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* Linear Estimation :-

Def^n :- Linear Estimator

- The Estimator which are linear funct^{ns} of the obs^{ns} are Known as Linear Estimator.

* Concept of Unbiased Linear Estimators of Linear funct^{ns} of parameters :-

- Sample Mean $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is an unbiased estimator of pop^n mean μ
  i.e. $\boxed{\hat{\mu} = \bar{y}}$
  which is a linear unbiased estimator.

Linear Models

- $s^2 = \frac{1}{n-1}\Sigma(y_i - \bar{y})^2$ is an unbiased estimator of pop^n variance σ^2 ie. $\hat{\sigma}^2 = s^2$
  But s^2 is not a linear estimator but a quadratic estimator of pop^n Var. σ^2.

- If we have more than one unbiased linear estimator then to select the best estimator we use the concept of BLUE (Best Linear Unbiased Estimator)

- The principle which is used to make this selection is to choose that linear estimator from among all unbiased linear estimators which has the smallest variance.

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- The sampling distⁿ of that estimator will have the maximum concentration around the unknown true parametric function. Such an estimator is known as Minimum Variance Unbiased Linear Estimator or BLUE.

- This Minimum Variance Unbiased Linear Estimators have variances and covariances which again lead lend themselves to unbiased estimation.

- The Minimum Variance approach to the estimation of the parameters in the linear model was given by Markov (1900).

- The Least Square Method which is a Practical Method of estimating unknown parameter in a linear model was published independently by Gauss (1809) and Legendre (1806) in book on astronomical problems. The combined results is the famous Gauss - Markov Theorem.

* **Gauss - Markov Linear Model :-**

- Consider a set of 'n' independent r.v's $y_1, y_2, \ldots, y_n$ with a common variance σ^2 whose expectation are linear $funct^{ns}$ with known coefficients (a_{ij}'s) of p unknown parameters $\beta_1, \beta_2, \ldots, \beta_p$ $(p<n)$

Thus,

$$\left.\begin{aligned} E(y_i) &= a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{ip}\beta_p \\ Var(y_i) &= \sigma^2 \quad \forall\, i = 1, 2, \ldots, n \\ Cov(y_i, y_j) &= 0 \quad , \quad i \neq j \end{aligned}\right\} \quad \text{--- (1)}$$

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Eq^n (1) is called the Gauss - Markov Linear Model.
Eq^n (1) can be represented in matrix form as

Let $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$; $\underline{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}$

and matrix of the known Coefficient as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

Eq^n (1) can be written as

$$\left.\begin{aligned} E(\underline{y}) &= A\underline{\beta} \\ D(\underline{y}) &= \sigma^2 I \end{aligned}\right\} \quad \text{———} \quad (2)$$

where $D(\underline{y})$ stands for Dispersion Matrix and I is the identity matrix of order n.

- An alternative way of representing eq^n (2) is using column vector $\underline{e}$ of independent errors $e_1, e_2, \ldots, e_n$ as

$$\left.\begin{aligned} &\underline{y} = A\underline{\beta} + \underline{e} \\ &E(\underline{e}) = \underline{0} , \quad D(\underline{e}) = \sigma^2 I \end{aligned}\right\} \quad \text{———} \quad (3)$$

where $\underline{0}$ is a null vector

- The unknown parameters β_j's in the Model are called effects.

* **Fixed Effect Model :-**

- In linear estimation, the effects are all fixed quantities (parameters) and such a model where all effects are unknown parameters is called Fixed

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Effect Model or Model I. Sometimes, one of the β_j's is a constant with $a_{ij} = 1$ for that j and all $i = 1, 2, \ldots, n$. Such an effect is called General Effect or an additive constant.

* Random Effect Model :-

- If effects are random in a model then it is called Random Effect Model.

* Mixed Effect Model :-

- If some effects are fixed and some effects are random then it is called Mixed Effect Model.

- Gauss Markov Linear Model may be classified into three broad areas depending on the nature of the values taken by coefficient a_{ij}'s.

→ Model in which a_{ij}'s are indicator variables taking values 1 or 0 then such a model is called ANOVA Model.

Design

→ A Model in which a_{ij}'s takes the values as indep. variables then we have a regression model.

Regression

→ A Model in which a_{ij}'s take indicator as well as indep. variable then the model is Analysis of Covariance Model.

*

Def^n :-

Def^n :-

Theorem

Proof :-

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* Concept of Estimable funct^{ns} is due to R.C. Bose

Def^n: A linear funct^{ns} of the parameters $\beta_1, \beta_2, \ldots, \beta_p$ is called a Parametric funct^n.
Thus $l_1\beta_1 + l_2\beta_2 + \cdots + l_p\beta_p = \underline{l}'\underline{\beta}$ is a parametric funct^n where $\underline{l}' = (l_1, l_2, \ldots, l_p)$ is known.

Def^n: An estimable funct^n is a parametric funct^n $\underline{l}'\underline{\beta}$ for which an unbiased linear estimator exists.
Thus if $\exists$ a vector $\underline{c} = (c_1, c_2, \ldots, c_n)'$ of constants $\ni$ $E(\underline{c}'\underline{y}) = \underline{l}'\underline{\beta}$ indep. of the parameters then the parametric funct^n $\underline{l}'\underline{\beta}$ is such to be linearly estimable or to be an estimable funct^n

Theorem (1): A necessary and sufficient condition for a parametric funct^n or a linear parametric funct^n $\underline{l}'\underline{\beta}$ of the parameters to be linearly estimable is
rank (A) = rank $\begin{pmatrix} A \\ \underline{l}' \end{pmatrix}$ where $\begin{pmatrix} A \\ \underline{l}' \end{pmatrix}$ is the matrix obtained from A by adjoining the row vector $\underline{l}'$.

Proof: From the def^n, we see that $\underline{l}'\underline{\beta}$ is estimable iff $\exists$ a vector of constants $\underline{c}$ $\ni$

$$E[\underline{c}'\underline{y}] = \underline{l}'\underline{\beta} \quad \forall \ \underline{\beta}$$

But $$E[\underline{c}'\underline{y}] = \underline{c}'E(\underline{y}) = \underline{c}'A\underline{\beta}$$

So $\underline{l}'\underline{\beta}$ is estimable iff $\underline{c}'A\underline{\beta} = \underline{l}'\underline{\beta}$ identically in $\underline{\beta}$ iff $\underline{c}'A = \underline{l}'$ —— (i) —— (*)

To find an unbiased estimator for $\underline{l}'\underline{\beta}$ we have to solve (i) for an unknown vector $\underline{c}$

And the condition for existence of eq^n (i) is

$$\text{rank}(A) = \text{rank}\begin{pmatrix} A \\ \underline{l}' \end{pmatrix}$$

Corollary-(1): If rank (A) = P every linear function of the parameters is linearly estimable.

Proof :- Whatever may be $\underline{l}$ giving parametric funct^n $\underline{l}'\underline{\beta}$

$$\text{rank}\begin{pmatrix} A \\ \underline{l}' \end{pmatrix} \geq \text{rank}(A)$$

But $\text{rank}\begin{pmatrix} A \\ \underline{l}' \end{pmatrix}$ cannot exceed P,

Since $\begin{pmatrix} A \\ \underline{l}' \end{pmatrix}$ has only 'P' columns.

Hence if rank (A) = P, then $\text{rank}\begin{pmatrix} A \\ \underline{l}' \end{pmatrix} = \text{rank}(A)$ so by thm (1)

every $\underline{l}'\underline{\beta}$ is estimable.

Corollary-(2) :- Every linearly estimable parametric funct^n is of the form $\underline{b}'A\underline{\beta}$ or $b_1 E(y_1) + b_2 E(y_2) + \cdots + b_n E(y_n)$ where $\underline{b}' = (b_1, b_2 \ldots b_n)$

Proof :- If $\underline{l}'\underline{\beta}$ is estimable then by $(*)$ $\underline{l}'$ is linearly dependent on the rows of A, so that $\underline{l}'$ is of the form $\underline{b}'A$.

Hence $\underline{l}'\underline{\beta} = \underline{b}'A\underline{\beta}$

(r.v)

Def^n :- A linear funct^n of $y_1, y_2, \ldots, y_n$ is said to belong to error if the expectation vanishes (becomes zero) independently of the parameters.

- If $\underline{d}' = (d_1, d_2, \ldots, d_n)$, then $\underline{d}'\underline{y}$ belong to error iff

$$E(\underline{d}'\underline{y}) = \underline{d}' E(\underline{y}) = \underline{d}' A \underline{\beta} = 0 \quad \forall \ \underline{\beta}$$

ie. iff $\underline{d}' A = 0$ or $A'\underline{d} = 0$

Theorem (2) :- A necessary and sufficient condition for the linear funct^n $\underline{d}'\underline{y}$ to belong to error is that $\underline{d}'$ is orthogonal to the vector space $V(A')$ generated by the row vectors of A'.

- The vector space $V(B)$ which is orthogonal to the space $V(A')$ is called the Error space

and

- The vector space $V(A')$ which contains the coefficient vectors of the best estimators of estimable funct^n is known as Estimation Space.

Theorem (3) :- If $\underline{l}'\underline{\beta}$ is any estimable linear funct^n of the parameters $\beta_1, \beta_2, \ldots, \beta_p$ then

(i) $\exists$ a unique linear funct^n $\underline{c}'\underline{y}$ of the r.v's $y_1, y_2, \ldots, y_n \ni \underline{c} \in V(A')$ and $E(\underline{c}'\underline{y}) = \underline{l}'\underline{\beta}$.

(ii) $Var(\underline{c}'\underline{y})$ is less than the var of any other linear unbiased estimator of $\underline{l}'\underline{\beta}$.

Proof :- (i) Since $\underline{l}'\underline{\beta}$ is estimable, $\exists$ a linear funct^n $\underline{b}'\underline{y}$ of r.v.'s $\ni E(\underline{b}'\underline{y}) = \underline{l}'\underline{\beta}$

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Now we can uniquely resolve b' into c' and d' $\ni$ $c' \in V(A')$ and $d' \in V(B)$ which is orthogonal to $V(A')$

Hence $\underline{b}'\underline{y} = \underline{c}'\underline{y} + \underline{d}'\underline{y}$

where $\underline{d}'\underline{y}$ belong to error

Also,

$$E(\underline{b}'\underline{y}) = E(\underline{c}'\underline{y}) + \underbrace{E(\underline{d}'\underline{y})}_{\text{zero } (0)}$$

$$\Rightarrow E[\underline{c}'\underline{y}) = E(\underline{b}'\underline{y}) = \underline{l}'\underline{\beta}$$

$$(\because E(\underline{d}'\underline{y}) = 0)$$

Thus $\exists$ a linear funct^n $\underline{c}'\underline{y}$ with $\underline{c}' \in V(A')$ $\ni$ $E(\underline{c}'\underline{y}) = \underline{l}'\beta$

Now to show that this is unique.

If possible let there be another row vector $\underline{c}_0' \in V(A')$ $\ni$

$$E(\underline{c}_0'\underline{y}) = \underline{l}'\underline{\beta}$$

Define the row vector $\underline{c}_1' = \underline{c}' - \underline{c}_0'$

Then $E(\underline{c}_1'\underline{y}) = E(\underline{c}'\underline{y}) - E(\underline{c}_0'\underline{y})$

Thus $\underline{c}_1'$ belongs to error space and being a linear combination $\underline{c}'$ and $\underline{c}_0'$ also belongs to estimation space but this is impossible unless $\underline{c}_1'$ is a null vector has non-null vector cannot lie into two orthogonal space.

$\underline{c}_1'$ is a null vector implying $\underline{c}' = \underline{c}_0'$. Thus $\underline{c}'$ which lies in $V(A')$ for which $E(\underline{c}'\underline{y}) = \underline{l}'\beta$

(ii) Let $\underline{b}'\underline{y}$ be any arbitrary unbiased linear estimator of $\underline{l}'\beta$.

then $Var(\underline{b}'\underline{y}) = \underline{b}' \, Var(\underline{y}) \, \underline{b}$

$$= \underline{b}' \sigma^2 \underline{b}$$

$= (\underline{c}' + \underline{d}')(\underline{c} + \underline{d})\sigma^2$

$= \underline{c}'\underline{c}\sigma^2 + \underline{d}'\underline{d}\sigma^2$

$= var(\underline{c}'\underline{y}) + Var(\underline{d}'\underline{y})$

Hence,

$Var(\underline{b}'\underline{y}) \geq Var(\underline{c}'\underline{y})$

and equality holds iff $Var(\underline{d}'\underline{y}) = 0$

ie. $\underline{d}'\underline{d} = 0$ or $\underline{d}'$ is a null vector.

$\Rightarrow$ $\underline{c}'\underline{y}$ has minimum variance.

Corollary :- The best estimator of any estimable funct^n $\underline{l}'\beta$ must be of the form $\underline{q}'A'\underline{y}$ where $\underline{q}' = (q_1, q_2, ---, q_p)$ is a row vector and satisfies the eq^n $\underline{q}'A'A = \underline{l}'$

Proof :- The coefficient vector $\underline{c}'$ of the best estimator lies in $V(A')$

Hence $\underline{c}' = \underline{q}'A'$ for a suitable $\underline{q}'$ and the best estimator is of the form $\underline{c}'\underline{y} = \underline{q}'A'\underline{y}$

Since $E(\underline{q}'A'\underline{y})$ must be $\underline{l}'\underline{\beta}$

$\Rightarrow$ $\underline{q}'A'A = \underline{l}'$

Corollary :- Let $\underline{l}_i'\underline{\beta}$ for $i = 1, 2, ---, k$ be 'k' estimable parametric funct^n and let $T_i = \underline{c}_i'\underline{y}_i$ for $i = 1, 2, --k$ be their estimators. Then the best estimator of $\sum b_i \underline{l}_i'\underline{\beta}$ is $T = \sum b_i T_i$

Proof :-

$E(T) = E(\sum b_i T_i)$

$= \sum b_i E(T_i)$

$= \sum b_i E(\underline{c}_i'\underline{y}_i)$

$= \sum b_i \underline{l}_i'\underline{\beta}$

T_i is an unbiased estimator of $\sum b_i \underline{l}_i'\underline{\beta}$.

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* Least Square Estimators :

- Let $b_1, b_2, \ldots, b_p$ denotes any set of p known quantities which can be used as estimates of $\beta_1, \beta_2, \ldots, \beta_p$.

$\hat{\beta}_1 = b_1, \ldots, \hat{\beta}_p = b_p$.

Def^n :- A set of measurable funct^n $\underline{y}$, say $\hat{\beta}_1 = \hat{\beta}_1(\underline{y})$, $\hat{\beta}_2 = \hat{\beta}_2(\underline{y}), \ldots, \hat{\beta}_p = \hat{\beta}_p(\underline{y})$ such that the values $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p$ minimize the sum of squares of the deviation $y_1, y_2, \ldots, y_n$ from their expectation

ie. $S = (\underline{y} - A\underline{\beta})'(\underline{y} - A\underline{\beta})$ is called a set of least square estimators of the known parameters $\beta_1, \beta_2, \ldots, \beta_p$ of the linear model.

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* Least Square Estimators and Normal eq^{ns} :-

$$S = [\underline{y} - A\underline{\beta}]'[\underline{y} - A\underline{\beta}]$$

To show that the minimum value of S is attained when $\hat{\underline{\beta}}$ is a sol^n of a set of eq^{ns} which are called Normal eq^{ns}.

We have $\underline{\beta}' A' \underline{y} = \underline{y}' A \underline{\beta} = \sum_{\alpha, j} a_{\alpha j} \beta_j y_\alpha$; $j = 1, 2, \ldots p$
$\alpha = 1, 2, \ldots n$

Hence,

$$\frac{d}{d\beta_j}(\underline{\beta}' A' \underline{y}) = \frac{d}{d\beta_j}(\underline{y}' A \underline{\beta})$$

$$= \sum_{\alpha, j} a_{\alpha j} y_\alpha = \boxed{\underline{d}_j' \underline{y}}$$

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where $d_1, d_2, \dots, d_p$ are the column vector of A.

Let $A'A = C = (C_{ij})$
where C is a symmetric matrix of order p.
Now,

$$\underline{\beta}' A' A \underline{\beta} = \underline{\beta}' C \underline{\beta} = \sum C_{ij} \beta_i \beta_j \quad ; \; i, j = 1, 2, \dots, p$$

Hence,

$$\frac{d}{d\beta_j} (\underline{\beta}' A' A \underline{\beta}) = \frac{d}{d\beta_j} (\underline{\beta}' C \underline{\beta})$$

$$= 2 \sum_i C_{ij} \beta_i$$

$$= 2 \underline{C}_j' \underline{\beta}$$

where

$$\underline{C}_j' = (C_{1j}, C_{2j}, \dots, C_{pj})$$

$$S = [\underline{y} - A\underline{\beta}]' [\underline{y} - A\underline{\beta}]$$

$$\frac{\partial S}{\partial \beta} = 0$$

$$S = [\underline{y}'\underline{y} - \underline{y}' A \underline{\beta} - A' \underline{\beta}' \underline{y} - A' \underline{\beta}' A \underline{\beta}]$$

$$\frac{\partial S}{\partial \beta} = 0 \Rightarrow -2 \frac{\partial}{\partial \beta_j} (\underline{\beta}' A' \underline{y}) + \frac{\partial}{\partial \beta_j} (\underline{\beta}' A' A \underline{\beta}) = 0$$

$$\Rightarrow -2 \underline{d}_j' \underline{y} + 2 \underline{C}_j' \underline{\beta} = 0 \quad ; \; j = 1, 2, \dots, p$$

$$\Rightarrow -\underline{d}_j' \underline{y} + \underline{C}_j' \underline{\beta} = 0 \quad \text{——— } (*)$$

Eq^n $(*)$ are normal eq^{ns} and are equivalent to

$$\left.\begin{aligned} A' A \underline{\beta} &= A' \underline{y} \\ &\text{or} \\ C \underline{\beta} &= A' \underline{y} \end{aligned}\right\} \text{——— } (**)$$

where $C = A'A$

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The normal eq^{ns} always admit a solution since $\underline{A'y}$ lies in the vector space generator by the columns of C.

Let $\underline{\hat{\beta}}$ be a sol^{ns} of these eq^{ns}

$$\underline{\hat{\beta}} = (A'A)^{-1} A'\underline{y}$$

NOTE :- Every solution of the normal eq^{ns} is a set of least square estimators and every set of least square estimators satisfies the normal equations.

Result :- The sol^{n} of normal eq^{ns} $\underline{\beta} = \underline{\hat{\beta}}$ gives an extreme value of S and thus extreme value is the minimum value of S.

ie $\underline{\beta} = \underline{\hat{\beta}}$ minimizes S.

Proof :- Consider $(\underline{y} - A\underline{\beta})'(\underline{y} - A\underline{\beta})$

$$= (\underline{y} - A\underline{\hat{\beta}} + A(\underline{\hat{\beta}} - \underline{\beta}))'(\underline{y} - A\underline{\hat{\beta}} + A(\underline{\hat{\beta}} - \underline{\beta}))$$

$$= (\underline{y} - A\underline{\hat{\beta}})'(\underline{y} - A\underline{\hat{\beta}}) + (\underline{\hat{\beta}} - \underline{\beta})'A'A(\underline{\hat{\beta}} - \underline{\beta})$$

$$\geq (\underline{y} - A\underline{\hat{\beta}})'(\underline{y} - A\underline{\hat{\beta}}) \quad \text{———} \quad (***)$$

($\because$ Since the quadratic form $[A(\underline{\hat{\beta}} - \underline{\beta})'][A(\underline{\hat{\beta}} - \underline{\beta})]$ cannot be negative)

The equality holds only when $\underline{\beta} = \underline{\hat{\beta}}$

$\Rightarrow \underline{\beta} = \underline{\hat{\beta}}$ minimizes S.

Further if $\underline{\beta}$ and $\hat{\underline{\beta}}$ are any two sol^{ns} of (*) then $(\underline{y} - A\hat{\underline{\beta}})'(\underline{y} - A\hat{\underline{\beta}}) = (\underline{y} - A\hat{\hat{\underline{\beta}}})'(\underline{y} - A\hat{\hat{\underline{\beta}}})$ this along with (***) show that every sol^{ns} of the normal eq^{ns} is a set of LS. estimators.

* Gauss - Markov Theorem :-

The best estimator of the estimable linear funct^{ns} $\underline{l}'\underline{\beta}$ of the parameters is $\underline{l}'\hat{\underline{\beta}}$ where $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p$ are a set of LSE of $\beta_1, \beta_2, \ldots, \beta_p$.

In other words, LSE of $\underline{l}'\underline{\beta}$ is identical with min. variance linear unbiased estimator of $\underline{l}'\underline{\beta}$.

Proof :- From Corollary (1) we have BLUE of $\cancel{\underline{\beta}}$.

$$\underline{l}'\underline{\beta} = \underline{q}'A'\underline{y} = \underline{q}'A'A\hat{\underline{\beta}} = \underline{l}'\hat{\underline{\beta}}$$

Since from the normal eq^{ns} we have

$$A'A\hat{\underline{\beta}} = A'\underline{y}$$

Since $\underline{q}'$ satisfies $\underline{q}'A'A = \underline{l}'$ and $\underline{l}'\hat{\underline{\beta}} = \underline{q}'A'\underline{y}$

$\underline{l}'\hat{\underline{\beta}}$ is the same for all LSE ~~of~~ $\hat{\underline{\beta}}'$ of $\underline{\beta}$ obtained by solving Normal eq^{n}.

* Variance and Covariance of LSE :-

The normal eq^{ns} are given by

$$A'A\underline{\beta} = A'\underline{y}$$

Let $A'A = C$

$\Rightarrow \quad C\underline{\beta} = \underline{Q}$ where $\underline{Q} = A'\underline{y}$ (say)

$E(\underline{y}) = A\underline{\beta}$

$$E(A'\underline{y}) = E(\underline{Q}) = A'E(\underline{y}) = A'A\underline{\beta}$$
$$= E(\underline{Q}) = E(\underline{Q})$$

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$\Rightarrow \quad E(\underline{q}) = C\underline{\beta}$

$E(\hat{\underline{q}}) = \underline{q}$

Let $\bar{C}$ be a g-inverse of C then a sol^n of Normal eq^{ns} is given by

$$\hat{\underline{\beta}} = \bar{C}\underline{q}$$

$$D(\hat{\underline{\beta}}) = D(\bar{C}\underline{q})$$

$$D(\underline{q}) = D(C\underline{\beta}) = D(A'\underline{y})$$
$$= A' D(\underline{y}) A$$
$$= A'A\sigma^2$$
$$= C\sigma^2$$

$$D(\hat{\underline{\beta}}) = D(\bar{C}\underline{q}) = (\bar{C}) D(\underline{q}) (\bar{C})'$$
$$= (\bar{C}\cdot C)\sigma^2 \bar{C}$$
$$= \sigma^2 \bar{C} = \boxed{\sigma^2 (A'A)^-}$$

Remark

Theorem :- If $\underline{l}_1'\hat{\underline{\beta}}$ and $\underline{l}_2'\hat{\underline{\beta}}$ LSE of two estimable fun^n $\underline{l}_1'\underline{\beta}$ and $\underline{l}_2'\underline{\beta}$ respectively then *

$$Var(\underline{l}_1'\hat{\underline{\beta}}) = \sigma^2 \underline{l}_1' \bar{C} \underline{l}_1$$
$$Cov(\underline{l}_1'\hat{\underline{\beta}}, \underline{l}_2'\hat{\underline{\beta}}) = \sigma^2 \underline{l}_1' \bar{C} \underline{l}_2$$

where $\bar{C}$ is a g-inverse of $C = A'A$

Proof :- $Var(\underline{l}_1'\hat{\underline{\beta}}) = \underline{l}_1' Var(\hat{\underline{\beta}}) \underline{l}_1 = \boxed{\sigma^2 \underline{l}' \bar{C} \underline{l}}$

$$Cov(\underline{l}_1'\hat{\underline{\beta}}, \underline{l}_2'\hat{\underline{\beta}}) = \underline{l}_1' Cov(\hat{\underline{\beta}}, \hat{\underline{\beta}}) \underline{l}_2$$
$$= \underline{l}_1' Var(\hat{\underline{\beta}}) \underline{l}_2$$

$$= \boxed{\sigma^2 \underline{l}_1' \bar{C} \underline{l}_2}$$

Remark: If $(A'A) = C$ is a non-singular matrix then rank $(A) = p$

$\Rightarrow C^{-1}$ exists and a unique sol^n of $\underline{\beta} = \hat{\underline{\beta}}$ is obtained $(A'A)\underline{\beta} = A'\underline{y}$

$$\Rightarrow \hat{\underline{\beta}} = (A'A)^{-1}A'\underline{y}$$

$$\because E(\hat{\underline{\beta}}) = E\left[(A'A)^{-1}A'\underline{y}\right]$$
$$= (A'A)^{-1}A'E(\underline{y})$$
$$= (A'A)^{-1}(A'A)\underline{\beta}$$

$$\boxed{E(\hat{\underline{\beta}}) = \underline{\beta}}$$

$$D(\hat{\underline{\beta}}) = D\left[(A'A)^{-1}A'\underline{y}\right]$$
$$= \left[(A'A)^{-1}A'\right]' D(\underline{y}) \left[(A'A)^{-1}A'\right]$$
$$= \left((A'A)^{-1}A'\right)' \sigma^2 (A'A)^{-1}A'$$
$$= (A'A)^{-1}\sigma^2 \qquad (A'A \text{ is symmetric})$$

* **Linear Estimation with linear restrictions on parameters :-**

- When the parameters are subject to a set of consistent linear restrictions $\underline{P}'\underline{\beta} = \underline{L}$ with rank $(P) = \underline{P}_S$, the appropriate linear model is

$$\left.\begin{aligned} E(\underline{y}) &= A\underline{\beta} \\ D(\underline{y}) &= \sigma^2 I \\ \underline{P}'\underline{\beta}' &= \underline{L} \end{aligned}\right\}$$

In this case there are two method to estimate the parameters. First we eliminate some of parameters in the obs^n eq^n with the help of the parameters in linear restrictions and obtained a different eq^n in linear set of observational eq^{ns} with less parameters and having non restrictions on these fewer parameters.

- This case is then similar to model

$$\left.\begin{array}{l} E(\underline{y}) = A\beta \\ D(\underline{y}) = \sigma^2 I \end{array}\right\}$$

and in this approach, let $\underline{\beta}_0 + F\underline{\beta}^*$ be a general sol^n of $\underline{p}'\underline{\beta} = \underline{L}$ where $\underline{\beta}_0$ is a particular a sol^n and $\underline{p}'\underline{F}' = \underline{0}$.

- β^* being the arbitrary vector of fewer parameters (s in no.) with $\underline{z} = \underline{y} - A\underline{\beta}_0$

$$E(\underline{z}) = AF\underline{\beta}^*$$

$$D(\underline{z}) = \sigma^2 I$$

Thus with $\underline{z}$ the model reduces to the original model with fewer parameters $\beta_1^*, \beta_2^*, \dots \beta_s^*$ and no restrictions $\underline{\beta}^*$'s.

To estimate,

$$\underline{l}'\beta = \underline{l}'(\beta_0 + F\underline{\beta}^*) = \underline{l}'\beta_0 + \underline{l}'F\underline{\beta}^*$$

we need to consider only $\underline{l}'F\underline{\beta}^*$.

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- Second approach is to minimize the S.S. $(\underline{y} - A\beta)'(\underline{y} - A\beta)$ such that the conditions $\underline{p}'\underline{\beta} = \underline{L}$ using a lagrangian multiplier λ.

$$S^2 = (\underline{y} - A\beta)'(\underline{y} - A\underline{\beta}) + \lambda(\underline{p}'\underline{\beta} - \underline{L})$$

The normal eq^n are

$$S^2 = (\underline{y}'\underline{y} - \underline{y}'A\underline{\beta} - \underline{\beta}'A'\underline{y} + \underline{\beta}'A'A\underline{\beta} + \lambda(\underline{p}'\underline{\beta} - \underline{L})$$

$$\frac{\partial S^2}{\partial \beta} = 0 \rightarrow 2\underline{\beta}'A'A + \underline{P}'\underline{\lambda} - 2A'\underline{Y} = 0$$

$$A'A\underline{\beta} + \underline{P}'\underline{\lambda} = A'\underline{Y}$$

$$P\underline{\lambda}$$

$$\frac{\partial S^2}{\partial \lambda} = 0 \Rightarrow P'\underline{\beta} = \underline{L}$$

$$\begin{pmatrix} A'A & \underline{P} \\ \underline{P}' & \underline{O} \end{pmatrix} \begin{pmatrix} \underline{\beta} \\ \lambda \end{pmatrix} = \begin{pmatrix} A'Y \\ \underline{L} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \hat{\beta} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} A'A & \underline{P} \\ \underline{P}' & 0 \end{pmatrix}^{-1} \begin{pmatrix} A'Y \\ \underline{L} \end{pmatrix}$$

$$\Rightarrow \hat{\underline{\beta}} = (A'A)^{-1}A'\underline{Y} + \underline{P}^{-1}\underline{L}$$

$$\hat{\lambda} = (\underline{P}')^{-1}A'\underline{Y}.$$

* **Linear Estimation with correlated variables :-**

- We consider a model, it is more general then the Gauss - Markov Linear Model

$$\left.\begin{aligned} E(\underline{Y}) &= A\underline{\beta} \\ D(\underline{Y}) &= \sigma^2 \underline{B} \end{aligned}\right\} \text{———} (1)$$

where σ^2 is an unknown positive constant and B is a known matrix B is symmetrical and we assume that $|B| \neq 0$.

- The above model can be reduced to the original Gauss - Markov model as follows :-
Since 'B' is assumed to be non - singular matrix of order 'n' $\exists$ a n.s. matrix H of order 'n' $\ni$ $H'BH = I$

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Then we consider the transformation
$$\underline{y}^* = H'\underline{y}$$

$$E(\underline{y}^*) = E(H'\underline{y}) = H'E(\underline{y}) = H'A\underline{\beta} = A^*\underline{\beta} \quad (\text{say})$$

$$D(\underline{y}^*) = D(H'\underline{y}) = H'D(\underline{y})H$$
$$= H'\sigma^2 BH$$
$$= \sigma^2 I \qquad (2)$$

There exist a linear transformation which ~~requires~~ reduces the case of correlated variable to the earlier model with the matrix of coefficient
$$A^* = H'A$$
and $\text{rank}(A^*) = \text{rank}(H'A) = \text{rank}(A)$

So all the results for model (2) (the original model) will also be true above model.

Remark:- However it is simple to used the actual obs^n $(y_1, y_2, \ldots, y_n)$ instead of transformed obs^n $(y_1^*, y_2^*, \ldots, y_n^*)$

LSE is found by minimising the following S.S in terms of transformed variables.
$$(\underline{y}^* - A\underline{\beta}^*)'(\underline{y}^* - A\underline{\beta}^*)$$

Since $(\underline{y}^* - A\underline{\beta}^*) = H'\underline{y} - H'A\underline{\beta}$
$= H'(\underline{y} - A\underline{\beta})$ and $HH' = B^{-1}$

we have,
$$(\underline{y} - A\underline{\beta})'HH'(\underline{y} - A\underline{\beta})$$
$$= (\underline{y} - A\underline{\beta})'B^{-1}(\underline{y} - A\underline{\beta}) \quad \text{with } B^{-1} = b^{ij}$$

The above model can be expressed as

$$\sum_i \sum_j b^{ij} (y_i - a_{i1}\beta_1 - a_{i2}\beta_2 - \cdots a_{ip}\beta_p) \times (y_j - a_{j1}\beta_1 - a_{j2}\beta_2 - \cdots a_{jp}\beta_p)$$

is called the Weighted Sum of Squares.

- LSE of β^* are then found by minimizing the weighted S.S. in the case of the correlated variables.

Remarks :- (1) If the variables are uncorrelated but do not have a constant variance σ^2 then B will be a diagonal matrix with say $b_1, b_2, \ldots, b_n$ as diagonal elements. Here b_i will be proportional to the $Var(y_i)$.

The weighted S.S. in this case, in order to get LSE of β is

$$\sum_i (y_i - a_{i1}\beta_1 - a_{i2}\beta_2 - \cdots a_{ip}\beta_p)^2 / b_i$$

if all $\frac{1}{b_i}$ are equal i.e. case of constant variance, the model reduces original Gauss - Markov Model.

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(1) Let y_1, y_2, y_3 be an independent r.v. with a common variance σ^2 and $E(y_1) = \theta_1 + \theta_3$, $E(y_2) = \theta_1 + 2\theta_2 + \theta_3$, $E(y_3) = \theta_1 + \theta_2$, obtain LSE of $2\theta_1 - \theta_2 - \theta_3$

→ $E(\underline{y}) = A\underline{\theta}$

$D(\underline{y}) = \sigma^2 I$

(∵ Normal eq^n

$(A'A)\hat{\underline{\beta}} = A'\underline{y}$

$\hat{\theta} = (A'A)^{-1} A'\underline{y}$)

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$$(A'A)^{-1} = \left[\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & \cancel{0}\,1 & \cancel{0}\,1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right]^{-1}$$

$$= \frac{1}{8} \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}$$

$$(A'A)^{-1} A'\underline{y} = \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & \cancel{0}\,1 & \cancel{0}\,1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} y_1 + y_2 + y_5 + y_6 \\ y_3 + y_4 + y_5 + y_6 \end{bmatrix}$$

$$\hat{\theta}_1 = \frac{1}{2}\left[(y_1 + y_2 + y_5 + y_6) - \frac{1}{2}(y_3 + y_4 + y_5 + y_6) \right]$$

$$\hat{\theta}_2 = \frac{1}{2}\left[-(y_1 + y_2 + y_5 + y_6) + \frac{1}{2}(y_3 + y_4 + y_5 + y_6) \right]$$

(3) y_1, y_2, y_3 are uncorrelated random variables with common variance σ^2 and $E(y_1) = \theta_1 + \theta_2$, $E(y_2) = \theta_1 + \theta_3$, $E(y_3) = \theta_1 + \theta_2$. Show that $l_1\theta_1 + l_2\theta_2 + l_3\theta_3$ is estimable iff $l_1 = l_2 + l_3$

$$\left(\begin{array}{l} \theta_1 + 2\theta_2 + \theta_3 \\ V(y_1 - 2y_2 + 3y_3) \\ V(y_1) + 4(y_2) + 9(y_3) \\ = \sigma^2 + 4\sigma^2 + 9\sigma^2 = 14\sigma^2 \end{array} \right)$$

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$\rightarrow$ $\underline{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

According to Thm,

$\underline{l}'\underline{\theta}$ is estimable iff $\text{rank}(A') = \text{rank}(A', \underline{l})$

or

$\text{rank}(A'A) = \text{rank}(A'A, \underline{l})$

$$A'A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(A'A, \underline{l}) = \begin{pmatrix} 3 & 2 & 1 & l_1 \\ 2 & 2 & 0 & l_2 \\ 1 & 0 & 1 & l_3 \end{pmatrix}$$

$$3R_3 - R_1 = \begin{bmatrix} 0 & -2 & 2 & 3l_3 - l_1 \\ 2 & 2 & 0 & l_2 \\ 1 & 0 & 1 & l_3 \end{bmatrix}$$

$$2R_3 - R_2 = \begin{bmatrix} 0 & -2 & 2 & 3l_3 - l_1 \\ 0 & -2 & 2 & 2l_3 - l_2 \\ 1 & 0 & 1 & l_3 \end{bmatrix}$$

$$R_2 - R_1 = \begin{bmatrix} 0 & 0 & 0 & l_1 - l_2 - l_3 \\ 0 & -2 & 2 & l_1 - 2l_3 \\ 1 & 0 & 1 & l_3 \end{bmatrix}$$

$$\text{rank}(A'A, \underline{l}) = \boxed{2}$$

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* Simultaneous estimation of Parametric functions :-

Consider the general step $(\underline{y}, X\underline{\beta}, I)$, with or without restrictions on the parameter $\underline{\beta}$.
Let $\underline{p}_1'\hat{\underline{\beta}}, \underline{p}_2'\hat{\underline{\beta}}, \ldots, \underline{p}_k'\hat{\underline{\beta}}$ be the individual lse of the parametric functions $\underline{p}_1'\underline{\beta}, \underline{p}_2'\underline{\beta}, \ldots, \underline{p}_k'\underline{\beta}$. Let A be the dispersion matrix of the estimators $\underline{p}_1'\hat{\underline{\beta}}, \underline{p}_2'\hat{\underline{\beta}}, \ldots, \underline{p}_k'\hat{\underline{\beta}}$. Then we have the following optimum property of LSE.

(i) Let $\underline{L}_1'\underline{y}, \underline{L}_2'\underline{y}, \ldots, \underline{L}_k'\underline{y}$ be any unbiased estimators of $\underline{p}_1'\underline{\beta}, \underline{p}_2'\underline{\beta}, \ldots, \underline{p}_k'\underline{\beta}$ and let the dispersion matrix of estimators be B then $B - A$ is non-negative definite implying

(a) trace $B \geq$ trace A.

(b) $|B| \geq |A|$

(c) trace $\varphi B \geq$ trace φA where φ is non-negative definite matrix.

(d) Max. latent root of A.

- Consider the linear parametric functions $a_1\underline{p}_1'\underline{\beta} + a_2\underline{p}_2'\underline{\beta} + \cdots + a_k\underline{p}_k'\underline{\beta}$ where LSE is $a_1\underline{p}_1'\hat{\underline{\beta}} + a_2\underline{p}_2'\hat{\underline{\beta}} + \cdots + a_k\underline{p}_k'\hat{\underline{\beta}}$

$$Var(\underline{a}'\underline{p}'\hat{\underline{\beta}}) = \underline{a}' \, Var(\underline{p}'\hat{\underline{\beta}})\, \underline{a} = \underline{a}'A\underline{a}$$

where $\underline{a}' = (a_1, a_2, \ldots a_k)$

$$V(a_1\underline{L}_1'\underline{y} + a_2\underline{L}_2'\underline{y} + \cdots + a_k\underline{L}_k'\underline{y}) = \underline{a}'B\underline{a}$$

Then for all $\underline{a}$, we have

$$\underline{a}'B\underline{a} \geq \underline{a}'A\underline{a}$$

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Prerequisite

- Consider the indep variables

$$y_i \sim N(x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{im}\beta_m, \sigma^2) \quad \text{---} \quad 1)$$

$; i = 1, 2, \ldots, n$

where x_{ij} are known coefficient and β_i are unknown parameters.

- In matrix notation, $\underline{y}$ stands for col^n vector of the variables y_i, $\underline{\beta}$ for the parameters β_i and $X = (x_{ij})$ for matrix of coefficient then

$$\Sigma(y_i - x_{i1}\beta_1 - \cdots - x_{im}\beta_m)^2$$
$$= (\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta})$$

Hence the pdf of $y_1, y_2, \ldots, y_n$ can be written as

$$c\, e^{-(\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta})/2\sigma^2} \quad \text{---} \quad (2)$$

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* **First Fundamental Thm :-**

Let $R_0^2 = \min_{\beta} (\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta})$

then

$$R_0^2 \sim \sigma^2 \chi^2_{(n-r)} \quad \text{where } r \text{ is the rank of } X.$$

Proof:- $(\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta})$ is minimum when $X\underline{\beta}$ is the projection of $\underline{y}$ on $\underline{M}(X)$ (mangi-fold of X)

[→ A linear subspace or a linear manifold in a vector space $\mathcal{V}$ is any subset of vectors M

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closed under addition and scalar multiplication if $\underline{x} \in M$, $\underline{y} \in M$ then $(c\underline{x} + d\underline{y}) \in M$ for any pair of scalars c and d. Any such subset M is itself a vector space.

- All linear combination of a given fixed set S of vectors $\underline{\alpha}_1, \underline{\alpha}_2, \ldots, \underline{\alpha}_k$ is a subspace called the linear manifold M(S) spanned by S. This is the smallest subspace containing S.]

- A projection is a linear transformation P from a vector space to itself $\ni$ $P^2 = P$.
i.e. whenever P is applied twice to any value it gives the same result as if it were applied once.

- The mapping $P; \underline{x} \rightarrow \underline{x}_1$ $(P\underline{x} = \underline{x}_1)$ is called the Projection of x and P is called the projection operator.

Thm: But Projection of any vector on M(x) is obtained through an operator a matrix P which is symmetric, idempotent $(P^2 = P)$ and of rank = rank (x) = r.

Proof

Thus the projection y on M(x) equal $P\underline{y}$.
Therefore $\underline{y} - P\underline{y}$ is perpendicular

chi-sq^n dist^n $\nearrow$

$$R_0^2 = (\underline{y} - x\underline{\beta})'(\underline{y} - x\underline{\beta})$$
$$= (\underline{y} - P\underline{y})'(\underline{y} - P\underline{y})$$
$$= \underline{y}'(I - P)'(I - P)\underline{y}$$
$$= \underline{y}'(I - P)\underline{y}$$

$\nearrow$ (Standard Normal dist^n)

Thm

The matrix $(I - P)$ is also idempotent and

$\therefore$ rank $(I - P) = \text{trace } I - \text{trace } P$

$= n - r$

The dist^n of quadratic form

$$\underline{y}'(I - P)\underline{y} \sim \sigma^2 \cdot \chi^2_{(n-r,\lambda)} \quad \text{——} \quad (*)$$

where,

λ is non-centrality parameter.

$$\lambda = E\left[\frac{\underline{y}'(I-P)\underline{y}}{\sigma^2}\right].$$

$$\sigma^2 \lambda = E[\underline{y}'(I - \underline{P})\underline{y}]$$
$$= E(\underline{y}')(I - \underline{P})E(\underline{y})$$
$$= (\underline{\beta}'x')(I - \underline{P})(x\underline{\beta})$$

$$= \underline{\beta}'[x'x - x'Px]\underline{\beta} \qquad (\because Px = x)$$
$$= 0$$

Projection operator

$\Rightarrow$ The dist^n of

$\dfrac{\underline{y}'(I-P)\underline{y}}{\sigma^2} \sim$ Central χ^2 with $(n-r)$ df.

Thm (2) :- Second fundamental Thm :-

- Let H be a matrix of order $(m \times k)$ and rank $k \ni M(\mu) \subset M(x')$ and

$$R_1^2 = \min_{\underline{\beta}} (\underline{y} - x\underline{\beta})'(\underline{y} - x\underline{\beta}) \quad \text{such that}$$

$$H\underline{\beta} = \xi \text{ (given)}$$

(a) R_0^2 and $R_1^2 - R_0^2$ are independently distributed.

(b) $R_0^2 \sim \sigma^2 \chi^2_{(n-r)}$ and $R_1^2 - R_0^2$ as a non-central χ^2 on k d.f.

$R_1^2 = (y - X\beta_0)'(I - M)(y - X\beta_0)$

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c) If $\underline{H}'\underline{\beta} = \xi$ is true then

$$R_1^2 - R_0^2 \sim \sigma^2 \chi^2_{(k)}$$

$$\frac{R_1^2 - R_0^2}{k} \div \frac{R_0^2}{n-r} \sim F_{(k, n-r)}$$

Proof :- If $H'\underline{\beta} = \xi$ then $\underline{\beta} = \underline{\beta}_0 + \gamma$ where $\underline{\beta}_0$ is a particular sol^n of $H'\underline{\beta} = \xi$ and γ is a general sol^n of $H'\underline{\beta} = 0$

Hence,

$$\min_{H'\underline{\beta} = \xi} (\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta})$$

$$= \min_{H'\gamma = 0} (\underline{y} - X\beta_0 - X\gamma)'(\underline{y} - X\beta_0 - X\gamma)$$

But $X\gamma$ with the restriction $H'\gamma = 0$ is a subspace $s \subset M(X)$ with

$$\text{rank}(s) = \text{rank}[X' \vdots H] - \text{rank}(H)$$
$$= s \text{ (say)}$$

whether or not H satisfies the condition $M(H) \subset M(X')$

Let P be an operator for projecting $M(X)$ and U be the operator for projecting on s.

$$\text{rank}(P) = \text{rank}(X) = r$$
$$\text{rank}(U) = s$$

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$$R_0^2 = \underline{y}'(I-P)\underline{y}$$
$$= (\underline{y} - X\underline{\beta}_0)'(I-P)(\underline{y} - X\underline{\beta}_0)$$

Introduction of the factor $X\underline{\beta}_0$ in the expression of R_0^2 does not alter its value.

- Since in R_1^2 is $(I-U)$ is idempotent
rank $(I-U) = n-s$

$$R_1^2 \sim \sigma^2 . \chi^2(n-s, \lambda)$$

$$\sigma^2\lambda = E\left[(\underline{y} - X\underline{\beta}_0)'(I-U)(\underline{y} - X\underline{\beta}_0)\right]$$

$$= (X\underline{\beta} - X\beta_0)'(I-U)(X\underline{\beta} - X\beta_0)$$

$$= (X\underline{\beta}_0 + X\gamma - X\beta_0)'(I-U)(X\beta_0 + X\gamma - X\beta_0)$$

$$\left(\because \text{If } H'\underline{\beta} = \underline{\xi} \text{ is true then } \underline{\beta} = \beta_0 + \gamma \right)$$

$$= (X\gamma)'(I-U)(X\gamma)$$
$$= \gamma'X'X\gamma - \gamma'X'UX\gamma$$
$$= \gamma'X'X\gamma - \gamma'X'X\gamma \quad (\text{Since } UX\gamma = X\gamma)$$
$$= 0$$

Since $R_0^2 \sim \sigma^2\chi^2_{(n-r)}$ by 1st thm and $R_1^2 - R_0^2 \geq 0$ also $R_1^2 \sim \sigma^2\chi^2(n-s)$

$$R_1^2 - R_0^2 \sim \sigma^2 . \chi^2(r-s)$$

In the special case, $M(H) \subset M(X')$ and H is of rank k

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$$S = \text{rank}\,(X' : H) - \text{rank}\,(H)$$
$$= k$$

Hence $\quad R_1^2 - R_0^2 \sim \sigma^2 \chi^2_{(k)}$

Hence $$\frac{R_1^2 - R_0^2}{k} \div \frac{R_0^2}{n-r} \sim F_{(k,\, n-r)}$$

Theorem (3) :- Let $\underline{y}' = (\underline{y}_1' \vdots \underline{y}_2')$ with the corresponding partition of the expectation vector
$$\beta' X' = \beta' (\underline{X}_1' \vdots \underline{X}_2') = (\underline{\beta} \underline{X}_1' \vdots \underline{\beta} \underline{X}_2')$$
then the statistic

$$u = \frac{\min\limits_{\beta} (y_1 - \underline{X}_1 \underline{\beta})' (\underline{y}_1 - \underline{X}_1 \underline{\beta})}{\min\limits_{\beta} (\underline{y} - X\underline{\beta})' (\underline{y} - X\underline{\beta})}$$

$$\sim \beta \left(\frac{q - r}{2}, \frac{n - q - r + r}{2} \right)$$

where $\text{rank}\,(X_1) = r_1$ and $\text{rank}\,(X) = r$
and q – col^n in X_1.

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* Test of Hypothesis and Interval Estimation :-

– The LSE of a parametric funct^n is only a point estimate and no exact statement of prob. of its deviation from the true value of the parametric funct^n can be made without a specific dist^n for the variable $\underline{y}$ being considered we shall assume that

$$\underline{y} \sim N(X\underline{\beta}, \sigma^2 I)$$

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Let H' be matrix of order $k \times m$ of row 'k' such that its rows ~~of~~ ~~x~~ depends in the rows of X.

→ k Parametric funct^{ns} $H'\underline{\beta}$ are individually estimable under the set up $(\underline{y} - X\underline{\beta}, \sigma^2 I)$ in which case the LSE are $H'\hat{\underline{\beta}} = Z$

In terms of X, Y we have

$$H'\hat{\underline{\beta}} = H'(X'X)^{-1}X'\underline{y}$$

$$R_0^2 = \text{Error S.S.} = y'(I - (X'X)^{-1}X')y$$
$$= (\underline{y} - X\hat{\beta})'(\underline{y} - X\hat{\beta})$$
$$= (\underline{y} - X(X'X)^{-1}X'\underline{y})'(\underline{y} - X(X'X)^{-1}X'\underline{y})$$
$$= y'(I - X(X'X)^{-1}X')'\underline{y}(I - X(X'X)^{-1}X')$$

Idempotent matrix $(A'A = A)$

$$= y'(I - X(X'X)^{-1}X')y$$

then

$$H'(X'X)^{-1}X'(I - X(X'X)^{-1}X')$$
$$= H'[X'(X'X)^{-1} - (X'X)^{-1}X']$$
$$= H'[X'(X'X)^{-1} - X'(X'X)^{-1}]$$
$$= 0$$

⇒ $H'\hat{\beta}$ and R_0^2 are independently distributed

The dist^n of Z is a k-variate normal with mean $H'\beta$ and dispersion matrix $\sigma^2 D$ (say) and

$$R_0^2 \sim \sigma^2 \chi^2_{(n-r)}$$

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Let H' be matrix of order $k \times m$ of row 'k' such that its rows ~~of~~ depends in the rows of X.

$\Rightarrow$ k Parametric funct^{ns} $H'\underline{\beta}$ are individually estimable under the set up $(\underline{y} - X\underline{\beta}, \sigma^2 I)$ in which case the LSE are $H'\hat{\underline{\beta}} = Z$

In terms of X, Y we have

$$H'\hat{\underline{\beta}} = H'(X'X)^{-1}X'\underline{y}$$

$$R_0^2 = \text{Error S.S.} = y'(I - (X'X)^{-1}X')y$$
$$= (\underline{y} - X\hat{\beta})'(\underline{y} - X\hat{\beta})$$
$$= (\underline{y} - X(X'X)^{-1}X'\underline{y})'(\underline{y} - X(X'X)^{-1}X'\underline{y})$$
$$= y'(I - X(X'X)^{-1}X')'\underline{y}(I - X(X'X)^{-1}X')$$

Idempotent matrix $(A'A = A)$

$$= y'(I - X(X'X)^{-1}X')\underline{y}$$

then

$$H'(X'X)^{-1}X'(I - X(X'X)^{-1}X')$$
$$= H'[X'(X'X)^{-1} - (X'X)^{-1}X']$$
$$= H'[X'(X'X)^{-1} - X'(X'X)^{-1}]$$
$$= 0$$

$\Rightarrow$ $H'\hat{\underline{\beta}}$ and R_0^2 are independently distributed

The dist^n of Z is a k-variate normal with mean $H'\beta$ ~~exp~~ and dispersion matrix $\sigma^2 D$ (say) and

$$R_0^2 \sim \sigma^2 \chi^2_{(n-r)}$$

$\therefore$ i.e. $H'\hat{\beta} \sim N(H'\beta, \sigma^2 D)$

$(\because E(H'\hat{\beta}) = H'E(\hat{\beta}) = H'\beta$
$Var(H'\hat{\beta}) = H' Var(\hat{\beta}) H = \sigma^2 D$ as $H'H = D)$

* Single Parametric funct^n :-

- Given an estimable parametric funct^n $\theta = p'\beta$
Let u be its LSE with Var $p^2\sigma^2$ Then
$u \sim N(\theta, p^2\sigma^2)$ and $R_0^2 \sim \sigma^2 \chi^2_{(n-r)}$ and are independent so that $s^2 = \frac{R_0^2}{(n-r)}$

then

$$t = \frac{u-\theta}{p\sigma} \div \frac{s}{\sigma} = \frac{u-\theta}{ps} \sim S_{(n-r)} \quad \text{—— (i)}$$

which is student's dist^n on $(n-r)$ d.f

If t_α is the α prob point of $|t|$
i.e. $P[|t| > t_\alpha] = \alpha$ then

$$P\left[\frac{|u-\theta|}{ps} \le t_\alpha\right] = 1 - \alpha \quad \text{—— (ii)}$$

i.e.

$$P\left[-t_\alpha \le \frac{(u-\theta)}{ps} \le t_\alpha\right] = 1 - \alpha$$

$$\Rightarrow P\left[u - ps t_\alpha \le \theta \le u + ps t_\alpha\right] = 1 - \alpha \quad \text{—— (iii)}$$

Let the Null hypothesis be $H_0 : p'\beta = \theta_0$
(an assigned value)

Null hyp is rejected at α level of significance

Effect size is based on Non-centrality parameter

$\theta = p'\beta$

$H_0: \mu = 1 \quad \mu \neq 1$

$H_1: \mu = 2$

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if for an observed u we have

$$\frac{|u - \theta_0|}{ps} > t_\alpha \quad \text{—— (iv)}$$

The Power of the test is given by $(1-\beta)$ i.e. $P[|t| > t_{\alpha/2, n-r} / H_1]$. Here t has non-central t-distⁿ with λ as non-central parameter. The power of t is monotonically increasing functⁿ of $|\lambda|$.

$$|\lambda| = \frac{\text{Value of } \underline{p}'\beta \text{ under } H_1 - \text{Value of } \underline{p}'\beta \text{ under } H_0}{\sqrt{\text{Var of best estimator of } \underline{p}'\underline{\beta}}}$$

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If $Z \sim N(0,1)$ ($\because$ z and v are indep)

$V \sim \chi^2_{(\nu)}$

then non-central 't' is given by

$$t_{\nu,\delta} = \frac{Z + \delta}{\sqrt{\frac{V}{\nu}}}$$; δ is non-centrality parameter.

Ex:- Suppose we want to test Hyp $H_0: \mu = 70$

$H_1: \mu > 70$

for $n = 25$, $\alpha = 0.05$. Find Power of 't' when $\sigma = 10$?

$\rightarrow$ $$\delta = \frac{75 - 70}{\sigma/\sqrt{n}} = \frac{5}{10/5} = \boxed{2.5}$$

Statistics Kingdom
- Power T Z calculator

R → PWR package

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Effect size d is given by
$$\delta = d \cdot \sqrt{n}$$
$(\sigma/\sqrt{n})$

$$d = \frac{2.5}{5} = 0.5$$

$$\Rightarrow \boxed{d = \frac{\mu - \mu_0}{\sigma}}$$

(4) Let $Y_1, Y_2, \dots, Y_n$ be indep obs^n from $N(\mu, \sigma^2)$
Test the hyp. $H_0: \mu = \mu_0$ (given)
this case $x\beta$ is the μ

→ In this case
$$SSE = \underset{\mu}{Min} \sum (y_i - \mu)^2$$
$$= \sum_{i=1}^{n} (y_i - \bar{y})^2 \quad \text{has } (n-1) \text{ d.f.}$$

The normal eq^n is
$$\sum Y_i = n\mu$$
$$\Rightarrow \hat{\mu} = \frac{\sum Y_i}{n} = \bar{Y}$$

Also $V(\bar{y}) = V\left(\frac{\sum Y_i}{n}\right) = \frac{1}{n^2} \sum V(Y_i)$
$$= \frac{\sum \sigma^2}{n^2} = \frac{n\sigma^2}{n^2} = \boxed{\frac{\sigma^2}{n}}$$

Thus appropriate test statistic for testing H_0 is
$$t = \frac{(\bar{Y} - \mu_0) \cdot \sqrt{n}}{\sqrt{MSSE}} \quad \text{which is } t \text{ with } (n-1) \text{ d.f.}$$

Reject H_0 if $|t|_{cal} > t_{tab}$ where
$$t_{tab} = t_{(\alpha/2,\, n-1)}$$

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The Power of the t-test wrt $H_1 : \mu = \mu_1$ is

$$P\left[\frac{\sqrt{n}\,|\bar{Y} - \mu_0|}{\sqrt{MSSE}} > t_{(\alpha/2,\, n-1)} \;\middle|\; H_1 \right]$$

where t has non-central t-dist^n with non-centrality parameter

$$\boxed{\lambda = \frac{|\mu_1 - \mu_0|\sqrt{n}}{\sigma}}$$

Ex:- Let $Y_1, Y_2, \ldots, Y_{n_1}$ be a r.s. from $N(\mu_1, \sigma^2)$ and $Y_1', Y_2', \ldots, Y_{n_2}'$ be a r.s from $N(\mu_2, \sigma^2)$ the two samples being mutually indep. Test the hyp. $H_0 : \mu_1 = \mu_2$ $\left(H_0 : P'\beta = 0, \; \mu_1 - \mu_2 = 0 \right)$

→ $E(Y_i) = \mu_1 \qquad i = 1, 2, \ldots, n$

$E(Y_i') = \mu_2 \qquad i = 1, 2, \ldots, n$

$$SSE = \lim_{\mu_1, \mu_2} \left[\sum_{i=1}^{n_1} (Y_i - \mu_1)^2 + \sum_{i=1}^{n_2} (Y_i' - \mu_2)^2 \right]$$

The normal eq^n are

$$n_1 \mu_1 = \sum Y_i = n_1 \bar{Y}$$
$$n_2 \mu_2 = \sum Y_i' = n_2 \bar{Y}$$

Hence

$$SSE = \sum_{i=1}^{n_1} (Y_i - \bar{Y})^2 + \sum_{i=1}^{n_2} (Y_i' - \bar{Y}')^2$$

with $(n_1 + n_2 - 2)$ d.f.

Variance for LSE of parametric $(\mu_1 - \mu_2)$ funct^n is

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$$V(\bar{y} - \bar{y}') = Var(\bar{y}) + Var(\bar{y}')$$
$$= \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}$$
$$= \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

Hence, the appropriate 't' statistic is

$$t = \frac{\bar{y} - \bar{y}'}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) MSSE}} \quad \text{with d.f. } (n_1 + n_2 - 2)$$

($\because$ Fisher's t)

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More than one parametric function

- Let us consider 'k' indep estimable linear parameter function

$$\theta_1 = \underline{H}_1'\underline{\beta}, \quad \theta_2 = \underline{H}_2'\underline{\beta}, \ldots, \theta_k = \underline{H}_k'\underline{\beta}$$

- In matrix notation, it may be written as

$$\underline{\theta} = \underline{H}'\underline{\beta}$$

where H' is a $k \times m$ matrix and $\underline{\theta}$ is the col^n vector $\theta_1, \theta_2, \ldots, \theta_k$

- The LSE of $(\theta_1, \theta_2, \ldots, \theta_k)$ are represented by $(z_1, z_2, \ldots, z_k) = z'$ and its dispersion matrix by $\sigma^2 D$. Then

$$E(\underline{z}) = \underline{\theta}, \quad D(\underline{z}) = \sigma^2 D$$

we have

$$(\underline{z} - \underline{\theta})' D^{-1} (\underline{z} - \underline{\theta}) \sim \sigma^2 \chi^2_{(k)} \text{ df.}$$

and

$$R_0^2 \sim \sigma^2 \chi^2_{(n-r)} \quad \text{are indep.}$$

Hence,

$$F = \frac{(\underline{z} - \underline{\theta})' D^{-1} (\underline{z} - \underline{\theta})}{k} \div \frac{R_0^2}{(n-r)}$$

$$\sim F_{(k,\, n-r)}$$

* **Test of multiple hypothesis :-**

(ANOVA)

- Let it be required to test the null hyp. that 'k' Parametric funct^n have the assigned values as

$$H_0 : \underline{H}_1' \underline{\beta} = \theta_{10} , \underline{H}_2' \underline{\beta} = \theta_{20} , \ldots , \underline{H}_k' \underline{\beta} = \theta_{k0}$$

which may be written in matrix notation as

$$\boxed{\underline{H}' \underline{\beta} = \theta_0}$$

Let $z_1 , z_2 , \ldots , z_k$ be LSE of $\underline{H}_1' \underline{\beta} , \underline{H}_2' \underline{\beta} , \ldots , \underline{H}_k' \underline{\beta}$

$(z - \theta_0)$ is the vector of deviations

Def^n lse and assigned values.

If infact θ_0 is not true, the deviations $(\underline{z} - \underline{\theta}_0)$ are likely to be large.

Let us consider the compound deviation (a single measure of deviations)

$$(\underline{z} - \underline{\theta}_0)' D^{-1} (\underline{z} - \underline{\theta}_0)$$

Now,

$$E \left[\frac{(\underline{z} - \underline{\theta}_0)' D^{-1} (\underline{z} - \underline{\theta}_0)}{k} \right]$$

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$$= \frac{\sigma^2}{k} \text{ trace}(D^{-1}D) + \frac{1}{k} E(\underline{z} - \underline{\theta}_0)' D^{-1} E(\underline{z} - \underline{\theta}_0)$$

D matrix > 0

$$= \frac{\sigma^2}{k} \cdot k + \frac{1}{k} \left[(\underline{H}'\underline{\beta} - \underline{\theta}_0)' D^{-1} (\underline{H}'\underline{\beta} - \underline{\theta}_0) \right]$$

$(\because E(\underline{z}) = \underline{\theta} = \underline{H}'\underline{\beta}$
$D(\underline{z}) = \sigma^2 D)$

$= \sigma^2$ if the null hyp is true, i.e. $\underline{H}'\underline{\beta} = \underline{\theta}_0$
$> \sigma^2$ if the null hyp is not true

(Since $(\underline{H}'\underline{\beta} - \underline{\theta}_0)' D^{-1} (\underline{H}'\underline{\beta} - \underline{\theta}_0)$ is positive definite)

Also,

$$E(s^2) = \frac{R_0^2}{n-r} = \sigma^2$$

Hence the test statistics becomes.

$$F_{cal} = \frac{(\underline{z} - \underline{\theta}_0)' D^{-1} (\underline{z} - \underline{\theta}_0)}{k} \div \frac{R_0^2}{(n-r)}$$

Reject H_0 if $F_{cal} > F_{tab}$ where $F_{tab} = F_{(k, n-r)}$

* Simultaneous Confidence Interval :-

It follows that

$$P\left\{ \frac{(\underline{z} - \underline{\theta})' D^{-1} (\underline{z} - \underline{\theta})}{k s^2} \le F_\alpha \right\} = 1 - \alpha \quad \text{—— (1)}$$

where $\underline{\theta} = \underline{H}'\underline{\beta}$ stands for 'k' indep. estimable parametric functions of $\underline{\beta}$.

Expression (1) is called $1-\alpha$ Confidence region of $\underline{\theta} = H'\underline{\beta}$.

Let it be represented ~~re-seperated~~ by C. Then simultaneous C.I. for funct^{ns} $g_i(\underline{\theta})$ $i = 1, 2, \ldots, k$ with Confidence Coefficients possible greater than $1-\alpha$ are given by

$$\left[\min_{\underline{\theta} \in C} g_i(\underline{\theta}) \,,\; \max_{\underline{\theta} \in C} g_i(\underline{\theta}) \right] \;;\quad i = 1, 2, \ldots, k$$

For any Particular $g(\underline{\theta})$

$$P\left[\min_{\underline{\theta} \in C} g(\underline{\theta}) \,,\; \max_{\underline{\theta} \in C} g(\underline{\theta}) \right] \geq 1-\alpha .$$

* Second Method :- (Using Cauchy - Schwartz inequality)

– If $\underline{u}$ and $\underline{A}$ are col^{n} vectors and B is a positive definite matrix then

$$\underline{A}' B^{-1} \underline{A} = \max_{\underline{u}} \frac{(\underline{u}'\underline{A})^2}{\underline{u}' B \underline{u}}$$

Let $\underline{A} = (\underline{z} - \underline{\theta})$, $B = D$

then we have

$$\frac{(\underline{z} - \underline{\theta})' D^{-1} (\underline{z} - \underline{\theta})}{k s^2} = \frac{1}{k s^2} \max_{\underline{u}} \frac{[\underline{u}'(\underline{z} - \underline{\theta})]^2}{\underline{u}' D \underline{u}} \qquad \text{—(1)}$$

then using (1) we have

LSD = Part of Post Hoc Test

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$$P\left\{ \max_{\underline{u}} \frac{|\underline{u}'(\underline{z}-\underline{\theta})|}{\sqrt{\underline{u}'D\underline{u}}} \le s\sqrt{kF_\alpha} \right\} = 1-\alpha.$$

$$\left(\because P\left\{ \sqrt{\frac{(\underline{z}-\underline{\theta})'D^{-1}(\underline{z}-\underline{\theta})}{ks^2}} \le \sqrt{F_\alpha} \right\} = 1-\alpha \right.$$

$$\left. P\left\{ \max_{\underline{u}} \frac{|\underline{u}'(\underline{z}-\underline{\theta})|}{\sqrt{k}\,s\sqrt{\underline{u}'D\underline{u}}} \le \sqrt{F_\alpha} \right\} \right)$$

$$\Rightarrow P\left\{ |\underline{u}'(\underline{z}-\underline{\theta})| \le s\sqrt{kF_\alpha\,\underline{u}'D\underline{u}} \text{ for all } \underline{u} \right\} = 1-\alpha$$

$$\Rightarrow P\left\{ -s\sqrt{kF_\alpha\,\underline{u}'D\underline{u}} \le \underline{u}'(\underline{z}-\underline{\theta}) \le s\sqrt{kF_\alpha\,\underline{u}'D\underline{u}} \right\} = 1-\alpha$$

$$\Rightarrow P\left\{ -s\sqrt{kF_\alpha\,\underline{u}'D\underline{u}} - \underline{u}'\underline{z} \le \underline{u}'\underline{\theta} \le s\sqrt{kF_\alpha\,\underline{u}'D\underline{u}} - \underline{u}'\underline{z} \right\} = 1-\alpha$$

$$\Rightarrow P\left\{ \underline{u}'\underline{z} + s\sqrt{kF_\alpha\,\underline{u}'D\underline{u}} \ge \underline{u}'\underline{\theta} \ge \underline{u}'\underline{z} - s\sqrt{kF_\alpha\,\underline{u}'D\underline{u}} \right\} = 1-\alpha$$

($100(1-\alpha)\%$ C.I. for $\boxed{\underline{u}'\underline{\theta} = \underline{u}'H'\underline{\beta}}$

* **Tukey's test :-**

- Suppose that following and analysis of variance in which we have rejected the null hyp of equal treatment means (For one-way ANOVA) we wish to test all Pairwise mean comparison

$H_0 : \mu_i = \mu_j$

$H_1 : \mu_i \neq \mu_j \quad \forall\ i \neq j$

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- Tukey propose a procedure for testing hyp. for which overall significance level is exactly α when sample sizes are equal and at the most α when the sample sizes are unequal.

- The procedure can also be used to construct C.Is on the differences in all pairs of means for this intervals the simultaneously Confidence level $100(1-\alpha)\%$ when the sample sizes are equal and atleast $100(1-\alpha)\%$ when the sample sizes are not equal.

- Tukey procedure makes use of the distn of students t and their range statistics

$$q = \frac{\bar{y}_{max} - \bar{y}_{min}}{\sqrt{MSSE/n}}$$

when $\bar{y}_{max} \leftarrow$ largest sample mean
$\bar{y}_{min} \leftarrow$ smallest sample mean out of a group of p samples.

$q_{\alpha}(p, f)$ is the upper α percentage points of p and f is the no. of d.f. associated with MSSE.

- For equal sample sizes, Tukey's test declares two mean significantly different if the absolute value of their sample difference exceeds T_{α}.

$$|\bar{y}_i - \bar{y}_j| > T_{\alpha}.$$

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$$T_\alpha = q_\alpha(p,f)\sqrt{\frac{MSSE}{n}}$$

100 (1 − α) % C.I. for all pairs of Means are given by

$$\bar{y}_{i\cdot} - \bar{y}_{j\cdot} - q_\alpha(p,f)\sqrt{\frac{MSSE}{n}} \le \mu_i - \mu_j$$
$$\le \bar{y}_{i\cdot} - \bar{y}_{j\cdot} + q_\alpha(p,f)\sqrt{\frac{MSSE}{n}} \quad ; i \ne j$$

when sample sizes are not equal then,

$$T_\alpha = \frac{q_\alpha(p,f)}{\sqrt{2}}\left(\sqrt{MESS\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}\right)$$

and 100 (1 − α) % C.I. is given by

$$\bar{y}_{i\cdot} - \bar{y}_{j\cdot} - \frac{q_\alpha(p,f)}{\sqrt{2}}\left(\sqrt{MESS\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}\right) \le \mu_i - \mu_j$$

$$\le \bar{y}_{i\cdot} - \bar{y}_{j\cdot} - \frac{q_\alpha(p,f)}{\sqrt{2}}\left(\sqrt{MESS\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}\right) ; i \ne j$$

Unequal Sample Size Version is sometimes called Tukey - Kramer Procedure.

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* Scheffe's test : (For any type of treatment contrast)

- In many situation, experimentor may not know in advance which contrast they wish to compare or they may be interested in more than $a(a-1)$ possible comparisons.

- In many exploratory experiment, the comparision of interest are discovered only after preliminary examination of the data.

- Scheffe's has proposed a method for comparing any and / or possible contrast between treatment means. In the scheffe's method, Type-I Error is atmost α for any of the possible comparision.

- Suppose a set of 'm' contrast in the treatment means

$$V_u = C_{1u}\mu_1 + C_{2u}\mu_2 + \cdots + C_{au}\mu_a \quad , u = 1, 2, \ldots, m$$

of interest have been determined.

- The corresponding contrast in the treatment average $\bar{y}_{i.}$ is

$$C_u = C_{1u}\bar{y}_{1.} + C_{2u}\bar{y}_{2.} + \cdots + C_{au}\bar{y}_{a.} \quad , u = 1, 2, \ldots, m$$

and the SE of the contrast is

$$S_{C_u} = \sqrt{MSSE \sum_{i=1}^{a} \left(\frac{(C_{iu})^2}{n_i}\right)}$$

where n_i is the no. of obs^n in i^{th} treatment

The critical value against which C_u can be compared is

$$S_{\alpha, u} = S_{C_u}\sqrt{(a-1)\, F_{(\alpha, a-1, N-a)}}$$

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If $|C_u| > S_{\alpha,u}$ the hyp that the contrast $V_u = 0$ is rejected

- The Scheffe's procedure can also be used to form C.I. for all possible contrast among treatment means, the resulting interval $C_u - S_{\alpha,u} \leq T_u \leq C_u + S_{\alpha,u}$ are simultaneous C.I.

- Although for pairwise treatment comparision Scheffe's method can be applied but it is not the most sensitive procedure for such comparisions.

Ex:

% weight of cotton	Observed tensile strength (lb/in^2) 1	2	3	4	5	Total $y_{i.}$	Avg $\bar{y}_{i.}$
15	7	7	15	11	9	49	9.8
20	12	17	12	18	18	77	15.4
25	14	18	18	19	19	88	17.6
30	19	25	22	19	23	108	21.6
35	7	10	11	15	11	54	10.8
						376	75.2

$$SS_{Total} = \Sigma\Sigma y_{ij}^2 - \frac{y_{..}^2}{N} = 6292 - \frac{141376}{25}$$

$$= 6292 - 5655.04$$

$$= \boxed{636.96}$$

$$SS_{Treatment} = \frac{1}{n}\Sigma y_{i.}^2 - \frac{y_{..}^2}{N} = 6130.8 - 5655.04$$

$$= \boxed{475.76}$$

Source Vari

Treatm

Erro

Total

$SSE = SS_{total} - SS_{treat} = \boxed{161.2}$

Hypothesis:- $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$

H_1 : Atleast one $\mu_i \neq 0$

ANOVA.

Source of Variation	d.f.	S.S.	MSS	F_{cal}	F_{tab}
Treatment	$a-1=4$	475.76	118.94	14.76	$F_{(4,20,0.05)}$
Error	20	161.2	8.06		$= 2.87$.
Total	$N-1=24$	636.96			

Conclusion:- $F_{cal} > F_{tab}$ $(14.76 > 2.87)$ so we reject our H_0 at 5% LOS

Hypothesis:- $H_0 : \mu_i = \mu_j$

$H_1 : \mu_i \neq \mu_j$

$$q = \frac{\bar{y}_{max} - \bar{y}_{min}}{\sqrt{\frac{MSSE}{n}}} = \frac{11.8}{1.270} = \boxed{9.291}$$

$$T_\alpha = q_\alpha(p,f)\sqrt{\frac{MSSE}{n}}$$

$$= q_{0.05}(5,20)\sqrt{\frac{8.06}{5}}$$

$$= 4.23 \times 1.270$$

$$= \boxed{5.3721}$$

(* → significantly different)

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$\bar{y}_{1.} - \bar{y}_{2.} = +5.6$ *	$\bar{y}_{2.} - \bar{y}_{4.} = 6.2$ *
$\bar{y}_{1.} - \bar{y}_{3.} = +7.8$ *	$\bar{y}_{2.} - \bar{y}_{5.} = 4.6$
$\bar{y}_{1.} - \bar{y}_{4.} = 11.8$ *	$\bar{y}_{3.} - \bar{y}_{4.} = 4$
$\bar{y}_{1.} - \bar{y}_{5.} = 1$	$\bar{y}_{3.} - \bar{y}_{5.} = 6.8$ *
$\bar{y}_{2.} - \bar{y}_{3.} = 2.2$	$\bar{y}_{4.} - \bar{y}_{5.} = 10.8$ *

NOTE :-

Test the Contrast :-

$$\gamma = \mu_1 + \mu_3 - \mu_4 - \mu_5$$

$$\Rightarrow \mu_1 + \mu_3 = \mu_4 + \mu_5$$

$H_0 : \gamma = 0$

$H_1 : \gamma \neq 0$

$$C_u = C_{1u}\bar{y}_{1.} + C_{2u}\bar{y}_{2.} + \cdots + C_{au}\bar{y}_{a.}$$

$$C_1 = \bar{y}_{1.} + \bar{y}_{3.} - \bar{y}_{4.} - \bar{y}_{5.} = -5$$

$$S_{C_1} = \sqrt{MSSE \sum_{i=1}^{a}\left(\frac{(C_{iu})^2}{n_0}\right)} = \sqrt{8.06\left(\frac{1+1+1+1}{5}\right)}$$

$$= \sqrt{6.448} = \boxed{2.54}$$

$$S_{\alpha;u} = S_{C_1}\sqrt{(a-1)\,F(\alpha, a-1, N-a)}$$

$$= 2.5\sqrt{4 \times 2.87} = 2.5 \times 3.39 = \boxed{8.47}$$

Test Criteria :- $|C_1| > 8.47$

Conclusion :- Since $|C_1| \not> S_{\alpha,u}$ $(5 \not> 8.47)$ we do not reject our H_0 at 5% LOS and hence $\gamma \neq 0$

*

NOTE :- If multiple test are performed on a given sample the value of α changes since sampling dist^n for t assumes only one t-test from any given sample.

- The true α level given multiple test or comparisions can be estimate as $1-(1-\alpha)^c$ where c = total no. of comparision, contrast or test ~~are~~ performed.

(eg. $\alpha = 0.05$, $c = 10$
$1-(1-0.05)^{10} = \boxed{0.4}$)

* Power of F-test :-

- O.C - Curve is a plot of Type-II error Prob. of a Statistical test for a particular sample size versus a parameter that reflects the extend to which the null hyp. is False.

- This curve can be used to guide the experimental in selecting the no. of replicates so that the design will be sensitive to imp. potential differences in the treatment.

- We consider the Prob. of Type-II error of the fixed effect model for the case of equal sample sizes per treatment.

$$\beta = 1 - P\{\text{Reject } H_0 / H_0 \text{ is false}\}$$

$$= 1 - P\{F_{cal} > F_{\alpha, a-1, N-a} / H_0 \text{ is false}\} \quad (*)$$

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To evaluate the prob, we need to know the distⁿ of the test statistics Fcal if null hyp is False.

- If H_0 is false, the statistic

$$F_{cal} = \frac{MSS_{treat}}{MSS_E} \sim \text{non-central } F$$

(zero)

with $(a-1, N-a)$ d.f. and non-centrality parameter δ, if $\delta = 0$ then non-central F becomes usual (central) F distⁿ.

- OC curve are used to evaluate the prob. statement in the (*).

- This curves plot the prob. of Type-II error (β) against a parameter Φ where

$$\Phi = \frac{n \sum_{i=1}^{a} \tau_i^2}{a\sigma^2}$$ ($\because a$ = no. of treatment)

The quantity Φ is related to non-centrality parameter δ.

Curves are available for $\alpha = 0.5$ and $\alpha = 0.01$

- One way to determine Φ is to choose the actual values of the treatment means for which we would like to reject the null hyp with high prob.

Thus if $\mu_1, \mu_2, \dots, \mu_a$ are the specified treatment means we find τ_i as

$$\tau_i = \mu_i - \bar{\mu} \quad \text{where} \quad \bar{\mu} = \frac{1}{a}\sum_{i=1}^{a} \mu_i$$

- We also required an estimate of σ^2. Sometimes this is available from prior experience or preliminary test or a judgement estimate and alternate approach is to select a sample size such that the difference betn any two treatments means if it exceeds a specified value, the null hyp. should be rejected.

- If difference betn any two treatment means is as large as D then

$$\boxed{\Phi = \frac{nD^2}{2a\sigma^2}}$$

— x — x —

if no. of obs^n in each cell are same → Balanced layout.

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UNIT - 3

* Random Effect Model :-

- If the effects in the linear model are all r.v's except for the additive constant if any which is a fixed quantity is called a Random Effect Model.

- The effects become random give to the sampling of the levels of the factors included. It is also called Variance - Component Model.

- Thus we have Variance - Component Model for one - way layout, two - way layout or multi - way layout. The complete p - way classification or layout is called Balanced if the no. of obs^{ns} in the different cells are equal.

- The one - way classification is balanced if no. of obs^{ns} under the categories are same.

* General Random Effect Model in the balanced case :-

Let an observable r.v. $y_{ijk \cdots m}$ for a balanced case be such that

$$y_{ijk \cdots m} = \mu + a_i + b_{ij} + c_{ijk} \cdots + e_{ijk \cdots m} \quad (1)$$

where μ is a constant, the r.v's $a_i, b_{ij}, c_{ijk} \cdots, e_{ijk \cdots m}$ are completely indep

Source
1
2
:
k

$a_1, a_2\ a_3 \sim N(0, \sigma_a^2)$

$b_1\ b_2\ b_3 \sim N(0, \sigma_b^2)$

$$\& \quad a_i \sim N(0, \sigma_a^2)$$
$$b_{ij} \sim N(0, \sigma_b^2)$$
$$c_{ijk} \sim N(0, \sigma_c^2)$$
$$\vdots$$
$$e_{ijk\cdots m} \sim N(0, \sigma_e^2)$$

$$E(y_{ijk\cdots m}) = \mu$$

$$D(y_{ijk\cdots m}) = Var(\mu + a_i + b_{ij} + \cdots + e_{ijk\cdots m})$$
$$= \sigma_a^2 + \sigma_b^2 + \cdots \sigma_e^2$$

- Diagonal elements of the dispersion matrix of Y's are all same. Then $\sigma_a^2, \sigma_b^2, \ldots, \sigma_e^2$ are the components of the variance of the obs^{ns} so they are called Variance - component and hence the model is called Variance - Component Model.

- The Dispersion Matrix is positive definite if it satisfies the condition of Model (1).

ANOVA (table 1)

Source	d.f.	S.S.	M.S.S.	E(MSS)
1	f_1	S_1	s_1^2	σ_1^2
2	f_2	S_2	s_2^2	σ_2^2
⋮	⋮	⋮	⋮	⋮
k	f_k	S_k	s_k^2	σ_k^2
	$\sum_{i=1}^{k} f_i$	$\sum_{i=1}^{k} S_i$		

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- Generally each σ_i^2 will be a linear funct^n, a Variance - component $\sigma_a^2, \sigma_b^2, \ldots, \sigma_e^2$

<u>Thm (1)</u> :- Let the model be (1) and be balanced and the appropriate ANOVA table be given as table (1) then

$\frac{S_1}{\sigma_1^2}, \frac{S_2}{\sigma_2^2}, \ldots, \frac{S_K}{\sigma_K^2}$ are mutually independently distributed as central χ^2 - with $f_1, f_2, \ldots, f_K$ d.f respectively. Since σ_i^2 are linear funct^{ns} of the variance components we can obtain unbiased estimators of the variance components by equating each E(MSS) to the corresponding MSS of the table (1).

- These estimators have the following properties

<u>Thm (2)</u> :- Let the model be (1) and be balanced then the estimators of the variance components obtained by equating each E(MSS) to the corresponding MSS of table (1) are minimum variance unbiased.

<u>Thm (3)</u> :- Let the model be (1) and be balanced with the corresponding ANOVA given in table (1) then the test of

$H_0 : \sigma_i^2 = \sigma_j^2 \quad (i \neq j)$

$H_1 : \sigma_i^2 > \sigma_j^2$ is given by

$$F = \frac{s_i^2}{s_j^2} \sim F_{central} \text{ with } (f_i, f_j) \text{ d.f.}$$

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* **Power :-**

- Power of the test H_0 depends on $H_1 : \frac{\sigma_i^2}{\sigma_j^2} = \lambda < 1$ and is given by

$$\beta(\lambda) = P\left[\frac{s_i^2}{s_j^2} > F(\alpha, f_i, f_j) \Big/ H_1\right]$$

$$= P\left[\frac{s_i^2}{s_j^2}\lambda > \lambda F(\alpha, f_i, f_j) \Big/ H_1\right]$$

$$= \int_{\lambda F(\alpha, f_i, f_j)}^{\infty} f(u)\, du$$

where f is the pdf of central F distⁿ with (f_i, f_j) df.

- Thus, in the variance component model, both the test and the power are given by Central F - distⁿ whereas in fixed Effect Model, power is given by Non-Central F-distⁿ.

* **One - Way classification : Random Effect Model**

- Consider the following balanced one-way classification random effect Model

$$y_{ij} = \mu + a_i + e_{ij}, \quad i = 1, 2, \ldots, p$$
$$j = 1, 2, \ldots, n$$

where a_i and e_{ij} are completely indep. & $a_i \sim N(0, \sigma_a^2)$ & $e_{ij} \sim N(0, \sigma_e^2)$
μ is additive constant.

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Here, a_i is random effect due to i^{th} classification and,

e_{ij} are errors.

To test,

$H_0 : \sigma_a^2 = 0$ against $H_1 : \sigma_a^2 > 0$

which means testing the homogeneity of the effects of the classification.

we have,

$$\sum_i \sum_j (y_{ij} - y_{oo})^2 = \sum_i \sum_j (y_{ij} - y_{io} + y_{io} - y_{oo})^2$$

$$= \sum_i \sum_j (y_{ij} - y_{io})^2 + \sum_i \sum_j (y_{io} - y_{oo})^2$$

$$= n \sum_i (y_{io} - y_{oo})^2 + \sum_i \sum_j (y_{ij} - y_{io})^2$$

$$= SS(\text{bet}^n \text{ classes}) + SS(\text{within classes})$$

$$= SSB + SSE$$

let $y_{io} - y_{oo} = \mu + a_i + e_{io} - \mu - a_o - e_{oo}$

$$= a_i + e_{io} - a_o - e_{oo}$$

where $a_o = \frac{\sum a_i}{p}$, $e_{io} = \sum_j \frac{e_{ij}}{n}$, $e_{oo} = \sum_i \frac{e_{io}}{p}$

Thus,

$$SSB = n \sum_i (a_i + e_{io} - a_o - e_{oo})^2$$

where

$$a_i + e_{io} \sim N(0, \sigma_a^2 + \sigma_e^2/n)$$

$$\left(\because Var(a_i + e_{io}) = Var(a_i) + Var(e_{io}) = Var(a_i) + Var\left(\frac{\sum e_{ij}}{n}\right) \right.$$

$$= Var(a_i) + \frac{1}{n^2} \sum Var(e_{ij})$$

$$\left. = \sigma_a^2 + \frac{n \sigma_e^2}{n^2} = \sigma_a^2 + \frac{1}{n} \sigma_e^2 \right)$$

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So

$$\frac{n\sum_{i=1}^{p}(a_i + e_{io} - a_o - e_{oo})^2}{(n\sigma_a^2 + \sigma_e^2)} \sim \chi^2_{(p-1)\,d.f.}$$

ie, SSB is $(\sigma_e^2 + n\sigma_a^2) \cdot \chi^2_{(p-1)}$

$$E[MSSB] = E\left[\frac{SSB}{p-1}\right] = \sigma_e^2 + n\sigma_a^2$$

Similarly,

$$SSE = \sum_i \sum_j (y_{ij} - y_{io})^2$$

$$= \sum_i \sum_j (\mu + a_i + e_{ij} - \mu - a_i - e_{io})^2$$

$$= \sum_i \sum_j (e_{ij} - e_{io})^2$$

SSE and e_{ij} are independently distributed and $e_{ij} \sim N(0, \sigma_e^2)$

So for different i

$\frac{\sum_{j=1}^{n}(e_{ij} - e_{io})^2}{\sigma_e^2}$ are independently distributed as $\chi^2_{(n-1)}$

Hence,

$$\frac{SSE}{\sigma_e^2} = \frac{\sum_i \sum_j (e_{ij} - e_{io})^2}{\sigma_e^2} \sim \chi^2_{p(n-1)}$$

* To show SSB and SSE are indep :-

We first show that

$u_i = a_i + e_{io} - a_o - e_{oo}$ & $v_{ij}' = e_{ij}' - e_{io}'$

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have zero Covariance.

$$Cov(u_i, v'_{ij}) = E[u_i, v'_{ij}] - E[u_i]E[v'_{ij}]$$

$$E[u_i] = a_i - a_0$$
$$E[v'_{ij}] = 0$$

$$\Rightarrow Cov(u_i, v'_{ij}) = E[u_i, v'_{ij}]$$

$$E[u_i, v'_{ij}] = E\left[\{(a_i - a_0) + (e_{i0} - e_{00})\} \times \{(e_{ij} - e_{i0})\}\right]$$

$$= E\left[\{(a_i - a_0)(e_{ij} - e_{i0})\} + \{(e_{i0} - e_{00})(e_{ij} - e_{i0})\}\right]$$

(Since a_i and e_{ij} are indep.
$$E\left[\{(a_i - a_0)(e_{ij} - e_{i'0})\}\right] = 0 \;)$$

$$= E\left[(e_{i0} - e_{00})(e_{ij} - e_{i'0})\right]$$

$$= E\left[(e_{i0}\, e_{ij}) - e_{i0}\, e_{i'0} - e_{00}\, e_{i'j} + e_{00}\, e_{i'0}\right]$$

$e_{ij} \sim N(0, \sigma_e^2)$
$V(e_{ij}) = E[e_{ij}^2] - [E[e_{ij}]]^2$
$\Rightarrow V(e_{ij}) = \sigma_e^2 = E[e_{ij}^2]$

$$= \delta_{ii'} \frac{\sigma_e^2}{n} - \delta_{ii'} \frac{\sigma_e^2}{n} - \frac{\sigma_e^2}{np} + \frac{\sigma_e^2}{np} = \boxed{0}$$

(where
$\delta_{ii'} = 1$ if $i = i'$
$\delta_{ii'} = 0$ if $i \neq i'$)

$\Rightarrow$ u_i and v'_{ij} are independent
Thus SSB and SSE being funct^{ns} of indep. quantities are also indep.
Hence

$$\frac{SSE}{\sigma_e^2} \sim \chi^2_{p(n-1)} \;\&$$

$\dfrac{SSB}{\sigma_e^2 + n\sigma_a^2} \sim \chi^2_{(p-1)}$ and are independently distributed.

ANOVA

Source	d.f.	S.S.	M.S.S.	E(MSS)
Bet^n classes	$p-1$	SSB	MSSB	$\sigma_e^2 + n\sigma_a^2$
Within classes	$p(n-1)$	SSE	MSSE	σ_e^2
Total	$np-1$	Total SS		

The test $H_0 : \sigma_a^2 = 0$ is given by

$$F_{cal} = \frac{MSSB}{MSSE}$$

we reject H_0 if $F_{cal} > F_{tab}$ at level α.

The power of the test against $H_1 : \lambda = \dfrac{\sigma_e^2}{\sigma_e^2 + n\sigma_a^2} < 1$ is given by

$$\beta(\lambda) = \int_{\lambda \cdot F(\alpha, (p-1), p(n-1))} f(u)\, du$$

where f is pdf of central F-dist^n.