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UNIT - 1

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* Multivariate Data :-

- Let there be a sample of size 'n' drawn randomly from a popⁿ and from each sample object the values of 'p' variable are observed

- Let us assume that each variable is measured on 'k' ocassions and the value of the j^{th} variable measured on k^{th} ocassions from i^{th} object be

$$x_{ijl} = \left(i = 1, 2, \ldots, n \;,\; j = 1, 2, \ldots, p \;,\; l = 1, 2, \ldots, k \right)$$

Object	Ocassion	Values of the variable						
		1	2	---	j	---	p
1	1	x_{111}	x_{121}	---	x_{1j1}	---	x_{1p1}
	2						
	⋮	⋮					
	k	x_{11k}	x_{12k}	---	x_{1jk}	---	x_{1pk}
		⋮					
2	1						
	2	⋮					
	⋮						
	k						
n	1	x_{n11}	x_{n21}	---	x_{nj1}	---	x_{np1}
	2	x_{n12}	x_{n22}	---	x_{nj2}	---	x_{np2}
	⋮	⋮					
	k	x_{n1k}	x_{n2k}	---	x_{njk}	---	x_{npk}

- The above elements x_{ijl} values in 'nk' rows and 'p' colⁿ is known as super matrix [cattel (1952)]

- In practice most of the multivariate data are recorded on only one occassion $k = 1$ and the observed data are arranged in 'n' rows and 'p' colms

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1j} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2j} & \cdots & x_{2p} \\ \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nj} & \cdots & x_{np} \end{bmatrix}_{n \times p}$$

The X matrix of order $(n \times p)$ can also be written as

$$X = (x_{ij}) = \begin{bmatrix} \underline{X}_1' \\ \underline{X}_2' \\ \vdots \\ \underline{X}_i' \\ \vdots \\ \underline{X}_n' \end{bmatrix} = \begin{bmatrix} \underline{X}_{(1)} & \underline{X}_{(2)} & \cdots & \underline{X}_{(p)} \end{bmatrix}$$

where

$$\underline{X}_i' = \begin{bmatrix} x_{i1} & x_{i2} & \cdots & x_{ip} \end{bmatrix}$$

$$\underline{X}_j' = \begin{bmatrix} x_{1j} & x_{2j} & \cdots & x_{nj} \end{bmatrix}$$

- Here $\underline{X}_i$ is a vector of values of p-variables observed from i^{th} object $(i = 1, 2, \dots, n)$
$\underline{X}_j$ is a vector of values of j^{th} variable observed from n objects $(j = 1, 2, \dots, p)$

- The Collected information is usually represented by a data vector $X = [\underline{X}_1 \ \underline{X}_2 \ \cdots \ \underline{X}_p]$

Ex :- Socioeconomic status $(x=0)$

Object	No. of new born children (x_1)	No. of dead children under age5 (x_2)	Level of Education	
			M (x_3)	F (x_4)

* **Data Summary :-**

* **Mean vector :-** The Mean of j^{th} variable of 'n' objects is

$$\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij} \qquad j = 1, 2, \ldots, p$$

Mean of all p - variables is represented by $\bar{X}$.
where $\bar{X} = [\bar{X}_1 \ \bar{X}_2 \ \ldots \ \bar{X}_p]'$

In matrix notation, $\bar{X} = n^{-1} X' \underline{1}$,
where, $\underline{1} = [1 \ 1 \ \ldots \ 1]'$

- $\bar{X}$ is called Centroid (Mean)

- The Mean vector represented above are called Sample Mean vector.

- let the j^{th} element of data vector $X = [X_1 \ X_2 \ldots X_p]$ $(j = 1, 2, \ldots, p)$ is a continuous r.v. with mean $\mu_j = E(x_j)$ &
Variance $\sigma_{jj} = E[x_j - \mu]^2$ so that random vector X follows Multivariate with
mean vector $E(X) = \mu = [\mu_1 \ \mu_2 \ \ldots \ \mu_p]'$ and
Var - Cov Matrix $\Sigma = E[(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})']$

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$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & & & \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}_{p \times p}$$

Dispersion $\rightarrow$ Matrix.

- Here $\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$ $i \neq j = 1, 2, \ldots p$ are called Covariances of i^{th} & j^{th} variable and $\mu \leftarrow$ pop^n mean vector.

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* <u>Sample Var-Cov Matrix</u> :-

$$S = (S_{ij}) = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ S_{21} & S_{22} & \cdots & S_{2p} \\ \vdots & & & \\ S_{p1} & S_{p2} & \cdots & S_{pp} \end{bmatrix}_{p \times p}$$

$$S_{jj} = \frac{1}{n} \sum_{j=1}^{n} (x_{jl} - \bar{x}_j)^2, \quad S_{ij} = \frac{1}{n} \sum_{i=1}^{n} (x_{il} - \bar{x}_i)(x_{jl} - \bar{x}_j)$$

$$S = n^{-1} X'X - \bar{X}\bar{X}'$$

* <u>Data Transformation</u> :-

- Multivariate Data are large in volume and this are analyse to infer about the parameter vector or pop^n.

- The variables under study are measured in different units and this differentials in unit show variations in the amount of variances of variables. Hence, Data Transformation is needed which provides variables of unique variances.

The transformation is taken in
(1) Linear Combination of variables.
(2) Scaling transformation
(3) Mahalanobis transformation
(4) Orthogonal transformation
(5) Principal Component transformation
(6) Lower triangular transformation.

(1) Linear Combination of variables

- let there be 'p' variables in the data vector $\underline{X}$, where $E(\underline{x}) = \underline{\mu}$, $Var(\underline{x}) = \Sigma$
- Assume sample Var-Cov Matrix is S

- let $\underline{a} = [a_1 \; a_2 \; --- \; a_p]'$ be a vector of known elements. Then $\underline{Y} = \underline{a}'\underline{x}$ is a linear combination of variables.

Sample Mean $= \bar{Y} = \underline{a}'\bar{\underline{x}}$
Sample Variance $= \underline{a}' S \underline{a}$

- The i^{th} value of Y is
$$Y_i = a_1 x_{i1} + a_2 x_{i2} + --- + a_p x_{ip} \; ; \; i = 1, 2, ---, n$$

- The above transformation of variables is 1-dimensional. In practice, multi-dimensional transformation has wide use so Multi-dim. transformation is given by
$$Y = XA' + \underline{1}\underline{b}'$$
where A is a $(q \times p)$ matrix of known elements

$\underline{b}$ is a col^n vector of 'q' known elements.

$\underline{\bar{Y}} = A\underline{\bar{X}} + \underline{b}$

(2) <u>Scaling transformation</u>

- This transformation changes the variable to a new one of unique variance.

- let $D = diag(\Sigma_j)$, where $\Sigma_j = \sqrt{\Sigma_{jj}}$; $j = 1, 2, \ldots p$

- The random vector X transforms to a new random vector Y by scaling transformation if

$$\underline{Y} = D^{-1}(\underline{X} - \underline{\mu})$$

$$E(Y) = D^{-1} E[\underline{X} - \underline{\mu}] = D^{-1}[E(X) - \underline{\mu}] = 0$$

$$V(\underline{Y}) = D^{-1} E[\underline{X} - \underline{\mu}][\underline{X} - \underline{\mu}]' D^{-1} = D^{-1}\Sigma D^{-1}$$

(3) <u>Mahalanobis transformation</u>

- If $\Sigma > 0$, then Σ^{-1} has a unique symmetric positive definite square root $\Sigma^{-1/2}$

- Mahalanobis transformation is defined by

$$Y = \Sigma^{-1/2}(\underline{X} - \bar{X}) \text{ where}$$

$$E(\underline{Y}) = 0 \quad \& \quad V(\underline{Y}) = I$$

so that the transformation eliminates the correlation bet^n the variables and standardises the variance of each variable

4) <u>Orthogonal transformation.</u>

- let Γ be an orthogonal matrix of order $p \times p$

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such that $\Gamma'\Gamma = \Gamma\Gamma' = I$
Then the transformation of a random vector $\underline{x}$ to a new vector $\underline{y} = \Gamma \underline{x}$ is called Orthogonal Transformation

$E(\underline{y}) = \Gamma \underline{\mu}$
$V(\underline{y}) = \Gamma \Sigma \Gamma'$

(5) Principal Component transformation

- let $\lambda_1 > \lambda_2 \cdots > \lambda_p > 0$ be the eigen values of the Cov Matrix Σ where $Var(\underline{x}) = \Sigma$, & $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be the corresponding eigen vectors. Then $\Gamma = [\Gamma_1 \Gamma_2 \cdots \Gamma_p]$ is a matrix formed with eigen vectors of Σ. If λ_i's are distinct & greater than 0, Γ will be an orthogonal matrix in such a case Σ can be written as $\Sigma = \Gamma \Lambda \Gamma'$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$
then $\underline{y} = \Gamma'(\underline{x} - \underline{\mu})$ is Principal Component transformation of x and y.

$E(\underline{y}) = 0$, $Var(\underline{y}) = \Gamma' \Sigma \Gamma = \Lambda$

(6) Lower Triangular Transformation

For any matrix Σ one can determine a lower triangular matrix C

$$C = \begin{bmatrix} C_{11} & 0 & 0 & \cdots & 0 \\ C_{21} & C_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ C_{p1} & C_{p2} & C_{p3} & \cdots & C_{pp} \end{bmatrix}_{p \times p}$$

Such that
$$\Sigma = CC'$$

Then the transformation of $\underline{x}$ to

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$\underline{y} = C^{-1}(\underline{x} - \underline{\mu})$ is called Lower Triangular Transformation.

* **Multivariate Normal distⁿ :-**

- let X be a p-dimensional random vector such that $X = [X_1 \; X_2 \; --- \; X_p]$, the distⁿ of X is called Multivariate Normal distⁿ if the joint density functⁿ of X is

$$f(x) = f(x_1, x_2, ..., x_p)$$

$f(x) > 0$

$$= [2\pi \Sigma]^{1/2} \exp\left[-\frac{1}{2}(\underline{x} - \underline{\mu})\Sigma^{-1}(\underline{x} - \underline{\mu})'\right]$$

$$= \frac{|V|^{1/2}}{(2\pi)^{p/2}} \exp\left[-\frac{1}{2}(\underline{x} - \underline{\mu})' V (\underline{x} - \underline{\mu})'\right]$$

H.W

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where $V = (v_{ij}) = \Sigma^{-1}$ is a positive definite matrix.
Here $X \sim N_p(\underline{\mu}, \Sigma)$; $-\infty < \mu_j < \infty$

$-\infty < X_j < \infty$; $j = 1, 2, ... p$

- The function $f(x)$ is a density functⁿ since V is +ve so $\frac{V^{1/2}}{(2\pi)^{p/2}}$ is positive. Also, any power of exponent is positive. Now, we need to show

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} --- \int_{-\infty}^{\infty} f(x_1, x_2, ..., x_p)\, dx_1\, dx_2 --- dx_p = 1$$

let us make a transformation $Z = X - \underline{\mu}$ such that $z_j = x_j - \mu_j$, $j = 1, 2, ..., p$.
The transformation of X to Z gives Jacobian as unity since $\frac{dx_j}{dz_k} = 0$ & $\frac{dx_j}{dz_j} = 1$
$(j \neq k)$

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$$\therefore f(z) = \frac{|V|^{1/2}}{(2\pi)^{p/2}} e^{-\frac{1}{2} z' V z}$$

Now

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z)\,dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{|V|^{1/2}}{(2\pi)^{p/2}} e^{-\frac{1}{2} z' V z}\, dz_1 \cdots dz_p$$

Here the transformation of X to Z is one-to-one.

H.W – Consider an orthogonal transformation of Z to T by $Z = PT$ where P is an orthogonal matrix.

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– The pdf of a univariate normal dist^n is of the form

$$f(x) = k e^{-\frac{1}{2} a (x-l)^2}$$

$$= k e^{-\frac{1}{2} (x-l) a (x-l)} \quad \text{if } x \in R$$

where $l \in R$, $a > 0$, $k > 0$

generalising concept, the pdf of a multivariate normal is taken as

$$f(\underline{x}) = k e^{-\frac{1}{2} (\underline{x} - \underline{l})' A (\underline{x} - \underline{l})}$$

if $\underline{x} \in R^p$, $\underline{l} \in R^p$, A is P.d matrix & $k > 0$ (Positive definite).

– Our objective is to find the constant $\underline{l}$ and A in terms of the moments of a Multivariate Normal dist^n.

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- Let $\underline{x}$ be a Multivariate Normal Variate with the pdf $f(\underline{x}) = K e^{-\frac{1}{2}(\underline{x}-\underline{\ell})' A (\underline{x}-\underline{\ell})}$, $x \in R^p$.

Now $\int_{R^p} f(\underline{x})\, d\underline{x} = 1$

$$\Rightarrow K \int_{R^p} e^{-\frac{1}{2}(\underline{x}-\underline{\ell})' A (\underline{x}-\underline{\ell})}\, d\underline{x} = 1$$

(let $\underline{y} = A'(\underline{x} - \underline{\ell})$

A is P.d. Matrix $\exists$ a non-singular matrix $P \ni PP' = A$ or $P'AP = I_p$)

$$\Rightarrow K \int_{R^p} e^{-\frac{1}{2}\underline{y}'\underline{y}} \cdot \frac{1}{|J|}\, d\underline{y} = 1$$

$$\left(|J| = \left| J\left(\frac{\underline{y}}{\underline{x}}\right) \right| = |P'| = \sqrt{|PP'|} = \sqrt{|A|} \right)$$

$$\Rightarrow \frac{K}{\sqrt{|A|}} \int_{R^p} e^{-\frac{1}{2}\underline{y}'\underline{y}}\, d\underline{y} = 1$$

$$\Rightarrow \frac{K}{\sqrt{|A|}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\Sigma y_i^2}\, dy_1\, dy_2 \cdots dy_p = 1$$

$$\Rightarrow \frac{K}{\sqrt{|A|}} \prod_{i=1}^{p} \left\{ \int_{-\infty}^{\infty} e^{-\frac{1}{2}y_i^2}\, dy_i \right\} = 1$$

$$\Rightarrow \frac{K}{\sqrt{|A|}} \cdot \left(\sqrt{2\pi}\right)^p = 1 \qquad \left(\because Z \sim N(0,1) \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = 1 \right)$$

$$\Rightarrow \boxed{K = \frac{\sqrt{|A|}}{\left(\sqrt{2\pi}\right)^p}}$$

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Now the pdf of $\underline{y}$ is

$$g(\underline{y}) = \frac{1}{(\sqrt{2\pi})^p} e^{-\frac{1}{2}\underline{y}'\underline{y}}, \quad \underline{y} \in R^p$$

$$E(y_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ y_i \cdot \frac{1}{(\sqrt{2\pi})^p} e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2} dy_1 \, dy_2 \cdots dy_p \right\}$$

$$= \left(\int_{-\infty}^{\infty} y_i \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}y_i^2} dy_i \right) \prod_{j \neq i} \left\{ \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}y_j^2}}{\sqrt{2\pi}} dy_j \right\}$$

$$\boxed{E(y_i) = 0}$$

$$Cov(y_i, y_j) = E(y_i y_j) - E(y_i)E(y_j)$$
$$= E(y_i y_j) \quad (\because E(y_i) = 0)$$

$$\Rightarrow Cov(y_i, y_j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} y_i y_j \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^p e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2} dy_1 \, dy_2 \cdots dy_p$$

$$= \int_{-\infty}^{\infty} \left(y_i^2 \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^p e^{-\frac{1}{2}y_i^2} \right) \left(\prod_{j \neq i} \left\{ \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right)^p e^{-\frac{1}{2}y_j^2} dy_j \right\} \right) \text{ if } i = j$$

$$= \left\{ \left(\int_{-\infty}^{\infty} y_i \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i^2} dy_i \right) \left(\int_{-\infty}^{\infty} y_j \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_j^2} dy_j \right) \left(\prod_{k \neq i \neq j} \cdots \right) \right\} \text{ if } i \neq j$$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

($i = j$ means variance & $i \neq j$ means mean)

$$\therefore E(\underline{y}) = 0 \quad \& \quad Disp(\underline{y}) = I_p$$

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Since $E(\underline{y}) = 0$

$\underline{y} = P'(\underline{x} - \underline{l})$

$E(\underline{y}) = E[P'(\underline{x} - \underline{l})] = P'[E(\underline{x}) - \underline{l}] = 0$

$\Rightarrow E(\underline{x}) - \underline{l} = 0$

$\Rightarrow E(\underline{x}) = \underline{l} = \underline{\mu}$ (say)

$D(\underline{y}) = I_p$

$D[P'(\underline{x} - \underline{l})] = I_p$

$\Rightarrow P' D(X) P = I_p$

$\Rightarrow D(\underline{x}) = (P')^{-1} P^{-1}$

$= (PP')^{-1} = A^{-1}$

$\Rightarrow A = \Sigma^{-1}$ where $\Sigma = D(\underline{x})$

Hence $K = \dfrac{1}{(\sqrt{2\pi})^p |\Sigma|}$

Def^n:- A random vector $X_{(p\times 1)}$ is said to have a multivariate Normal dist^n with mean vector $E(\underline{x}) = \underline{\mu}$ & dispersion matrix $= \Sigma$ if the pdf of $\underline{x}$ is given by

$$f_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})} \quad ; \underline{x} \in R^p$$

we write $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ (density funct^n of p-variate Normal)

* Remarks :-

(1) The pdf of $f_{\underline{x}}(\underline{x})$ is maximum if the exponent is minimum i.e. $(\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu})$ is minimum.

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Note that $(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu}) > 0$ if $\underline{x} \neq \underline{\mu}$

$= 0$ if $\underline{x} = \underline{\mu}$

since Σ^{-1} is positive definite.

Hence $\underline{x} = \underline{\mu}$ is the mode of the distⁿ.

Let $\varphi(\underline{x}) = (\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})$

then $\frac{\partial \varphi(\underline{x})}{\partial \underline{x}} = 0$ at $\underline{x} = \underline{\mu}$

(2) Note that the exponent

$$\varphi(\underline{x}) = (\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})$$
$$= \underline{x}'\Sigma^{-1}\underline{x} - \underline{x}'\Sigma^{-1}\underline{\mu} - \underline{\mu}'\Sigma^{-1}\underline{x} + \underline{\mu}'\Sigma^{-1}\underline{\mu}$$
$$= \underline{x}'\Sigma^{-1}\underline{x} - 2\underline{x}'\Sigma^{-1}\underline{\mu} + \underline{\mu}'\Sigma^{-1}\underline{\mu}$$

Let $\varphi^*(\underline{x}) = \sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij}x_i x_j + \sum_{i=1}^{p} l_i x_i + e_i$

be the exponent of a multivariate distⁿ pdf.

Note that $\varphi^*(\underline{x}) = \underline{x}'A\underline{x} + \underline{x}'\underline{l} + e$

Comparing we get

$$A = \Sigma^{-1} \Rightarrow \boxed{\Sigma = A^{-1}}$$

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Thm (1): If $\underline{X}_{p\times 1} \sim N_p(\underline{\mu}, \Sigma)$ then for a non-singular matrix $P_{p\times p}$

$$P'\underline{X} \sim N_p(P'\underline{\mu}, P'\Sigma P)$$

Proof: The pdf of $\underline{x}$ is

$$f(\underline{x}) = \frac{1}{(2\pi)^{p/2}\sqrt{|\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})}, \quad \underline{x} \in \mathbb{R}^p$$

let $\underline{Y} = P'\underline{X} \Rightarrow \underline{X} = (P')^{-1}\underline{Y}$

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$$|J| = \left|\frac{\partial \underline{y}}{\partial \underline{x}}\right| = ||P'|| = \sqrt{|P'P|}$$

The pdf of $\underline{y}$ is

$$g(\underline{y}) = \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} e^{-\frac{1}{2}\{(P')^{-1}\underline{y} - \underline{\mu}\}' \Sigma^{-1} \{(P')^{-1}\underline{y} - \underline{\mu}\}} \cdot \frac{1}{\sqrt{|P'P|}}$$

$$= \frac{1}{(\sqrt{2\pi})^p \sqrt{|P'\Sigma P|}} e^{-\frac{1}{2}(\underline{y} - P'\underline{\mu})' P^{-1} \Sigma^{-1} (P')^{-1} (\underline{y} - P'\underline{\mu})}$$

$$= \frac{1}{(\sqrt{2\pi})^p \sqrt{|P'\Sigma P|}} e^{-\frac{1}{2}(\underline{y} - P'\underline{\mu})' (P'\Sigma P)^{-1} (\underline{y} - P'\underline{\mu})}$$

$$\therefore \underline{y} \sim N_p(P'\underline{\mu}, P'\Sigma P)$$

* <u>Mgf :-</u>

The mgf of a p-dimensional r.v. $\underline{X}$ is defined as $M_{\underline{X}}(\underline{t}) = E[e^{\underline{t}'\underline{x}}]$ provided it exists, where $\underline{t}$ belongs to a region containing the origin as an interior point.

m (2) :- If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ then the mgf of $\underline{X}$ is $M_{\underline{X}}(\underline{t}) = e^{\underline{t}'\underline{\mu} + \underline{t}'\Sigma\underline{t}}$ where $\underline{t}$ belongs to a region containing the origin as an interior point.

Proof: $$M_{\underline{X}}(\underline{t}) = E[e^{\underline{t}'\underline{x}}]$$

$$= \int_{R^p} e^{\underline{t}'\underline{x}} f(\underline{x})\, d\underline{x}$$

$$= \int_{R^p} e^{\underline{t}'\underline{x}} \cdot \frac{1}{\sqrt{(2\pi)^p}\sqrt{|\Sigma|}} e^{-\frac{1}{2}(\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu})} d\underline{x}$$

$$= \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} \int_{R^p} e^{-\frac{1}{2}\{\underline{x}'\Sigma^{-1}\underline{x} - \underline{x}'\Sigma^{-1}\underline{\mu} - \underline{\mu}'\Sigma^{-1}\underline{x} + \underline{\mu}'\Sigma^{-1}\underline{\mu}\}} \cdot e^{\underline{t}'\underline{x}} \, d\underline{x}$$

($\because$ $\underline{x}'$ and $\underline{x}$ are symmetric.

$$= \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} \int_{R^p} e^{-\frac{1}{2}\{\underline{x}'\Sigma^{-1}\underline{x} - 2\underline{x}'\Sigma^{-1}\underline{\mu} + \underline{\mu}'\Sigma^{-1}\underline{\mu} - 2\underline{t}'\underline{x}\}} \, d\underline{x}$$

$$= \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} \int_{R^p} e^{-\frac{1}{2}\{\underline{x}'\Sigma^{-1}\underline{x} - 2\underline{x}'(\Sigma^{-1}\underline{\mu} + \underline{t}') + \underline{\mu}'\Sigma^{-1}\underline{\mu}\}} \, d\underline{x}$$

$$= \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} \int_{R^p} e^{-\frac{1}{2}\{\underline{x}'\Sigma^{-1}\underline{x} - 2\underline{x}'(\Sigma^{-1}(\underline{\mu} + \Sigma\underline{t})) + \underline{\mu}'\Sigma^{-1}\underline{\mu}\}} \, d\underline{x}$$

$$= \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} \int_{R^p} e^{-\frac{1}{2}(\underline{x} - \underline{\mu} - \Sigma\underline{t})^{-1'}\Sigma^{-1}(\underline{x} - \underline{\mu} - \Sigma\underline{t}) + \underline{t}'\underline{\mu} - \underline{t}'\Sigma\underline{t}} \, d\underline{x}$$

$$(\because \underline{x}'\Sigma^{-1}\underline{x} - 2\underline{x}\Sigma^{-1}(\underline{\mu} + \Sigma\underline{t}) + \underline{\mu}'\Sigma^{-1}\underline{\mu}$$
$$= \underline{x}'\Sigma^{-1}\underline{x} - 2\underline{x}\Sigma^{-1}(\underline{\mu} + \Sigma\underline{t}) + (\underline{\mu} + \Sigma\underline{t})'\Sigma^{-1}(\underline{\mu} + \Sigma\underline{t}) + \underline{\mu}'\Sigma^{-1}\underline{\mu} - (\underline{\mu} + \Sigma\underline{t})'\Sigma^{-1}(\underline{\mu} + \Sigma\underline{t})$$
$$= (\underline{x} - \underline{\mu} - \Sigma\underline{t})^{-1'}\Sigma^{-1}(\underline{x} - \underline{\mu} - \Sigma\underline{t}) - \underline{\mu}'\Sigma^{-1}\underline{t} - \underline{t}'\Sigma\Sigma^{-1}\underline{\mu} - \underline{t}'\Sigma\Sigma^{-1}\Sigma\underline{t}$$
$$= (\underline{x} - \underline{\mu} - \Sigma\underline{t})'\Sigma^{-1}(\underline{x} - \underline{\mu} - \Sigma\underline{t}) - 2\underline{t}'\underline{\mu} - \underline{t}'\Sigma\underline{t})$$

$$= e^{\underline{t}'\underline{\mu} + \frac{1}{2}\underline{t}'\Sigma\underline{t}} \underbrace{\int_{R^p} \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} e^{-\frac{1}{2}(\underline{x} - \underline{\mu} - \Sigma\underline{t})'\Sigma^{-1}(\underline{x} - \underline{\mu} - \Sigma\underline{t})} \, d\underline{x}}_{= 1 \ (\because \text{pdf})}$$

$$= e^{\underline{t}'\underline{\mu} + \frac{1}{2}\underline{t}'\Sigma\underline{t}}$$

Ex :- Prove thm (1) using mgf technique

$$M_{\underline{x}}(\underline{t}) = E[e^{\underline{t}'\underline{x}}]$$
$$= E[e^{\underline{t}'P'\underline{x}}] = E[e^{(P\underline{t})'\underline{x}}]$$

let $\underline{u} = P\underline{t}$

$$M_{\underline{x}}(\underline{t}) = E[e^{\underline{u}'\underline{x}}]$$
$$= e^{\underline{u}'\underline{\mu} + \frac{1}{2}\underline{u}'\Sigma\underline{u}}$$
$$= e^{\underline{t}'P'\underline{\mu} + \frac{1}{2}\underline{t}'P'\Sigma P\underline{t}}$$

$$= e^{\underline{t}'(P'\underline{\mu}) + \frac{1}{2}\underline{t}'(P'\Sigma P)\underline{t}}$$

which is mgf of $N_p(P'\underline{\mu}, P'\Sigma P)$

Theorem (3) :- Let $\underline{X}_{p\times1} \sim N_p(\underline{\mu}, \Sigma)$ & $B_{q\times p}$ ($q \le p$) be a matrix of rank q then

$$\underline{Y}_{q\times1} = B\underline{X} \sim N_q(B\underline{\mu}, B\Sigma B')$$

Proof:- $M_{\underline{y}}(t) = E[e^{\underline{t}'\underline{y}}] = E[e^{\underline{t}'B\underline{x}}]$
$$= E[e^{(B'\underline{t})'\underline{x}}]$$

let $\underline{u} = B't$

$$= E[e^{\underline{u}'\underline{x}}]$$
$$= e^{\underline{u}'\underline{\mu} + \frac{1}{2}\underline{u}'\Sigma\underline{u}}$$
$$= e^{(B'\underline{t})'\underline{\mu} + \frac{1}{2}(B'\underline{t})'\Sigma(B'\underline{t})}$$
$$= e^{\underline{t}'(B\underline{\mu}) + \frac{1}{2}\underline{t}'(B\Sigma B')\underline{t}}$$

which is mgf of $N_q(B\underline{\mu}, B\Sigma B')$

$\therefore$ By uniqueness of mgf

$$\underline{Y} \sim N_q(B\underline{\mu}, B\Sigma B')$$

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Ex :- Prove thm (1) using mgf technique

$$M_x(t) = E[e^{t'x}]$$
$$= E[e^{t'P'x}] = E[e^{(Pt)'x}]$$

let $u = Pt$

$$M_x(t) = E[e^{u'x}]$$
$$= e^{u'\mu + \frac{1}{2}u'\Sigma u}$$
$$= e^{t'P'\mu + \frac{1}{2}t'P'\Sigma P t}$$
$$= e^{t'(P'\mu) + \frac{1}{2}t'(P'\Sigma P)t}$$

which is mgf of $N_p(P'\mu, P'\Sigma P)$

Theorem (3) :- Let $X_{p\times 1} \sim N_p(\mu, \Sigma)$ & $B_{q\times p}$ $(q \le p)$ be a matrix of rank q then
$$Y_{q\times 1} = BX \sim N_q(B\mu, B\Sigma B')$$

Proof:- $M_y(t) = E[e^{t'y}] = E[e^{t'Bx}]$
$$= E[e^{(B't)'x}]$$

let $u = B't$

$$= E[e^{u'x}]$$
$$= e^{u'\mu + \frac{1}{2}u'\Sigma u}$$
$$= e^{(B't)'\mu + \frac{1}{2}(B't)'\Sigma(B't)}$$
$$= e^{t'(B\mu) + \frac{1}{2}t'(B\Sigma B')t}$$

which is mgf of $N_q(B\mu, B\Sigma B')$

$\therefore$ By uniqueness of mgf

$$Y \sim N_q(B\mu, B\Sigma B')$$

Theorem
Proof
Corollary
Proof

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Theorem (4) :- If $\underline{x}_{p\times 1} \sim N_p(\underline{\mu}, \Sigma)$, then we can write $\underline{x} = \underline{\mu} + P\underline{y}$ where $PP' = \Sigma$ & $\underline{y} \sim N_p(0, I_p)$

Proof :- Pdf of $\underline{x}$ is

$$f(\underline{x}) = \frac{1}{(\sqrt{2\pi})^p \sqrt{|\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})}, \quad x \in R^p$$

Since Σ is Positive - definite $\exists$ a non - singular $P \ni PP' = \Sigma$

let $\underline{y} = \frac{\underline{x} - \underline{\mu}}{P} \Rightarrow \underline{y} = P^{-1}(\underline{x} - \underline{\mu})$

Note that $|J| = \left| \frac{\partial y}{\partial x} \right| = ||P^{-1}|| = \sqrt{|P^{-1}|'|P^{-1}|} = \sqrt{|(PP')^{-1}|} = \sqrt{|\Sigma^{-1}|}$

$$g(\underline{y}) = \frac{1}{(\sqrt{2\pi})^p} \cdot \frac{1}{\sqrt{|\Sigma|}} e^{-\frac{1}{2}(P\underline{y})'(PP')^{-1}(P\underline{y})} \cdot \sqrt{|\Sigma|}$$

$$= \frac{1}{(\sqrt{2\pi})^p} e^{-\frac{1}{2}\underline{y}'\underline{y}}$$

$\Rightarrow \underline{y} \sim N_p(\underline{0}, I_p)$ where $\underline{x} = \underline{\mu} + P\underline{y}$ & $I = P'P$

Corollary :- If $\underline{x}_{p\times 1} \sim N_p(\underline{\mu}, \Sigma)$ then $(\underline{x} - \underline{\mu})'\Sigma^{-1}(\underline{x} - \underline{\mu}) \sim \chi^2_p$

Proof :- If $\underline{x} \sim N_p(\underline{\mu}, \Sigma)$ then we have $\underline{x} - \underline{\mu} = P\underline{y}$ where $PP' = \Sigma$ & $\underline{y} \sim N_p(\underline{0}, I_p)$

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Prof :-

Pdf of $\underline{y}$

$$g(\underline{y}) = \frac{1}{(\sqrt{2\pi})^p} e^{-\frac{1}{2}\underline{y}'\underline{y}} = \frac{1}{(\sqrt{2\pi})^p} e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2}$$

Now

$$(\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu})$$
$$= (\underline{y}'P') \Sigma^{-1} (\underline{y}P)$$
$$= \underline{y}'(P'P)\Sigma^{-1}\underline{y}$$
$$= \underline{y}'\underline{y} = \sum_{i=1}^{p} y_i^2 \leftarrow \left(\text{Quadratic form of } \chi^2 \text{ dist}^n\right)$$
$$= \chi^2_{(p)} \text{ df.}$$

Let $\underline{x} \sim N_p(\underline{\mu}, \Sigma)$

Consider the following partition

$$\underline{x}_{(p\times1)} = \begin{pmatrix} \underline{x}_{(1)p_1\times1} \\ \underline{x}_{(2)} \end{pmatrix}, \quad \underline{\mu}_{p\times1} = \begin{pmatrix} \underline{\mu}_{(1)} \\ \underline{\mu}_{(2)} \end{pmatrix},$$

$$Disp(\underline{x}) = \Sigma_{p\times p} = \begin{bmatrix} \Sigma_{11}(p_1\times p_1) & \Sigma_{12} \\ \Sigma_{21}\,\boxed{(\Sigma_{12})'} & \Sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} Cov(x_{(1)}, \underline{x}_{(1)}) & Cov(x_{(1)}, x_{(2)}) \\ Cov(x_{(2)}, x_{(1)}) & Cov(x_{(2)}, x_{(2)}) \end{bmatrix}$$

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Theorem (5) :- If $\underline{x} \sim N_p(\underline{\mu}, \Sigma)$, then the necessary and sufficient condition for $\underline{x}_{(1)}$ & $\underline{x}_{(2)}$ to be Independent is that $\Sigma_{12} = 0$.

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Proof: Let $\Sigma_{12} = \mathrm{cov}(\underline{X}_{(1)}, \underline{X}_{(2)}) = 0$

Then the pdf of $\underline{x}_{(p\times 1)}$ is $N_p(\underline{\mu}, \Sigma)$

$$= \frac{1}{(2\pi)^{p/2}\sqrt{|\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})}, \quad \underline{x} \in R.$$

Now $\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$ $(\because \Sigma_{12} = 0)$

Hence $|\Sigma| = |\Sigma_{11}| \cdot |\Sigma_{22}|$ &

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}$$

Now the exponent becomes

$(\underline{x} - \underline{\mu}')'\Sigma^{-1}(\underline{x}-\underline{\mu})$

$$= \begin{pmatrix} \underline{x}_{(1)} - \underline{\mu}_{(1)} \\ \underline{x}_{(2)} - \underline{\mu}_{(2)} \end{pmatrix}' \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} \underline{x}_{(1)} - \underline{\mu}_{(1)} \\ \underline{x}_{(2)} - \underline{\mu}_{(2)} \end{pmatrix}$$

$$= (\underline{x}_{(1)} - \underline{\mu}_{(1)})'\Sigma_{11}^{-1}(\underline{x}_{(1)} - \underline{\mu}_{(1)}) + (\underline{x}_{(2)} - \underline{\mu}_{(2)})'\Sigma_{22}^{-1}(\underline{x}_{(2)} - \underline{\mu}_{(2)})$$

Hence $f(\underline{x} \mid \underline{\mu}, \Sigma)$

$$= \frac{1}{(\sqrt{2\pi})^{p_1}\sqrt{|\Sigma_{11}|}} e^{-\frac{1}{2}(\underline{x}_{(1)} - \underline{\mu}_{(1)})'\Sigma_{11}^{-1}(\underline{x}_{(1)} - \underline{\mu}_{(1)})}$$

$$\times \frac{1}{(\sqrt{2\pi})^{p-p_1}\sqrt{|\Sigma_{22}|}} e^{-\frac{1}{2}(\underline{x}_{(2)} - \underline{\mu}_{(2)})'\Sigma_{22}^{-1}(\underline{x}_{(2)} - \underline{\mu}_{(2)})}$$

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Hence $\underline{X}_{(1)}$ and $\underline{X}_{(2)}$ are independent

$$\underline{X}_{(1)} \sim N_{p_1}(\underline{\mu}_{(1)}, \Sigma_{11})$$
&
$$\underline{X}_{(2)} \sim N_{p-p_1}(\underline{\mu}_{(2)}, \Sigma_{22})$$

Conversely,

let $\underline{X}_{(1)}$ and $\underline{X}_{(2)}$ are independent

$$Cov(x_i, x_j) = E[(x_i - \mu_i)(x_j - \mu_j)] \qquad i = 1, 2, \ldots, p_1, \; j = p_1+1, \ldots, p$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j) f(x_1, \ldots, x_p)\, dx_1\, dx_2 \cdots dx_p$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ (x_i - \mu_i) f_1(x_1, x_2, \ldots, x_{p_1})\, dx_1\, dx_2 \cdots dx_{p_1} \right\}$$

$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ (x_j - \mu_j) f_2(x_{p_1+1}, x_{p_1+2}, \ldots, x_p)\, dx_{p_1+1} \cdots dx_p \right\}$$

$$= \{E(x_i) - \mu_i\}\{E(x_j) - \mu_j\}$$

$$= 0 \qquad (\because E(x_i) = \mu_i \;\&\; E(x_j) = \mu_j)$$

$$\boxed{\sigma_{ij} = 0} \quad \forall \; i = 1, \ldots, p_1, \; j = p_1+1, p_1+2, \ldots, p$$

$$\Sigma_{12} = \Sigma_{21} = 0$$

Hence the proof.

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Theorem (6): If $\underline{X}_{p\times 1} \sim N_p(\underline{\mu}, \Sigma)$, then any sub-vector is also a multivariate normal with mean vector and dispersion matrix obtained by taking the corresponding components $\underline{\mu}$ & Σ. In particular $\underline{X}_{(2)} \sim N_{p-p_1}(\underline{\mu}_{(2)}, \Sigma_{22})$

Proof: From previous Theorem,

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ then

$\underline{X}_{(1)} \sim N_{p_1}(\underline{\mu}_{(1)}, \Sigma_{11})$ and

$\underline{X}_{(2)} \sim N_{p-p_1}(\underline{\mu}_{(2)}, \Sigma_{22})$ if $\Sigma_{12} = 0$

Consider the transformation

$$\underline{Y}_{(1)} = \underline{X}_{(1)} + M\underline{X}_{(2)}$$

$\underline{Y}_{(2)} = \underline{X}_{(2)}$ and M is $\ni$ $Cov(\underline{Y}_{(1)}, \underline{Y}_{(2)}) = 0$

$$\Rightarrow E[\underline{Y}_{(1)} - E(\underline{Y}_{(1)})][\underline{Y}_{(2)} - E(\underline{Y}_{(2)})]' = 0$$

$$\Rightarrow E[\underline{X}_{(1)} + M\underline{X}_{(2)} - E(\underline{X}_{(1)} + M\underline{X}_{(2)})][\underline{X}_{(2)} - E(\underline{X}_{(2)})]' = 0$$

$$\Rightarrow E[\underline{X}_{(1)} + M\underline{X}_{(2)} - \underline{\mu}_{(1)} - M\underline{\mu}_{(2)}][\underline{X}_{(2)} - \underline{\mu}_{(2)}]' = 0$$

$$\Rightarrow E[(\underline{X}_{(1)} - \underline{\mu}_{(1)}) + M(\underline{X}_{(2)} - \underline{\mu}_{(2)})][\underline{X}_{(2)} - \underline{\mu}_{(2)}]' = 0$$

$$\Rightarrow E[\underline{X}_{(1)} - \underline{\mu}_{(1)}][\underline{X}_{(2)} - \underline{\mu}_{(2)}]' + ME[\underline{X}_{(2)} - \underline{\mu}_{(2)}][\underline{X}_{(2)} - \underline{\mu}_{(2)}]' = 0$$

$$\Rightarrow Cov(\underline{X}_{(1)}, \underline{X}_{(2)}) + M\,Cov(\underline{X}_{(2)}, \underline{X}_{(2)}) = 0$$

$$\Rightarrow \Sigma_{12} + M\Sigma_{22} = 0$$

$$\Rightarrow M = -\Sigma_{12}\Sigma_{22}^{-1}$$

Hence the transformation becomes

$\underline{Y}_{(1)} = \underline{X}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{X}_{(2)}$ and $\underline{Y}_{(2)} = \underline{X}_{(2)}$

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$$\Rightarrow \underline{Y} = \begin{pmatrix} \underline{Y}_{(1)} \\ \underline{Y}_{(2)} \end{pmatrix} = \begin{pmatrix} I_{p_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-p_1} \end{pmatrix} \begin{pmatrix} \underline{X}_{(1)} \\ \underline{X}_{(2)} \end{pmatrix}$$

$\Rightarrow \underline{Y} = P\underline{X}$ (say) where

$P = \begin{pmatrix} I_{p_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-p_1} \end{pmatrix}$ is non-singular matrix

Hence by theorem

$$\underline{Y}_{p\times 1} = P\underline{X} \sim N_p(P\underline{\mu}, P\Sigma P')$$

Note that

$$P\underline{\mu} = \begin{pmatrix} I_p & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-p_1} \end{pmatrix} \begin{pmatrix} \underline{\mu}_{(1)} \\ \underline{\mu}_{(2)} \end{pmatrix} = \begin{pmatrix} \underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)} \\ \underline{\mu}_{(2)} \end{pmatrix}$$

$$P\Sigma P' = \begin{pmatrix} I_{p_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-p_1} \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{bmatrix} I_{p_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{p-p_1} \end{bmatrix}'$$

$$= \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$$

Hence

$$\begin{pmatrix} \underline{Y}_{(1)} \\ \underline{Y}_{(2)} \end{pmatrix} \sim N_p\left(\begin{pmatrix} \underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)} \\ \underline{\mu}_{(2)} \end{pmatrix}, \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \right)$$

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Here $Cov(\underline{Y}_{(1)}, \underline{Y}_{(2)}) = 0$

Hence we have,

$$\underline{Y}_{(2)} = \underline{X}_{(2)} \sim N_{p-p_1}(\underline{\mu}_{(2)}, \Sigma_{22})$$

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Ex:- (1) If $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$, find the distⁿ of

$$\begin{bmatrix} X_1 - X_2 \\ X_2 - X_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$= AX$$

$$A\underline{X} \sim N(A\underline{\mu}, A\Sigma A')$$

$$A\underline{\mu} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_3 \end{bmatrix}$$

$$A\Sigma A' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{21} & \sigma_{12} - \sigma_{22} & \sigma_{13} - \sigma_{23} \\ \sigma_{21} - \sigma_{31} & \sigma_{22} - \sigma_{32} & \sigma_{23} - \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{21} - \sigma_{12} + \sigma_{22} & \sigma_{12} - \sigma_{22} - \sigma_{13} + \sigma_{23} \\ \sigma_{21} - \sigma_{31} - \sigma_{22} + \sigma_{32} & \sigma_{22} - \sigma_{32} - \sigma_{23} + \sigma_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - 2\sigma_{12} + \sigma_{22} & \sigma_{12} - \sigma_{22} - \sigma_{13} + \sigma_{23} \\ \sigma_{21} - \sigma_{31} - \sigma_{22} + \sigma_{32} & \sigma_{22} - 2\sigma_{23} + \sigma_{33} \end{bmatrix}$$

$\because$ $\sigma_{12} = \sigma_{21}$ & $\sigma_{23} = \sigma_{32}$

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Ex:- (2) If $\underline{X} \sim N_5(\underline{\mu}, \Sigma)$ find the distⁿ of $\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$

we set

$$\underline{X}_{(1)} = \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}, \quad \underline{\mu}_{(1)} = \begin{bmatrix} \mu_2 \\ \mu_4 \end{bmatrix}, \quad \Sigma_{11} = \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix}$$

Now

$$\underline{X} = \begin{bmatrix} X_2 \\ X_4 \\ \hline X_1 \\ X_3 \\ X_5 \end{bmatrix} \begin{matrix} \leftarrow \underline{X}_{(1)} \\ \\ \\ \leftarrow \underline{X}_{(2)} \\ \\ \end{matrix} \qquad \underline{\mu} = \begin{bmatrix} \mu_2 \\ \mu_4 \\ \hline \mu_1 \\ \mu_3 \\ \mu_5 \end{bmatrix}$$

$$\Sigma = \left[\begin{array}{cc|ccc} \sigma_{22} & \sigma_{24} & \sigma_{12} & \sigma_{23} & \sigma_{25} \\ \sigma_{24} & \sigma_{44} & \sigma_{14} & \sigma_{34} & \sigma_{45} \\ \hline \sigma_{12} & \sigma_{14} & \sigma_{11} & \sigma_{13} & \sigma_{15} \\ \sigma_{32} & \sigma_{34} & \sigma_{13} & \sigma_{33} & \sigma_{35} \\ \sigma_{25} & \sigma_{45} & \sigma_{15} & \sigma_{35} & \sigma_{55} \end{array}\right]$$

Σ_{11} (2×2), Σ_{12} (2×3), Σ_{21} or Σ_{12}' (3×2), Σ_{22} (3×3)

$$\underline{X}_{(1)} \sim N_2\left(\begin{bmatrix} \mu_2 \\ \mu_4 \end{bmatrix}, \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix}\right)$$

Ex:- (3) Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$

with

$$\Sigma = \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 1 & 3 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

Σ_{11}, Σ_{12}, Σ_{21} or Σ_{12}', Σ_{22}

Are X_1 and X_2 independent? What about (X_1, X_2) & X_3?

$\rightarrow$ $Cov(X_1, X_2) \neq 0 \Rightarrow X_1$ and X_2 are not Indep.

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \end{bmatrix}$$

Since

$Cov((X_1, X_2), X_3) = \Sigma_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so we can conclude

that X_3 is indep of X_1 as well as X_2.

Ex:- (4) :- Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$ with $\underline{\mu} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$ and

$$\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which of the following r.v.'s are indep. ? Explain.

(i) X_1 and X_2 $\longrightarrow$ Not indep.

(ii) X_2 and X_3 $\longrightarrow$ Indep

(iii) (X_1, X_2) and X_3 $\longrightarrow$ Independent.

(iv) $\dfrac{(X_1 + X_2)}{2}$ and X_3.

(v) Also find the dist^n of $3X_1 - 2X_2 + X_3$

$\rightarrow$ (iv) let $Y = \dfrac{X_1 + X_2}{2}$

$$Var(Y, X_3) = Var(Y) + Var(X_3) + 2Cov(Y, X_3)$$

$$= Var\left(\frac{X_1 + X_2}{2}\right) + 2 + 0$$

$$= \frac{1}{4} Var(X_1 + X_2) + 2$$

$$= \frac{1}{4}\left[Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)\right] + 2$$

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$$= \frac{1}{4}\left[1 + 5 - 4\right] + 2 = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$$

(v) $3X_1 - 2X_2 + X_3 \sim N(-7, 55)$

Mean $\to 3(-3) - 2(1) + 4$

$-9 - 2 + 4 = -7$

$Var(3X_1 - 2X_2 + X_3)$

$= 9\,Var(X_1) + 4\,Var(X_2) + Var(X_3)$

$- 12\,Cov(X_1, X_2) - 4\,Cov(X_2, X_3) + 6\,Cov(X_1, X_3)$

$= 9(1) + 4(5) + 2 - 12(-2) - 4(0) + 6(0)$

$= 9 + 20 + 2 + 24$

$= 55$

* Conditional Distribution :-

Theorem (7) :- If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then the conditional distⁿ of $\underline{X}_{(1)}$ given $\underline{X}_{(2)} = \underline{x}_{(2)}$ is given by $N_{p_1}(\underline{\mu}_{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_{(2)} - \underline{\mu}_{(2)}), \Sigma_{11\cdot 2})$

where

$$\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Proof :- Consider the following transformation

$$\underline{Y}_{(1)} = \underline{X}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$\underline{Y}_{(2)} = \underline{X}_{(2)}$$

Then,

$$\begin{pmatrix} \underline{Y}_{(1)} \\ \underline{Y}_{(2)} \end{pmatrix} \sim N_p\left[\begin{pmatrix} \underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)} \\ \underline{\mu}_{(2)} \end{pmatrix}, \begin{pmatrix} \Sigma_{11\cdot 2} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}\right]$$

Hence the pdf of $\begin{pmatrix} \underline{Y}_{(1)} \\ \underline{Y}_{(2)} \end{pmatrix}$ is

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$$f(\underline{y}_{(1)}, \underline{y}_{(2)}) = f(\underline{y}_{(1)} / \underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)}, \Sigma_{11\cdot 2}) \cdot f(\underline{y}_{(2)} / \underline{\mu}_{(2)}, \Sigma_{22}) \quad (*)$$

($\because$ $\underline{y}_{(1)} \sim N_{p_1}(\underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)}, \Sigma_{11\cdot 2})$
and
$\underline{y}_{(2)} \sim N_{p-p_1}(\underline{\mu}_{(2)}, \Sigma_{22})$
independently as $Cov(\underline{y}_{(1)}, \underline{y}_{(2)}) = 0$)

The pdf of $\begin{pmatrix} \underline{x}_{(1)} \\ \underline{x}_{(2)} \end{pmatrix}$ will be obtained from (*) by replacing $\underline{y}_{(1)}$ by $x_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{x}_{(2)}$ and $\underline{y}_{(2)}$ by $\underline{x}_{(2)}$

Since the jacobian of the transformation is unity
Hence the pdf of $\begin{pmatrix} \underline{x}_{(1)} \\ \underline{x}_{(2)} \end{pmatrix}$ is

$$f(\underline{x}_{(1)}, \underline{x}_{(2)}) = f(\underbrace{\underline{x}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{x}_{(2)}}_{z} / \underbrace{\underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)}}_{\mu}, \Sigma_{11\cdot 2})$$
$$\times f(\underline{x}_{(2)} / \underline{\mu}_{(2)}, \Sigma_{22})$$

Hence the conditional pdf of $\underline{X}_{(1)}$ given $\underline{X}_{(2)} = \underline{x}_{(2)}$ is

$$f(\underline{x}_{(1)} / \underline{x}_{(2)}) = \frac{f(\underline{x}_{(1)}, \underline{x}_{(2)})}{f(\underline{x}_{(2)})}$$

$$= \frac{1}{(\sqrt{2\pi})^{p_1}\sqrt{|\Sigma_{11\cdot 2}|}} e^{-\frac{1}{2}[(\underline{z}-\underline{\mu})'\Sigma_{11\cdot 2}^{-1}(\underline{z}-\underline{\mu})]}$$

$$= \frac{1}{(\sqrt{2\pi})^{p_1}\sqrt{|\Sigma_{11\cdot 2}|}} e^{-\frac{1}{2}\left[\{(\underline{x}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{x}_{(2)}) - (\underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)})\}'\Sigma_{11\cdot 2}^{-1} \times \{(\underline{x}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{x}_{(2)}) - (\underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_{(2)})\}\right]}$$

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$$= \frac{1}{(\sqrt{2\pi})^{p_1}\sqrt{|\Sigma_{11\cdot 2}|}} e^{-\frac{1}{2}\left[\{\underline{x}_{(1)} - \underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_{(2)} - \underline{\mu}_{(2)})\}' \Sigma_{11\cdot 2}^{-1} \times \{\underline{x}_{(1)} - \underline{\mu}_{(1)} - \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_{(2)} - \underline{\mu}_{(2)})\}\right]}$$

$$\boxed{\underline{X}_{(1)} / \underline{X}_{(2)} = \underline{x}_{(2)} \sim N_{p_1}\left(\underline{\mu}_{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_{(2)} - \underline{\mu}_{(2)}), \Sigma_{11\cdot 2}\right)}$$

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* **Remark** :-

(1) $E[\underline{X}_{(1)} / \underline{X}_{(2)} = \underline{x}_{(2)}] = \underline{\mu}_{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_{(2)} - \underline{\mu}_{(2)})$ is a linear function of $\underline{x}_{(2)}$.

Hence the regression of $\underline{X}_{(1)}$ and $\underline{X}_{(2)}$ is linear.

Also, the dispersion of $\underline{X}_{(1)}$ ~~and~~ given $\underline{X}_{(2)} = \underline{x}_{(2)}$ is $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ is indep. of $\underline{X}_{(2)}$

Hence, the conditional distⁿ of $\underline{X}_{(1)} / \underline{X}_{(2)} = \underline{x}_{(2)}$ is Homoscedastic.

Ex :- (5) :- $\underline{X} = (X_1, X_2, X_3) \sim N_3(\underline{0}, \Sigma)$ where

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

Show that for any $c > 0$

$$P[(x_2^2 + c)\rho^2 - 2(x_1x_2 + x_2x_3)\rho + x_1^2 + x_2^2 + x_3^2 - c \leq 0]$$

$$= \int_0^c \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \cdot y^{1/2}\, dy$$

Solⁿ:

→ Since $\underline{X} \sim N_3(\underline{0}, \Sigma)$

$$(\underline{X} - \underline{0})' \Sigma^{-1} (\underline{X} - \underline{0}) \sim \chi^2_{(3)}$$

$$\underline{X}' \Sigma^{-1} \underline{X} \sim \chi^2_{(3)}$$

$$\boxed{\begin{aligned}\left(\frac{x-\mu}{\sigma}\right)^2 &\sim \chi^2_{(1)}\\ (\underline{x}-\underline{\mu})'\,\Sigma^{-1}(\underline{x}-\underline{\mu}) &\sim \chi^2_{(3)}\end{aligned}}$$

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$$\Rightarrow P[\underline{x}'\Sigma^{-1}\underline{x} \le c] = \int_0^c f_{\chi^2_{(3)}}(y)\,dy$$

$$= \int_0^c \frac{1}{\sqrt{2\pi}} e^{-y/2}\, y^{1/2}\, dy$$

$\frac{1}{\sqrt{2\pi}} e^{-\chi^2/2} (\chi^2)^{\frac{n-1}{2}}$
↳ Chi-square dist^n
Here $n = 3$ & $\chi^2 = y$

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(1) Suppose ~~$\underline{X}_{(1)}, \underline{X}_{(2)}$~~ $(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_p)' \sim N_p(\underline{\mu}, \Sigma)$
where $\underline{\mu} = (1, 1, \dots, 1)'$, $\Sigma = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1\\ 1 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 \cdots & 3 & 3 \\ \vdots & & & & \\ 1 & 2 & 3 \cdots & (p-1) & p\end{bmatrix}$

then Show that

$\varphi = (x_2 - x_1)^2 + (x_3 - x_2)^2 + \dots + (x_p - x_{p-1})^2$ has a χ^2 - dist^n.

Sol^n: let $Y_i = X_i - X_{i-1}$, $i = 1, 2, \dots, p$

$$\therefore \underline{Y}_{(p-1)\times 1} = \begin{pmatrix} Y_2 \\ Y_3 \\ \vdots \\ Y_p \end{pmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots -1 & 1 \end{bmatrix}_{(p-1)\times p} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}_{p\times 1}$$

$$= B_{(p-1)\times p}\, \underline{X}_{p\times 1} \quad \text{(say)}$$

Note that $\rho(B) = (p-1)$

Hence $\underline{Y} = B\underline{X} \sim N_{p-1}(B\underline{\mu}, B\Sigma B')$

$\big(\because E(\underline{Y}) = E(B\underline{X}) = BE(\underline{X}) = B\underline{\mu}$,
$D(\underline{Y}) = D(B\underline{X}) = BD(\underline{X})B' = B\Sigma B'\big)$

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$$E(B\underline{x}) = E[B(x_i - x_{i-1})] = B[E(x_i) - E(x_{i-1})]$$
$$= B[\mu - \mu]$$
$$= \boxed{0}$$

$$B\Sigma B' = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots \\ 1 & 2 & 3 & \cdots & (p-1) & p \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(p-1)\times(p-1)}$$

$$\underline{y} \sim N_{p-1}(0, I_{p-1})$$
$\Rightarrow$ $y_2, y_3, \ldots, y_p$ are iid $N_1(0,1)$

$$Q = y_2^2 + y_3^3 + \cdots + y_p^2$$
$$= \sum_{i=2}^{p} y_i^2 \sim \chi^2_{(p-1)\,df.}$$

Theorem (8):- $\underline{x}_{p\times 1} \sim N_p(\underline{\mu}, \Sigma)$ iff
$\underline{l}'\underline{x} \sim N_1(\underline{l}'\underline{\mu}, \underline{l}'\Sigma\underline{l})$, $\underline{l}_{p\times 1} \in R^p$

Proof:- If : Let $\underline{l}'\underline{x} \sim N_1(\underline{l}'\underline{\mu}, \underline{l}'\Sigma\underline{l})$
Then the mgf of $\underline{l}'\underline{x}$ is $E[e^{t(\underline{l}'\underline{x})}]$

$$= e^{t(\underline{l}'\underline{\mu}) + \frac{1}{2}t^2(\underline{l}'\Sigma\underline{l})} \quad \forall t$$

putting $t = 1$
$$= E[e^{\underline{l}'\underline{x}}] = e^{\underline{l}'\underline{\mu} + \frac{1}{2}\underline{l}'\Sigma\underline{l}} \quad \forall \underline{l}$$
= mgf of p-variate Normal
ie. $N_p(\underline{\mu}, \Sigma)$

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By uniqueness property of mgf

$\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$

Only if :- let $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$

Consider a vector $k\underset{\sim}{t}$ where k is a constant

The mgf of $\underset{\sim}{X}$ is $E\left[e^{(k\underset{\sim}{t})'\underset{\sim}{X}}\right] = e^{(k\underset{\sim}{t})'\mu + \frac{1}{2}(k\underset{\sim}{t})'\Sigma(k\underset{\sim}{t})}$, $\forall t \in R^p$

~~$\Leftrightarrow E\left[e^{k(t'x)}\right] = e^{k(\underset{\sim}{t}'\underset{\sim}{\mu}) + \frac{1}{2}k^2(\underset{\sim}{t}'\Sigma\underset{\sim}{t})}$~~

$\Leftrightarrow E\left[e^{k(\underset{\sim}{t}'\underset{\sim}{X})}\right] = e^{k(\underset{\sim}{t}'\underset{\sim}{\mu}) + \frac{1}{2}k^2(\underset{\sim}{t}'\Sigma\underset{\sim}{t})}$

$\Leftrightarrow$ mgf of $\underset{\sim}{t}'\underset{\sim}{X} = e^{k(\underset{\sim}{t}'\underset{\sim}{\mu}) + \frac{1}{2}k^2(\underset{\sim}{t}'\Sigma\underset{\sim}{t})} \quad \forall t \in R^p$

$=$ mgf of $N_1(\underset{\sim}{t}'\underset{\sim}{\mu}, \underset{\sim}{t}'\Sigma\underset{\sim}{t})$

Hence

$\underset{\sim}{t}'\underset{\sim}{X} \sim N(\underset{\sim}{t}'\underset{\sim}{\mu}, \underset{\sim}{t}'\Sigma\underset{\sim}{t}) \quad \forall t \in R^p$.

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* MLE of the parameters of Multivariate Normal Distⁿ :-

Theorem (a) :- If $\underset{\sim}{X}_n$, $n = 1, 2, \ldots, N$ be a random sample of size n from a $N_p(\underset{\sim}{\mu}, \Sigma)$ Then the MLE of its parameters $\underset{\sim}{\mu}$ and Σ are given by $\hat{\underset{\sim}{\mu}} = \bar{\underset{\sim}{X}}$ and $\hat{\Sigma} = \frac{A}{N}$ where

$$A = \sum_{n=1}^{N} (\underset{\sim}{X}_n - \bar{\underset{\sim}{x}})(\underset{\sim}{X}_n - \bar{\underset{\sim}{x}})'$$

Proof :- The density function of the n^{th} obsⁿ is given by

$$f(\underset{\sim}{x}_n) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left[\frac{-1}{2}(\underset{\sim}{x}_n - \underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{x}_n - \underset{\sim}{\mu})\right]$$

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Then the likelihood funct^n corresponding to (1) is given by

$$L(\underline{\mu}, \Sigma) = \prod_{n=1}^{N} f(\underline{x}_n)$$

$$= \prod_{n=1}^{N} \left[\frac{1}{(2\pi)^{P/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2} (\underline{x}_n - \underline{\mu})' \Sigma^{-1} (\underline{x}_n - \underline{\mu}) \right] \right]$$

$$= \frac{1}{(2\pi)^{NP/2}} \cdot \frac{1}{|\Sigma|^{N/2}} \exp\left[-\frac{1}{2} \sum_{n=1}^{N} (\underline{x}_n - \underline{\mu})' \Sigma^{-1} (\underline{x}_n - \underline{\mu}) \right] \quad (2)$$

Taking log on both sides of (2), we get

$$\ln L(\underline{\mu}, \Sigma) = -\frac{NP}{2} \ln(2\pi) - \frac{N}{2} \ln|\Sigma| - \frac{1}{2} \sum_{n=1}^{N} (\underline{x}_n - \underline{\mu})' \Sigma^{-1} (\underline{x}_n - \underline{\mu}) \quad (3)$$

Consider,

$$\sum_{n=1}^{N} (\underline{x}_n - \underline{\mu})' \Sigma^{-1} (\underline{x}_n - \underline{\mu})$$

$$= \sum_{n=1}^{N} (\underline{x}_n - \underline{\bar{x}} + \underline{\bar{x}} - \underline{\mu})' \Sigma^{-1} (\underline{x}_n - \underline{\bar{x}} + \underline{\bar{x}} - \underline{\mu}) \quad \text{where } \bar{x} = \frac{1}{n} \sum_{n=1}^{N} x_n$$

$$= \sum_{n=1}^{N} (\underline{x}_n - \underline{\bar{x}})' \Sigma^{-1} (\underline{x}_n - \underline{\bar{x}}) + \underbrace{\sum_{n=1}^{N} (\underline{x}_n - \underline{\bar{x}})' \Sigma^{-1} (\underline{x}_n - \underline{\mu})}_{\longrightarrow = 0}$$

$$+ \sum_{n=1}^{N} (\underline{\bar{x}} - \underline{\mu})' \Sigma^{-1} (\underline{x}_n - \underline{\bar{x}}) + \underbrace{\sum_{n=1}^{N} (\underline{\bar{x}} - \underline{\mu})' \Sigma^{-1} (\underline{\bar{x}} - \underline{\mu})}_{\longrightarrow = 0}$$ (Property of AM)

Since $\sum_{n=1}^{N} (x_n - \underline{\bar{x}})' \Sigma^{-1} (\underline{\bar{x}} - \underline{\mu}) = 0$ & $\sum_{n=1}^{N} (\underline{\bar{x}} - \underline{\mu}) \Sigma^{-1} (\underline{\bar{x}} - \underline{\mu}) = 0$

Also the term
$(\underline{x}_n - \underline{\bar{x}})' \Sigma^{-1} (\underline{x}_n - \underline{\bar{x}})$ is a scalar term and the trace of scalar is the same scalar.

$$= \sum_{n=1}^{N} \text{trace}\, (\underline{x}_n - \underline{\bar{x}})(\underline{x}_n - \underline{\bar{x}})' \Sigma^{-1} + N (\underline{\bar{x}} - \underline{\mu})' \Sigma^{-1} (\underline{\bar{x}} - \underline{\mu})$$

$(\because \text{trace}(AB) = \text{trace}(BA))$

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$$= \text{trace}\left[\sum_{n=1}^{N}(\underline{x}_n-\underline{\bar{x}})(\underline{x}_n-\underline{\bar{x}})'\right]\Sigma^{-1} + N(\underline{\bar{x}}-\underline{\mu})'\Sigma^{-1}(\underline{\bar{x}}-\underline{\mu}) \quad \text{——— (4)}$$

Using (4) in (3),

$$\ln L(\underline{\mu},\Sigma) = -\frac{NP}{2}\ln(2\pi) - \frac{N}{2}\ln|\Sigma| - \frac{1}{2}\text{trace}\left[\sum_{n=1}^{N}(x_n-\bar{x})(x_n-\bar{x})'\right]$$

$$\times\Sigma^{-1} - \frac{N}{2}(\underline{\bar{x}}-\underline{\mu})'\Sigma^{-1}(\underline{\bar{x}}-\underline{\mu}) \quad \text{——— (5)}$$

$$\left(\ln|\Sigma| = \ln\frac{1}{|\Sigma|^{-1}}\right)$$

$$\frac{\partial \ln L(\underline{\mu},\Sigma)}{\partial \underline{\mu}} = 0$$

$$\Rightarrow -\frac{N}{2}\frac{\partial}{\partial \underline{\mu}}(\underline{\bar{x}}-\underline{\mu})'\Sigma^{-1}(\underline{\bar{x}}-\underline{\mu}) = 0$$

$$\Rightarrow -\frac{N}{2}\left[(\underline{\bar{x}}-\underline{\mu})'\Sigma^{-1}(-1) + (-1)\Sigma^{-1}(\underline{\bar{x}}-\underline{\mu})\right] = 0$$

$$\Rightarrow -\frac{N}{2}\left[2\Sigma^{-1}(\underline{\bar{x}}-\underline{\mu})(-1)\right] = 0$$

$$\Rightarrow \underline{\bar{x}}-\underline{\mu} = 0$$

$$\Rightarrow \boxed{\hat{\underline{\mu}} = \underline{\bar{x}}}$$

$$\left(\because \Sigma^{-1}(\underline{\bar{x}}-\underline{\mu}) = 0 \quad \text{But } \Sigma^{-1}\neq 0 \Rightarrow \underline{\bar{x}}-\underline{\mu} = 0\right)$$

Using (5),

$$\frac{\partial \ln L(\mu,\Sigma)}{\partial \Sigma^{-1}} = 0$$

$$\Rightarrow \frac{N}{2}(\Sigma^{-1})^{-1} - \frac{1}{2}\left(\sum_{n=1}^{N}(\underline{x}_n-\underline{\bar{x}})(\underline{x}_n-\underline{\bar{x}})'\right) = 0 \quad (\because \hat{\underline{\mu}} = \bar{x})$$

$$\Rightarrow \hat{\Sigma} = \frac{1}{N}\left[\sum_{n=1}^{N}(\underline{x}_n-\underline{\bar{x}})(\underline{x}_n-\underline{\bar{x}})'\right]$$

H.W Take $\boxed{\mu = \mu_0}$

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* Elliptically Contoured Distⁿ :- (ECD) :-

- ECD are a class of multivariate prob. distⁿˢ that generalize the multi-variate Normal distⁿ. They are characterized by the property that they contours of constant prob. density are ellipsoids.

- A key feature of ECD is that they retain many of the desirable properties of Normal distⁿ such as Linearity Marginal and Conditional distⁿ while also allowing for heavier or lighter tails.

- A random vector $\underline{x}$ is said to have ECD with parameters $\underline{\mu}$, Σ and g if its pdf can be expressed as

$$f(x) = k\,|\Sigma|^{-1/2}\, g\left[(\underline{x}-\underline{\mu})'\,\Sigma^{-1}(\underline{x}-\underline{\mu})\right]$$

where

$\underline{\mu} \leftarrow$ location parameter
$\Sigma \leftarrow$ positive definite shape matrix
$g \leftarrow$ generator funcⁿ that determines the specific distⁿ within the class.
$k \leftarrow$ normalization constant.

- A special case of ECD is spherically symmetric distⁿ where the density function depends on the distance from the center

$$g(x) = h(\|x\|) \text{ for some function } h.$$

↳ Norm of Matrix.

- If X has an ECD and $Y = TX + b$, where T is a non-singular matrix, then Y also has an ECD.

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- The most well-known example is Multivariate Normal distⁿ where

$$g(u) = \exp\left(-\frac{u}{2}\right)$$

* <u>Applications</u>:-

→ Financial Modelling
→ Robust Statistics
→ Spatial Statistics
→ Machine Learning

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(1) Let $\underline{X} \sim N_p(\underline{0}, \sigma^2 I_p)$ and $P_1(m\times p)$ is a matrix $\ni P_1P_1' = I_m$ then show that $\underline{Y} = P_1\underline{X} \sim N_m(\underline{0}, \sigma^2 I_m)$ is independently distributed with $\frac{1}{\sigma^2}(\underline{X}'\underline{X} - \underline{Y}'\underline{Y}) \sim \chi^2_{p-m}$

<u>Proof</u>:- Note that $P_1P_1' = I_m$, here $P_1(m\times p)$ is semi-orthogonal matrix. Hence we can find $P_2(p-m\times p) \ni PP' = I_p$

where $P_{(p\times p)} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$

Let $\underline{Z} = P\underline{X} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\underline{X} = \begin{pmatrix} P_1\underline{X} \\ P_2\underline{X} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ \underline{W} \end{pmatrix}$

where $W = P_2\underline{X}$

Since P is non-singular

$$Z = P\underline{X} \sim N_p(\underline{0}, \sigma^2 I_p)$$

$$\Rightarrow \begin{pmatrix} \underline{Y} \\ \underline{W} \end{pmatrix} \sim N_p(\underline{0}, \sigma^2 I_p)$$

$\Rightarrow \underline{Y}_{(m\times 1)} \sim N_m(\underline{0}, \sigma^2 I_m)$ and

$\underline{W} \sim N_{(p-m)}(\underline{0}, \sigma^2 I_{p-m})$ independently

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Now, $\underline{z}'\underline{z} = \underline{x}'P'P\underline{x}$
$= \underline{x}'\underline{x}$

$$\Rightarrow \underline{x}'\underline{x} = \underline{z}'\underline{z} = \begin{pmatrix} \underline{y} \\ \underline{W} \end{pmatrix}' \begin{pmatrix} \underline{y} \\ \underline{W} \end{pmatrix}$$
$$= \begin{pmatrix} \underline{y}' & \underline{W}' \end{pmatrix} \begin{pmatrix} \underline{y} \\ \underline{W} \end{pmatrix}$$
$$= \underline{y}'\underline{y} + \underline{W}'\underline{W}$$

$$\Rightarrow \cancel{\underline{W}\underline{W}} \quad \underline{W}'\underline{W} = \underline{x}'\underline{x} - \underline{y}'\underline{y}$$

Since $\underline{W} \sim N_{p-m}(\underline{0}, \sigma^2 I_{p-m})$

$$\Rightarrow \frac{\underline{W}'\underline{W}}{\sigma^2} \sim \chi^2_{(p-m)}$$

$$\Rightarrow \frac{1}{\sigma^2}(\underline{x}'\underline{x} - \underline{y}'\underline{y}) \sim \chi^2_{(p-m)}$$

which is independently distⁿ with
$\underline{y} \sim N_m(\underline{0}, \sigma^2 I_m)$

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WISHART DISTRIBUTION :-

Def^n :- A $p \times p$ random matrix W is called a Wishart matrix based on 'm' d.f. if it can be expressed as

$$W = \sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha' \quad \alpha \text{ follows p-variate Normal dist}^n.$$

where $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_m$ are some 'm' indep. $N_p(0, \Sigma)$ random vectors and we say that

$W \sim$ Wishart dist^n on 'm' d.f. with parameters Σ and we write

$W \sim W_p(\Sigma, m)$ ↳ (p-variate)

— In particular in the univariate case when $p=1$ in which $\Sigma = \sigma^2$ (say) is the common variance of the 'm' normal pop^n 'w' on 'm' d.f. is S.S. for 'm' indep. $N(0, \sigma^2)$ variates

$$W = \sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha' = \underline{Y}_1 \underline{Y}_1' + \underline{Y}_2 \underline{Y}_2' + \cdots + \underline{Y}_m \underline{Y}_m'$$

$$= \sum_{i=1}^{m} y_i^2 \quad (\text{taking } p = 1)$$

$$\Rightarrow \frac{\sum_{i=1}^{m} y_i^2}{\sigma^2} = \frac{w}{\sigma^2} \sim \chi^2_{(m)} \text{ d.f.}$$

$X \sim N(\mu, \sigma^2)$

$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$

$Z^2 \sim \chi^2_{(1)}$

$\Sigma Z^2 \sim \chi^2_{(m)}$

Note :- This is why multivariate Wishart dist^n is generally regarded as Multivariate generalization of the χ^2 dist^n.

$$\left(Y \sim N(0, \sigma^2) \rightarrow \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{y}{\sigma} \right)^2 \right] \right.$$

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* **Basic Properties of Wishart distⁿ :-**

Theorem (1) :- If $W \sim W_p(\Sigma, m)$ and if c is a positive constant then $cW \sim W_p(c\Sigma, m)$

Proof :- Since $W \sim W_p(\Sigma, m)$, there are m-indep. p - component random vectors $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_m$ each distributed as $N_p(\underline{0}, \Sigma)$ such that $\sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha'$

Evidently we have $cW = c \sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha'$

$$= \sum_{\alpha=1}^{m} (\sqrt{c}\,\underline{Y}_\alpha)(\sqrt{c}\,\underline{Y}_\alpha')$$

$\sqrt{c}\,\underline{Y}_1, \sqrt{c}\,\underline{Y}_2, \ldots, \sqrt{c}\,\underline{Y}_m$ are iid as $N(\underline{0}, c\Sigma)$

$\because \underline{Y}_\alpha \sim N_p(\underline{0}, \Sigma)$
$E(\sqrt{c}\,\underline{Y}_\alpha) = \sqrt{c} \cdot 0 = 0$
$D(\sqrt{c}\,\underline{Y}_\alpha) = (\sqrt{c})^2 D(\underline{Y}_\alpha)$
$= c \cdot \Sigma$

$\Rightarrow cW \sim W_p(c\Sigma, m)$

Theorem (2) :- If $W \sim W_p(\Sigma, m)$ and if A is $r \times p$ matrix of rank R, then $AWA' \sim W(A\Sigma A', m)$

Proof :- As $W \sim W_p(\Sigma, m)$ and if A is $r \times p$ matrix of rank 'r' Also
$W = \sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha'$, there are '$m$' indep. $N_p(\underline{0}, \Sigma)$

random vectors $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_m$.

Let $\underline{Z}_\alpha = A\underline{Y}_\alpha \quad (\alpha = 1, 2, \ldots, m)$

then $\underline{Z}_\alpha'$s are indep $N_r(0, A\Sigma A')$

$(\because E(\underline{Z}_\alpha) = E(A\underline{Y}_\alpha) = 0$

$D(\underline{Z}_\alpha) = D(A\underline{Y}_\alpha) = AD(\underline{Y}_\alpha)A' = A\Sigma A'$

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so that $r \times r$ random matrix

$$\sum_{\alpha=1}^{m} \underline{Z}_{r} \underline{Z}_{r}' = \sum_{\alpha=1}^{m} (A\underline{Y}_{\alpha})(A\underline{Y}_{\alpha})'$$

$$= \sum_{\alpha=1}^{m} A(\underline{Y}_{\alpha}\underline{Y}_{\alpha}')A'$$

$$= A\left(\sum_{\alpha=1}^{m} \underline{Y}_{\alpha}\underline{Y}_{\alpha}'\right)A'$$

$$= AWA'$$

$\sum \underline{Z}_{r}\underline{Z}_{r}'$ is a $r \times r$ ~~random~~ wishart matrix on 'm' d.f.

i.e $\sum \underline{Z}_{\alpha}\underline{Z}_{\alpha}' \sim W_{r}(A\Sigma A', m)$

$(\because D(AWA') = AD(W)A' = A\Sigma A')$

Theorem (3) :- Let $W \sim W_p(\Sigma, m)$ and let $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$ where Σ_{11} and W_{11} are $k \times k$ then

(i) $W_{11} \sim W_k(\Sigma_{11}, m)$ and $W_{22} \sim W_{p-k}(\Sigma_{22}, m)$

(ii) W_{11} and W_{22} are indep. iff $\Sigma_{12} = 0$

Proof :- (i) Since $W \sim W_p(\Sigma, m)$

$W = \sum_{\alpha=1}^{m} \underline{Y}_{\alpha}\underline{Y}_{\alpha}'$ $\underline{Y}_{\alpha}$'s are some indep $N_p(\underline{0}, \Sigma)$ random vectors

Now partitioning $\underline{Y}_{\alpha}$ as

$\underline{Y}_{\alpha}' = (\underline{Y}_{\alpha}^{(1)'}, \underline{Y}_{\alpha}^{(2)'})$ $\alpha = 1, 2, \dots, m$

where each $\underline{Y}_{\alpha}^{(1)}$ has 'k' components and $\underline{Y}_{\alpha}^{(2)}$ has $(p-k)$ components.

$\underline{Y}_{\alpha}^{(1)}$ are iid as $N_k(\underline{0}, \Sigma_{11})$ and

$\underline{Y}_{\alpha}^{(2)}$ are iid as $N_{p-k}(\underline{0}, \Sigma_{22})$

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Moreover we have

$$W_{11} = \sum_{\alpha=1}^{m} \underline{Y}_{\alpha}^{(1)} \underline{Y}_{\alpha}^{(1)'} \quad \text{and} \quad W_{22} = \sum_{\alpha=1}^{m} \underline{Y}_{\alpha}^{(2)} \underline{Y}_{\alpha}^{(2)'}$$

Hence $W_{11} \sim W_K(\Sigma_{11}, m)$ and

$W_{22} \sim W_{p-K}(\Sigma_{22}, m)$

(ii) Further more for each α, $\underline{Y}_{\alpha}^{(1)}$ and $\underline{Y}_{\alpha}^{(2)}$ are independent iff $\Sigma_{12} = 0$

Hence (ii) follows from the independence of $\underline{Y}_{\alpha}'s$

Corollary :- Let $W \sim W_p(\Sigma, m)$ then

(i) Every principal sub-matrix of W has a Wishart distⁿ on 'm' d.f.

(ii) For a given $K \leq p$ the principal submatrix $W_{11}, W_{22}, \ldots, W_{KK}$ of W are mutually indep iff $\Sigma_{ij} = 0 \;\; \forall \; i \neq j$ where Σ_{ij}'s are the sub matrix of the properly partitioned Σ.

(iii) For each $i = 1, 2, \ldots, p$, $\dfrac{W_{ii}}{\sigma_{ii}} \sim \chi^2_{(n)}$ where

W_{ij}'s and σ_{ij}'s are ~~iid~~ (i, j)th element of W and Σ respectively.

Theorem (4) :- If $W_1 \sim W_p(\Sigma, m_1)$ and $W_2 \sim W_p(\Sigma, m_2)$ independently then $W_1 + W_2 \sim W_p(\Sigma, m_1 + m_2)$

Proof :- Since $W_1 \sim W_p(\Sigma, m_1)$ and $W_2 \sim W_p(\Sigma, m_2)$ independently, there are $m_1 + m_2$ indep $N_p(\underline{0}, \Sigma)$ random vector $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_{m_1+m_2}$ such that

$$W_1 = \sum_{\alpha=1}^{m_1} \underline{Y}_{\alpha} \underline{Y}_{\alpha}' \quad \text{and} \quad W_2 = \sum_{\alpha=m_1+1}^{m_2} \underline{Y}_{\alpha} \underline{Y}_{\alpha}'$$

Consequently,

$$W_1 + W_2 = \sum_{\alpha=1}^{m_1} \underline{Y}_\alpha \underline{Y}_\alpha' + \sum_{\alpha=m_1+1}^{m_2} \underline{Y}_\alpha \underline{Y}_\alpha'$$

$$= \sum_{\alpha=1}^{m_1+m_2} \underline{Y}_\alpha \underline{Y}_\alpha'$$

Hence $W_1 + W_2 \sim W_p(\Sigma, m_1 + m_2)$

Corollary :- If $W_1, W_2, \dots, W_k$ are independently distributed as $W_p(\Sigma, m_1)$, $W_p(\Sigma, m_2)$, ---, $W_p(\Sigma, m_k)$ resp.

then

$$W_1 + W_2 + \dots + W_k \sim W_p(\Sigma, m_1 + m_2 + \dots + m_k)$$

* Remarks :-

(1) Thm (i) is a reproductive property of the Wishart dist^n. As we have seen that two independent Wishart matrices with same Σ parameter can combine to give birth to a third Wishart Matrix with a same ~~pam~~ parameter and d.f. equal to the sum of the d.f. of the parent matrices.

(2) Wishart dist^n can be regarded as Multivariate Generalization of chi - square dist^n.

(3) From the def^n of Wishart Matrix $W = \sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha'$

$$E(W) = E\left(\sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha'\right) \qquad \underline{Y}_\alpha \sim N_p(\underline{0}, \Sigma)$$

$$= \sum_{\alpha=1}^{m} \left[E(\underline{Y}_\alpha \underline{Y}_\alpha')\right]$$

$$= \left[E(\underline{Y}_1 \underline{Y}_1') + E(\underline{Y}_2 \underline{Y}_2') + \dots + E(\underline{Y}_m \underline{Y}_m')\right]$$

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$= V(\underline{Y}_1) + V(\underline{Y}_2) + \cdots + V(\underline{Y}_m)$ $\quad$ $\left(\because V(\underline{Y}_\alpha) = E(\underline{Y}_\alpha \underline{Y}_\alpha') - E(\underline{Y}_\alpha) \dots \right.$

$= \boxed{m\Sigma}$ $\quad$ $\Rightarrow E(\underline{Y}_\alpha \underline{Y}_\alpha') = V(\underline{Y}_\alpha)$

$\left. (\because E(\underline{Y}_\alpha) = 0 \right)$ (i)

(4) Assumption $m \geq p$ is invariably made in the application of Wishart distⁿ inorder to have a non-singular distⁿ.

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* Characteristic function of a Wishart distⁿ :- (ii)

Theorem (1) :- The Characteristic functⁿ of Wishart $W_p(\Sigma, m)$ distⁿ

$$\Phi(t) = \left(\frac{|\Sigma^{-1}|}{|\Sigma^{-1} - 2iT|} \right)^{m/2}$$

where T is a symmetric matrix of real no.s.

Proof :- Suppose $W \sim W_p(\Sigma, m)$ then there are 'm' indep. normal $N_p(\underline{0}, \Sigma)$ random vectors $\underline{Y}_1, \underline{Y}_2, \ldots, \underline{Y}_m \ni$

$$W = \sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha'$$

Now Since Σ^{-1} is positive definite for a given real symmetric matrix, T, we find a non-singular matrix $P \ni P^{-1}\Sigma^{-1}P = I$

$$P'TP = \mathrm{diag}(d_1, d_2, \ldots, d_p) = D \text{ (say)}$$

where $d_1, d_2, \ldots, d_p$ are the roots of the determinants eqⁿˢ.

$$|T - d\Sigma^{-1}| = 0$$

If $Z_\alpha = P^{-1}\underline{Y}_\alpha$, $\alpha = 1, 2, \ldots, m$

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we see that

(i) $z_1, z_2, \ldots, z_m$ are distributed independently as $N_p(\underline{0}, I)$.

$E(\underline{z}_\alpha) = E(P^{-1}\underline{y}_\alpha) = P^{-1}E(\underline{y}_\alpha) = \boxed{0}$

$D(\underline{z}_\alpha) = D(P^{-1}\underline{y}_\alpha) = P^{-1}E(\underline{y}_\alpha)P^{-1}$ → symmetric

$= P^{-1}\Sigma P^{-1} = \boxed{I}$

(ii) trace TW = trace $\left(T\sum_{\alpha=1}^{m}\underline{y}_\alpha \underline{y}_\alpha'\right)$

Since $Tr\, AB$ = trace BA
trace $a = a$ for all scalars.

$$= \sum_{\alpha=1}^{m}\underline{y}_\alpha' T\underline{y}_\alpha = \sum_{\alpha=1}^{m}\underline{y}_\alpha' P'TP\underline{y}_\alpha$$

$$= \sum_{\alpha=1}^{m} z_\alpha' P'TP\underline{z}_\alpha \quad \left(\because (Pz_\alpha')T(Pz_\alpha)' = z_\alpha' P' T P' z_\alpha\right)$$

$$= \sum_{\alpha=1}^{m} \underline{z}_\alpha' D\underline{z}_\alpha$$

$$= \sum_{\alpha=1}^{m}\sum_{j=1}^{p} d_j z_{j\alpha}^2$$

$$\left(\because \sum_{\alpha=1}^{m}\left[d_1^2 z_{1\alpha}^2 + d_2^2 z_{2\alpha}^2 + \cdots + d_p^2 z_{p\alpha}^2\right)\right.$$

where $z_{j\alpha}$ denotes the j^{th} component of z_α
$(j = 1, 2, \ldots, p, \ \alpha = 1, \ldots, m)$

The C.F. of W which is the joint C.F. of elements of W is

$$E\left[e^{i\,tr\,TW}\right] = E\left[e^{i\,tr\,T\sum_{\alpha=1}^{m}\underline{y}_\alpha \underline{y}_\alpha'}\right]$$

$$= E\left[e^{i\sum_j\sum_\alpha d_j z_{j\alpha}^2}\right]$$

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$$= \prod_{\alpha} \prod_{j} E\left[e^{i d_j z_{j\alpha}^2}\right] \quad \left(\because E[e^{x+y}] = E[e^x . e^y] = E[e^x] E[e^y]\right)$$

($\because$ since z_α's are indep. and so

Since each $z_{j\alpha} \sim N(0,1)$ and hence each $z_{j\alpha}^2 \sim \chi^2_{(1)}$

$$= \prod_{\alpha} \prod_{j} (1 - 2i d_j)^{-1/2} \quad (\text{Using C.F. of } \chi^2 \text{ dist}^n)$$

$$= \prod_{\alpha} \left| 1 - 2i \sum_j d_j \right|^{-1/2}$$

$$= \prod_{\alpha} | 1 - 2iD |^{-1/2}$$

$$= | P' \Sigma^{-1} P - 2i P' T P |^{-m/2}$$

$$= \left[|P'P| \, |\Sigma^{-1} - 2iT| \right]^{-m/2}$$

$$= \left[\frac{|\Sigma|}{|\Sigma^{-1} - 2iT|^{-1}} \right]^{-m/2}$$

$$= \left[\frac{|\Sigma^{-1}|}{|\Sigma^{-1} - 2iT|} \right]^{m/2}$$

Hence the proof.

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* **Gram Matrix :-**

- A real symmetric matrix A is called a gram matrix if it is expressible as $A = BB'$ for some matrix B in which case B is called a grammian component of A.

Theorem

Proof :

$\cdot e^y]$
$[e^y]$
$\sim \chi^2_{(1)}$

Theorem :- Suppose $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ are independently distributed as $N_p(B\underline{a}_1, \Sigma), \ldots, N_p(B\underline{a}_n, \Sigma)$ respectively where B is some $p \times k$ matrix $n \geq p+k$ and $A = (\underline{a}_1\, \underline{a}_2 \ldots \underline{a}_n)$ is a given $k \times n$ matrix of rank k. Also let $X = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$, $H = AA'$, $G = XA'H^{-1}$ then

$$XX' - GHG' \sim W_p(\Sigma, n-k)$$

Proof :- Since A is full row rank, the gram matrix H is p.d. and hence we can find a non-singular matrix $P \ni$

$$\begin{aligned} I &= PHP' \\ &= P(AA')P' \\ &= (PA)(PA)' \end{aligned}$$

Showing that the matrix $Q = PA$ is some orthogonal. Thus, we can find an $(n-k) \times n$ matrix R of rank $(n-k) \ni (Q' \mid R')$ is orthogonal in which case $QR' = 0$

Evidently, since P is non-singular we have $AR' = 0$

Consider the orthogonal transformation

$$\underline{y} = X(Q' \mid R') = (XQ' \mid XR')$$

let
$$XQ' = \underline{y}^{(1)} = (\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_k)$$
$$XR' = \underline{y}^{(2)} = (\underline{y}_{k+1}, \underline{y}_{k+2}, \ldots, \underline{y}_n)$$

$$E(\underline{y}^{(1)}) = E(XQ') = E(X)Q'$$

Since $\underline{x}_1 \sim N_p(B\underline{a}_1, \Sigma)$

Also $X = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$

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$$E(\underline{y}^{(1)}) = E(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)\,\varphi'$$
$$= E(\underline{x}_1)\,E(\underline{x}_2) \cdots E(\underline{x}_n)\,\varphi'$$
$$= (B\underline{a}_1)(B\underline{a}_2) \cdots (B\underline{a}_n)\,\varphi'$$
$$= B(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)\,\varphi'$$
$$= BA\varphi'$$

Since $\varphi = PA \Rightarrow A = P^{-1}\varphi$.

$$E(\underline{y}^{(1)}) = BP^{-1}\varphi\varphi'$$

$$\Rightarrow E(\underline{y}^{(1)}) = BP^{-1} \quad (\text{Since } \varphi'\varphi = I)$$

$$E(\underline{y}^{(2)}) = E(XR') = E(X)R'$$
$$= BAR'$$
$$= 0 \quad (\because AR' = 0)$$

$$D(\underline{y}) = D[X(\varphi' \mid R')]$$
$$= E\left[(X - E(X))(\varphi' \mid R')\begin{pmatrix}\varphi \\ \hline R\end{pmatrix}(X - E(X))'\right]$$
$$= E[(X - E(X))(X - E(X))']$$
$$= D(X)$$
$$= D(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$$
$$= \text{diag}(\Sigma, \Sigma, \ldots, \Sigma) \quad (\because \text{Since } \underline{x}_\alpha\text{'s are independent})$$

$$D(\underline{x}_\alpha) = \Sigma \quad \forall \; \alpha = 1, 2, \ldots, n$$

Note that $D(\underline{y})$ is $P_n \times P_n$

$$\therefore \quad D(\underline{y}_\alpha) = \Sigma \quad \forall \; \alpha = 1, 2, \ldots, n$$

It follows that $\underline{y}_{k+1}, \underline{y}_{k+2}, \ldots, \underline{y}_n$ are independently distributed as $N_p(\underline{0}, \Sigma)$ each

Rema

Finally we have,

$X^{*}X' - GHG'$

$$= \underline{Y}\,(Q' \,\vdots\, R')^{-1} \begin{pmatrix} Q \\ \hdashline R \end{pmatrix}^{-1} \underline{Y}' - XA'H^{-1}\underbrace{HH^{-1}}_{I}AX'$$

$$= \underline{Y}\,\underline{Y}' - XA'H^{-1}AX' \qquad (\because Q' \text{ and } R' \text{ are orthogonal})$$

$$= \underline{Y}\,\underline{Y}' - XQ'(P^{-1})'H^{-1}P^{-1}QX'$$

$$= \underline{Y}\,\underline{Y}' - XQ'(PHP')^{-1}QX' \qquad (\text{Since } A = P^{-1}Q)$$

Since $PHP' = I$

$$\to = \underline{Y}\,\underline{Y}' - XQ'QX'$$

$$= \underline{Y}\,\underline{Y}' - \underline{Y}^{(1)}\underline{Y}^{(1)'} \qquad (\text{Since } XQ' = \underline{Y}^{(1)})$$

$$= \underline{Y}^{(1)}\underline{Y}^{(1)'} + \underline{Y}^{(2)}\underline{Y}^{(2)'} - \underline{Y}^{(1)}\underline{Y}^{(1)'}$$

$$= \underline{Y}^{(2)}\underline{Y}^{(2)'} \sim W_p(\Sigma, n-k)$$

Remark :- $GHG' = \underline{Y}^{(1)}\underline{Y}^{(1)'} = \sum_{\alpha=1}^{k} \underline{Y}_{\alpha}^{(1)}\underline{Y}_{\alpha}^{(1)'}$ and columns of $\underline{Y}_{\alpha}^{(1)}$ and $\underline{Y}_{\alpha}^{(1)'}$ are independently distributed as $N_p(\nu_1, \Sigma), \ldots, N_p(\nu_k, \Sigma)$ respectively where $\nu_{\alpha}' = B\underline{f}_{\alpha}$, $\underline{f}_{\alpha}$ being the α^{th} column of P^{-1}, $\alpha = 1, 2, \ldots, k$.

- The dist^n of such a matrix $\underline{Y}^{(1)}\underline{Y}^{(1)'}$ is called a non-central Wishart dist^n on k degrees of freedom with non-centrality parameter

$$\delta^2 = \sum_{\alpha=1}^{k} \nu_{\alpha}'\Sigma^{-1}\nu_{\alpha}$$

and it is denoted as

$$GHG' \sim W_{p,\delta}(\Sigma, k)$$

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Corollary (1) :- $XX' - GHG'$ and G are independent.

→ This follows as col^n of $\underline{y}_\alpha^{(1)}$'s are independently distributed.

Corollary (2) :- If $B = 0$ then

$$GHG' = \underline{y}^{(1)} \underline{y}^{(1)'} \sim W_p(\Sigma, k)$$

Theorem :- Suppose $W \sim W_p(\Sigma, m)$ and let W & Σ be partitioned respectively as $\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$ & $\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

where W_{11} and Σ_{11} are $k \times k$ then for a given (fixed) W_{22} the random matrix

$$W_{11\cdot2} = W_{11} - W_{12} \Sigma_{22}^{-1} W_{21} \sim W_k(\Sigma_{11\cdot2}, m-p+k)$$

where $\Sigma_{11\cdot2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

Proof :- By definition of Wishart matrix, we have $W = \sum_{\alpha=1}^{m} \underline{y}_\alpha \underline{y}_\alpha'$, for some 'm' indep. $N_p(\underline{0}, \Sigma)$

random vectors $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_m$

Now partitioning each $\underline{y}_\alpha$ as

$$\underline{y}_\alpha = (\underline{y}_\alpha^{(1)}, \underline{y}_\alpha^{(2)}), \quad \alpha = 1, 2, \ldots, m$$

where each $\underline{y}_\alpha^{(1)}$ has 'k' and each $\underline{y}_\alpha^{(2)}$ has remaining components.

So,

$$W_{11} = \sum_{\alpha=1}^{m} \underline{y}_\alpha^{(1)} \underline{y}_\alpha^{(1)'} \quad \& \quad W_{12} = \sum_{\alpha=1}^{m} \underline{y}_\alpha^{(1)} \underline{y}_\alpha^{(2)'} = W_{21}$$

$$W_{22} = \sum_{\alpha=1}^{m} \underline{y}_\alpha^{(2)} \underline{y}_\alpha^{(2)'}$$

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Evidently holding w_{22} fixed means holding $\underline{y}_\alpha^{(2)} = \underline{y}_\alpha^{(2)}$ (say) fixed ; $\alpha = 1, 2, \ldots m$

Now for $\alpha = 1, 2, \ldots, m$, the conditional distⁿ of $\underline{y}_\alpha^{(1)}$ holding $\underline{y}_\alpha^{(2)} = \underline{y}_\alpha^{(2)}$ fixed
i.e. $N_k(\Sigma_{12} \Sigma_{22}^{-1} \underline{y}_\alpha^{(2)}, \Sigma_{11 \cdot 2})$

Moreover, since $\underline{y}_\alpha$ are independent, so are $\underline{y}_\alpha^{(1)}$ disregarding of what conditions the corresponding $\underline{y}_\alpha^{(2)}$ are subjected to.

The 'n' conditional distⁿ of $\underline{y}_\alpha^{(1)}$ satisfy all the conditions of theorem (1) with $n = m$

$$X = (\underline{y}_1^{(1)}, \underline{y}_2^{(1)}, \ldots \underline{y}_m^{(1)})$$

$$B = \Sigma_{12} \Sigma_{22}^{-1}$$

$$A = (\underline{y}_1^{(2)}, \underline{y}_2^{(2)}, \ldots, \underline{y}_n^{(2)}), \quad \Sigma = \Sigma_{11 \cdot 2}$$

so that we have,

$$H = \sum_{\alpha=1}^{m} \underline{y}_\alpha^{(2)} \underline{y}_\alpha'^{(2)} = w_{22}$$

$$G = \underline{y}^{(1)} A' w_{22}^{-1} = w_{12} w_{22}^{-1}$$

$$XX' = \underline{y}^{(1)} y^{(1)'} = w_{11}$$

$$GHG' = w_{12} w_{22}^{-1} w_{21}$$

$$XX' - GHG' = w_{11} - w_{12} w_{22}^{-1} w_{21} = w_{11 \cdot 2}$$

Hence the proof.

NOTE: Conditional distⁿ of a Wishart sub-matrix given another Wishart sub-matrix is again Wishart.

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Def^n :- Let us call a Wishart $W_p(I, m)$ dist^n, a standard Wishart dist^n and it is denoted by $\Phi_{p.m}$.

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* **Derivation of Pdf of Wishart dist^n** :-

Theorem :- If $W \sim W_p(\Sigma, m)$ then its joint pdf is

$$f(W) = \frac{|\Sigma|^{-m/2} \, |W|^{(m-p-1)/2} \, e^{-\frac{1}{2} tr W \Sigma^{-1}}}{2^{mp/2} \, \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{m-i+1}{2}\right)} \quad \text{—— (1)}$$

Proof :- As W is Wishart $W_p(\Sigma, m)$ it is expressible as $W = \sum_{\alpha=1}^{m} \underline{Y}_\alpha \underline{Y}_\alpha'$ where $\underline{Y}_\alpha$'s are indep $N_p(\underline{0}, \Sigma)$ random vectors, moreover Σ is p.d. we can find a n.s. lower triangular Matrix $T \ni$
$T \Sigma T' = I$ —— (2)
Then the random vectors
$\underline{Z} = T \underline{Y}_\alpha$ —— (3), $\alpha = 1, 2, \ldots, m$ are ~~where~~ ~~$\underline{y}$~~ independently and identically distributed as $N_p(\underline{0}, \Sigma)$ So that the $p \times p$ random matrix

$$\Phi_{p.m} = TWT' = T \Sigma \underline{Y}_\alpha \underline{Y}_\alpha' T'$$
$$= \Sigma (T \underline{Y}_\alpha)(T \underline{Y}_\alpha)'$$
$$= \sum_{\alpha=1}^{m} \underline{Z}_\alpha \underline{Z}_\alpha' \quad \text{—— (4)}$$

This has a standard Wishart dist^n on m d.f.

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$$E[\Phi] - E\left[\sum_{\alpha=1}^{m} Z_\alpha Z_\alpha'\right] = 0$$

$$\left(\because \sum_{\alpha=1}^{m} [E(Z_\alpha Z_\alpha')] = 0\right)$$

$$D(\Phi) = D\left[\sum_{\alpha=1}^{m} Z_\alpha Z_\alpha'\right] = D\left[\sum_{\alpha=1}^{m} T Y_\alpha Y_\alpha' T'\right]$$

$$= T\, D(\Sigma Y_\alpha Y_\alpha')\, T'$$

$$= T \Sigma T' \quad \left(\because D(\Sigma Y_\alpha Y_\alpha') = \Sigma \text{ as } W = \Sigma Y_\alpha Y_\alpha' \sim W_p(\Sigma, m)\right)$$

Hence $\Phi_{pm} \sim W_p(\Sigma, I)$

Let $\underline{u}_i'$ denote the i^{th} row of $p \times m$ matrix

$$Z = (Z_1, Z_2 \cdots Z_m), \quad i = 1, 2, \ldots, p$$

Since $\underline{u}_i \neq 0$ $\exists$ a random orthogonal matrix P

$$\left.\begin{array}{l} \ni \ \underline{u}_1' P = (\underline{0}', w_1), \quad w_1 = \sqrt{\underline{u}_1' \underline{u}_1} \\ \underline{u}_i' P = (\underline{v}_i', w_i), \quad i = 1, 2, \ldots, p \end{array}\right\} \quad \text{—— (5)}$$

then $(p-1) \times (p-1)$ gram matrix $\underline{v}\underline{v}'$ where $v' = (\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_p)$ is standard wishart $\Phi_{(p-1)(m-1)}$ matrix and

$$|\Phi_{pm}| = w_1^2 \cdot |\Phi_{(p-1)(m-1)}| \quad \text{—— (6)}$$

Also since 'p' is orthogonal we have

$$\sum_{i=1}^{p} \underline{u}_i' \underline{u}_i = \sum_{i=1}^{p} w_i^2 + \sum_{i=1}^{p} \underline{v}_i' \underline{v}_i \quad \text{—— (7)}$$

$$\Rightarrow \sum_{i=1}^{p} w_i^2 = \operatorname{tr} \sum_{\alpha=1}^{m} \underline{z}_\alpha \underline{z}_\alpha' - \operatorname{tr} vv' \quad \left(\text{since } \sum \underline{u}_i' \underline{u}_i = \operatorname{tr} zz' \right)$$

$$= \operatorname{tr} \Phi_{pm} - \operatorname{tr} \Phi_{(p-1)(m-1)} \quad \text{——— (8)}$$

Moreover by our transformation (5)

(i) the r.v's $w_2, \dots, w_p$ are independently $N(0, 1)$ variates.

(ii) $w_1^2 \sim \chi^2_{(m)}$ (since $\underline{u}_i \sim N_m(0, I)$)

(iii) $w_1, w_2, \dots, w_p$ and $\Phi_{(p-1)(m-1)}$ are mutually independent.

Hence the joint pdf of $w_1^2; w_2, \dots, w_p$ and $\Phi_{(p-1)(m-1)}$ is

$$= \frac{1}{2^{m/2} \Gamma m/2} (w_1^2)^{m/2 - 1} e^{-w_1^2/2} \times \frac{1}{(2\pi)^{(p-1)/2}} e^{-(w_2^2 + w_3^3 + \dots + w_p^2)/2}$$

$$\times \; g_1\left(\Phi_{(p-1)(m-1)} \right) \quad \text{——— (9)}$$

where $g_1(\Phi_{(p-1)(m-1)})$ denotes the pdf of $\Phi_{(p-1)(m-1)}$

Besides since 'P' is orthogonal by (5) we have

$$\Phi_{pm} = zz' = (zP)(zP)' \quad \text{——— (10)}$$

and hence ϕ_{ij} denotes the (i, j)th element of Φ_{pm}.

we have,

$$\phi_{11} = w_1^2, \quad \phi_{ij} = w_i w_j, \text{ whenever } i = 1 \text{ and } j = 2, \dots, p$$

$$= v_i v_j, \text{ whenever } i, j \geq 2$$

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Now consider the inverse transformation from $w_1^2, w_2, \dots, w_p, \Phi_{(p-1)(m-1)}$ to Φ_{pm}

To evaluate the jacobian of the transformation we work out the partial differential table as shown below.

	Φ_{11}	Φ_{12}	Φ_{13}	$\dots$	Φ_{1p}	Φ_{22}	$\dots$	Φ_{2p}
w_1^2	1	0	$\dots$		0	0	$\dots$	0
w_2	0	w_1	$\dots$		0	0	$\dots$	0
w_3	0	0	w_1		0	0	$\dots$	0
$\vdots$								
w_p	0	0	0	$\dots$	w_1	0	$\dots$	0
$\underline{v}_2'\underline{v}_2$	0	0	0	$\dots$	0	1	$\dots$	0
$\vdots$								
$\underline{v}_p'\underline{v}_p$	0	0	0	$\dots$	0	0	$\dots$ 0	1

$w_1^2 \to \phi_{11}$

$w_1 w_2 \to \phi_{12}$

$w_1 w_3 \to \phi_{13}$

$\vdots$

$w_1 w_p \to \phi_{1p}$

$w_{22} = v_2 v_2' \to \phi_{22}$

$w_{23} = v_2 v_3' \to \phi_{23}$

$\vdots$

$w_{2p} = v_2 v_p' \to \phi_{2p}$

$\vdots$

$w_{pp} = v_p v_p' \Rightarrow \phi_{pp}$

Clearly the jacobian of the transformation is

$$w_1^{-(p-1)} = \frac{1}{|J|}$$

The pdf of Φ_{pm} is then

$$g(\Phi_{pm}) = \frac{\Phi_{11}^{(m-p-1)/2}}{2^{m/2}\Gamma_{m/2}\,(2\pi)^{(p-1)/2}} \left[e^{-\frac{1}{2}\left(tr\,\phi_{pm} - tr\,\phi_{(p-1)(m-1)}\right)} \cdot g_1\left(\phi_{(p-1)(m-1)}\right)\right] \quad (12)$$

(since $\frac{1}{|J|} = w_1^{-(p-1)}$ and since $w_1 = \sqrt{\Phi_{11}}$ $\Phi^{(m-p-1)/2}$

$$\Rightarrow \frac{1}{|J|} = (\Phi_{11})^{\frac{-(p-1)}{2}}$$

since (9), $(w_1^2)^{m/2-1} = (\Phi_{11})^{m/2-1}$

But following (6) we have

$$\phi_{11} = \omega_1^2 = \frac{|\phi_{pm}|}{|\phi_{(p-1)\times(m-1)}|} \qquad (13)$$

Hence substituting for ϕ_{11} in (12)

$$|\Phi_{pm}|^{-(m-p-1)/2} g(\Phi_{pm}) e^{tr\,\phi_{pm}/2}$$

$$= \frac{|\Phi_{(p-1)\times(m-1)}|^{-(m-p-1)/2} g_1(\Phi_{(p-1)\times(m-1)}) e^{\frac{1}{2} tr\,\phi_{(p-1)\times(m-1)}}}{2^{m/2}\Gamma(m/2)(2\pi)^{(p-1)/2}} \qquad (14)$$

$$\left(\because (\phi_{11})^{(m-p-1)/2} = \left(\frac{|\phi_{pm}|}{|\phi_{(p-1)\times(m-1)}|}\right)^{(m-p-1)/2} \right)$$

Similarly it can be shown that

$$|\Phi_{(p-1)(m-1)}|^{-(m-p-1)/2} g(\Phi_{(p-1)(m-1)}) e^{tr\,\phi_{(p-1)(m-1)}/2}$$

$$= \frac{|\Phi_{(p-2)(m-2)}|^{-(m-p-1)/2}}{2^{(m-1)/2}\Gamma\left(\frac{m-1}{2}\right)(2\pi)^{(p-2)/2}} \cdot g_2(\Phi_{(p-2)(m-2)}) e^{tr(\Phi_{(p-2)(m-2)})/2}$$

where g_2 is the pdf of $\Phi_{p-2,\,m-2}$.

Continuing in this way we get

$$|\Phi_{pm}|^{-\frac{(m-p-1)}{2}} g(\Phi_{pm}) e^{\frac{1}{2} tr\,\Phi_{pm}}$$

$$= \frac{1}{2^{m/2 + \frac{(m-1)}{2} + \cdots + \frac{(m-p-1)}{2}} (2\pi)^{\frac{(p-1)}{2} + \frac{(p-2)}{2} + \cdots + \frac{1}{2}} \Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m-1}{2}\right)\cdots\Gamma\left(\frac{m-p-1}{2}\right)}$$

which gives

$$g(\Phi_{pm}) = \frac{|\Phi_{pm}|^{(m-p-1)/2}}{2^{mp/2}\, \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{m-i-1}{2}\right)}$$

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– Finally let us consider the inverse transformation from T_{p_1} to ω. Recalling that $\Phi_{pm} = T\omega T'$ and T is lower triangular and denoting by t_{ij} and w_{ij} the $(i,j)^{th}$ element of T and ω respectively we have

$$\Phi_{ij} = \sum_{r<i} \sum_{s<j} t_{ir} w_{rs} t_{sj} \qquad \text{———} \quad (17)$$

Now the partial differential table for the said transformation is as follows :-

	ϕ_{11}	ϕ_{12}	---	ϕ_{1p}	ϕ_{22}	---	ϕ_{2p}
w_{11}	t_{11}^2	0	-	-	-	-	-
w_{12}	-	$t_{11}t_{12}$	-	-	-	-	-
⋮	⋮	0		—	—		-
w_{1p}	0	0		$t_{11}t_{pp}$	-		-
w_{22}	0	0		-	t_{22}^2		-
⋮	⋮	⋮		⋮	⋮		⋮
w_{2p}	0	0		⋮	⋮		t_{pp}^2

The required Jacobian transformation is

$$J = (t_{11}, t_{12}, \ldots, t_{pp})^{p+1}$$

$$= |\Sigma|^{-\frac{(p+1)}{2}} \qquad \text{———} \quad (18) \qquad (\text{by } (2))$$