

# Proofs without words I

Exercises in METAPOST

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March 2021 —



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Geometry and Algebra

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# Geometry and Algebra

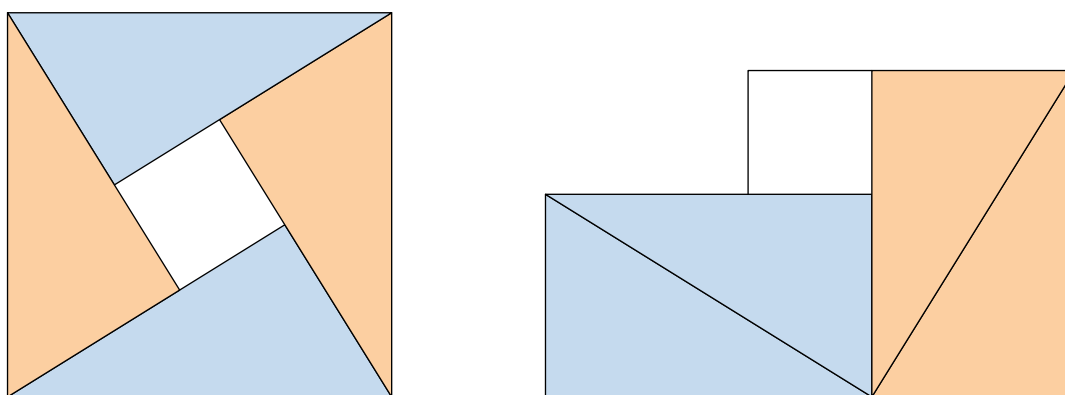
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## The Pythagorean theorem I



— adapted from the *Chou pei san ching*

## The Pythagorean theorem II



*Behold!*

— Bhāskara (12th century)

## The Pythagorean theorem III



— based on Euclid's proof

## The Pythagorean theorem IV



— H. E. Dudeney (1917)

## The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a+b)^2$$

$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)



## The Pythagorean theorem VI

$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2 + b^2 = c^2$$



— Michael Hardy

**A Pythagorean theorem:  $aa' = bb' + cc'$**



$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$

$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$

$$\therefore aa' = a(x + y) = bb' + cc'.$$

— Enzo R. Gentile

## The rolling circle squares itself



— Thomas Elsner

## On trisecting an angle



— Rufus Isaacs

## Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

— Eric Kincanon

# Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

**The vertex angles of a star sum to  $180^\circ$**



— Fouad Nakhli

## Viviani's theorem I

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



*This shows a particular example, with C'GQ collinear, rather than the general case*

— Samuel Wolf



## **Viviani's theorem II**

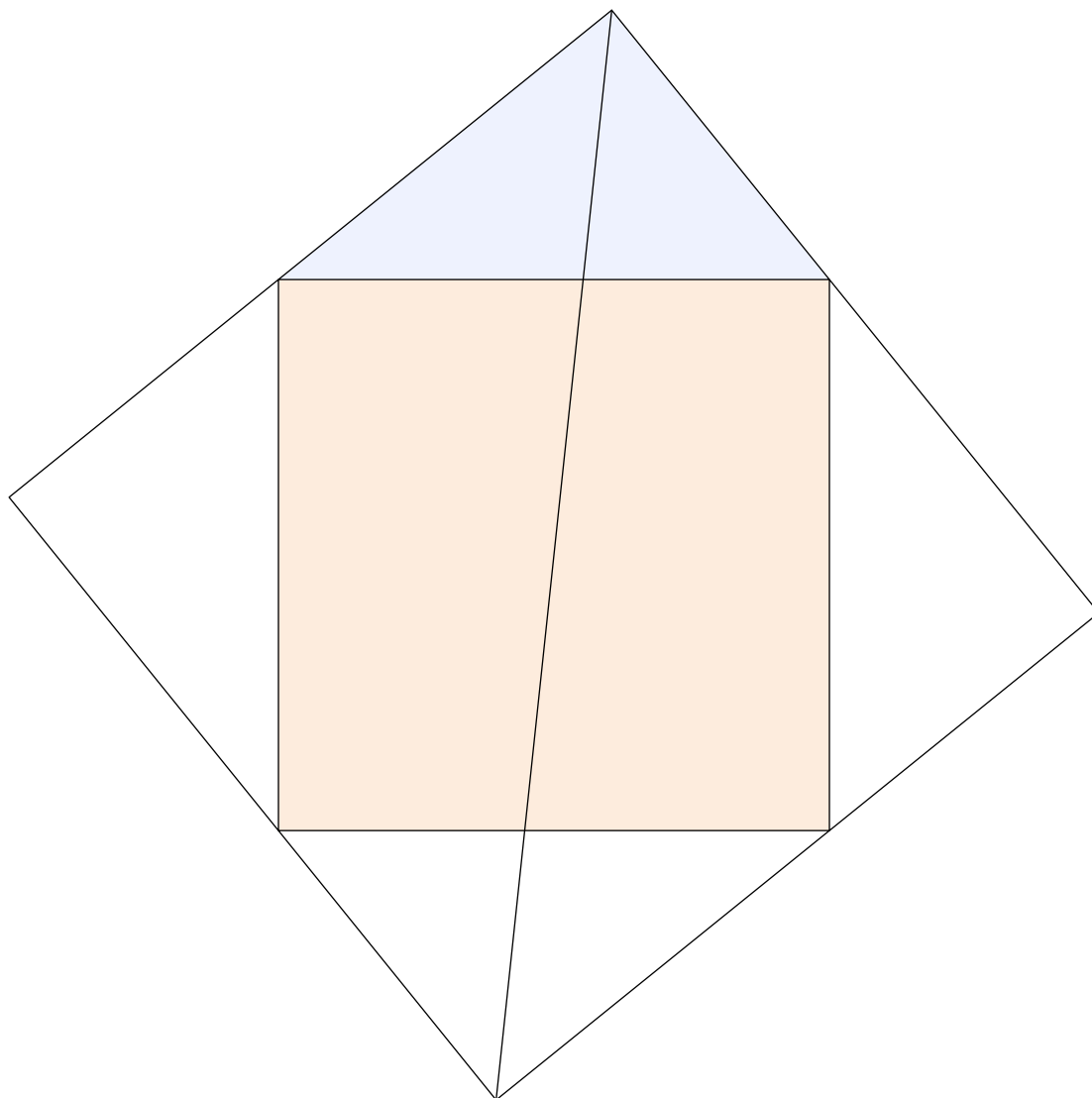
The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

## **A theorem about right angles**

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse



— Roland H. Eddy

# Area and the projection theorem of a right triangle



$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

## Chords and tangents of equal length

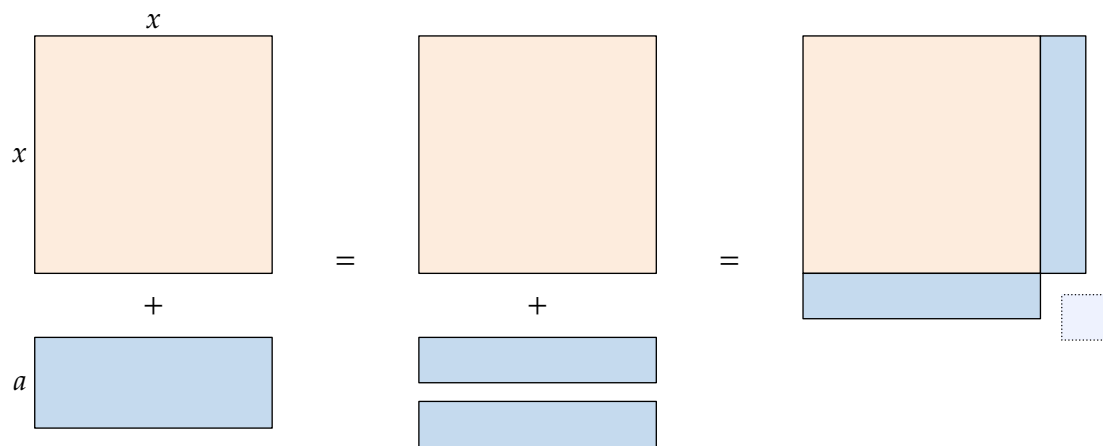
If circle  $C_1$  passes through the center  $O$  of circle  $C_2$ , the length of the common chord  $\overline{PQ}$  is equal to the tangent segment  $\overline{PR}$ .



— Roland H. Eddy

## Completing the square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

## Algebraic areas I

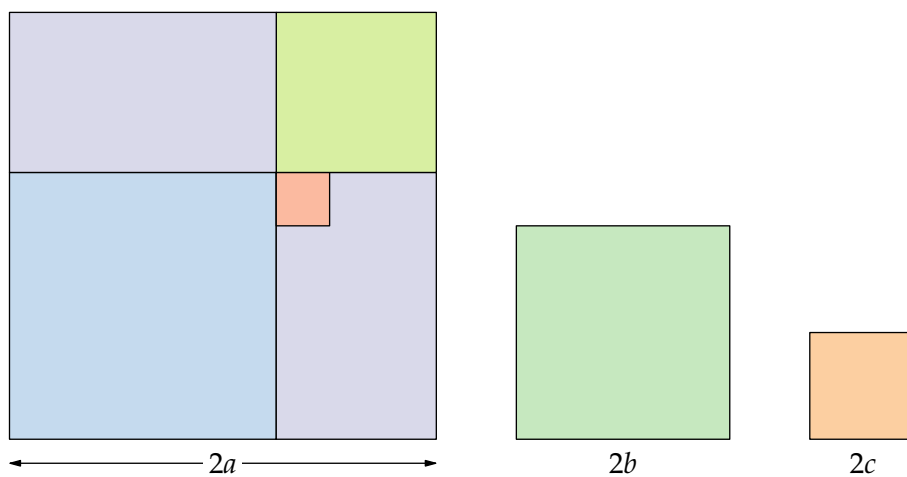
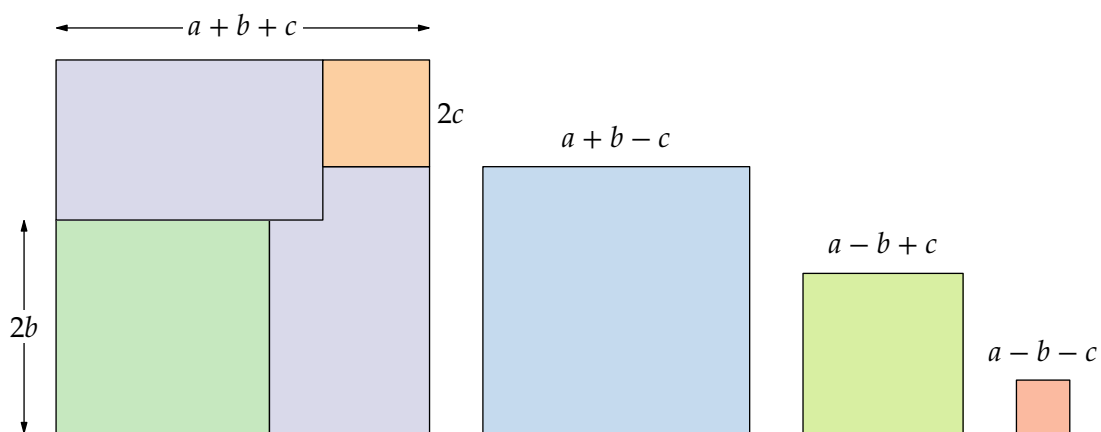
$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$



— Shirley Wakin

## Algebraic areas II

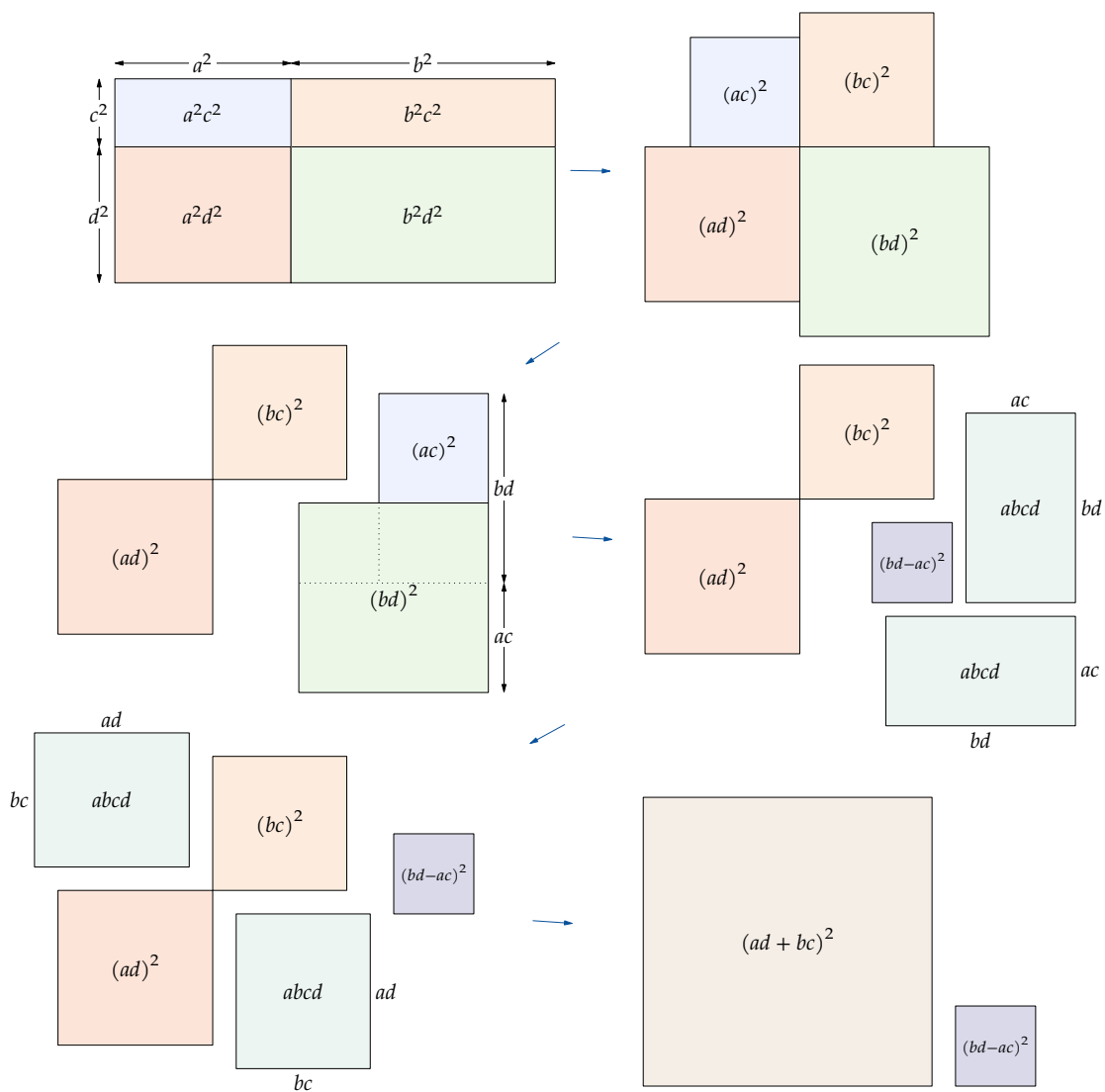
$$(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$



— Sam Pooley and K. Ann Drude

## Sum of squares identity

$$(a^2 + b^2)(c^2 + d^2) = (ab + bc)^2 + (bd - ac)^2$$



— Diophantus of Alexandria