Module - 2

Joint Probability and Markov Chain

Expectation: Let X be the discrete random variable taking the values $x_1, x_2, x_3, \dots x_n$ having probability function $f(x)$ then the expectation of X denoted by $E(X)$ (or) μ_X is

$$E(X) = \sum x f(x)$$

$E(Y)$ (or) μ_Y is

$$E(Y) = \sum y g(y)$$

$$E(XY) = \sum xy J_{i,j}$$

Variance (V): $V(X)$ (or) $\sigma^2_X = E(X^2) - [E(X)]^2$

but $E(X^2) = \sum x^2 f(x)$

$V(Y)$ (or) $\sigma^2_Y = E(Y^2) - [E(Y)]^2$

but $E(Y^2) = \sum y^2 g(y)$

Standard deviation (σ): $\sigma_X = \sqrt{V(X)}$ called standard deviation of X.

$\sigma_Y = \sqrt{V(Y)}$ SD of Y

Covariance (cov): $\mathrm{cov}(X,Y) = E(XY) - E(X).E(Y)$

Correlation ($\rho(X,Y)$): $\rho(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\sigma_X . \sigma_Y}$

NOTE: If X and Y are independent random variables, then covariance of $(X,Y) = 0$

$\rho(X,Y) = 0$

$E(XY) = E(X).E(Y)$

1. The joint distribution of two random variables X and Y is as follows

x \ y	-4	2	7
1	1/3	1/4	1/8
5	1/4	1/8	1/8

find i) $E(x)$ and $E(y)$
ii) $E(xy)$
iii) σ_x and σ_y
iv). $cov(x, y)$
v). $\rho(x, y)$.

i). ~~$E(x) = \Sigma x f(x)$~~ Distribution of x

Distribution of y

x	1	5
$f(x)$	$1/2$	$1/2$

y	-4	2	7
$g(y)$	$3/8$	$3/8$	$1/4$

i). $E(x) = \Sigma x f(x)$

$$= 1 \times \frac{1}{2} + 5 \times \frac{1}{2}$$

$$\boxed{E(x) = 3}$$

$E(y) = \Sigma y g(y)$

$$= -4 \times \frac{3}{8} + 2 \times \frac{3}{8} + 7 \times \frac{1}{4}$$

$$\boxed{E(y) = 1}$$

ii). $E(xy) = \Sigma xy J_{ij}$

$$= \left(1 \times -4 \times \frac{1}{3}\right) + \left(1 \times 2 \times \frac{1}{4}\right) + \left(1 \times 7 \times \frac{1}{8}\right) + \left(5 \times -4 \times \frac{1}{4}\right) +$$

$$\left(5 \times 2 \times \frac{1}{8}\right) + \left(5 \times 7 \times \frac{1}{8}\right)$$

$$\boxed{E(xy) = \frac{3}{2}}$$

iii) $cov(xy) = E(xy) - E(x).E(y)$

$$= \frac{3}{2} - 3(1)$$

$$= \frac{-3}{2}$$

iv) $V(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \Sigma x^2 f(x)$

$= 1 \times \frac{1}{2} + 25 \times \frac{1}{2}$

$E(X^2) = 13$

$V(X) = 13 - 3^2 = E(X^2) - [E(X)]^2$

$V(X) = 4$

$\sigma_X = \sqrt{V(X)}$

$= \sqrt{4}$

$\boxed{\sigma_X = 2}$

$V(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \Sigma y^2 f(y)$

$= 16 \times \frac{3}{8} + 4 \times \frac{3}{8} + 49 \times \frac{1}{4}$

$E(Y^2) = \frac{79}{4}$

$V(Y) = E(Y^2) - [E(Y)]^2$

$= \frac{79}{4} - 1 = \frac{75}{4}$

$\sigma_Y = \sqrt{V_Y}$

$= \sqrt{\frac{75}{4}} = \frac{5\sqrt{3}}{2}$

$= 4.3301$

$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$

$= \frac{-3/2}{2 \times 4.3301}$

$= -0.1732$

2.

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

find i). $E(X)$ and $E(Y)$ ii). $E(XY)$

iii) σ_X and σ_Y

iv) $\text{cov}(X,Y)$

v) $\rho(X,Y)$

Distribution of x

x	1	2
$f(x)$	0.6	0.4

Distribution of y

y	-2	-1	4	5
$g(y)$	0.3	0.3	0.1	0.3

i). $E(X) = \sum x f(x)$

$= 1 \times 0.6 + 2 \times 0.4$

$\boxed{E(X) = 1.4}$

$E(Y) = \sum y g(y)$

$= -2 \times 0.3 + (-1 \times 0.3) + 4(0.1) + 5(0.3)$

$\boxed{E(Y) = 1}$

ii). $E(XY) = \sum xy J_{ij}$

$= (1 \times -2 \times 0.1) + (1 \times -1 \times 0.2) + (1 \times 4 \times 0) + (1 \times 5 \times 0.3) + (2 \times -2 \times 0.2) + (-1 \times 2 \times 0.1) + (2 \times 4 \times 0.1) + (2 \times 5 \times 0)$

$\boxed{E(XY) = 0.9}$

iii). ~~cov~~ σ_x and σ_y

$V(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 f(x)$

$= 1 \times 0.6 + 4 \times 0.4$

$E(X^2) = 2.2$

$V(X) = 2.2 - 1.4^2$

$\boxed{V(X) = 0.24}$

$\sigma_x = \sqrt{V(X)}$

$= \sqrt{0.24}$

$\boxed{\sigma_x = 0.49}$

$V(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \sum y^2 g(y)$

$= 4 \times 0.3 + 1 \times 0.3 + 16 \times 0.1 + 25 \times 0.3$

$\cancel{V(Y) = 10.6}$

$E(Y^2) = 10.6$

$V(Y) = 10.6 - 1^2$

$\boxed{V(Y) = 9.6}$

$\sigma_y = \sqrt{9.6}$

$\boxed{\sigma_y = 3.09}$

iv $COV(X,Y) = E(XY) - E(X).E(Y)$

$= 0.9 - (1.4)(1)$

$\boxed{COV(X,Y) = -0.5}$

$$\rho(x,y) = \frac{cov(xy)}{\sigma_x \sigma_y}$$

$$= \frac{-0.5}{0.49 \times \cancel{0.}63.09}$$

$$\boxed{\rho(x,y) = -0.3}$$

→ Stochastic process

Definition 1: Probability vector: A vector 'V' $(V = v_1, v_2, v_3, \ldots v_n)$ is called a probability vector if each and every component of 'v' must be non-negative and their sum is unity.

Eg: $v = (0,1) = 0+1 = 1$

$v_1 = (1/2, 1/2) = \frac{1}{2} + \frac{1}{2} = 1$

$v_2 = (1/3, 1/3, 1/3) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

$v_3 = (0,1,0,0) = 1$

$w = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = \frac{2}{4} + \frac{1}{2} = 1$

→ Stochastic matrix: A square matrix A is said to be stochastic if every row is in the form of probability of probability vector.

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} 1+0=1 \\ 0+1=1 \end{matrix}$ $B = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} = \begin{matrix} 1/2+1/2=1 \\ 0+1=1 \end{matrix}$

$C = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix} = \begin{matrix} 1/2+1/2+0=1 \\ 1/2+1/2=1 \\ 1+0+0=1 \end{matrix}$

→ Regular stochastic matrix: A stochastic matrix A is said to be a regular stochastic matrix if all the entries of some A^n are positive.

Eg. $A = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1/4 & 3/4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 1/8 + 3/4 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/8 & 7/8 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1/8 & 7/8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/16 & 1/16 + 7/8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/16 & \frac{15}{16} \\ 0 & 1 \end{bmatrix}$$

until non-zero elements

Eg: $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

Thus, A is regular stochastic matrix of order $n=2$

NOTE: Properties of regular stochastic matrix

1. Let A has unique fixed point such that $Ax = x$
2. A has a unique fixed probability vector v such that $vA = v$.
3. $A^2, A^3, A^4, \ldots A$ approaches the matrix V whose rows are each the fixed probability vector v.

4. If u is any probability vector, then the sequence $uA, uA^2, \dots$ approaches the unique probability vector v

Problems:

1. Find the unique fixed probability of the regular stochastic matrix $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

$\Rightarrow$ Let $v = (x, y)$ where $x + y = 1$ such that $vA = v$

$$(x, y)\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ 1/2 & 1/2 \end{bmatrix} = (x, y)$$

$$\left[\frac{3x}{4} + \frac{y}{2}, \frac{x}{4} + \frac{y}{2}\right] = (x, y)$$

$$\frac{3x}{4} + \frac{y}{2} = x \qquad \frac{x}{4} + \frac{y}{2} = y.$$

WKT $x + y = 1$

$y = 1 - x$

$$\frac{3x}{4} + \frac{(1-x)}{2} = x$$

$$\frac{3x + 2(1-x)}{4} = x$$

$3x + 2 - 2x = 4x$

$3x - 6x = -2$

$-3x = -2$

$$\boxed{x = \frac{2}{3}}$$

$y = 1 - x$

$y = 1 - \frac{2}{3}$

$$\boxed{y = \frac{1}{3}}$$

$v(x, y) = \left(\frac{2}{3}, \frac{1}{3}\right)$

is the unique fixed probability vector

2. Find the unique fixed probability of the regular stochastic matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

$v = (x, y, z)$ where $x+y+z = 1$

$vA = v$

$$[x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$\left[0 + \frac{y}{6} + 0, \; x + \frac{y}{2} + \frac{2z}{3}, \; 0 + \frac{y}{3} + \frac{z}{3}\right] = [x, y, z]$$

$$\Rightarrow \left[\frac{y}{6}, \; x + \frac{y}{2} + \frac{2z}{3}, \; \frac{y}{3} + \frac{z}{3}\right] = \left[\frac{x}{6}, y, z\right]$$

$\frac{y}{6} = x$, $x + \frac{y}{2} + \frac{2z}{3} = y$, $\frac{y}{3} + \frac{z}{3} = z$

$y = 6x$ $\qquad \frac{6x + 3y + 4z}{6} = y$ $\qquad y + z = 3z$

WKT $x + y + z = 1$

$6x + \cancel{3y}$ $\qquad y + z = 3z$ $\qquad \cancel{x+y}$

$\Rightarrow y + \frac{y}{3} + \frac{z}{3} = 3z$ $\qquad \frac{6x + 3y + 4z}{6} = y.$

$\Rightarrow \frac{3y + y + 3}{3} = 3z$ $\qquad 6x + 3(6x) + 4z = 6y.$

$6x + 18x + 4z = 6y.$

$= 3y + y + 3 = 3z$ $\qquad 24x + 4z = 6y.$

$= 4y + 3 = 3z$ $\qquad 24x + 4z = 6(6x)$

$= \frac{4y + 3}{3} = z$ $\qquad 24x + 4z = 36x$

$4z = 36x - 24x$

$= \frac{4y + 3}{3} = 3x$ $\qquad 4z = 12x$

$z = \frac{12x}{4}$ $\qquad \boxed{z = 3x}$

$\Rightarrow \cancel{4y + 3 = 9x}$

$y = 6x \qquad \dfrac{6x + 3y + 4z}{6} = y \qquad \dfrac{y+z}{3} = z$

$y = 6x \qquad 6x + 3y + 4z = 6y \qquad y + z = 3z$

$\boxed{y = 6x} \qquad 6x - 3y + 4z = 0 \qquad y - 2z = 0$

$x + y + z = 1$

$z = 1 - x - y$

$z = 1 - x - 6x$

$\boxed{z = 1 - 7x}$

$6x - 3y + 4z = 0$

$6x - 3(6x) + 4(1 - 7x) = 0$

$6x - 18x + 4 - 28x = 0$

$6x - 40x + 4 = 0$

$40x = 4$

$\boxed{x = \dfrac{1}{10}}$

$\Rightarrow \boxed{y = \dfrac{6}{10}} \qquad z = 1 - 7\left(\dfrac{1}{10}\right)$

$z = 1 - \dfrac{7}{10}$

$\boxed{z = \dfrac{3}{10}}$

3. $P = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$

$\Rightarrow v(a, b, c, d) =$ where $a + b + c + d = 1$

$vp = v.$

$$(a,b,c,d)\begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a,b,c,d]$$

$$\Rightarrow \left[\frac{b}{2}+\frac{c}{2}+\frac{d}{2}, \frac{a}{2}+\frac{c}{2}+\frac{d}{2}, \frac{a}{4}+\frac{b}{4}, \frac{a}{4}+\frac{b}{4}\right] = [a,b,c,d]$$

$$\frac{b}{2}+\frac{c}{2}+\frac{d}{2} = a \qquad \frac{a}{2}+\frac{c}{2}+\frac{d}{2} = b \qquad \frac{a}{4}+\frac{b}{4} = c \qquad \frac{a}{4}+\frac{b}{4} = d$$

$$\Rightarrow b+c+d = 2a \qquad a+c+d = 2b \qquad a+b = 4c \qquad a+b = 4d$$

$$a+b+c+d = 1$$

$$b+c+d = 2a \qquad a+c+d = 2b \qquad a+b = 4c \qquad a+b = 4d$$

$$b+c+d = 1-a \qquad a+c+d = 1-b \qquad \frac{1}{3}+\frac{1}{3} = 4c \qquad \frac{2}{3} = 4d$$

$$1-a = 2a \qquad 1-b = 2b \qquad \frac{2}{3} = 4c \qquad 2 = 12d$$

$$1 = 3a \qquad 1 = 3b \qquad 2 = 12c \qquad \boxed{d = \frac{1}{6}}$$

$$\boxed{\frac{1}{3} = a} \qquad \boxed{b = \frac{1}{3}} \qquad \boxed{\frac{1}{6} = c}$$

$$v(a,b,c,d) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

4. Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix also find unique fixed probability vector

$$[x, y, z]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x, y, z]$$

$$\Rightarrow \left[\frac{z}{2}, x + \frac{z}{2}, y\right] = [x, y, z]$$

$\frac{z}{2} = x$, $x + \frac{z}{2} = y$, $y = z$

$x + y + z = 1$

$z = 2x$; $x + \frac{z}{2} = y$; $y = z$

$y = z$

$z = 2x$; $x + \frac{2x}{2} = y$

$x + x = y$

$2x = y$

$x = \frac{y}{2}$

$x + y + z = 1$

$\Rightarrow \frac{y}{2} + y + y = 1$ $\qquad$ $y = z$

$\boxed{z = \frac{2}{5}}$

$\frac{y}{2} + 2y = 1$

$y + 4y = 2$ $\qquad$ $z = 2x$

$5y = 2$ $\qquad$ $\frac{2}{5} = 2x$

$\boxed{y = \frac{2}{5}}$ $\qquad$ $\boxed{x = \frac{1}{5}}$

$$v(x, y, z) = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$$

regular stochastic matrix $P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0+0+0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P^2.P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = P^3.P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P^4.P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

Thus we observed P^5 all the enteries are positive.
Hence P is regular stochastic matrix.

5.* Show that the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$ is a regular stochastic matrix?

⑥* Find the unique FPV $P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

→ Markov's chain

* Transition probability matrix [tpm] : Denoted by P

$$= [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ P_{31} & P_{32} & \dots & P_{3n} \end{bmatrix}$$

* $p^{(0)} = [P_1^{[0]}, P_2^{[0]}, P_3^{[0]} \dots P_n^{[0]}]$ Denoted by initial probability distribution at the start of the process

* $p^{(n)} = [P_1^{(n)}, P_2^{(n)}, \dots P_m^{(n)}]$ Denoted by probability distribution of end of n terms.

* $p^{(1)} = p^{(0)} . P = [P_1(1), P_2(1)]$

$p^{(2)} = p^{(0)} . P^2 = [P_1(2), P_2^{(2)}]$

$p(3) = p^{(0)} . P^3 = [P_1(3), P_2(3)]$

Markov chain is irreducable if the associated transition probability matrix regular ~~stostachi~~ stochastic matrix.

$$P^2 = P.P = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} \end{bmatrix} \cancel{P_{13}^{(2)}}$$

$P_{21}^{(2)} \longrightarrow$ Probability of moving a_2 to state a_1 from two steps

$$P^3 = P^2 P = \begin{bmatrix} P_{11}^{(3)} & P_{12}^{(3)} \\ P_{21}^{(3)} & P_{22}^{(3)} \end{bmatrix}$$

→ Problems

1. The transition matrix P of a markov chain is given by $P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with the initial probability distribution $p(0) = [1/4, 3/4]$ find the following

i). $P_{21}^{(2)}$

ii). $P_{12}^{(2)}$

iii). $p^{(2)}$

iv). $p_1^{(2)}$

$$\Rightarrow P^2 = P \cdot P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}^{(2)}, & P_{12}^{(2)} \\ P_{21}^{(2)}, & P_{22}^{(2)} \end{bmatrix} \quad = P_{12}^{(2)} = \frac{3}{8} \quad P_{21}^{(2)} = \frac{9}{16}$$

$$\Rightarrow p^{(2)} = p^{(0)} \cdot p^2 = \left(\frac{1}{4}, \frac{3}{4}\right)\begin{bmatrix} 3/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \left[\frac{37}{64}, \frac{27}{64}\right]$$

$$= [p_1^{(2)}, p_2^{(2)}] \quad p_1^{(2)} = \frac{37}{64}$$

2. The t.p.m of a markov chain is given by ~~P = [1/2 0~~

$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$ and the initial probability distribution $p(0) = [1/2, 1/2, 0]$

Find

$\Rightarrow P_{13}^{(2)}, P_{23}^{(2)}, p^{(2)}, p_1^{(2)}$

$$\Rightarrow p^2 = p \cdot p = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$$

$= P_{13}^{(2)} = 3/8$

$P_{23}^{(3)} = 1/2$

$$p^{(2)} = p^{(0)} \cdot p^2 = [1/2, 1/2, 0] \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$p^{(2)} = \left[\frac{7}{16}, \frac{1}{8}, \frac{7}{16}\right] = \left[P_1^{(2)}, P_2^{(2)}, P_3^{(2)}\right]$$

$$P_1^{(2)} = \frac{7}{16}$$

3. The t.p.m of Markov chain $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducable, then ST probability vector

⇒ Irreducable means we have to show the given matrix is regular stochastic matrix.

$$P^2 = P.P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Thus P^2 all the entries are non-negative without zero. Hence P^2 is regular stochastic matrix. Hence markov chain is irreducible.

To show probability vector $v = (x, y, z)$ where $x+y+z=1$ such that $vP = v$

$$[x, y, z] \cdot \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x, y, z]$$

$$\left[\frac{y}{2} + \frac{z}{2} \quad \frac{2x}{3} + \frac{z}{2} \quad \frac{x}{3} + \frac{y}{2}\right] = (x, y, z)$$

$$\frac{y}{2} + \frac{z}{2} = x \; ; \; \frac{2x}{3} + \frac{z}{2} = y \quad \frac{x}{3} + \frac{y}{2} = z$$

4. Student's study habits are as follows. If he studies one night, he is 70% sure not to study in the the next night. On the otherhand, if he doesnot study one night, he is 60% sure not to study in the next night. In the long run how often does he study?

$\Rightarrow$ Here state space A = study B = not study
The tpm

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array} \qquad P = \frac{1}{10}\begin{bmatrix} 3 & 7 \\ 4 & 6 \end{bmatrix}$$

$V = (x, y) \quad x+y=1$

$VP = V$

$$[x, y]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x, y]$$

$$(x, y)\frac{1}{10}\begin{bmatrix} 3 & 7 \\ 4 & 6 \end{bmatrix} = [x, y]$$

$$(x, y)\begin{bmatrix} 3 & 7 \\ 4 & 6 \end{bmatrix} = 10[x, y]$$

$\Rightarrow 3x+4y \quad 7x+6y = 10[x, y]$

$= 3x+4y = 10x \quad 7x+6y = 10y.$

$x+y=1 \quad x = y-1$

$3x+4y = 10x$

$4y - 7x = 0 \quad 3(y-1)+4y = 10(y-1)$

$3y - 3 + 4y = 10y - 10$

$7y - 10y = -10+3$

$-13y = -7$

$$\boxed{y = \frac{7}{13}}$$

$x+y=1 \quad x+\frac{7}{11} = 1 \quad \boxed{x = \frac{4}{11}}$

Thus, the probability of the student the study after long time is $\frac{4}{11}$

5. A man smoking habits are as follows. If he smokes filter cigarettes for one week, he switches to non-filter cigarettes the next week with probability 0.2.

On the otherhand, if he smokes non-filter cigarettes one week there is a probability of 0.7 that he will smoke non-filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes?

states space A, B A = filter cigarette B - non filter cigarette

The tr tpm
$$P = \begin{array}{c|cc} & A & B \\ A & 0.8 & 0.2 \\ & 0.3 & 0.7 \end{array}$$

$$P = \frac{1}{10}\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix} \qquad x+y=1 \qquad V=(x,y)$$
$$VP = V$$

$$(x,y)\begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix}\frac{1}{10} = [x,y]$$

$$\Rightarrow 8x+3y,\ 2x+7y = [10x, 10y]$$

$$= \cancel{8x+3y = 10x} \qquad \cancel{2x+7y = 10y} \qquad 8x+3y=10x$$
$$\cancel{3y-2x=0} \qquad \cancel{2x-3y=0} \qquad 2x+7y=10y$$
$$\cancel{x+y=1} \qquad \cancel{y=x-1} \qquad x+y=1 \quad y\ x=y-$$
$$\cancel{3(x-1)+2x=0} \qquad \cancel{x+y=1}$$
$$\cancel{3x-3-2x=0} \qquad \cancel{3+y=1}$$
$$\cancel{x-3=0} \qquad \cancel{y=}$$
$$\boxed{x=3}$$

$$8(y-1)+3y = 10(y-1) \qquad y= \ x+y=1$$
$$8y-8+3y = 10y-10 \qquad x+\frac{2}{5}=1$$
$$-\cancel{2y}\,5y = 2$$
$$\boxed{y=\frac{2}{5}} \qquad \boxed{x=\frac{3}{5}}$$

65. In the long run, he will smoke $\frac{3}{5}$.

6. Three boys A, B, C are throwing ball to each other. A always throws ball to B and B always throws ball to C, C is just as likely to throw the ball B as to A. If C was the first person to throw the ball. Find the probability that after three throws
i). A has the ball
ii). B has the ball
iii). C has the ball

⇒ State space A, B, C
t.p.m matrix is
$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

Given, C was the first person to throw the ball, hence initial probability vector $p^{(0)} = (0, 0, 1)$. Since probability after three throws is $p^{(3)} = p^{(0)} P^{(3)}$

$$p^{(3)} = (0,0,1) \cdot \begin{pmatrix} 0/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right] = \left[p_A^{(3)}, p_B^{(3)}, p_C^{(3)}\right]$$

1. A is 1/4   2. B is 1/4   3. C is 1/2.

7. Two boys B_1, B_2 and two girls G_1, G_2 are throwing ball from one to other. Each boy throws the ball to the other boy with probability 1/2 and to each girl with probability 1/4. On the other hand, girl throws the ball to each boy with probability 1/2 and never to the other girl. In the long run, how often does each

receive the ball.

$\Rightarrow$ State space B_1, B_2, G_1, G_2

t.p.m matrix is

$$P = \begin{array}{c|cccc} & B_1 & B_2 & G_1 & G_2 \\ \hline B_1 & 0 & 1/2 & 1/4 & 1/4 \\ B_2 & 1/2 & 0 & 1/4 & 1/4 \\ G_1 & 1/2 & 1/2 & 0 & 0 \\ G_2 & 1/2 & 1/2 & 0 & 0 \end{array}$$

$\Rightarrow$ state $VP = V$

$$(\cancel{x, y, w, x, y, z})\ (a, b, c, d) \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a, b, c, d]$$

$$= \left[\frac{b}{2} + \frac{c}{2} + \frac{d}{2}, \frac{a}{2} + \frac{c}{2} + \frac{d}{2}, \frac{a}{4} + \frac{b}{4}, \frac{a}{4} + \frac{b}{4}\right]$$

$$= \frac{b}{2} + \frac{c}{2} + \frac{d}{2} = a \; ; \; \frac{a}{2} + \frac{c}{2} + \frac{d}{2} = b \quad \frac{a}{4} + \frac{b}{4} = c \quad \frac{a}{4} + \frac{b}{4} = c$$

$$v = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right).$$

8. A gambler's luck follows a pattern. If he wins a game then the probability of winning the next game is 0.6. However, if he losses a game, the probability of the losing next game is 0.7. There is a even chance of winning the first game. If so i). what is the probability of he wining the second game. ii). What is the probability of he winning the third game. & iii). In the long run. How often he will wins

$w \rightarrow$ win $\quad l \rightarrow$ Lose

$$P = \begin{array}{c|cc} & w & l \\ w & 0.6 & 0.4 \\ l & 0.5 & 0.7 \end{array}$$

$$P = \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 5 & 7 \end{bmatrix}$$

Probability of winning is 6, 4, 5, 7 and probability of winning the same.

i). $p(0)$ = is the probability winning the first game

ii). $p(1)$ is the probability of second time.

iii) $p(2)$ is the probability of winning third game

$$p^{(1)} = p^{(0)}.P = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \frac{1}{2}[1, 1]\ \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

$$= \frac{1}{20}[1, 1]\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

$$= \frac{1}{20}[9, 11] = \left(\frac{9}{20}, \frac{11}{20}\right)$$

$$p^{(2)} = p^{(0)}.p^2 = \left(\frac{1}{2}, \frac{1}{2}\right)\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}\frac{1}{100}$$

$$= \left(\frac{1}{2}, \frac{1}{2}\right)\begin{bmatrix} 0.48 & 0.52 \\ 0.29 & 0.61 \end{bmatrix}$$

$$p^{(2)} = \left[\frac{87}{200}, \frac{113}{200}\right]\frac{1}{100} = \left[p_w^{(2)}, p_L^{(2)}\right]$$

$$\boxed{P^{(2)}_{10} = \frac{87}{200}}$$

iii). $V(x,y) = V$

$$[x, y]\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}\frac{1}{10} = [x, y]$$

$6x + 3y = 10x$ $\quad$ $4x + 7y = 10y$.

$x + y = 1$

9. Each year a man trades his car for a new car in 3 brands of the popular companies maruthi limited. If he has a standard he trades it for zen. He trades it for esteem. If he has a esteem he is just as likely to trade for a new esteem or for a zen or a standard. In 1996 he bought his first car which was esteem i). Find the probability that he has

a). 1998 esteem

b) 1998 standard

c) 1999 zen

d) 1999 esteem

ii). In the long run, how often he has a esteem?

$\Rightarrow$ state space $\Rightarrow$ A, B, C $\quad$ A – standard $\quad$ B – Zen

C – Esteem

$$t.p.m = P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/3 & 1/3 & 1/3 \end{array}$$

In 1996, as the first year ∴ 1998, is to be second year and 1999 third year. Hence we need to calculate $P^{(2)}$ and $P^{(3)}$

$$P^{(2)} = P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^2 = \begin{array}{c} A \\ \\ \\ \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

~~1998 Esteem~~
~~1999 Standard~~

1998 Esteem $= a_{33}^{(2)} = 4/9$

1998 Standard $= a_{31}^{(2)} = 1/9$

$$P^3 = P^2.P = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 4/9 & 4/9 & 4/9 \\ \cancel{\frac{4491}{7500}} \; 4/27 & \cancel{\frac{8833}{22500}} \; 7/27 & \frac{16}{27} \end{bmatrix}$$

~~$V = (x, y, z)$~~

1999 Zen $= a_{32}^{(3)} = \frac{7}{27}$

1999 Esteem $= a_{33}^{(3)} = \frac{16}{27}$

$v = (x, y, z) \neq x + y + z = 1$

$vP = v$

$$(x, y, z)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$\left[\frac{z}{3}, \ x + \frac{z}{3}, \ y + \frac{z}{3}\right] = [x, y, z]$$

$\frac{z}{3} = x \qquad x + \frac{z}{3} = y \qquad 3y + \frac{z}{3} = 3z$

$z = 3x \qquad 3x + z = 3y \qquad 3y + z = 3z$

$x + y + z = 1 \qquad y = 1 - x - z$

$z = 3x$ and $y = 1 - x - z$

$3x + z = 3y$

$3x + 3x = 3(1 - x - z)$

$6x = 3 - 3x - 3z$

$6x = 3 - 3x - 3x\,(3x)$

$6x = 3 - 3x - 9x$

$6x = 3 - 10x$

$6x + 12x = 3$

$18x = 3$

$\boxed{x = \frac{3}{18}}$

$y = 1 - x - z$

$y = 1 - \frac{3}{18} - \frac{1}{2}$

$\boxed{y = \frac{1}{3}}$

$x + y + z = 1$

$\frac{3}{18} + \frac{1}{3} + z = 1$

$\boxed{z = \frac{1}{2}}$

Define probability vector, stochastic matrix, regular stochastic matrix, stationary distribution and observing state of Markov chain.

10. Every year, a man trades his car for a new car. If he has a Maruthi, he trades it for a Ambassador, If he has Ambassador he trades it for Santro. However, if he had a Santro, he is just as likely to trade it for new Santro as to trade it for a Maruthi or an Ambassador. In 2000, he bought his first car which was a Santro i Find the probability that he has 2002 Santro ii) 2002 Maruthi iii) 2003 Ambassador iv) 2003 Santro.

$\Rightarrow$ State space $\Rightarrow$ A; B, C; A $\rightarrow$ Maruthi B - Ambassador
C $\rightarrow$ Santro.

$$\text{t.p.m} \quad P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

In 2000, he bought his first case which was Santro.
2002 2nd year, 2003 3rd year : P $P^{(2)}$ and $P^{(3)}$

$$\Rightarrow P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^2 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix} \end{array}$$

$$\boxed{a_{33}^{(2)} = \frac{4}{9}} \Rightarrow \text{2002 Santro}$$

$\boxed{a_{31}^{(2)} = \frac{1}{9}}$ ⇒ 2002 Maruthi

$$p^{(3)}\ P^3 = P^2 \cdot P = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{array}{c|ccc} & A & B & C \\ A & 1/3 & 1/3 & 1/3 \\ B & 1/9 & 4/9 & 4/9 \\ C & 4/27 & 7/27 & 16/27 \end{array}$$

2003 Ambassador $= a_{32}^{(3)} = \frac{7}{27}$

2003 Santro $= a_{33}^{(3)} = \frac{16}{27}$

11. A salesman consists three cities, he never sales in the same city on successive days, If he sale in the city A then the next day ~~then~~ he sales in city B. However, ~~A~~ if he sales in either B or C then the next day he is twice as likely to sale in city A as in the other city. In the long run, how often does he sale in ~~the~~ each of the cities.

⇒ State space ⇒ A, B, C city A → A city B → B city C → C

$$\text{t.p.m}\quad P = \begin{array}{c|ccc} & A & B & C \\ A & 0 & 1 & 0 \\ B & 2/3 & 0 & 1/3 \\ C & 2/3 & 1/3 & 0 \end{array}$$

$V = (x, y, z)$

where $x + y + z = 1$

$VP = V$

$$(x, y, z) \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = [x, y, z]$$

$$\Rightarrow \left[\frac{2y}{3}+\frac{2z}{3}, \frac{x+z}{3}, \frac{y}{3}\right] = [x, y, z]$$

$$\frac{2y+2z}{3} = x, \quad \frac{3x+z}{3} = y \;;\; y = 3z$$

$$2y+2z = 3x, \quad 3x+z = 3y \;;\; y = 3z$$

$$x+y+z = 1 \qquad x = 1-y-z$$

$$2y+2z = 3(1-y-z)$$

$$= 2y+2z-3+3y+3z = 0$$

$$5y+5z-3 = 0$$

$$5(3z)+5z-3 = 0$$

$$15z+15z-3 = 0$$

$$20z = 3$$

$$\boxed{z = \frac{3}{20}}$$

$$y = 3z$$

$$y = 3\left(\frac{3}{20}\right)$$

$$\boxed{y = \frac{9}{20}}$$

$$\Rightarrow x = 1-y-z$$

$$x = 1-\frac{9}{20}-\frac{3}{20}$$

$$\boxed{x = \frac{2}{5}}$$

$$v(x,y,z) = \left(\frac{2}{5}, \frac{3}{20}, \frac{9}{20}\right)$$

12. A habitual gambler is a member of two clubs A and B. If he visits either of the clubs everyday for playing cards. He never visit club A on two consecutive days. But, if he visits club B on a particular day, then the next day he is an likely visit club B or club A. Find the matrix of the Markov chain also i). ST that the matrix regular stochastic matrix and find the unique fixed propbability matrix ii). If the person had

visited club B on Monday, find the probability that he visits club A on Thursday.

$\Rightarrow$ States space = A, B

$$\text{t.p.m } P = \begin{array}{c|cc} & A & B \\ \hline A & 0 & 1 \\ B & 1/2 & 1/2 \end{array}$$

To S.T P is regular stochastic matrix

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

Since all the entries of P^2 are positive, hence P is a regular stochastic matrix.

$\Rightarrow$ to find unique probability matrix

$V=(x,y)$ $\quad x+y=1 \quad x=y-1$

$VP=V$

$$(x,y)\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = [x,y]$$

$\Rightarrow \frac{y}{2}=x \;;\; x+\frac{y}{2}=y.$ $\qquad x=y-1$

$y=2x$ $\qquad \cancel{2x+y=2y}$ $\qquad 2x+y=2y$

$y=2x$ $\qquad \cancel{2x+2x=2.(2x)}=0$ $\qquad 2x+y=2y$

$\Rightarrow \cancel{\frac{y}{2}=x}$ $\qquad \cancel{2x+2x=4x}$ $\qquad 2(y-1)+y=2y$

$4x=y=0$ $\qquad 2y-2+y=2y$

$\cancel{\frac{2}{2}=x}$ $\qquad 3y-2y-2=0$

$\cancel{y=2}$ $\quad \frac{y}{2}=2x$ $\qquad y=\frac{1}{3}$

$\cancel{x=}$ $\qquad \boxed{\frac{2}{3}=x}$ $\qquad \cancel{2x}+$

Let Suppose Monday is the first day

If the person visited club B on Monday then the probability that he visits club after 3 days is equalent to find

$a_{21}^{(3)}$ from P^3

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$$

$$\boxed{a_{21}^{(3)} = \frac{3}{8}}$$

$\cancel{3/4}$