1 Equations

1.1 Compressible

$$\rho_{,t} + \sum_{i=1}^{3} \left[\rho u_i\right]_{,i} = 0 \tag{1}$$

$$\{pu_j\}_{,t} + \sum_{i=1}^{3} \{\rho u_i u_j\}_{,i} + P_{,j} = \sum_{i=1}^{3} \{\tau_{ij}\}_{,i} + b_j \tag{2}$$

$$\{\rho e_{tot}\}_{,t} + \sum_{i=1}^{3} \{\rho u_i e_{tot}\}_{,i} \tag{3}$$

$$e_{tot} = e + \frac{u_i u_i}{2} \tag{4}$$

$e = c_v T$
$\tau_{ij} = \mu S_{ij}$
$S_{ij} = (U_{i,j} + u_{j,i} + \frac{\lambda}{\mu} \sum_{k=1}^{3} u_{k,k} \delta_{ij}$
$q_i = -\chi T_{,i}$
$P = \rho RT$

$$\vec{u} = \begin{Bmatrix} \rho \\ \rho \vec{u_i} \\ \rho e_{tot} \end{Bmatrix} \tag{5}$$

$$\vec{F_i} = \begin{Bmatrix} \rho u_i \\ \rho u_i u_j \\ \rho u_i e_{tot} \end{Bmatrix} + \begin{Bmatrix} 0 \\ P\delta_{ij} \\ Pu_i \end{Bmatrix} - \begin{Bmatrix} 0 \\ \tau_{ij} \\ u_j \tau_{ij} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0_j \\ q_i \end{Bmatrix} \tag{6}$$

$$\vec{F_i^{adv}} = \begin{Bmatrix} \rho u_i \\ \rho u_i u_j \\ \rho u_i e_{tot} \end{Bmatrix} + \begin{Bmatrix} 0 \\ P\delta_{ij} \\ Pu_i \end{Bmatrix} \tag{7}$$

$$\vec{F_i^{diff}} = \; - \begin{Bmatrix} 0 \\ \tau_{ij} \\ u_j \tau_{ij} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0_j \\ q_i \end{Bmatrix} \tag{8}$$

$$\vec{\mathscr{F}} = \vec{u_{,t}} + \vec{F_{i,i}} = \begin{Bmatrix} 0 \\ b_j \\ b_j u_k \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0_j \\ r \end{Bmatrix} \tag{9}$$

$$\vec{\mathscr{F}} = \frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{F}_1}{\partial x_1} + \frac{\partial \vec{F}_2}{\partial x_2} + \frac{\partial \vec{F}_3}{\partial x_3} \tag{10}$$

1.2 Work Flow

The first thing we have is $\vec{u}_{,t} + \vec{F}_{i,i} - \vec{\mathscr{F}} = 0$ which is the **residual form** of the PDE
Then we introduce a weight vector and contract it with the residual which gives $\vec{W} \cdot (\vec{u}_{,t} + \vec{F}_{i,i} - \vec{\mathscr{F}}) = 0$
We then integrate this over the domain.

$$\int_{\Omega} \vec{W} \cdot (\vec{u}_{,t} + \vec{F}_{i,i} - \vec{\mathscr{F}}) d\Omega = 0 \tag{*}$$

We now look for solutions to (*) such that any $\vec{W}$ is true. We also will use Integration by Parts (IBP) to move derivative from $\vec{F}_i$ to $\vec{W}$. Disributing the weight vector and carrying out IBP yields

$\int_{\Omega}(\vec{W} \cdot \vec{u}_{,t} + \vec{W}_i \cdot \vec{F}_i - \vec{W} \cdot \vec{\mathscr{F}}) d\Omega = 0$
This is known as the Weak Galerkin Form
Next we will assume solutions of the form $\vec{u}(x) = \sum_{A=1}^{n_{np}} N_A(x)\vec{u}_A$ for this we use $n_{np} \rightarrow$ number of nodes points, $\vec{u}_A \rightarrow \vec{u}$ at node A, and $N_a \rightarrow$ shape functions
We mimic this form for the weight vectors $\vec{W}(x) = \sum_{A=1}^{n_{np}} N_B(x)\vec{W}_B$

1.3 New Part

$$\int_{\Omega} \vec{W} \cdot (\vec{u}_{,t} + \vec{F}_{i,i} - \vec{\mathscr{F}}) d\Omega = 0 \tag{1}$$

$$\vec{u}(x) = \sum_{A=1}^{n_{np}} N_A(x)\vec{u}_A \tag{2}$$

$$\vec{W}(x) = \sum_{A=1}^{n_{np}} N_B(x)\vec{W}_B \tag{3}$$

Substituting (2) and (3) into (1) gives

$$\int_{\Omega} \left\{ \sum_{B=1}^{n_{np}} N_B \vec{W}_B \cdot \sum_{A=1}^{n_{np}} N_A \vec{u}_{A,t} - \sum_{B=1}^{n_{np}} N_{B,i} \vec{W}_B \vec{F}_i - \sum_{B=1}^{n_{np}} N_B \vec{W}_B \vec{\mathscr{F}} \right\} d\Omega + \int_{\Gamma} N_B W_B \vec{F}_i n_i d\Gamma = 0 \tag{4}$$

This is the Weak form with explicit piecewise polynomial substitution for $\vec{u}$ and $\vec{W}$.If we factor out the sum over weights, we have

$$\sum_{B=1}^{n_{np}} \vec{W}_B \left\{ \int_{\Omega} \left\{ N_B \cdot \left[\sum_{A=1}^{n_{np}} \vec{u}_{A,t} - \vec{\mathscr{F}} \right] - N_{B,i} \vec{F}_i \right\} d\Omega + \int_{\Gamma} N_B \vec{F}_i n_i d\Gamma \right\} = 0 \tag{5}$$

This reduces to $\sum_{B=1}^{n_{np}} \vec{W}_B \cdot \vec{G}_B$ where $\vec{G}_b = 0 \, \forall B$. This gives $5n_{np}$ equations for $5n_{np}$ unknowns, coming from the fact that $\vec{u}_A$ and $\vec{u}_{A,t}$ represent $5n_{np}$ ODEs

Note: $\vec{u}$ is the solution vector
$\vec{W}(x)$ is the weight function

1.4 Computational localization

Computationally we must localize the problem to be able to calculate anything. This is mathematically represented as

$$\begin{gathered}
\sum_{B=1}^{n_{np}} \vec{W}_B \left\{ \int_{\Omega} \left\{ N_B \cdot \left[\sum_{A=1}^{n_{np}} \vec{u}_{A,t} - \vec{\mathscr{F}} \right] - N_{B,i} \vec{F}_i \right\} d\Omega + \int_{\Gamma} N_B \vec{F}_i n_i d\Gamma \right\} = 0 \\
\sum_{e=1}^{n_{el}} \sum_{b=1}^{n_{np}} \vec{W}_b \left\{ \int_{\Omega^e} \left\{ N_b \cdot \left[\sum_{a=1}^{n_{np}} \vec{u}_{a,t} - \vec{\mathscr{F}} \right] - N_{b,i} \vec{F}_i \right\} d\Omega^e + \int_{\Gamma^e} N_b \vec{F}_i n_i d\Gamma^e \right\} = 0
\end{gathered} \tag{1}$$

From this we can see that

$$\vec{G}^e_B = \int_{\Omega^e} \left\{ N_b \cdot \left[\sum \vec{u}_{a,t} - \vec{\mathscr{F}} \right] - N_{b,i}\vec{F}_i \right\} d\Omega^e + \int_{\Gamma^e} N_b \vec{F}_i n_i d\Gamma^e \Bigg\} = 0 \tag{2}$$

Note: that for each local node b of the e^{th} element we have a $\vec{G}_B$ of length 5
$N_b(\xi)$ always exists in a mapped (parent) domain, i.e. $\xi \ \epsilon\ [0,1]$
$\vec{G}^e_b$ represents the residual of the e^{th} element.

1.5 Stabilization

Now we need to add a stability term, as Galerikin's method is stable for diffusion but is unstable for advention. New Galerkin form = Old Galerkin form + $\sum_{e=1}^{n_{el}} \int_{\Omega^e} \hat{\mathscr{L}}^T \vec{W} \cdot \overset{\leftrightarrow}{\tau} \{\mathscr{L}\vec{u} - \mathscr{F}\} d\Omega^e$. Here $\mathscr{L}\vec{u} - \mathscr{F}$ is the full unsteady compressible Navier-Stokes in strong form. We call this a Petrov-Galerkin Method (which is still a weighted residual method).
$\hat{\mathscr{L}}$ and $\mathscr{L}$ are differential operators.
$\mathscr{L}\vec{u} \equiv \vec{u}_{,t} + \vec{F}_{i,i} \ \therefore \mathscr{L}\vec{u} - \mathscr{F} = 0 \implies$ Which is the PDE residual
$\vec{F}^{adv}_{i,i} = \frac{\partial \vec{F}_i}{\partial \vec{u}} \frac{\partial \vec{u}}{\partial \vec{x}_i} = \overset{\leftrightarrow}{A}_i \vec{u}_i$ we do this to make $\mathscr{L}$ appear linear. **Note:** $\overset{\leftrightarrow}{A}_i(\vec{u})$ is a 5x5 matrix for each i
$\vec{F}^{diff}_{i,i} \equiv -\left[K_{ij}\vec{u}_{,j}\right]_{,i} \quad \therefore\ \vec{F}^{diff}_i \equiv -K_{ij}\vec{u}_{,j}$ **Note:** K_{ij} represents 9 5x5 matrices

In quasi-linear form:

$$\mathscr{L}\vec{u} \equiv \vec{u}_{,t} + \overset{\leftrightarrow}{A}_i \vec{u}_i - \left[K_{ij}\vec{u}_{,j}\right]_i = \left\{ \frac{\partial}{\partial t} + \overset{\leftrightarrow}{A}_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}\left[K_{ij}\frac{\partial}{\partial x_j}\right]\right\} \cdot \vec{u} \tag{1}$$

$$\mathscr{L} = \frac{\partial}{\partial t} + \overset{\leftrightarrow}{A}_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}\left[K_{ij}\frac{\partial}{\partial x_j}\right] \tag{2}$$

$\hat{\mathscr{L}}$ can be defined in different ways to various methods. We wiill study SUPG

- Galerkin Least Squares (GLS) $\quad \hat{\mathscr{L}} = \mathscr{L}$
- Streamwise Upwind Petrov-Galerkin (SUPG) $\quad \hat{\mathscr{L}} = \overset{\leftrightarrow}{A}_i \frac{\partial}{\partial x_i}$
- Douglas-Wang $\quad \hat{\mathscr{L}} = -\mathscr{L}^* = \frac{\partial}{\partial t} + \overset{\leftrightarrow}{A}_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}\left[K_{ij}\frac{\partial}{\partial x_j}\right]$

3. Is the element level residual with SUPG terms, 4. and 5. are expansions contained within 3

$$\vec{G}^e_B = \int_{\Omega^e} N_b \left[\sum \vec{u}_{a,t} - \vec{\mathscr{F}}\right] - N_{b,i}\vec{F}_i d\Omega^e + \int_{\Gamma^e} N_b \vec{F}_i n_i d\Gamma^e + \int_{\square} \hat{\mathscr{L}} N_b(\xi) \overset{\leftrightarrow}{\tau} \{\mathscr{L}\vec{u} - \mathscr{F}\} \mathscr{D}(\xi) d\square = 0 \tag{3}$$

$$\mathscr{L} N_b = \overset{\leftrightarrow}{A}_i \Big(\sum_{a=1}^{n_{en}} N_a \vec{u}_a \Big) N_{b,\xi_l} \xi_{l,i} = \overset{\leftrightarrow}{A}_i(\vec{u}) N_{b,i} \tag{4}$$

$$\mathscr{L}\vec{u} = \sum_{a=1}^{n_{en}} N_a \vec{u}^e_{a,i} + \overset{\leftrightarrow}{A}_i \Big(\sum_{a=1}^{n_{en}} N_a \vec{u}_a \Big) \sum_{c=1}^{n_{en}} \underbrace{N_{c,\xi_l} \xi_{l,i} \cdot \vec{u}^e_c}_{\vec{u}_i} \tag{5}$$

$\hat{\vec{G}}_b = \underset{e=1}{\overset{n_{el}}{\mathbb{A}}} \hat{\vec{G}}^e_b$

2 Variable Mapping

Math Notation	Code Var	Summary
n_{en}	nshg	shape function gradient
	ndof	Degrees of Freedom at a given node
$\vec{Y}_B$	Y(nshg,ndof)	Solution variable vector
$\vec{Y}_{A,t}$	ac	Time derivative of the solution vector
$\vec{Y}_{a,t}$	acl	Time derivative of solution vector locally
ν	rmu	viscosity
$\vec{G}_B$	res	Global Residual
$\vec{G}^e_b$	rl	Element level residual
$u_i u_i$	rk	advective velocity
N_a	shp	shape function
$N_{a,\xi}$	shgl	local gradient of shape function
e	npro	Number of elements in a computational block
$\underline{\underline{A}}_0 Y_{,t} - \not{\ell}$	ri	
n_{en}	nshl	number of local shape functions