

(34) Carathéodory Extension Theorem: -
(Only statement)

Statement: Suppose μ is a σ -finite measure defined on a field. Then there exists a unique measure $\bar{\mu}$ on the σ -field $\sigma(\mathcal{F})$, which is also σ -finite.

—x—
The above theorem will help to define Lebesgue measure and Lebesgue-Stieltjes measure

—x—
Lebesgue measure :-

Consider the measurable space $(\mathbb{R}, \mathcal{B})$. We want to define a measure on this space. Now to define a measure on \mathcal{B} is not practically possible, as we do not know the nature of \mathcal{B} . We know simply some types of Borel sets but not all. Hence we first define a measure on a field \mathcal{F} using above theorem extend it to \mathcal{B} .

Consider the field

$$\mathcal{F} = \{ I \mid I \text{ is a finite union of disjoint intervals of the type } (a, b], (-\infty, \alpha] \text{ \& } (\beta, \infty) \}$$

We know that \mathcal{F} is a field.

Define a set function on \mathcal{F} as follows.

$$\text{let } A \in \mathcal{F}. \text{ Then } A = \bigcup_{j=1}^N I_j,$$

where I_j 's are disjoint intervals of above types.

$$\text{Define } \mu(A) = \mu\left(\bigcup_{j=1}^N I_j\right) = \sum_{j=1}^N \mu(I_j)$$

$$\text{where } \mu\{(a, b]\} = b - a.$$

$$\mu(-\infty, \alpha] = \infty$$

$$\mu(\beta, \infty) = \infty$$

Then μ is a measure.

(35) μ is not finite.
but μ is surely σ -finite

Define $A_n = (n, n+1]$ then $\mu(A_n) = n+1 - n = 1 < \infty$
 $\mathbb{R} \subset \bigcup_{n=-\infty}^{\infty} A_n$

Thus μ is σ -finite.

Then by Caratheodory extension theorem,
 \exists a unique measure λ on $\mathcal{B} = \sigma(\mathcal{F})$
such that μ is a restriction of λ on \mathcal{F}
OR λ is extension of μ on \mathcal{B} .

$$\mathbb{R} \quad \lambda\{(a, b]\} = \mu\{(a, b]\} = b - a.$$

This measure λ is known as the
'Lebesgue measure'.

Sets which ~~are~~ can be measured by λ
are known as Lebesgue measurable sets.

→ Is every Borel set Lebesgue measurable?

The answer is 'Yes'.

Lebesgue measure of some Borel sets:-

(i) $B \in \mathcal{B}$ where B is a singleton set

$$\text{say } B = \{x\}, \quad x \in \mathbb{R}$$

$$\text{Now } \{x\} = \lim_{n \rightarrow \infty} (x - \frac{1}{n}, x]$$

$$\text{ii } \lambda(B) = \lambda\{x\} = \lambda\left\{\lim_{n \rightarrow \infty} (x - \frac{1}{n}, x]\right\}$$

$$= \lim_{n \rightarrow \infty} \lambda(x - \frac{1}{n}, x]$$

$$= \lim_{n \rightarrow \infty} x - (x - \frac{1}{n})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$= 0$$

(36)

Thus Lebesgue measure of every singleton set is zero.

$$(2) \text{ Let } B = [a \ b]$$

$$= \{a\} \cup (a \ b]$$

$$\text{ii } \lambda(B) = \lambda[\{a\} \cup (a \ b)]$$

$$= \lambda\{a\} + \lambda(a \ b]$$

$$= 0 + b - a$$

$$= b - a$$

$$(3) \text{ Let } B = [a \ b)$$

$$= \{a\} \cup (a \ b)$$

$$= \lim_{n \rightarrow \infty} [\{a\} \cup (a, b - \frac{1}{n}]]$$

$$\text{ii } \lambda(B) = \lambda\{a\} + \lim_{n \rightarrow \infty} \lambda(a, b - \frac{1}{n})$$

$$= 0 + \lim_{n \rightarrow \infty} (b - \frac{1}{n} - a)$$

$$= b - a$$

$$(4) \text{ Let } B = (a \ b)$$

$$\text{ii } = \lim_{n \rightarrow \infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

$$\text{ii } \lambda(B) = \lim_{n \rightarrow \infty} \lambda[a + \frac{1}{n}, b - \frac{1}{n}]$$

$$= \lim_{n \rightarrow \infty} (b - \frac{1}{n} - a - \frac{1}{n})$$

$$= b - a$$

Thus, whatever type of interval, we have, its Lebesgue measure is $b - a$.

In general

$$\lambda(A - B) = \lambda(A) - \lambda(B) \text{ if } B \subset A,$$

provided both $\lambda(A) \neq \lambda(B)$ are not $+\infty$.

—x—

(37) let $B = \text{Set of rationals}$
 $= \{r_1, r_2, \dots\}$

then $B = \bigcup_k A_k$ where $A_k = \{r_k\}$

i) $\lambda(B) = \lambda\left(\bigcup_k A_k\right) = \sum_k \lambda(A_k) = 0$

* The Lebesgue measure of the set of rationals is zero.

e.g. Let $B = \text{set of rationals between } (3, 10]$.

then $\lambda(B) = 0$

Let $B_1 = \text{Set of irrationals between } (3, 10]$

then

$$B_1 = (3, 10] - \{\text{the set of rationals between } (3, 10]\}$$

i) $\lambda(B_1) = \lambda\{(3, 10]\} - 0$

$$= 7$$

let $B_2 = \text{set of irrationals in } (-\infty, \infty)$

then $\lambda(B_2) = \infty$.

Remark: For whatever Borel set we want to find the Lebesgue measure, try to write that set in the form $(a, b]$.

(38) Lebesgue - Stieltjes measure (L-S measure)

Let F be a function on real line, such that F is non-decreasing, right continuous, $F(-\infty) = 0$ & $F(+\infty) = 1$.

(In probability theory, we call such a function as distribution function.)

Let \mathcal{F} be the field as defined above.

$\mathcal{F} = \{I \mid I \text{ is finite union of disjoint intervals of the type } (a, b], (-\infty, a], (b, \infty)\}$

Define a measure μ on \mathcal{F} as follows.

$$\mu\{(a, b]\} = F(b) - F(a) \geq 0 \quad (\because F \text{ is non-decreasing}).$$

$$\begin{aligned}\mu(\emptyset) &= \lim_{b \downarrow a} F(b) - F(a) \\ &= F(a^+) - F(a) \\ &= 0 \quad (\because F \text{ is right continuous})\end{aligned}$$

$$\mu(\beta, \infty) = \lim_{n \rightarrow \infty} \mu(\beta, n]$$

& for $I \in \mathcal{F}$,

$$\mu(I) = \mu\left(\bigcup_{j=1}^K I_j\right) = \sum_{j=1}^K \mu(I_j)$$

where I_j 's are intervals of above type.

$$\begin{aligned}\mu(\mathbb{R}) &= F(+\infty) - F(-\infty) \\ &= 1 - 0 \\ &= 1\end{aligned}$$

$\Rightarrow \mu$ is a finite measure & hence μ is σ -finite.

Hence by Caratheodory extension theorem, \exists a unique measure μ_F on \mathcal{B} , $\exists \mu_F\{(a, b]\} = F(b) - F(a)$.

This measure μ_F is known as the

'Lebesgue - Stieltjes measure' or L-S measure corresponding to F .

(39)

Ex: let $F = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } 0 \leq x < 1 \\ 3/2 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2. \end{cases}$

Note that $F(0^-) = 0$, $F(0) = F(0^+) = 1/2$.

$\Rightarrow x=0$ is a point of discontinuity &

all $x \neq 0$, $x \in \mathbb{R}$ is a continuity point.

Let us find L-S measure of different sets.

Let $B = \{x\}$, $x \in \mathbb{R}$.

Let $x=0$ i.e. $B = \{0\}$.

$$\text{then } \mu_F(B) = \mu_F\{0\}$$

$$= \lim_{n \rightarrow \infty} \mu_F(0 - 1/n, 0]$$

$$= \lim_{n \rightarrow \infty} F(0) - F(0 - 1/n)$$

$$= F(0) - F(0^-)$$

$$= 1/2 - 0$$

$$= 1/2$$

Let $x \neq 0$ say $x = 3/4$, $B = \{3/4\}$

$$\mu_F(B) = \mu_F\{3/4\}$$

$$= \lim_{n \rightarrow \infty} \mu_F(3/4 - 1/n, 3/4]$$

$$= \lim_{n \rightarrow \infty} [F(3/4) - F(3/4 - 1/n)]$$

$$= F(3/4) - \lim_{n \rightarrow \infty} F(3/4 - 1/n)$$

$$= F(3/4) - F(3/4^-)$$

$$= 1/2 - 1/2$$

$$= 0$$

Similarly if $x = 4$, $\mu_F\{4\} = 0$.

Thus the LS measure of every continuity pt. of F is zero.

& For points of discontinuity,

L-S measure = height of jump at that point.

(40)

Next, suppose $B = [a \ b]$

$$= \{a\} \cup (a \ b)$$

$$\begin{aligned} \text{i) } \mu_F(B) &= \mu_F\{a\} + \mu_F(a \ b) \\ &= \mu_F\{a\} + F(b) - F(a) \end{aligned}$$

$$\begin{aligned} \text{Let } B &= (a \ b) \\ &= [a \ b] - \{b\} \end{aligned}$$

$$\text{i) } \mu_F(B) = \mu_F[a \ b] - \mu_F\{b\}$$

$$\begin{aligned} \text{Let } B &= [a \ b] \\ &= (a \ b) + \{a\} - \{b\} \\ \text{i) } \mu_F(B) &= \mu_F(a \ b) + \mu_F\{a\} - \mu_F\{b\} \\ &= F(b) - F(a) + \mu_F\{a\} - \mu_F\{b\}. \end{aligned}$$

So e.g. let $B = [0 \ 3]$

$$\begin{aligned} &= \{0\} \cup (0 \ 3) \\ \text{i) } \mu_F(B) &= \mu_F\{0\} + (F(3) - F(0)) \\ &= \frac{1}{2} + 1 - \frac{1}{2} \\ &= 1 \end{aligned}$$

$$\text{let } B = (0, 3) = [0 \ 3] - \{3\}$$

$$\begin{aligned} \text{i) } \mu_F(B) &= \mu_F[0, 3] - \mu_F\{3\} \\ &= F(3) - F(0) - 0 \\ &= 1 - \frac{1}{2} - 0 \\ &= \frac{1}{2} \end{aligned}$$

$$B = \left(\frac{3}{2}, \frac{7}{2}\right]$$

$$\mu_F(B) = F\left(\frac{7}{2}\right) - F\left(\frac{3}{2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

—x—

(41) Let $F(x) = \begin{cases} 0 & x < -3 \\ 1/4 & -3 \leq x < 0 \\ \frac{x+1}{2} & 0 \leq x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$

Find L-S measure of the following sets.

$A = \{0, 1/2\}$, $B = [0, 1/2]$, $C = (-3, 0)$, $D = (0, 1/2)$
 $E = [-3, 0]$.

Here we note that

x	$F(x^-)$	$F(x)$	$F(x^+)$
-3	0	$1/4$	$1/4$
0	$1/4$	$1/2$	$1/2$
$1/2$	$3/4$	1	1

Thus $x = -3, 0, 1/2$ are points of discontinuity & all $x \in \mathbb{R}$ except $x = -3, 0, 1/2$ are continuity points of F .

i) $\mu_F \{-3\} = \frac{1}{4} - 0 = 1/4$

$\mu_F \{0\} = \frac{1}{2} - \frac{1}{4} = 1/4$

$\mu_F \{1/2\} = 1 - 3/4 = 1/4$

ii) $\mu_F(A) = \mu_F \{0, 1/2\} = \mu_F \{0\} + \mu_F \{1/2\}$
 $= 1/4 + 1/4$
 $= 1/2$

$\mu_F(B) = \mu_F [0, 1/2]$
 $= \mu_F \{0\} + \mu_F (0, 1/2]$
 $= \frac{1}{4} + F(1/2) - F(0)$
 $= \frac{1}{4} + 1 - \frac{1}{2}$
 $= 3/4$

$\mu_F(C) = \mu_F(-3, 0)$
 $= \mu_F \{(-3, 0] - \{0\}\}$
 $= \mu_F(-3, 0] - \mu_F \{0\}$
 $= F(0) - F(-3) - 1/4$
 $= 1/2 - 1/4 - 1/4$
 $= 0$

$$(42) \mu_F(\mathbb{D}) = \mu_F(0, 1/2)$$

$$= \mu_F(0, 1/2] - \mu_F\{1/2\}$$

$$= F(1/2) - F(0) - 1/4$$

$$= 1 - \frac{1}{2} - \frac{1}{4}$$

$$= 1/4$$

$$\mu_F(\mathbb{E}) = \mu_F([-3, 0])$$

$$= \mu_F\{-3\} + \mu_F(-3, 0]$$

$$= 1/4 + F(0) - F(-3)$$

$$= 1/4 + 1/2 - 1/4$$

$$= 1/2$$

$$\text{let } F(x) = \begin{cases} 0 & , \text{if } x < 0 \\ 1/2 & , \text{if } 0 \leq x < 1 \\ 1 - \frac{1}{2}e^{-x+1} & , \text{if } x \geq 1. \end{cases}$$

Find L-S measure of the sets $\{0\}$, $\{0, 1\}$, $[0, 1]$, $(0, 1)$.

Here $F(0^-) = 0$, $F(0) = F(0^+) = 1/2$.

$x=0$ is the only point of discontinuity & $\forall x \neq 0$ is continuity point of F .

$$i) \mu_F\{0\} = \frac{1}{2} - 0 = 1/2 \text{ \& } \mu_F\{x\} = 0 \text{ } \forall x \neq 0.$$

$$ii) \mu_F\{0, 1\} = \frac{1}{2} + 0 = 1/2$$

$$\mu_F[0, 1] = \mu_F\{0\} + \mu_F(0, 1]$$

$$= \frac{1}{2} + F(1) - F(0)$$

$$= \frac{1}{2} + \frac{1}{2} - 1/2$$

$$= 1/2$$

$$\mu_F(0, 1) = \mu_F(0, 1] - \mu_F\{1\}$$

$$= F(1) - F(0) - 0$$

$$= 1/2 - 1/2$$

$$= 0 \rightarrow x \rightarrow$$

(43) Let $\Omega = \mathbb{R}$

let $\mathcal{A} = (-\infty, 0] \in \mathcal{C} = \{\Omega, \emptyset, A, A'\}$

then \mathcal{C} is a field.

Define μ on \mathcal{C} as $\mu(B) = \begin{cases} 1 & \text{if } 4 \in B \\ 0 & \text{if } 4 \notin B \end{cases}$

then $\mu(-\infty, 0] = 1$, $\mu(\emptyset) = 0$, $\mu(A) = 0$ & $\mu(A') = 1$.

Now consider a set $C = (-5, 2) \subset A$.

but $\mu(A) = 0$

$\Rightarrow \mu(C) = 0$ $\because C \notin \mathcal{C}$ & μ is defined only for sets in \mathcal{C} & not for any other sets.

In fact, here $\mu(C)$ is not defined.

So to rectify this, define a σ -field \mathcal{C}^* as

$$\mathcal{C}^* = \sigma[\mathcal{C} \cup \{\text{all subsets of } A\}]$$

[to get rid of the problem that subsets of set of measure zero are not measurable, we need to define \mathcal{C}^* as above.]

Def: A measure μ is called a complete measure if $\mu(A) = 0$, $B \subset A \Rightarrow B$ is a measurable set & hence $\mu(B) = 0$.

Thus $(\mathbb{R}, \mathcal{A}, \mu)$ is known as complete measure space, if for mble set $A \in \mathcal{A}$ of measure zero, all subsets of A are measurable.

A set of measure zero i.e. $\mu(A) = 0$ is known as a null set (or μ null set).

Remark: Any measure space, can be completed by the procedure of defining \mathcal{C}^* .

—x—