STA2101C01 : Measure Theory Unit - 1

(4 Credit course).

Let Ω be some abstract space.

Let $A \subset \Omega$, $B \subset \Omega$. Then we know how to define $A \cup B$, $A \cap B$, A^c, B^c etc. (set theory).

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets i.e.

$A_n \subset \Omega \quad \forall n = 1, 2, \cdots$

& the seq^n. is $\{A_1, A_2, A_3, \cdots\}$.

Our interest is to study the limiting behaviour of the sequence $\{A_n\}$.

Recall - the concept of $\lim a_n$ where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.

First we will define limit superior and limit inferior.

So let $\{A_n\}$ be a seq^n. of sets of Ω.

i.e $A_n \subset \Omega \quad \forall n = 1, 2 \cdots$

then

~~lim inf An~~

$$\lim \sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$$

and

$$\lim \inf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$$

Note that $\lim \sup A_n$ and $\lim \inf A_n$, both are sets & subsets of Ω.

Result 1 : $\lim \inf A_n \subset \lim \sup A_n$.

Proof : Let $\omega \in \lim \inf A_n$,

to prove $\omega \in \lim \sup A_n$.

So let $\omega \in \lim \inf A_n$

i.e $\omega \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

(2) let $\bigcap_{n=k}^{\infty} A_n = C_k$

then $\omega \in \bigcup_{k=1}^{\infty} C_k$

$\Leftrightarrow \omega \in C_k$ for some k

$\Leftrightarrow \omega \in \bigcap_{n=k}^{\infty} A_n$ for some k

$\Leftrightarrow \omega \in A_n \ \forall n \geq k$, for some positive integer k. —— (1)

Now to prove $\omega \in \limsup A_n$

i.e $\omega \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

$= \bigcap_{r=1}^{\infty} \bigcup_{n=r}^{\infty} A_n$

let $\bigcup_{n=r}^{\infty} A_n = B_r$ (say)

Hence $\Leftrightarrow$ to prove $\omega \in \bigcap_{r=1}^{\infty} B_r$

i.e $\Leftrightarrow$ to prove $\omega \in B_r \ \forall r$

i.e $\Leftrightarrow$ to prove $\omega \in \bigcup_{n=r}^{\infty} A_n \quad \forall r \geq 1$ —— (2)

Let r be any fixed positive integer.

to prove $\omega \in \bigcup_{n=r}^{\infty} A_n$.

Suppose $r < k$ $\qquad r < r+1 < \cdots < k < k+1 < \cdots$

From (1) we know that $\qquad k \longrightarrow$

$\omega \in A_n \quad \forall n \geq k$

$\Rightarrow \omega \in \bigcup_{n=k}^{\infty} A_n \subset \bigcup_{n=r}^{\infty} A_n$

$\Rightarrow \omega \in \bigcup_{n=r}^{\infty} A_n$

$k < k+1 < \cdots < r < \cdots$

If $k \leq r$, again from (1),

$\omega \in A_n \quad \forall n \geq k$

$\Rightarrow \omega \in A_r, A_{r+1}, \ldots$

$\Rightarrow \omega \in \bigcup_{n=r}^{\infty} A_n$

Thus in either case, $\omega \in \bigcup_{n=r}^{\infty} A_n$. Since r is arbitrary, $\omega \in \bigcup_{n=r}^{\infty} A_n \ \forall r \geq 1$.

Thus (2) holds. Hence the proof.

—x—

(3)

Thm: $\limsup A_n = \{\omega \in \Omega \mid \omega \in A_n$ for an infinite no. of values of $n\}$

$= \{\omega \in \Omega \mid \omega \in A_n$ infinitely often (i.o.)$\}$

Proof:

We know that $\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

let $\omega \in \limsup A_n$

$\Leftrightarrow \omega \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

$\Leftrightarrow \omega \in \bigcup_{n=k}^{\infty} A_n, \ \forall k \geq 1$

$\Leftrightarrow \omega \in A_n$ for some $n \geq k$, $\forall k \geq 1$

Thus $\forall k \geq 1$, $\exists$ an integer $n \geq k$ s.t. $\omega \in A_n$

$\Leftrightarrow \omega \in A_n$ infinitely often.

—x—

Thm: $\liminf A_n = \{\omega \in \Omega \mid \omega \in$ all A_n except possibly a finite no. of them$\}$

proof: We know that

$\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

let $\omega \in \liminf A_n$

$\Leftrightarrow \omega \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

$\Leftrightarrow \omega \in \bigcap_{n=k}^{\infty} A_n$ for some $k \geq 1$

$\Leftrightarrow \omega \in$ all A_n $\forall n \geq k$

where k is some positive integer.

$\Leftrightarrow \omega \in$ all A_n, except possibly a finite no. of them.

—x—

4

Def: If for a sequence of sets $\{A_n\}$, $\liminf A_n = \limsup A_n$, we say that $\lim A_n$ exists and

$$\lim A_n = \liminf A_n = \limsup A_n$$

—x—

Result: $(\liminf A_n)^c = \limsup A_n^c$

$$(\liminf A_n)' = \limsup A_n'$$

proof: $(\liminf A_n)' = \left(\bigcup_{K=1}^{\infty} \bigcap_{n=K}^{\infty} A_n\right)'$

$$= \bigcap_{K=1}^{\infty} \left(\bigcap_{n=K}^{\infty} A_n\right)'$$

$$= \bigcap_{K=1}^{\infty} \bigcup_{n=K}^{\infty} A_n'$$

$$= \limsup A_n'$$

Similarly, $(\limsup A_n)' = \liminf A_n'$

—x—

Ex: Suppose A & $B \subset \Omega$.

Define $A_n = \begin{cases} A & \text{if } n \text{ is even} \\ B & \text{if } n \text{ is odd.} \end{cases}$

Check whether $\lim A_n$ exists or not?

Here $\limsup A_n = \bigcap_{K=1}^{\infty} \bigcup_{n=K}^{\infty} A_n$

$$= \bigcap_{K=1}^{\infty} (A \cup B)$$

$$= A \cup B$$

& $\liminf A_n = \bigcup_{K=1}^{\infty} \bigcap_{n=K}^{\infty} A_n$

$$= \bigcup_{K=1}^{\infty} (A \cap B)$$

$$= A \cap B$$

In general $\liminf A_n \neq \limsup A_n$

If $A = B$, then only $\liminf A_n = \limsup A_n$

Thus $\lim A_n$ exists only when $A = B$ —x—

(5) Let $\{A_n\}$ be a sequence of disjoint sets.
Does $\lim A_n$ exists ?
Clearly there is now ω which belongs to an infinite no. of A_n's.

$\Rightarrow \lim\sup A_n = \phi$

but $\lim\inf A_n \subset \lim\sup A_n$

$\Rightarrow \lim\inf A_n = \phi$.

Thus $\lim A_n = \lim\inf A_n = \lim\sup A_n = \phi$.

—x—

Def: Monotone sequence:

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets of Ω.

$\{A_n\}$ is said to be a monotone increasing sequence if $A_n \subset A_{n+1}$ $\forall n \geq 1$ i.e

$A_1 \subset A_2 \subset A_3 \subset \cdots\cdots$ Notation: $A_n \uparrow$

Similarly, $\{A_n\}$ is said to be a monotone decreasing sequence if $A_n \supset A_{n+1}$ $\forall n \geq 1$ i.e

$A_1 \supset A_2 \supset A_3 \supset \cdots\cdots$ Notation: $A_n \downarrow$.

—x—

Result:- If sequence $\{A_n\}$ is such that $A_n \uparrow$, then $\lim A_n = \bigcup_{n=1}^{\infty} A_n$ &

If $A_n \downarrow$, then $\lim A_n = \bigcap_{n=1}^{\infty} A_n$

Proof: Suppose $A_n \uparrow$ i.e $A_n \subset A_{n+1}$, $\forall n \geq 1$.

to prove that $\lim A_n = \bigcup_{n=1}^{\infty} A_n$.

Consider $\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

Consider
$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots\cdots$$
$$= A_2 \cup A_3 \cup \cdots \quad (\because A_1 \subset A_2)$$
$$= A_3 \cup A_4 \cup \cdots \quad (\because A_2 \subset A_3)$$
$$\vdots$$
$$= \bigcup_{n=k}^{\infty} A_n$$

Thus $\bigcup_{n=k}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n \quad \forall k \geq 1$

(6) Hence

$$\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$$

$$= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_n$$

$$= \bigcup_{n=1}^{\infty} A_n. \quad \text{—} ①$$

next, consider

$$\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$$

$$= \bigcup_{k=1}^{\infty} A_k \qquad (\text{since } A_1 \subset A_2 \subset \cdots)$$

$$= \bigcup_{n=1}^{\infty} A_n \quad \text{—} ②$$

From ① & ②, $\liminf A_n = \limsup A_n$ & hence $\lim A_n$ exists & $\lim A_n = \bigcup_{n=1}^{\infty} A_n$

Now suppose $A_n \downarrow$, to prove $\lim A_n = \bigcap_{n=1}^{\infty} A_n$.

Consider $\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

Now $A_n \downarrow \Rightarrow A_1 \supset A_2 \supset A_3 \supset \cdots$

$\Rightarrow \bigcup_{n=k}^{\infty} A_n = A_k$

& hence $\limsup A_n = \bigcap_{k=1}^{\infty} A_k$ — ③

Further $\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$

now consider $\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots$

but $A_1 \supset A_2 \supset A_3 \supset \cdots$

$\Rightarrow \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=2}^{\infty} A_n = \bigcap_{n=3}^{\infty} A_3 = \cdots \bigcap_{n=k} A_n$

& hence $\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} A_n$

$$= \bigcap_{n=1}^{\infty} A_n \quad \text{—} ④$$

From ③ & ④, we have

$\lim A_n = \liminf A_n = \limsup A_n = \bigcap_{n=1}^{\infty} A_n$

—x—

Indirect proof using first part of the result –

Suppose $A_n \downarrow$, i.e $A_1 \supset A_2 \supset A_3 \supset \cdots$

to prove $\lim A_n = \bigcap_{n=1}^{\infty} A_n$.

Since $A_1 \supset A_2 \supset A_3 \supset \cdots$

$\Rightarrow A_1^c \subset A_2^c \subset A_3^c \subset \cdots$

i.e $A_1' \subset A_2' \subset A_3' \subset \cdots$

Thus $\{A_n'\} \uparrow$

then using the first part of the result

$$\liminf A_n' = \limsup A_n' = \bigcup_{n=1}^{\infty} A_n'$$

Taking compliments, we have

$$(\liminf A_n')' = (\limsup A_n')' = \left(\bigcup_{n=1}^{\infty} A_n'\right)'$$

$$\Rightarrow \limsup A_n = \liminf A_n = \bigcap_{n=1}^{\infty} A_n = \lim A_n$$

hence the result.

—x—

Examples:

1) let $A_n = [-\frac{1}{n}, \frac{1}{n})$, $n \geq 1$.

Check whether $\lim A_n$ exists or not!

$A_1 = [-1, 1)$

$A_2 = [-\frac{1}{2}, \frac{1}{2})$

$A_3 = [-\frac{1}{3}, \frac{1}{3})$

$\vdots$

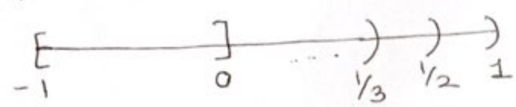
Note that $A_n \downarrow$ & hence $\lim A_n = \bigcap_{n=1}^{\infty} A_n = \{0\}$.

—x—

For each of the following seq^n. of sets, check whether $\lim A_n$ exists or not?

2) $A_n = [-1, 1/n)$, $\forall n \geq 1$

$A_1 = [-1, 1)$



$A_2 = [-1, 1/2)$

$A_3 = [-1, 1/3)$

We note that $A_n \downarrow$ &

$$\lim A_n = \bigcap_{n=1}^{\infty} A_n = [-1, 0].$$

—x—

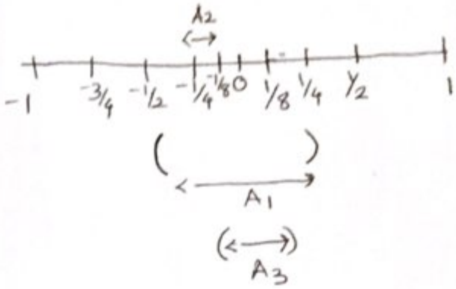
(8)

$$A_n = \left(-2^{-n}, \left(-\tfrac{1}{2}\right)^{n+1}\right)$$

$$= \left(-\frac{1}{2^n}, \left(-\frac{1}{2}\right)^{n+1}\right), \quad n \geq 1.$$

$A_1 = \left(-\frac{1}{2}, \frac{1}{4}\right)$

$A_2 = \left(-\frac{1}{4}, -\frac{1}{8}\right)$



$A_3 = \left(-\frac{1}{8}, \frac{1}{16}\right)$

$A_4 = \left(-\frac{1}{16}, -\frac{1}{32}\right)$

We observe that

$\lim\sup A_n = \{\omega \in \Omega \mid \omega \in A_n \text{ i.o}\}$
$= \{0\}$.

$\&$ $\lim\inf A_n = \phi$

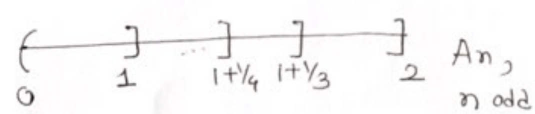
$A_{2n+1} \to \{0\}$
$\&$ A_{2n} are disjoint sets

Hence $\lim A_n$ does not exist.

—x—

Let $A_n = \begin{cases} (0, 1-\frac{1}{n}), & \text{if } n \text{ is even} \\ (0, 1+\frac{1}{n}], & \text{if } n \text{ is odd.} \end{cases}$

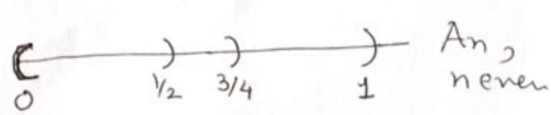
$A_1 = (0, 2]$



$A_2 = (0, 1/2)$

$A_3 = (0, 1+1/3]$

$A_4 = (0, 3/4)$



We note that as n increases

$A_n \to (0, 1)$ for n even

$\&$ $A_n \to (0, 1]$ for n odd

i.e.

$A_{2n} \to (0, 1)$

$A_{2n+1} \to (0, 1]$

$\Rightarrow \lim\inf A_n = (0, 1)$ $\&$ $\lim\sup A_n = (0, 1]$

Hence limit does not ~~ext~~ exist.

—x—

(9) $A_{2n+1} = \left(-\frac{1}{2n+1}, 1\right)$, $A_{2n} = \left[0, 1-\frac{1}{2n}\right]$.

$A_1 = (-1, 1)$ $\qquad$ $A_2 = \left[0, \frac{1}{2}\right]$

$A_3 = \left(-\frac{1}{3}, 1\right)$ $\qquad$ $A_4 = \left[0, \frac{3}{4}\right]$

$A_5 = \left(-\frac{1}{5}, 1\right)$ $\qquad$ $A_6 = \left[0, \frac{5}{6}\right]$

$\vdots$

we note that

$A_{2n+1} \to [0, 1)$

& $A_{2n} \to [0, 1)$

Thus $\liminf A_n = \limsup A_n = [0, 1) = \lim A_n$

Thus $\lim A_n$ exists.

——x——