Integration:-

Integration of a mble function w.r.t. a measure μ:-

Let $f : \Omega \to \overline{\mathbb{R}}$ be a measurable function.

To define $\int_\Omega f\,d\mu$ where μ is a measure defined on $(\Omega, \mathcal{A})$.

So let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

let $f : \Omega \to \overline{\mathbb{R}}$. We will define $\int_\Omega f\,d\mu$ when

(i) f is non-ve simple function

(ii) f is non-ve mble function

(iii) f is any mble function.

(i) Let f be a non-ve simple function.

i.e $f = \sum_{j=1}^{n} a_j I_{A_j}$, $a_j \geq 0$ & distinct,

$A_i \cap A_j = \phi \;\; \forall i \neq j$ &

$\bigcup_{j=1}^{n} A_j = \Omega$, $A_j \in \mathcal{A} \;\; \forall j$

Def: Integral of a non-ve simple function $f = \sum_{j=1}^{n} a_j I_{A_j}$ is defined as

$$\int_\Omega f\,d\mu = \sum_{j=1}^{n} a_j\, \mu(A_j)$$

(ii) Let f be non-ve mble function.

We know that if $f \geq 0$, mble, then $\exists \{f_n\}$ such that $f_n \geq 0$, f_n simple & $0 \leq f_n \uparrow f$

then

Def

$$\boxed{\int_\Omega f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu}$$

let f be any mble function, then

Def. $\int f\,d\mu$ is said to exist, if

$$\int f^+ d\mu < \infty \quad \text{or} \quad \int f^- d\mu < \infty$$

and whenever it exists,

$$\int_\Omega f\,d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

[check for existance is necessary first]

[Note: $\int_\Omega f\,d\mu$ does not exist if $\int_\Omega f^{\pm} d\mu = \infty$.

e.g. let $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$

& $E(X) = \int x f(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} + \int_{0}^{\infty}\right]$$

$$-\infty \qquad \infty$$

Thus $\int x f(x)\,dx$ does not exist.]

Def: A mble function f is said to be integrable if both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite & then $\int f\,d\mu < \infty$.

Remark: It is necessary to show that the definition of integrals are unambiguous.

Properties of integrals :—

If $A \in \mathcal{A}$ & $A \subset \Omega$ then $\int_A f\,d\mu = \int_\Omega f I_A\,d\mu$

Let $\int f\,d\mu$, $\int g\,d\mu$ & $\int f\,d\mu + \int g\,d\mu$ exists.

(i.e either both finite or both $+\infty$ or $-\infty$ or one of them finite)

Then, we have the following properties of integrals.

(A) Linearity property:-

(i) $\int_{\Omega} (f+g)\,d\mu = \int_{\Omega} f\,d\mu + \int_{\Omega} g\,d\mu$

(ii) $\int_{A\cup B} f\,d\mu = \int_{A} f\,d\mu + \int_{B} f\,d\mu$, $A, B \in \mathcal{A}$, $A \cap B = \phi$

(iii) For a constant c, $\int cf\,d\mu = c\int f\,d\mu$

(B) Order preserving property.

(i) $f \geq 0 \Rightarrow \int f\,d\mu \geq 0$

(ii) $f \geq g \Rightarrow \int f\,d\mu \geq \int g\,d\mu$

(iii) $f = g$ a.e $\Rightarrow \int f\,d\mu = \int g\,d\mu$

—x—

Step I : Establish Def. 1 is full proof

II : " properties A & B for non-ve simple function

Step III : " Def 2 is full proof

IV : " properties A & B for non-ve mble function

V : " Def 3 is full proof

VI : " properties A & B for any mble function

—x—

Step I : Let f be a non-ve simple function. Suppose we write

$$f = \sum_{i=1}^{m} a_i I_{A_i} \quad \& \quad f = \sum_{j=1}^{n} b_j I_{B_j}$$

then $\int f\,d\mu = \sum_{i=1}^{m} a_i \mu(A_i)$ & also $\int f\,d\mu = \sum_{j=1}^{n} b_j \mu(B_j)$

To prove

$$\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{j=1}^{n} b_j \mu(B_j)$$

proof: Note $\omega \in A_i \cap B_j \Rightarrow a_i = b_j$

Consider $\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{i=1}^{m} a_i \mu(A_i \cap \Omega)$

$$= \sum_{i=1}^{m} a_i \mu\left(A_i \cap \left(\bigcup_{j=1}^{n} B_j\right)\right)$$

$$= \sum_{i=1}^{m} a_i \, \mu\left[\bigcup_{j=1}^{n} (A_i \cap B_j)\right]$$

$$= \sum_{i=1}^{m} a_i \sum_{j=1}^{n} \mu(A_i \cap B_j) \quad \because A_i \cap B_j \subset A_i \text{ are disjoint sets.}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \, \mu(A_i \cap B_j) \quad — ①$$

Similarly

$$\sum_{j=1}^{n} b_j \, \mu(B_j) = \sum_{j=1}^{n} \sum_{i=1}^{m} b_j \, \mu(A_i \cap B_j) \quad — ②$$

but $a_i = b_j$ on $A_i \cap B_j$

$$\Rightarrow \int f \, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \, \mu(A_i \cap B_j) = \sum_{j=1}^{n} \sum_{i=1}^{n} b_j \, \mu(A_i \cap B_j)$$

$$\text{i.e.} \quad \sum_{i=1}^{m} a_i \, \mu(A_i) = \sum_{j=1}^{n} b_j \, \mu(B_j)$$

Thus the definition of integral of a non-ve simple function is full proof.

—x—

Step II : A (i)

To prove $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

let $f = \sum_{i=1}^{m} a_i I_{A_i}$ & let $g = \sum_{j=1}^{n} b_j I_{B_j}$

then $f + g = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i + b_j) I_{A_i \cap B_j}$

$$\therefore \int (f+g) \, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i + b_j) \, \mu(A_i \cap B_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \, \mu(A_i \cap B_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} b_j \, \mu(A_i \cap B_j)$$

$$= \sum_{i=1}^{m} a_i \, \mu\left(A_i \cap \bigcup_{j=1}^{n} B_j\right) + \sum_{j=1}^{n} b_j \, \mu\left(\bigcup_{i=1}^{m} A_i \cap B_j\right)$$

$$= \sum_{i=1}^{m} a_i \, \mu(A_i) + \sum_{j=1}^{n} b_j \, \mu(B_j)$$

$$= \int f \, d\mu + \int g \, d\mu$$

—x—

A (ii) To prove $\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$

where $A \cap B = \phi$

Proof:

$\int_{A\cup B} f\,d\mu = \int f\, I_{A\cup B}\, d\mu$, but $I_{A\cup B} = I_A + I_B$

$\because A \cap B = \phi$.

$\therefore \int_{A\cup B} f\,d\mu = \int f\,(I_A + I_B)\,d\mu$

$= \int f I_A\,d\mu + \int f I_B\,d\mu$

$= \int_A f\,d\mu + \int_B f\,d\mu$

—x—

A (iii) Let C be a constant

To prove $\int cf\,d\mu = c\int f\,d\mu$

If $c = 0$ then $cf = 0$ & hence both sides are zero.

If $c > 0$ $(0 < c < \infty)$ then $f = \sum_{i=1}^{m} a_i I_{A_i}$

$\therefore cf = \sum_{i=1}^{m} (ca_i) I_{A_i}$

then $\int cf\,d\mu = \sum_{i=1}^{m} (ca_i)\mu(A_i)$

$= c\sum_{i=1}^{m} a_i\,\mu(A_i)$

$= c\int f\,d\mu$

—x—

(B) (i) To prove if $f \geq 0 \Rightarrow \int f\,d\mu \geq 0$

$f \geq 0 \Rightarrow$ all a_i's ≥ 0 also μ is non-ve

$\Rightarrow \int f\,d\mu = \sum_{i=1}^{m} a_i\,\mu(A_i) \geq 0$

(ii) $f \geq g \Rightarrow \int f\,d\mu \geq g\,d\mu$

$f \geq g \Rightarrow f - g \geq 0$

Now $f = f - g + g$

$\therefore \int f\,d\mu = \int (f-g)\,d\mu + \int g\,d\mu$

$\geq \int g\,d\mu$

(75) (iii) If $f=g$ a.e, then $\int f d\mu = \int g d\mu$

Let $A = \{f=g\}$ then $f=g$ a.e $\Rightarrow \mu(A')=0$

Now consider

$$\int (f-g)\, d\mu = \int_A (f-g) d\mu + \int_{A'} (f-g) d\mu$$

$$= 0 + \int (f-g).I_{A'}\, d\mu$$

$$= 0 + 0 \quad \because A_i \cap B_j \cap A' \subset A' \;\&\; \mu(A_i\cap B_j \cap A')=0$$

$\Rightarrow \int (f-g) d\mu = 0$

$\therefore \int f d\mu = \int (f-g+g) d\mu$

$$= \int (f-g) d\mu + \int g\, d\mu$$

$$= \int g\, d\mu$$

hence proved.

—x—

Step III : TP Def. (2) is full proof.

We accept this result without proof.

(~~A part of proof is as follows~~).

[Result :- Let $\{X_n\}$ be a non-decreasing sequence of non-ve simple functions s.t. $X_n \to X$. Let $X \geq Y$, where Y is non-ve simple f^n then $\lim \int X_n d\mu \geq \int Y d\mu$]

Assuming the above result holds, we can prove Def. (2) is full proof or unambiguous.

Def 2 : $0 \le f_n \uparrow f$, f_n simple $\forall n$ then

$$\int f d\mu = \lim_n \int f_n d\mu$$

To prove this def is full proof, if possible,

suppose $0 \le f_n \uparrow f$, f_n simple

$0 \le g_n \uparrow f$, g_n simple

To prove $\lim\limits_{n\to\infty} \int f_n d\mu = \lim\limits_{n\to\infty} \int g_n d\mu$

Note that $\forall$ fixed $K \geq 1$,

$$f \geq g_K, \quad g_K \text{ simple}$$

$\therefore$ by above result,

$$\lim_{n\to\infty} \int f_n d\mu \geq \int g_K d\mu \quad \forall K \geq 1$$

let $K \to \infty$ then

$$\lim_{n\to\infty} \int f_n d\mu \geq \lim_{K\to\infty} \int g_K d\mu \quad — (1)$$

On similar lines, we can prove that

$$\lim_{n\to\infty} \int g_n d\mu \geq \lim_{m\to\infty} \int f_m d\mu \quad — (2)$$

From (1) & (2) we have

$$\lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \int g_n d\mu$$

Thus $\int f d\mu$ is unambiguously defined if $f \geq 0$.

Step IV : To prove properties of integral of non-ve mble f^n:-

A(i) $\int (f+g) d\mu = \int f d\mu + \int g d\mu$

where f & $g \geq 0$, mble f^ns.

Recall $0 \leq f_n \uparrow f$, f_n simple, $0 \leq g_n \uparrow g$, g_n simple

then $0 \leq f_n + g_n \uparrow f+g$, $f_n + g_n$ simple

$$\therefore \int (f+g) d\mu = \lim_{n\to\infty} \int (f_n + g_n) d\mu$$

$$= \lim_{n\to\infty} \left[\int f_n d\mu + \int g_n d\mu \right]$$

$$= \lim_{n\to\infty} \int f_n d\mu + \lim_{n\to\infty} \int g_n d\mu$$

$$= \int f d\mu + \int g d\mu$$

hence proved.

A(ii) $\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$, $A \cap B = \phi$.

In fact, all properties A(ii), A(iii) & B(i), B(ii) B(iii) hold for non-ve mble f^n.

(can be proved on similar lines as A(i))

—x—

(77) Def 3 : Integral of a mble f^n.

Step V :

Let f be a mble function.

$\int f\,d\mu$ is said to exist if $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$. Then $\int f\,d\mu = \int f^+ d\mu - \int f^- d\mu$

This definition is unambiguous by itself, because $\int f^+ d\mu$ & $\int f^- d\mu$ are unambiguously defined.

Properties : Let f and g be any mble function s.t. $\int f\,d\mu$, $\int g\,d\mu$ & $\int f d\mu + \int g\,d\mu$ exists.

$\int f\,d\mu$ can be $\pm\infty$, $\int g\,d\mu$ can be $\pm\infty$ but

$\int f\,d\mu + \int g\,d\mu$ exists

$\Rightarrow$ $\int f\,d\mu$ & $\int g\,d\mu$ both can be $+\infty$

OR " " " $-\infty$

but not one $+\infty$ & the other $-\infty$.

Then A (i) $\int_\Omega (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$

Proof : Define

$A_1 = \{\omega \mid f \geq 0,\ g \geq 0\}$

$A_2 = \{\omega \mid f \geq 0,\ g < 0,\ f+g \geq 0\}$

$A_3 = \{\omega \mid f \geq 0,\ g < 0,\ f+g < 0\}$

$A_4 = \{\omega \mid f < 0,\ g < 0,\ f+g < 0\}$

$A_5 = \{\omega \mid f < 0,\ g \geq 0,\ f+g \geq 0\}$

& $A_6 = \{\omega \mid f < 0,\ g \geq 0,\ f+g < 0\}$

then $A_i \cap A_j = \phi \ \forall i \neq j$ & $\bigcup_i A_i = \Omega$

We shall prove that

$$\int_{A_K} (f+g)\,d\mu = \int_{A_K} f\,d\mu + \int_{A_K} g\,d\mu, \quad K = 1, 2, \cdots 6.$$

(78) For set A_1 & A_4 i.e for $k=1$ & 4, the result is obvious.

Suppose $k=2$

Write $f+g = f-(-g)$

i.e $f = f+g+(-g)$ on A_2

Here f, $f+g$ & $-g$ all are non-ve mble f^n's & hence

$$\int_{A_2} f\,d\mu = \int_{A_2}(f+g)\,d\mu + \int_{A_2} -g\,d\mu$$

$$= \int_{A_2}(f+g)\,d\mu - \int_{A_2} g\,d\mu$$

i.e $$\int_{A_2}(f+g)\,d\mu = \int_{A_2} f\,d\mu + \int_{A_2} g\,d\mu$$

On similar lines, we can prove that

$$\int_{A_K}(f+g)\,d\mu = \int_{A_K} f\,d\mu + \int_{A_K} g\,d\mu \quad \text{for } k=3,5,6.$$

$\therefore$ adding all such results, we have (& rearranging the terms)

$$\sum_{K=1}^{6}\int_{A_K}(f+g)\,d\mu = \sum_{K=1}^{6}\int_{A_K} f\,d\mu + \sum_{K=1}^{6}\int g\,d\mu$$

$$\Rightarrow \int_{\Omega}(f+g)\,d\mu = \int_{\Omega} f\,d\mu + \int_{\Omega} g\,d\mu$$

On similar lines, we can prove the remaining properties

—x—

<u>Integrability property</u> :—

A mble f^n f is said to be integrable if

$\int f^+ d\mu < \infty$ & $\int f^- d\mu < \infty$

In that case $\int f\,d\mu = \int f^+ d\mu - \int f^- d\mu < \infty$.

Now $|f| = f^+ + f^-$

$\therefore \int |f|\,d\mu = \int f^+ d\mu + \int f^- d\mu < \infty$

Note that in general $\int |f|\,d\mu$ always exists.

$\because$ it is either $< \infty$ or $= +\infty$

(79) Thus

i) f is integrable iff $\int |f| d\mu < \infty$.

ii) If $|f| \leq g$, g integrable then f is integrable

proof: $|f| \leq g \Rightarrow \int |f| d\mu \leq \int g \, d\mu < \infty$

$\Rightarrow \int |f| d\mu < \infty$

$\Rightarrow f$ is integrable

iii) $f = g$ a.e $\Rightarrow \int f d\mu = \int g d\mu$

Let $N = \{\omega \mid f = g\}$ then $N' = \{\omega \mid f \neq g\} \in$

$\mu(N') = 0$

Now

$$\int_{\Omega} f d\mu = \int_N f d\mu + \int_{N'} f d\mu = \int_N f d\mu$$

Similarly $$\int_{\Omega} g d\mu = \int_N g d\mu + \int_{N'} g d\mu = \int_N g d\mu$$

$\because \mu(N') = 0$

but on N, $f = g \Rightarrow \int_N f d\mu = \int_N g d\mu$

$$\Rightarrow \int_{\Omega} f d\mu = \int_{\Omega} g d\mu$$

—x—

Remark: The 2nd definition uses sequence of non-ve simple function $\{f_n\}$ s.t. $f_n \uparrow f$, then $\int f_n d\mu \to \int f d\mu$ where f is non-ve mble f^n.

What will happen if $\{f_n\}$ is a seq^n. of non-ve mble functions ?

The answer is in the following famous theorem.

Convergence Theorems :