Convergence :-

Let $(\Omega, \mathcal{A}, \mu)$ be fixed measure space.
We consider mble functions on this space.
We want to study the convergence patten of f_n to f.

We say $f_n(\omega) \to f(\omega)$, $\forall \omega \in \Omega$

(Remember, $f_n(\omega)$ $\forall n$ & $f(\omega)$ are real numbers)

So, by the concept of convergence of sequences of real numbers,

We say $f_n(\omega) \to f(\omega)$, if $\forall \varepsilon > 0$, $\exists N$ large enough s.t. $|f_n(\omega) - f(\omega)| < \varepsilon$

$$\forall n \geq N(\varepsilon, \omega)$$

(We assume both f_n and f are finite valued)

The above condition can also be written as

$$\underbrace{\bigcap_{\varepsilon > 0}}_{\forall \varepsilon > 0} \underbrace{\bigcup_{N=1}^{\infty}}_{\text{for some } N \text{ large}} \underbrace{\bigcap_{n \geq N}^{\infty}}_{\forall n \geq N} \{\omega \mid |f_n(\omega) - f(\omega)| < \varepsilon\}$$

& the above set is known as set of Convergence.

We will discuss two types of convergences.

1) Convergence almost everywhere
2) Convergence in measure.

Convergence almost everywhere :-

A set N is said to hold almost everywhere, if $\mu(N')=0$ i.e N' is a μ-null set.

In general, a concept is said to hold almost everywhere, if the measure of set on which concept doesnot hold is zero.

e.g. 1) A function f is said to be a.e. finite valued if $\mu\{\omega \mid |f(\omega)|=\infty\}=0$

2) Two functions f and g are said to be equivalent a.e. if

$$\mu\{\omega \mid f(\omega)\neq g(\omega)\}=0$$

Henceforth we consider $f:\Omega\to\mathbb{R}$ which are finite a.e. i.e. even though the f^n f takes values $+\infty$ or $-\infty$, the μ measure of such sets is zero.

So let $\{f_n\}$ be a seq^n of a.e. finite valued mble functions.

<u>Def</u> : A seq^n of mble function $\{f_n\}$ converges to a mble f^n f a.e. if

$$\mu\{\omega \mid f_n(\omega)\not\to f(\omega)\}=0$$

i.e if $\omega\in N$, $f_n\to f$

& if $\omega\in N'$, $f_n\not\to f$ (N' has measure zero).

Criteria for a.e. convergence:-

<u>Result</u> : $f_n\to f$ a.e. iff

$$\lim_{k\to\infty}\mu\left(\bigcup_{n=k}^{\infty}|f_n-f|\geq\epsilon\right)=0 \quad \forall\,\epsilon>0,$$

provided $\mu\left(\bigcup_{n=k}^{\infty}|f_n-f|\geq\epsilon\right)<\infty$ for some k.

<u>Proof</u> :

Proof :

We know that

$f_n \to f$ a.e.

iff $\mu[\omega \mid f_n(\omega) \nrightarrow f(\omega)] = 0$.

Now consider

$$\{\omega \mid f_n(\omega) \to f(\omega)\} = \bigcap_{\epsilon>0} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{\omega \mid |f_n(\omega) - f(\omega)| < \epsilon\}$$

This is the set of convergence (N)

then $f_n \to f$ a.e. if $\mu(N') = 0$

i.e iff $\mu\left[\bigcup_{\epsilon>0} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon\right] = 0$

i.e iff $\mu\left[\bigcap_{k=1}^{\infty} \underbrace{\bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon}_{B_K}\right] = 0 \quad \forall \epsilon > 0$

then $B_K \downarrow$ & $\lim B_K = \bigcap_{k=1}^{\infty} B_k$

i.e iff $\mu\left[\lim_{k\to\infty} B_K\right] = 0 \quad \forall \epsilon > 0$

i.e. iff $\mu\left[\lim_{k\to\infty} \bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon\right] = 0 \quad \forall \epsilon > 0$

i.e iff $\lim_{k\to\infty} \mu\left[\bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon\right] = 0 \quad \forall \epsilon > 0$

provided $\mu\left[\bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon\right] < \infty$ for some k.

This is known as the a.e. convergence criteria.

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Def: $\{f_n\}$ converges mutually a.e. if

$$\mu\{\omega \mid |f_m(\omega) - f_n(\omega)| \not\to 0\} = 0 \text{ as } m, n \to \infty$$

Remark: Proceeding as above, the criteria for convergence mutually a.e. is

$$\mu\left[\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} |f_{n+k} - f_n| \geq \varepsilon\right] = 0.$$

Remark: $f_n \to f$ a.e $\Longleftrightarrow \{f_n\}$ converges mutually a.e.

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Convergence in measure:—

Def: A sequence of mble function $\{f_n\}$ is said to converge in measure to a mble f^n f if $\forall \varepsilon > 0$,

$$\mu[\omega \mid |f_n(\omega) - f(\omega)| > \varepsilon] \to 0 \text{ as } n \to \infty.$$

[Note that nothing can be said about compliment because we don't know whether μ is finite or not also what is value of $\mu(\Omega)$.

Notation $f_n \xrightarrow{\mu} f$

Def: A sequence of mble functions $\{f_n\}$ converges mutually in measure if

$$\forall \varepsilon > 0, \ \mu[|f_m - f_n| \geq \varepsilon] \to 0 \text{ as } m, n \to \infty.$$

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Suppose $f_n \xrightarrow{\mu} f$. Now consider.

$$\begin{aligned}
\mu[|f_m - f_n| \geq \varepsilon] &= \mu[|f_m - f + f - f_n| \geq \varepsilon] \\
&\leq \mu[|f_m - f| + |f_n - f| \geq \varepsilon] \\
&\leq \mu\left[|f_m - f| \geq \tfrac{\varepsilon}{2} \text{ or } |f_n - f| \geq \tfrac{\varepsilon}{2}\right] \\
&\leq \mu\left[|f_m - f| \geq \tfrac{\varepsilon}{2}\right] + \mu\left[|f_n - f| \geq \varepsilon/2\right]
\end{aligned}$$

$\to 0$ as $m, n \to \infty$

$\Rightarrow \{f_n\}$ converges mutually in measure.

Thus if $f_n \xrightarrow{\mu} f \Rightarrow \{f_n\}$ converges mutually in measure.

Conversely, if $\{f_n\}$ converges mutually in measure,

$\Rightarrow f_n \xrightarrow{\mu}$ to some mble f^n f.

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To study relation between two types of convergence:

Result 1: a.s. convergence $\Rightarrow$ convergence in measure provided the measure is finite.

Proof: Suppose $\{f_n\}$ is a seq^n of mble functions such that $f_n \xrightarrow{a.e.} f$ & let μ be a finite measure.

$\Rightarrow$ ~~By~~ criteria of a.s. convergence holds

$$\Rightarrow \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0.$$

$$\& \quad \mu\left(\bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon\right) < \infty \quad \because \mu \text{ is finite.}$$

Now $\lim\limits_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} |f_n - f| \geq \epsilon\right) = 0$

$\Rightarrow \mu\left(|f_n - f| \geq \epsilon\right) \to 0$ as $n \to \infty$

$\Rightarrow f_n \xrightarrow{\mu} f$

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If the measure is not finite, a.e. convergence <u>may</u> or <u>may not</u> imply convergence in measure.

Ex. I. Let $\Omega = [0, \infty)$

Define $f_n = \begin{cases} 1 & \text{if } \omega \in (0, 1/n) \\ 0 & \text{ow} \end{cases}$

Fix a $\omega \in \Omega$ i.e $\omega \geq 0$

$\forall \omega \in \Omega$, $f_n(\omega) = 0$ for n large.

i.e $f_n \to 0$ a.e.

$\therefore \mu[\omega \mid f_n \not\to 0] = 0$

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Consider

$$\mu\,[\,|f_n - 0| \geq \epsilon\,] = \mu\,[\,f_n = 1\,]$$
$$= \mu\,[\,(0 \;\; 1/n)\,]$$

Let μ be the Lebesgue measure λ & we know that this measure is not finite.

$$= \lambda\,[(0 \;\; 1/n)]$$
$$= \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow f_n \xrightarrow{\mu} 0$$

Thus A.S. convergence $\Rightarrow$ convergence in measure & the measure is not finite.

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Ex 2: Let $\Omega = [0, \infty)$

$$f_n = \begin{cases} 1 & \text{if } \omega \in [n, n+1] \\ 0 & \text{ow} \end{cases}$$

Fix a $\omega \in \Omega$

then $\forall \omega \in \Omega$, $f_n(\omega) = 0$ for n large

Thus $f_n(\omega) \to 0$

Thus $f_n \to 0$ a.e.

Now let $\epsilon > 0$. Consider

$$\mu\,[\,|f_n - 0| \geq \epsilon\,] = \mu\,[\,f_n = 1\,]$$
$$= \mu\,[\,[n, n+1]\,]$$

let μ be the Lebesgue measure λ.

then

$$= \lambda\,[n, n+1]$$
$$= 1 \not\to 0$$

Thus $X_n \overset{\mu}{\not\to} 0$.

Thus a.s. convergence $\not\Rightarrow$ convergence in measure where μ is not finite.

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(68) Convergence in measure may not imply convergence a.e.

<u>Ex</u>: Let $\Omega = [0, \infty)$

Define

$$X_{nk} = \begin{cases} 1 & \text{if } \omega \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \\ 0 & \text{o/w.} \end{cases}$$

$k \leq n$, $n = 1, 2 \ldots$

let

$Y_1 = X_{11}$, $Y_2 = X_{21}$, $Y_3 = X_{22}$,

$Y_4 = X_{31}$, $Y_5 = X_{32}$, $Y_6 = X_{33}$, ... and so on.

Thus $\{Y_m\}$ is a countable seq^n. of r.v.s.

What happens to $\{Y_m\}$ as $m \to \infty$.

[given an m, $\exists$ an n & k s.t. $Y_m = X_{nk}$]

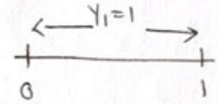
i.e for each $m \geq 1$, $\exists$ n_0, k_0 s.t.

$X_{n_0 k_0} = Y_m$ & hence

$$Y_m = \begin{cases} 1 & \text{if } \omega \in \left[\frac{k_0 - 1}{n_0}, \frac{k_0}{n_0}\right] \\ 0 & \text{o/w} \end{cases}$$

let $n = 1$,

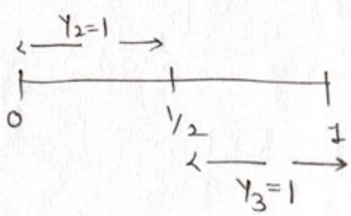
$$X_{11} = Y_1 = \begin{cases} 1 & \text{if } \omega \in [0\ 1] \\ 0 & \text{o/w} \end{cases}$$



$n = 2$

$X_{21} = Y_2$

$X_{22} = Y_3$



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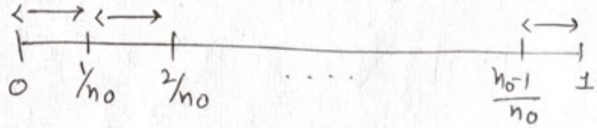
In general,

$n = n_0$



(69) Let $\omega \geq 0$ be fixed. $\{\Omega = [0, \infty)\}$

If $\omega > 1$, all $y_m = 0$

Suppose $0 \leq \omega \leq 1$, then clearly for each n, there is only one X_{nk} which takes value 1 at ω & all other X_{nj}'s are zero.

Thus $\{Y_m(\omega)\}$ has a value 1 at several places and zero at a lot of other places.

Hence $X_{nk} \nrightarrow$ to any no. at any $\omega \in [0\ 1]$.

Thus X_{nk} converges nowhere.

Hence X_{nk} doesnot converge a.e.

but $\{X_{nk}\}$ converges in measure.

let μ be the Lebesgue measure

$\because \mu[|X_{nk} - 0| \geq \varepsilon] = \lambda[\{X_{nk} = 1]$

$$= \lambda\left[\frac{k-1}{n}, \frac{k}{n}\right]$$

$$= \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$\Rightarrow X_{nk} \xrightarrow{\mu} 0$ but $X_n \overset{a.e}{\nrightarrow}$.

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So under what conditions convergence in measure implies convergence a.e.

Thm: Let $\{f_n\}$ be a seq^n of finite valued mble functions which converges mutually in measure. Then $\exists$ a subsequence $\{n_k\}$ such that $\{f_{n_k}\}$ converges a.e to some finite valued function f.

Also $f_{n_k} \xrightarrow{\mu} f$

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