(UC Berkeley Math 53) Multivariate Calculus Review Problems 13.1 to 16.1

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1 Background

The following questions are based on Stewart's "Calculus: Early Transcendentals", 8th Edition sections 13.1 to 16.1. These questions came out of my tutoring sessions with students taking Math 53 at UC Berkeley (Multivariate Calculus) but could apply to any Calculus III course.

If you find this useful, you can reach out to me via email for private tutoring sessions. You can also let me know if you have any questions or concerns about the solutions. -Oliver

2 Practice Problems

1. **Vector Functions:** Given the vector function $\mathbf{r}(t) = \langle \sin(t), e^t, \ln(t) \rangle$, find $\mathbf{r}'(t)$ and evaluate it at $t = \pi$.

2. **Partial Derivatives:** Find the first-order partial derivatives of the function $f(x, y) = x^2y + \sin(xy) + e^{x-y}$.

3. **Multiple Integrals:** Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$.

4. **Vector Calculus (Line Integral):** Compute the line integral of $\mathbf{F}(x, y) = \langle y, x \rangle$ along the curve C given by $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ from $t = 0$ to $t = 1$.

5. **Vector Functions (Arc Length):** Find the arc length of the curve $\mathbf{r}(t) = \langle t^3, \sin(t), e^{2t} \rangle$ from $t = 0$ to $t = 1$.

6. **Partial Derivatives (Chain Rule):** If $z = z(x, y)$ is given by $x^2 + xy + y^2 = z^2$, find $\frac{\partial z}{\partial x}$ at the point $(1, 1, z)$.

7. **Vector Calculus (Divergence and Curl):** For the vector field $\mathbf{F}(x, y, z) = (x^2y,\, y^2z,\, z^2x)$, find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

8. **Volume Under a Paraboloid:** Calculate the volume under the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

9. **Volume Between Spheres** Find the volume between two spheres, where the inner sphere is defined by $x^2 + y^2 + z^2 = 1$ and the outer sphere by $x^2 + y^2 + z^2 = 4$.

10. **Vector Functions (Curvature):** Calculate the curvature of the curve given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln(\cos(t)) \rangle$.

11. **Vector Calculus (Surface Integral):** Find the surface integral $\iint_S (x^2 + y^2 + z^2)\, dS$ over the sphere of radius 2 centered at the origin.

12. **Level Surface and Gradient:** Consider the function $f(x, y, z) = x^2 + y^2 - z$.

    (a) Sketch the level surface $f(x, y, z) = 0$.

    (b) Find the gradient vector ∇f at the point $(1, -1, 2)$.

    (c) Find the directional derivative of f at $(1, -1, 2)$ in the direction of the vector $\mathbf{v} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$.

13. **Maximizing the Function Subject to a Circle Constraint:** Consider the function $f(x, y) = x + 3y$, and suppose we want to maximize it subject to the constraint $x^2 + y^2 = 10$.

14. **Closest Point on a Plane to the Origin:** Find the point on the plane $2x - 2y + z = 5$ that is closest to the origin.

15. **Open-Ended Prompt:** A farmer has a flat piece of land next to a river. She wants to create a rectangular enclosure to plant vegetables, using the river as one of the sides so that fencing is only needed on the other three sides. If she has 300 meters of fencing available, what dimensions should she use to create the largest possible area for planting?

3 Solutions

1. Vector Function Differentiation

Given the vector function $\mathbf{r}(t) = \langle \sin(t), e^t, \ln(t) \rangle$, find $\mathbf{r}'(t)$ and evaluate it at $t = \pi$.

Solution

The derivative of a vector function is found by differentiating each component function individually. Thus, we have:

$$\begin{aligned} \mathbf{r}'(t) &= \left\langle \frac{d}{dt}[\sin(t)], \frac{d}{dt}[e^t], \frac{d}{dt}[\ln(t)] \right\rangle \\ &= \left\langle \cos(t), e^t, \frac{1}{t} \right\rangle. \end{aligned}$$

Evaluating at $t = \pi$, we get:

$$\mathbf{r}'(\pi) = \left\langle \cos(\pi), e^{\pi}, \frac{1}{\pi} \right\rangle = \left\langle -1, e^{\pi}, \frac{1}{\pi} \right\rangle.$$

Therefore, the derivative of the vector function $\mathbf{r}(t)$ at $t = \pi$ is $\left\langle -1, e^{\pi}, \frac{1}{\pi} \right\rangle$.

2. Partial Derivatives

Find the first-order partial derivatives of the function $f(x, y) = x^2 y + \sin(xy) + e^{x-y}$.

Solution

To find the first-order partial derivatives, we differentiate with respect to each variable while treating the other variable as a constant.

- For the partial derivative with respect to x, use the product rule for $x^2 y$ and the chain rule for $\sin(xy)$ and e^{x-y}.

$$\frac{\partial f}{\partial x} = 2xy + y\cos(xy) + e^{x-y}$$

- For the partial derivative with respect to y, use the constant multiple rule for $x^2 y$, the product rule and chain rule for $\sin(xy)$, and the exponent rule for e^{x-y}.

$$\frac{\partial f}{\partial y} = x^2 + x\cos(xy) - e^{x-y}$$

3. Multiple Integrals

Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$.

Solution

The integral can be evaluated by integrating with respect to x first, then y. For the inner integral, apply the power rule for x^2 and the constant multiple rule for y^2.

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy &= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_0^{\sqrt{1-y^2}} dy \\
&= \int_0^1 \left(\frac{(1-y^2)^{3/2}}{3} + y^2 \sqrt{1-y^2} \right) dy.
\end{aligned}$$

The outer integral is then evaluated using a suitable substitution, such as $u = 1 - y^2$, followed by the power rule and other antiderivative techniques.

4. Vector Calculus (Line Integral

Compute the line integral of $\mathbf{F}(x,y) = \langle y, x \rangle$ along the curve C given by $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ from $t = 0$ to $t = 1$.

Solution

A line integral is evaluated by parametrizing the curve, substituting into the vector field, and then integrating with respect to the parameter t.

- First, find $\mathbf{r}'(t)$:
$$\mathbf{r}'(t) = \frac{d}{dt}\langle t^2, t^3 \rangle = \langle 2t, 3t^2 \rangle$$
- Next, substitute $\mathbf{r}(t)$ into $\mathbf{F}(x,y)$ and compute the dot product with $\mathbf{r}'(t)$:
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle t^3, t^2 \rangle \cdot \langle 2t, 3t^2 \rangle = 2t^4 + 3t^4 = 5t^4$$
- Integrate the result with respect to t from 0 to 1:
$$\int_0^1 5t^4 \, dt = \left[\frac{5t^5}{5} \right]_0^1 = 1$$

Therefore, the value of the line integral is 1.

5. Arc Length of a Vector Function

Given the vector function $\mathbf{r}(t) = \langle t^3, \sin(t), e^{2t} \rangle$, find the arc length of the curve from $t = 0$ to $t = 1$.

Solution

The arc length S of a curve given by the vector function $\mathbf{r}(t)$ is calculated as:

$$S = \int_a^b \|\mathbf{r}'(t)\| \, dt$$

where $\|\mathbf{r}'(t)\|$ is the magnitude of the derivative of $\mathbf{r}(t)$.

First, we find $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \langle 3t^2, \cos(t), 2e^{2t} \rangle$$

The magnitude of $\mathbf{r}'(t)$ is:

$$\|\mathbf{r}'(t)\| = \sqrt{(3t^2)^2 + (\cos(t))^2 + (2e^{2t})^2}$$

The arc length is then:

$$S = \int_0^1 \sqrt{9t^4 + \cos^2(t) + 4e^{4t}} \, dt$$

This becomes too difficult to do by hand but it shows you what steps to take to tackle these kind of problems!

6. Chain Rule for Partial Derivatives

Consider the function $z(x, y)$ given implicitly by $x^2 + xy + y^2 = z^2$. We want to find $\frac{\partial z}{\partial x}$ at the point $(1, 1, z)$.

Solution

Implicit differentiation with respect to x gives:

$$2x + y = 2z\frac{\partial z}{\partial x}$$

Solving for $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} = \frac{2x + y}{2z}$$

At the point $(1, 1, z)$, we first determine z by substituting $x = 1$ and $y = 1$ into the original equation, yielding $z = \sqrt{3}$. Thus, at $(1, 1, \sqrt{3})$:

$$\frac{\partial z}{\partial x} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

7. Vector Calculus (Divergence and Curl)

For the vector field $\mathbf{F}(x, y, z) = (x^2y,\ y^2z,\ z^2x)$, find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Solution

To find the divergence $\nabla \cdot \mathbf{F}$, we take the dot product of the del operator with $\mathbf{F}$. This results in the sum of the partial derivatives of the vector field's components.

$$\nabla \cdot \mathbf{F} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(y^2z)}{\partial y} + \frac{\partial(z^2x)}{\partial z} = 2xy + 2yz + 2zx$$

To find the curl $\nabla \times \mathbf{F}$, we take the cross product of the del operator with $\mathbf{F}$, treating the del operator as if it were a vector with partial derivatives as its components.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix} = \mathbf{i}\left(\frac{\partial(z^2x)}{\partial y} - \frac{\partial(y^2z)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial(x^2y)}{\partial z} - \frac{\partial(z^2x)}{\partial x}\right) + \mathbf{k}\left(\frac{\partial(y^2z)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\right)$$

After computing the partial derivatives, we get:

$$\nabla \times \mathbf{F} = \mathbf{i}(0 - y^2) - \mathbf{j}(0 - z^2) + \mathbf{k}(0 - x^2) = (-y^2\mathbf{i}, -z^2\mathbf{j}, -x^2\mathbf{k})$$

The divergence of $\mathbf{F}$ is $2xy + 2yz + 2zx$, and the curl of $\mathbf{F}$ is $(-y^2, -z^2, -x^2)$.

8. Triple Integral for Volume Under a Paraboloid

Calculate the volume under the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

Solution

To find the volume under the paraboloid, we set up the triple integral in Cartesian coordinates as:

$$V = \int\int\int_D dz\, dy\, dx$$

where D is the projection of the paraboloid onto the xy-plane.

Overview of Steps

Triple integrals look threatening... just break it down and move through 'em!

- Express the Equation in Cylindrical Coordinates
- Determine the Bounds
- Set up Triple Integral
- w.r.t. z
- w.r.t. r
- w.r.t. theta

Trig-time!
In cylindrical coordinates,

$$x = r\cos(\theta), y = r\sin(\theta), z = 4 - x^2 - y^2$$

And

$$(\cos(\theta))^2 - (\sin(\theta))^2 = 1$$

so

$$z = 4 - r^2$$

And the bounds
r is the radius of the circle in the xy-plane. We will integrate from 0 to 2.
The bounds for z start from 0 and go to the height of the paraboloid above any point (r, theta):

$$4 - r^2$$

This is how you conceptualize this problem. Thinking about it this way - like a circular 3D object along the z-axis with a radius expanding from 0 to 2 - may help you master these sort of questions. You got this!

High-level View

In a nutshell:

Since z ranges from the xy-plane (0) up to the surface of the paraboloid, the limits for z are 0 to $4 - x^2 - y^2$. The projection D is a circle with radius 2 centered at the origin, so we use polar coordinates to express x and y, with r ranging from 0 to 2 and θ from 0 to 2π:

$$V = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\, dz\, dr\, d\theta$$

Now we integrate with respect to z first:

$$V = \int_0^{2\pi} \int_0^2 [rz]_0^{4-r^2}\, dr\, d\theta = \int_0^{2\pi} \int_0^2 r(4 - r^2) dr\, d\theta$$

Evaluating the integral over r:

$$V = \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta = \int_0^{2\pi} \left(2(4) - \frac{16}{4}\right) d\theta = \int_0^{2\pi} 4 d\theta$$

And finally, the integral over θ:

$$V = [4\theta]_0^{2\pi} = 8\pi$$

The volume under the paraboloid is 8π cubic units.

9. Triple Integral for Volume Between Spheres

Find the volume between two spheres, where the inner sphere is defined by $x^2 + y^2 + z^2 = 1$ and the outer sphere by $x^2 + y^2 + z^2 = 4$.

Solution

The volume between the two spheres can be determined by the difference between the volume of the outer sphere and the volume of the inner sphere. Using spherical coordinates, where $\rho^2 = x^2 + y^2 + z^2$, we set up the triple integral as:

$$V = \int\int\int_E \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

The region E is bounded by $\rho = 1$ and $\rho = 2$. The angular limits are 0 to 2π for θ and 0 to π for ϕ:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

We integrate over ρ first:

$$V = \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3}\right]_1^2 \sin(\phi)\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \left(\frac{8}{3} - \frac{1}{3}\right) \sin(\phi)\, d\phi\, d\theta$$

$$V = \int_0^{2\pi} \int_0^{\pi} \frac{7}{3} \sin(\phi)\, d\phi\, d\theta$$

Evaluating the integral over ϕ:

$$V = \int_0^{2\pi} \left[-\frac{7}{3} \cos(\phi) \right]_0^{\pi} d\theta = \int_0^{2\pi} \frac{14}{3} d\theta$$

And finally, the integral over θ:

$$V = \left[\frac{14}{3} \theta \right]_0^{2\pi} = \frac{28\pi}{3}$$

The volume between the two spheres is $\frac{28\pi}{3}$ cubic units.

10. Vector Functions (Curvature)

Calculate the curvature of the curve given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln(\cos(t)) \rangle$.

Solution

The curvature κ of a curve defined by the vector function $\mathbf{r}(t)$ is given by

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

First, we find the derivatives of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \langle \cos(t), -\sin(t), -\tan(t) \rangle$$

$$\mathbf{r}''(t) = \langle -\sin(t), -\cos(t), -\sec^2(t) \rangle$$

Then, we calculate the cross product of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & -\sin(t) & -\tan(t) \\ -\sin(t) & -\cos(t) & -\sec^2(t) \end{vmatrix}$$

$$= \left(\sin^2(t) + \cos^2(t)\tan(t), \cos(t)\tan(t) + \sin(t)\sec^2(t), -\cos^2(t) - \sin^2(t) \right)$$

Since $\sin^2(t) + \cos^2(t) = 1$, this simplifies to

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle \tan(t), \tan(t) + \sec^2(t), -1 \rangle$$

The magnitude of this cross product is

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{\tan^2(t) + (\tan(t) + \sec^2(t))^2 + 1}$$

The magnitude of $\mathbf{r}'(t)$ is

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2(t) + \sin^2(t) + \tan^2(t)}$$

$$= \sqrt{1 + \tan^2(t)} = \sec(t)$$

Finally, the curvature is

$$\kappa(t) = \frac{\sqrt{\tan^2(t) + (\tan(t) + \sec^2(t))^2 + 1}}{\sec^3(t)}$$

Note: Sometimes questions ask you to evaluate this at certain points.

11. Vector Calculus (Surface Integral)

Find the surface integral $\iint_S (x^2+y^2+z^2)\, dS$ over the sphere of radius 2 centered at the origin.

Solution

The function to integrate is the squared distance from the origin, which for a sphere of radius 2 is always 4. The surface element dS on a sphere of radius r is $r^2 \sin(\phi)\, d\phi\, d\theta$, where ϕ is the polar angle and θ is the azimuthal angle.

The surface integral over the sphere is then

$$\iint_S 4\, dS = 4 \iint_S dS$$

$$= 4 \int_0^{2\pi} \int_0^{\pi} (2)^2 \sin(\phi)\, d\phi\, d\theta$$

$$= 16 \int_0^{2\pi} \int_0^{\pi} \sin(\phi)\, d\phi\, d\theta$$

$$= 16 \int_0^{2\pi} [-\cos(\phi)]_0^{\pi}\, d\theta$$

$$= 16 \int_0^{2\pi} [-(-1) - (-1)]\, d\theta$$

$$= 16 \int_0^{2\pi} 2\, d\theta$$

$$= 32\, [\theta]_0^{2\pi}$$

$$= 32(2\pi - 0)$$

$$= 64\pi$$

Thus, the surface integral over the sphere is 64π.

12. Level Surface and Gradient

(a) Level Surface

The level surface of the function $f(x, y, z) = x^2 + y^2 - z$ for the level $c = 0$ is the set of all points (x, y, z) where $x^2 + y^2 - z = 0$. This corresponds to a paraboloid opening downwards along the z-axis.

(b) Gradient Vector

The gradient of f is found by taking the partial derivatives with respect to x, y, and z:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 2y, -1)$$

At the point $(1, -1, 2)$, the gradient vector is:

$$\nabla f(1, -1, 2) = (2 \cdot 1, 2 \cdot (-1), -1) = (2, -2, -1)$$

Note: Directional Derivatives

Vectors are great when you get the hang of them. One of their use cases is with directional derivatives. But what is a directional derivative?

A **directional derivative** of a function in the direction of a vector tells us how much the function is changing at some point in the specific direction (of the vector). Let's express with an example.

Imagine you are hiking up a mountain. The steepness of the slope you're walking on can change depending on the direction you take. If you walk straight to the peak, you are walking in the *direction of the largest derivative.* Or, in other words, that direction has the largest derivative. That simple! If you zig-zag up the mountain, you are walking in a *direction with a smaller derivative.*

(c) Directional Derivative

The directional derivative of f at $(1, -1, 2)$ in the direction of $\mathbf{v}$ is given by:

$$D_{\mathbf{v}} f = \nabla f \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Quick Note: Unit Vector

How to tell v is a unit vector?

Magnitude of a unit vector is 1. So, if the magnitude of v = 1, then it is a unit vector.

Dividing by magnitude is a way to normalize a vector.

Quick Note: Magnitude of Vector

The magnitude is found by squaring each of the vector's terms, adding those squared values together, and then taking the square root of the result. If that is 1, it is already normalized.

I.e. it is already a unit vector.

Solution, Cont.

Since $\mathbf{v}$ is already a unit vector, we have:

$$D_{\mathbf{v}}f = \nabla f(1,-1,2) \cdot \mathbf{v} = (2,-2,-1) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

13. Maximizing the Function Subject to a Circle Constraint

Consider the function $f(x,y) = x + 3y$, and suppose we want to maximize it subject to the constraint $x^2 + y^2 = 10$.

Solution

The problem requires us to find the maximum of $f(x,y)$ on the circle $x^2 + y^2 = 10$. We introduce a Lagrange multiplier λ and set up the following system of equations:

$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y), \\ \nabla f(x,y) &= \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \nabla g(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, \\ x^2 + y^2 &= 10. \end{aligned}$$

Quick Note: Gradients, Conceptually

What actually is a gradient? A gradient is just a slope in more than 2-dimensions. Calc I (*2D slope*) only had from left-to-right or from right-to-left directions. With multivariate Calc, our slopes can go in all directions. We call these 3D slopes, **gradients**.

Quick Note: Gradients, Practically

Gradients are found by taking the **partial derivatives** of the functions they come from. This is the same way you take the derivative of the function for slope.

Solution, Cont.

This gives us the following system to solve:

$$\begin{aligned} 1 &= 2\lambda x, \\ 3 &= 2\lambda y, \\ x^2 + y^2 &= 10. \end{aligned}$$

Dividing the first two equations, we get $\frac{1}{3} = \frac{x}{y}$, or $y = 3x$. Substituting into the constraint equation, we get $x^2 + (3x)^2 = 10$, or $10x^2 = 10$, thus $x = \pm 1$. This gives $y = \pm 3$.

We substitute these values back into $f(x, y)$ to determine the maximum:

For $x = 1, y = 3$, $f(1, 3) = 1 + 3(3) = 10$. For $x = -1, y = -3$, $f(-1, -3) = -1 - 3(3) = -10$.

The maximum value of f subject to the constraint is therefore 10, at the point $(1, 3)$.

14. Finding the Closest Point on a Plane to the Origin

Find the point on the plane $2x - 2y + z = 5$ that is closest to the origin.

Solution

The distance D from any point (x, y, z) to the origin is given by $D = \sqrt{x^2 + y^2 + z^2}$. To minimize this distance, we minimize the function $f(x, y, z) = x^2 + y^2 + z^2$.

Using a Lagrange multiplier λ, we have the system:

$$\begin{aligned} &\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \\ &\nabla f(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \quad \nabla g(x, y, z) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \\ &2x - 2y + z = 5. \end{aligned}$$

We obtain the following equations:

$$\begin{aligned} 2x &= 2\lambda, \\ 2y &= -2\lambda, \\ 2z &= \lambda, \\ 2x - 2y + z &= 5. \end{aligned}$$

Solving these equations, we get $x = \lambda, y = -\lambda, z = \frac{\lambda}{2}$. Substituting these into the plane equation, $2\lambda + 2\lambda + \frac{\lambda}{2} = 5$, or $\frac{9}{2}\lambda = 5$, which gives us $\lambda = \frac{10}{9}$.

Therefore, the point on the plane closest to the origin is $\left(\frac{10}{9}, -\frac{10}{9}, \frac{5}{9}\right)$.

15. Open-Ended Problem

A farmer has a flat piece of land next to a river. She wants to create a rectangular enclosure to plant vegetables, using the river as one of the sides so that fencing is only needed on the other three sides. If she has 300 meters of fencing available, what dimensions should she use to create the largest possible area for planting?

Solution

For the love of word problems!

Let's denote the sides perpendicular to the river as x and the side parallel to the river as y. The area A of the rectangle is $A = xy$, and we want to maximize this area subject to the constraint given by the available fencing, which is $2x + y = 300$.

Quick Note: Lagrange, Conceptually

There is nothing like the open-ended-ness of a word problem in math to completely throw you off just when you thought you finally got the hang of something...

But these sort of problems force you to creatively (**conceptually**) picture the problem and how to address it before you even do a single partial derivative.

If you want to minimize or maximize some function over an area / region / constraint, you need to setup a *Lagrange Multiplier*. Directional Derivatives show you how your function changes in a specified direction. Lagrange shows you what the largest or smallest value of that function is over a specified region.

Solution, Cont.

Lagrange Multiplier Setup At the heart of it all is the Lagrange multiplier λ. Set up the system of equations as follows:

$$\begin{aligned} \nabla A &= \lambda \nabla g, \\ \nabla A &= \begin{bmatrix} y \\ x \end{bmatrix}, \quad \nabla g = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \\ 2x + y &= 300. \end{aligned}$$

Solving the System This gives us the equations:

$$\begin{aligned} y &= 2\lambda, \\ x &= \lambda, \\ 2x + y &= 300. \end{aligned}$$

Finding the Dimensions Substitute $x = \lambda$ and $y = 2\lambda$ into the constraint equation:

$$\begin{aligned} 2(\lambda) + 2\lambda &= 300, \\ 4\lambda &= 300, \\ \lambda &= 75. \end{aligned}$$

Thus, $x = 75$ meters and $y = 2(75) = 150$ meters.

Maximized Area The dimensions that give the largest area, with 300 meters of fencing and one side along the river, are 75 meters for the sides perpendicular to the river and 150 meters for the side parallel to the river. This will give an area of:

$$A = 75 \times 150 = 11250 \text{ square meters.}$$

This is the largest area the farmer can enclose with 300 meters of fencing while using the river as one side of the enclosure.

4 Feedback and Tutoring

As stated above, these questions have come about from my experience as a private tutor. I am a PhD Candidate at UC Berkeley with the thesis *Deepening Ecology through Artificial Intelligence (AI)*. I have nearly 700 hours tutored on Wyzant (tutoring website / platform) with over 150 5-star reviews. I will most likely be tutoring full-time (or near full-time) for the next couple of years. Please feel free to email me with any questions, concerns, or corrections you feel are needed to this document.

I would be happy to help you with your educational journey. I specialize in Data Science (ML, DL, RL, AI, NLP, LLM), Bayesian statistics, Python, and much more. I also have a website that is partially complete / in development that will have blog posts, video series on specialized topics, and courses that I will create and implement. The site can be found at: **olivertutor.me**

5 Bonus Questions

Problem: Evaluate the triple integral $\int\int\int_E x^2 + y^2 \, dV$, where E is the region inside the cylinder $x^2 + y^2 = 4$ and between $z = 0$ and $z = 5$

Solution:

This problem is also best approached in cylindrical coordinates. We set up the integral with r from 0 to 2, θ from 0 to 2π, and z from 0 to 5.

The integral becomes:

$$\int_0^{2\pi} \int_0^2 \int_0^5 (r^2)\, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^3 \left(z \Big|_0^5 \right) dr$$

$$
\begin{aligned}
&= 5\int_0^{2\pi} d\theta \int_0^2 r^3 dr \\
&= 5\int_0^{2\pi} d\theta \left(\frac{1}{4} r^4 \Big|_0^2 \right) \\
&= 5 \cdot 2\pi \cdot 4 = 40\pi.
\end{aligned}
$$

The value of the integral is 40π.