

(34) Caratheodory Extension Theorem:-

(Only statement)

Statement: Suppose μ is a σ -finite measure defined on a field. Then there exists a unique measure $\bar{\mu}$ on the σ -field $\sigma(\mathcal{F})$, which is also σ -finite.

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The above theorem will help to define Lebesgue measure and Lebesgue-Stieltjes measure

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Lebesgue measure :-

Consider the measurable space $(\mathbb{R}, \mathcal{B})$.

We want to define a measure on this space.

Now to define a measure on \mathcal{B} is not practically possible, as we do not know the nature of \mathcal{B} . We know simply some types of Borel sets but not all. Hence we first define a measure on a field & using above theorem extend it to \mathcal{B} .

Consider the field

$\mathcal{F} = \{ I \mid I \text{ is a finite union of disjoint intervals of the type } (a, b], (-\infty, a] \text{ & } (\beta, \infty) \}$

We know that \mathcal{F} is a field.

Define a set function on \mathcal{F} as follows.

Let $A \in \mathcal{F}$. ~~If A is~~ Then $A = \bigcup_{j=1}^N I_j$,

where I_j 's are disjoint intervals of above types.

Define $\mu(A) = \mu\left(\bigcup_{j=1}^N I_j\right) = \sum_{j=1}^{\infty} \mu(I_j)$

where $\mu\{(a, b]\} = b - a$.

$$\mu(-\infty, a] = \infty$$

$$\mu(\beta, \infty) = \infty$$

Then μ is a measure.

(35) μ is not finite.
but μ is surely σ -finite

Define $A_n = [n, n+1]$ then $\mu(A_n) = n+1-n = 1 < \infty$

$$\text{f } R \subset \bigcup_{n=-\infty}^{\infty} A_n$$

Thus μ is σ -finite.

Then by Caratheodory extension theorem,
 \exists a unique measure λ on $\mathcal{B} = \sigma(\mathcal{F}_P)$
such that μ is a restriction of λ on \mathcal{F}
OR λ is extension of μ on \mathcal{B} .

$$\text{f } \lambda\{(a, b]\} = \mu\{(a, b]\} = b-a.$$

This measure λ is known as the
'Lebesgue measure'.

Sets which ~~can~~ can be measured by λ
are known as Lebesgue measurable sets.

→ Is every Borel set Lebesgue measurable?
The answer is 'Yes'.

Lebesgue measure of some Borel sets:-

(i) $B \in \mathcal{B}$ where B is a singleton set

$$\text{say } B = \{x\}, x \in R$$

$$\text{Now } \{x\} = \lim_{n \rightarrow \infty} (x - \frac{1}{n}, x]$$

$$\text{i) } \lambda(B) = \lambda\{x\} = \lambda\left\{\lim_{n \rightarrow \infty} (x - \frac{1}{n}, x]\right\}$$

$$= \lim_{n \rightarrow \infty} \lambda(x - \frac{1}{n}, x]$$

$$= \lim_{n \rightarrow \infty} x - (x - \frac{1}{n})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$= 0$$

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Thus Lebesgue measure of every singleton set is zero.

(2) Let $B = [a, b]$

$$= \{a\} \cup (a, b]$$

$$\therefore \lambda(B) = \lambda[\{a\} \cup (a, b)]$$

$$= \lambda\{a\} + \lambda(a, b]$$

$$= 0 + b-a$$

$$= b-a$$

(3) Let $B = [a, b)$

$$= \{a\} \cup (a, b)$$

$$= \lim_{n \rightarrow \infty} [\{a\} \cup (a, b - \frac{1}{n})]$$

$$\therefore \lambda(B) = \lambda\{a\} + \lim_{n \rightarrow \infty} \lambda(a, b - \frac{1}{n})$$

$$= 0 + \lim_{n \rightarrow \infty} (b - \frac{1}{n} - a)$$

$$= b-a$$

(4) Let $B = (a, b)$

$$\therefore = \lim_{n \rightarrow \infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

$$\therefore \lambda(B) = \lim_{n \rightarrow \infty} \lambda[a + \frac{1}{n}, b - \frac{1}{n}]$$

$$= \lim_{n \rightarrow \infty} (b - \frac{1}{n} - a - \frac{1}{n})$$

$$= b-a$$

Thus, whatever type of interval, we have,
its Lebesgue measure is $b-a$.

In general

$$\lambda(A-B) = \lambda(A) - \lambda(B) \quad \text{if } B \subset A,$$

provided both $\lambda(A)$ & $\lambda(B)$ are not $+\infty$.

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(37) Let $B = \text{Set of rationals}$

$$= \{r_1, r_2, \dots\}$$

then $B = \bigcup_k A_k$ where $A_k = \{r_k\}$

$$\therefore \lambda(B) = \lambda\left(\bigcup_k A_k\right) = \sum_k \lambda(A_k) = 0$$

* The Lebesgue measure of the set of rationals is zero.

e.g. Let $B = \text{set of rationals between } (3, 10)$.

then $\lambda(B) = 0$

Let $B_1 = \text{set of irrationals between } (3, 10]$

then

$$B_1 = (3, 10] - \{\text{the set of rationals betn } (3, 10]\}$$

$$\therefore \lambda(B_1) = \lambda((3, 10]) - 0$$

$$= 7$$

let $B_2 = \text{set of irrationals in } (-\infty, \infty)$

then $\lambda(B_2) = \infty$.

Remark: For whatever Borel set we want to find the Lebesgue measure, try to write that set in the form $(a \ x \ b)$.

(38) Lebesgue-Stieltjes measure (L-S measure)

Let F be a function on real line, such that F is non-decreasing, right continuous, $F(-\infty) = 0$ & $F(+\infty) = 1$.

(In probability theory, we call such a function as distribution function.)

Let \mathcal{F}_F be the field as defined above.

$\mathcal{F}_F = \{I \mid I \text{ is finite union of disjoint intervals of the type } (a, b], (-\infty, a], (b, \infty)\}$

Define a measure μ on \mathcal{F}_F as follows.

$$\mu\{(a, b]\} = F(b) - F(a) \geq 0 \quad (\because F \text{ is non-decreasing})$$

$$\begin{aligned} \mu(\emptyset) &= \lim_{b \downarrow a} F(b) - F(a) \\ &= F(a^+) - F(a) \\ &= 0 \quad (\because F \text{ is right continuous}) \end{aligned}$$

$$\mu(\beta, \infty) = \lim_{n \rightarrow \infty} (\beta, n]$$

& for $I \in \mathcal{F}_F$,

$$\mu(I) = \mu(\bigcup_{j=1}^K I_j) = \sum_{j=1}^K \mu(I_j)$$

where I_j 's are intervals of above type.

$$\begin{aligned} \mu(\mathbb{R}) &= F(+\infty) - F(-\infty) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

$\Rightarrow \mu$ is a finite measure & hence μ is σ -finite.

Hence by Caratheodory extension theorem, \exists a unique measure μ_F on \mathcal{B} , $\exists \mu_F\{(a, b]\} = F(b) - F(a)$.

This measure μ_F is known as the 'Lebesgue-Stieltjes measure' or L-S measure corresponding to F .

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Ex: let $F = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \leq x < 1 \\ \frac{3}{2} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2. \end{cases}$

Note that $F(0^-) = 0$, $F(0) = F(0^+) = \frac{1}{2}$.

$\Rightarrow x=0$ is a point of discontinuity &
all $x \neq 0$, $x \in \mathbb{R}$ is a continuity point.

Let us find L-S measure of different sets.

Let $B = \{x\}$, $x \in \mathbb{R}$.

Let $x=0$ i.e. $B = \{0\}$.

$$\text{then } \mu_F(B) = \mu_F\{0\}$$

$$= \lim_{n \rightarrow \infty} \mu_F(0^{-\frac{1}{n}}, 0]$$

$$= \lim_{n \rightarrow \infty} F(0) - F(0^{-\frac{1}{n}})$$

$$= F(0) - F(0^-)$$

$$= \frac{1}{2} - 0$$

Let $x \neq 0$ say $x = \frac{3}{4}$, $B = \{\frac{3}{4}\}$

$$\mu_F(B) = \mu_F\{\frac{3}{4}\}$$

$$= \lim_{n \rightarrow \infty} \mu_F(\frac{3}{4}-n, \frac{3}{4}]$$

$$= \lim_{n \rightarrow \infty} [F(\frac{3}{4}) - F(\frac{3}{4}-n)]$$

$$= F(\frac{3}{4}) - \lim_{n \rightarrow \infty} F(\frac{3}{4}-n)$$

$$= F(\frac{3}{4}) - F(\frac{3}{4}^-)$$

$$= \frac{1}{2} - \frac{1}{2}$$

$$= 0$$

Similarly if $x = 4 \Rightarrow \mu_F\{4\} = 0$.

Thus the L-S measure of every continuity pt.

of F is zero.

& for points of discontinuity,
L-S measure = height of jump at that point.

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Next, suppose $B = [a \ b]$

$$= \{a\} \cup (a \ b]$$

$$\begin{aligned}\text{i) } \mu_F(B) &= \mu_F\{a\} + \mu_F(a \ b] \\ &= \mu_F\{a\} + F(b) - F(a)\end{aligned}$$

$$\text{Let } B = (a \ b)$$

$$= (a \ b] - \{b\}$$

$$\text{i) } \mu_F(B) = \mu_F(a \ b] - \mu_F\{b\}$$

$$\text{Let } B = [a \ b)$$

$$= (a \ b] + \{a\} - \{b\}$$

$$\begin{aligned}\text{i) } \mu_F(B) &= \mu_F(a \ b] + \mu_F\{a\} - \mu_F\{b\} \\ &= F(b) - F(a) + \mu_F\{a\} - \mu_F\{b\}.\end{aligned}$$

$$\text{so e.g. let } B = [0 \ 3]$$

$$= \{0\} \cup (0 \ 3]$$

$$\begin{aligned}\text{i) } \mu_F(B) &= \mu_F\{0\} + (F(3) - F(0)) \\ &= \frac{1}{2} + 1 - \frac{1}{2}\end{aligned}$$

$$\text{let } B = (0, 3) = (0 \ 3] - \{3\}$$

$$\begin{aligned}\text{i) } \mu_F(B) &= \mu_F(0, 3] - \mu_F\{3\} \\ &= F(3) - F(0) - 0 \\ &= 1 - \frac{1}{2} - 0\end{aligned}$$

$$B = \left(\frac{3}{2}, \frac{7}{2}\right] = V_2$$

$$\mu_F(B) = F\left(\frac{7}{2}\right) - F\left(\frac{3}{2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

—x—

(41) Let $F(x) = \begin{cases} 0 & x < -3 \\ \frac{1}{4} & -3 \leq x < 0 \\ \frac{x+1}{2} & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$

Find L-S measure of the following sets.

$$A = \{0, \frac{1}{2}\}, B = [0, \frac{1}{2}], C = (-3, 0), D = (0, \frac{1}{2})$$

$$E = [-3, 0].$$

Here we note that

x	$F(x^-)$	$F(x)$	$F(x^+)$
-3	0	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{3}{4}$	1	1

Thus $x = -3, 0, \frac{1}{2}$ are points of discontinuity & all $x \in \mathbb{R}$ except $x = -3, 0, \frac{1}{2}$ are continuity points of F .

$$\text{i) } \mu_F \{-3\} = \frac{1}{4} - 0 = \frac{1}{4}$$

$$\mu_F \{0\} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\mu_F \{\frac{1}{2}\} = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\text{ii) } \mu_F(A) = \mu_F \{0, \frac{1}{2}\} = \mu_F \{0\} + \mu_F \{\frac{1}{2}\}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

$$\begin{aligned} \mu_F(B) &= \mu_F [0, \frac{1}{2}] \\ &= \mu_F \{0\} + \mu_F (0, \frac{1}{2}) \\ &= \frac{1}{4} + F(\frac{1}{2}) - F(0) \\ &= \frac{1}{4} + 1 - \frac{1}{2} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mu_F(C) &= \mu_F (-3, 0] \\ &= \mu_F \{(-3, 0]\} - \{0\} \\ &= \mu_F (-3, 0] - \mu_F \{0\} \\ &= F(0) - F(-3) - \frac{1}{4} \\ &= \frac{1}{2} - \frac{1}{4} - \frac{1}{4} \\ &= 0 \end{aligned}$$

$$\begin{aligned}
 (42) \quad \mu_F([0, 1]) &= \mu_F([0, \frac{1}{2}]) \\
 &\quad + \mu_F([\frac{1}{2}, 1]) - \mu_F\{\frac{1}{2}\} \\
 &= F(\frac{1}{2}) - F(0) - \frac{1}{4} \\
 &= 1 - \frac{1}{2} - \frac{1}{4} \\
 &= \frac{1}{4} \\
 \mu_F([0, 1]) &= \mu_F\{[-3, 0]\} \\
 &= \mu_F\{-3\} + \mu_F([-3, 0]) \\
 &= \frac{1}{4} + F(0) - F(-3) \\
 &= \frac{1}{4} + \frac{1}{2} - \frac{1}{4} \\
 &= \frac{1}{2}
 \end{aligned}$$

Let $F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } 0 \leq x < 1 \\ 1 - \frac{1-x+1}{2}e^{-x+1}, & \text{if } x \geq 1. \end{cases}$

Find L-S measure of the sets $\{0\}$, $\{0, 1\}$,

$$[0, 1], (0, 1),$$

$$\text{Here } F(0^-) = 0, F(0) = F(0^+) = \frac{1}{2}.$$

& $x=0$ is the only point of discontinuity &
 $\forall x \neq 0$ is continuity point of F .

$$\text{i) } \mu_F\{0\} = \frac{1}{2} - 0 = \frac{1}{2} \neq \mu_F\{x\} = 0 \quad \forall x \neq 0.$$

$$\text{ii) } \mu_F\{0, 1\} = \frac{1}{2} + 0 = \frac{1}{2}$$

$$\begin{aligned}
 \mu_F[0, 1] &= \mu_F\{0\} + \mu_F(0, 1] \\
 &= \frac{1}{2} + F(1) - F(0) \\
 &= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mu_F(0, 1) &= \mu_F(0, 1] - \mu_F\{1\} \\
 &= F(1) - F(0) - 0 \\
 &= \frac{1}{2} - \frac{1}{2} \\
 &= 0
 \end{aligned}$$

(43)

Let $\sigma = \mathbb{R}$ Let $A = (-\infty, 0]$ & $\mathcal{C} = \{-2, \emptyset, A, A'\}$ then \mathcal{C} is a field.Define μ on \mathcal{C} as $\mu(B) = \begin{cases} 1 & \text{if } B \in B \\ 0 & \text{if } B \notin B \end{cases}$ then $\mu(-2) = 1$, $\mu(\emptyset) = 0$, $\mu(A) = 0$ & $\mu(A') = 1$.Now consider a set $C = (-5, 2) \subset A$.but $\mu(A) = 0$ $\Rightarrow \mu(C) = 0 \because C \notin \mathcal{C}$ & μ is defined only for sets in \mathcal{C} & not for any other sets.In fact, here $\mu(C)$ is not defined.So to rectify this, define a σ -field \mathcal{C}^* as

$$\mathcal{C}^* = \sigma [\mathcal{C} \cup \{\text{all subsets of } A\}]$$

[to get rid of the problem that subsets of sets of measure zero are not measurable, we need to define \mathcal{C}^* as above.]Def: A measure μ is called a complete measure if $\mu(A) = 0$, $B \subset A \Rightarrow B$ is a measurable set & hence $\mu(B) = 0$.Thus $(\mathbb{R}, \mathcal{A}, \mu)$ is known as complete measure space, if for mble set $A \in \mathcal{A}$ of measure zero, all subsets of A are measurable.A set of measure zero i.e $\mu(A) = 0$ is known as a null set (or a null set).Remark: Any measure space, can be completed by the procedure of defining \mathcal{C}^* .