Measurable functions:-

Recall : $\mathbb{R} : (-\infty \;\; \infty) \rightarrow$ σ-field : $\mathcal{B}$

$\overline{\mathbb{R}} : [-\infty, \infty]$: extended real line.

Then $\overline{\mathcal{B}}$: σ-field on $\overline{\mathbb{R}}$.

$\overline{\mathcal{B}}$ is generated in the same way as $\mathcal{B}$.

Now as we discussed earlier, set functions defined on σ-field are not convenient to work with, so we define functions on Ω.

So Consider $f : \Omega \rightarrow \overline{\mathbb{R}}$.

We will study different types of functions.

Def: A function $f: \Omega \rightarrow \overline{\mathbb{R}}$ is called an indicator function, if for some $A \subset \Omega$, $(A \in \mathcal{A}^*)$

$$f(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

i.e

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Def: A function $f : \Omega \rightarrow \overline{\mathbb{R}}$ is called a simple function, if it takes a finite number of distinct values $c_1, c_2, \cdots c_n$ on sets $A_1, A_2, \cdots A_n$, ~~s.t.~~ where $A_1, \cdots A_n$ are subsets of Ω s.t.

$$A_i \cap A_j = \phi \;\; \forall i \neq j \;\; \& \;\; \bigcup_{k=1}^{n} A_k = \Omega$$

Thus $f(\omega) = c_k$ if $\omega \in A_k$, $k = 1 \cdots n$

Another way of writing f is

$$f = \sum_{k=1}^{n} c_k \cdot I_{A_k}$$

or

$$f(\omega) = \sum_{k=1}^{n} c_k \cdot I_{A_k}(\omega)$$

$$\Rightarrow f(\omega) = c_k \text{ if } \omega \in A_k$$

Remark: 1) Indicator function is a special case of simple function.

2) In above definition, any of the c_k may be $+\infty$ or $-\infty$.

50 <u>Def</u>: A function $f: \Omega \to \overline{R}$ is called an elementary function if it takes a countable no. of distinct values.

i.e $f(\omega) = C_K$ if $\omega \in A_K$ $K = 1, 2, \ldots$

where $A_i \cap A_j = \phi$ $\forall i \neq j$, & $\bigcup_{K=1}^{\infty} A_K = \Omega$.

—x—

<u>Result</u>: If f and g are two simple functions. Then all ~~simple~~ arethmetic operations on simple functions result in simple functions provided the resulting function is well defined. Thus $f+g$, $f-g$, $f \times g$ & $\frac{f}{g}$ are all simple function, provided we are not coming accross the terms like $\infty - \infty$, $\frac{\infty}{\infty}$, $\frac{0}{\infty}$, $0 \times \infty$ etc.

—x—

Thus we can define a no. of function. We are interested in function having specific property, & we call such function as measurable functions.

Measurable function:

<u>Def</u>: Let $(\Omega, \mathcal{A})$ be fixed.
(D) A function $f: \Omega \to \overline{\mathcal{R}}$ is called a measurable function if $f^{-1}(B) \in \mathcal{A}$ $\forall B \in \overline{\mathcal{B}}$.

i.e. inverse image under f of a Borel set is in $\mathcal{A}$ i.e. a measurable set.

<u>Remark</u>: This is a discriptive definition, since it only discribies the concept, but donot help to constrnct a mble function

<u>Def</u> (D'): (Discriptive type)

A function $f: \Omega \to \overline{R}$ is called a measurable function if $f^{-1}(C) \in \mathcal{A}$ $\forall C \in \mathcal{C}$ where $\sigma(\mathcal{C}) = \mathcal{B}$.

e.g. $\mathcal{C} = \{ I \mid I = (a\ b],\ -\infty \leq a \leq b \leq \infty \}$

or $\mathcal{C} = \{ I \mid I = (-\infty, \alpha),\ -\infty \leq \alpha \leq \infty \}$

Agian, this is a discriptive definition

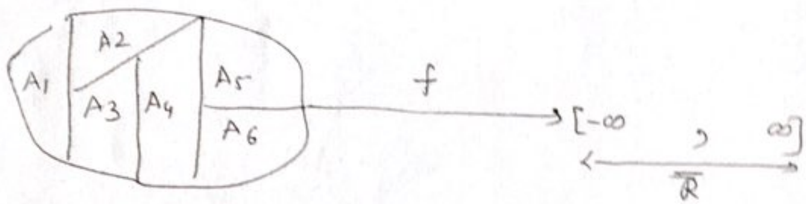
(51) Properties of functions measurable according to def D' :—

1) Suppose f is a simple function. Then f is measurable (D').

Proof : Let $f = \sum_{j=1}^{n} c_j I_{A_j}$ be a simple function,

where $A_1, \cdots A_n \in \mathcal{A}$, $A_i \cap A_j = \phi$, $\bigcup_{j=1}^{n} A_j = \Omega$.

Now $\mathcal{A}$ is a σ-field. So all possible unions/intersections/compliments of $A_1, \cdots A_n$ are in $\mathcal{A}$.



then if $\mathcal{C} = \{ I \mid I = (a\ b],\ -\infty \le a \le b \le \infty \}$

$$f^{-1}\{(a\ b]\} = \{ \omega \mid a < f(\omega) \le b \}$$

$$= \bigcup_{\{j \mid c_j \in (a\ b]\}} A_j \in \mathcal{A}$$

Thus $f^{-1}(C) \in \mathcal{A} \ \forall\ C \in \mathcal{C}$ s.t. $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$

Hence f is measurable according to definition (D').

2) Let $f_1, f_2, \cdots$ be functions mble (D').

Then $g = \max_k f_k$ is mble (D').

Let $\mathcal{C} = \{ I \mid I = [-\infty, \alpha] \}$

Then $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$.

$$g^{-1}(I) = \{ \omega \mid \max f_k(\omega) \le \alpha \}$$

$$= \{ \omega \mid f_k(\omega) \le \alpha \ \ \forall k \}$$

$$= \bigcap_k \{ \omega \mid f_k(\omega) \le \alpha \}$$

(52)

$$= \bigcap_k f_k^{-1}(I) \in \mathcal{A}$$

$\because f_k$ is mble $(\mathcal{D}')$.

$\Rightarrow \max_k f_k$ is mble $(\mathcal{D}')$.

3) If f is mble $(\mathcal{D}')$, $-f$ is mble $(\mathcal{D}')$.

let $g = -f$. Let

$$\mathcal{C} = \{ I \mid I = [-\infty\ \alpha],\ -\infty \le \alpha \le \infty \}$$

& consider

$$g^{-1}(I) = \{ \omega \mid g(\omega) \in [-\infty\ \alpha] \}$$

$$= \{ \omega \mid g(\omega) \le \alpha \}$$

$$= \{ \omega \mid -f(\omega) \le \alpha \}$$

$$= \{ \omega \mid f(\omega) \geq -\alpha \}$$

$$= f^{-1}[-\alpha, \infty] \in \mathcal{A} \quad (\because \mathcal{A} \text{ is a } \sigma \text{ field})$$

$\Rightarrow g^{-1}(I) \in \mathcal{A}\ \forall I \in \mathcal{C}$ where $\sigma(\mathcal{C}) = \overline{\mathcal{B}}$

$\Rightarrow g$ is mble $(\mathcal{D}')$.

—x—

On similar limes, we can prove that $\min f_k$, $\limsup f_n$, $\liminf f_n$ and $\lim f_n$ if it exists are all mble $(\mathcal{D}')$, if each f_n is mble $(\mathcal{D}')$.

—x—

Constructive Definition:-

Def: (C) A function $f : \Omega \to \overline{R}$ is called a measurable function if $\exists$ a sequence $\{f_n\}$ of simple functions s.t. $f_n(\omega) \to f(\omega)$; Here each f^n

f_n is $\quad f_n = \sum_{k=1}^{n_k} C_{n,k}\, I_{A_{n,k}}$

(each f_n is a simple function)

—x—

53

Result: All the three definition are equivalent.

Proof: Suppose f is mble (D).

i.e $f^{-1}(B) \in \mathcal{A} \ \forall B \subset \bar{\mathcal{B}}$

since $\mathcal{C} \subset \bar{\mathcal{B}}$ & $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$

$\Rightarrow f^{-1}(C) \in \mathcal{A} \ \forall C \in \mathcal{C}$ with $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$

$\Rightarrow f$ is mble (D').

Conversely, suppose f is mble (D').

i.e $f^{-1}(C) \in \mathcal{A} \ \forall C \in \mathcal{C}$ where $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$. (1)

To prove that $f^{-1}(B) \in \mathcal{A} \ \forall B \in \bar{\mathcal{B}}$.

i.e to prove $f^{-1}(\bar{\mathcal{B}}) \subseteq \mathcal{A}$.

Now, we know that

$\sigma[f^{-1}(\mathcal{C})] = f^{-1}(\sigma(\mathcal{C})) = f^{-1}(\bar{\mathcal{B}})$

From (1), we have

$f^{-1}(C) \in \mathcal{A} \ \forall C \in \mathcal{C}$

$\Rightarrow \sigma(f^{-1}(\mathcal{C})) \subset \mathcal{A}$ ($\because \mathcal{A}$ is a σ-field).

$\Rightarrow f^{-1}(\bar{\mathcal{B}}) \subset \mathcal{A}$

Thus f is mble (D).

Thus we proved that Def $(D) \equiv$ Def (D').

—x—

Now suppose that $\{f_n\}$ is a seqⁿ of simple functions s.t. $f_n \to f$

(i.e f is mble according to def C)

(Recall if f_n is mble (D') ($\because$ it is a simple fⁿ)

so $\lim f_n = f$ is mble (D').

& f is mble (D) ($\because D' \equiv D$)

Thus $C \Rightarrow D'$ and D.

Finally, let f be mble (D'/D).

To prove f is mble according 'C'.

(54)

Define $f_n(\omega)$ as follows.

$$f_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \le f(\omega) < \frac{k}{2^n}, \\ & k = -n2^n+1, \cdots, n2^n \\ -n & \text{if } f(\omega) < -n \\ n & \text{if } f(\omega) \ge n \end{cases}$$

The function f_n takes $2n2^n+1$ values.

Let us write the f^n for say $n=1, 2$.

when $n=1$, range for k is

$-1, 0, 1, 2$

then $\frac{k-1}{2^n}$ will be $-1, -\frac{1}{2}, 0, \frac{1}{2}$

then

$$f_1(\omega) = \begin{cases} -1 & \text{if } f(\omega) < -1 \\ -1 & \text{if } -1 \le f(\omega) < -1/2 \\ -1/2 & \text{if } -1/2 \le f(\omega) < 0 \\ 0 & \text{if } 0 \le f(\omega) < 1/2 \\ 1/2 & \text{if } 1/2 \le f(\omega) < 1 \\ 1 & \text{if } f(\omega) \ge 1 \end{cases}$$

Similarly, let us write $f_2(\omega)$.

Here $n=2$. Hence

$k = -n2^n+1, \cdots n2^n$ becomes

$k = -7, -6, -5, \cdots -1, 0, 1, 2 \cdots 8$

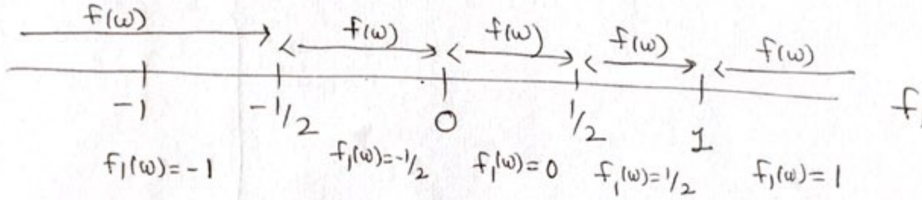
So that

$$\frac{k-1}{2^n} = \frac{-8}{4}\ \frac{-7}{4}\ \frac{-6}{4} \cdots \frac{-2}{4}\ \frac{-1}{4}\ 0\ \frac{1}{4} \cdots \frac{7}{4}$$

& hence $f_2(\omega)$ is

$$f_2(\omega) = \begin{cases} -2 & \text{if } f(\omega) < -2 \\ -8/4\,(=-2) & \text{if } -2 \le f(\omega) < -7/4 \\ -7/4 & \text{if } -7/4 \le f(\omega) < -6/4 \\ \vdots & \\ -1/4 & \text{if } -1/4 \le f(\omega) < 0 \\ 0 & \text{if } 0 \le f(\omega) < 1/4 \\ 1/4 & \text{if } 1/4 \le f(\omega) < 2/4 \\ \vdots & \\ 7/4 & \text{if } 7/4 \le f(\omega) < 8/4 = 2 \\ 2 & \text{if } f(\omega) \geq 2 \end{cases}$$

In general $f_n(\omega)$ takes $2n2^n+1$ distinct values.
So each f_n is a simple function.



while defining $f_2(\omega)$, we make still smaller intervals bet^n $(-2, 2)$ & ~~defi~~ approximate $f(\omega)$ by $f_2(\omega)$.

$\Rightarrow$ as n increases, the difference bet^n $f(\omega)$ & $f_n(\omega)$ becomes smaller and smaller.

Thus being simple function, each f_n is a mble function.

If $f(\omega) = +\infty$ then $f_n(\omega) = n \to \infty$ as $n \to \infty$

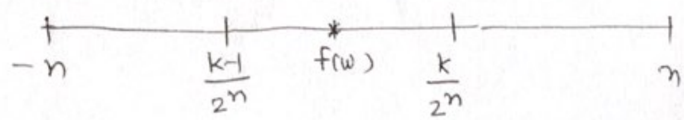
If $f(\omega) = -\infty$ then $f_n(\omega) = -n \to -\infty$ as $n \to \infty$

Thus in either case $f_n \to f$.

If $-\infty < f(\omega) < \infty$, then $\exists$ n (large) s.t.

$$-n \le f(\omega) < n$$

then $\exists$ k s.t $\frac{k-1}{2^n} \le f(\omega) < \frac{k}{2^n}$



In that case,

$$|f_n(\omega) - f(\omega)| < \frac{1}{2^n} \to 0 \quad \text{as } n \to \infty$$

Thus $f_n \to f$ in all cases.

i.e f is a limit of seq^n. of simple functions $\{f_n\}$. $\Rightarrow f$ is mble (C).

Thus all the three definitions ~~are~~ of mble f^n are equivalent.

—x—

Remark: let $f : (\Omega, \mathcal{A}_1) \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ be a mble function.

Let $\mathcal{A}_2$ be another σ- field. Then f may not be measurable w.r.t $(\Omega, \mathcal{A}_2)$.

e.g. Let $\Omega = \mathbb{R}$ &

$$\mathcal{A}_1 = \{\Omega, \phi, (-\infty\ 0], (0, \infty)\} \ \&$$

$$\mathcal{A}_2 = \{\Omega, \phi, (-\infty, 5), [5, \infty)\}$$

Define $f(\omega) = \begin{cases} 1 & \text{if } \omega \le 0 \\ 2 & \text{if } \omega > 0. \end{cases}$

Then

$$f(\omega) = 1 \cdot I_{(-\infty, 0]} + 2 \cdot I_{(0, \infty)}$$

$$= 1 \cdot I_A + 2\, I_{A^c}$$

Note A & $A^c \in \mathcal{A}_1$ hence f is mble $(\Omega, \mathcal{A}_1)$.

but A & $A^c \notin \mathcal{A}_2 \Rightarrow f$ is not mble $(\Omega, \mathcal{A}_2)$

for Any σ- field not containing A & A^c, f will not be mble w.r.t. that σ- field.

Now suppose we have another σ- field say $\mathcal{A}$ s.t. $\mathcal{A}_1 \subset \mathcal{A}$, then f is surely mble w.r.t. $(\Omega, \mathcal{A})$. We know that $\mathcal{A}_1 \subset \mathcal{P}(\Omega)$, $\Rightarrow$ f is mble w.r.t $(\Omega, \mathcal{P}(\Omega))$.

Thus f is mble w.r.t. every σ- field containing $\mathcal{A}$.

So we need to choose the smallest one.

Def: The smallest σ-field w.r.t. which a given f^n is mble is called the σ-field ~~generated~~ induced by f on Ω.

In the above example, where f takes only 2 distinct values on A & A^c, the σ-field ~~gen~~ induced by f is $\{\phi, \Omega, A, A^c\}$.

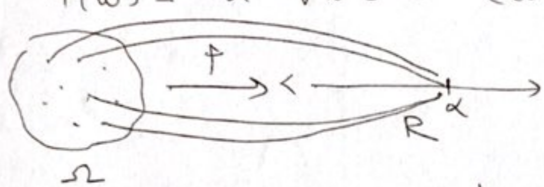
—x—

Let $\mathcal{A} = \{\Omega, \phi\}$: trivial σ-field

To find which function are mble w.r.t. this σ-field.

Define

$$f(\omega) = \alpha \quad \forall \omega \in \Omega \quad \text{(constant function)}$$



To check the condition $f^{-1}(C) \in \mathcal{A}$

$$f^{-1}(C) = \begin{cases} \phi & \text{if } \alpha \notin C \\ \Omega & \text{if } \alpha \in C \end{cases}$$

Thus the smallest σ-field induced by f is $\{\Omega, \phi\}$

Conclusion: The only f^ns mble w.r.t. trivial σ-field are constant function, which takes only one value.

—x—

Now $\{\phi, \Omega\} \subset$ every σ-field $\Rightarrow$ Constant functions are mble w.r.t. every σ-field.

—x—

(58) The earlier theorem is for mble f^n f, which takes values over $\mathbb{R}$ or $\overline{\mathbb{R}}$.

The following theorem is specifically for non-ve mble function.

Thm: Given a non-ve mble f^n f, $\exists$ a non-decreasing sequence of non-ve simple functions f_n s.t. $f_n \uparrow f$.

i.e. $\exists \{f_n\}$, f_n simple $\forall n$ s.t. $0 \le f_n \uparrow f$.

Proof: Define

$$f_n(\omega) = \frac{k}{2^n} \quad \text{if } \frac{k}{2^n} \le f(\omega) < \frac{k+1}{2^n},$$

$$k = 0, 1, 2 \cdots n2^n$$

$$= n \quad \text{if } f(\omega) \ge n$$

Let us observe say $f_1(\omega)$, $f_2(\omega)$, ...

For $n=1$, $k = 0, 1, 2$

$$f_1(\omega) = \begin{cases} 0 & \text{if } 0 \le f(\omega) < \frac{1}{2} \\ 1/2 & \text{if } 1/2 \le f(\omega) < 1 \\ 1 & \text{if } f(\omega) \ge 1 \end{cases}$$

For $n=2$, $k = 0, 1, 2, \cdots 8$

$$f_2(\omega) = \begin{cases} 0 & \text{if } 0 \le f(\omega) < \frac{1}{4} \\ 1/4 & \text{if } 1/4 \le f(\omega) < 2/4 = 1/2 \\ 2/4 = 1/2 & \text{if } 2/4 \le f(\omega) < 3/4 \\ 3/4 & \text{if } 3/4 \le f(\omega) < 4/4 \\ 4/4 & \text{if } 4/4 \le f(\omega) < 5/4 \\ 5/4 & \text{if } 5/4 \le f(\omega) < 6/4 \\ 6/4 & \text{if } 6/4 \le f(\omega) < 7/4 \\ 7/4 & \text{if } 7/4 \le f(\omega) < 8/4 \\ 2 & \text{if } f(\omega) \ge 2 \end{cases}$$

Now observe that $f_1(\omega) \le f_2(\omega)$ $\forall \omega$

Further each f_n is a simple f^n $\forall n$

(59) &

$$\left| f_n(\omega) - f(\omega) \right| \leq \frac{1}{2^n} \quad \text{when } f(\omega) < n$$

& when $f(\omega) \geq n$, $f_n(\omega) = n \to \infty$ as $n \to \infty$

Thus in either case, as $n \to \infty$ $f_n(\omega) \uparrow f(\omega)$

Hence the theorem

—x—

Result: A mble function can also be obtained as a limit of a sequence of elementary functions. Further this convergence is uniform if f is bounded.

—x—

Basically we are interested in three types of measurable function.

1) Non-ve simple function:
   (Takes non-ve, finitely many distinct values)
2) Non-ve mble function ($f \geq 0$)
3) Any mble function (mble f^n which takes values over $\overline{\mathbb{R}}$)

Let f be any measurable f^n. Then f can be written as

$$f = \underbrace{f^+}_{\text{+ve part of } f} - \underbrace{f^-}_{\text{-ve part of } f}$$

where $f^+ = \max(f, 0) = \begin{cases} f & \text{if } f \geq 0 \\ 0 & \text{if } f < 0 \end{cases}$

&

$f^- = -\min(f, 0) = \begin{cases} 0 & \text{if } f \geq 0 \\ -f & \text{if } f < 0 \end{cases}$

Note that $f^+ \geq 0$ & $f^- \geq 0$

Now since f is mble also '0' being constant function always mble $\Rightarrow f^+ = \max(f, 0)$ is mble

Simly $-f$ is also mble $\Rightarrow f^- = -\min(-f, 0)$ is mble

(60) The converse is non-necessarily true.

i.e. If f^+ and f^- are mble, f may not be mble.

Ex: Suppose f_1 and f_2 be mble f^n ($\mathcal{A}$).

$\Rightarrow \quad \{g_n\} \to f_1 \quad g_n$ simple $\forall n$

& $\{h_n\} \to f_2$, h_n simple $\forall n$

$\Rightarrow \{g_n + h_n\}$ simple $\forall n$

$\Rightarrow g_n + h_n \to f_1 + f_2$

& hence $f_1 + f_2$ is mble ($\mathcal{A}$).

$\alpha f_1 + \beta f_2$ is mble ($\mathcal{A}$), $\alpha, \beta \in R$

$f_1 \cdot f_2$ is mble ($\mathcal{A}$)

$\max(f_1, f_2)$ mble ($\mathcal{A}$)

$\min(f_1, f_2)$ mble ($\mathcal{A}$)

All ordered functions are mble ($\mathcal{A}$).

—x—

Ex: $\exists$ a f^n $f: \Omega \to R \ni f^2$ is mble $\mathcal{A}$ but f is not mble.

Let $\Omega = R$ & $\mathcal{A} = \{\phi, \Omega\}$

Define $f^2(\omega) = 1 \quad \forall \omega \in \Omega$

then $f(\omega) = \begin{cases} 1 & \text{if } \omega > 0 \\ -1 & \text{if } \omega \leq 0 \end{cases}$

then f is not mble ($\mathcal{A}$)

—x—

Ex: Define $|X| = \begin{cases} X & \text{if } X \geq 0 \\ -X & \text{if } X < 0 \end{cases}$

i.e $|X| = X^+ + X^-$.

Remark:- Instead of f notation 'X' is used.

Suppose X is mble $\Rightarrow X^+$ & X^- are mble $\Rightarrow |X|$ is mble.

Is the converse true? i.e.

If $|X|$ is mble, then is X mble?

No - not necessarily.

Thus the question of interest is

" Is $g(X)$ mble if X is mble?

The answer is in the following theorem.

(61) 1) Let f be a mble f^n, $f : \Omega \to R$.
Let $g : R \to R$ be a continuous f^n.
Then $g(f) : \Omega \to R$ is also a mble function.

—x—

2) Define a class of function $\mathcal{C}$ as
$\mathcal{C} = \{ g \mid$ either g is continuous or is a limit of sequence of continuous functions $\}$
A function h is called a baire function if $h \in \mathcal{C}$.

Result: A baire function of a mble f^n is also mble
i.e f mble, $h \in \mathcal{C} \Rightarrow h(f)$ is also mble

—x—

3) Borel function:-
A function $g : R \to R$ is called a borel function if $g^{-1}(B)$ is a Borel set.

Result: Borel function of a mble function is mble.
e.g. if f is mble $\Rightarrow f^2$ is mble
or say e^f is mble.

—x—