(UC Berkeley Math 53) Multivariate Calculus Review Problems 13.1 to 16.1

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March 2024

1 Background

The following questions are based on Stewart's "Calculus: Early Transcendentals", 8th Edition sections 13.1 to 16.1. These questions came out of my tutoring sessions with students taking Math 53 at UC Berkeley (Multivariate Calculus) but could apply to any Calculus III course.

If you find this useful, you can reach out to me via email for private tutoring sessions. You can also let me know if you have any questions or concerns about the solutions. -Oliver

2 Practice Problems

1. **Vector Functions:** Given the vector function $\mathbf{r}(t) = \langle \sin(t), e^t, \ln(t) \rangle$, find $\mathbf{r}'(t)$ and evaluate it at $t = \pi$.

2. **Partial Derivatives:** Find the first-order partial derivatives of the function $f(x, y) = x^2y + \sin(xy) + e^{x-y}$.

3. **Multiple Integrals:** Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$.

4. **Vector Calculus (Line Integral):** Compute the line integral of $\mathbf{F}(x, y) = \langle y, x \rangle$ along the curve C given by $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ from $t = 0$ to $t = 1$.

5. **Vector Functions (Arc Length):** Find the arc length of the curve $\mathbf{r}(t) = \langle t^3, \sin(t), e^{2t} \rangle$ from $t = 0$ to $t = 1$.

6. **Partial Derivatives (Chain Rule):** If $z = z(x, y)$ is given by $x^2 + xy + y^2 = z^2$, find $\frac{\partial z}{\partial x}$ at the point $(1, 1, z)$.

7. **Vector Calculus (Divergence and Curl):** For the vector field $\mathbf{F}(x, y, z) = (x^2y,\ y^2z,\ z^2x)$, find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

8. **Volume Under a Paraboloid:** Calculate the volume under the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

9. **Volume Between Spheres** Find the volume between two spheres, where the inner sphere is defined by $x^2 + y^2 + z^2 = 1$ and the outer sphere by $x^2 + y^2 + z^2 = 4$.

10. **Vector Functions (Curvature):** Calculate the curvature of the curve given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln(\cos(t)) \rangle$.

11. **Vector Calculus (Surface Integral):** Find the surface integral $\iint_S (x^2 + y^2 + z^2)\, dS$ over the sphere of radius 2 centered at the origin.

3 Solutions

1. Vector Function Differentiation

Given the vector function $\mathbf{r}(t) = \langle \sin(t), e^t, \ln(t) \rangle$, find $\mathbf{r}'(t)$ and evaluate it at $t = \pi$.

Solution

The derivative of a vector function is found by differentiating each component function individually. Thus, we have:

$$\begin{aligned} \mathbf{r}'(t) &= \left\langle \frac{d}{dt}[\sin(t)], \frac{d}{dt}[e^t], \frac{d}{dt}[\ln(t)] \right\rangle \\ &= \left\langle \cos(t), e^t, \frac{1}{t} \right\rangle. \end{aligned}$$

Evaluating at $t = \pi$, we get:

$$\mathbf{r}'(\pi) = \left\langle \cos(\pi), e^\pi, \frac{1}{\pi} \right\rangle = \left\langle -1, e^\pi, \frac{1}{\pi} \right\rangle.$$

Therefore, the derivative of the vector function $\mathbf{r}(t)$ at $t = \pi$ is $\left\langle -1, e^\pi, \frac{1}{\pi} \right\rangle$.

2. Partial Derivatives

Find the first-order partial derivatives of the function $f(x, y) = x^2 y + \sin(xy) + e^{x-y}$.

Solution

To find the first-order partial derivatives, we differentiate with respect to each variable while treating the other variable as a constant.

- For the partial derivative with respect to x, use the product rule for $x^2 y$ and the chain rule for $\sin(xy)$ and e^{x-y}.

$$\frac{\partial f}{\partial x} = 2xy + y\cos(xy) + e^{x-y}$$

- For the partial derivative with respect to y, use the constant multiple rule for x^2y, the product rule and chain rule for $\sin(xy)$, and the exponent rule for e^{x-y}.
$$\frac{\partial f}{\partial y} = x^2 + x\cos(xy) - e^{x-y}$$

3. Multiple Integrals

Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$.

Solution

The integral can be evaluated by integrating with respect to x first, then y. For the inner integral, apply the power rule for x^2 and the constant multiple rule for y^2.

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy &= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_0^{\sqrt{1-y^2}} dy \\ &= \int_0^1 \left(\frac{(1-y^2)^{3/2}}{3} + y^2\sqrt{1-y^2} \right) dy. \end{aligned}$$

The outer integral is then evaluated using a suitable substitution, such as $u = 1 - y^2$, followed by the power rule and other antiderivative techniques.

4. Vector Calculus (Line Integral

Compute the line integral of $\mathbf{F}(x, y) = \langle y, x \rangle$ along the curve C given by $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ from $t = 0$ to $t = 1$.

Solution

A line integral is evaluated by parametrizing the curve, substituting into the vector field, and then integrating with respect to the parameter t.

- First, find $\mathbf{r}'(t)$:
$$\mathbf{r}'(t) = \frac{d}{dt}\langle t^2, t^3 \rangle = \langle 2t, 3t^2 \rangle$$
- Next, substitute $\mathbf{r}(t)$ into $\mathbf{F}(x, y)$ and compute the dot product with $\mathbf{r}'(t)$:
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle t^3, t^2 \rangle \cdot \langle 2t, 3t^2 \rangle = 2t^4 + 3t^4 = 5t^4$$
- Integrate the result with respect to t from 0 to 1:
$$\int_0^1 5t^4\, dt = \left[\frac{5t^5}{5} \right]_0^1 = 1$$

Therefore, the value of the line integral is 1.

5. Arc Length of a Vector Function

Given the vector function $\mathbf{r}(t) = \langle t^3, \sin(t), e^{2t} \rangle$, find the arc length of the curve from $t = 0$ to $t = 1$.

Solution

The arc length S of a curve given by the vector function $\mathbf{r}(t)$ is calculated as:

$$S = \int_a^b \|\mathbf{r}'(t)\| \, dt$$

where $\|\mathbf{r}'(t)\|$ is the magnitude of the derivative of $\mathbf{r}(t)$.

First, we find $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \langle 3t^2, \cos(t), 2e^{2t} \rangle$$

The magnitude of $\mathbf{r}'(t)$ is:

$$\|\mathbf{r}'(t)\| = \sqrt{(3t^2)^2 + (\cos(t))^2 + (2e^{2t})^2}$$

The arc length is then:

$$S = \int_0^1 \sqrt{9t^4 + \cos^2(t) + 4e^{4t}} \, dt$$

This becomes too difficult to do by hand but it shows you what steps to take to tackle these kind of problems!

6. Chain Rule for Partial Derivatives

Consider the function $z(x, y)$ given implicitly by $x^2 + xy + y^2 = z^2$. We want to find $\frac{\partial z}{\partial x}$ at the point $(1, 1, z)$.

Solution

Implicit differentiation with respect to x gives:

$$2x + y = 2z\frac{\partial z}{\partial x}$$

Solving for $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} = \frac{2x + y}{2z}$$

At the point $(1, 1, z)$, we first determine z by substituting $x = 1$ and $y = 1$ into the original equation, yielding $z = \sqrt{3}$. Thus, at $(1, 1, \sqrt{3})$:

$$\frac{\partial z}{\partial x} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

7. Vector Calculus (Divergence and Curl)

For the vector field $\mathbf{F}(x, y, z) = (x^2y,\ y^2z,\ z^2x)$, find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Solution

To find the divergence $\nabla \cdot \mathbf{F}$, we take the dot product of the del operator with $\mathbf{F}$. This results in the sum of the partial derivatives of the vector field's components.

$$\nabla \cdot \mathbf{F} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(y^2z)}{\partial y} + \frac{\partial(z^2x)}{\partial z} = 2xy + 2yz + 2zx$$

To find the curl $\nabla \times \mathbf{F}$, we take the cross product of the del operator with $\mathbf{F}$, treating the del operator as if it were a vector with partial derivatives as its components.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix} = \mathbf{i}\left(\frac{\partial(z^2x)}{\partial y} - \frac{\partial(y^2z)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial(x^2y)}{\partial z} - \frac{\partial(z^2x)}{\partial x}\right) + \mathbf{k}\left(\frac{\partial(y^2z)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\right)$$

After computing the partial derivatives, we get:

$$\nabla \times \mathbf{F} = \mathbf{i}(0 - y^2) - \mathbf{j}(0 - z^2) + \mathbf{k}(0 - x^2) = (-y^2\mathbf{i}, -z^2\mathbf{j}, -x^2\mathbf{k})$$

The divergence of $\mathbf{F}$ is $2xy + 2yz + 2zx$, and the curl of $\mathbf{F}$ is $(-y^2, -z^2, -x^2)$.

8. Triple Integral for Volume Under a Paraboloid

Calculate the volume under the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

Solution

To find the volume under the paraboloid, we set up the triple integral in Cartesian coordinates as:

$$V = \int\int\int_D dz\, dy\, dx$$

where D is the projection of the paraboloid onto the xy-plane. Since z ranges from the xy-plane (0) up to the surface of the paraboloid, the limits for z are 0 to $4 - x^2 - y^2$. The projection D is a circle with radius 2 centered at the origin, so we use polar coordinates to express x and y, with r ranging from 0 to 2 and θ from 0 to 2π:

$$V = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\, dz\, dr\, d\theta$$

Now we integrate with respect to z first:

$$V = \int_0^{2\pi} \int_0^2 [rz]_0^{4-r^2} \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r(4 - r^2) dr \, d\theta$$

Evaluating the integral over r:

$$V = \int_0^{2\pi} \left[4r - \frac{r^3}{3} \right]_0^2 d\theta = \int_0^{2\pi} \left(8 - \frac{8}{3} \right) d\theta = \int_0^{2\pi} \frac{16}{3} d\theta$$

And finally, the integral over θ:

$$V = \left[\frac{16}{3} \theta \right]_0^{2\pi} = \frac{32\pi}{3}$$

The volume under the paraboloid is $\frac{32\pi}{3}$ cubic units.

9. Triple Integral for Volume Between Spheres

Find the volume between two spheres, where the inner sphere is defined by $x^2 + y^2 + z^2 = 1$ and the outer sphere by $x^2 + y^2 + z^2 = 4$.

Solution

The volume between the two spheres can be determined by the difference between the volume of the outer sphere and the volume of the inner sphere. Using spherical coordinates, where $\rho^2 = x^2 + y^2 + z^2$, we set up the triple integral as:

$$V = \int \int \int_E \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

The region E is bounded by $\rho = 1$ and $\rho = 2$. The angular limits are 0 to 2π for θ and 0 to π for ϕ:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

We integrate over ρ first:

$$V = \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3} \right]_1^2 \sin(\phi) \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \left(\frac{8}{3} - \frac{1}{3} \right) \sin(\phi) \, d\phi \, d\theta$$

$$V = \int_0^{2\pi} \int_0^{\pi} \frac{7}{3} \sin(\phi) \, d\phi \, d\theta$$

Evaluating the integral over ϕ:

$$V = \int_0^{2\pi} \left[-\frac{7}{3} \cos(\phi) \right]_0^{\pi} d\theta = \int_0^{2\pi} \frac{14}{3} d\theta$$

And finally, the integral over θ:

$$V = \left[\frac{14}{3}\theta\right]_0^{2\pi} = \frac{28\pi}{3}$$

The volume between the two spheres is $\frac{28\pi}{3}$ cubic units.

10. Vector Functions (Curvature)

Calculate the curvature of the curve given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln(\cos(t)) \rangle$.

Solution

The curvature κ of a curve defined by the vector function $\mathbf{r}(t)$ is given by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

First, we find the derivatives of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \langle \cos(t), -\sin(t), -\tan(t) \rangle$$

$$\mathbf{r}''(t) = \langle -\sin(t), -\cos(t), -\sec^2(t) \rangle$$

Then, we calculate the cross product of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & -\sin(t) & -\tan(t) \\ -\sin(t) & -\cos(t) & -\sec^2(t) \end{vmatrix}$$

$$= \left(\sin^2(t) + \cos^2(t)\tan(t), \cos(t)\tan(t) + \sin(t)\sec^2(t), -\cos^2(t) - \sin^2(t)\right)$$

Since $\sin^2(t) + \cos^2(t) = 1$, this simplifies to

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle \tan(t), \tan(t) + \sec^2(t), -1 \rangle$$

The magnitude of this cross product is

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{\tan^2(t) + (\tan(t) + \sec^2(t))^2 + 1}$$

The magnitude of $\mathbf{r}'(t)$ is

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2(t) + \sin^2(t) + \tan^2(t)}$$

$$= \sqrt{1 + \tan^2(t)} = \sec(t)$$

Finally, the curvature is

$$\kappa(t) = \frac{\sqrt{\tan^2(t) + (\tan(t) + \sec^2(t))^2 + 1}}{\sec^3(t)}$$

Note: Sometimes questions ask you to evaluate this at certain points.

11. Vector Calculus (Surface Integral)

Find the surface integral $\iint_S (x^2+y^2+z^2)\,dS$ over the sphere of radius 2 centered at the origin.

Solution

The function to integrate is the squared distance from the origin, which for a sphere of radius 2 is always 4. The surface element dS on a sphere of radius r is $r^2 \sin(\phi)\,d\phi\,d\theta$, where ϕ is the polar angle and θ is the azimuthal angle.

The surface integral over the sphere is then

$$\iint_S 4\,dS = 4\iint_S dS$$

$$= 4\int_0^{2\pi}\int_0^{\pi} (2)^2 \sin(\phi)\,d\phi\,d\theta$$

$$= 16\int_0^{2\pi}\int_0^{\pi} \sin(\phi)\,d\phi\,d\theta$$

$$= 16\int_0^{2\pi} [-\cos(\phi)]_0^{\pi}\ d\theta$$

$$= 16\int_0^{2\pi} [-(-1)-(-1)]\ d\theta$$

$$= 16\int_0^{2\pi} 2\,d\theta$$

$$= 32\,[\theta]_0^{2\pi}$$

$$= 32(2\pi - 0)$$

$$= 64\pi$$

Thus, the surface integral over the sphere is 64π.