(22) or let $E = (a, b)$, $F = [a, b)$

& define, 2) $\psi(E)$ = length of interval E

so that $\psi(E) = b-a$, $\psi(F) = b-a$.

or say 3) $\psi(E) = \begin{cases} 1 & \text{if } E \text{ is non-empty} \\ 0 & \text{if } E \text{ is empty.} \end{cases}$

Thus we can define a variety of set function on $\mathcal{A}$.

All sets function are of not much use. We have some special properties associated with set functions & we prefer those set functions which have these properties.

Def: Finitely additive set function:-

A set function ψ is said to be finitely additive, if $\psi(A_1 \cup A_2) = \psi(A_1) + \psi(A_2)$, where $A_1 \& A_2 \in \mathcal{A}$ & $A_1 \cap A_2 = \phi$.

Remark: [The above definition is for finite no. of sets i.e If $A_1, \ldots A_K \in \mathcal{A}$ s.t. $A_i \cap A_j = \phi \ \forall i \neq j$, then $\psi(\bigcup_{i=1}^{K} A_i) = \sum_{i=1}^{K} \psi(A_i)$

Remark: It is not necessary that the set function is to be defined on a σ-field. It can also be defined over some non-empty collection $\mathcal{C}$.

Remark: Since ψ takes values in $\overline{\mathbb{R}} = [-\infty, \infty]$, ψ must not take both $\pm\infty$. At the most, ψ can be takes $+\infty$ or $-\infty$ but not both.

e.g. the third set function defined above is not finitely additive.

e.g. let $E_1 \& E_2$ be two non-empty subsets & $E_1 \cap E_2 = \phi$.

then $E_1 \cup E_2$ is also non-empty.

$\therefore \psi(E_1 \cup E_2) = 1$, $\psi(E_1) = 1$, $\psi(E_2) = 1$

$\therefore \psi(E_1) + \psi(E_2) = 2$ and

$\psi(E_1 \cup E_2) \neq \psi(E_1) + \psi(E_2)$

—×—

(23) Countablly additive set function:-

Def: A set function ψ is said to be countablly additive if

$$\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$$

where $A_n \in \mathcal{A}\ \forall n$ & $A_i \cap A_j = \phi\ \forall i \neq j$

i.e. $\{A_n\}$ is a sequence of disjoint sets.

Another name ; σ- additive set function

Remark: 1) $0 \cdot \infty \to$ undefined

2) $-\infty + (-\infty) = -\infty$ 3) $\infty + \infty = \infty$

$\infty - \infty \to$ undefined.

Hence the set function can assume values only $+\infty$ or $-\infty$ but not both. Otherwise $\psi(A) + \psi(B)$ is not well defined.

—x—

Def: Finite Set function:

A set function ψ is said to be a finite set function, if $|\psi(A)| < \infty\ \forall A \in \mathcal{A}$.

Henceforth, we consider only non-ve valued set functions i.e. $\psi(A) \in [0, \infty]$.

So we can revise the above definition as follows.

A set function ψ is said to be a finite set function, if $\psi(A) < \infty\ \forall A \in \mathcal{A}$.

e.g. The set function in example 3 above.

—x—

σ- finite set function:

Def: A non-ve set function ψ is said to be σ-finite if set $E \in \mathcal{A}$, $\exists$ a seq^n $\{A_n\}$, $A_n \in \mathcal{A}\ \forall n$ such that $\bigcup_n A_n \supset E$ &

$\psi(A_n) < \infty\ \forall n$.

—x—

Remark: A finite set function is σ- finite but a σ- finite set function need not be finite.

(24) e.g. set f^n in example 3 is finite & hence σ- finite.

The first two function are σ- finite but not finite.

e.g. in ①,

$\psi(E)$ = no. of elements in E

let $E = \{1, 2, 3, \cdots\}$ & let $A_n = \{n\}$,

then $\psi(A_n) = 1 \quad \forall n$

& $E \subset \bigcup_{n=1}^{\infty} A_n$ or $\bigcup_{n=1}^{\infty} A_n \supset E$

Thus ψ is σ- finite, but ψ is not finite $\because \psi(E)$ is not finite.

—x—

Continuity theorems for additive set function:-

Def: A set function ψ is said to be continuous from below if $A_n \uparrow A \Rightarrow \psi(A_n) \uparrow \psi(A)$.

Similarly,

Def: A set function ψ is said to be continuous from above if $A_n \downarrow A \rightarrow \psi(A_n) \downarrow \psi(A)$.

Def: A set f^n ψ is continuous at A if $A_n \rightarrow A \Rightarrow \psi(A_n) \rightarrow \psi(A)$.

Thm: 1) A σ- additive set function ψ defined on a σ- field is finitely additive if $\psi(\phi) = 0$

Converesly,

if ψ is finitely additive and is continuous from below, then ψ is σ- additive.

Proof: Suppose ψ is σ- additive & $\psi(\phi) = 0$.

Now ψ σ- additive $\Rightarrow \psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$,

where $A_i \cap A_j = \phi \quad \forall i \neq j$,

let $A_n = \phi \quad \forall n \geq 3$, then $A_i \cap A_j = \phi \quad \forall i \neq j$

& $\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \psi(A_1) + \psi(A_2) + \psi(\phi) + \psi(\phi) + \cdots$

i.e $\psi(A_1 \cup A_2) = \psi(A_1) + \psi(A_2)$

$\Rightarrow \psi$ is finitely additive.

Conversely suppose ψ is finitely additive & is continuous from below, then to prove ψ is σ-additive.

Now ψ is finitely additive

$$\Rightarrow \quad \psi\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} \psi(B_k), \quad B_i \cap B_j = \phi \quad \forall\, i \neq j.$$

Let $\{A_n\}$ be a sequence of disjoint sets.

To prove that $\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$.

Define $D_k = \bigcup_{n=1}^{k} A_n \uparrow \bigcup_{n=1}^{\infty} A_n = A$ say.

Since ψ is continuous from below

$$\Rightarrow \psi(D_k) \uparrow \psi(A)$$

i.e. $\psi\left(\bigcup_{n=1}^{k} A_n\right) \uparrow \psi(A)$

i.e. $\lim_{k \to \infty} \psi\left(\bigcup_{n=1}^{k} A_n\right) = \psi\left(\bigcup_{n=1}^{\infty} A_n\right)$

i.e. $\lim_{k \to \infty} \sum_{n=1}^{k} \psi(A_n) = \psi\left(\bigcup_{n=1}^{\infty} A_n\right)$

i.e. $\sum_{n=1}^{\infty} \psi(A_n) = \psi\left(\bigcup_{n=1}^{\infty} A_n\right)$

i.e. $\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$

$\Rightarrow \psi$ is σ- additive.

—x—

<u>Thm</u> : If ψ is finite, finitely additive and ~~&~~ ψ is continuous at ϕ, then ψ is σ-additive.

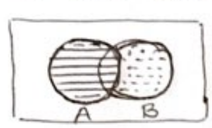
To prove this theorem, we will use following result.

<u>Result</u> : For any sequence $\{A_n\}_{n=1}^{\infty}$, $\bigcup_{n=1}^{\infty} A_n$ can always be written as union of disjoint ~~events~~ sets.

<u>Proof</u> : Let us first consider the case of two sets.

$A \cup B = A \cup (B \cap A')$

where A & $B \cap A'$ are disjoint sets.



(26) Consider a seq^n $\{A_n\}$ of sets.

Define $B_1 = A_1$, $B_2 = A_2 \cap A_1'$, $B_3 = A_3 \cap A_1' \cap A_2'$

$$B_n = A_n \cap A_{n-1}' \cap \cdots \cap A_1$$

then $\{B_n\}$ is a sequence of disjoint sets & $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

Hence the result.

———— X ————

Proof of the theorem:—

Suppose ψ is finite i.e $\psi(A) < \infty$ $\forall A \in \mathcal{A}$.

Also let ψ be finitely additive & ψ is continuous at ϕ.

To prove that ψ is σ-additive.

Let $\{A_n\}$ be a seq^n of disjoint sets.

To prove $\psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$.

Let $A = \bigcup_{n=1}^{\infty} A_n$ and

$$B_K = A - \bigcup_{n=1}^{K} A_n$$

then $B_K \downarrow \phi$

Now $\psi(B_K) = \psi(A) - \psi\left(\bigcup_{n=1}^{K} A_n\right)$

$= \psi(A) - \sum_{n=1}^{K} \psi(A_n)$ ($\because$ A_n's are disjoint & ψ is finitely additive)

$\therefore \lim_{K\to\infty} \psi(B_K) = \psi(A) - \lim_{K\to\infty} \sum_{n=1}^{K} \psi(A_n)$

but $B_K \downarrow \phi \Rightarrow \lim_{K\to\infty} \psi(B_K) \downarrow \psi(\phi)$

Now $\psi(\phi)$ has to be equal to zero, because ψ is finitely additive

$\therefore \psi(A_1 \cup \phi) = \psi(A_1) + \psi(\phi)$

$\Rightarrow \psi(A_1) = \psi(A_1) + \psi(\phi)$

$\Rightarrow \psi(\phi) = 0$

Thus $\lim_{K\to\infty} \psi(B_K) = \psi(\phi) = 0$

$\Rightarrow \psi(A) = \lim_{K\to\infty} \sum_{n=1}^{K} \psi(A_n)$

27

$$\Rightarrow \psi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \psi(A_n)$$

$\Rightarrow \psi$ is σ-additive.

—— x ——

Now we define a set function with specific properties.

Measure:

Def: Let $(\Omega, \mathcal{A})$ be a measurable space. A set function μ is said to be a measure if

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \geq 0$

(iii) μ is σ-additive (or countablly additive)

—— x ——

Measure space:

The triplet $(\Omega, \mathcal{A}, \mu)$ is known as 'Measure space'.

—— x ——

Properties of a measure:

1) If $A \subset B$, $A, B \in \mathcal{A}$ then $\mu(A) \leq \mu(B)$

Proof: since $A \subset B$,

$$B = A \cup (B \cap A')$$

& A, $B \cap A'$ are disjoint sets.

$\therefore \mu(B) = \mu(A) + \mu(B \cap A')$

Note that μ is non-negative

$\Rightarrow \mu(B \cap A') \geq 0$

$\Rightarrow \mu(B) \geq \mu(A)$ OR $\mu(A) \leq \mu(B)$

—— x ——

2. If $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$

Proof: Let $A_n \uparrow A$ i.e $A_1 \subset A_2 \subset A_3 \subset \cdots$

Define $B_1 = A_1$

$B_2 = A_2 \cap A_1'$

$\vdots$

$B_j = A_j \cap A_{j-1}' \cap \cdots \cap A_1'$

(28), then

$$A_n = B_1 \cup B_2 \cup \cdots \cup B_n$$

and B_j's are disjoint sets.

$$\therefore \mu(A_n) = \mu\left(\bigcup_{j=1}^{n} B_j\right)$$

$$= \sum_{j=1}^{n} \mu(B_j) \quad \text{—} \ (1)$$

Now consider

$$A = \bigcup_{n=1}^{\infty} A_n \quad (\because A_n \uparrow)$$

$$= \bigcup_{n=1}^{\infty} B_n \text{, where } B_n\text{'s are disjoint sets}$$

$$\therefore \mu(A) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

(because μ is σ-additive)

$$\therefore \mu(A) = \lim_{K\to\infty} \sum_{n=1}^{K} \mu(B_n)$$

$$= \lim_{K\to\infty} \mu\left(\bigcup_{n=1}^{K} B_n\right)$$

$$= \lim_{K\to\infty} \mu(A_K)$$

$$= \lim_{n\to\infty} \mu(A_n)$$

Thus $\mu(A_n) \uparrow \mu(A)$ i.e $\left[\mu(\lim A_n) = \lim \mu(A_n) \text{ when } A_n \uparrow.\right]$

(remember if $A \subset B$, $\mu(A) \le \mu(B)$)

———x———

3. If $A_n \downarrow A$ and μ is a finite measure then $\mu(A_n) \downarrow \mu(A)$.

Proof: $A_n \downarrow A \Rightarrow A_1 \supset A_2 \supset A_3 \supset \cdots$

$$\Rightarrow \mu(A_n) \geq \mu(A_{n+1}) \ \forall n \geq 1.$$

Consider the set

$$B_n = A_1 - A_n \uparrow$$

then by property (2) above,

$$\lim_n \mu(B_n) = \mu(\lim B_n)$$

i.e. $\lim \left[\mu(A_1 - A_n)\right] = \mu\left[\lim (A_1 - A_n)\right]$

i.e. $\mu(A_1) - \lim\limits_{n\to\infty} \mu(A_n) = \mu\left[A_1 - \lim\limits_{n\to\infty} A_n\right]$

i.e. $\mu(A_1) - \lim\limits_{n\to\infty} \mu(A_n) = \mu(A_1) - \mu\left(\lim\limits_{n\to\infty} A_n\right)$

$\because$ μ is finite

$$\Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu\left(\lim_{n\to\infty} A_n\right)$$

$$\Rightarrow \mu(A_n) \downarrow \mu(A)$$

—×—

Note: It is sufficient if $\mu(A_K) < \infty$ for some $k \geq 1$. Then in place of A_1, choose the set A_K & above proof hold.

—×—

What will happen if μ is not finite?

Example: Let $\Omega = \mathbb{R}$, $\mathcal{A} = \mathcal{B}$

Consider the sequence $A_n = \{n, n+1, \cdots\}$, $n \geq 1$ & define $\mu(A_n)$ = no. of elements in A_n.

then $A_n \downarrow \phi$

$\mu(A_n) = \infty \quad \forall n$

$\Rightarrow \mu(A_n) \to +\infty$ but $\mu(\phi) = 0$

Thus $\mu(A_n) \not\downarrow 0$.

Thus $\mu(A_K) < \infty$ for some $K \geq 1$ is a necessary condition for the above result.

—×—

Note: Above properties are specifically for monotone sequences. The following properties are for any general seq $\{A_n\}$.

(30) Q. Let $\{A_n\}$ be a sequence of mble sets.
then we know that

$$\limsup A_n = \bigcap_{K=1}^{\infty} \bigcup_{n=K}^{\infty} A_n = A \text{ (say) } \&$$

$$\liminf A_n = \bigcup_{K=1}^{\infty} \bigcap_{n=K}^{\infty} A_n = B \text{ (say)}.$$

Then

(a) $\liminf \mu(A_n) \geq \mu(\liminf A_n)$

(b) $\limsup \mu(A_n) \leq \mu(\limsup A_n)$
if μ is a finite measure

Proof :

(a) [Recall if $\{\alpha_n\}$ & $\{\beta_n\}$ are seq^n of real nos &
$\alpha_n \leq \beta_n$
$\Rightarrow \liminf \alpha_n \leq \liminf \beta_n$],

Consider $\liminf A_n = \bigcup_{K=1}^{\infty} \bigcap_{n=K}^{\infty} A_n$

$$= \bigcup_{K=1}^{\infty} B_K$$

where $B_K = \bigcap_{n=K}^{\infty} A_n$, then $B_K \uparrow$

$$\therefore \mu(\liminf A_n) = \mu\left(\bigcup_{K=1}^{\infty} B_K\right)$$
$$= \mu(\lim B_K)$$
$$= \lim \mu(B_K) \quad \text{(from earlier result)} \quad \text{--- (1)}$$

Now $B_K \subset A_K$

$\Rightarrow \mu(B_K) \leq \mu(A_K)$

$\Rightarrow \liminf \mu(B_K) \leq \liminf \mu(A_K)$

i.e. $\liminf \mu(A_K) \geq \liminf \mu(B_K)$
$$= \lim \mu(B_K)$$
$$= \mu(\liminf A_n)$$

Thus
$$\liminf \mu(A_n) \geq \mu(\liminf A_n)$$

——— x ———

(31) (b) We know that

$$\lim\sup A_n = (\lim\inf A_n')'$$
$$= \Omega - \lim\inf A_n'$$

$\therefore$ $$\mu(\lim\sup A_n) = \mu(\Omega) - \mu(\lim\inf A_n')$$
$$\geq \mu(\Omega) - \lim\inf \mu(A_n')$$
(by part (a))

[Recall : $\alpha - \lim\inf \alpha_n$
$= \lim\sup(\alpha - \alpha_n)$]

Hence

$\cancel{\mu(\lim\sup A_n) \geq \lim\sup[\mu(\Omega) - \lim\inf A_n']}$

$$\mu(\lim\sup A_n) \geq \lim\sup[\mu(\Omega) - \mu(A_n')]$$
$$= \lim\sup[\mu(A_n)] \quad \because \mu \text{ is finite.}$$

Thus

$$\mu(\lim\sup A_n) \geq \lim\sup \mu(A_n)$$

— x —

Give an independent proof of (b) & hence prove (a).

<u>Proof</u>: Consider $\lim\sup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$

$$= \bigcap_{k=1}^{\infty} B_k$$

where $B_k = \bigcup_{n=k}^{\infty} A_n \downarrow$

$\therefore$ $$\mu(\lim\sup A_n) = \mu\left(\bigcap_{k=1}^{\infty} B_k\right)$$
$$= \mu(\lim B_k)$$
$$= \lim \mu(B_k) \quad \text{(using earlier result } \because \mu \text{ is finite)}$$

Now $A_k \subset B_k$

$\Rightarrow \mu(A_k) \leq \mu(B_k)$

$\Rightarrow \lim\sup \mu(A_k) \leq \lim\sup \mu(B_k)$

Hence

$$\limsup \mu(A_k) \le \limsup \mu(B_k)$$
$$= \lim \mu(B_k)$$
$$= \mu(\lim B_k)$$
$$= \mu(\limsup A_k)$$

Hence

$$\limsup \mu(A_n) \le \mu(\limsup A_n)$$

Now

$$\liminf A_n = (\limsup A_n')'$$
$$= \Omega - \limsup A_n'$$

$$\therefore \mu(\liminf A_n) = \mu(\Omega) - \mu(\limsup A_n')$$
$$\le \mu(\Omega) - \limsup \mu(A_n')$$
$$= \liminf \mu(\Omega - A_n')$$
$$= \liminf \mu(A_n)$$

hence the proof.

—— x ——

5. If $A_n \to A$ and μ is a finite measure then

$$\mu(A) = \mu(\lim A_n) = \lim \mu(A_n).$$

Proof: Recall since $A_n \to A$

$$\Rightarrow \liminf A_n = \limsup A_n = \lim A_n = A.$$

Hence

$$\mu(A) = \mu(\liminf A_n) \le \liminf \mu(A_n) \le \limsup \mu(A_n)$$
$$\le \mu(\limsup A_n) = \mu(A)$$

$\Rightarrow$ equality holds everywhere

$$\Rightarrow \mu(A) = \liminf \mu(A_n) = \limsup \mu(A_n)$$
$$= \lim \mu(A_n)$$

Thus $\mu(\lim A_n) = \lim \mu(A_n)$

—— x ——