Monotone Convergence Theorem:- (MCT)

Statement: Suppose $0 \le f_n \uparrow f$, where f_n is a mble f^n $\forall n$.

Then $0 \le \int f_n d\mu \uparrow \int f d\mu$

Proof: Since $f_n \ge 0 \Rightarrow \int f_n d\mu \ge 0$ &

$$\int f_n d\mu \le \int f_{n+1} d\mu$$

Now for some k, $f_k \ge 0 \Rightarrow \exists$ a sequence $\{f_{kn}\}$ of non-ve simple functions s.t. $0 \le f_{kn} \uparrow f_k$ as $n \to \infty$

Thus

$$\begin{array}{llllll} f_{11} & f_{12} & \cdots & f_{1n} & \cdots & \to f_1 \\ f_{21} & f_{22} & \cdots & f_{2n} & \cdots & \to f_2 \\ \vdots & & & & & \\ f_{k1} & f_{k2} & \cdots & f_{kn} & \cdots & \to f_k \\ \vdots & & & & & \\ f_{n1} & f_{n2} & \cdots & f_{nn} & \cdots & \to f_n \end{array}$$

Then $0 \le f_{kn} \le f_k \le f_n \quad \forall k \le n$

Let $\boxed{X_n = \max_{k \le n} f_{kn}}$, which is a simple f^n $\forall n$

then $0 \le f_{kn} \le X_n \le f_n \quad \forall k \le n$ — ①

and $0 \le \int f_{kn} d\mu \le \int X_n d\mu \le \int f_n d\mu$ — ②

Let $n \to \infty$ in ① & ②, then

$$0 \le f_k \le \lim_{n\to\infty} X_n \le f \quad \forall k \quad — ③$$

and $0 \le \int f_k d\mu \le \lim_{n\to\infty} \int X_n d\mu \le \lim_{n\to\infty} \int f_n d\mu \quad \forall k$ ④

Now let $k \to \infty$ in ③ & ④, we have

$$0 \le f \le \lim_{n\to\infty} X_n \le f \quad \text{and} \quad — ⑤$$

$$0 \le \lim_{k\to\infty} \int f_k d\mu \le \lim_{n\to\infty} \int X_n d\mu \le \lim_{n\to\infty} \int f_n d\mu \quad — ⑥$$

From ⑤, we have $\lim X_n = f$ & from ⑥, we have

$$\lim_{n\to\infty} \int X_n d\mu = \lim_{n\to\infty} \int f_n d\mu$$

but X_n's are simple f^n s.t. $\lim_{n\to\infty} X_n = f$

$$\therefore \lim_{n\to\infty} \int X_n d\mu = \int \lim_{n\to\infty} X_n d\mu = \int f d\mu$$

$\Rightarrow \lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$

Hence the proof.

—— × ——

Applications of MCT

let $f_n \geq 0$ be mble f^ns for every n.

then $\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \left(\sum_{n=1}^{\infty} f_n\right) d\mu$

i.e. indefinite integral is σ-additive.

Proof : Let $g_K = \sum_{1}^{K} f_n$, then $0 \leq g_K \uparrow \sum_{n=1}^{\infty} f_n$.

Each g_K is mble & so is $\sum_{n=1}^{\infty} f_n$.

($\because$ limit of sequence of mble function is mble).

Then by MCT,

$$\lim_{K\to\infty} \int g_K \, d\mu = \int \left(\lim_{K\to\infty} g_K\right) d\mu$$

i.e. $$\lim_{K\to\infty} \int \left[\sum_{1}^{K} f_n\right] d\mu = \int \left(\sum_{n=1}^{\infty} f_n\right) d\mu$$

$$\Rightarrow \lim_{K\to\infty} \sum_{1}^{K} \int f_n \, d\mu = \int \left(\sum_{n=1}^{\infty} f_n\right) d\mu$$

$$\Rightarrow \sum_{n=1}^{\infty} \int f_n \, d\mu = \int \left(\sum_{n=1}^{\infty} f_n\right) d\mu$$

i.e. countable sum can be taken inside integral.

—×—

Examples : (i) Compute $\lim_{n\to\infty} \int_1^2 \frac{n}{1+nx^2} dx$

Ans : First we note that $f_n = \frac{n}{1+nx^2}$ is such that if $n < m$ then $f(n) \leq f(m)$ i.e each f_n is a mble f^n s.t. $0 \leq f_n \uparrow$.

[e.g. let $x = 1.5$,

n	$f_n = \frac{n}{1+n(1.5)^2}$
1	0.3076
2	0.3636
3	0.3870

and so on.]

$\therefore$ By MCT, $$\lim_{n\to\infty} \int_1^2 \frac{n}{1+nx^2} dx = \int_1^2 \lim_{n\to\infty} \frac{n}{1+nx^2} dx$$

$$= \int_1^2 \lim_{n\to\infty} \frac{1}{\frac{1}{n}+x^2} dx$$

$$= \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

—×—

(82)

Fatou's Lemma :—

(a) Suppose $f_n \geq g$, g mble & $f_n \; \forall n$ are mble & g is integrable. Let $\int f_n d\mu$ exists $\forall n$, then

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu$$

(b) Suppose $f_n \leq h$, $f_n \forall n$ & h are mble, h is integrable

then $\int (\limsup f_n) \, d\mu \geq \limsup \int f_n \, d\mu$

(c) Suppose $g \leq f_n \leq h$ & $\lim f_n = f$, where h and g are integrable

then $\lim \int f_n \, d\mu = \int f \, d\mu$

Remark : In MCT, we have an $\uparrow$ seqn of mble fn f_n. But what is the seqn is not $\uparrow$. Then Fatou's lemma is applicable.

Proof : (a) Assume first that $g \geq 0$.

Recall $\liminf f_n = \lim\limits_{k \to \infty} \inf\limits_{n \geq k} f_n$

let $g_k = \inf\limits_{n \geq k} f_n$

$\therefore \liminf f_n = \lim\limits_{k \to \infty} g_k$, where $0 \leq g_k \uparrow$

Hence by MCT,

$$\lim_{k \to \infty} \int g_k \, d\mu = \int \lim_{k \to \infty} g_k \, d\mu$$

$$= \int \liminf f_n \, d\mu$$

i.e. $\int \liminf f_n d\mu = \lim\limits_{k \to \infty} \int g_k \, d\mu$

$$= \lim_{k \to \infty} \int \left(\inf_{n \geq k} f_n \right) d\mu \quad \text{— (1)}$$

but $\int \left(\inf\limits_{n \geq k} f_n \right) d\mu \leq \int f_k \, d\mu \quad \forall k$

$\therefore \liminf \int \left(\inf\limits_{n \geq k} f_n \right) d\mu \leq \liminf \int f_k \, d\mu$ — (2)

but from (1), $\lim\limits_{k \to \infty} \int \left(\inf\limits_{n \geq k} f_n \right) d\mu$ exists

$\Rightarrow \liminf \int \left(\inf\limits_{n \geq k} f_n \right) d\mu = \lim\limits_{k \to \infty} \int \left(\inf\limits_{n \geq k} f_n \right) d\mu$

$$= \int \liminf f_n \, d\mu \quad \text{(3)}$$

(83) using (3) in (2), we have

$\int (\liminf f_n) d\mu \le \liminf \int f_n \, d\mu$ if $g \geq 0$

Suppose $g < 0$, then $-g > 0$ &

define $f_n^* = f_n - g \geq 0$

Hence applying previous argument to f_n^*, we have

$\int \liminf f_n^* \, d\mu \le \liminf \int f_n^* \, d\mu$

i.e $\int \liminf (f_n - g) d\mu \le \liminf \int (f_n - g) d\mu$

i.e $\int (\liminf f_n - g) d\mu \le \liminf \left[\int f_n d\mu - \int g \, d\mu\right]$

i.e $\int \liminf f_n \, d\mu - \int g \, d\mu \le \liminf \int f_n d\mu - \int g \, d\mu$

& $\because g$ is integrable $\Rightarrow \int g \, d\mu < \infty$

$\Rightarrow \int \liminf f_n \, d\mu \le \liminf \int f_n d\mu$

Thus (a) is proved.

—x—

(b) Proof using part (a)

To prove $\int \limsup f_n d\mu \geq \limsup \int f_n d\mu$

if $f_n \le h$, h integrable.

Note that $-f_n \geq -h$ & $-h$ is integrable

$\therefore$ By part (a),

$\int \liminf (-f_n) d\mu \le \liminf \int (-f_n) d\mu$

i.e $\int - \limsup f_n \, d\mu \le - \limsup \int f_n d\mu$

i.e $\int \limsup f_n \, d\mu \geq \limsup \int f_n d\mu$

Thus (b) is proved.

—x—

(c) Combining (a) & (b) we have

$\int \liminf f_n \, d\mu \le \liminf \int f_n d\mu$

$\le \limsup \int f_n d\mu \le \int \limsup f_n \, d\mu.$

But $\lim f_n = f$ exists

$\Rightarrow \liminf f_n = \limsup f_n = \lim f_n = f$

$\Rightarrow \int \liminf f_n \, d\mu = \int \limsup f_n \, d\mu = \int \lim f_n \, d\mu$

$= \int f \, d\mu$

$\Rightarrow$ equality holds everywhere

$$\Rightarrow \quad \liminf \int f_n\,d\mu = \limsup \int f_n\,d\mu = \lim \int f_n\,d\mu$$
$$= \int \lim f_n\,d\mu = \int f\,d\mu$$

thus $\lim \int f_n\,d\mu = \int \lim f_n\,d\mu$

—x—

Note :- 1) The result in (c) holds even when ~~fn are~~ $f_n \to f$ a.e. $[\text{i.e } \mu[f_n \not\to f] = 0]$

2) If no convergence is specified, we consider it as pointwise convergence.

—x—

Lebesgue's Dominated convergence Theorem (LDCT) :-

Statement : Let $\{f_n\}$ be a sequence of mble functions s.t. $|f_n| \le g$ a.e., where g is integrable. Let $f_n \to f$ a.e or in measure, then $\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$ — (1)

Remark : In fact $\int f_n\,d\mu \to \int f\,d\mu$

$$\Rightarrow \left| \int f_n\,d\mu - \int f\,d\mu \right| \to 0$$

but $\left| \int f_n\,d\mu - \int f\,d\mu \right| = \left| \int (f_n - f)\,d\mu \right|$

$$\le \int |f_n - f|\,d\mu$$

Hence if $\int |f_n - f|\,d\mu \to 0$ — (2)

$$\Rightarrow \left| \int f_n\,d\mu - \int f\,d\mu \right| \to 0$$

i.e $\int f_n\,d\mu \to \int f\,d\mu$

i.e $\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$ which is (1).

Thus (2) $\Rightarrow$ (1)

Hence it is enough to prove (2) holds.

Proof : Write $|f_n - f| = g_n$

since $|f_n| \le g \Rightarrow |g_n| \le 2g$

& $g_n \to 0$ a.e. or in measure

(85)

write $|f_n - f| = g_n$

Since $|f_n| \leq g$

$\Rightarrow |g_n| \leq 2g$

and $g_n \to 0$ a.e or in measure

First consider the case of convergence a.e.

If $f_n \to f$ a.e. i.e $g_n \to 0$ a.e,

then by part (c) of Fatou's lemma,

$\int g_n \, d\mu \to 0$

i.e $\int |f_n - f| \, d\mu \to 0$

i.e. (2) holds.

Thus we need to consider only the case $g_n \to 0$ in measure.

To prove $\int g_n \, d\mu \to 0$ as $n \to \infty$.

Now $g_n \geq 0$

$\Rightarrow \int g_n \, d\mu \geq 0$

It is sufficient to prove that

$\limsup \int g_n \, d\mu = 0$ — (3)

[because then $\liminf \int g_n \, d\mu = 0$ &
hence $\lim \int g_n \, d\mu = 0$]

Let $\limsup \int g_n \, d\mu = c > 0$ [if possible]

Let n' be a sequence of +ve integers

s.t. $C_{n'} = \int g_{n'} \, d\mu \to c$ as $n \to \infty$

Recall the following result

[If $h_n \xrightarrow{\mu} h$, then $\exists\, n_k$ s.t. $h_{n_k} \to h$ a.e]

here $g_{n'} \to 0$ in measure $g_{n'} \xrightarrow{\mu} 0$

$\Rightarrow \exists$ a further subsequence

$\{n''\} \subset \{n'\}$ s.t.

(86)

$$g_{n''} \to 0 \text{ a.e.}$$

$\therefore$ By Fatou's lemma,

$$C_n'' = \int g_{n''}\, d\mu \to \int 0\, d\mu = 0 \quad — (*)$$

but $\{C_n''\} \subset \{C_n'\}$ which converges to c

$$\Rightarrow C_n'' \to c \quad —— (**)$$

$(*)$ & $(**)$ together implies

$$c \equiv 0$$

$$\Rightarrow \limsup \int g_n\, d\mu = 0$$

$$\Rightarrow \lim \int g_n\, d\mu = 0$$

$$\Rightarrow \lim \int |f_n - f|\, d\mu = 0$$

$\Rightarrow$ (2) holds.

—×—

Extensions (Applications):

$f_n : \Omega \to R$

If f is continuous

$f(x, \omega) \to f(x_0 \; \omega)$ as $x \to x_0$

$f_n(\omega) \to f(\omega)$ is equivalent to

$f(x, \omega) \to f(x_0, \omega)$ as $x \to x_0$

(i.e $n \to \infty$ is replaced by $x \to x_0$ along some arbitrary set)

Thus the above result hold with f_n replaced by $f(x)$ and $n \to \infty$ replaced by $x \to x_0$.

New form of DCT :—

(i) If $|f(x)| \le g$, g integrable &

$$f(x) \to f(x_0) \text{ as } x \to x_0$$

then $\int f(x) \longrightarrow \int f(x_0)$

Application 1:

(ii) Suppose for ($x \in T$ some arbitrary set in which $x \to x_0$),

$\frac{d}{dx} f(x)$ exists at x_0 &

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \le Y, \quad Y \text{ integrable}$$

then $$\left(\frac{d}{dx} \int f(x) \right)_{x_0} = \int \left(\frac{d}{dx} f(x) \right)_{x_0}$$

Proof:

LHS $$\left. \frac{d}{dx} \int f(x) \right|_{x_0} = \lim_{x \to x_0} \frac{\int f(x) - \int f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \int \left(\frac{f(x) - f(x_0)}{x - x_0} \right) d\mu$$

by LDCT,

$$= \int \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] d\mu$$

$$= \int \left[\frac{d}{dx} F(x, \omega) \right]_{x = x_0} d\mu$$

= RHS.

In the above, the result is written for at single point x_0. It is extended in the following way.

(iii) let $f(x, \omega) : \Omega \to R$ $\quad x \in [a, b]$
Suppose on this finite interval,

$$\frac{df(x)}{dx} \text{ exists } \& \left| \frac{df(x)}{dx} \right| \leq Y,$$

Y integrable, then on $[a, b]$,

$$\frac{d}{dx} \int F(x) \, d\mu = \int \frac{d}{dx} f(x) \, d\mu$$
$$\forall x \in [a, b]$$

(iv) Let $f(x, \omega)$ be a continuous f^n of x for each $x \in [a, b]$ &
$|F(x)| \leq Y$, Y integrable, then
$\forall x \in [a, b]$

$$\int_a^x \int_\Omega f(t, \omega) \, d\mu \, dt = \int_\Omega \int_a^x f(t, \omega) \, dt \, d\mu \quad (*)$$

(89) $\int \cdots dx$: Reymann - Integral

<u>Proof</u>: let $G(x) = $ LHS of (*)

& $H(x) = $ RHS of (*)

Note :- $F(t) = \int_0^t f(x)dx$

If f is continuous &

$F'(t) = f(t)$

Now
$$G(x) = \int_a^x \int_\Omega f(t, \omega)\, d\mu\, dt$$

since f is continuous,

$$\Rightarrow G'(x) = \int_\Omega f(x, \omega)\, d\mu$$

By the previous result

$$\frac{d}{dx} H(x) = \frac{d}{dx} \int \left[\int_a^x f(t, \omega)\, dt \right] d\mu$$

$$= \int \frac{d}{dx} \left[\int_a^x f(t, \omega)\, dt \right] d\mu$$

$$= \int f(x, \omega)\, d\mu$$

$$= G'(x)$$

since $G(a) = H(a) = 0$

& $G'(x) = H'(x)$

$$\Rightarrow G(x) = H(x) \quad \forall x \in [a\ b]$$

Further,
if the above assumption hold for every finite interval and

$$\int_{-\infty}^{\infty} |f.(x)|\, dx \leq Z \text{ integrable, then}$$

$$\int_{-\infty}^{\infty} \left(\int f(x) \, dy \right) dx = \int \left[\int_{-\infty}^{\infty} f(x)\, dx \right] d$$

The integration w.r.t. x are Reimann integral.

[Here we allow $a \to -\infty$ & $x \to +\infty$]

—x—

(91) Results.

If f is integrable, then f is finite a.e.

i.e $\mu[|f| = \infty] = 0$.

Proof: Let $f_n = \begin{cases} f & \text{if } |f| < n \\ n & \text{if } f \geq n \\ -n & \text{if } f \leq -n \end{cases}$

then $|f_n| \leq n \quad \forall n$

& $f_n \to f \quad \forall \omega$.

Let $A_n = \{|f| < n\}$

then

$$\int |f| \, d\mu = \int_{A_n} |f| \, d\mu + \int_{A_n'} |f| \, d\mu$$

Now on A_n', $|f| \geq n$

$\therefore \int |f| \, d\mu \geq \int_{A_n} |f| \, d\mu + n \, \mu(A_n')$

Since $\int |f| \, d\mu < \infty$ (as f is integrable), it is necessary that $\mu(A_n') \to 0$

i.e $\mu[|f| = \infty] = \mu[\lim A_n'] = 0$

$[\because \mu[|f| = \infty] = \mu[\lim A_n'] = \mu[\lim\inf A_n']$

$\leq \lim\inf \mu(A_n')$

$= 0 \quad \because \mu(A_n') \to 0]$

Hence $\mu[|f| = \infty] = 0$

$\Rightarrow f$ is finite a.e.

—x—

If $f \geq 0$, $\mu(E) \to 0$ then $\int_E f \, d\mu \to 0$

OR $\mu(E) < \delta \Rightarrow \int_E f \, d\mu < \epsilon$

Proof: Define $f_n = \begin{cases} f & \text{if } f < n \\ n & \text{if } f \geq n \end{cases}$

then $f_n \leq n \; \forall n$ & $f_n \to f$

Consider $\int_E f \, d\mu = \int_E f_n \, d\mu + \int_E f \, d\mu - \int_E f_n \, d\mu$

Now $f_n \leq n \; \forall n \Rightarrow f_n \leq n$ on E

$\therefore \int_E f_n \, d\mu \leq \int_E n \, d\mu = \int n I_E \, d\mu = n \, \mu(E)$

$\therefore \int_E f\,d\mu \leq n\,\mu(E) + \varepsilon/2$ for n large

[$\because f_n \to f$, f integrable & $f \geq 0$

$\Rightarrow \int f_n\,d\mu \to \int f\,d\mu$

$\Rightarrow [\int_E f_n\,d\mu - \int_E f\,d\mu] \leq \varepsilon/2$ for n large]

Now if $\mu(E) \nless \frac{\varepsilon}{n}$, then

$n\,\mu(E) \nless \varepsilon$

$\Rightarrow \int_E f\,d\mu = \infty$ for n large

$\Rightarrow \int_\Omega f\,d\mu = \infty$

$\Rightarrow f$ is not integrable

which is a contradiction.

$\Rightarrow \mu(E) < \frac{\varepsilon}{n}$

$\Rightarrow \mu(E) < \delta \Rightarrow \int_E f\,d\mu < \varepsilon$

—x—

Counting measure

Let $Z^+ = \{0, 1, 2, \ldots\}$ i.e set of all non-ve integers.

Let $\mathcal{A}$: σ field of subsets of Z^+.

Let $A \in \mathcal{A}$

& define $\gamma(A) =$ no. of elements in A if A is finite

$= +\infty$ if A is not finite.

Then γ is a measure & is known as counting measure.

—x—

Remark: To define discrete distributions, we use counting measure & to define continuous distributions, we use Lebesgue or Lebesgue-Stiltjes measure.

—x—