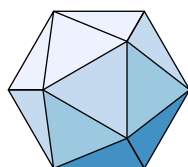


Proofs without words II

More exercises in METAPOST

Toby Thurston

March 2020



Contents

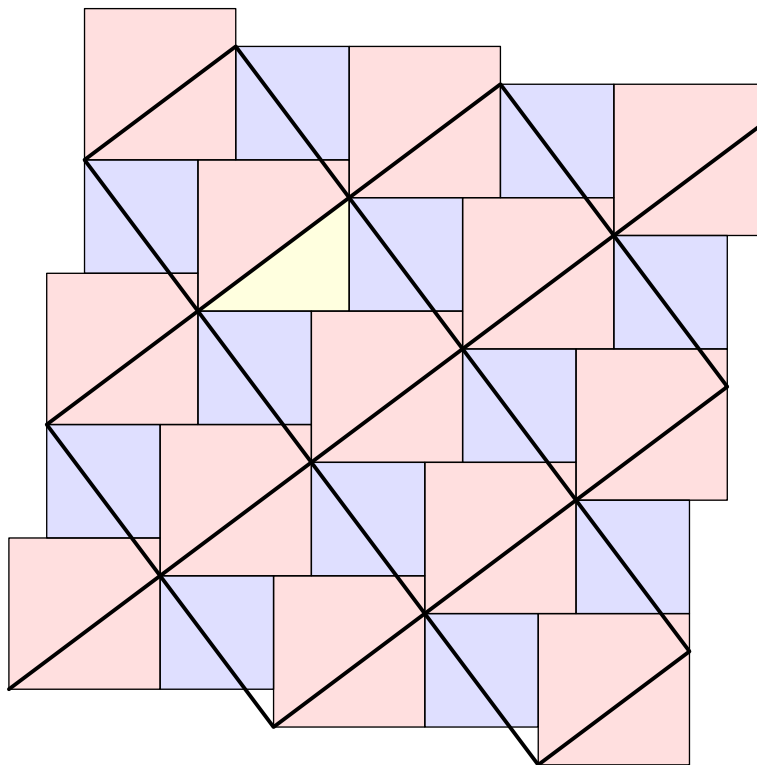
Geometry and Algebra	3
Trigonometry, Calculus, & Analytic Geometry	38
Inequalities	68
Integer sums	79

Geometry and Algebra

The Pythagorean theorem VII	5
The Pythagorean theorem VIII	6
The Pythagorean theorem IX	7
The Pythagorean theorem X	8
The Pythagorean theorem XI	9
The Pythagorean theorem XII	10
A generalization from Pythagoras	11
A theorem of Hippocrates of Chios (circa 440 BC)	12
The area of a right triangle with acute angle $\pi/12$	13
A right angle inequality	14
The inradius of a right triangle	15
The product of the perimeter of a triangle and its inradius is twice the area of the triangle	16
Four triangles with equal area	17
The triangle of medians has $3/4$ the area of the original triangle	18
Heptasection of a triangle	19
A Golden Section problem from the <i>Monthly</i>	20
Tiling with squares and parallelograms	21
The area of a quadrilateral I	22
The area of a quadrilateral II	23
A square within a square	24
Areas and perimeters of regular polygons	25
The area of a Putnam octagon	26
A Putnam dodecagon	27
The area of a regular dodecagon	28
Fair allocation of a pizza	29
A three-circle theorem	30
A constant chord	31
A Putnam area problem	32
The area under a polygonal arch	33
The length of a polygonal arch	34
The volume of a frustum of a square pyramid	35

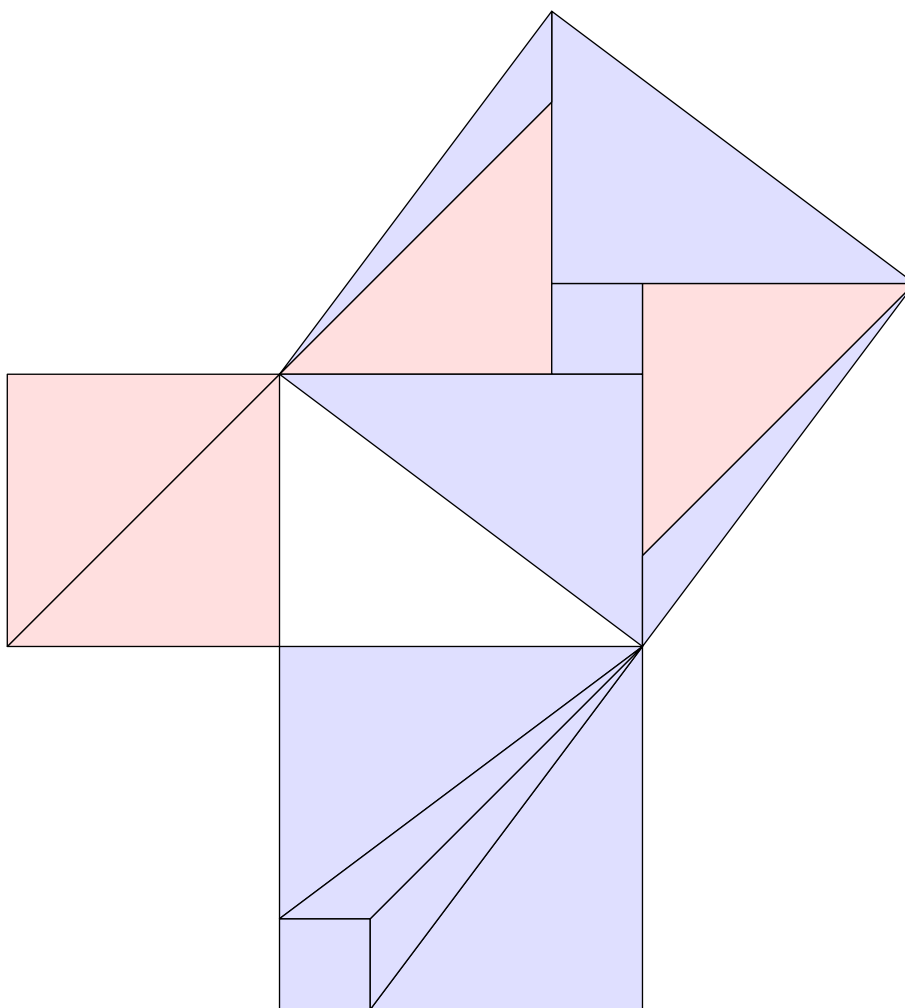
The product of four (positive) numbers in arithmetic progression is always the difference of two squares	36
Algebraic areas III: Factoring the sum of two squares	37

The Pythagorean theorem VII



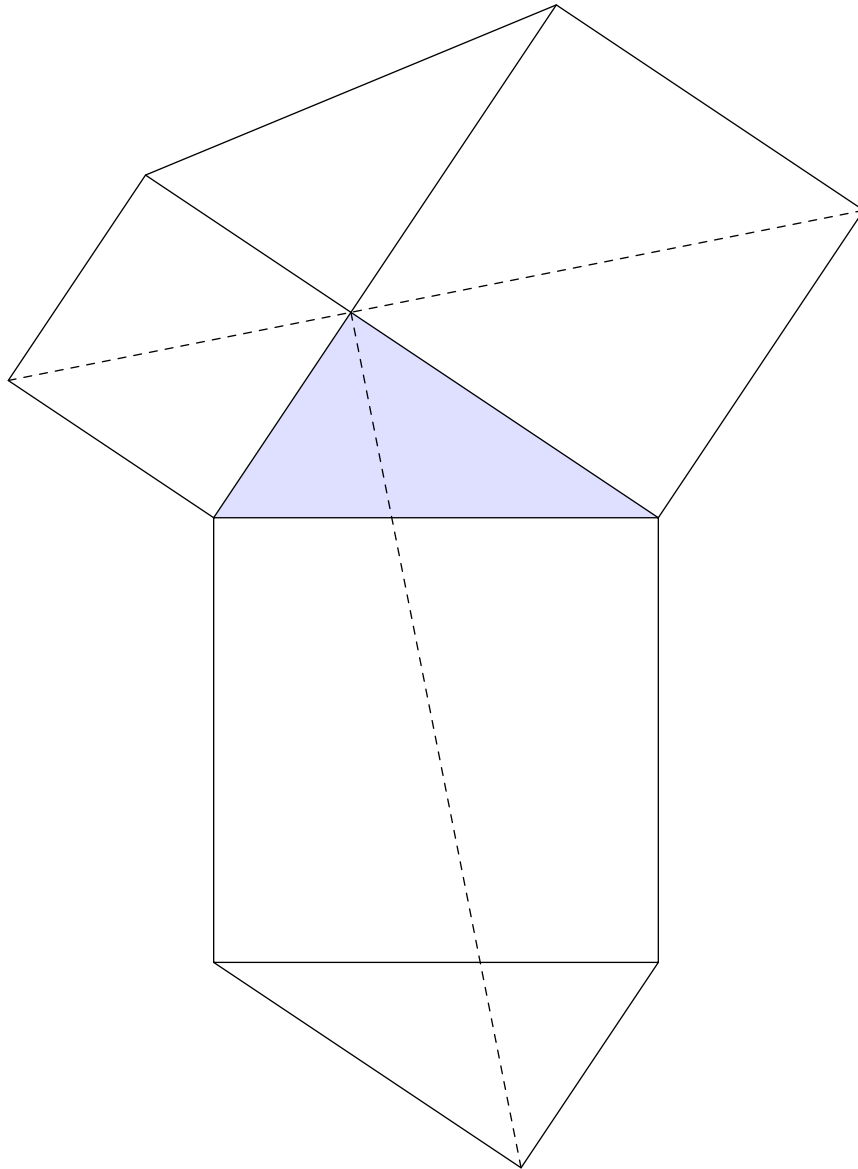
— Annairizi of Arabia (circa 900)

The Pythagorean theorem VIII



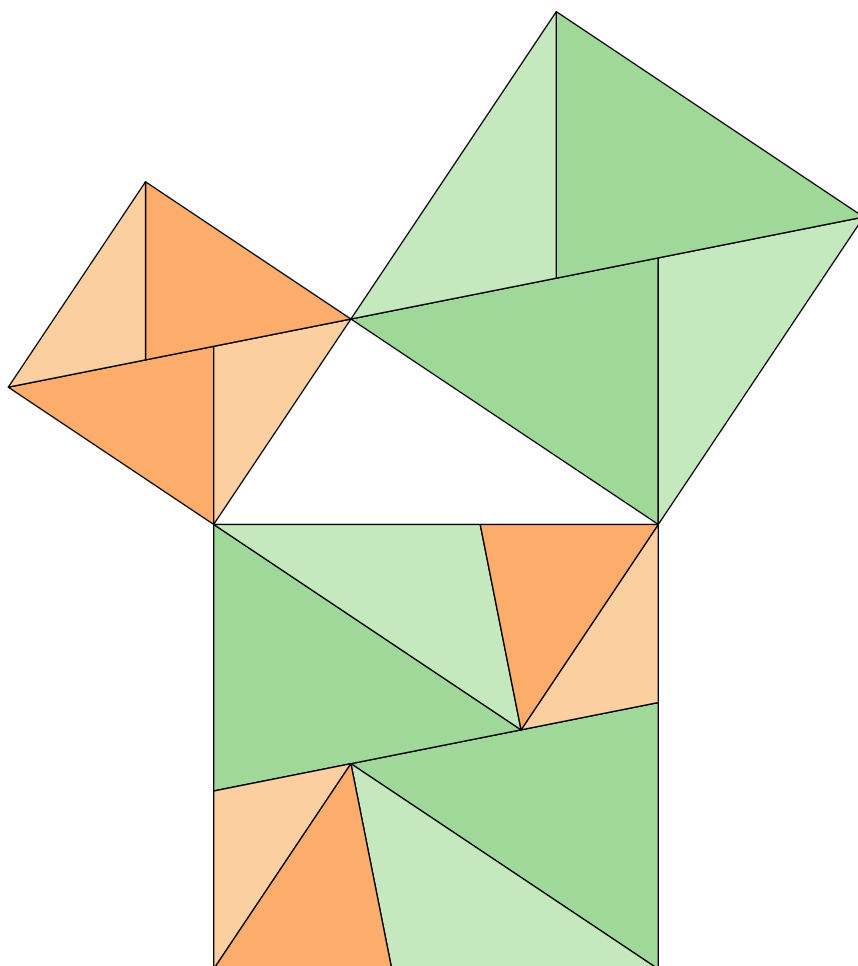
— Liu Hui (3rd century A.D.)

The Pythagorean theorem IX



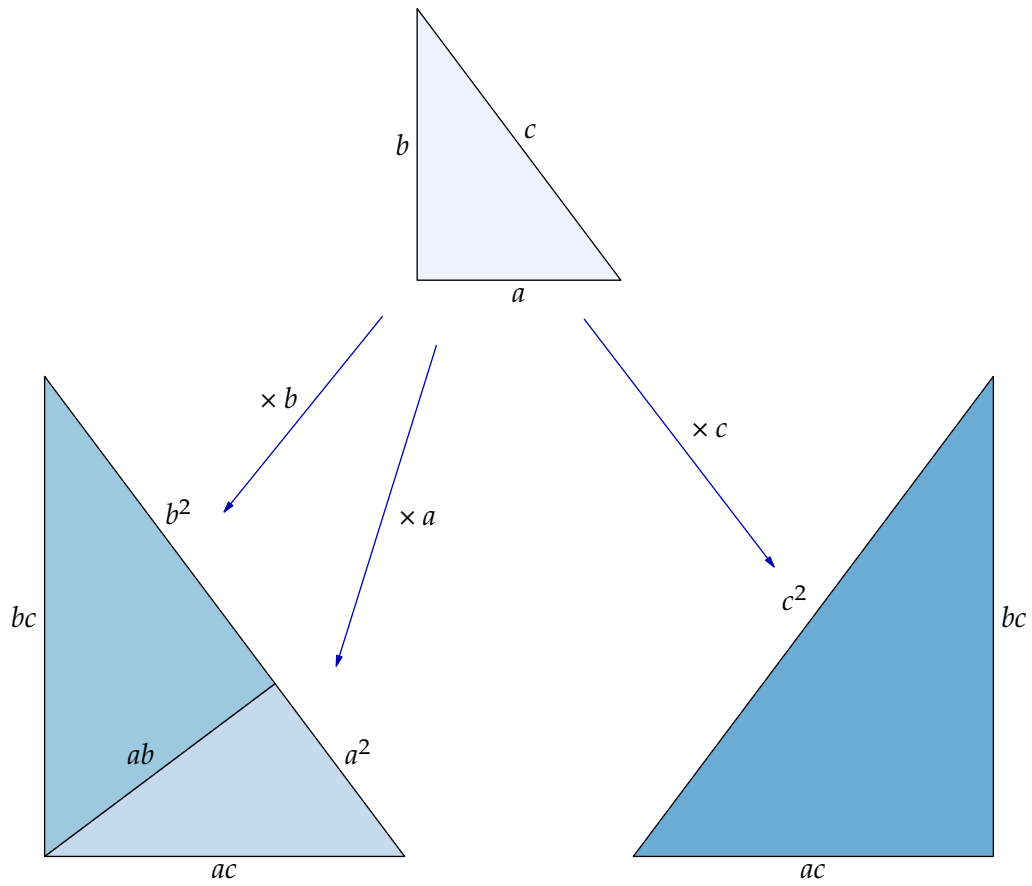
— Leonardo da Vinci (1452–1519)

The Pythagorean theorem X



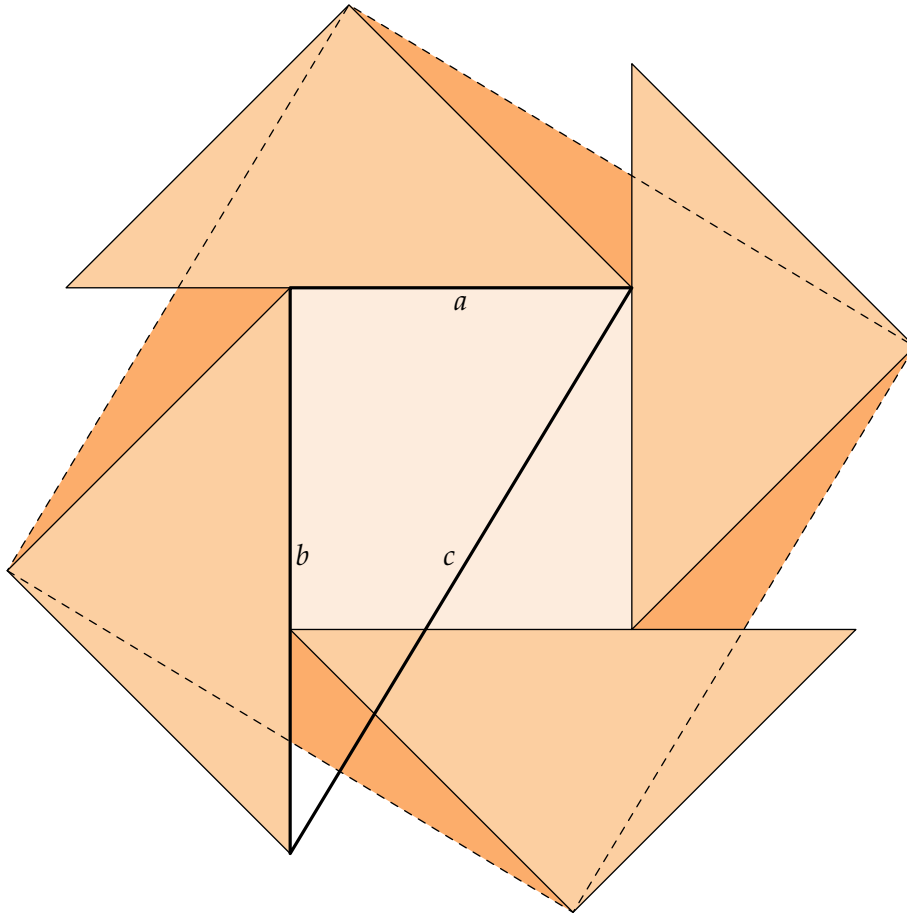
— J. E. Böttcher

The Pythagorean theorem XI



— Frank Burk

The Pythagorean theorem XII

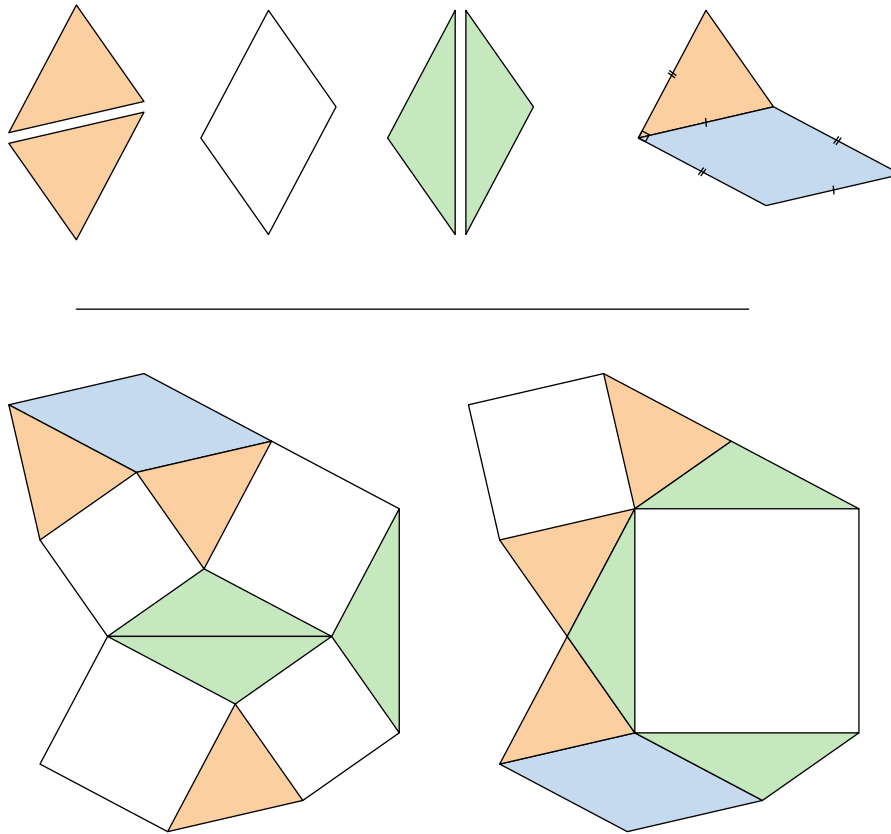


$$a^2 + b^2 = c^2$$

— Poo-Sung Park

A generalization from Pythagoras

The sum of the area of two squares, whose sides are the lengths of two diagonals of a parallelogram, is equal to the sum of the area of four squares, whose sides are its four sides.



COROLLARY: The Pythagorean theorem (when the parallelogram is a rectangle).

— David S. Wise

A theorem of Hippocrates of Chios (circa 440 BC)

The combined area of the lunes constructed on the legs of a given right angle triangle is equal to the area of the triangle.

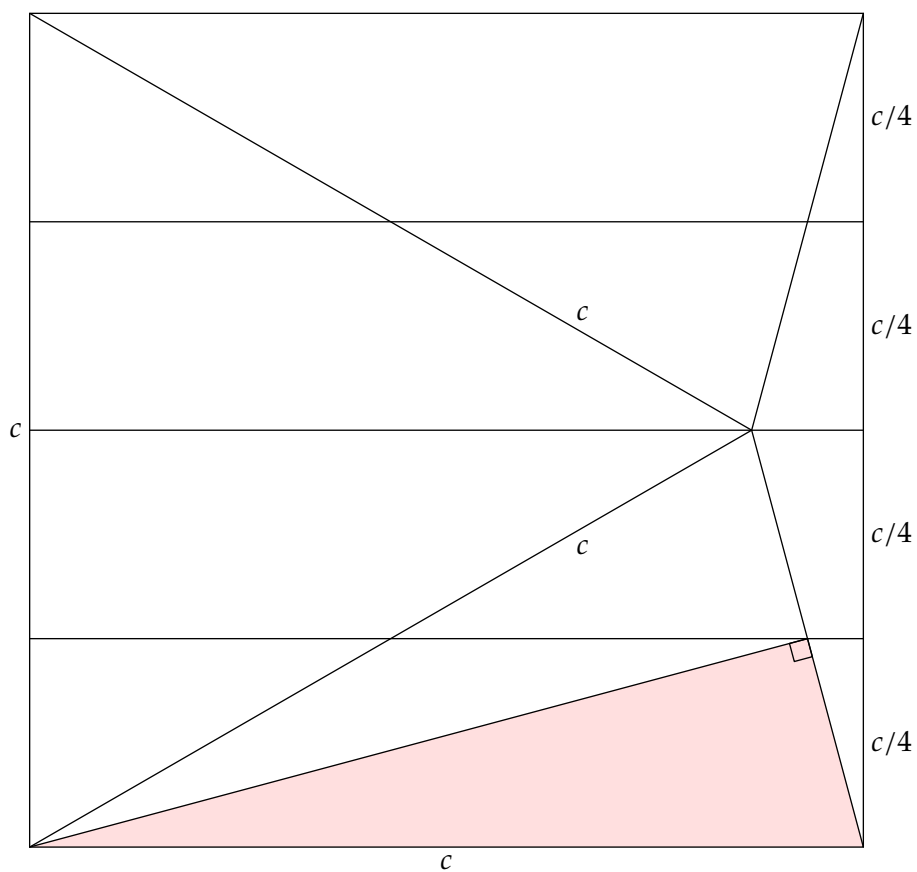


$$\begin{aligned}
 A_1 + A_2 &= A_3 \\
 (L_1 + S_1) + (L_2 + S_2) &= T + S_1 + S_2 \\
 L_1 + L_2 &= T
 \end{aligned}$$

— Eugene A. Margerum and Michael M. McDonnell

The area of a right triangle with acute angle $\pi/12$

The area of a right triangle is $\frac{1}{8}(\text{hypotenuse})^2$ if and only if one acute angle is $\pi/12$.



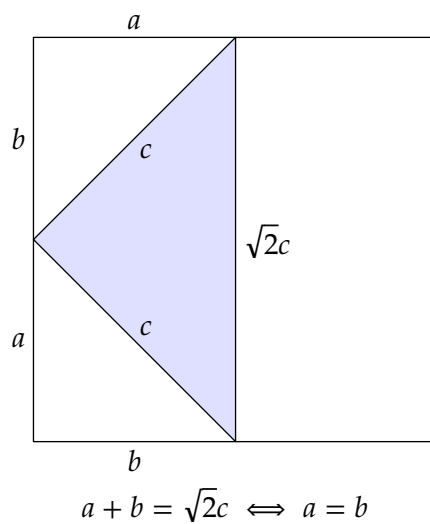
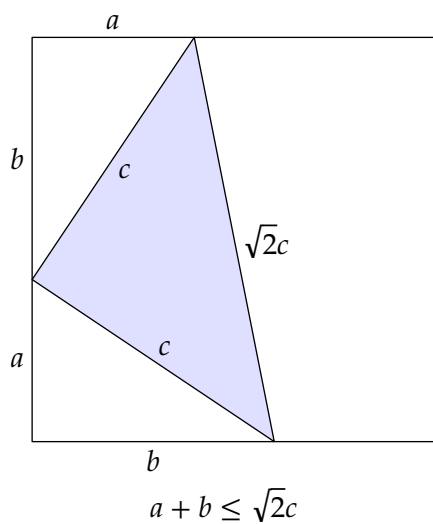
— Klara Pinter

A right angle inequality

Let c be the hypotenuse of a right triangle whose other two sides are a and b . Prove that

$$a + b \leq \sqrt{2}c.$$

When does equality hold?



— Canadian Mathematical Olympiad 1969

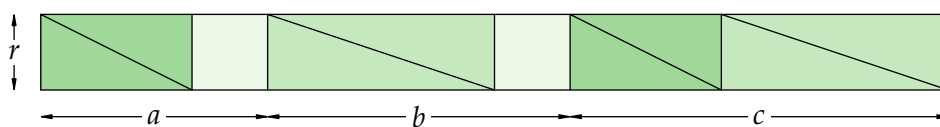
The inradius of a right triangle



$$\text{I. } r = \frac{ab}{a+b+c}$$

$$\text{II. } r = \frac{a+b-c}{2}$$

$$\text{I. } ab = r(a+b+c)$$



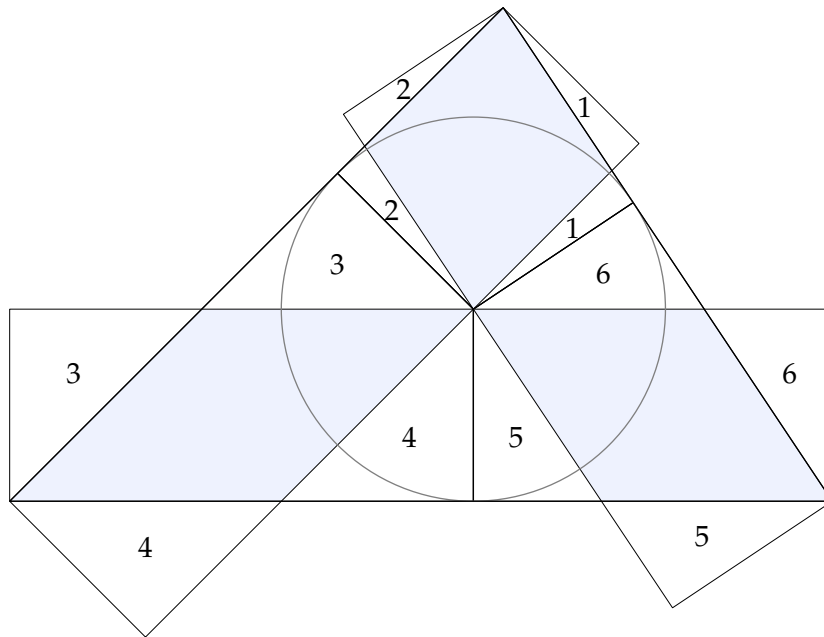
— Liu Hui (3rd century A.D.)

$$\text{II. } c = a + b - 2r$$



The product of the perimeter of a triangle and its inradius is twice the area of the triangle

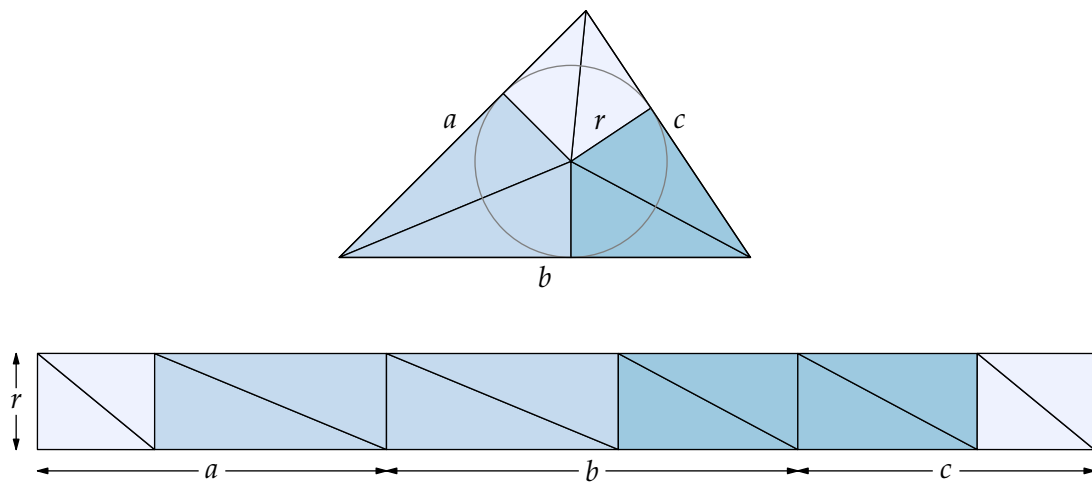
I.



NOTE: Regions bearing the same number are equal in area.

— Grace Lin

II.



Four triangles with equal area



— Steven L. Snover

The triangle of medians has $3/4$ the area of the original triangle



$$\frac{3}{4} \text{area}(\triangle abc) = \text{area}(\triangle m_a m_b m_c)$$

— Norbert Hungerbühler

Heptasection of a triangle

If the one-third points on each side of a triangle are joined to opposite vertices, the resulting central triangle is equal in area to one-seventh that of the initial triangle.



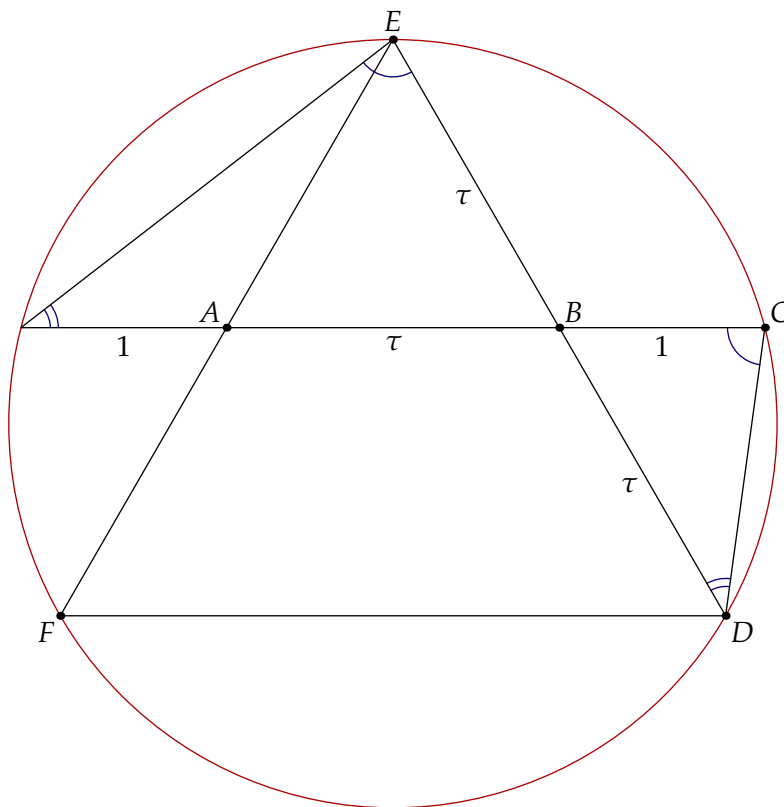
— William Johnston and Joe Kennedy

A Golden Section problem from the *Monthly*

(Problem E3007, *American Mathematical Monthly*, 1983, p.482)

Let A and B be the midpoints of the sides EF and ED of an equilateral triangle DEF . Extend AB to meet the circumcircle (of DEF) at C . Show that B divides AC according to the golden section.

SOLUTION:



$$\tau^2 = \tau + 1$$

— Jan van de Craats

Tiling with squares and parallelograms

If squares are constructed eternally on the sides of the parallelogram, their centres form a square.



— Alfinio Flores

The area of a quadrilateral I

The area of a quadrilateral is less than or equal to half the product of the lengths of its diagonals, with equality if and only if the diagonals are perpendicular.

I. Convex quadrilaterals



$$\begin{aligned}\text{Area} &= \frac{1}{2} \overline{AC} \cdot (h + k) \\ &\leq \frac{1}{2} \overline{AC} \cdot \overline{BD}\end{aligned}$$

II. Concave quadrilaterals



$$\begin{aligned}\text{Area} &= \frac{1}{2} \overline{AC} \cdot (h - k) \\ &\leq \frac{1}{2} \overline{AC} \cdot \overline{BD}\end{aligned}$$

— David B. Sher, Ronald Skurnick, and Dean C. Nataro

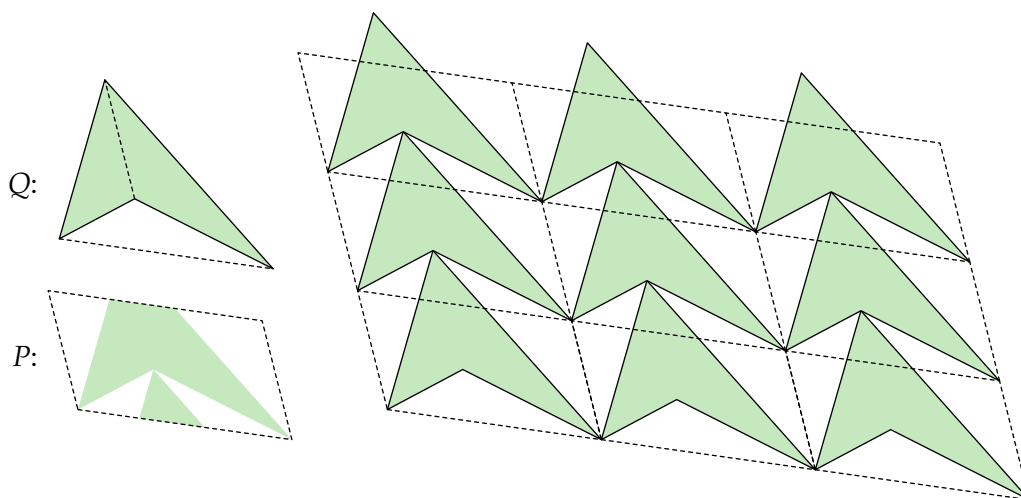
The area of a quadrilateral II

The area of a quadrilateral Q is equal to one-half the area of a parallelogram P whose sides are parallel to and equal in length to the diagonals of Q .

I. Q convex



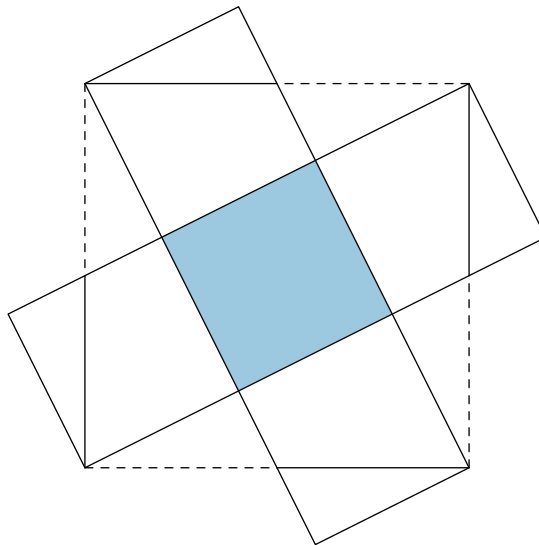
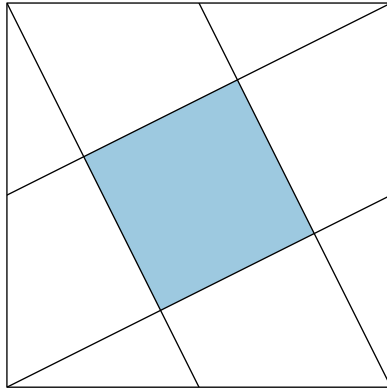
II. Q concave



$$\text{area}(Q) = \frac{1}{2} \text{area}(P)$$

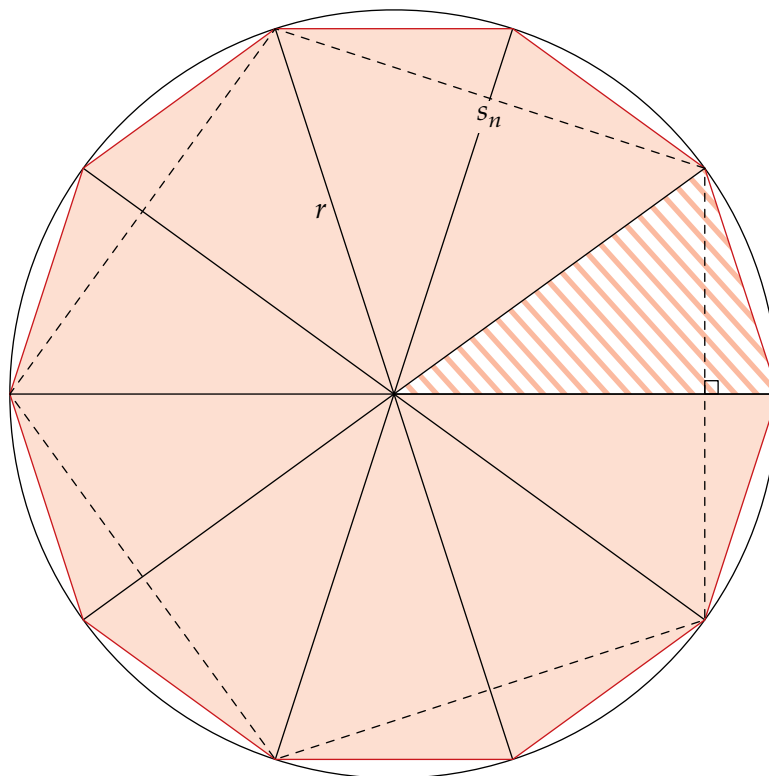
A square within a square

If lines from the vertices of a square are drawn to the mid-points of adjacent sides (as shown in the figure), then the area of the smaller square so produced is one-fifth that of the given square.



Areas and perimeters of regular polygons

The area of a regular $2n$ -gon inscribed in a circle is equal to one-half the radius of the circle times the perimeter of a regular n -gon similarly inscribed ($n \geq 3$).



$$\begin{aligned}\frac{1}{2n} \text{area}(P_{2n}) &= \frac{1}{2} \cdot r \cdot \frac{1}{2}s_n \\ \text{area}(P_{2n}) &= \frac{1}{2}r \cdot ns_n \\ &= \frac{1}{2}r \cdot \text{perimeter}(P_n)\end{aligned}$$

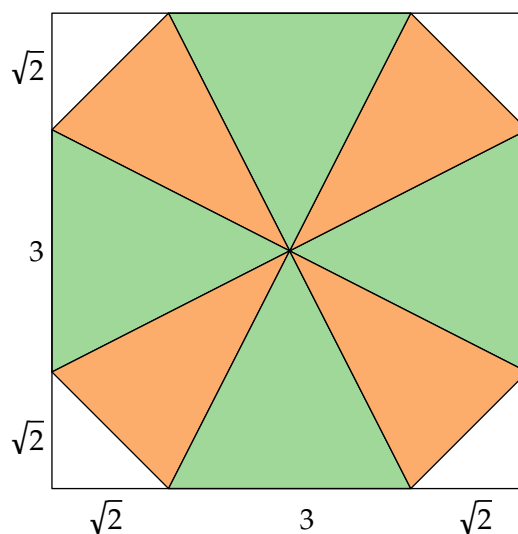
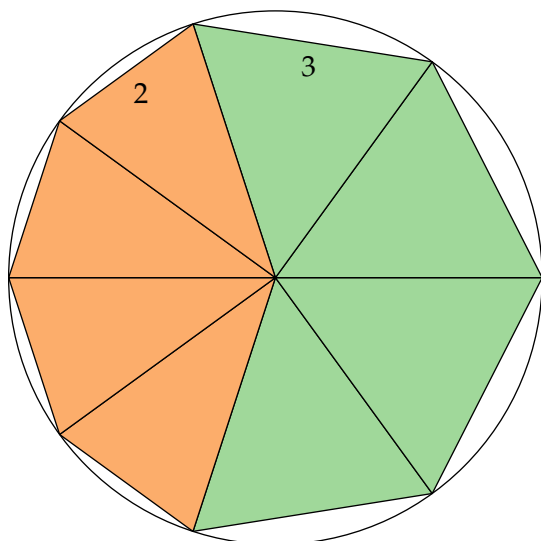
COROLLARY [Bhāskara, *Litāvatī* (India, 12th century AD)]: The area of a circle is equal to one-half the product of its radius and circumference.

The area of a Putnam octagon

(Problem B1, 39th Annual William Lowell Putnam Mathematical Competition, 1978).

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$, with r , s , and t positive integers.

SOLUTION:



$$A = (3 + 2\sqrt{2})^2 - 4 \cdot \frac{1}{2} (\sqrt{2})^2 = 9 + 6\sqrt{2} + 6\sqrt{2} + 8 - 4 = 13 + 12\sqrt{2}$$

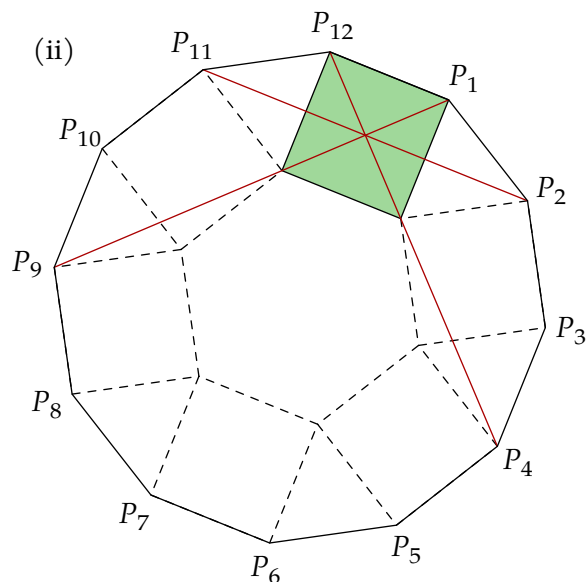
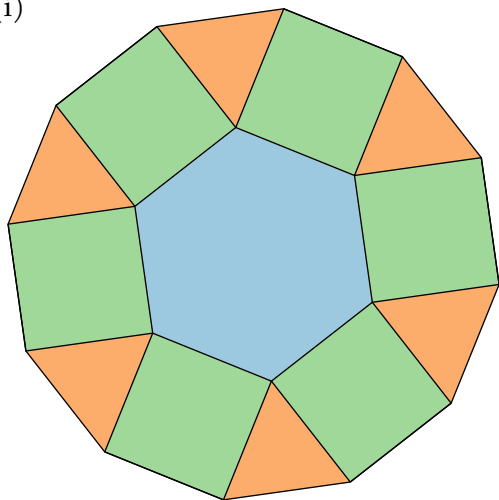
A Putnam dodecagon

(Problem I-1, 24th Annual William Lowell Putnam Mathematical Competition, 1963)

- (i) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.
- (ii) Let P_1, P_2, \dots, P_{12} be the successive vertices of a regular dodecagon. Discuss the intersection(s) of the three diagonals P_1P_9 , P_2P_{11} , and P_4P_{12} .

SOLUTION:

(i)



The area of a regular dodecagon

A regular dodecagon with circumradius one has area three.



— J. Kürshák

Fair allocation of a pizza

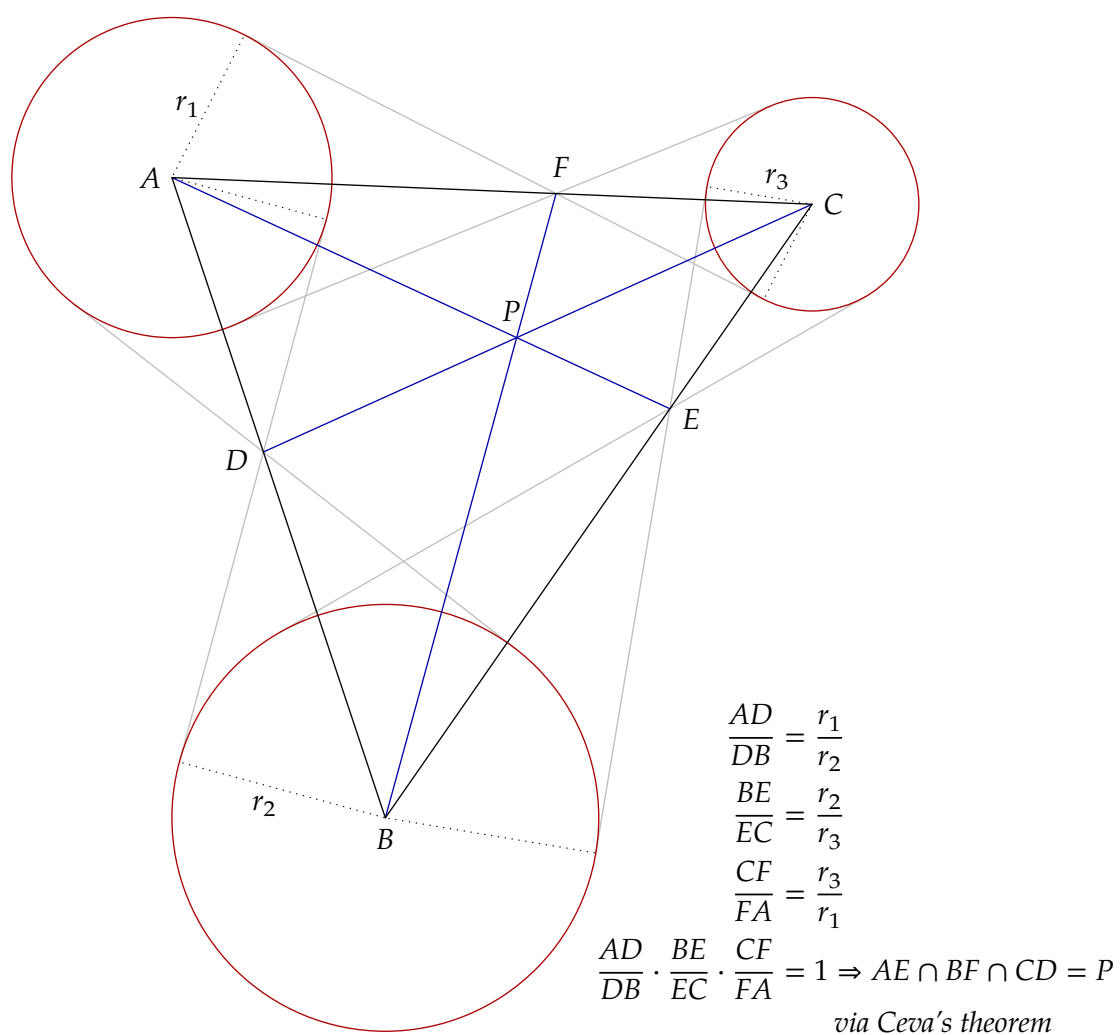
THE PIZZA THEOREM: If a pizza is divided into eight pieces by making cuts at 45° angles through an arbitrary point in the pizza, then the sums of the areas of alternate slices are equal.

PROOF:



A three-circle theorem

Given three non-intersecting, mutually external circles, connect the intersection of the internal common tangents of each pair of circles with the centre of the other circle. Then the resulting three line segments are concurrent.



— R. S. Hu

A constant chord

Suppose two circles Q and R intersect in A and B . A point P on the arc of Q which lies outside R is projected through A and B to determine chord CD of R . Prove that no matter where P is chosen on its arc, the length of chord CD is always the same.



$$\angle C'AC = \angle P'AP = \angle P'BP = \angle D'BD$$

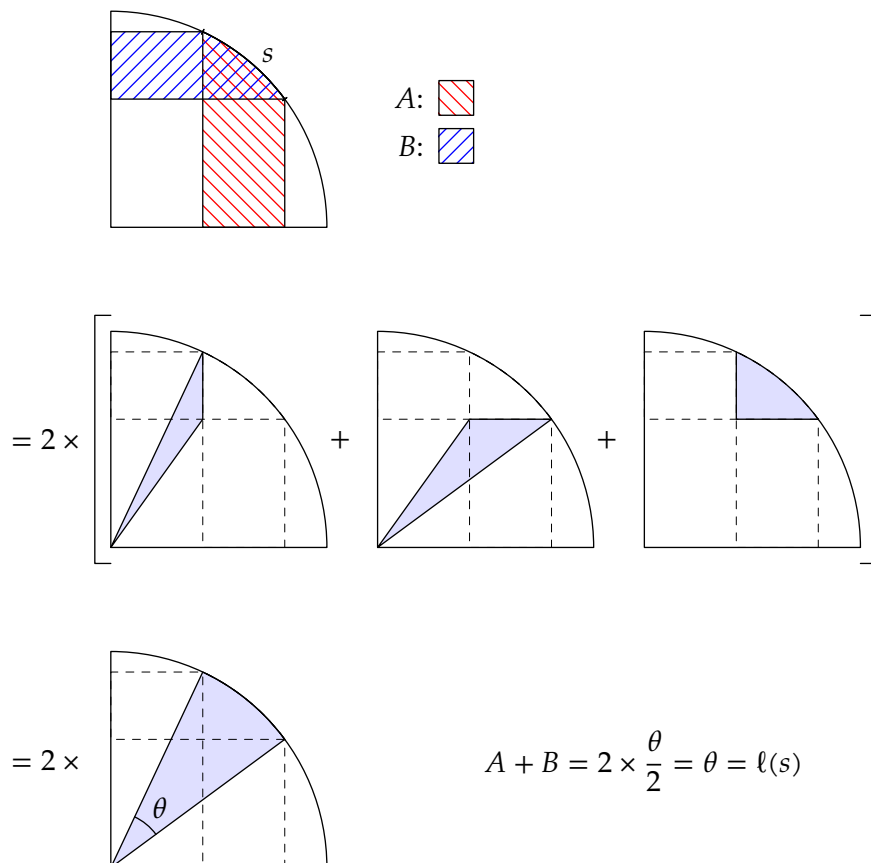
$$\widehat{C'C} = \widehat{D'D}, \quad \widehat{C'D'} = \widehat{CD}$$

$$C'D' = CD$$

A Putnam area problem

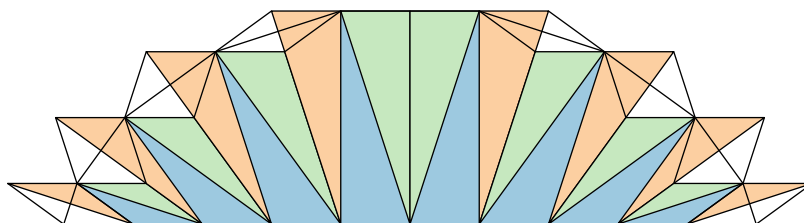
Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x -axis, and let B be the area of the region lying to the right of the y -axis and to the left of s . Prove that $A + B$ depends only on the arc length, and not on the position, of s .

SOLUTION:



The area under a polygonal arch

The area under a polygonal arch generated by one vertex of a regular n -gon rolling along a straight line is three times the area of the polygon.

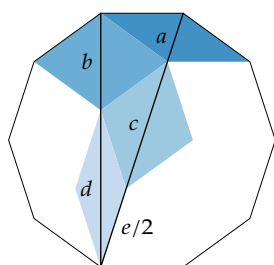
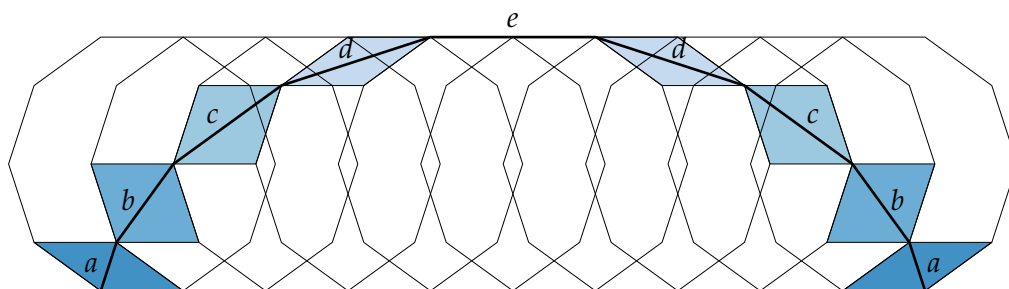


COROLLARY: The area under one arch of a cycloid is three times the area of the generating circle.

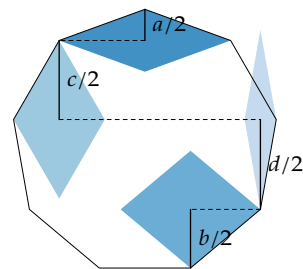
— Philip R. Mallinson

The length of a polygonal arch

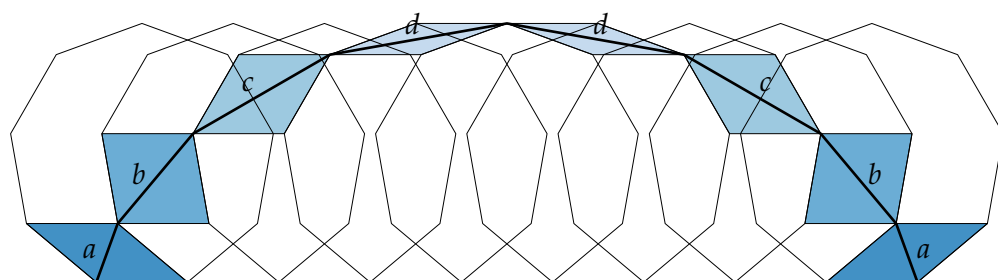
The length of the polygonal arch generated by one vertex of a regular n -gon rolling along a straight line is four times the length of the in-radius plus four times the length of the circum-radius of the n -gon.



Even $n...$



Odd $n...$



COROLLARY: The arc length of one arch of a cycloid is eight times the radius of the generating circle.

— Philip R. Mallinson

The volume of a frustum of a square pyramid



$$P_4 = 3P_5$$

$$P_1 + P_3 = 2P_2 + 4P_4 \Rightarrow P_1 + P_2 + P_3 = 3P_2 + 12P_5 = 3(P_2 + 4P_5) = 3P$$

$$\therefore V = \frac{h}{3} (a^2 + ab + b^2)$$

— Sidney J. Kung

The product of four (positive) numbers in arithmetic progression is always the difference of two squares

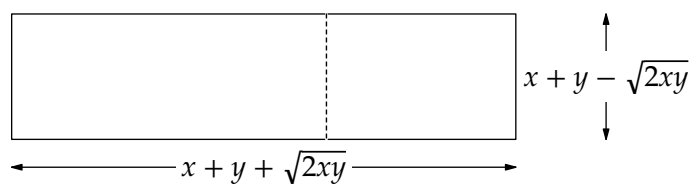
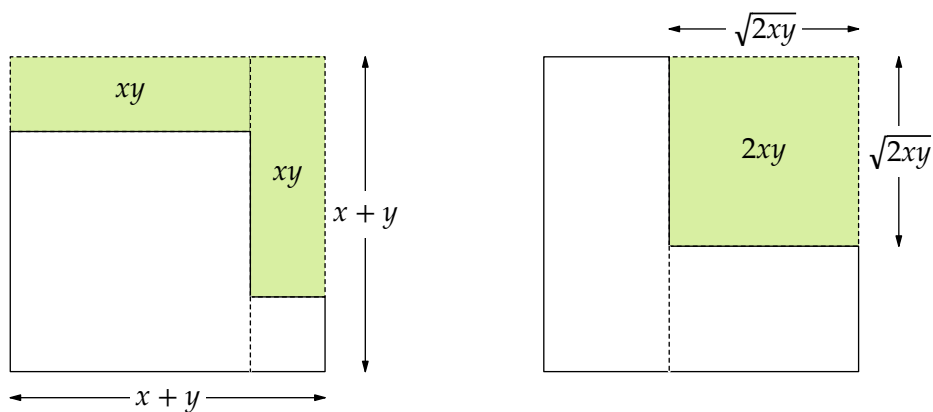
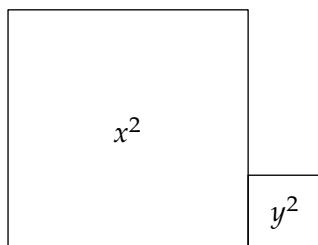


$$a(a+d)(a+2d)(a+3d) = (a^2 + 3ad + d^2)^2 - (d^2)^2$$

— RBN

Algebraic areas III: Factoring the sum of two squares

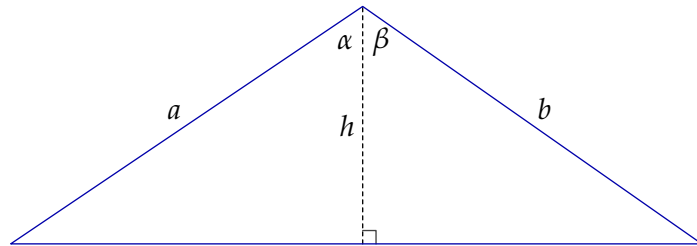
$$x^2 + y^2 = (x + \sqrt{2xy} + y)(x - \sqrt{2xy} + y)$$



Trigonometry, Calculus, & Analytic Geometry

Sine of the sum - II	39
Sine of the sum - III	40
Cosine of the sum	41
Geometry of addition formulas	42
Geometry of subtraction formulas	43
The difference identity for tangents I	44
The difference identity for tangents II	45
One figure, six identities	46
The double-angle formulas II	48
The double-angle formulas III (via the laws of sines and cosines)	49
The sum-to-product identities I	50
The difference-to-product identities I	51
The sum-to-product identities II	52
The difference-to-product identities II	53
Adding like sines	54
A complex approach to the laws of sines and cosines	55
Eisenstein's duplication formula	56
A familiar limit for e	57
A common limit	58
Geometric evaluation of a limit	59
The derivative of the inverse sine	60
The logarithm of a product	61
An integral of a sum of reciprocal powers	62
The arctangent integral	63
The method of last resort — Weierstrass substitution	64
The trapezoidal rule — for increasing functions	65
Construction of a hyperbola	66
The focus and directrix of an ellipse	67

Sine of the sum - II



$$\alpha, \beta \in (0, \pi/2) \implies h = a \cos \alpha = b \cos \beta$$



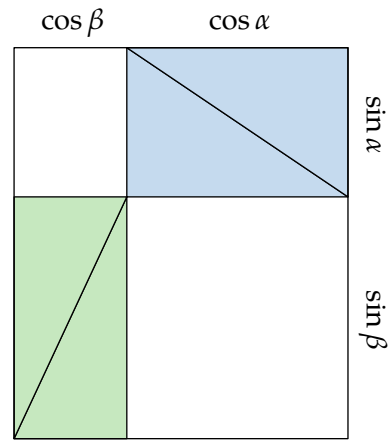
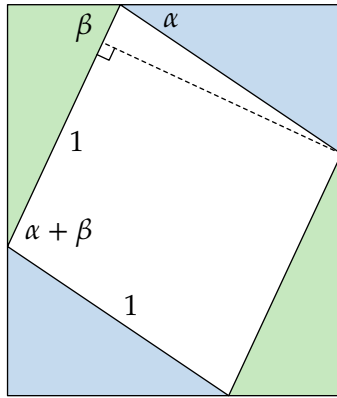
$$\begin{aligned} \frac{1}{2}ab \sin(\alpha + \beta) &= \frac{1}{2}ah \sin \alpha + \frac{1}{2}bh \sin \beta \\ &= \frac{1}{2}ab \cos \beta \sin \alpha + \frac{1}{2}ba \cos \alpha \sin \beta \\ \therefore \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{aligned}$$

— Christopher Brüningsen

Sine of the sum – III

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

I.



II.



— Volker Priebe and Edgar A. Ramos

Cosine of the sum

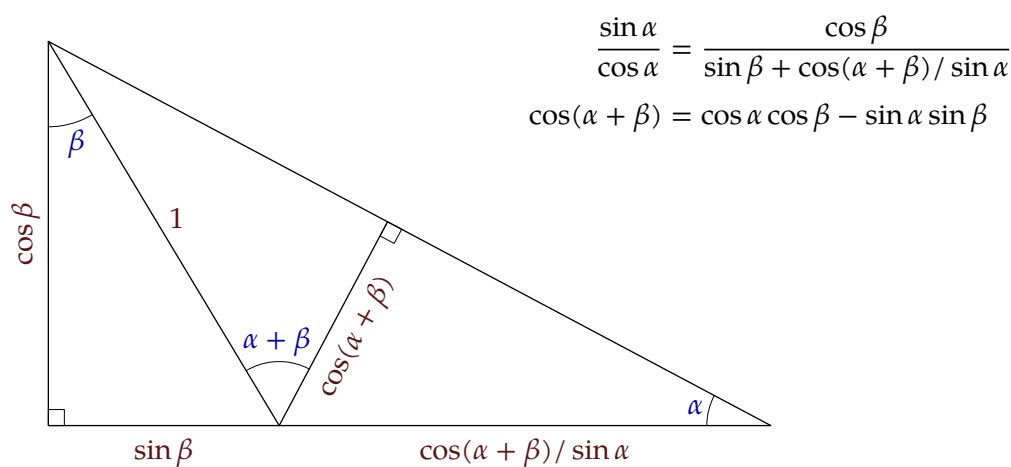


$$\frac{1}{2}ab \sin\left(\frac{\pi}{2} - (\alpha + \beta)\right) = \frac{1}{2}b \cos \alpha \cdot a \cos \beta - \frac{1}{2}b \sin \alpha \cdot a \sin \beta$$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

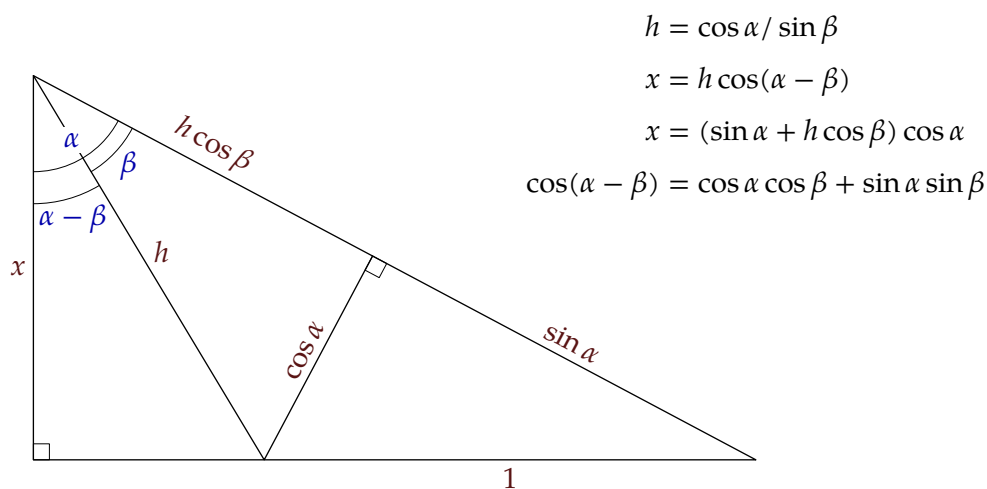
— Sidney H. Kung

Geometry of addition formulas



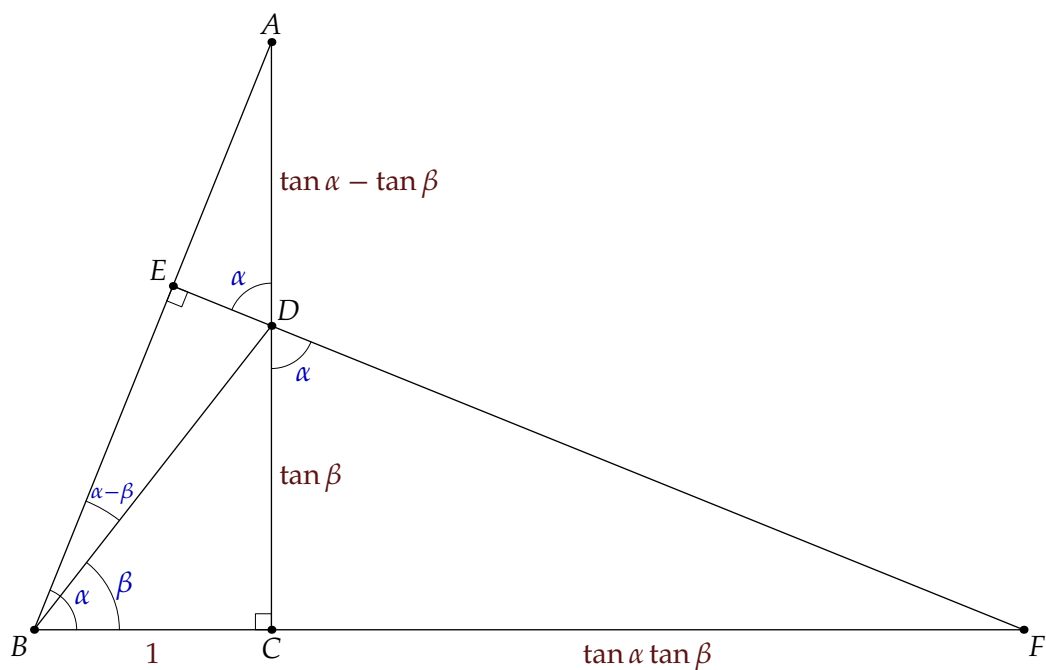
— Leonard M. Smiley

Geometry of subtraction formulas



— Leonard M. Smiley

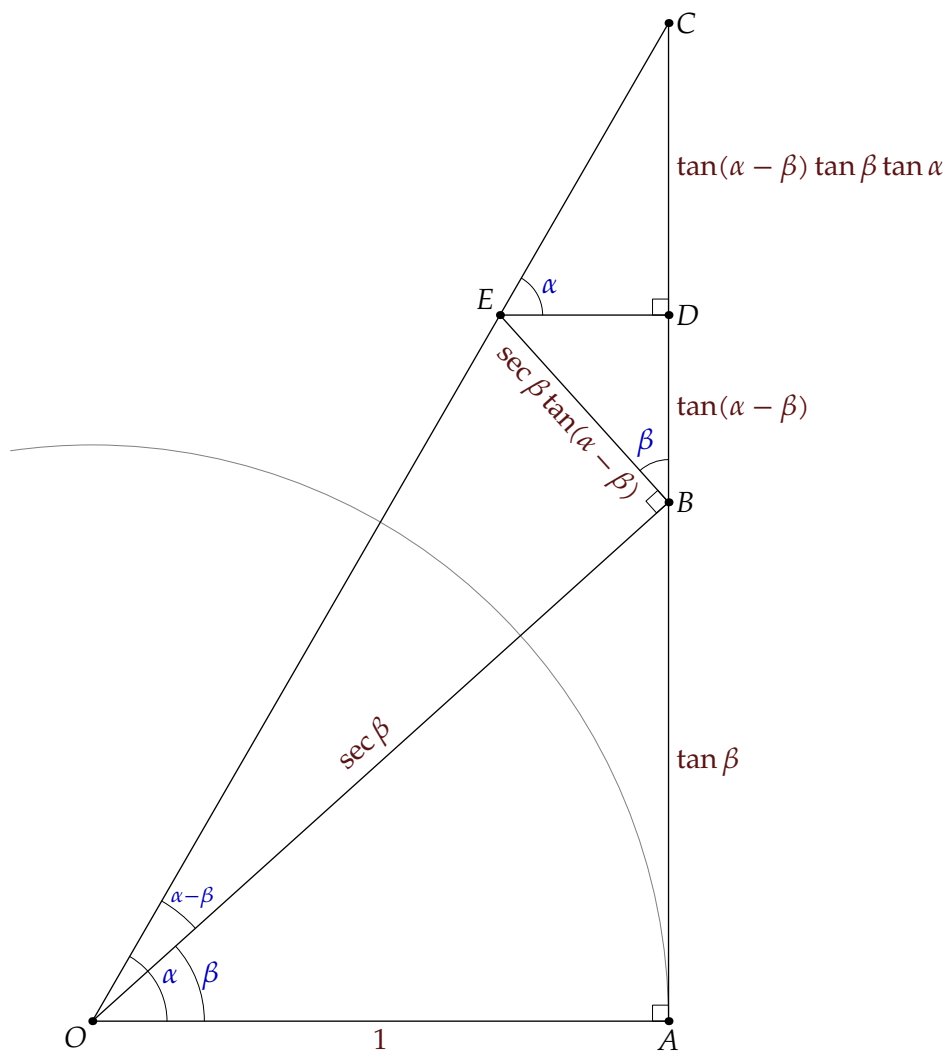
The difference identity for tangents I



$$\begin{aligned} \frac{BF}{BE} &= \frac{AD}{DE} \\ \therefore \tan(\alpha - \beta) &= \frac{DE}{BE} = \frac{AD}{BF} = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \end{aligned}$$

— Guanshen Ren

The difference identity for tangents II



$$AC - AB = BD + DC$$

$$\therefore \tan \alpha - \tan \beta = \tan(\alpha - \beta) + \tan \alpha \tan \beta \tan(\alpha - \beta)$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

— Fukuzo Suzuki

One figure, six identities



The figure

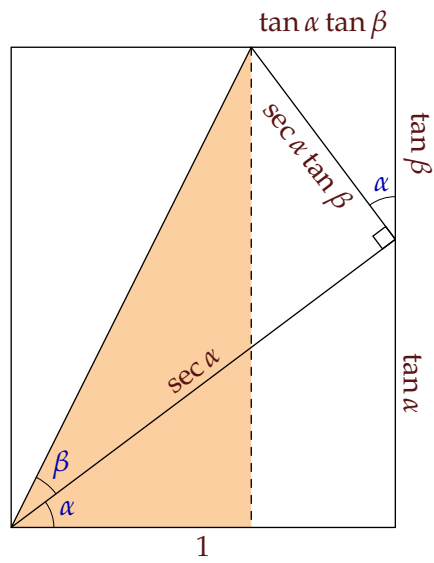
$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}$$



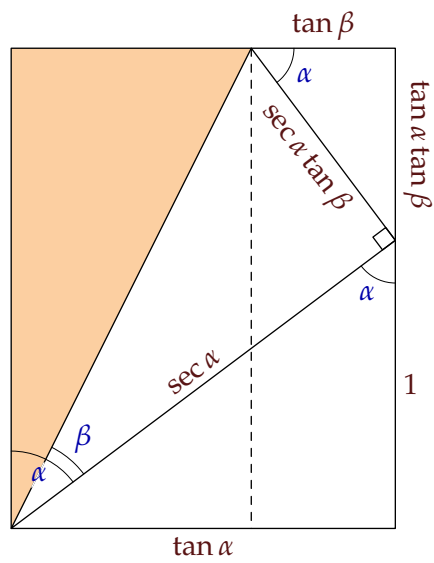
$$\begin{aligned}\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta\end{aligned}$$



$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 + \tan \alpha \tan \beta}$$



$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$



The double-angle formulas II



$$2 \sin \theta \cos \theta = \sin 2\theta$$



$$2 \cos^2 \theta = 1 + \cos 2\theta$$

— Yihnan David Gau

The double-angle formulas III (via the laws of sines and cosines)

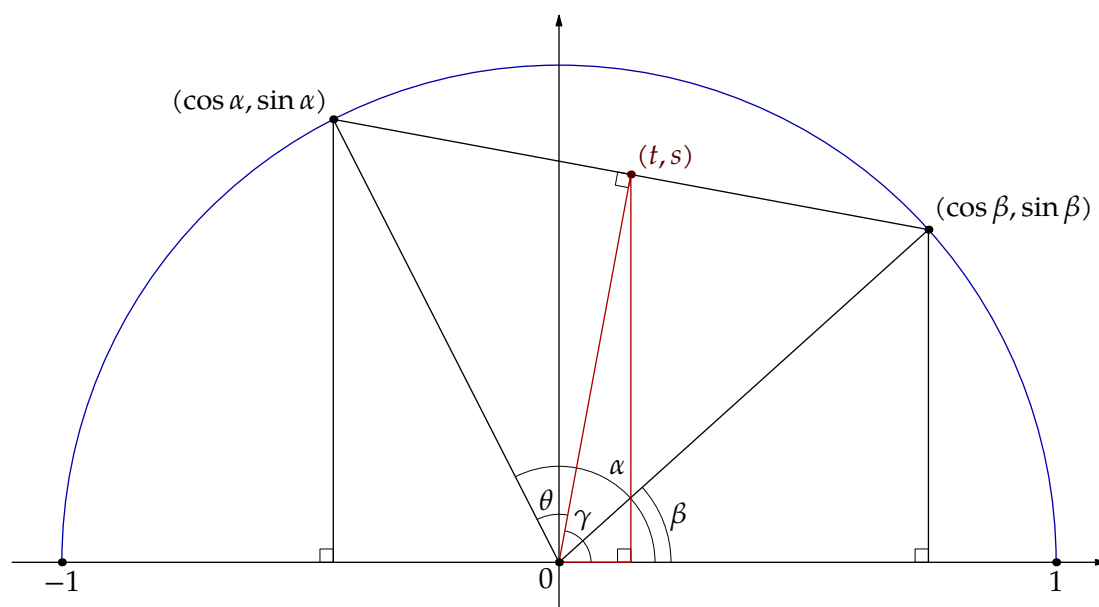


$$\frac{\sin 2\theta}{2 \sin \theta} = \frac{\sin(\pi/2 - \theta)}{1} = \cos \theta$$
$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$(2 \sin \theta)^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 2\theta$$
$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

— Sidney H. Kung

The sum-to-product identities I



$$\theta = \frac{\alpha - \beta}{2}, \quad \gamma = \frac{\alpha + \beta}{2}$$

$$\frac{\sin \alpha + \sin \beta}{2} = s = \cos \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

$$\frac{\cos \alpha + \cos \beta}{2} = t = \cos \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

— Sidney H. Kung

The difference-to-product identities I



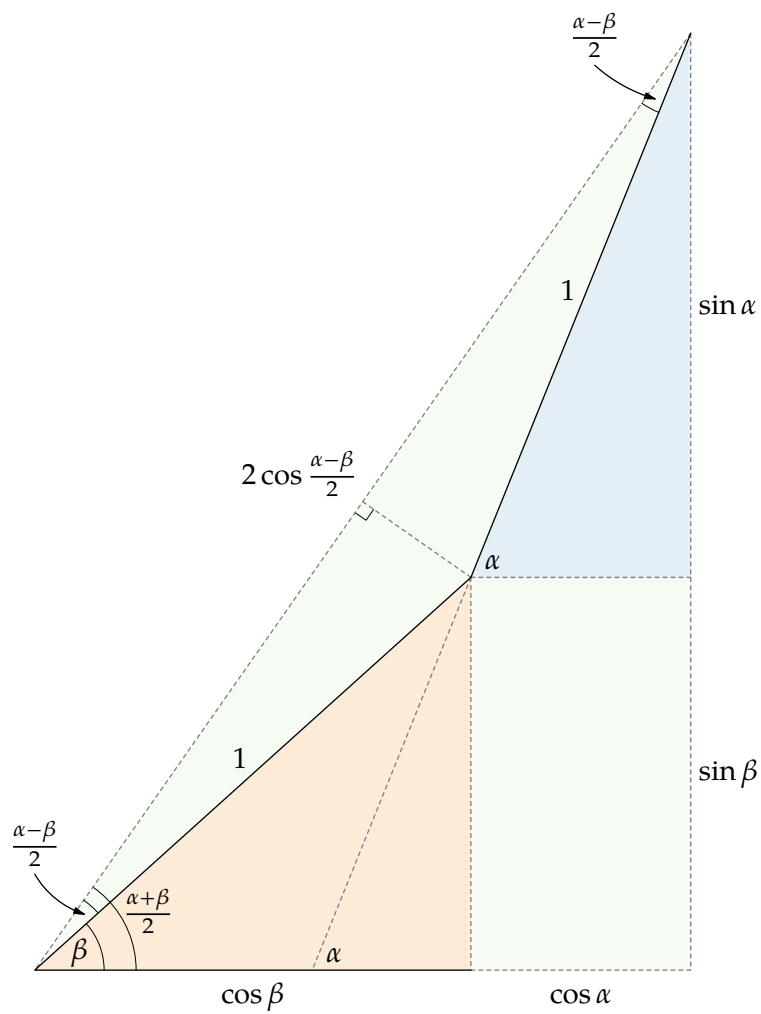
$$\theta = \frac{\alpha - \beta}{2}, \quad \gamma = \frac{\alpha + \beta}{2}$$

$$\sin \alpha - \sin \beta = v = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$\cos \beta - \cos \alpha = u = 2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

— Sidney H. Kung

The sum-to-product identities II



$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha + \sin \beta = 2 \cos \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

— Yukio Kobayashi

The difference-to-product identities II



$$\cos \beta - \cos \alpha = 2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

— Yukio Kobayashi

Adding like sines



$$R_\phi = \sqrt{A^2 + B^2 + 2AB \cos \phi}, \quad \tan \theta = \frac{B \sin \phi}{A + B \cos \phi}$$

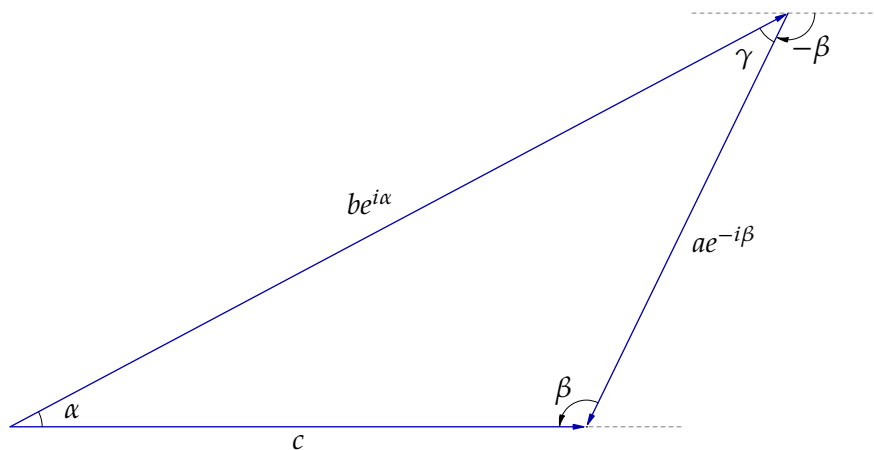
$$A \sin x + B \sin(x + \phi) = R_\phi \sin(x + \theta)$$

$$\phi = \pi/2 \Rightarrow \tan \theta = B/A$$

$$\therefore A \sin x + B \cos x = \sqrt{A^2 + B^2} \sin(x + \theta)$$

— Rick Mabry and Paul Deiermann

A complex approach to the laws of sines and cosines



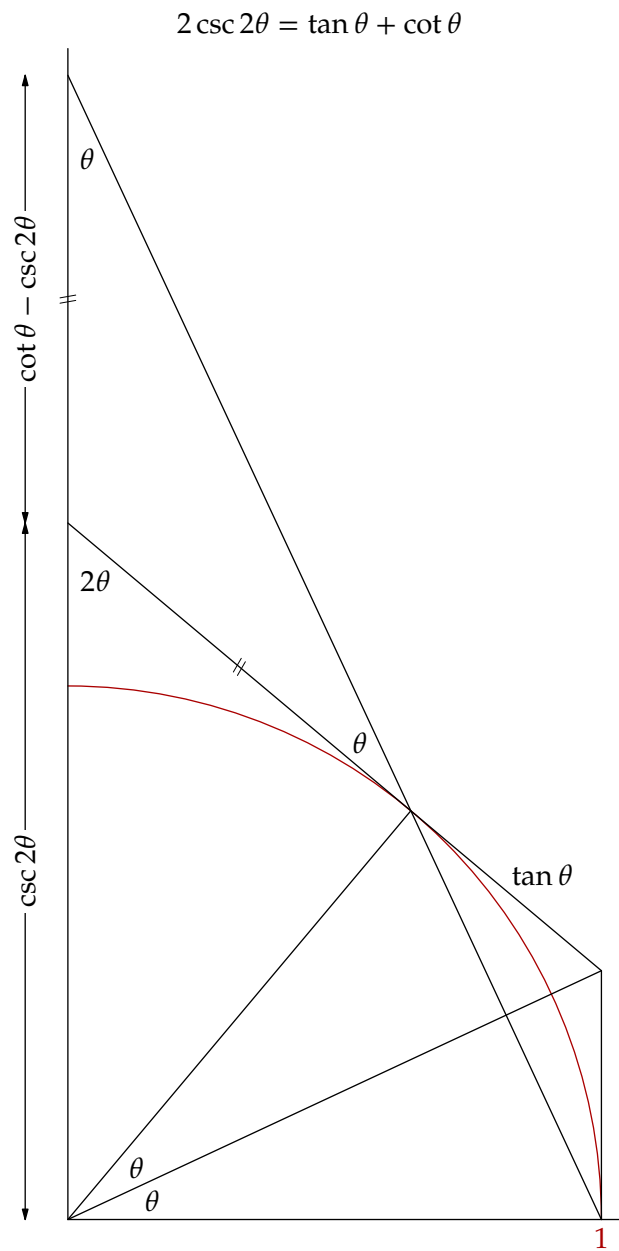
$$c = be^{i\alpha} + ae^{-i\beta} = (b \cos \alpha + a \cos \beta) + i(b \sin \alpha - a \sin \beta)$$

if c is real, then $b \sin \alpha - a \sin \beta = 0$, hence $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$

$$\begin{aligned} c^2 = |c|^2 &= (b \cos \alpha + a \cos \beta)^2 + (b \sin \alpha - a \sin \beta)^2 \\ &= a^2 + b^2 + 2ab \cos(\alpha + \beta) \\ &= a^2 + b^2 - 2ab \cos \gamma \end{aligned}$$

— William V. Grounds

Eisenstein's duplication formula



G. Eisenstein, *Mathematische Werke*, Chelsea, NY. 1975, p.411

A familiar limit for e

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$



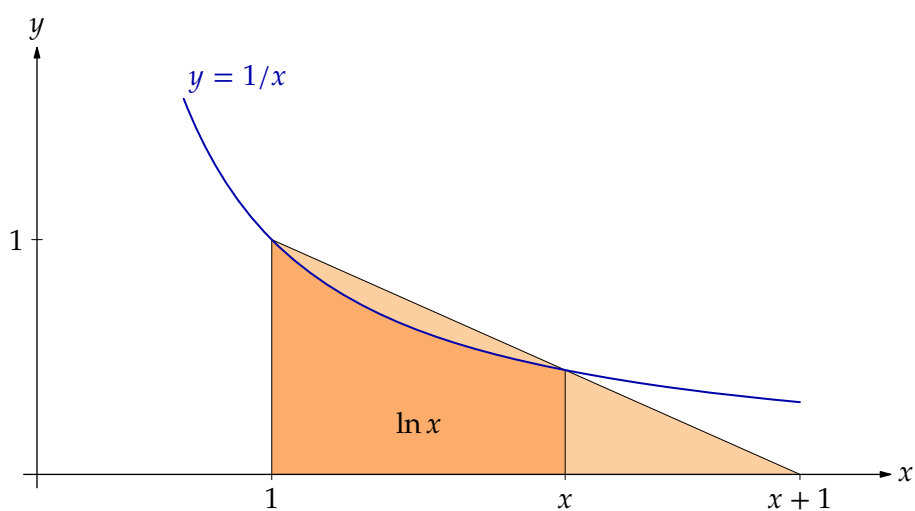
$$\frac{1}{n} \cdot \frac{n}{n+1} \leq \ln \left(1 + \frac{1}{n}\right) \leq \frac{1}{n} \cdot 1$$

$$\frac{n}{n+1} \leq n \cdot \ln \left(1 + \frac{1}{n}\right) \leq 1$$

$$\therefore \lim_{n \rightarrow \infty} \ln \left(\left(1 + \frac{1}{n}\right)^n \right) = 1$$

A common limit

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = 0$$



$$\ln x < \frac{1}{2}x$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^{x-\ln x}} = 0$$

— Alan H. Stein and Dennis McGavran

Geometric evaluation of a limit

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}} = 2$$



— Guanshen Ren

The derivative of the inverse sine



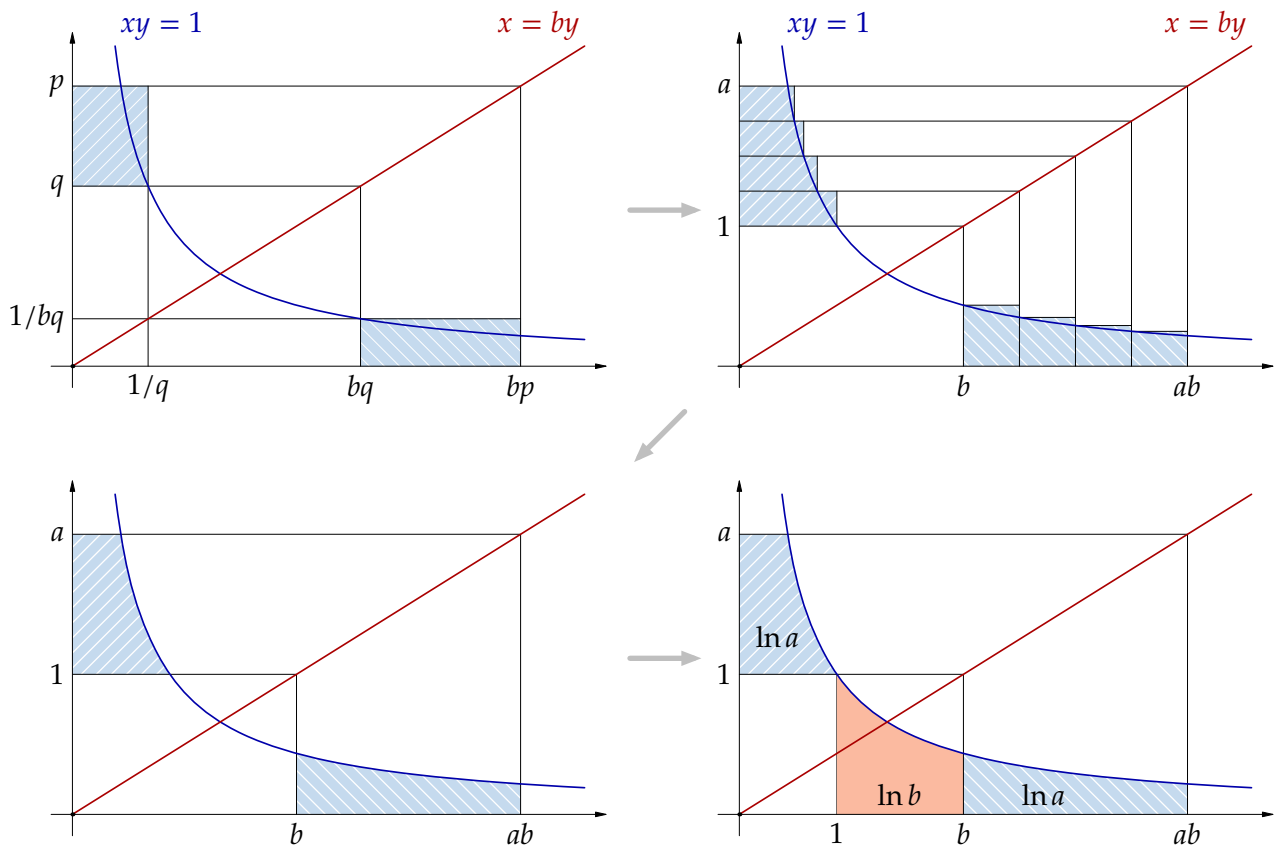
$$L = \sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

$$\therefore \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

— Craig Johnson

The logarithm of a product

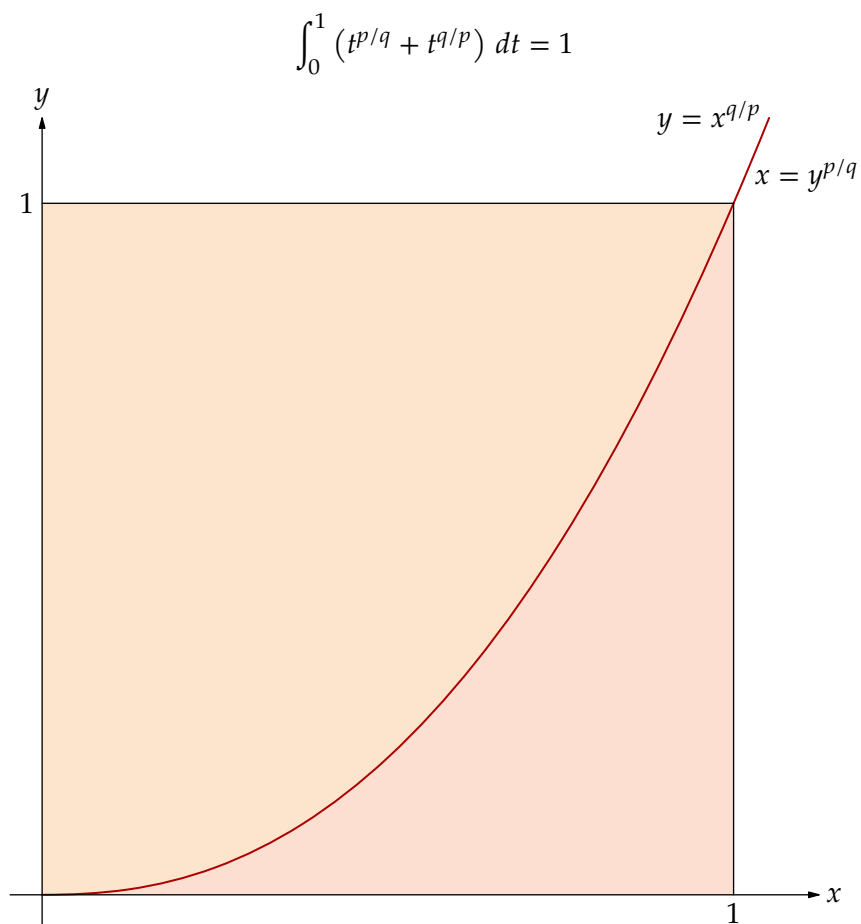
$$\ln ab = \ln a + \ln b$$



$$\text{Area}(\text{shaded}) = \text{Area}(\text{shaded})$$

— Jeffery Ely

An integral of a sum of reciprocal powers



— Peter R. Newbury

The arctangent integral



— Aage Bondesen

The method of last resort — Weierstrass substitution



$$u = \tan \frac{\theta}{2}, \quad DE = 2 \sin \frac{\theta}{2} = \frac{2u}{\sqrt{1+u^2}}$$

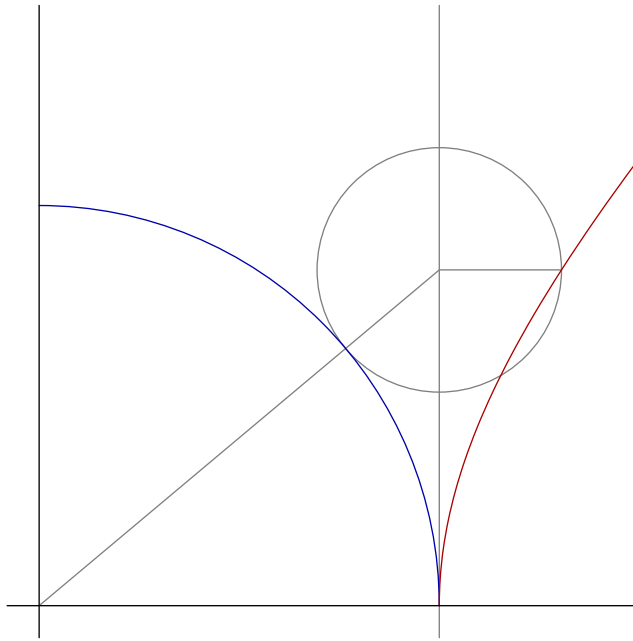
$$\frac{CE}{DE} = \frac{OA}{BA} \implies \sin \theta = \frac{2u}{1+u^2}$$

$$\frac{CD}{DE} = \frac{OB}{BA} \implies \cos \theta = \frac{1-u^2}{1+u^2}$$

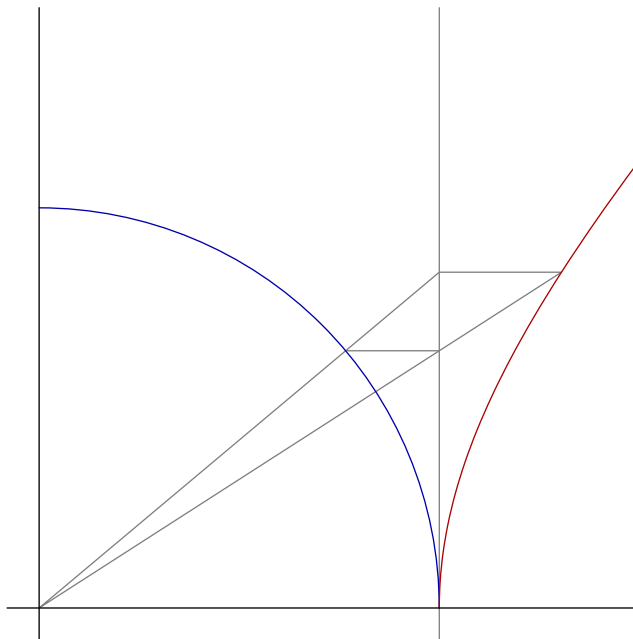
— Paul Deiermann

Construction of a hyperbola

I.



II.



— Ernest J. Eckert

The focus and directrix of an ellipse

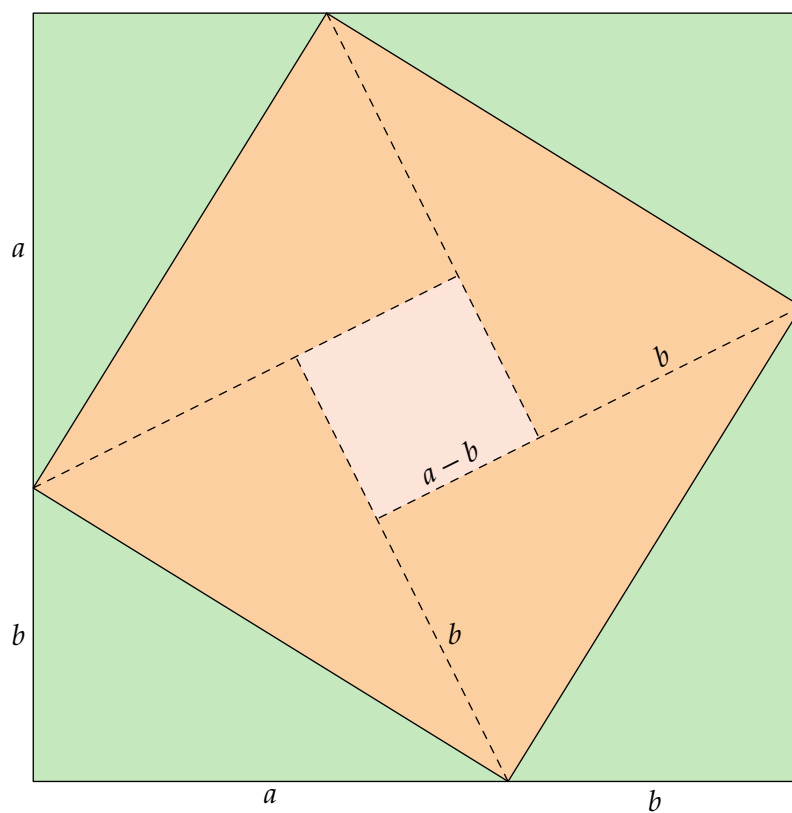


— Michel Bataille

Inequalities

The arithmetic mean – geometric mean inequality IV	69
The arithmetic mean – geometric mean inequality V	70
The arithmetic mean – geometric mean inequality VI	71
The arithmetic mean – geometric mean inequality for three positive numbers	72
The arithmetic-geometric-harmonic mean inequality	73
The arithmetic-logarithmic-geometric mean inequality	74
The mean of the squares exceeds the square of the mean	75
The Chebyshev inequality for positive monotone sequences	76
Jordan’s inequality	77
Young’s inequality	78

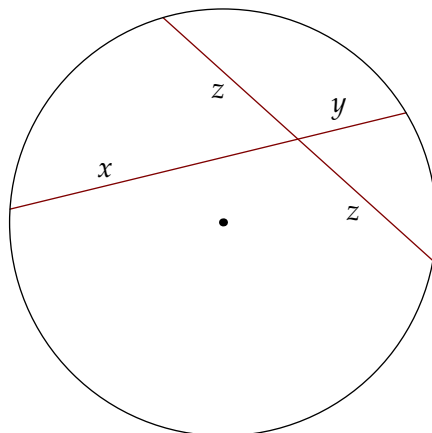
The arithmetic mean – geometric mean inequality IV



$$(a+b)^2 \geq 4ab \implies \frac{a+b}{2} \geq \sqrt{ab}$$

— Ayoub B. Ayoub

The arithmetic mean – geometric mean inequality V



$$z^2 = xy$$



$$d < c \implies x + y > 2\sqrt{xy}$$



$$d = c = 0 \implies x + y = 2\sqrt{xy}$$

— Sidney H. Kung

The arithmetic mean – geometric mean inequality VI



$$0 < a < b, 0 < t < 1 \Rightarrow (1-t)a + tb > a^{1-t}b^t$$

$$t = \frac{1}{2} \Rightarrow \frac{a+b}{2} > \sqrt{ab}$$

— Michael K. Brozinsky

The arithmetic mean – geometric mean inequality for three positive numbers

LEMMA: $ab + bc + ac \leq a^2 + b^2 + c^2$



THEOREM: $3abc \leq a^3 + b^3 + c^3$



— Claudi Alsina

The arithmetic-geometric-harmonic mean inequality

$$a, b > 0 \implies \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$



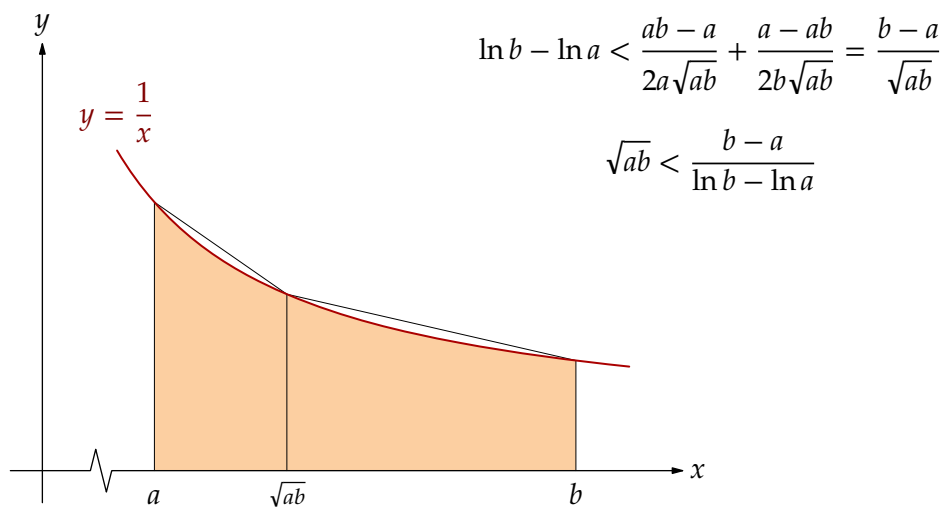
$$\overline{AM} = \frac{a+b}{2}, \quad \overline{GM} = \sqrt{ab}, \quad \overline{HM} = \frac{2ab}{a+b},$$

$$\overline{AM} \geq \overline{GM} \geq \overline{HM}.$$

— Pappus of Alexandria (circa A.D. 320)

The arithmetic-logarithmic-geometric mean inequality

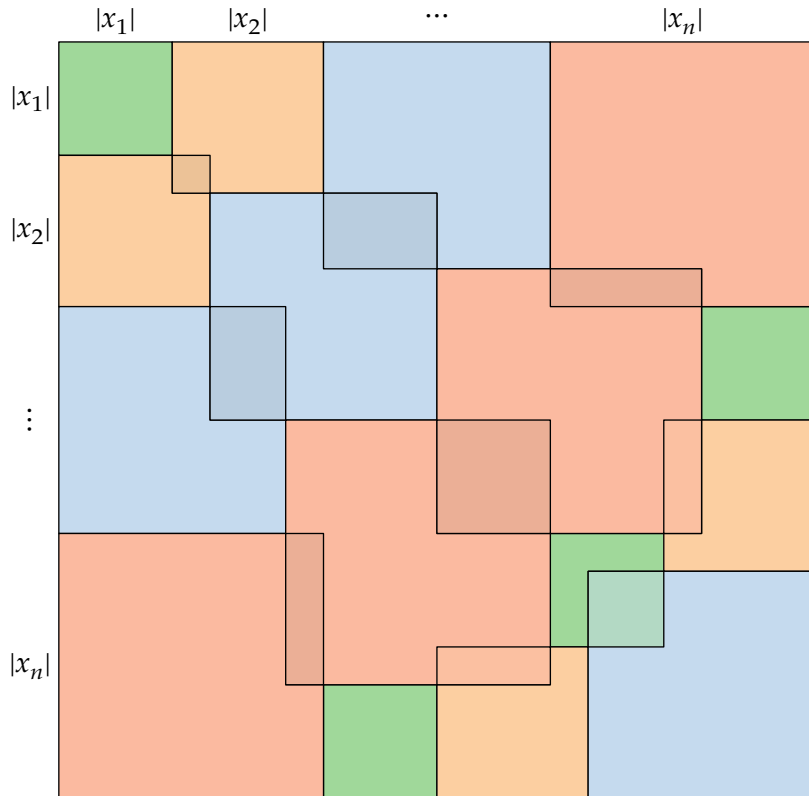
$$b > a > 0 \implies \frac{a+b}{2} > \frac{b-a}{\ln b - \ln a} > \sqrt{ab}$$



— RBN

The mean of the squares exceeds the square of the mean

$$\frac{1}{n} \sum_{i=1}^n x_i^2 \geq \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^2$$



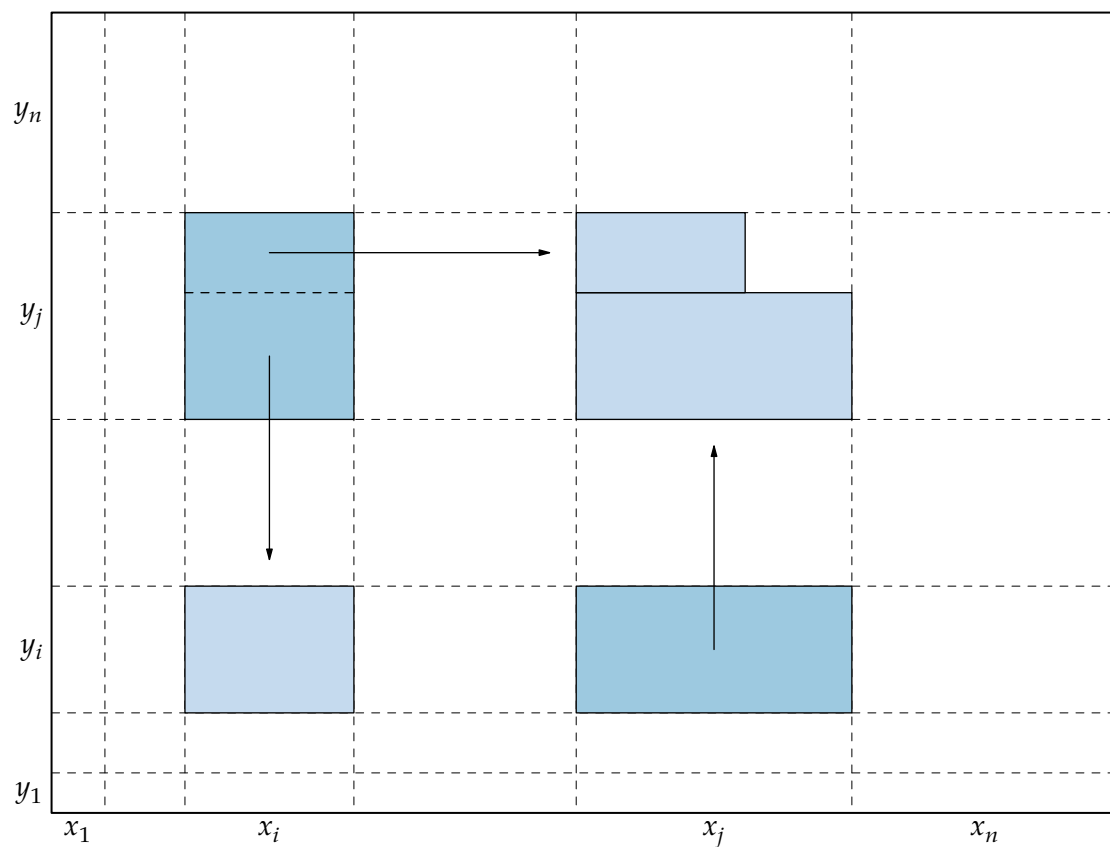
$$n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (|x_1| + |x_2| + \cdots + |x_n|)^2 \geq (x_1 + x_2 + \cdots + x_n)^2$$

$$\therefore \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \geq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^2$$

— RBN

The Chebyshev inequality for positive monotone sequences

$$\sum_{i=1}^n x_i \sum_{i=1}^n y_i \leq \sum_{i=1}^n x_i y_i$$



$$x_i < x_j \text{ \& } y_i < y_j \quad \Rightarrow \quad x_i y_j + x_j y_i \leq x_i y_i + x_j y_j$$

$$\therefore (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) \leq n(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

— RBN

Jordan's inequality

$$0 \leq x \leq \frac{\pi}{2} \Rightarrow \frac{2x}{\pi} \leq \sin x \leq x$$



$$\begin{aligned} OB = OM + MP &\geq OA \Rightarrow \widehat{PBQ} \geq \widehat{PAQ} \geq \overline{PQ} \\ &\Rightarrow \pi \sin x \geq 2x \geq 2 \sin x \\ &\Rightarrow \frac{2x}{\pi} \leq \sin x \leq x \end{aligned}$$

— Feng Yuefeng

Young's inequality

W. H. Young, "On classes of summable functions and their Fourier series", *Proc. Royal Soc. (A)*, 87 (1912) 225–229.

THEOREM: Let ϕ and ψ be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each others. Then for $a, b \geq 0$ we have

$$ab \leq \int_0^a \phi(x) dx + \int_0^b \psi(y) dy$$

with equality if and only if $b = \phi(a)$.

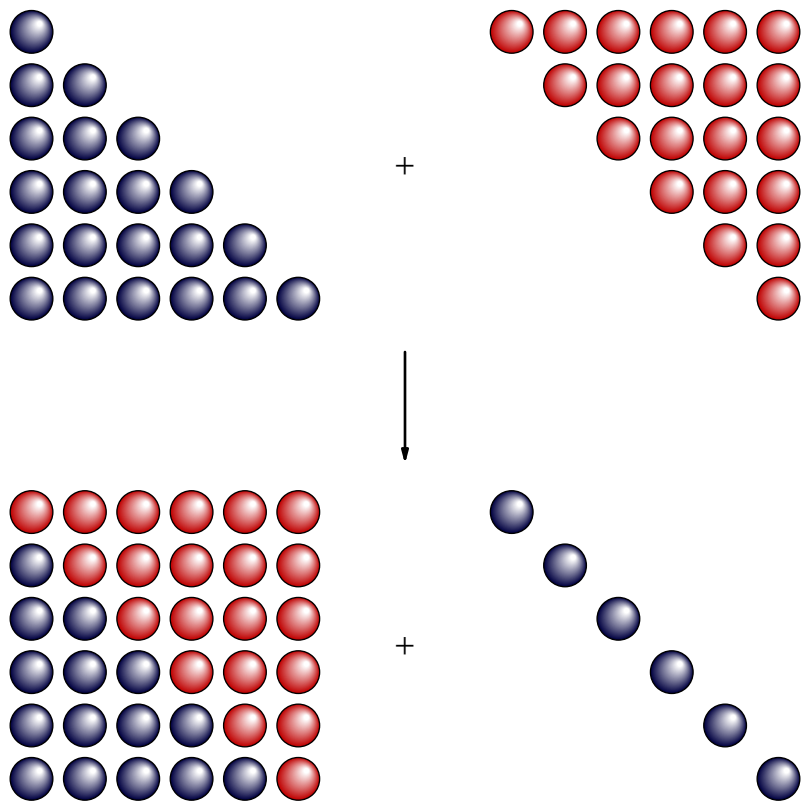
PROOF:



Integer sums

Sums of integers III	80
Sums of consecutive positive integers	81
Consecutive sums of consecutive integers II	82
Sums of squares VI	83
Sums of squares VII	84
Sums of squares VIII	85
Sums of squares IX (via centroids)	86
Sums of odd squares	87
Sums of sums of squares	88
Pythagorean runs	89
Sums of cubes VII	90
Sums of integers as sums of cubes	91
The square of any odd number is the difference between two triangular numbers	92
Triangular numbers mod 3	93
Counting triangular numbers IV: Counting cannonballs	94
Alternating sums of triangular numbers	95
The sum of the squares of consecutive triangular numbers is triangular	96
Recursion for triangular numbers	97
Identities for triangular numbers	98
More identities for triangular numbers	99
Identities for pentagonal numbers	100
Sums of octagonal numbers	101

Sums of integers III



$$1 + 2 + \cdots + n = \frac{1}{2} (n^2 + n)$$

— S. J. Barlow

Sums of consecutive positive integers

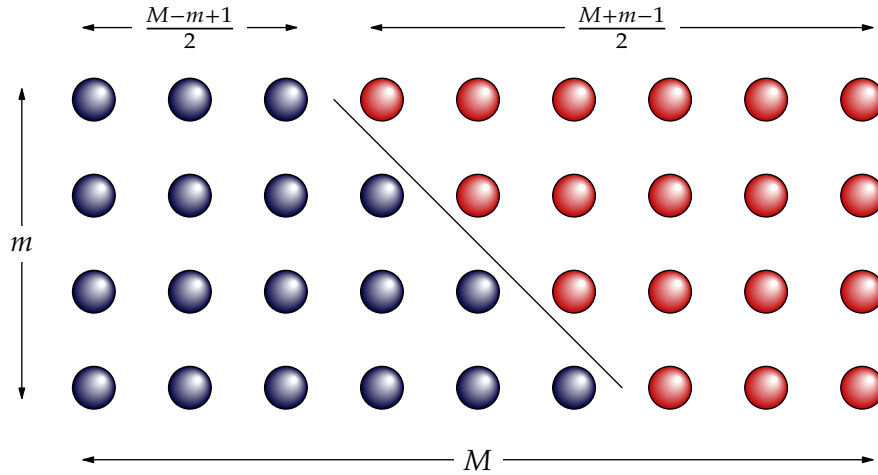
Every integer $N > 1$, not a power of two, can be expressed as the sum of two or more positive integers.

$$N = 2^n(2k + 1) \quad (n \geq 0, k \geq 1)$$

$$m = \min \{2^{n+1}, 2k + 1\}$$

$$M = \max \{2^{n+1}, 2k + 1\}$$

$$2N = mM$$

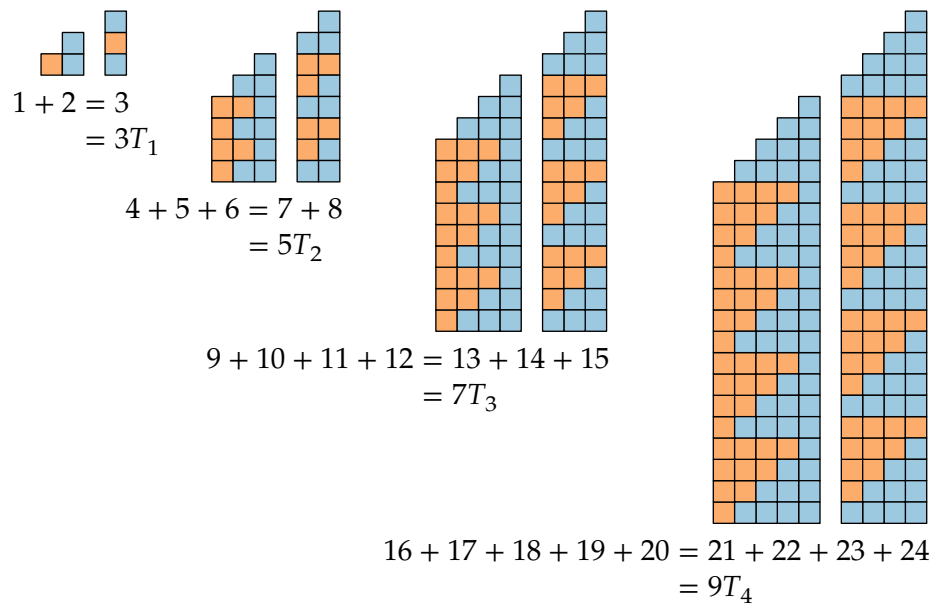


$$N = \left(\frac{M-m+1}{2} \right) + \left(\frac{M-m+1}{2} + 1 \right) + \cdots + \left(\frac{M+m-1}{2} \right)$$

— C. L. Frenzen

Consecutive sums of consecutive integers II

$$T_k = 1 + 2 + \cdots + k \quad \Rightarrow$$



$$\begin{aligned}
 n^2 + (n^2 + 1) + \cdots + (n^2 + n) &= (n^2 + n + 1) + \cdots + (n^2 + 2n) \\
 &= (2n + 1)T_n
 \end{aligned}$$

Sums of squares VI

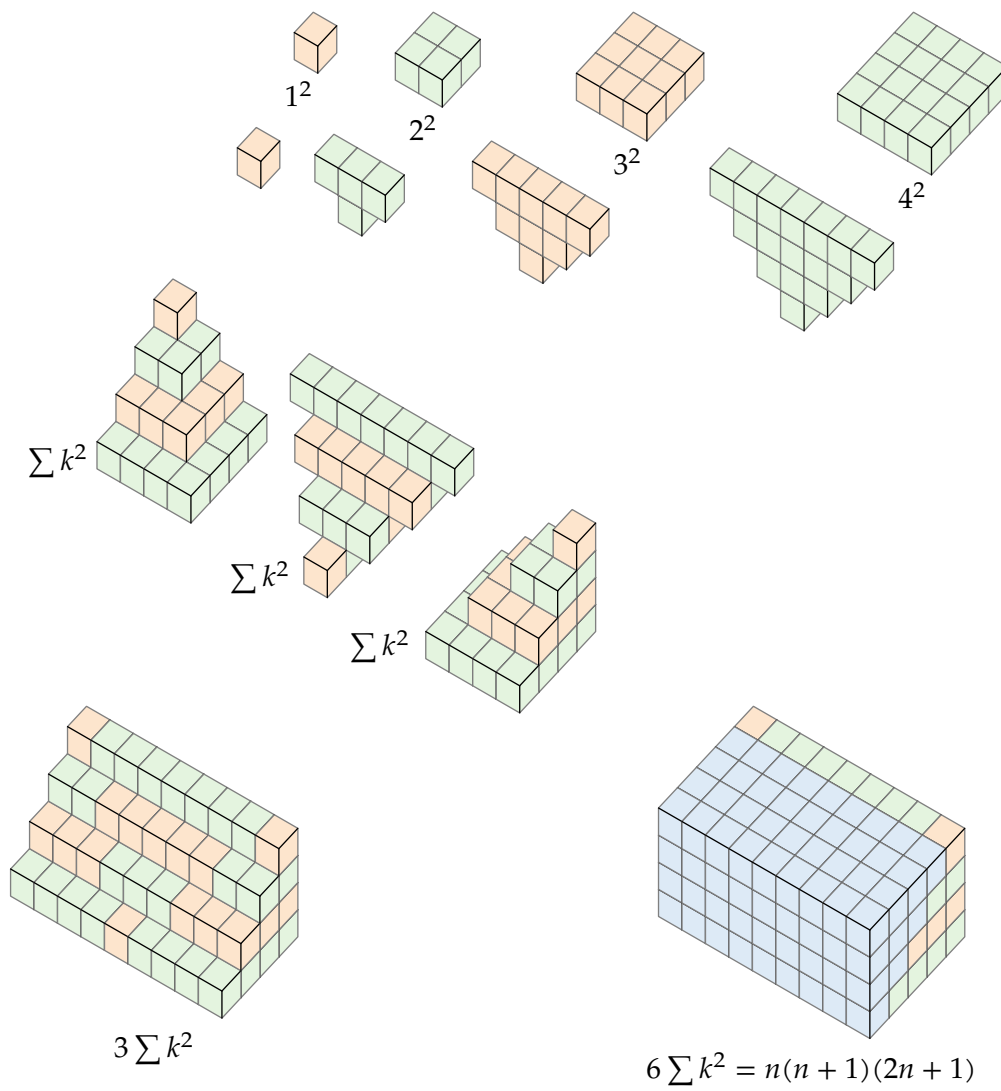


$$\begin{aligned}
 1^2 + 2^2 + \dots + n^2 &= \frac{1}{3}n^2 \times n + 4 \times \frac{n(n+1)}{2} \times \frac{1}{4} - 4 \times n \times \frac{1}{12} \\
 &= \frac{1}{6}n(n+1)(2n+1)
 \end{aligned}$$

— I. A. Sakmar

Sums of squares VII

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



— Nanny Wermuth and Hans-Jürgen Schuh

Sums of squares VIII

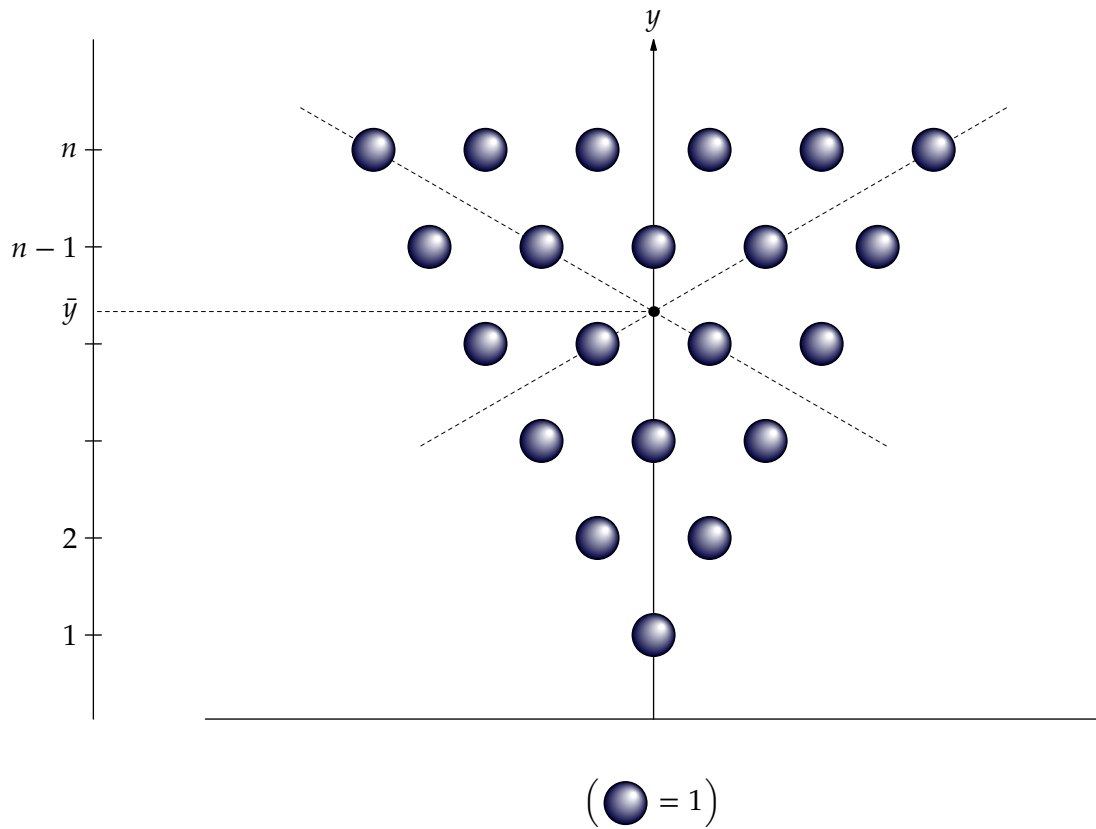
$$k^2 = 1 + 3 + \dots + (2k - 1) \quad \Rightarrow \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\begin{array}{c}
 \begin{array}{c}
 1 \\
 3 \quad 1 \\
 5 \quad 3 \quad \vdots \\
 \vdots \quad 5 \quad \vdots \quad 1 \\
 2n-3 \quad \vdots \quad 3 \quad 1 \\
 2n-1 \quad 2n-3 \quad \dots \quad 5 \quad 3 \quad 1
 \end{array}
 +
 \begin{array}{c}
 2n-1 \\
 2n-3 \quad 2n-3 \\
 \vdots \quad \vdots \\
 5 \quad \dots \quad 5 \quad 5 \\
 3 \quad 3 \quad \dots \quad 3 \quad 3 \\
 1 \quad 1 \quad 1 \quad \dots \quad 1 \quad 1
 \end{array}
 +
 \begin{array}{c}
 1 \\
 1 \quad 3 \\
 1 \quad 3 \quad 5 \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 1 \quad 3 \quad 5 \quad \dots \quad 2n-3 \quad 2n-1
 \end{array}
 \\
 =
 \begin{array}{c}
 2n+1 \\
 2n+1 \quad 2n+1 \\
 2n+1 \quad 2n+1 \quad 2n+1 \\
 \vdots \quad \vdots \\
 2n+1 \quad 2n+1 \quad 2n+1 \quad \dots \quad 2n+1 \\
 2n+1 \quad 2n+1 \quad 2n+1 \quad \dots \quad 2n+1 \quad 2n+1
 \end{array}
 \end{array}$$

$$3(1^2 + 2^2 + \dots + n^2) = (2n+1)(1 + 2 + \dots + n)$$

$$\therefore 1^2 + 2^2 + \dots + n^2 = \frac{2n+1}{3} \cdot \frac{n(n+1)}{2}$$

Sums of squares IX (via centroids)



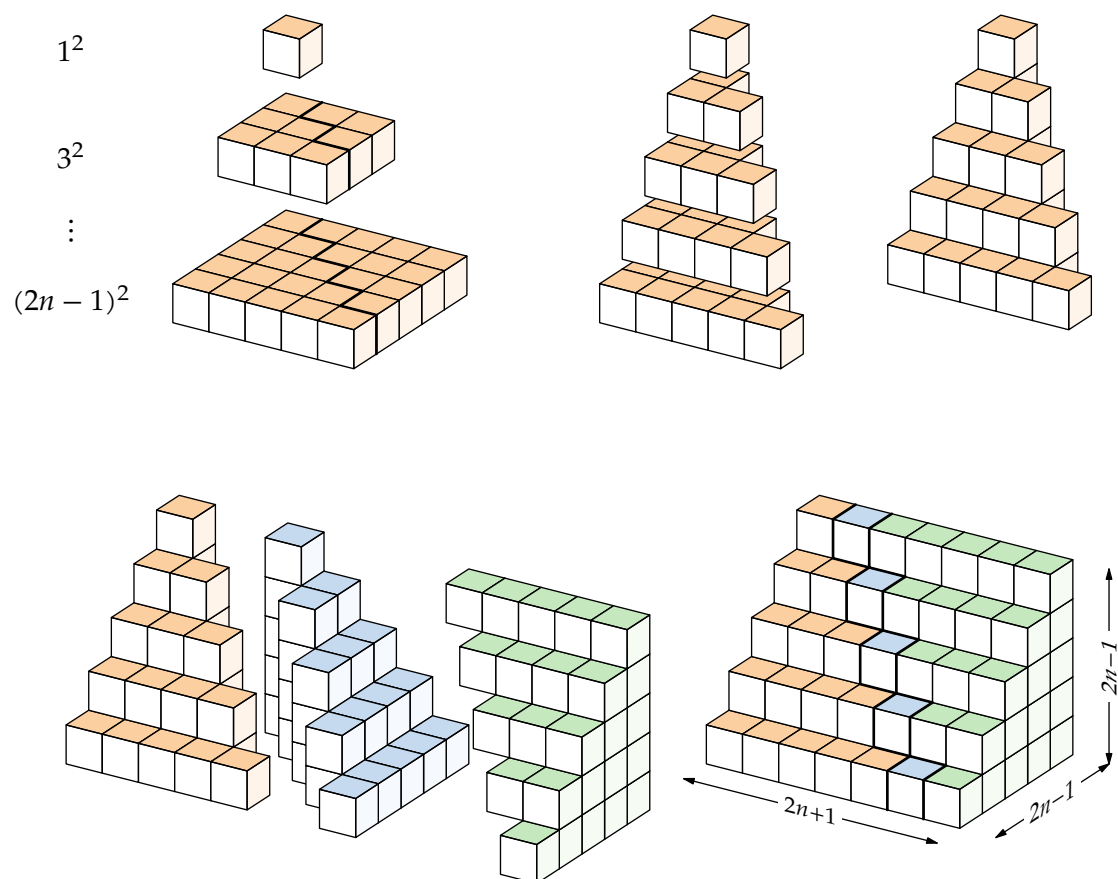
$$\bar{y} = 1 + \frac{2}{3}(n-1) = \frac{1 \cdot 1 + 2 \cdot 2 + \cdots + n \cdot n}{1 + 2 + \cdots + n}$$

$$\therefore 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{2} \cdot \frac{1}{3}(2n+1) = \frac{1}{6}n(n+1)(2n+1)$$

— Sidney H. Kung

Sums of odd squares

$$1^2 + 2^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

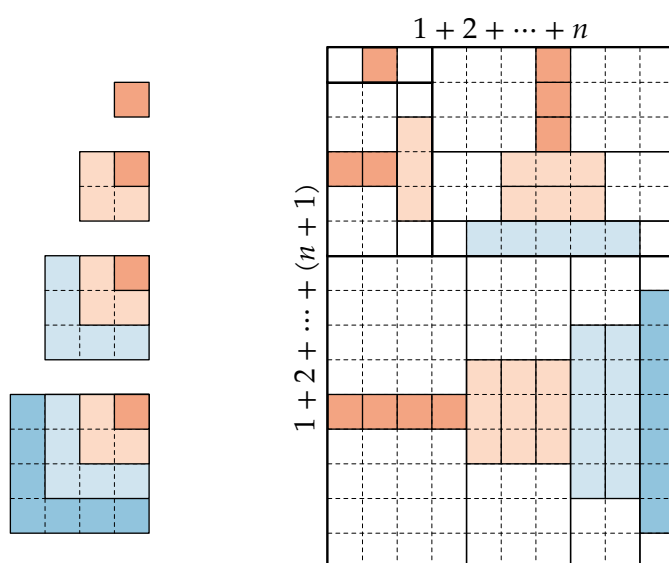


$$\begin{aligned} 3 \times (1^2 + 3^2 + \cdots + (2n-1)^2) &= (1 + 2 + \cdots + (2n-1)) \times (2n+1) \\ &= \frac{(2n-1)(2n)(2n+1)}{2} = n(2n-1)(2n+1) \end{aligned}$$

— RBN

Sums of sums of squares

$$\sum_{k=1}^n \sum_{i=1}^k i^2 = \frac{1}{3} \binom{n+1}{2} \binom{n+2}{2}$$



$$3(1^2) + 3(1^2 + 2^2) + 3(1^2 + 2^2 + 3^2) + \dots + 3(1^2 + 2^2 + \dots + n^2) = \binom{n+1}{2} \binom{n+2}{2}$$

— C. G. Wastun

Pythagorean runs

$$3^2 + 4^2 = 5^2$$

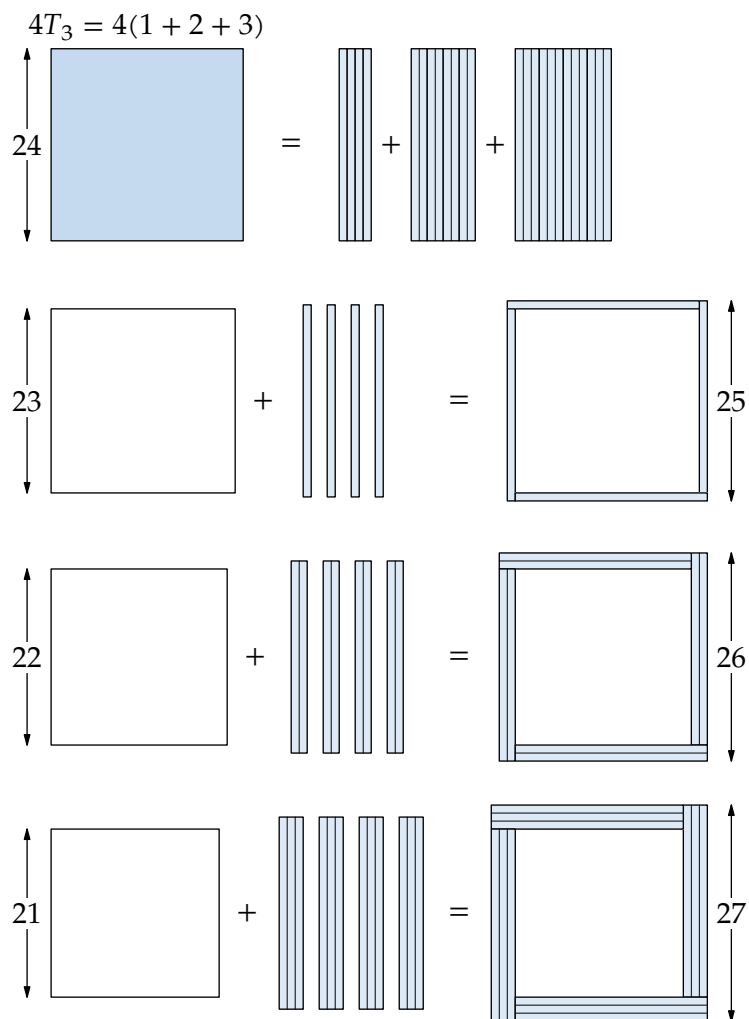
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

\vdots

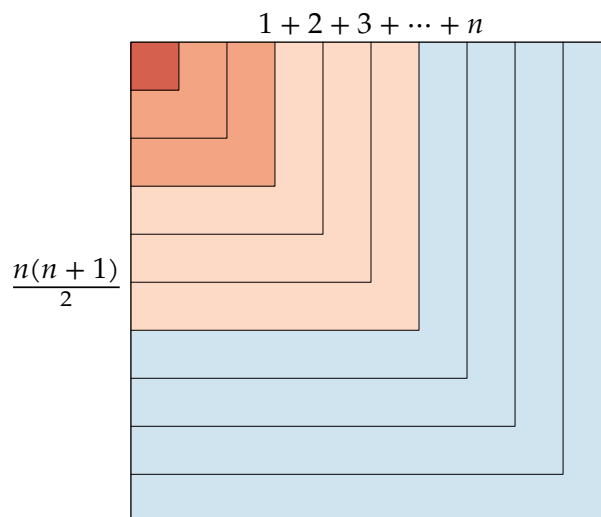
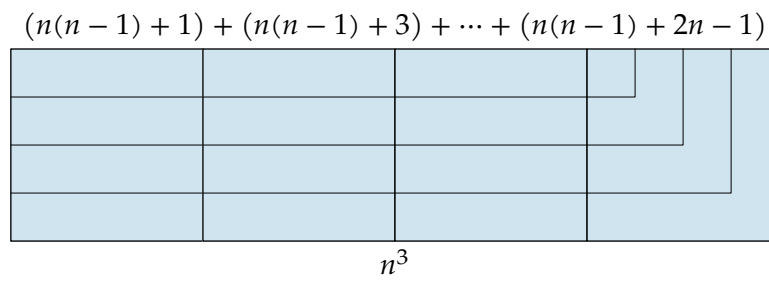
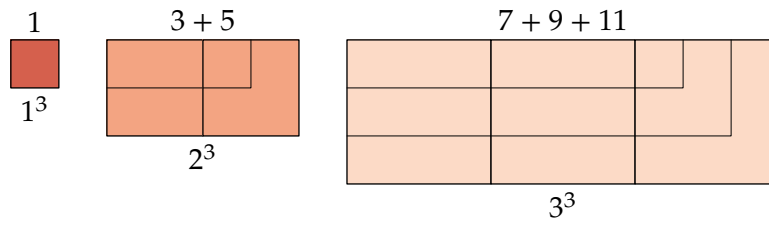
$$T_n = 1 + 2 + \cdots + n \Rightarrow (4T_n - n)^2 + \cdots + (4T_n)^2 = (4T_n + 1)^2 + \cdots + (4T_n + n)^2$$

e.g., $n = 3$:



— Michael Boardman

Sums of cubes VII



$$1^3 + 2^3 + \dots + n^3 = 1 + 3 + 5 + \dots + 2\frac{n(n-1)}{2} - 1 = \left(\frac{n(n-1)}{2}\right)^2$$

— Alfinio Flores

Sums of integers as sums of cubes

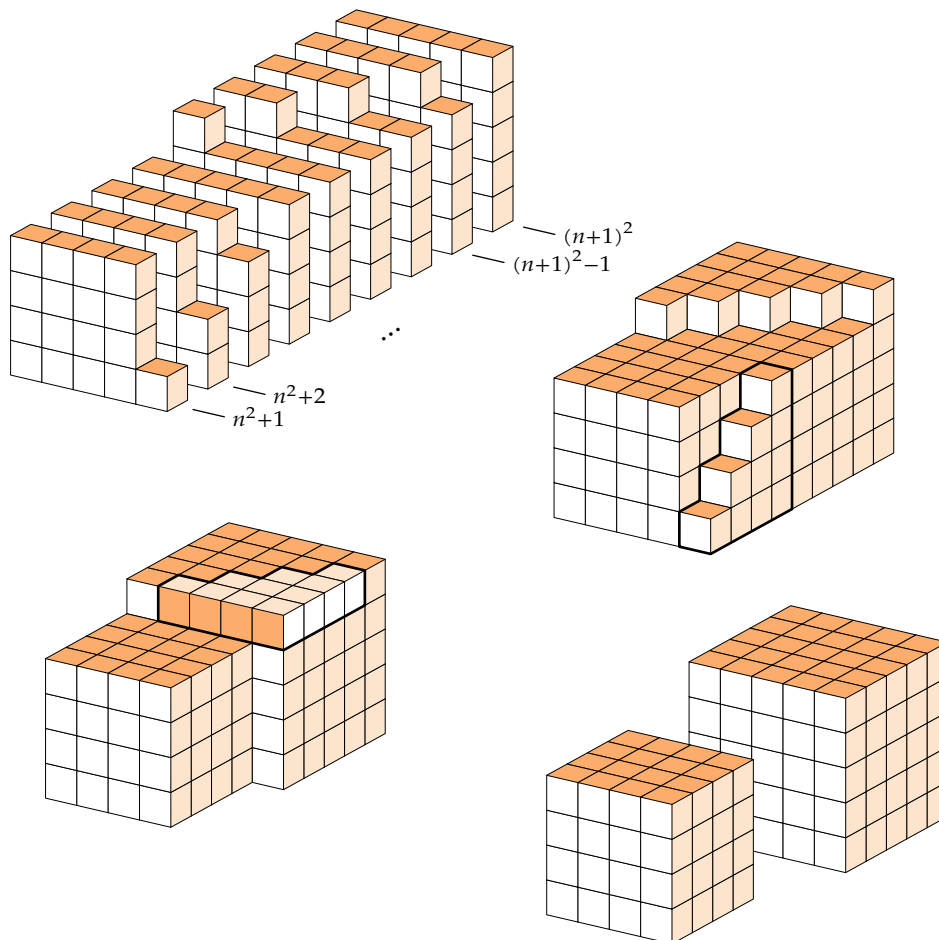
$$2 + 3 + 4 = 1 + 8$$

$$5 + 6 + 7 + 8 + 9 = 8 + 27$$

$$10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$$

\vdots

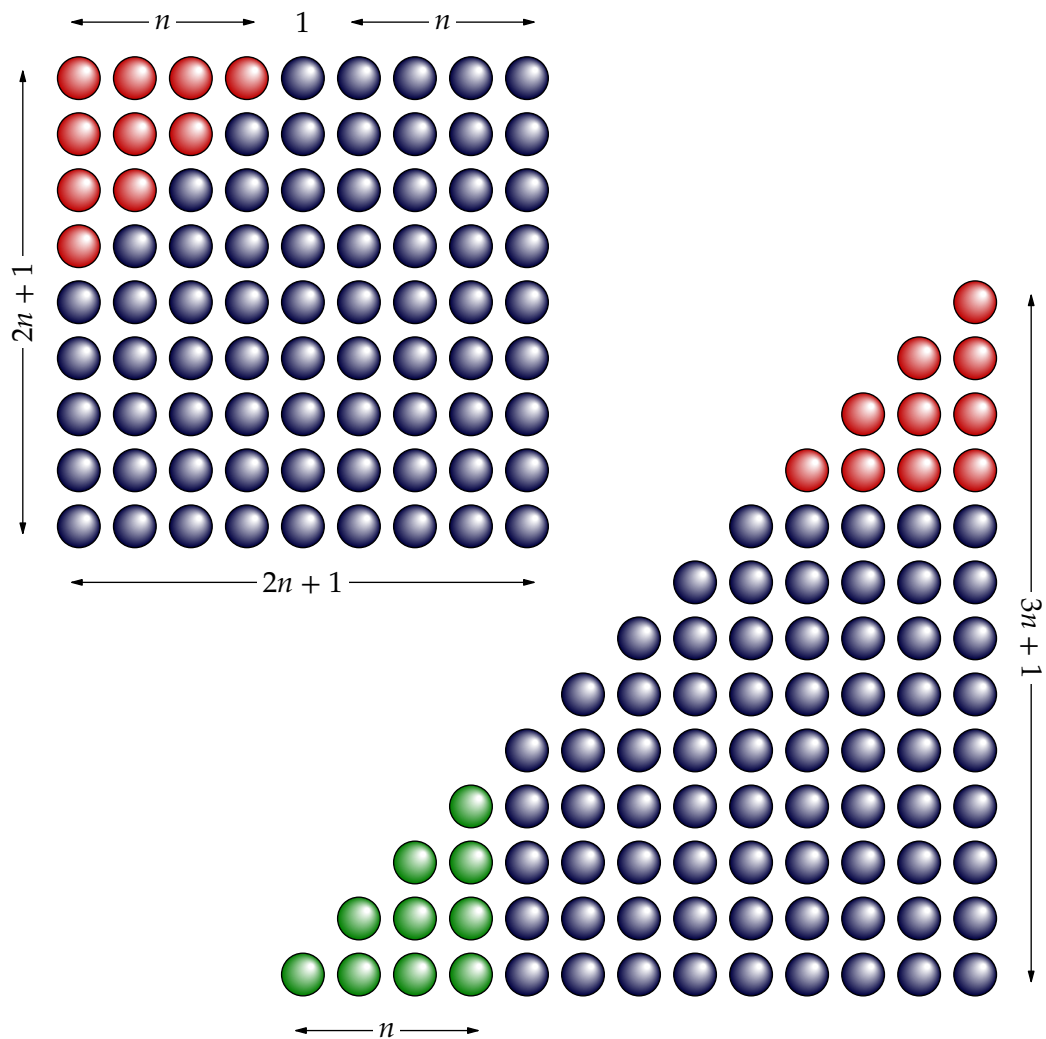
$$(n^2 + 1) + (n^2 + 2) + \cdots + (n + 1)^2 = n^3 + (n + 1)^3$$



— RBN

The square of any odd number is the difference between two triangular numbers

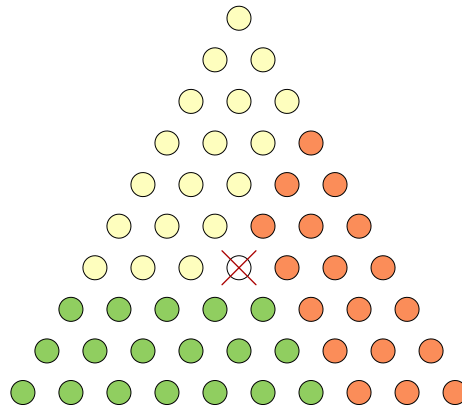
$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad (2n + 1)^2 = T_{3n+1} - T_n$$



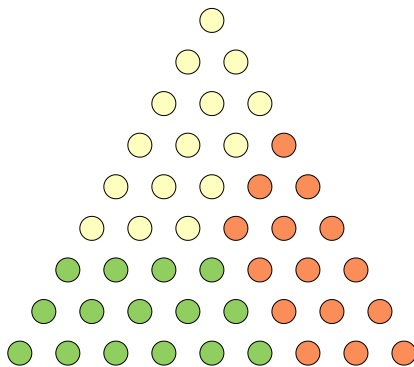
— RBN

Triangular numbers mod 3

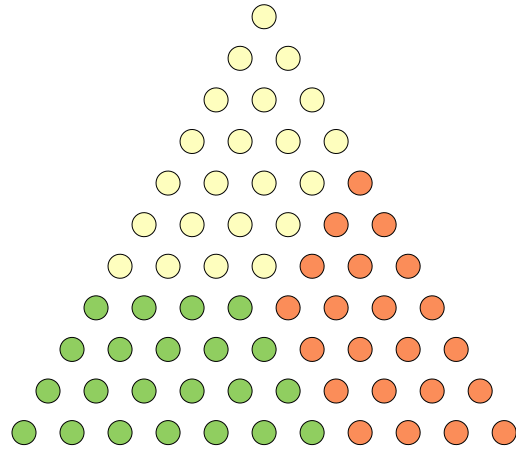
$$1 + 2 + \cdots + n = T_n \Rightarrow \begin{cases} T_n \equiv 1 \pmod{3}, & n \equiv 1 \pmod{3} \\ T_n \equiv 0 \pmod{3}, & n \not\equiv 1 \pmod{3} \end{cases}$$



$$T_{3k+1} = 1 + 3(T_{2k+1} - T_{k+1})$$



$$T_{3k} = 3(T_{2k} - T_k)$$



$$T_{3k+2} = 3(T_{2k+1} - T_k)$$

Counting triangular numbers IV: Counting cannonballs

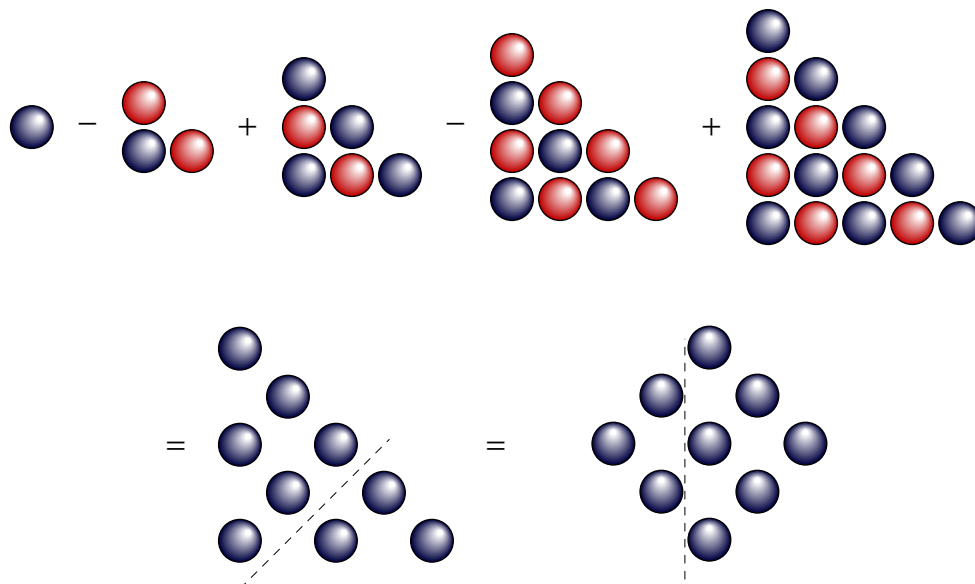
$$1 + 2 + \cdots + k = T_k \Rightarrow \sum_{k=1}^n T_k = \sum_{k=1}^n k(n - k + 1)$$



— Deanna B. Haunsperger and Stephen F. Kennedy

Alternating sums of triangular numbers

$$1 + 2 + \cdots + k = T_k \Rightarrow \sum_{k=1}^{2n-1} (-1)^{k+1} T_k = n^2$$



— RBN

The sum of the squares of consecutive triangular numbers is triangular

$$1 + 2 + \cdots + n = T_n \Rightarrow T_{n-1}^2 + T_n^2 = T_{n^2}$$



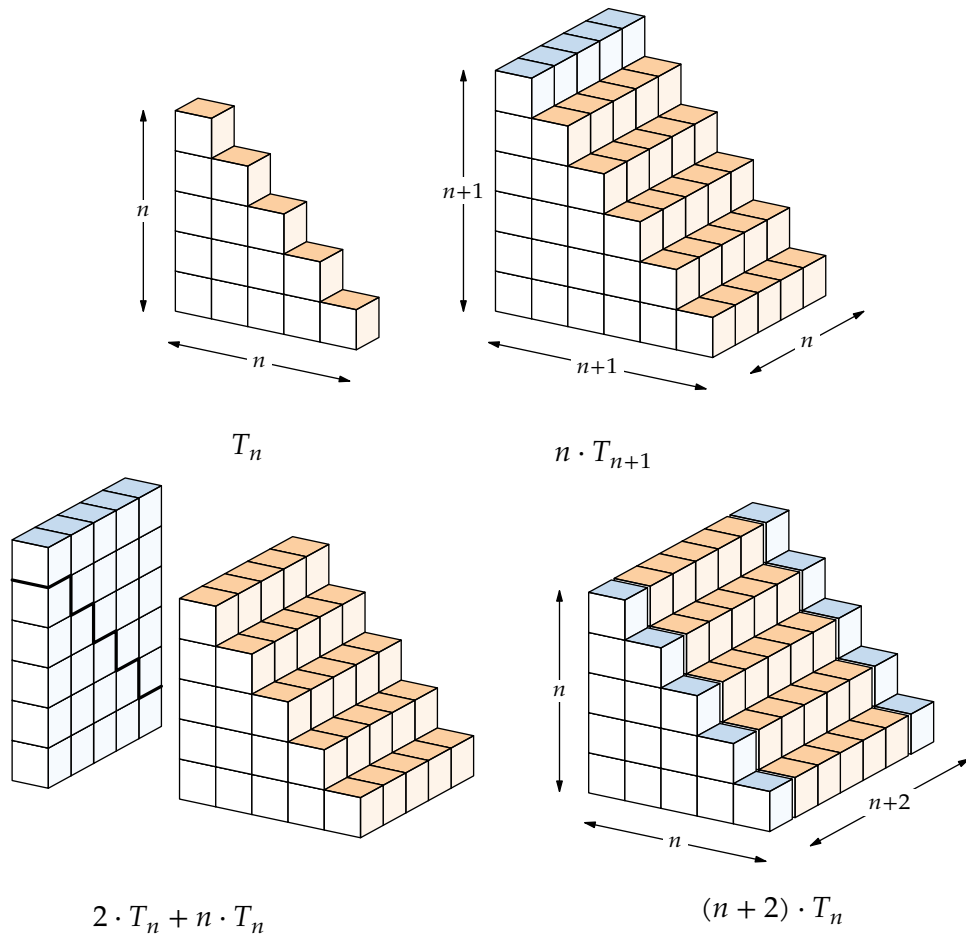
NOTE: This is a companion result to the more familiar $T_{n-1} + T_n = n^2 \rightarrow$



— RBN

Recursion for triangular numbers

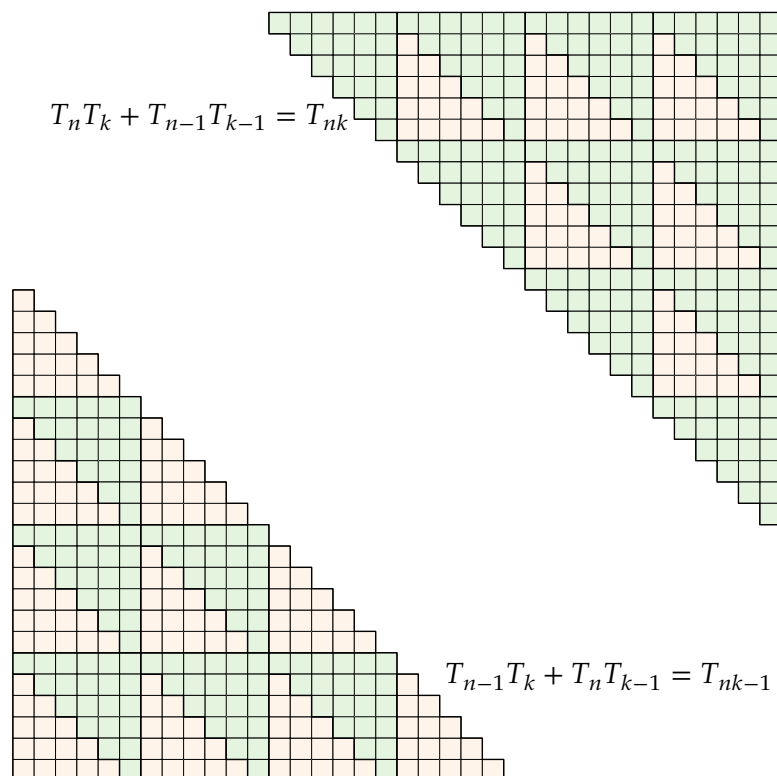
$$1 + 2 + \cdots + n = T_n \Rightarrow T_{n+1} = \frac{n+2}{n} T_n$$



$$n \cdot T_{n+1} = (n+2) \cdot T_n \Rightarrow T_{n+1} = \frac{n+2}{n} T_n$$

Identities for triangular numbers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



— RBN

More identities for triangular numbers

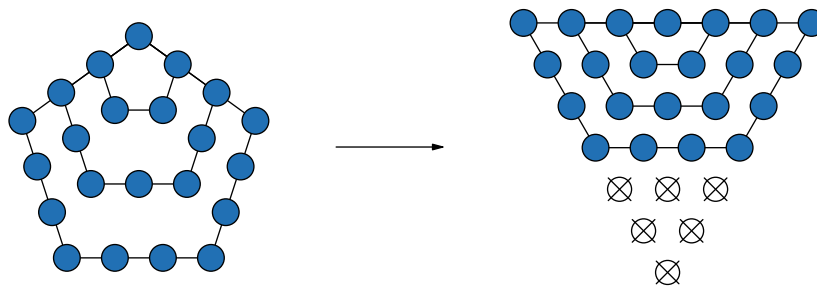
$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



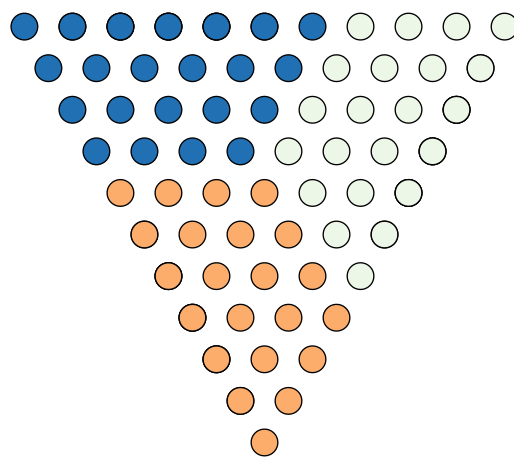
— James O. Chilaka

Identities for pentagonal numbers

$$\left. \begin{array}{l} P_n = 1 + 4 + 7 + \cdots + (3n - 2) \\ T_n = 1 + 2 + 3 + \cdots + n \end{array} \right\} \Rightarrow$$

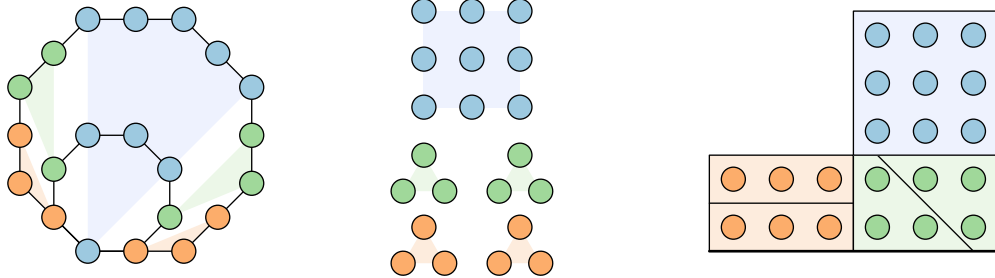


$$P_n = T_{2n-1} - T_{n-1}$$

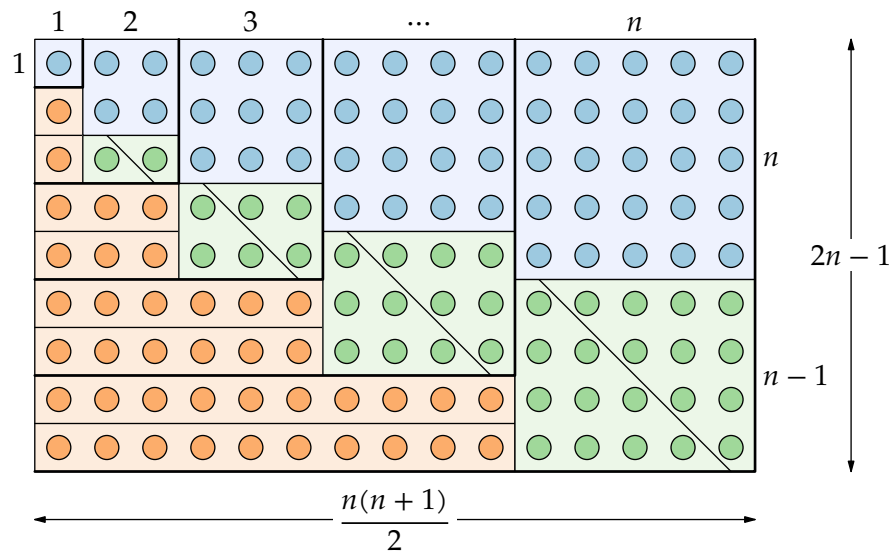


$$3P_n = T_{3n-1}$$

Sums of octagonal numbers



$$T_k = 1 + 2 + \dots + k \Rightarrow O_k = k^2 + 4T_{k-1}$$



$$\sum_{k=1}^n O_k = 1 + 8 + 21 + 40 + \dots + (n^2 + 4T_{n-1}) = \frac{n(n+1)(2n-1)}{2}$$

— James O. Chilaka