Fields and σ- Fields :—

Algebra and σ- Algebra:-

Let Ω be an abstract space.

Let $A, B, C \cdots$ be subsets of Ω.

let $\mathcal{C}$ denote some collection of subsets of Ω.

e.g. $\mathcal{C} = \{A, B, C\}$, $\mathcal{C} = \{\phi, \Omega\}$, $\mathcal{C} = \{\phi, A, B\}$ etc.

$\mathcal{C}$ can be even empty collection.

or $\mathcal{C} = \{\phi\}$ consists of one set ϕ, so this is a non-empty collection.

Field:

Def: A non-empty collection $\mathcal{C}$ of subsets of Ω is known as a Field or Algebra if it satisfies the following conditions

(i) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$ ie $A' \in \mathcal{C}$

(ii) $A, B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C}$.

In other words

A Non-empty collection $\mathcal{C}$ of subsets of Ω is known as a Field if it closed under compliments and finite unions.

—x—

Remarks: A ~~fild~~ field is a non-empty collection.

So let $A \in \mathcal{C}$ then $A' \in \mathcal{C}$

Also $\Rightarrow A \cup A' \in \mathcal{C}$

i.e. $\Omega \in \mathcal{C}$

also $\Omega' \in \mathcal{C}$ ie $\phi \in \mathcal{C}$

Thus

(i) Every field always contains ϕ and Ω.

(ii) $\mathcal{C} = \{\phi, \Omega\}$ is the smallest field.

(Also known as trivial field.)

(iii) If $A \in \mathcal{C} \Rightarrow A' \in \mathcal{C}$

Thus $\{\phi, \Omega, A, A'\}$ is also a field.

(Check ?)

(iv) Suppose $\mathcal{C}$ is a field & suppose $A, B \in \mathcal{C}$, then $A' \in \mathcal{C}$, $B' \in \mathcal{C}$

$\Rightarrow A \cup B,\ A \cup B',\ A' \cup B,\ A' \cup B' \in \mathcal{C}$

but $A' \cup B' = (A \cap B)' \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ and so on.

(11)

$\mathcal{C} = \{\phi, \Omega, A, B, A', B', A\cup B, A\cup B', A'\cup B, A'\cup B', A\cap B, \cdots\cdots\}$

All possible unions, intersection & compliments have to be in $\mathcal{C}$.

—x—

Alternate Def:- A non-empty collection $\mathcal{C}$ of subsets of Ω is a field if it is closed under compliments and finite intersections i.e.

$A \in \mathcal{C} \Rightarrow A' \in \mathcal{C}$ &

if $A, B \in \mathcal{C} \Rightarrow A\cap B \in \mathcal{C}$

—x—

Largest Field:-

Power set of $\Omega = \mathcal{P}(\Omega) = \{A \mid A \subset \Omega\}$.

—x—

Notation:- " $\mathcal{F}_0$ " for a field.

—x—

σ- field

Def: A non-empty collection of subsets of Ω, $(\mathcal{A})$ is called a σ- field (or σ- algebra) if

(i) $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$

(ii) $A_n \in \mathcal{A}, \forall n = 1, 2, \cdots$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

i.e. closed under compliments and countable unions.

Alternate def: A non-empty collection $\mathcal{A}$ of subsets of Ω is called a σ- field if it closed under compliments and countable intersections.

Properties of σ- field:-

Let $\mathcal{A}$ be a σ- field. Let $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$

Choose A_n s.t. $A_1 = A'$ & $A_2 = A_3 = \cdots = A$

then $\bigcup_{n=1}^{\infty} A_n = A' \cup A = \Omega \in \mathcal{A}$

hence $\Omega' = \phi \in \mathcal{A}$.

Thus every σ-field must contain ϕ and Ω.

In fact $\{\phi, \Omega\}$ is the smallest σ- field and $\mathcal{P}(\Omega)$ is the largest σ- field.

—x—

(12) Let $\mathcal{A}$ be a σ-field.

let $\{A_n\} \in \mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$.

also $A_n \in \mathcal{A} \Rightarrow A_n^c \in \mathcal{A}$

$$\Rightarrow \bigcup A_n^c \in \mathcal{A}$$

$$\Rightarrow \left(\bigcap A_n\right)^c \in \mathcal{A}$$

$$\Rightarrow \bigcap A_n \in \mathcal{A}$$

Thus $\mathcal{A}$ is closed under countable intersections.

—x—

$\lim\sup A_n$, $\lim\inf A_n \in \mathcal{A}$.

→ let $A_n \in \mathcal{A}$ $\forall n = 1, 2, \dots$

then $\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

let $B_k = \bigcup_{n=k}^{\infty} A_n$ then $B_k \in \mathcal{A}$ $\forall k$

$\therefore \bigcap_{k=1}^{\infty} B_k \in \mathcal{A}$

$\Rightarrow \lim\sup A_n \in \mathcal{A}$

Similarly, $\lim\inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n \in \mathcal{A}$

Further, if $\lim A_n$ exists, then $\lim A_n \in \mathcal{A}$

—x—

Let $\mathcal{A}$ be a σ-field. Then $\mathcal{A}$ is also a field.

→ It is sufficient to check that $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

Take $A_1 = A$ & $A_2 = A_3 = A_4 = \dots = B$, then

$A_n \in \mathcal{A}$ $\forall n$ &

$\bigcup_{n=1}^{\infty} A_n = A \cup B \in \mathcal{A}$

$\Rightarrow \mathcal{A}$ is a field.

Thus <u>every σ-field is also a field.</u>

But the converse is not true.

The following example will prove that every field is not necessarily a σ-field.

(13) Let $\Omega = \{1, 2, 3, \cdots\}$

Define

$\mathcal{F} = \{ A \subset \Omega \mid \text{either } A \text{ or } A^c \text{ is a finite set} \}$

Claim: 1) $\mathcal{F}$ is non-empty.

let $A \in \mathcal{F}$ $\Rightarrow A^c$ is finite if A is not finite

& A^c is not finite if A is finite.

Thus $\forall A \in \mathcal{F}$, $A^c \in \mathcal{F}$.

Thus $\mathcal{F}$ is closed under compliments.

Further let $A, B \in \mathcal{F}$. To prove $A \cup B \in \mathcal{F}$.

When A & $B \in \mathcal{F}$, then one of the following holds.

(i) A & B both finite $\Rightarrow A \cup B$ is finite.

(ii) A & B' are finite $\Rightarrow A' \cap B'$ are finite

$\Rightarrow (A \cup B)'$ is finite

$\Rightarrow A \cup B \in \mathcal{F}$

(iii) A' & B are finite. By similar arguments, $A \cup B \in \mathcal{F}$.

(iv) A' and B' are finite $\Rightarrow A' \cap B'$ is finite

$\Rightarrow (A \cup B)'$ is finite

$\Rightarrow A \cup B \in \mathcal{F}$

Thus $\mathcal{F}$ is also closed under finite unions.

Thus $\mathcal{F}$ is a field.

Now define A_n as

~~$A_n = \{2, 4, 6, \cdots\}$ if n is even~~ $\{2n\}$ $A_n = \{n\}$, if n even

& $A_n = \phi$ if n is odd

Then $A_n \in \mathcal{F}$ $\forall n$.

$\bigcup A_n = \{2, 4, 6, 8, \cdots\}$ not finite

$(\bigcup A_n)' = \{1, 3, 5, 7, \cdots\}$ not finite

Thus $\bigcup A_n \notin \mathcal{F}$

$\Rightarrow \mathcal{F}$ is not closed under countable unions

$\Rightarrow \mathcal{F}$ is not a σ-field.

—x—

(14) Thm: A finite field is a σ-field.

Proof: Let $\mathcal{F}$ be a finite field.

i.e $\mathcal{F}$ is a field containing finite no. of sets

So let $\mathcal{F} = \{C_1, \cdots C_N\}$ where N is finite.

Let $A_n \in \mathcal{F}$ $\forall n$, then A_n ~~is~~ are some or all of sets $C_1, \cdots C_N$ & hence

$\bigcup_{n=1}^{\infty} A_n$ is union of some or all sets $C_1 \cdots C_N$.

i.e $\bigcup_{n=1}^{\infty} A_n$ is a finite union

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ ($\because \mathcal{F}$ is a field).

Hence $\mathcal{F}$ is also a σ-field.

—x—

Thus only finite field is a σ-field.

—x—

Def: A non-empty collection $\mathcal{M}$ of subsets of Ω is called a monotone class if $\mathcal{M}$ is closed under limits of monotone sequences.

i.e $A_n \in \mathcal{M}$, $n = 1, 2 \cdots$ & $\{A_n\}$ is monotone then $\lim A_n \in \mathcal{M}$.

—x—

Result: Is a σ-field a monotone class?

Yes. Every σ-field is a monotone class, because σ-field is closed under limits of any type of sequence, if it exists.

The Converse is not always true.

i.e a monotone class is not always a σ-field.

let $\mathcal{C} = \{ I \mid I \text{ is an interval in } \mathbb{R} \}$

i.e any type of the interval $(a\ b)$, $(a\ b]$, $[a, b)$, $[a\ b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$

then $\mathcal{C}$ is a monotone class.

e.g. let $A_n = (a, b - \frac{1}{n}]$, $n \geq 1$

then $A_n \uparrow$ & $\lim A_n = (a, b) \in \mathcal{C}$

Here, we note that $\mathcal{C}$ is closed under limits of monotone sequences. Hence $\mathcal{C}$ is a monotone class. But if $(a, b) \in \mathcal{C}$, $(a\ b)' \notin \mathcal{C}$

$\Rightarrow$ $\mathcal{C}$ is not closed under compliments

$\Rightarrow$ $\mathcal{C}$ is not a σ- field.

—x—

<u>Thm</u> : A monotone field is a σ- field.

i.e a field which is a monotone class or a monotone class which is a field is also a σ- field.

Proof :- Let $\mathcal{C}$ be a monotone field.

Being a field, it is always closed under compliments.

Let $\{A_n \in \mathcal{C}$ $\forall n = 1, 2 \ldots$

To prove $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

Define $B_K = \bigcup_{n=1}^{K} A_n$

then $B_K \in \mathcal{C}$ $\forall K$

& $B_K \uparrow \Rightarrow \lim_{K \to \infty} B_K \in \mathcal{C}$

but $\lim_{K \to \infty} B_K = \bigcup_K B_K = \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$

$\Rightarrow$ $\mathcal{C}$ is closed under countable unions.

$\Rightarrow$ $\mathcal{C}$ is a σ- field.

—x—

(16) Thus,

Thm:- A field is a σ-field iff it is a monotone class. Also

Thm:- A monotone class is a σ-field iff it is a field.

—x—

let $\Omega = \mathbb{R}$, let $A_x = (x, \infty)$, $x \in \mathbb{R}$

Define $\mathcal{A}_x = \{\Omega, \phi, A_x, A_x'\}$

then $\mathcal{A}_x$ is a σ-field $\forall x \in \mathbb{R}$

Thus on a given space, we can define infinite no. of σ-fields.

So let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two σ-fields, of subsets of Ω.

clearly $\phi, \Omega \in (\mathcal{A}_1 \cap \mathcal{A}_2)$

Thus $\mathcal{A}_1 \cap \mathcal{A}_2$ is non empty.

let $A \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow A \in \mathcal{A}_1$ and $A \in \mathcal{A}_2$

$\Rightarrow A' \in \mathcal{A}_1$ and $A' \in \mathcal{A}_2$

$\Rightarrow A' \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow \mathcal{A}_1 \cap \mathcal{A}_2$ is closed under complimentation.

let $\{A_n\}_{n=1}^{\infty} \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow A_n \in \mathcal{A}_1 \; \forall n$ & $A_n \in \mathcal{A}_2 \; \forall n$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_1$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_2$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow \mathcal{A}_1 \cap \mathcal{A}_2$ is closed under countable unions.

Hence $\mathcal{A}_1 \cap \mathcal{A}_2$ is also a σ-field.

—x—

This idea can be extended to intersection of any number of σ-fields.

(17) Let $\{\mathcal{A}_t, t \in T\}$ be a collection of σ-fields of subsets of Ω.

(T can be finite, infinite, countable or uncountable)

Then $\bigcap_{t \in T} \mathcal{A}_t$ is also a σ-field

—x—

Is Union of two σ-fields also a σ-field?

No, not necessarily.

e.g. Let $\Omega = \mathbb{R}$, $A = [0, \infty)$, $B = (0, \infty)$

then let $\mathcal{A}_1 = \{\phi, \Omega, [0, \infty), (-\infty, 0)\}$

$\mathcal{A}_2 = \{\phi, \Omega, (0, \infty), (-\infty, 0]\}$

then

$\mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, \Omega, [0, \infty), (-\infty, 0), (0, \infty), (-\infty, 0]\}$

Now $(-\infty\ 0] \cap [0, \infty) = \{0\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$

$\Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-field.

—x—

Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-fields. Further suppose $\mathcal{A}_1 \cup \mathcal{A}_2$ is a σ-field, then what is the relation between $\mathcal{A}_1$ & $\mathcal{A}_2$?

The relation is either $\mathcal{A}_1 \subset \mathcal{A}_2$ or $\mathcal{A}_2 \subset \mathcal{A}_1$

Proof: If possible suppose $\mathcal{A}_1 \not\subset \mathcal{A}_2$ or $\mathcal{A}_2 \not\subset \mathcal{A}_1$

$\Rightarrow \exists$ set $B_1 \in \mathcal{A}_1$ but $B_1 \notin \mathcal{A}_2$

& $\exists$ set $B_2 \in \mathcal{A}_2$ but $B_2 \notin \mathcal{A}_1$

Now

$B_1 \in \mathcal{A}_1 \Rightarrow B_1 \in \mathcal{A}_1 \cup \mathcal{A}_2$

$B_2 \in \mathcal{A}_2 \Rightarrow B_2 \in \mathcal{A}_1 \cup \mathcal{A}_2$

Now

$B_1 \cup B_2 \in \mathcal{A}_1 \because B_2 \notin \mathcal{A}_1$

$B_1 \cup B_2 \in \mathcal{A}_2 \because B_1 \notin \mathcal{A}_2$

$\Rightarrow B_1 \cup B_2 \notin \mathcal{A}_1 \cup \mathcal{A}_2$

$\Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not closed under finite union

$\Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not a field.

(18) $\Rightarrow$ $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-field.
Which is a contradiction.
Thus our assumption must be wrong.
$\Rightarrow$ $\mathcal{A}_1 \cup \mathcal{A}_2$ is a σ-field iff
either $\mathcal{A}_1 \subset \mathcal{A}_2$ or $\mathcal{A}_2 \subset \mathcal{A}_1$.

—x—

Given a class $\mathcal{C}$ of subsets of Ω, find the smallest σ-field containing $\mathcal{C}$.

→ Note that power set of Ω always contain $\mathcal{C}$ & $P(\Omega)$ is a σ-field.

Now to find smallest σ-field containing $\mathcal{C}$, let us consider the collection of all σ-fields, that contain $\mathcal{C}$.

i.e. let $\{\mathcal{A}_t, t \in T\}$ be such that $\mathcal{A}_t \supset \mathcal{C}$, $\forall t \in T$.

Let $\mathcal{A} = \bigcap_{t \in T} \mathcal{A}_t \supset \mathcal{C}$

We know that $\mathcal{A}$ is also a σ-field.
& $\mathcal{A} \subset \mathcal{A}_t \ \forall t \in T$
Thus $\mathcal{A}$ is the smallest σ-field containing $\mathcal{C}$.

—x—

<u>Def</u>: σ-field generated by class $\mathcal{C}$.

Let $\mathcal{C}$ be a non-empty collection of subsets of Ω. Then the smallest σ-field containing the class $\mathcal{C}$ is called the σ-field generated by class $\mathcal{C}$ & is denoted by $\sigma(\mathcal{C})$.

[<u>Remark</u>: If $\mathcal{C}$ itself is a σ-field then $\sigma(\mathcal{C}) = \mathcal{C}$.

But if $\mathcal{C}$ is simply a non-empty collection, to reach up to σ-field containing $\mathcal{C}$, we need to add all possiblely required sets, such that it becomes a σ-field.

e.g. let $A \subset \Omega$ & $\mathcal{C} = \{A\}$
then $\sigma(\mathcal{C}) = \{A, \phi, \Omega, A'\}$
Thus $\mathcal{C} \subset \sigma(\mathcal{C})$

—x—

(19) Let $\mathcal{C} = \{A, B\}$, $A, B \subset \Omega$

then $\sigma(\mathcal{C}) = \{\phi, \Omega, A, B, A', B', A\cup B, A\cap B, A\cup B', A'\cup B,$
$A'\cup B', A\cap B', A'\cap B, A'\cap B', \ldots \}$

Borel σ-field :

Let $\Omega = \mathbb{R}$. Then the σ-field defined over real line is known as Borel σ-field & is denoted by $\mathcal{B}$.

How to generate Borel σ-field?

Define

$$\mathcal{F}_1 = \{A \mid A \text{ is a finite union of sets of the type } (a, b], (-\infty, \alpha], (\beta, \infty)\}$$

$$a \leq b,\ a, b, \alpha, \beta \in \mathbb{R}.$$

Then $\mathcal{F}_1$ is a field. why?

e.g. $\cancel{(4,5) \cap (5,\infty] = \phi}$

$(-\infty\ \ 5] \cup (5, \infty) = \mathbb{R} \in \mathcal{F}_1$

Let $a = 5$, $b = 5$ then $(5, 5] = \phi \in \mathcal{F}_1$

Thus $\phi, \mathbb{R} \in \mathcal{F}_1$.

Further $\mathcal{F}_1$ is closed under finite unions.

$\Rightarrow \mathcal{F}_1$ is a field.

Then the σ field generated by $\mathcal{F}_1$ i.e $\sigma(\mathcal{F}_1)$ is known as the Borel σ-field.

Thus $\mathcal{B} = \sigma(\mathcal{F}_1)$

Borel σ-field can also be generated as follows.

Define the following 4 classes of subsets of $\mathbb{R}$.

$\mathcal{C}_1 = \{I \mid I = (a, b),\ -\infty < a \leq b < \infty\}$

$\mathcal{C}_2 = \{I \mid I = [a, b],\ -\infty < a \leq b < \infty\}$

$\mathcal{C}_3 = \{I \mid I = (a, b],\ -\infty < a \leq b < \infty\}$

$\mathcal{C}_3 = \{I \mid I = [a, b),\ -\infty < a \leq b < \infty\}$

<u>Remark</u>: Any member of any class can be written as member of other class.

e.g. let $A_n = (a, b - \frac{1}{n}]$, $n = 1, 2 \ldots$

then $A_n \in \mathcal{C}_3$. Note that $A_n \uparrow (a, b)$

i.e $\lim_n A_n = (a, b) \in \mathcal{C}_1$

i.e. set in $\mathcal{C}_1$ can be written as limits of sets in $\mathcal{C}_3$.

Similarly

$$\underset{\in \mathcal{C}_3}{(\alpha, \beta]} = \lim_{n\to\infty} \underset{\mathcal{C}_1}{(\alpha, \beta + \tfrac{1}{n})}$$

OR

$$\underset{\in \mathcal{C}_2}{[a \;\; b]} = \lim_{n\to\infty} \underset{\in \mathcal{C}_1}{\left(a - \tfrac{1}{n}, \; b + \tfrac{1}{n}\right)}$$

Thus it doesnot matter, among $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ & $\mathcal{C}_4$ which one we choose.

Then $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2) = \sigma(\mathcal{C}_3) = \sigma(\mathcal{C}_4) = \mathcal{B}$.

[Smallest σ-field containing $\mathcal{C}_1$ OR σ-field generated by $\mathcal{C}_1$]

This is how Borel σ-field is generated

—x—

Extended real line:

Real line $(-\infty, \infty) = \mathbb{R}$

Extended real line $[-\infty, \infty] = \overline{\mathbb{R}}$

σ- field over $\mathbb{R}$: Borel σ-field ($\mathcal{B}$)

σ- field over $\overline{\mathbb{R}}$: extended Borel σ-field ($\overline{\mathcal{B}}$)

—x—

(21) Def: Set $B \in \mathcal{B}$ are called Borel sets.

Examples of Borel sets: open sets, closed set, semi-open or semiclosed sets, singleton sets, set of all rationals, set of all irrationals etc are are all Borel sets.

* Remark: Note that Borel sets are always subset of $\mathbb{R}$, but every subset of $\mathbb{R}$ need not be a Borel set.

* Remark: To discribe a Borel set is highly impossible. We can simply give a number of examples of Borel sets.

—x—

Let Ω be some abstract space & let $\mathcal{A}$ be a fixed σ-field of subsets of Ω.

Then $(\Omega, \mathcal{A})$ is known as a Measurable space.

Similarly, $(\mathbb{R}, \mathcal{B})$ is also a measurable space.

Let $(\Omega, \mathcal{A})$ be a measurable space. Sets belonging to $\mathcal{A}$ are known as measurable sets (mble set).

e.g. let $A, B \subset \Omega$.

let $\mathcal{A}_1 = \{\phi, \Omega, A, A'\}$

& $\mathcal{A}_2 = \{\phi, \Omega, B, B'\}$

then A is mble w.r.t. $\mathcal{A}_1$ but not mble w.r.t. $\mathcal{A}_2$.

Thus whenever a measurable space is defined, the σ-field has to be kept fixed. Then the mble sets are also fixed.

Now the question is how to measure a mble set?

For this, we have the concept of set function, which assigns every mble set some value over $\mathbb{R}$ or $\overline{\mathbb{R}}$.

Set function:

Def: A function $\psi : \mathcal{A} \to \overline{\mathbb{R}}$ is known as a set function.

e.g. let $(\mathbb{R}, \mathcal{B})$ be a mble space

let $B = \{5, 7\}$, $C = \{2, 4, 8\}$ etc

1) define $\psi(B)$ = no. of elements in $B \Rightarrow \psi(B) = 2$, $\psi(C) = 3$ etc.