## Absolute Continuity and Singularity:

**Def**. A set function $\psi$ is said to be $\mu$-continuous or absolutely continuous w.r.t. a measure $\mu$ if
$\mu(A) = 0 \Rightarrow \psi(A) = 0$, where $A$ is a mble set.

Notation:- $\psi << \mu$

**Def**:- A set function $\psi$ is said to be singular w.r.t. a measure $\mu$ if $\exists$ a mble set $N$ s.t.
$\mu(N) = 0$ (i.e $N$ is a $\mu$-null set) &
$\psi(A) = \psi(A \cap N)$ i.e $\psi(A \cap N') = 0$.
i.e. $\psi$ has all its mass concentrated on a $\mu$-null set.

Notation: $\psi \perp \mu$.

examples: All continuous distributions have their probability measures absolutely continuous w.r.t the Lebesgue measure.

Similarly All discrete distributions have their prob measure singular w.r.t. Lebesgue measure.

e.g. let $Z^+ = \{0, 1, 2, \ldots\}$
let $\mu_p$ denote prob. measure of Poisson distn
then $\mu_p(Z^+) = 1$
but $\lambda(Z^+) = 0$
$\Rightarrow \mu_p \perp \lambda$

Similarly if $\mu_B$ represents prob. measure of Binomial distn, then $\mu_B \perp \lambda$.

1) In fact, all discrete distributions are singular w.r.t. Lebesgue measure.

2) All continuous distributions (like Normal, Gamma, Beta, Uniform, t Dett all are absolutely Continuous w.r.t Lebesgue measure.

3) All discrete distribution are singular w.r.t. all continuous distribution.

4) All discrete distribution are absolutely continuous w.r.t. counting measure.

(94) $\mu_B << \mu_P$
(Binomial) (Poission)

$\phi_{Normal} << \phi_{Student's\ t}$

$\phi_{Student's\ t} << \phi_{Normal}$

Result :- If $\phi_1 << \phi_2$, $\phi_2 << \phi_3 \Rightarrow \phi_1 << \phi_3$

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Now, we like to define a set function using the concept of integral of a mble function.

So let us define set function $\nu(A)$ as

$$\nu(A) = \int_A f d\mu = \int f I_A d\mu .$$

Then $\nu(\cdot)$ is a set function defined on $\sigma$-field $\mathcal{A}$.

& is known as indefinite integral of f.

1) Now suppose first f is integrable, then we know that f is finite a.e. i.e $\mu[f=\infty]=0$.

$\Rightarrow \int f d\mu < \infty$ & hence $\nu(\Omega) < \infty$

$\Rightarrow \nu(A) < \infty \quad \forall A \in \mathcal{A}$.

i.e $\nu$ is finite $\Rightarrow \nu$ is $\sigma$-finite also.

If we assume $\int f d\mu$ exists then

$\int f^+ d\mu < \infty$ OR $\int f^- d\mu < \infty$.

Suppose $\int f^- d\mu < \infty$

Claim 1: $\nu$ is $\sigma$-additive (countably additive).

i.e to prove if $\{A_n\}$ is a seq$^n$ of disjoints sets in $\mathcal{A}$, $(A_i \cap A_j = \phi \quad \forall i \neq j)$, then $\nu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i)$

Consider $\nu(\bigcup_{i=1}^{\infty} A_i) = \int_{\cup A_n} f d\mu$

$$= \int f (I_{\cup A_n}) d\mu$$

$$= \int f^+ I_{\cup A_n} d\mu - \int f^- I_{\cup A_n} d\mu$$

$$= \int f^+ (\Sigma I_{A_n}) d\mu - \int f^- (\Sigma I_{A_n}) d\mu$$

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$$\int \sum f^+ I_{A_n} d\mu - \int \sum f^- I_{A_n} d\mu$$

$$= \sum \int f^+ I_{A_n} d\mu - \sum \int f^- I_{A_n} d\mu$$ (∵ Indefinite integrals are $\sigma$-additive) by MCT

$$= \sum \left[\int f^+ I_A d\mu - \int f^- I_{A_n} d\mu\right]$$

$$= \sum \int_{A_n} f\, d\mu$$

$$= \sum \nu(A_n)$$

$\Rightarrow \nu$ is $\sigma$-additive

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Claim 2: Let $f$ be almost everywhere finite valued i.e. $\mu\{f = \pm\infty\} = 0$. Let $\mu$ be $\sigma$-finite then $\nu$ is also $\sigma$-finite.

Proof: $\mu$ is $\sigma$-finite

$\Rightarrow \exists \{B_n\}$ s.t. $\Omega = \bigcup_{n=1}^{\infty} B_n$

& $\mu(B_n) < \infty$

To prove $\nu(B_n) < \infty$, $\nu(B_n) = \int_{B_n} f\, d\mu$

Now $f$ is a.e. finite

$\Rightarrow N = \{\omega \mid |f(\omega)| < \infty\}$ then $\mu(N') = 0$

Now

$$\nu(B_n) = \nu(B_n \cap N) + \nu(B_n \cap N')$$
$$= \nu(B_n \cap N)$$

Now

$$N = \{\omega \mid |f(\omega)| < \infty\}$$
$$= \sum_{k=-\infty}^{\infty} \{\omega \mid k \le f(\omega) < k+1\}$$

$$\therefore \nu(B_n) = \sum_{k=-\infty}^{\infty} \nu(B_n \cap \{k \le f(\omega) < k+1\})$$

$$\therefore |\nu(B_n)| = \left| \sum_{k=-\infty}^{\infty} \int_{B_n \cap \{k \le f(\omega) < k+1\}} f\, d\mu \right|$$

$$\le 2 \sum_{k=0}^{\infty} (k+1) \int_{B_n \cap \{k \le f(\omega) < k+1\}} d\mu$$

$$= 2 \sum_{k=0}^{\infty} (k+1)\, \mu[B_n \cap \{k \le f(\omega) \le k+1\}]$$

$< \infty$ $\because \mu$ is finite.

Thus $\nu$ is $\sigma$- finite

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<u>Remark</u> : The problem which arises is whether the above stated properties characterizes indefinite integrals? The aswer lies in the following two celebrated theorems.

<u>Lebesgue's decomposition theorem</u>:-

Let $\mu$ be a $\sigma$-finite measure. Let $\nu$ be a $\sigma$-finite and $\sigma$-additive set function on the same measure space. Then there exists a unique decomposition of $\nu$ as $\nu = \nu_1 + \nu_2$ where $\nu_1$ and $\nu_2$ satisfy,

$\nu_1 << \mu$ & $\nu_2 \perp \mu$.

Further $\exists$ a non-negative a.e. finite valued function $f$ which is determined upto an equivalence $\ni$ $\nu(A) = \int_A f d\mu$, $\forall A \in \mathcal{A}$.

$f$ is called Radon-Nikodym derivative of $\nu_1$ w.r.t. $\mu$.

<u>Radon-Nykodym theorem</u> :

Let $\mu$ be $\sigma$-finite measure. Let $\nu$ be a $\sigma$-finite and $\sigma$-additive measure. Let $\nu << \mu$. Then $\nu$ is the indefinite integral of same a.e. finite valued function $f$ i.e $\nu(A) = \int_A f d\mu$, $\forall A \in \mathcal{A}$.

→ Here $f$ is determined up to equivalence. $f$ is called the R-N derivative of $\nu$ w.r.t. $\mu$ i.e.

$$f = \frac{\partial \nu}{\partial \mu}.$$

e.g. Let $\mu_F$ be the L-S measure corresponding to a DF F.

Let $\lambda$ be the Lebesgue measure

Suppose ~~that~~ $\mu_F << \lambda$

then $\mu_F(-\infty, x] = \int_{(-\infty, x]} f d\lambda = \int_{-\infty}^{x} f(t)dt$:

$$\Rightarrow F(x) - F(-\infty) = \int_{-\infty}^{x} f(t)dt$$

i.e $F(x) = \int_{-\infty}^{x} f(t)\,dt$

then $\frac{d}{dx} F(x) = f(x)$

i.e $\frac{\partial}{\partial x} \mu_F = f(x)$

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The converse of the above theorems is also true.

Thus we have the following results which states both parts.

Thm: Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space. A set function $\nu$ which is absolutely continuous w.r.t $\mu$ can be expressed as an indefinite integral of a finite function $f$ i.e $\nu(A) = \int_A f\,d\mu$ iff $\nu$ is $\sigma$-additive and $\sigma$-finite

Further $f$ is integrable iff $\nu$ is finite

(f is determined up to an equivalence

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## Product space :

Suppose $\Omega_1$ and $\Omega_2$ are two abstract spaces with $\sigma$- fields $\mathcal{A}_1$ and $\mathcal{A}_2$ on them, & measures $\mu_1$ & $\mu_2$ respectively.

Thus $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are two measure spaces.

Define

$$\Omega_1 \times \Omega_2 = \{ (w_1, w_2) \mid w_1 \in \Omega_1, w_2 \in \Omega_2 \}$$

is called the product space of the two spaces $\Omega_1 \times \Omega_2$.

Rectangles $A_1 \times A_2$ in the product space is given by $A_1 \times A_2 = \{(w_1, w_2) \mid w_1 \in A_1, w_2 \in A_2\}$

$A_1 \subset \Omega_1, A_2 \subset \Omega_2, A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$

The question is to find appropriate $\sigma$- field of subsets of $\Omega_1 \times \Omega_2$ & to find measure on this $\sigma$- field.

Suppose $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ & consider

$$\mathcal{C} = \{ A_1 \times A_2 \mid A_i \in \mathcal{A}_i, i = 1, 2 \}$$

Is $\mathcal{C}$ is a $\sigma$- field of subsets of $\Omega_1 \times \Omega_2$?

Note that $\mathcal{C}$ is not closed under compliments / Unions. so we find the $\sigma$- field generated by $\mathcal{C}$.

[Recall:- $\sigma$ field generated by $\mathcal{C}$ is the smallest $\sigma$- field of subsets of $\Omega_1 \times \Omega_2$ containing $\mathcal{C}$].

We consider this $\sigma$- field of subsets of $\Omega_1 \times \Omega_2$ as the appropriate $\sigma$- field & call it product $\sigma$- field.

& give the notation as $\sigma(\mathcal{C}) = \mathcal{A}_1 \times \mathcal{A}_2$.

thus $A_1 \times A_2 \in \mathcal{A}_1 \times \mathcal{A}_2$.

Now define $\mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2)$

Let $B \in \mathcal{A}_1 \times \mathcal{A}_2$ be any mble set in $\Omega_1 \times \Omega_2$.
By Caratheodory extension theorem,
$\exists$ a measure $\bar{\mu}$ on the $\sigma$-field $\mathcal{A}_1 \times \mathcal{A}_2$
s.t. the above measure $\mu$ is a restriction of $\bar{\mu}$
on the rectangles.
$\bar{\mu}$ is called the extension of $\mu$.

$$\bar{\mu}(A_1 \times A_2) = \mu(A_1 \times A_2) = \mu_1(A_1) \times \mu_2(A_2).$$

Thus we have the measure space
$(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$, which is known as
product measure space.

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Section of sets:-
For any set $A \subset \Omega_1 \times \Omega_2$, define the section of
$A$ at $\omega_1$ as

$$\{\omega_2 \in \Omega_2 \mid (\omega_1, \omega_2) \in A\} = A_{\omega_1}$$

Thus $A_{\omega_1} \subset \Omega_2$

Similarly,

$$A_{\omega_2} = \{\omega_1 \in \Omega_1 \mid (\omega_1, \omega_2) \in A\} \subset \Omega_1$$

$$(A_1 \times A_2)_{\omega_1} = \begin{cases} A_2 & \text{if } \omega_1 \in A_1 \\ \phi & \text{if } \omega_1 \notin A_1 \end{cases}$$

Similarly

$$(A_1 \times A_2)_{\omega_2} = \begin{cases} A_1 & \text{if } \omega_2 \in A_2 \\ \phi & \text{if } \omega_2 \notin A_2 \end{cases}$$

Now let us define functions on the product measure space.

$$f(\omega_1, \omega_2) : \Omega_1 \times \Omega_2 \to R$$

$f$ is mble iff $f^{-1}(B) \in \mathcal{A}_1 \times \mathcal{A}_2 \;\; \forall B \in \mathcal{B}$.

So let $f: \Omega_1 \times \Omega_2 \to R$ be a mble function.

For fixed $\omega_1 \in \Omega_1$, section of $f$ at $\omega_1$ is defined by

$$f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2) : \Omega_2 \to R$$

Similarly, for fixed $\omega_2 \in \Omega_2$, section of $f$ at $\omega_2$ is defined as

$$f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2) ; \Omega_1 \to R$$

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Thm : Sections of mble sets are mble sets.

Proof : Recall sections of a rectangle $A_1 \times A_2$ at $\omega_1$ is either $A_2$ or $\phi$. Since $A_2$ & $\phi \in \mathcal{A}_2 \Rightarrow$ $(A_1 \times A_2)_{\omega_1}$ is a mble set.

Since all mble sets $A \in \mathcal{A}_1 \times \mathcal{A}_2$ are generated from rectangles $A_1 \times A_2$, it follows that $A_{\omega_1}$ is also a mble set. Hence the theorem.

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Thm : Sections of mble functions are also mble.

Proof : Let $f : \Omega_1 \times \Omega_2 \to R$ be a mble function.

$\Rightarrow f^{-1}(B) \in \mathcal{A}_1 \times \mathcal{A}_2 \quad \forall B \in \mathcal{B}.$

Consider $f_{\omega_1}(\omega_2) : \Omega_2 \to R$ (for fixed $\omega_1$)

To prove $f_{\omega_1}^{-1}(B) \in \mathcal{A}_2 \quad \forall B \in \mathcal{B}$.

Consider
$$\begin{aligned} f_{\omega_1}^{-1}(B) &= \{\omega_2 \mid f_{\omega_1}(\omega_2) \in B\} \\ &= \{\omega_2 \mid f(\omega_1, \omega_2) \in B\} \\ &= \{(\omega_1, \omega_2) \mid f(\omega_1, \omega_2) \in B\}_{\omega_1} \\ &= \{f^{-1}(B)\}_{\omega_1} \in \mathcal{A}_2 \end{aligned}$$

$\Rightarrow f_{\omega_1}$ is mble. Similarly we can prove $f_{\omega_2}$ is also mble

$\Rightarrow$ Sections of mble functions are also mble

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(101) Fubini's Theorem:-

Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ & $(\Omega_2, \mathcal{A}_2, \mu_2)$ be two σ- finite measure spaces.

Let $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$ be the product measure space. Let f be a function mble w.r.t. $\mathcal{A}_1 \times \mathcal{A}_2$. Let $f(\omega_1, \omega_2)$ be either non-ve or integrable w.r.t. measure $\mu_1 \times \mu_2$, then

$$\int_{\Omega_1 \times \Omega_2} f \, d(\mu_1 \times \mu_2) = \int_{\Omega_1} \left[ \int_{\Omega_2} f_{\omega_1}(\omega_2) \, d\mu_2 \right] d\mu_1$$

$$= \int_{\Omega_2} \left[ \int_{\Omega_1} f_{\omega_2}(\omega_1) \, d\mu_1 \right] d\mu_2 .$$

Further Section of X i.e $X_{\omega_1}(\cdot)$ & $X_{\omega_2}(\cdot)$ also integrable.

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<u>Remark</u> : (i) Iterated integrals are to be read from right to left.

(ii) The above result extends to product of arbitrary but finite no. of product measure spaces.

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