(x,w)\, du$$

By the previous result

$$\frac{d}{dx} H(x) = \frac{d}{dx} \int \left[\int_a^x f(t,w)\, dt \right] du$$

$$= \int \frac{d}{dx} \left[\int_a^x f(t,w)\, dt \right] du$$

$$= \int f(x,w)\, du = G'(x)$$

Since $G(a) = H(a) = 0$
and $G'(x) = H'(x)$
$\Rightarrow G(x) = H(x) \quad \forall x \in [a, b]$

Further if the above assumption hold for every finite interval and

$\int_{-\infty}^{\infty} |f(x)|\, dx \le Z$ integrable, then

$$\int_{-\infty}^{\infty} \left(\int F(x)\, du \right) dx = \int \left[\int_{-\infty}^{\infty} f(x)\, dx \right] du$$

The integration w.r.t x are Reimann integral.

[Here we allow $a \to -\infty$ & $x \to +\infty$]

<u>Result</u> If f is integrable, then f is finite a-e
i-e $\mu[|f| = \infty] = 0$

<u>Proof</u> let $f_n = \begin{cases} f & \text{if } |f| < n \\ n & \text{if } f \ge n \\ -n & \text{if } f \le -n \end{cases}$

then
$$|f_n| \le n \quad \forall n \quad \text{and} \quad f_n \to f \ \forall w$$

let $A_n = \{|f| < n\}$
then
$$\int |f|\, du = \int_{A_n} |f|\, du + \int_{A_n'} |f|\, du$$

Now on A_n', $|f| \ge n$

$\therefore \int |f| \, d\mu \geq \int_{A_n} |f| \, d\mu + n\,\mu(A_n')$

Since

$\int |f| \, d\mu < \infty$ (as f is integrable)

it is necessary that $\mu(A_n') \to 0$

i.e

$\mu[\,|f| = \infty\,] = \mu[\lim A_n'] = 0$

$[\because \mu[\,|f| = \infty\,] = \mu[\lim A_n']$
$= \mu[\liminf A_n']$
$\leq \liminf \mu[A_n']$
$= 0$

$[\because \mu(A_n') \to 0]$

Here $\mu[\,|f| = \infty\,] = 0$

$\Rightarrow$ f is finite a.e

—×—

If $f \geq 0$, $\mu(E) \to 0$ then $\int_E f \, d\mu \to 0$

OR $\mu(E) < \delta \Rightarrow \int_E f \, d\mu < \varepsilon$

Proof Define $f_n = \begin{cases} f & \text{if } f < n \\ n & \text{if } f \geq n \end{cases}$

then

$f_n \leq n \ \forall n$ and $f_n \to f$

Consider

$$\int_E f \, d\mu = \int_E f_n \, d\mu + \int_E f \, d\mu - \int_E f_n \, d\mu$$

Now $f_n \leq n \ \forall n \Rightarrow f_n \leq n$ on E

$$\therefore \int_E f_n \, d\mu \leq \int_E n \, d\mu = \int n I_E \, d\mu = n \mu(E)$$

$$\therefore \int_E f \, d\mu \leq n \mu(E) + \frac{\varepsilon}{2} \quad \text{for } n \text{ large}$$

$[\because f_n \to f$, f integrable & $f \geq 0$

$$\Rightarrow \int f_n \, d\mu \to \int f \, d\mu$$

$$\Rightarrow \left[\int_E f_n \, d\mu - \int_E f \, d\mu\right] \leq \varepsilon/2 \quad \text{for } n \text{ large}]$$

Now if $\mu(E) \not< \frac{\varepsilon}{n}$, then

$$n\mu(E) \not< \varepsilon$$

$$\Rightarrow \int_E f \, d\mu = \infty \quad \text{for } n \text{ large}$$

$$\Rightarrow \int_\Omega f \, d\mu = \infty$$

$\Rightarrow f$ is not integrable
which is a contradiction

$$\Rightarrow \mu(E) < \frac{\varepsilon}{n}$$

$$\Rightarrow \mu(E) < \delta \quad \Rightarrow \int_E f \, d\mu < \varepsilon$$

Discrete $dist^n \rightarrow$ Counting Measure
Continuous $dist^n \rightarrow$ Lebesgue Stiltjes Measure

Counting Measure

Let $Z^+ = \{0, 1, 2, \ldots\}$ i.e set of all non -ve integers.

let $\mathcal{A}$ - σ field of subsets of Z^+
let $A \in \mathcal{A}$
and
define $X(A)$ = no. of elements in A if A is finite
$= +\infty$ if A is not finite

Then ψ is a measure and is known as counting Measure

Remark:- To define discrete distribution, we have counting measure and to define continuous distribution, we use lebesgue or lebesgue - Stiltjes measure.

2018 - 5(a) - 4M

* Absolute continuity and singularity

Def: A set function ψ is said to be μ-continuous or absolutely continuous w.r.t a measure μ if $\mu(A) = 0 \Rightarrow \psi(A) = 0$, where A is a mble set
Notation :- $\psi << \mu$

Def :- A Set function ψ is said to be singular w.r.t a measure μ if $\exists$ a mble set N s.t
$\mu(N) = 0$ (i.e N is a μ null set) and
$\psi(A) = \psi(A \cap N)$ i.e $\psi(A \cap N') = 0$

i.e ψ has all its mass concentrated on a μ- null
$\psi \perp \mu$ - Notation

Notation : $\varphi \perp \mu$

Examples :- All continuous distribution have their probability measures absolutely continuous w.r.t the lebesgue measure.
Similarly All discrete distribution have their prob measure singular w.r.t lebesgue measure.

eg. let $Z^+ = \{0,1,2,\ldots\}$
let μ_p denote prob measure of Poisson distⁿ
then $\mu_p(Z^+) = 1$
but $\lambda(Z^+) = 0$

$$\Rightarrow \mu_p \perp \lambda$$

Similarly if μ_B represents prob. measure of Binomial distⁿ, then $\mu_B \perp \lambda$.

1) In fact, all discrete distribution are singular w.r.t lebesgue measure

2) All continuous distribution (like Normal, Gamma Beta, Uniform, t) etc all are absolutely Continuous w.r.t lebesgue measure.

3) All discrete distribution are singular w.r.t all continuous distribution

4) All discrete distribution are absolutely continuous w.r.t counting measure

$$\mathcal{U}_B << \mathcal{U}_P$$
(Binomial) (Poission)

$$\phi_{Normal} << \phi_{student's\ t}$$
$$\phi_{student's\ t} << \phi_{Normal}$$

Recall :- If $\varphi_1 << \varphi_2$, $\varphi_2 << \varphi_3$, $\Rightarrow \varphi_1 << \varphi_3$.

— x —

Now, we like to define a set function using the concept of integral of a m'ble function So let us define set function $\nu(A)$ as

$$\nu(A) = \int_A f\,du = \int f I_A\,du$$

Thm $\nu(\cdot)$ is a set function defined on σ-field $\mathcal{A}$ and is known as indefinite integral of f.

1) Now suppose first f is integrable, then we know that f is finite a.e i.e $\mu[f=\infty]=0$

$$\Rightarrow \int f\,du < \infty \text{ and hence } \nu(\Omega) < \infty$$
$$\Rightarrow \nu(A) < \infty \quad \forall A \in \mathcal{A}$$

i.e ν is finite $\Rightarrow \nu$ is σ-finite also

If we assume $\int f\,du$ exists then
$\int f^+\,du < \infty$ OR $\int f^-\,du < \infty$

Suppose $\int f^-\,du < \infty$

Claim I: ν is σ- additive (counting additive)
i.e to prove it $\{A_n\}$ is a seq^n of disjoints sets in A

$$\left[(A_i \cap A_j) = \phi \ \forall\, i \neq j \,),\ \text{then} \quad \nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i) \right] \text{to prove}$$

Consider $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \int_{\cup A_n} f \, d\mu$

$$= \int f \,(I_{\cup A_n})\, d\mu$$

$$= \int f^+ I_{\cup A_n} \, d\mu - \int f^- I_{\cup A_n} \, d\mu$$

$$= \int f^+ (\Sigma I_{A_n})\, d\mu - \int f^- (\Sigma I_{A_n})\, d\mu$$

$$\int \Sigma f^+ I_{A_n} \, d\mu - \int \Sigma f^- I_{A_n} \, d\mu$$

$$= \Sigma \int f^+ I_{A_n} \, d\mu - \Sigma \int f^- I_{A_n} \, d\mu$$ [$\because$ Indefinite integrals are σ-additive by MCT)

$$= \Sigma \left[\int f^+ I_A \, d\mu - \int f^- I_{A_n} \, d\mu \right]$$

$$= \Sigma \int_{A_n} f \, d\mu$$

$$= \nu(A_n)$$

$\Rightarrow$ ν is σ- additive.

Claim 2:- Let f be almost everywhere finite valued i.e $\mu\{f = \pm\infty\} = 0$. Let μ be σ-finite then ν is also σ-finite.

proof:- μ is σ-finite

$\Rightarrow \exists \{B_n\}$ s.t $\Omega = \bigcup_{n=1}^{\infty} B_n$

and $\mu(B_n) < \infty$

To prove $\nu(B_n) < \infty$, $\nu(B_n) = \int_{B_n} f \, d\mu$

Now f is a.e finite

$\Rightarrow N = \{\omega \mid |f(\omega)| < \infty\}$ then $\mu(N') = 0$

Now.

$$\nu(B_n) = \nu(B_n \cap N) + \nu(B_n \cap N')$$
$$= \nu(B_n \cap N)$$

Now $N = \{\omega \mid |f(\omega)| < \infty\}$

$$= \sum_{k=-\infty}^{\infty} \{\omega \mid k \leq f(\omega) < k+1\}$$

$$\therefore \nu(B_n) = \sum_{k=-\infty}^{\infty} \nu\left(B_n \cap \{k \leq f(\omega) < k+1\}\right)$$

$$\therefore |\nu(B_n)| = \left| \sum_{k=-\infty}^{\infty} \int_{B_n \cap \{k \leq f(\omega) < k+1\}} f \, d\mu \right|$$

$$\leq 2 \sum_{k=0}^{\infty} (k+1) \int_{B_n \cap \{k \leq f(\omega) < k+1\}} d\mu$$

$$= 2 \sum_{k=0}^{\infty} (k+1) \mu\left[B_n \cap \{k \leq f(\omega) \leq k+1\}\right]$$

$< \infty$ $\because$ μ is finite

Thus ν is σ-finite

— x —

Remark :- The problem which arises is whether the above stated properties characterizes indefinite integrals? The answer lies in the following two celebrated theorem.

* Lebesgue's decomposition theorem :- 2019 - 5(a)

Let μ be a σ-finite measure. Let ν be a σ-finite and σ-additive set function on the same measure space. Then there exists a unique decomposition of ν as $\nu = \nu_1 + \nu_2$ where ν_1 and ν_2 satisfy

$$\nu_1 << \mu \quad \text{and} \quad \nu_2 \perp \mu$$

Further $\exists$ a non-negative a.e finite valued function f which is determined upto an equivalence $\ni$ $\nu(A) = \int_A f \, d\mu$, $\forall A \in \mathcal{A}$.
f is called Radon - Nikodym derivates of ν_1 w.r.t μ.

* Radon - Nykodym theorem :- 2018 - 5(b) - 2M, 2016 - 4(b), 2019 - 5(a)

Let μ be σ-finite measure. Let ν be a σ-finite and σ-additive measure. Let $\nu << \mu$. Then ν is the indefinite integral of some a.e. finite valued function f

$$\text{i.e } \nu(A) = \int_A f \, d\mu \, , \quad \forall A \in \mathcal{A}$$

→ Here f is determined upto equivalence f is called the R-N derivate of ν w.r.t μ i.e

$$f = \frac{\partial \nu}{\partial \mu}$$

eg. let μ_F be the L-S measure corresponding to a DF F.
let λ be the lebesgue measure
Suppose $\mu_F << \lambda$
then $\mu_F(-\infty, x] = \int_{(-\infty,x]} f \, d\lambda = \int_{-\infty}^{x} f(t)\, dt$

$$\Rightarrow F(x) - F(-\infty) = \int_{-\infty}^{x} f(t)\, dt$$

i.e $F(x) = \int_{-\infty}^{x} f(t)\, dt$

then $\frac{d}{dx} F(x) = f(x)$

i.e $\frac{\partial}{\partial \cancel{x}} \mu_F = f(x)$

—x—

The converse of the above theorem is also true. Thus, we have the following results which states both parts.

Thm let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. A set function ν which is absolutely continuous w.r.t μ can be expressed as an indefinite integral of a finite function f i.e

$$\nu(A) = \int_A f \, d\mu \quad \underline{\text{iff}} \quad \nu \text{ is } \sigma\text{-additive and } \sigma\text{-finite}$$

Further f is integrable iff ν is finite
(f is determined upto an equivalence)

★ Product Space 2016 – 4(c) $\boxed{2018-5(c)-3M}$
2019 – 5(a)

Suppose Ω_1 and Ω_2 are two abstract space with σ-fields $\mathcal{A}_1$ and $\mathcal{A}_2$ on them, and measure μ_1 and μ_2 respectively.

Thus $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are two measure spaces.

Define $\Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) \mid \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ is called the product space of the two spaces $\Omega_1 \times \Omega_2$

Rectangles $A_1 \times A_2$ in the product space is given by

$$A_1 \times A_2 = \{(\omega_1, \omega_2) \mid \omega_1 \in A_1, \omega_2 \in A_2\}$$

$$A_1 \subset \Omega_1, \; A_2 \subset \Omega_2, \; A_1 \in \mathcal{A}_1, \; A_2 \in \mathcal{A}_2$$

The question is to find approximate σ-field of subsets of $\Omega_1 \times \Omega_2$ and to find measure on this σ-field

Suppose $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$ and consider

$$\mathcal{C} = \{A_1 \times A_2 \mid A_i \in \mathcal{A}_i, \; i = 1, 2\}$$

Is $\mathcal{C}$ is a σ-field of subsets of $\Omega_1 \times \Omega_2$?

Note that $\mathcal{C}$ is not closed under compliments/ Unions. So we find the σ-field generated by $\mathcal{C}$.

[Recall :- ~~B~~ σ-field generated by $\mathcal{C}$ is the smallest σ-field of subsets of $\Omega_1 \times \Omega_2$ containing $\mathcal{C}$]

We consider this σ-field of subsets of $\Omega_1 \times \Omega_2$ as the appropriate σ-field and call it product σ-field.
and give the notation as $\sigma(\mathcal{C}) = \mathcal{A}_1 \times \mathcal{A}_2$

thus $A_1 \times A_2 \in \mathcal{A}_1 \times \mathcal{A}_2$

Now define $\mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2)$

let $B \in \mathcal{A}_1 \times A_2$ be any mble set in $\Omega_1 \times \Omega_2$
By caratheodary extension theorem
$\exists$ a measure $\bar{\mu}$ on the σ-field $\mathcal{A}_1 \times \mathcal{A}_2$
S.t the above measure μ is a restriction of $\bar{\mu}$ on the rectangles

$\bar{\mu}$ is called the extension of μ

$\bar{\mu}(A_1 \times A_2) = \mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2)$
Thus we have the measure space $(\Omega_1 \times \Omega_2), \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$ which is known as product measure space.

—x—

Section of set (2016-4(i))

For any set $A \subset \Omega_1 \times \Omega_2$, define the section of A at ω_1 as

$$\{\omega_2 \in \Omega \mid (\omega_1, \omega_2) \in A\} = A_{\omega_1}$$

Thus $A_{\omega_1} \subset \Omega_2$

Similarly

$$A_{\omega_2} = \{\omega_1 \in \Omega_1 \mid (\omega_1, \omega_2) \in A\} \subset \Omega_1$$

$$(A_1 \times A_2)_{\omega_1} = \begin{cases} A_2 & \text{if } \omega_1 \in A_1 \\ \phi & \text{if } \omega_1 \notin A_1 \end{cases}$$

Similarly

$$(A_1 \times A_2)_{\omega_2} = \begin{cases} A_1 & \text{if } \omega_2 \in A_2 \\ \phi & \text{if } \omega_2 \notin A_2 \end{cases}$$

Now let us define function on the product measure space

$$f(\omega_1, \omega_2) ; \Omega_1 \times \Omega_2 \to R$$

F is mble iff $f^{-1}(B) \in \mathcal{A}_1 \times \mathcal{A}_2 \ \forall B \in \mathcal{B}$

So let $f : \Omega_1 \times \Omega_2 \to R$ be a mble function for fixed $\omega_1 \in \Omega_1$ section of f at ω_1 is defined by

$$F_{\omega_1}(\omega_2) = f(\omega_1, \omega_2) : \Omega_2 \to R$$

Similarly, for fixed $\omega_2 \in \Omega_2$, section of f at ω_2 is defined as

$$F_{\omega_2}(\omega_1) = f(\omega_1, \omega_2) : \Omega_1 \to R$$

Thm Section of mble sets are mble sets.

proof Recall section of a rectangle $A_1 \times A_2$ at ω_1 is either A_2 or ϕ. Since A_2 and $\phi \in \mathcal{A}_2$
$\Rightarrow (A_1 \times A_2)_{\omega_1}$ is a mble set

Since all mble sets $A \in \mathcal{A}_1 \times \mathcal{A}_2$ are generated form rectangle $A_1 \times A_2$, it follows that A_{ω_1} is also a mble set, Hence the theorem

Thm Section of mble function are also mble 2016-4(c)

Proof let $f: \Omega_1 \times \Omega_2 \to R$ be a mble function
$\Rightarrow f^{-1}(B) \in \mathcal{A}_1 \times \mathcal{A}_2 \quad \forall B \in \mathcal{B}$

Consider
$$f_{\omega_1}(\omega_2): \Omega_2 \to R \text{ (for fixed } \omega_1)$$
To prove
$$f_{\omega_1}^{-1}(B) \in \mathcal{A}_2 \quad \forall B \in \mathcal{B}$$
Consider
$$\begin{aligned} f_{\omega_1}^{-1}(B) &= \{\omega_2 \mid f_{\omega_1}(\omega_2) \in B\} \\ &= \{\omega_2 \mid f(\omega_1, \omega_2) \in B\} \\ &= \{(\omega_1, \omega_2) \mid f(\omega_1, \omega_2) \in B\}_{\omega_1} \\ &= \{f^{-1}(B)\}_{\omega_1} \in \mathcal{A}_2 \end{aligned}$$

$\Rightarrow f_{\omega_1}$ is mble Similarly we can prove f_{ω_2} is also mble

$\Rightarrow$ Section of mble function are also mble

Fubini's Theorem 2019 (5 (a))

let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be two σ- finite measure space.

let $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$ be the product measure space. let f be a function mble w.r.t $\mathcal{A}_1 \times \mathcal{A}_2$; let $f(\omega_1, \omega_2)$ be either non-ne or integerable w.r.t measure $\mu_1 \times \mu_2$ then

$$\int_{\Omega_1 \times \Omega_2} f \, d(\mu_1 \times \mu_2) = \int_{\Omega_1} \left[\int_{\Omega_2} f_{\omega_1}(\omega_2) \, d\mu_2 \right] d\mu_1$$

$$= \int_{\Omega_2} \left[\int_{\Omega_1} f_{\omega_2}(\omega_1) \, d\mu_1 \right] d\mu_2$$

further section of X i.e $X_{\omega_1}(\cdot)$ & $X_{\omega_2}(\cdot)$ also integrable.

— x —

Remark:

(1) Iterated integrals are to be read from right to left

(2) The above result extends to product of arbitrary but finite no. of product measure space.