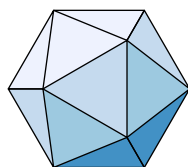


Proofs without words II

More exercises in METAPOST

Toby Thurston

March 2020 — July 2023



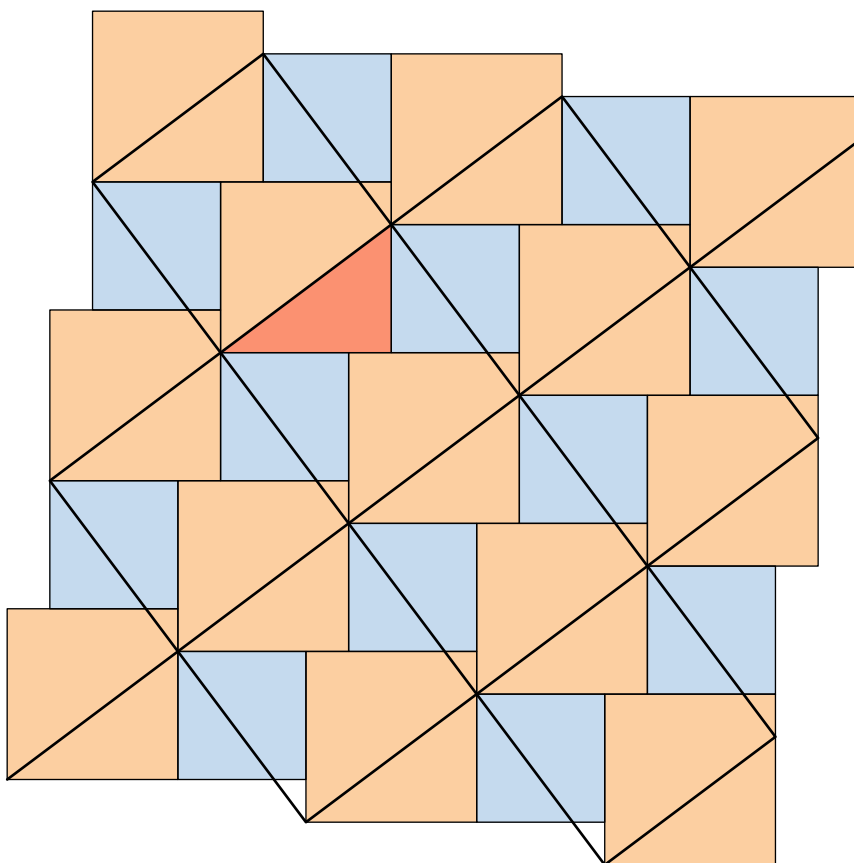
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Geometry and Algebra

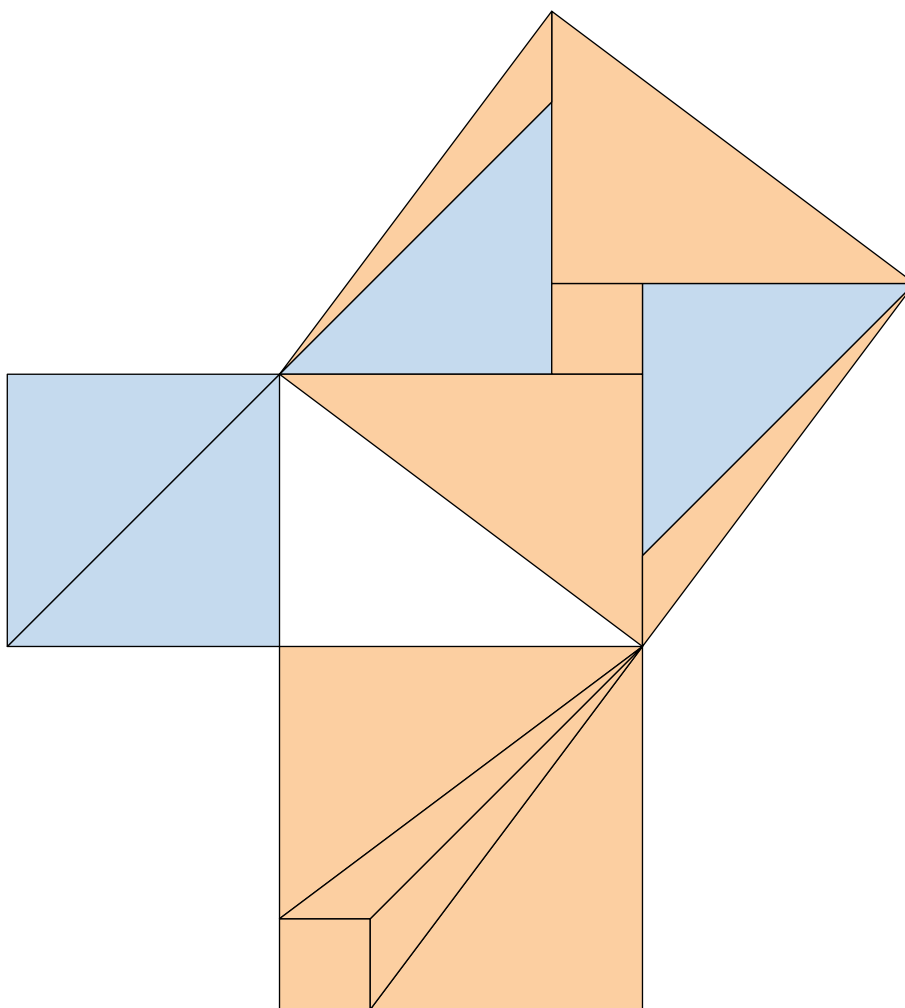
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The Pythagorean theorem VII



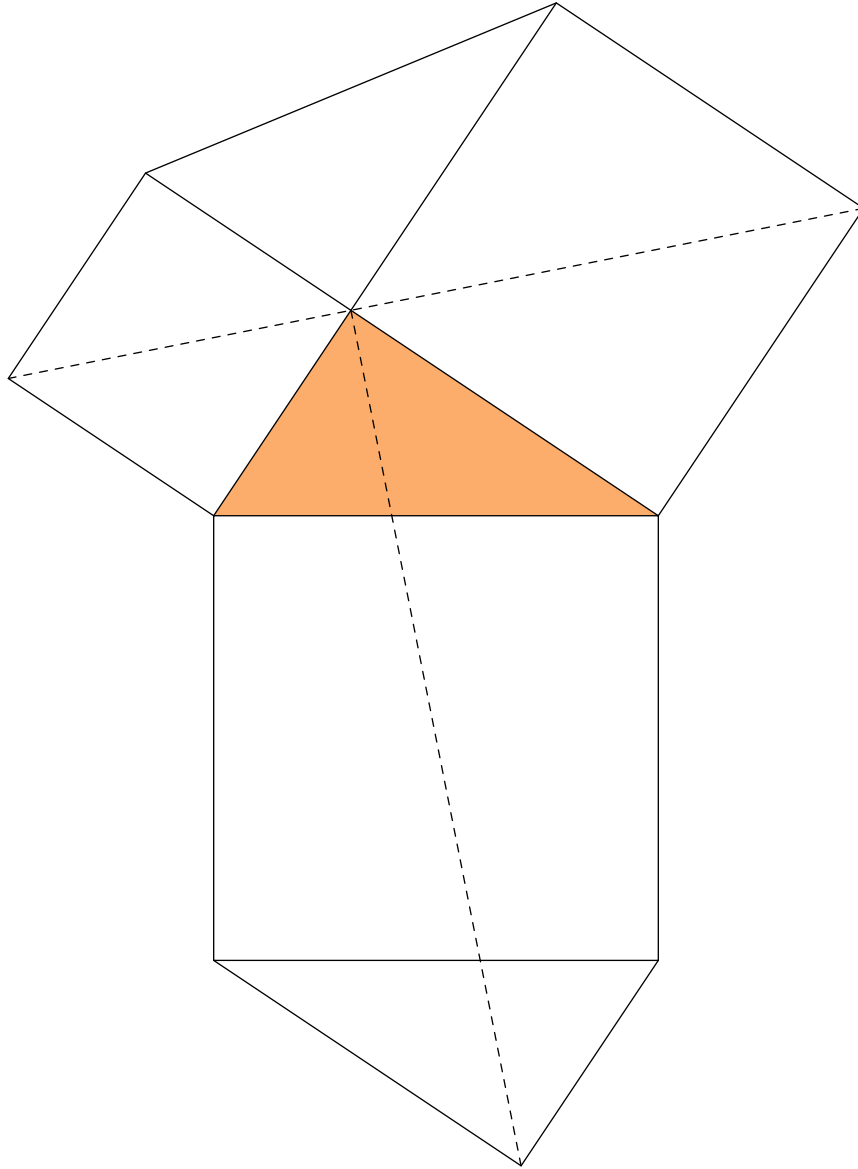
— Annairizi of Arabia (circa 900)

The Pythagorean theorem VIII



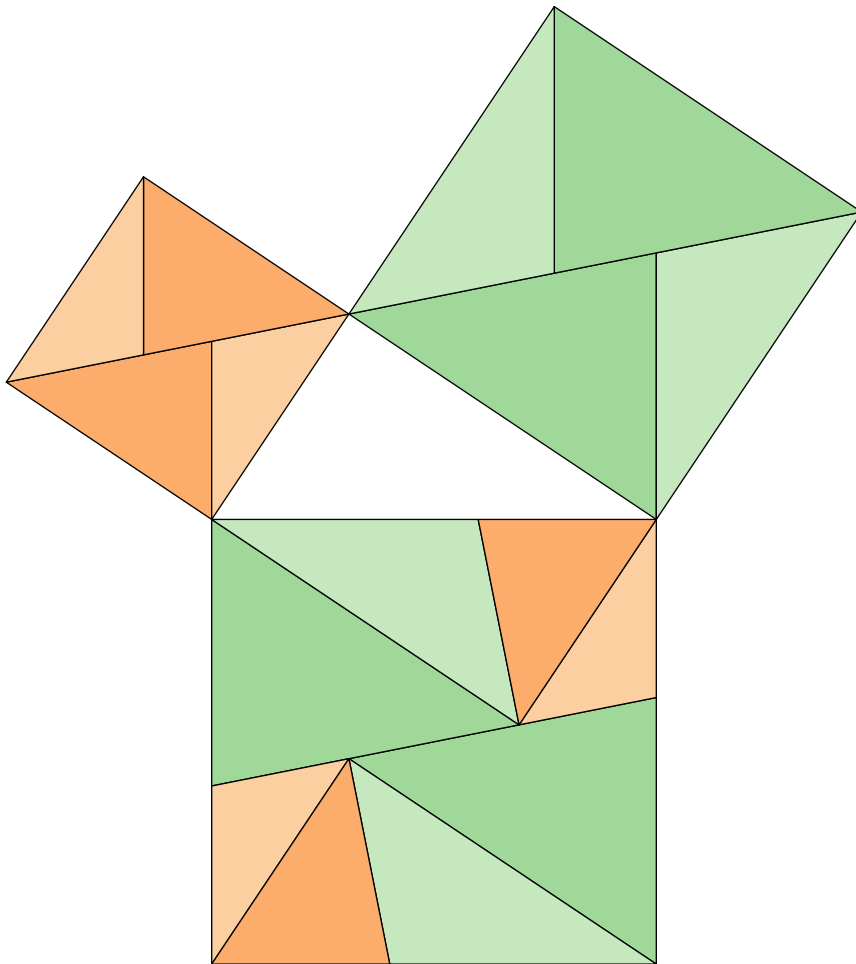
— Liu Hui (3rd century A.D.)

The Pythagorean theorem IX



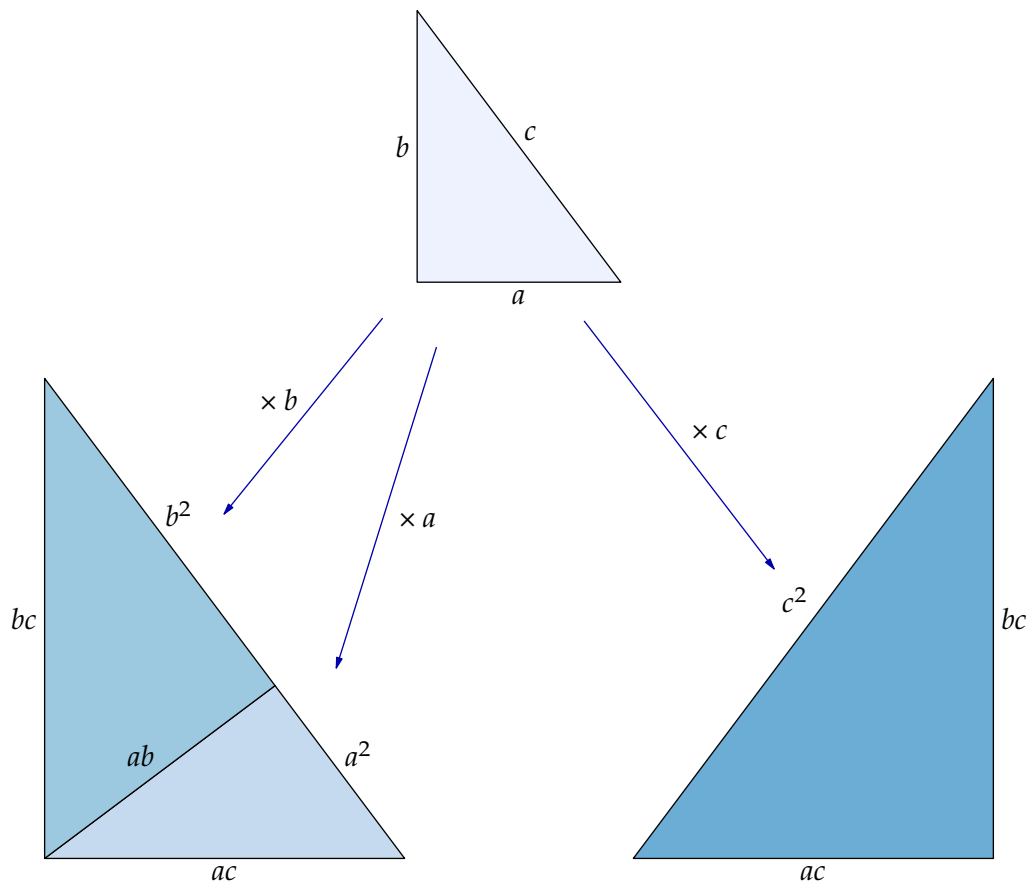
— Leonardo da Vinci (1452–1519)

The Pythagorean theorem X



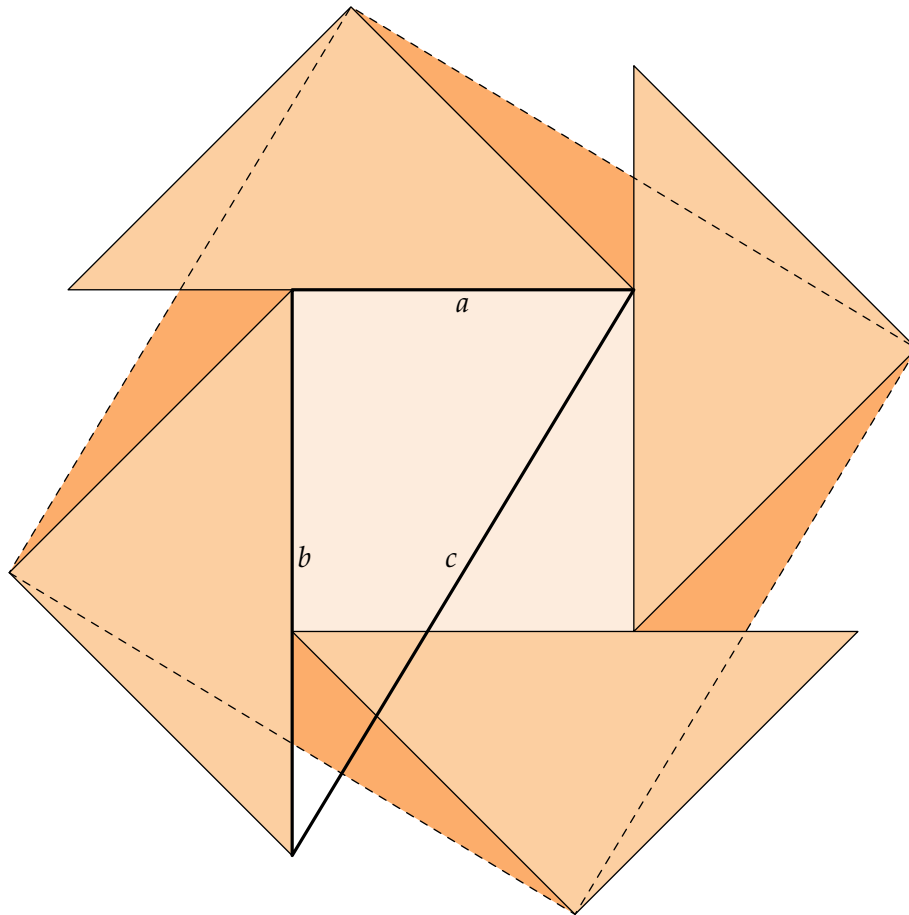
— J. E. Böttcher

The Pythagorean theorem XI



— Frank Burk

The Pythagorean theorem XII

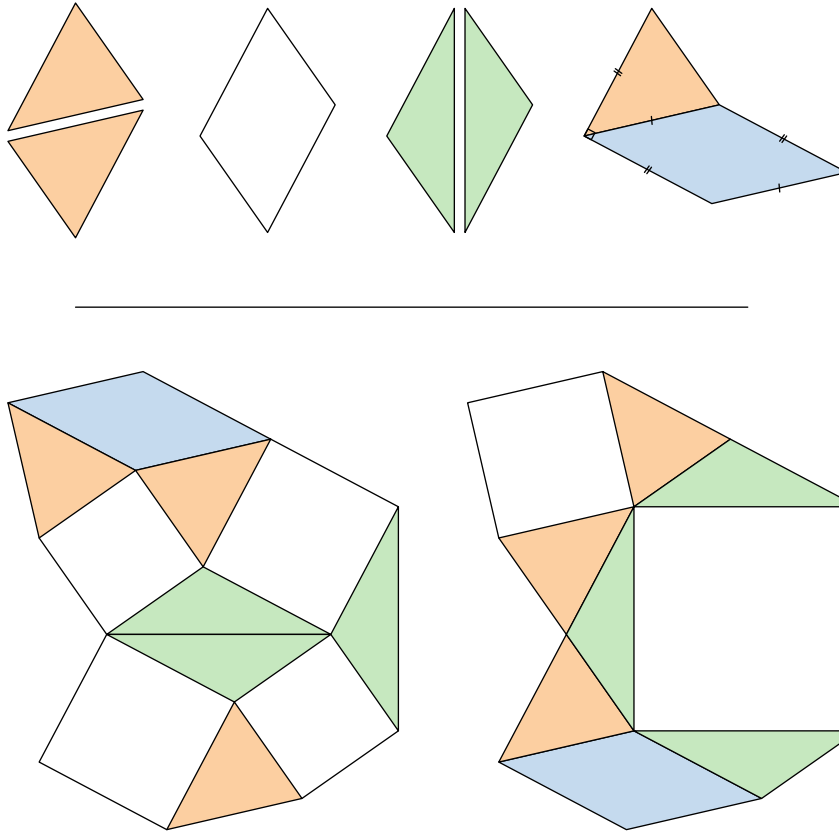


$$a^2 + b^2 = c^2$$

— Poo-Sung Park

A generalization from Pythagoras

The sum of the area of two squares, whose sides are the lengths of two diagonals of a parallelogram, is equal to the sum of the area of four squares, whose sides are its four sides.

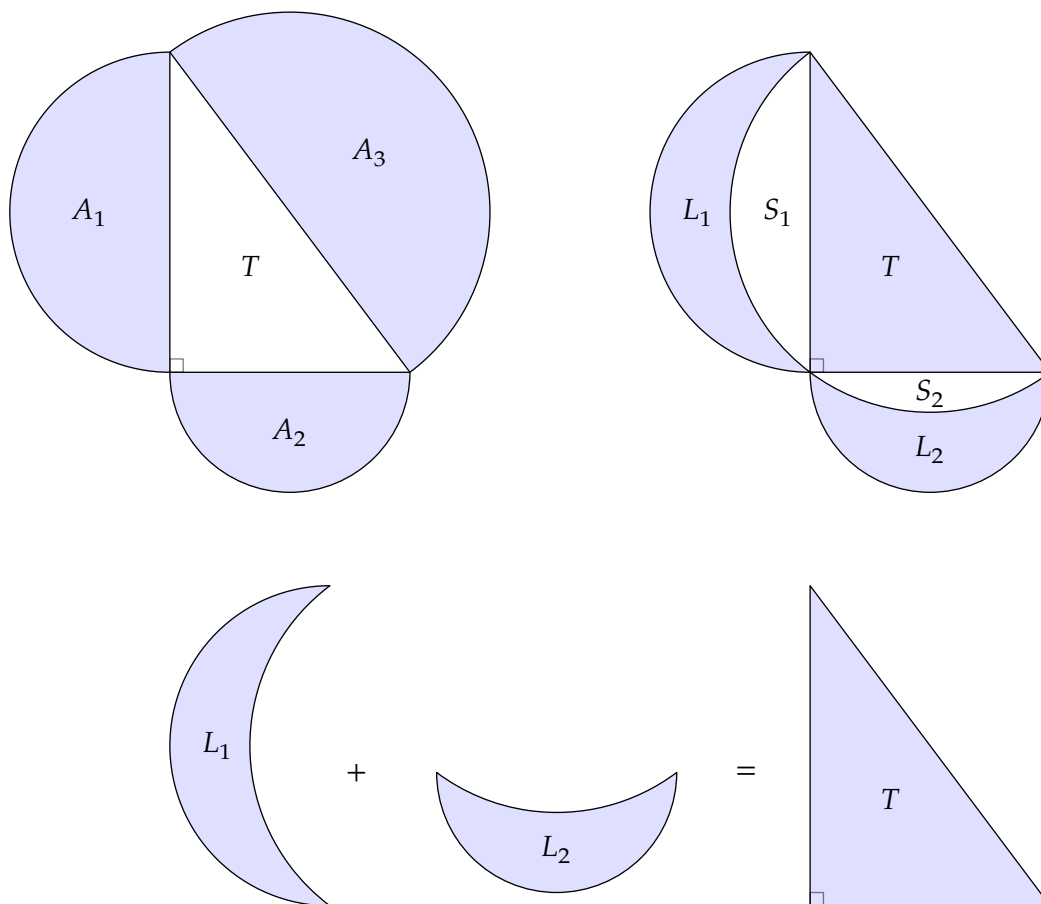


COROLLARY: The Pythagorean theorem (when the parallelogram is a rectangle).

— David S. Wise

A theorem of Hippocrates of Chios (circa 440 BC)

The combined area of the lunes constructed on the legs of a given right angle triangle is equal to the area of the triangle.

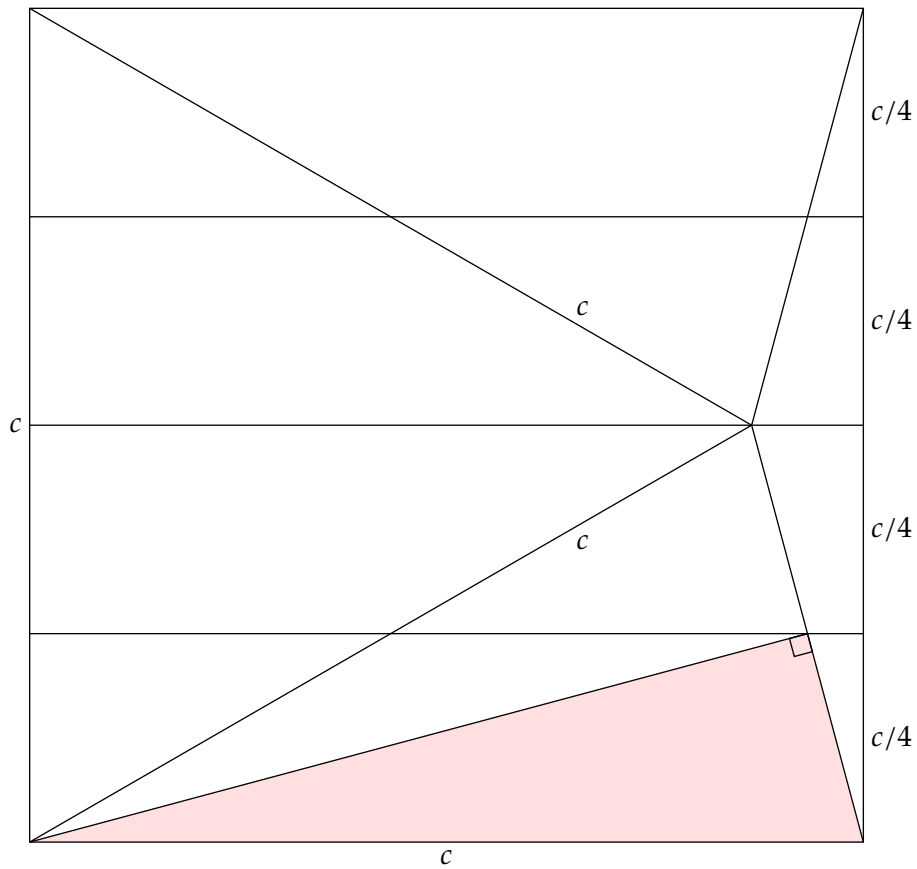


$$\begin{aligned}
 A_1 + A_2 &= A_3 \\
 (L_1 + S_1) + (L_2 + S_2) &= T + S_1 + S_2 \\
 L_1 + L_2 &= T
 \end{aligned}$$

— Eugene A. Margerum and Michael M. McDonnell

The area of a right triangle with acute angle $\pi/12$

The area of a right triangle is $\frac{1}{8}(\text{hypotenuse})^2$ if and only if one acute angle is $\pi/12$.



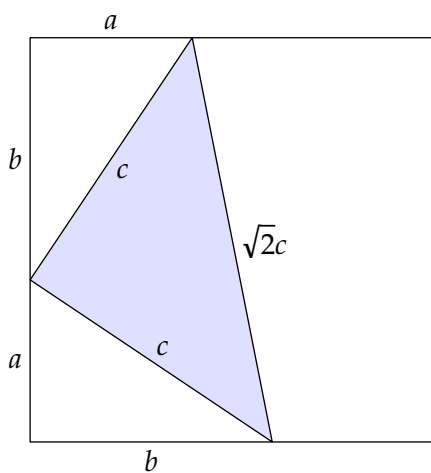
— Klara Pinter

A right angle inequality

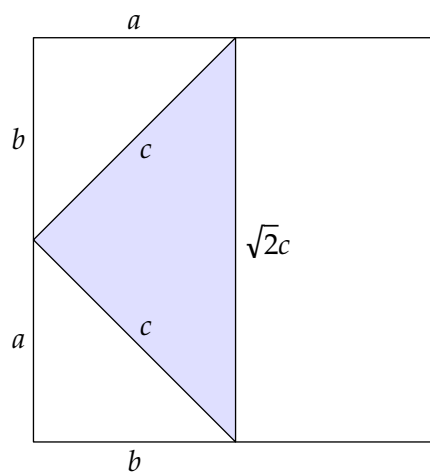
Let c be the hypotenuse of a right triangle whose other two sides are a and b . Prove that

$$a + b \leq \sqrt{2}c.$$

When does equality hold?



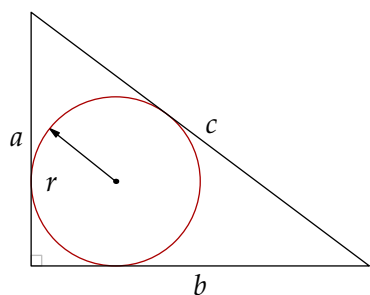
$$a + b \leq \sqrt{2}c$$



$$a + b = \sqrt{2}c \iff a = b$$

— Canadian Mathematical Olympiad 1969

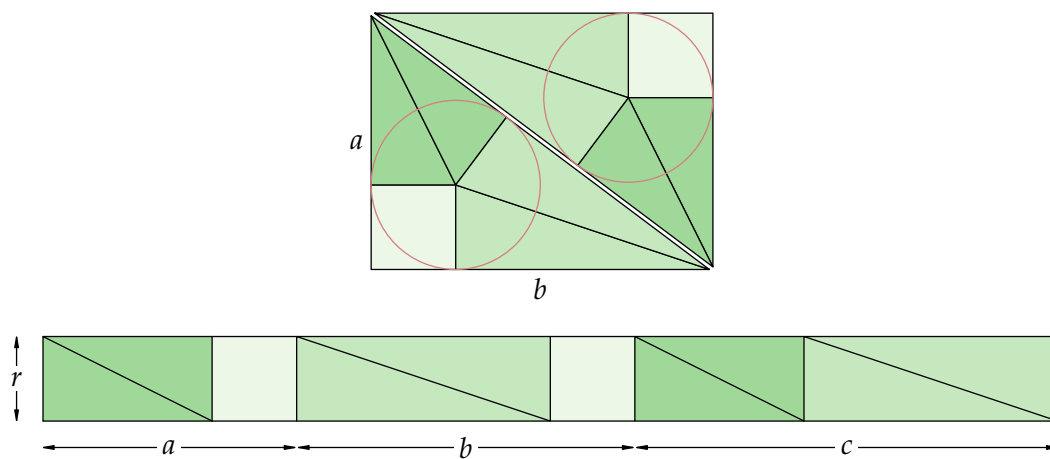
The inradius of a right triangle



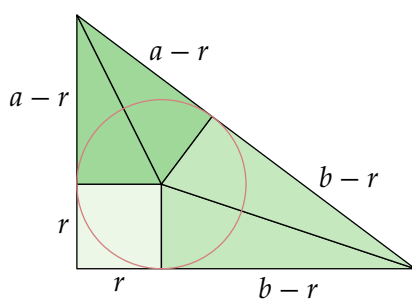
$$\text{I. } r = \frac{ab}{a+b+c}$$

$$\text{II. } r = \frac{a+b-c}{2}$$

$$\text{I. } ab = r(a+b+c)$$

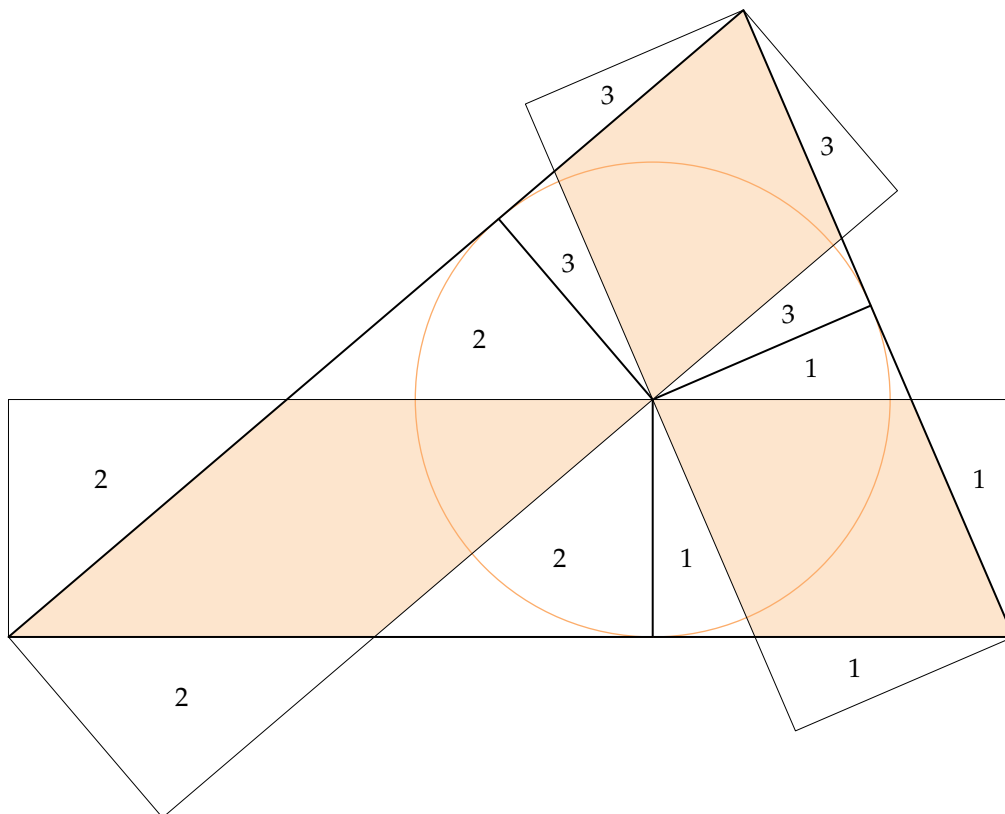


$$\text{II. } c = a + b - 2r$$



— Liu Hui (3rd century A.D.)

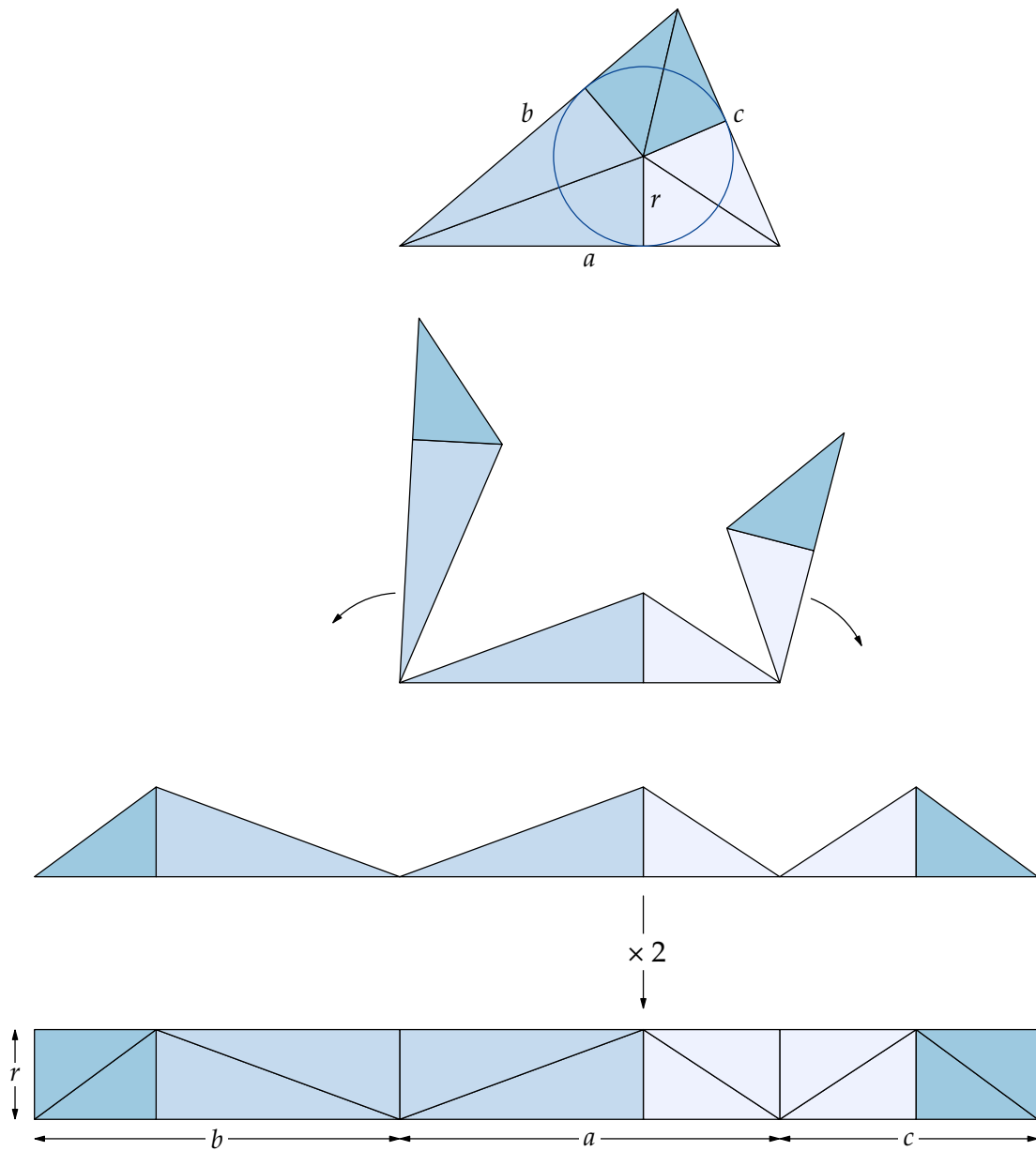
The product of the perimeter of a triangle and its inradius is twice the area of the triangle



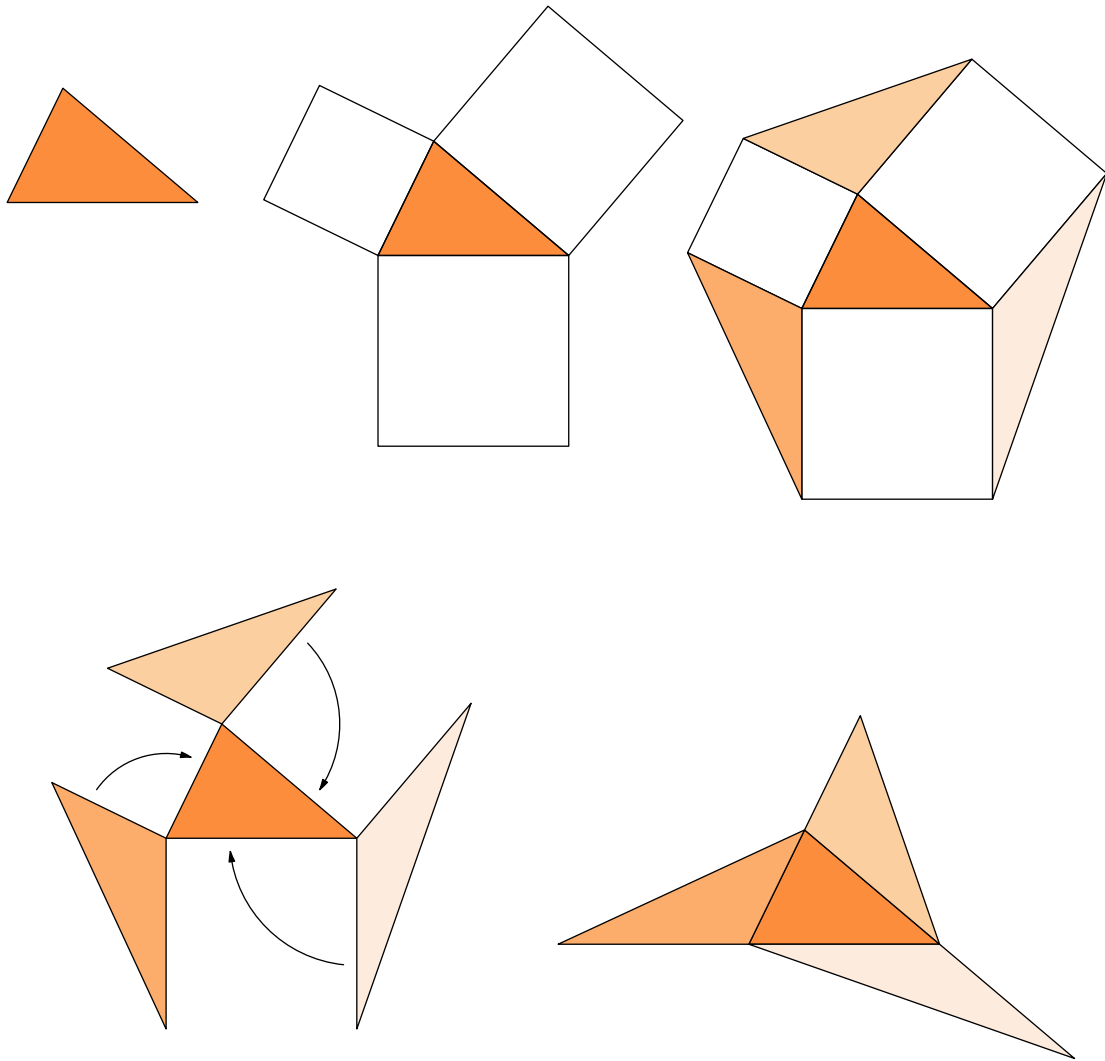
NOTE: Triangles marked with the same number are equal in area.

— Grace Lin

The product of the perimeter of a triangle and its inradius is twice the area of the triangle II



Four triangles with equal area



— Steven L. Snover

The triangle of medians has $3/4$ the area of the original triangle

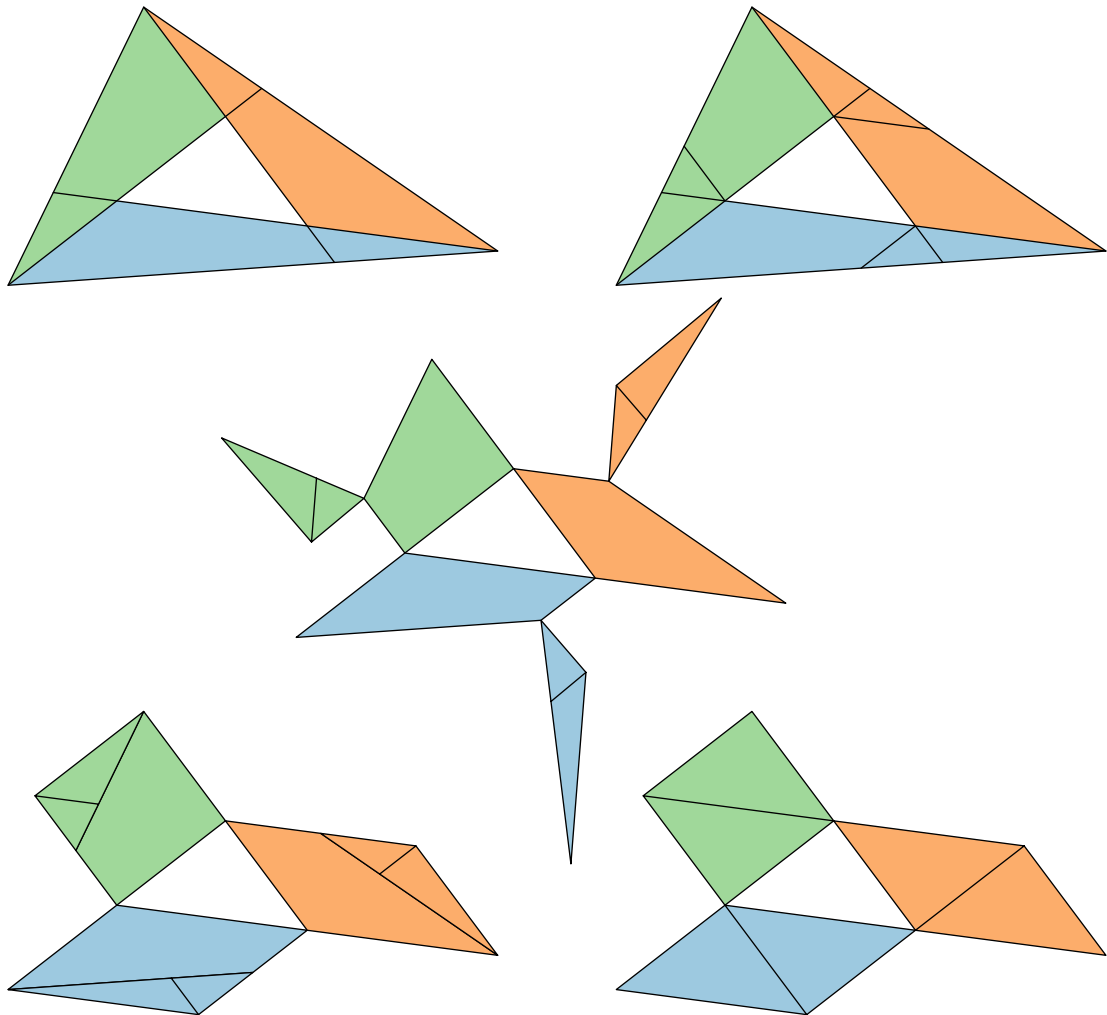


$$\frac{3}{4} \text{area}(\triangle abc) = \text{area}(\triangle m_a m_b m_c)$$

— Norbert Hungerbühler

Heptasection of a triangle

If the one-third points on each side of a triangle are joined to opposite vertices, the resulting central triangle is equal in area to one-seventh that of the initial triangle.



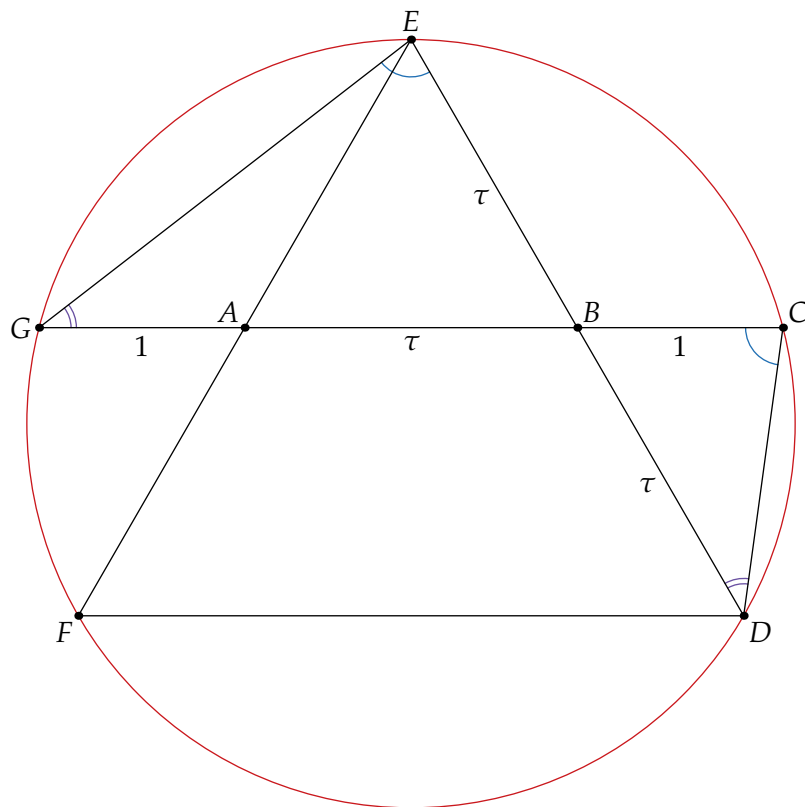
— William Johnston and Joe Kennedy

A Golden Section problem from the *Monthly*

(Problem E3007, *American Mathematical Monthly*, 1983, p.482)

Let A and B be the midpoints of the sides EF and ED of an equilateral triangle DEF . Extend AB to meet the circumcircle (of DEF) at C . Show that B divides AC according to the golden section.

SOLUTION:



$$\tau^2 = \tau + 1$$

— Jan van de Craats

Tiling with squares and parallelograms

If squares are constructed eternally on the sides of the parallelogram, their centres form a square.



— Alfinio Flores

The area of a quadrilateral I

The area of a quadrilateral is less than or equal to half the product of the lengths of its diagonals, with equality if and only if the diagonals are perpendicular.

I. Convex quadrilaterals



$$\begin{aligned}\text{Area} &= \frac{1}{2} \overline{AC} \cdot (h + k) \\ &\leq \frac{1}{2} \overline{AC} \cdot \overline{BD}\end{aligned}$$

II. Concave quadrilaterals



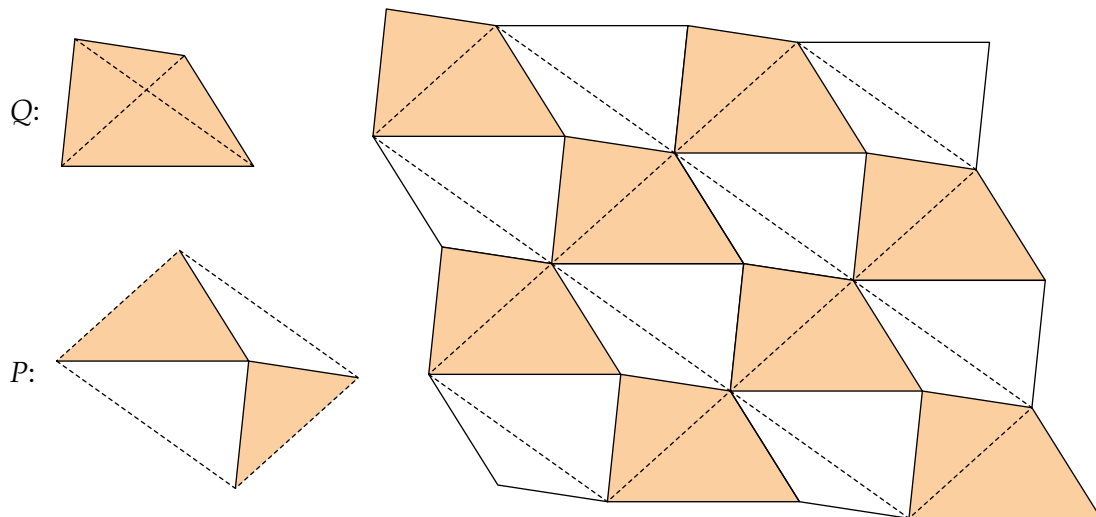
$$\begin{aligned}\text{Area} &= \frac{1}{2} \overline{AC} \cdot (h - k) \\ &\leq \frac{1}{2} \overline{AC} \cdot \overline{BD}\end{aligned}$$

— David B. Sher, Ronald Skurnick, and Dean C. Nataro

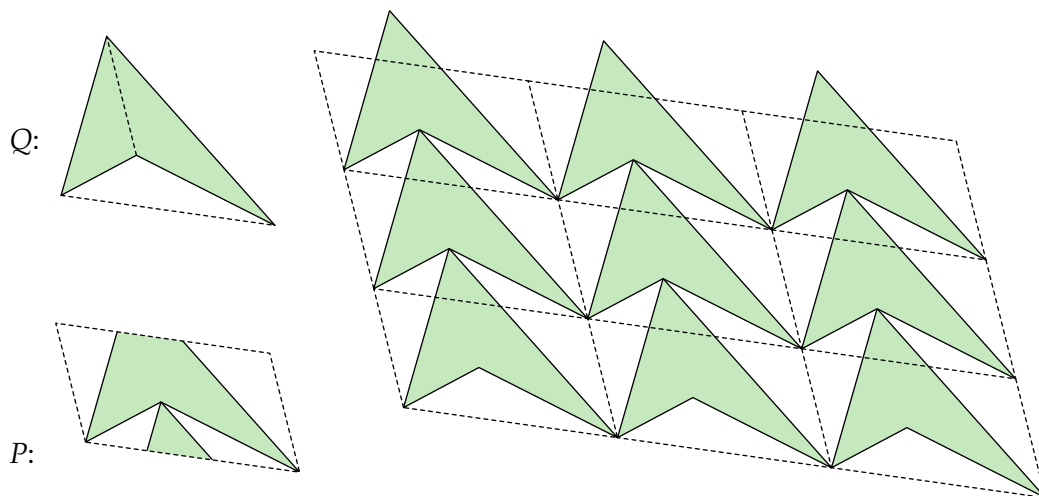
The area of a quadrilateral II

The area of a quadrilateral Q is equal to one-half the area of a parallelogram P whose sides are parallel to and equal in length to the diagonals of Q .

I. Q convex



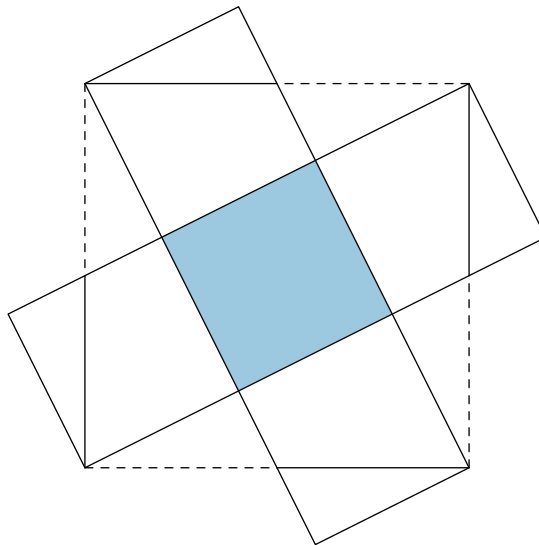
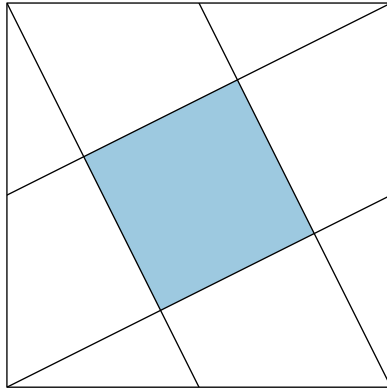
II. Q concave



$$\text{area}(Q) = \frac{1}{2} \text{area}(P)$$

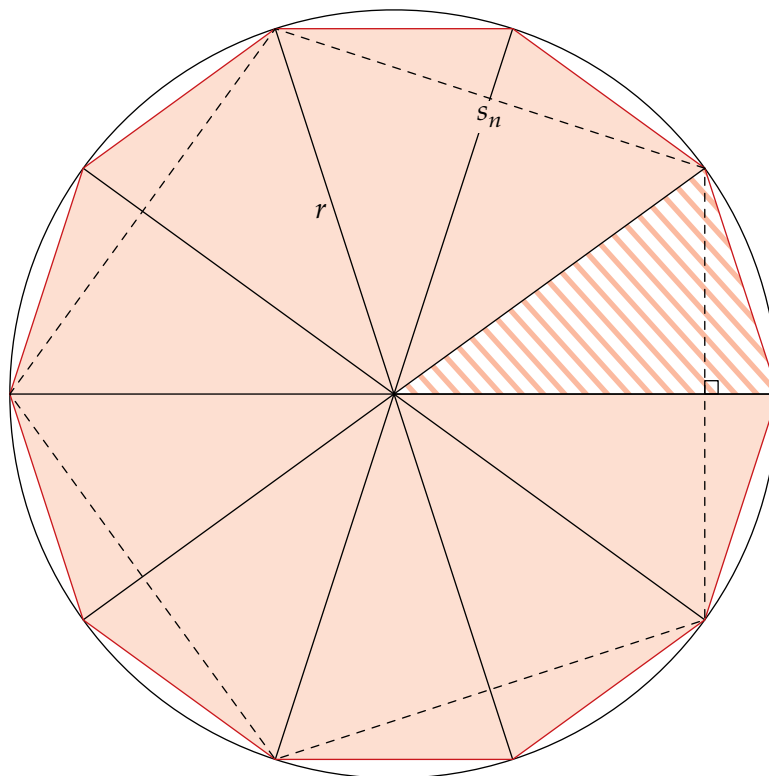
A square within a square

If lines from the vertices of a square are drawn to the mid-points of adjacent sides (as shown in the figure), then the area of the smaller square so produced is one-fifth that of the given square.



Areas and perimeters of regular polygons

The area of a regular $2n$ -gon inscribed in a circle is equal to one-half the radius of the circle times the perimeter of a regular n -gon similarly inscribed ($n \geq 3$).



$$\begin{aligned}\frac{1}{2n} \text{area}(P_{2n}) &= \frac{1}{2} \cdot r \cdot \frac{1}{2}s_n \\ \text{area}(P_{2n}) &= \frac{1}{2}r \cdot ns_n \\ &= \frac{1}{2}r \cdot \text{perimeter}(P_n)\end{aligned}$$

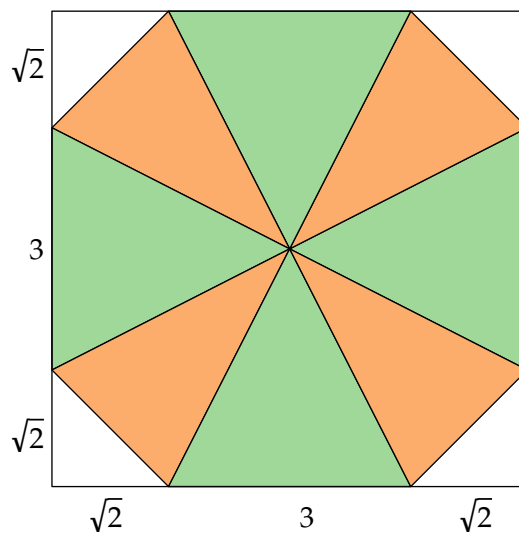
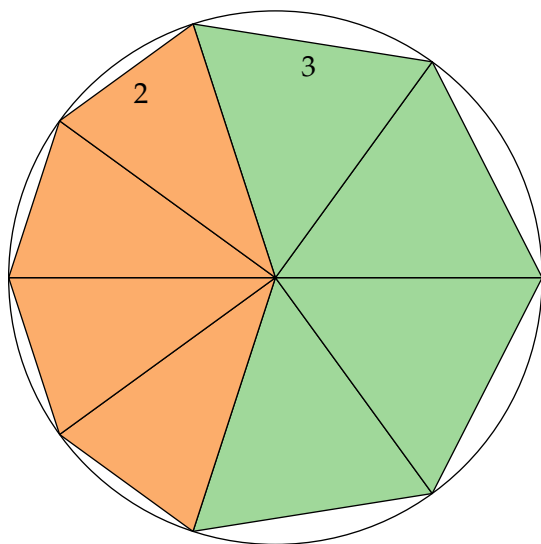
COROLLARY [Bhāskara, *Litāvatī* (India, 12th century AD)]: The area of a circle is equal to one-half the product of its radius and circumference.

The area of a Putnam octagon

(Problem B1, 39th Annual William Lowell Putnam Mathematical Competition, 1978).

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$, with r, s , and t positive integers.

SOLUTION:



$$A = (3 + 2\sqrt{2})^2 - 4 \cdot \frac{1}{2} (\sqrt{2})^2 = 9 + 6\sqrt{2} + 6\sqrt{2} + 8 - 4 = 13 + 12\sqrt{2}$$

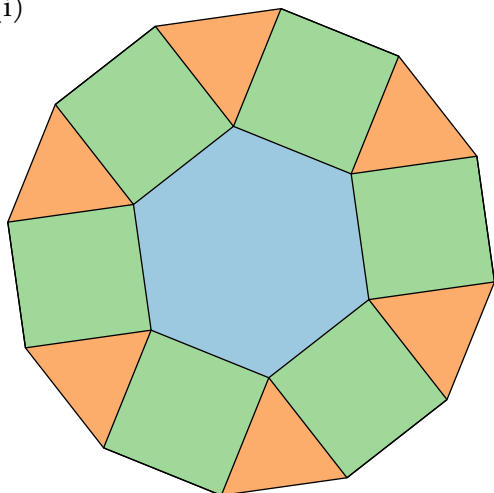
A Putnam dodecagon

(Problem I-1, 24th Annual William Lowell Putnam Mathematical Competition, 1963)

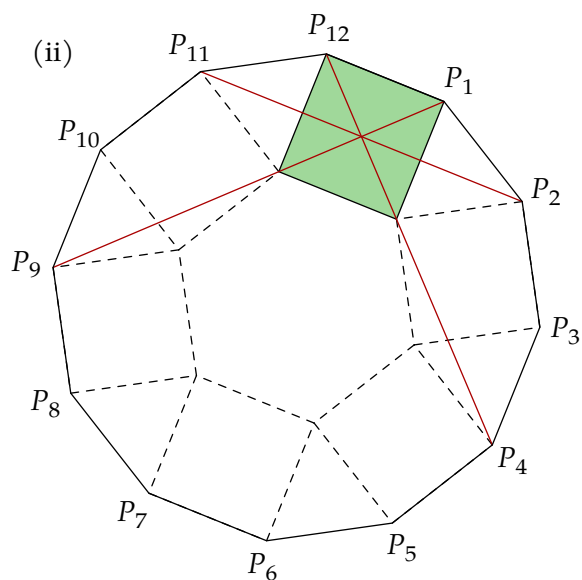
- (i) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.
- (ii) Let P_1, P_2, \dots, P_{12} be the successive vertices of a regular dodecagon. Discuss the intersection(s) of the three diagonals P_1P_9 , P_2P_{11} , and P_4P_{12} .

SOLUTION:

(i)



(ii)



The area of a regular dodecagon

A regular dodecagon with circumradius one has area three.

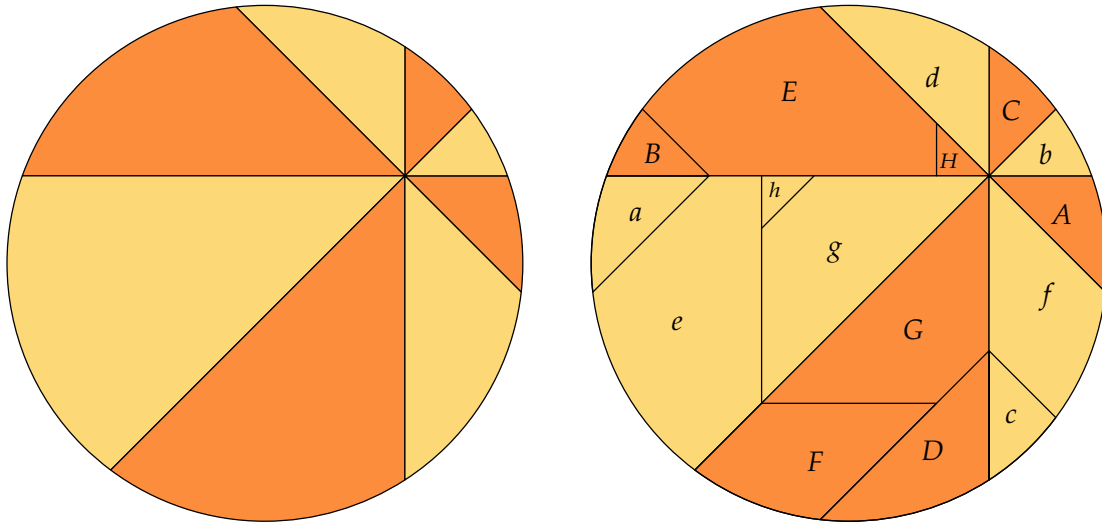


— J. Kürshák

Fair allocation of a pizza

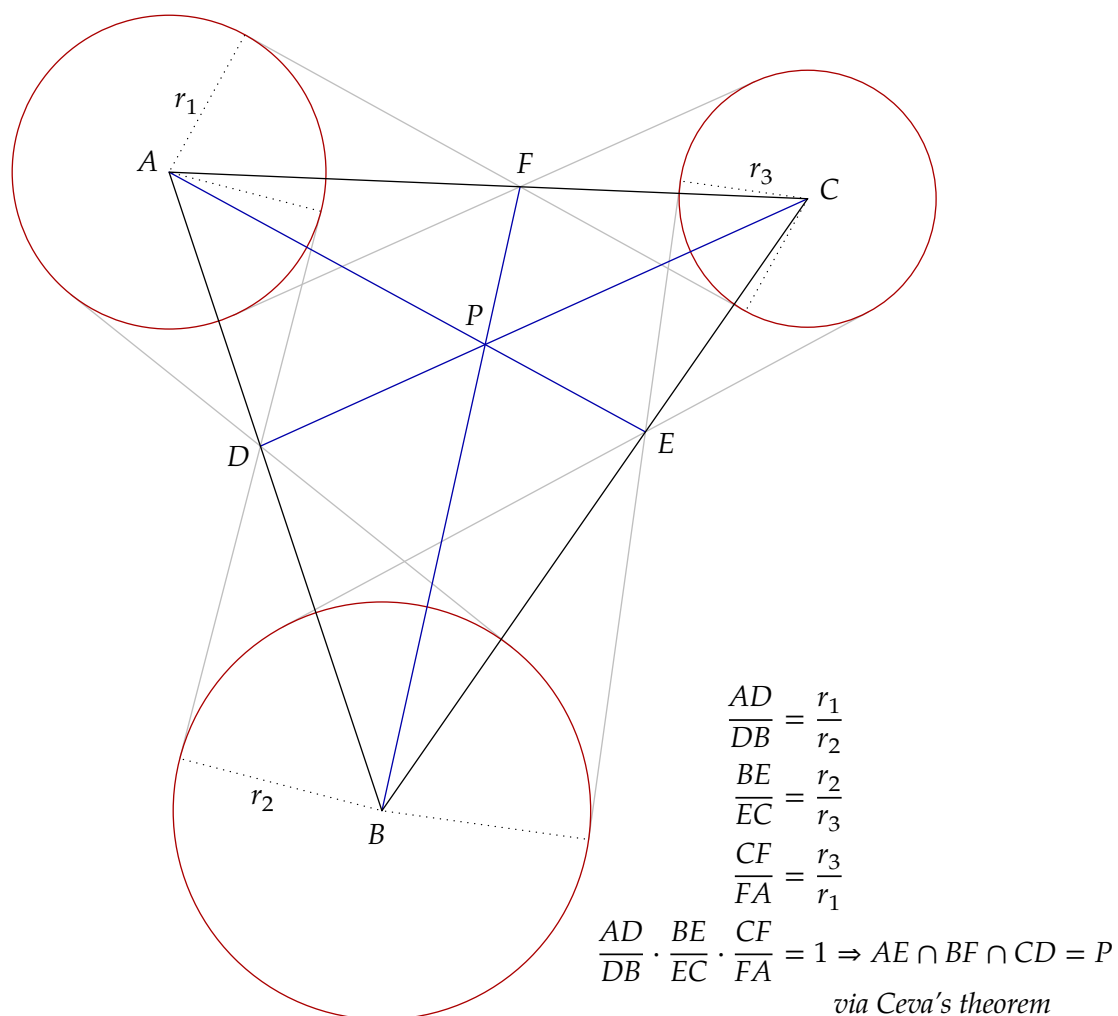
THE PIZZA THEOREM: If a pizza is divided into eight pieces by making cuts at 45° angles through an arbitrary point in the pizza, then the sums of the areas of alternate slices are equal.

PROOF:



A three-circle theorem

Given three non-intersecting, mutually external circles, connect the intersection of the internal common tangents of each pair of circles with the centre of the other circle. Then the resulting three line segments are concurrent.



— R. S. Hu

A constant chord

Suppose two circles Q and R intersect in A and B . A point P on the arc of Q which lies outside R is projected through A and B to determine chord CD of R . Prove that no matter where P is chosen on its arc, the length of chord CD is always the same.



$$\angle C'AC = \angle P'AP = \angle P'BP = \angle D'BD$$

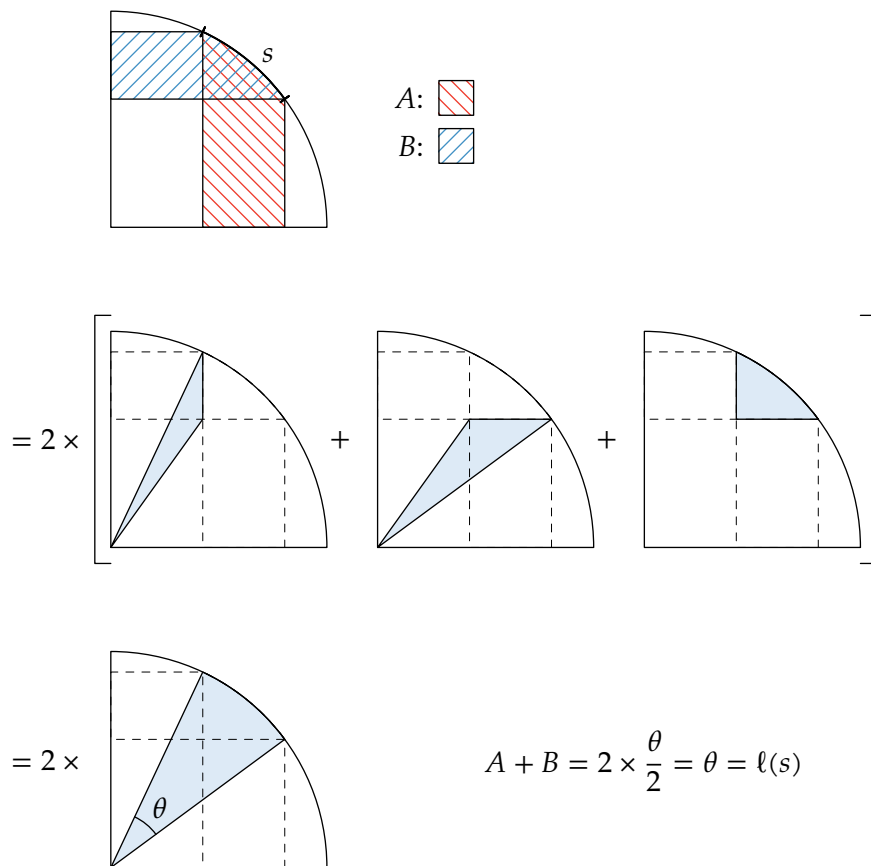
$$\widehat{C'C} = \widehat{D'D}, \quad \widehat{C'D'} = \widehat{CD}$$

$$C'D' = CD$$

A Putnam area problem

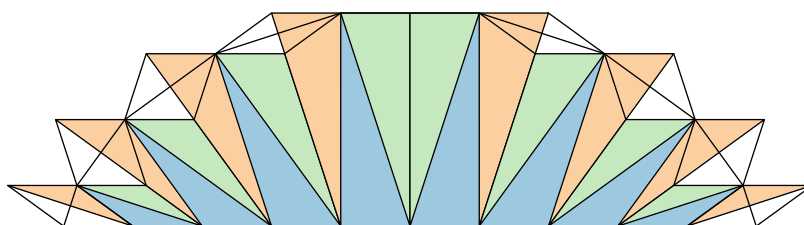
Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x -axis, and let B be the area of the region lying to the right of the y -axis and to the left of s . Prove that $A + B$ depends only on the arc length, and not on the position, of s .

SOLUTION:



The area under a polygonal arch

The area under a polygonal arch generated by one vertex of a regular n -gon rolling along a straight line is three times the area of the polygon.

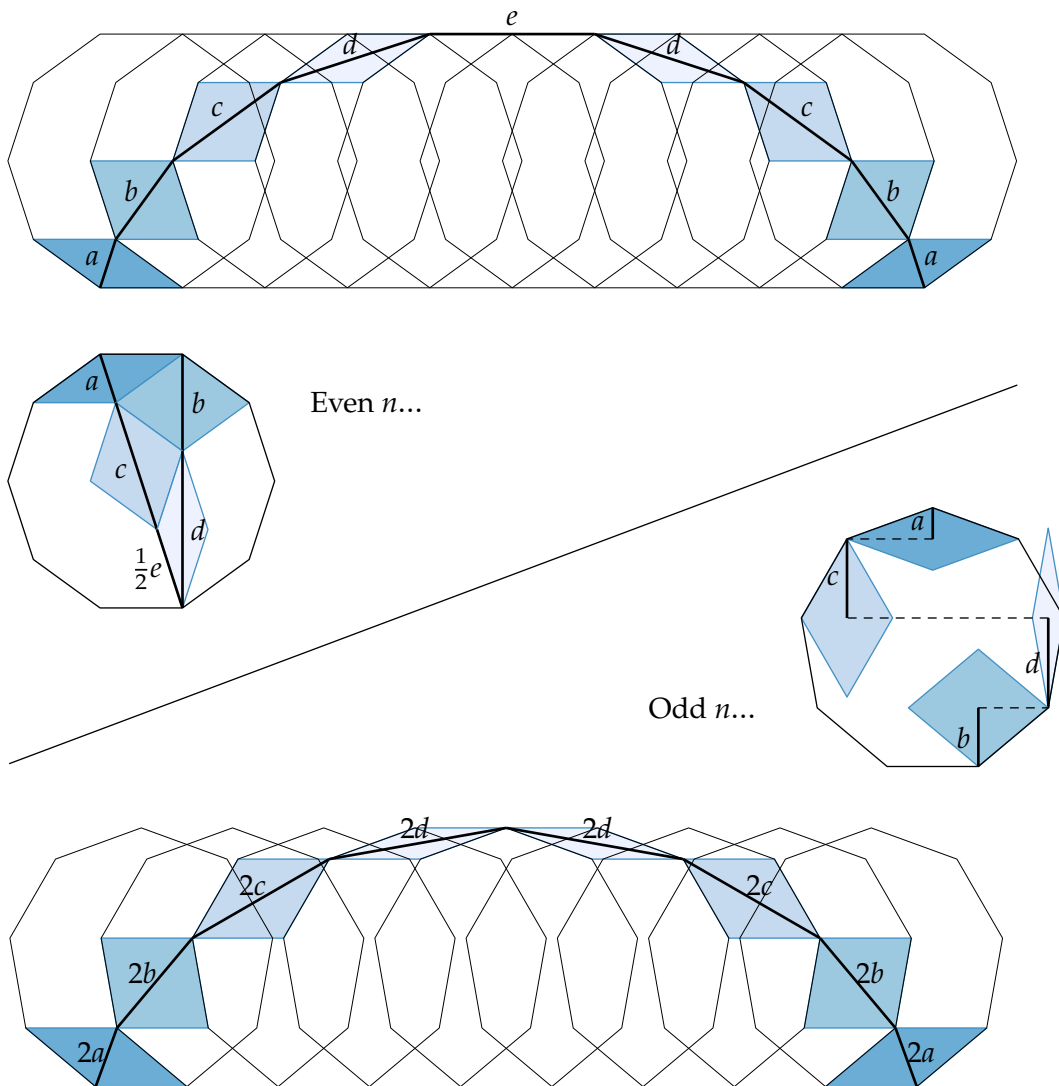


COROLLARY: The area under one arch of a cycloid is three times the area of the generating circle.

— Philip R. Mallinson

The length of a polygonal arch

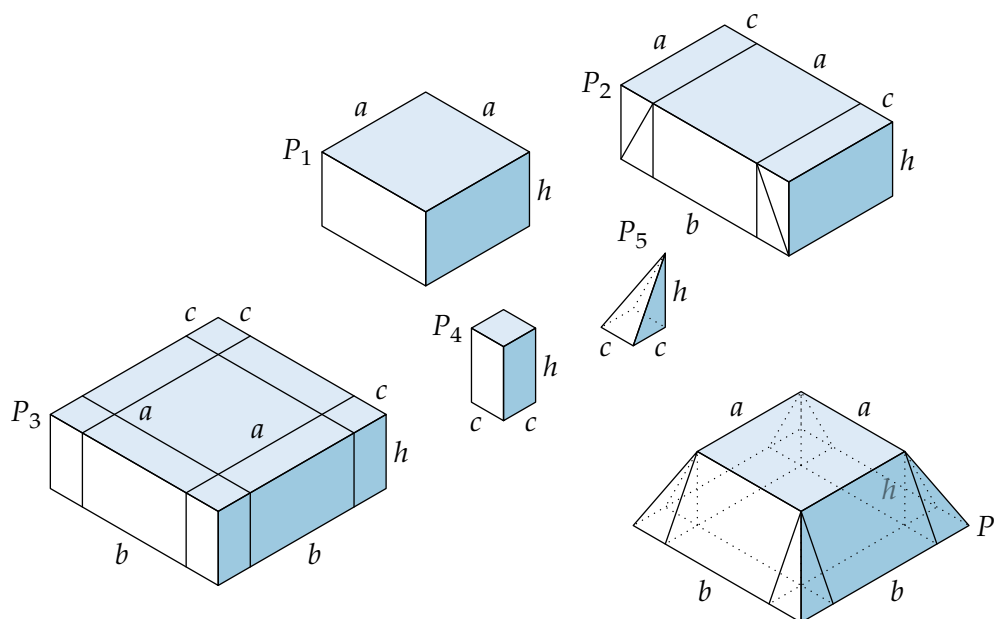
The length of the polygonal arch generated by one vertex of a regular n -gon rolling along a straight line is four times the length of the in-radius plus four times the length of the circum-radius of the n -gon.



COROLLARY: The arc length of one arch of a cycloid is eight times the radius of the generating circle.

— Philip R. Mallinson

The volume of a frustum of a square pyramid



$$P_4 = 3P_5$$

$$P_1 + P_3 = 2P_2 + 4P_4 \Rightarrow P_1 + P_2 + P_3 = 3P_2 + 12P_5 = 3(P_2 + 4P_5) = 3P$$

$$\therefore V = \frac{h}{3} (a^2 + ab + b^2)$$

— Sidney J. Kung

The product of four (positive) numbers in arithmetic progression is always the difference of two squares

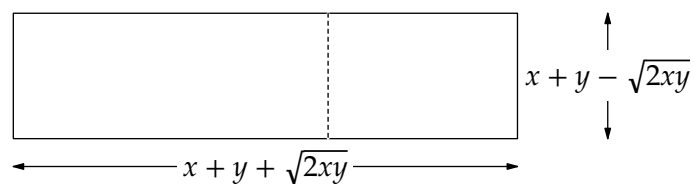
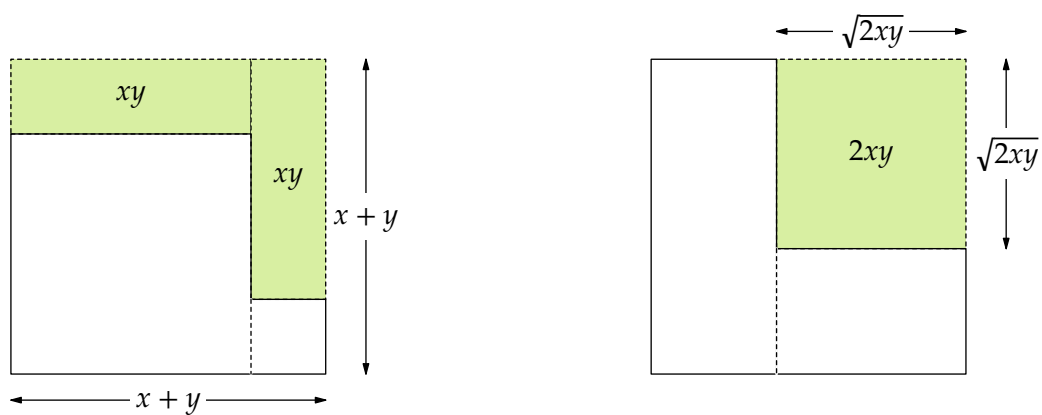
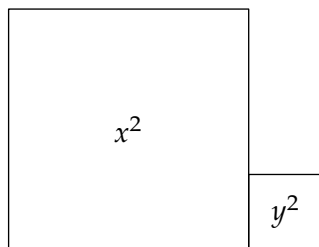


$$a(a+d)(a+2d)(a+3d) = (a^2 + 3ad + d^2)^2 - (d^2)^2$$

— RBN

Algebraic areas III: Factoring the sum of two squares

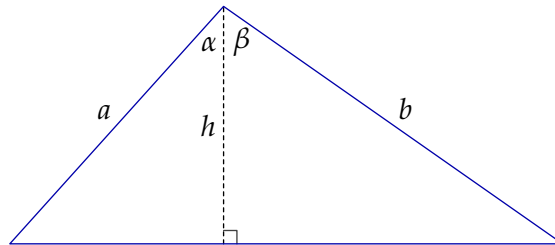
$$x^2 + y^2 = (x + \sqrt{2xy} + y)(x - \sqrt{2xy} + y)$$



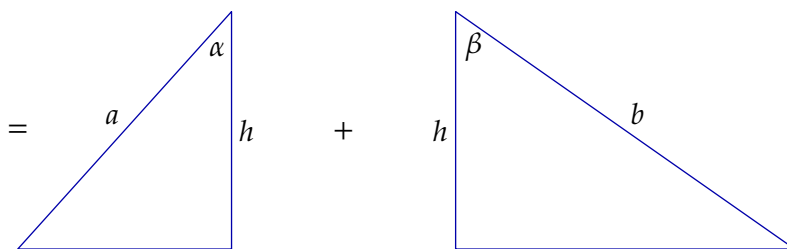
Trigonometry, Calculus, & Analytic Geometry

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Sine of the sum - II



$$\alpha, \beta \in (0, \pi/2) \implies h = a \cos \alpha = b \cos \beta$$



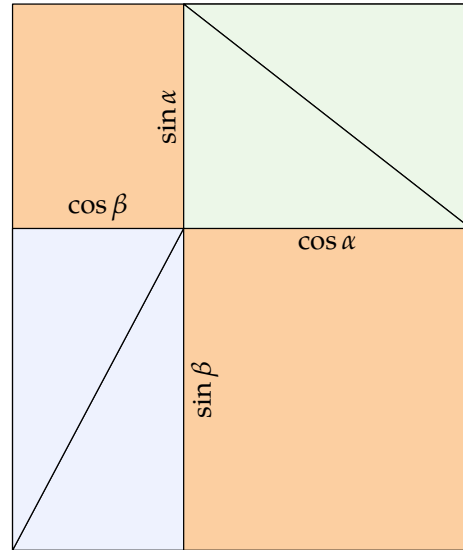
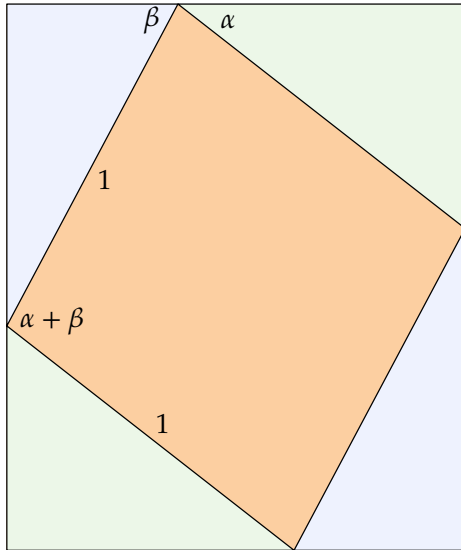
$$\begin{aligned} \frac{1}{2}ab \sin(\alpha + \beta) &= \frac{1}{2}ah \sin \alpha + \frac{1}{2}bh \sin \beta \\ &= \frac{1}{2}ab \cos \beta \sin \alpha + \frac{1}{2}ba \cos \alpha \sin \beta \\ \therefore \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{aligned}$$

— Christopher Brüningsen

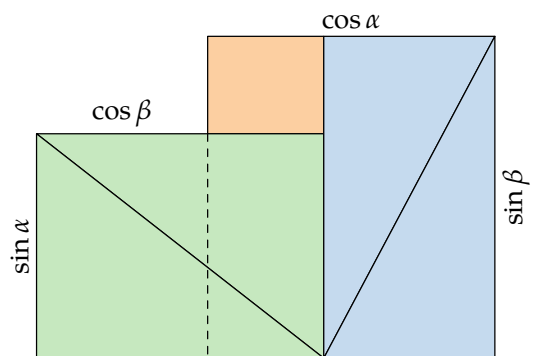
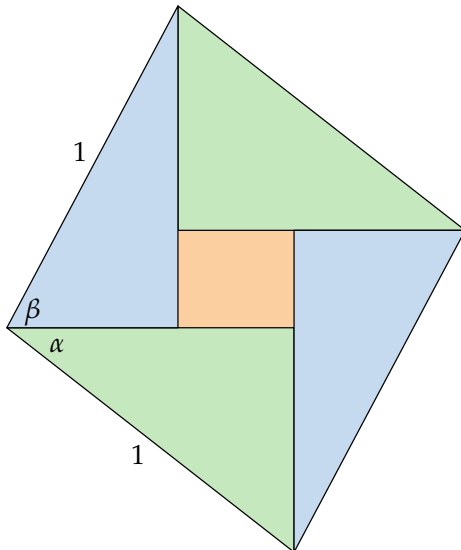
Sine of the sum – III

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

I.



II.



— Volker Priebe and Edgar A. Ramos

Cosine of the sum

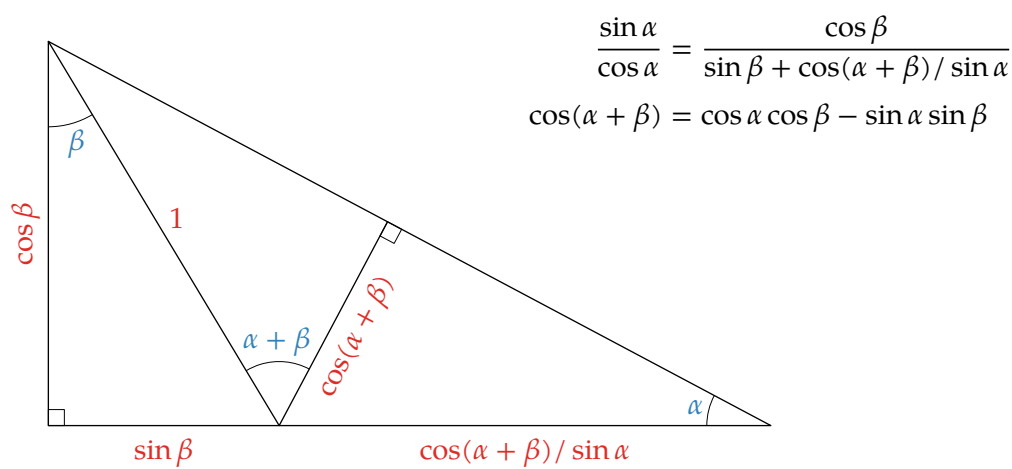
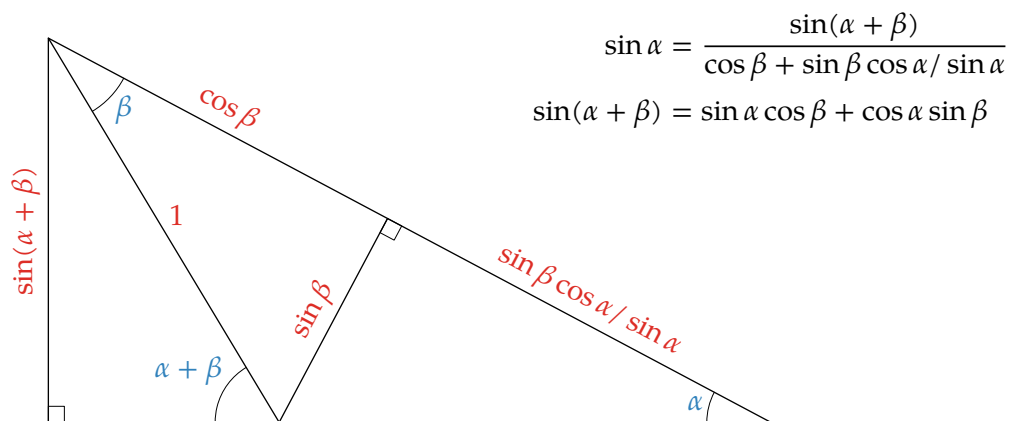


$$\frac{1}{2}ab \sin \left(\frac{\pi}{2} - (\alpha + \beta) \right) = \frac{1}{2}b \cos \alpha \cdot a \cos \beta - \frac{1}{2}b \sin \alpha \cdot a \sin \beta$$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

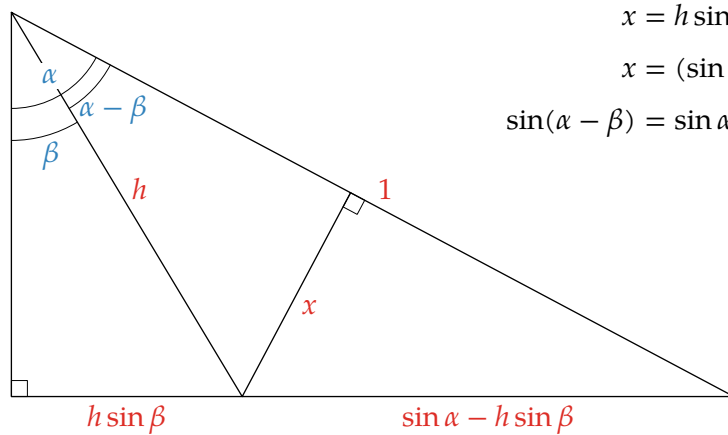
— Sidney H. Kung

Geometry of addition formulas



— Leonard M. Smiley

Geometry of subtraction formulas

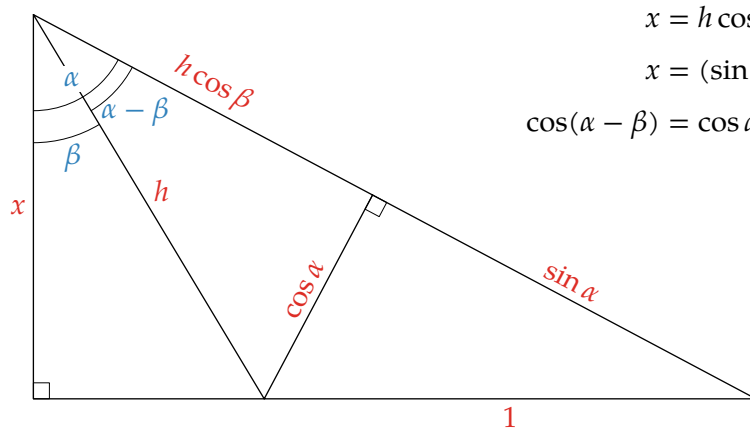


$$h = \cos \alpha / \cos \beta$$

$$x = h \sin(\alpha - \beta)$$

$$x = (\sin \alpha - h \sin \beta) \cos \alpha$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$h = \cos \alpha / \sin \beta$$

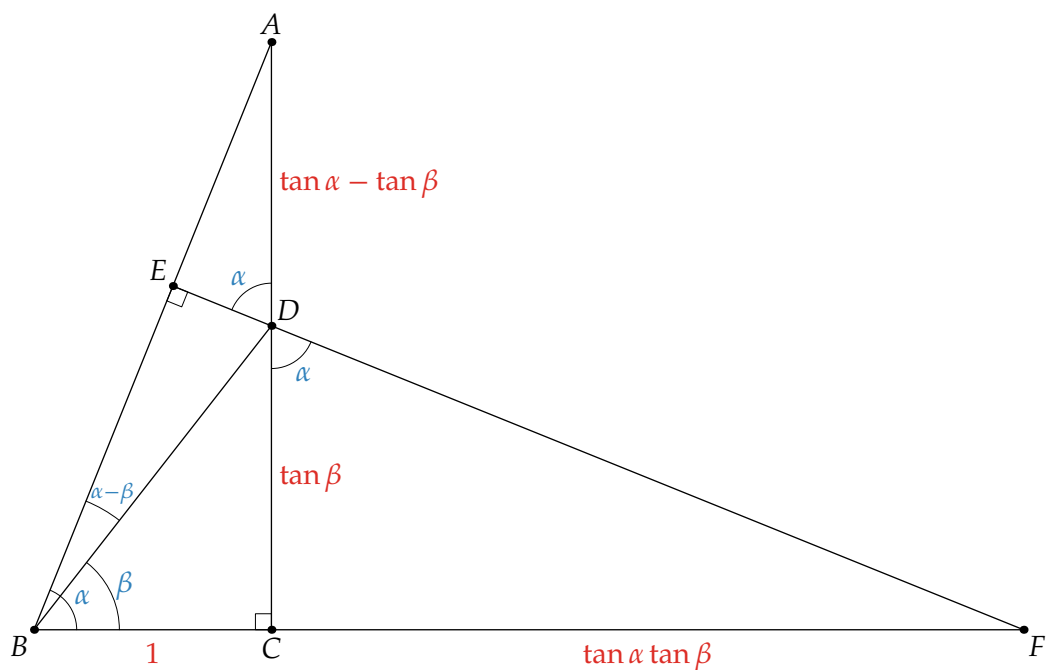
$$x = h \cos(\alpha - \beta)$$

$$x = (\sin \alpha + h \cos \beta) \cos \alpha$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

— Leonard M. Smiley

The difference identity for tangents I

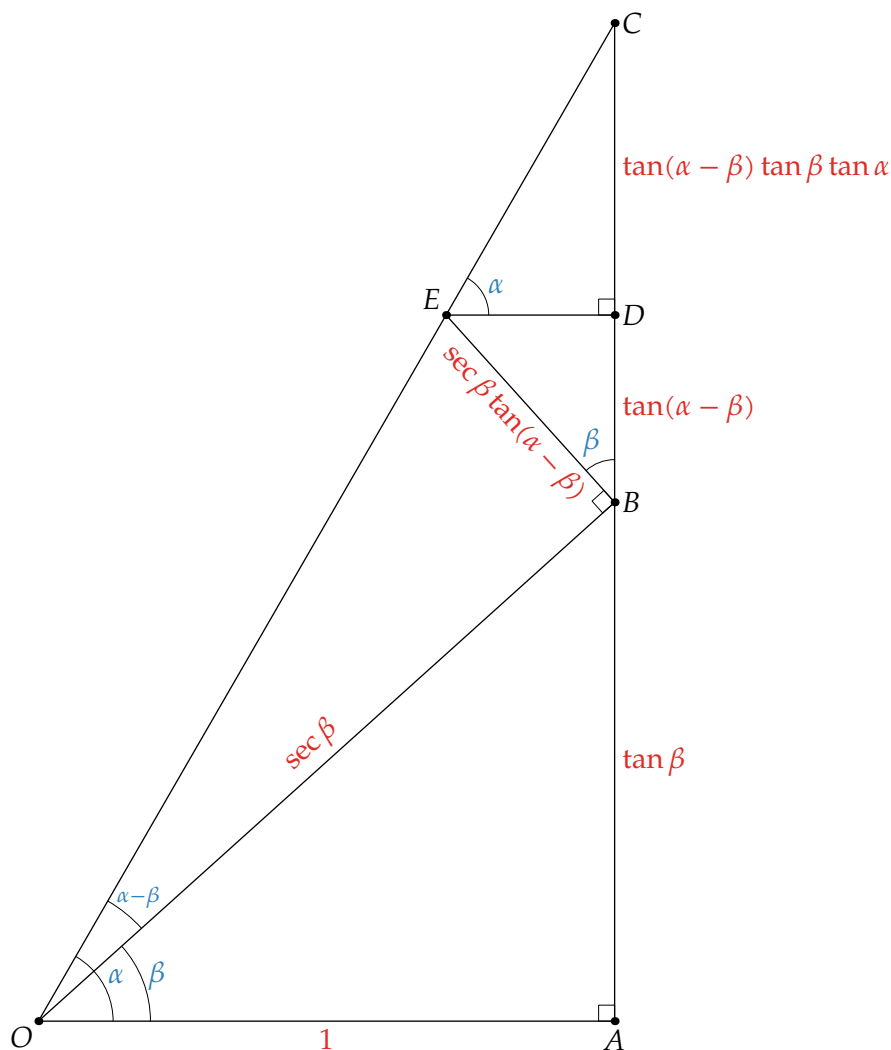


$$\frac{BF}{BE} = \frac{AD}{DE}$$

$$\therefore \tan(\alpha - \beta) = \frac{DE}{BE} = \frac{AD}{BF} = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

— Guanshen Ren

The difference identity for tangents II



$$AC - AB = BD + DC$$

$$\therefore \tan \alpha - \tan \beta = \tan(\alpha - \beta) + \tan \alpha \tan \beta \tan(\alpha - \beta)$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

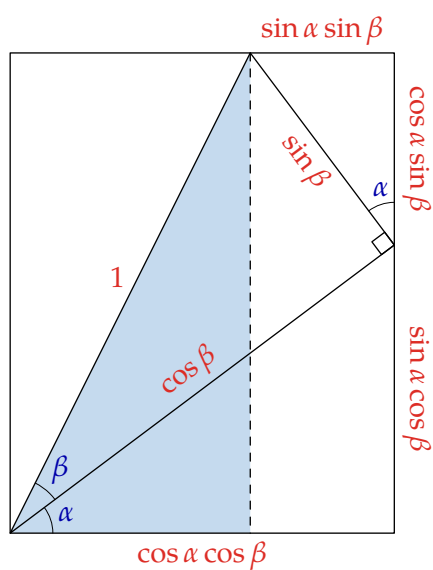
— Fukuzo Suzuki

One figure, six identities

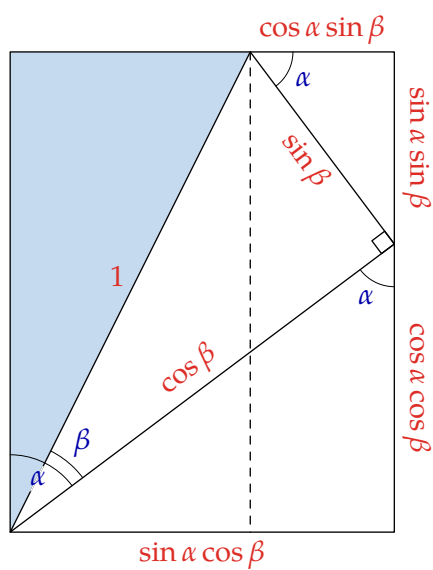


The figure

$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}$$

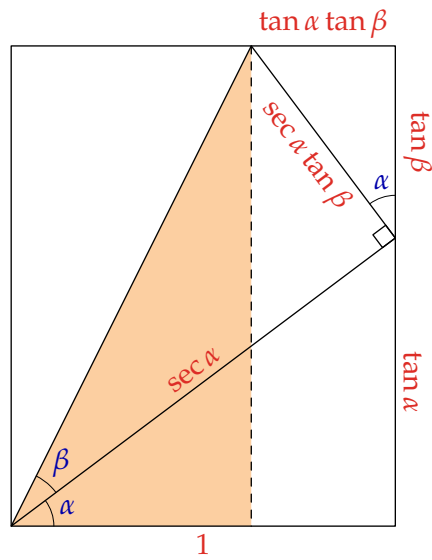


$$\begin{aligned}\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta\end{aligned}$$

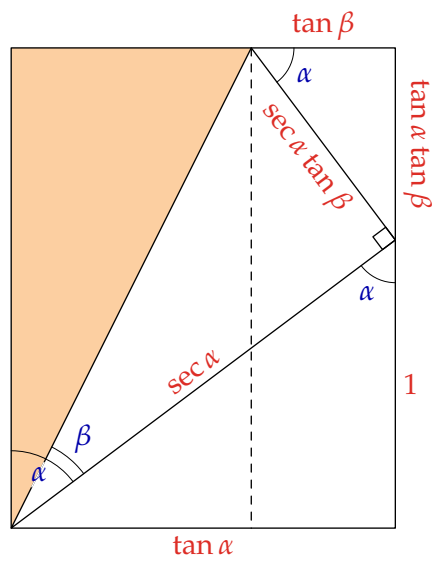


One figure, six identities

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 + \tan \alpha \tan \beta}$$

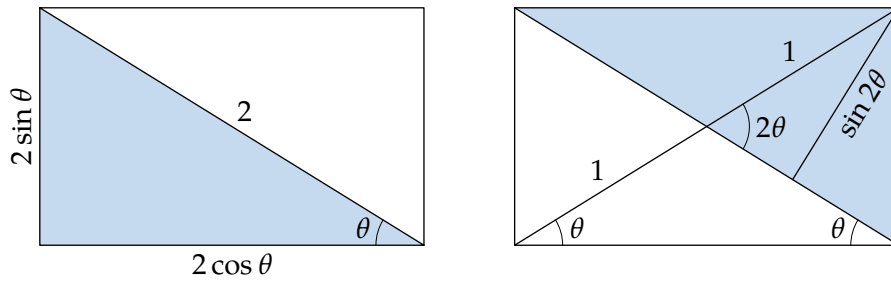


$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

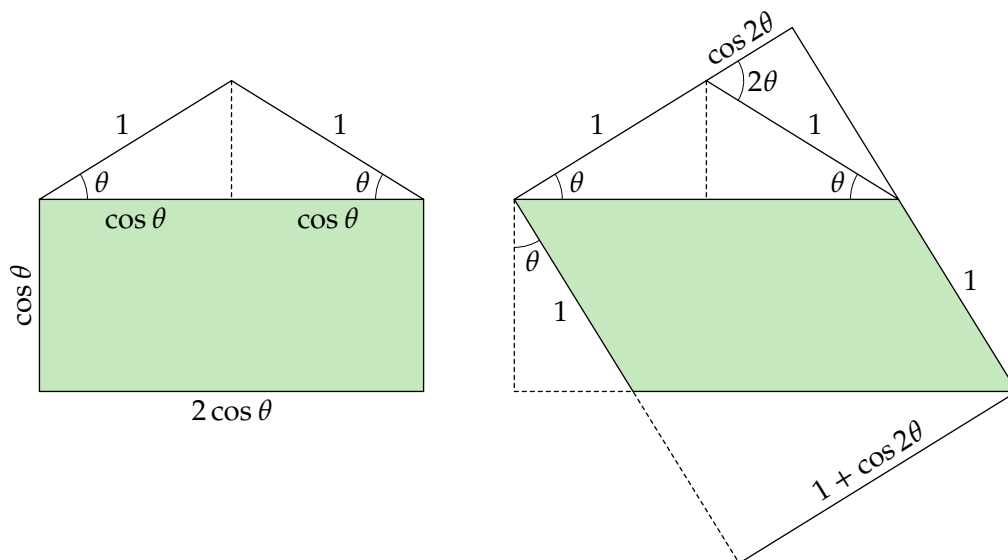


— RBN

The double-angle formulas II



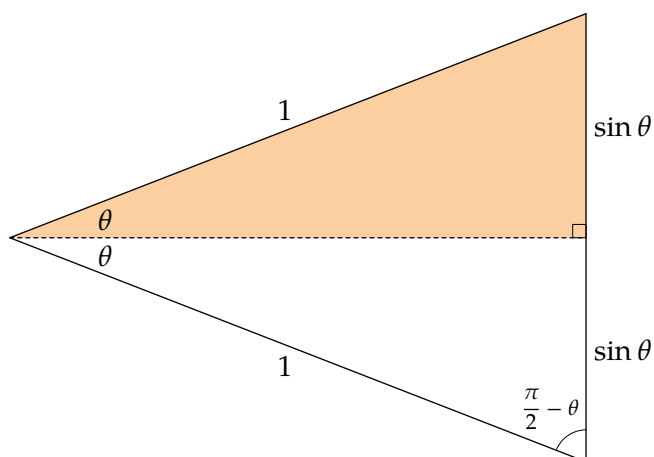
$$2 \sin \theta \cos \theta = \sin 2\theta$$



$$2 \cos^2 \theta = 1 + \cos 2\theta$$

— Yihnan David Gau

The double-angle formulas III (via the laws of sines and cosines)

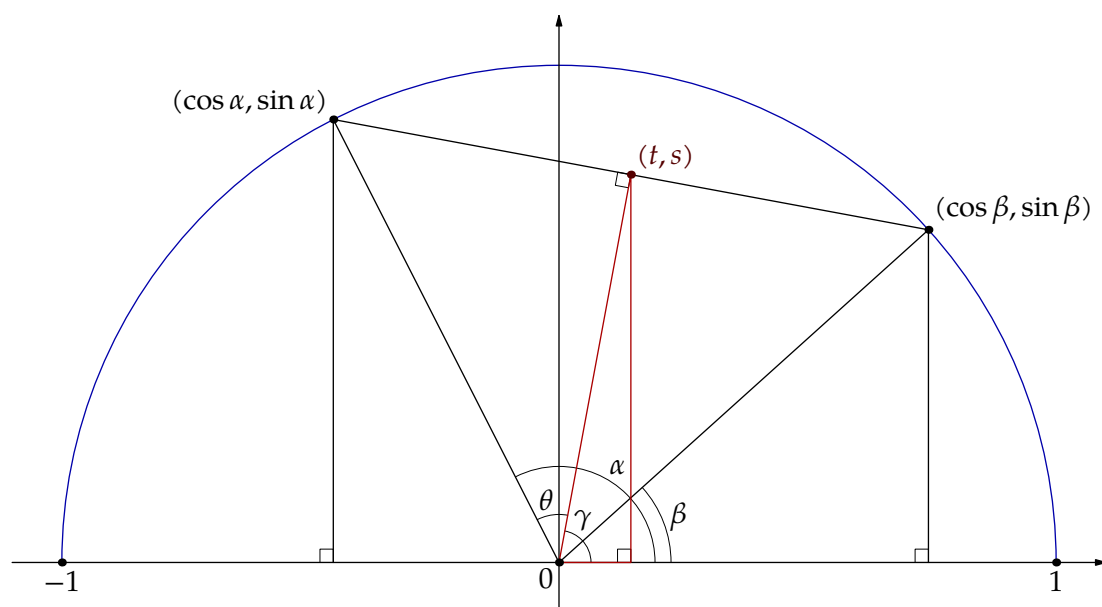


$$\frac{\sin 2\theta}{2 \sin \theta} = \frac{\sin(\pi/2 - \theta)}{1} = \cos \theta$$
$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$(2 \sin \theta)^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 2\theta$$
$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

— Sidney H. Kung

The sum-to-product identities I



$$\theta = \frac{\alpha - \beta}{2}, \quad \gamma = \frac{\alpha + \beta}{2}$$

$$\frac{\sin \alpha + \sin \beta}{2} = s = \cos \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

$$\frac{\cos \alpha + \cos \beta}{2} = t = \cos \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

— Sidney H. Kung

The difference-to-product identities I



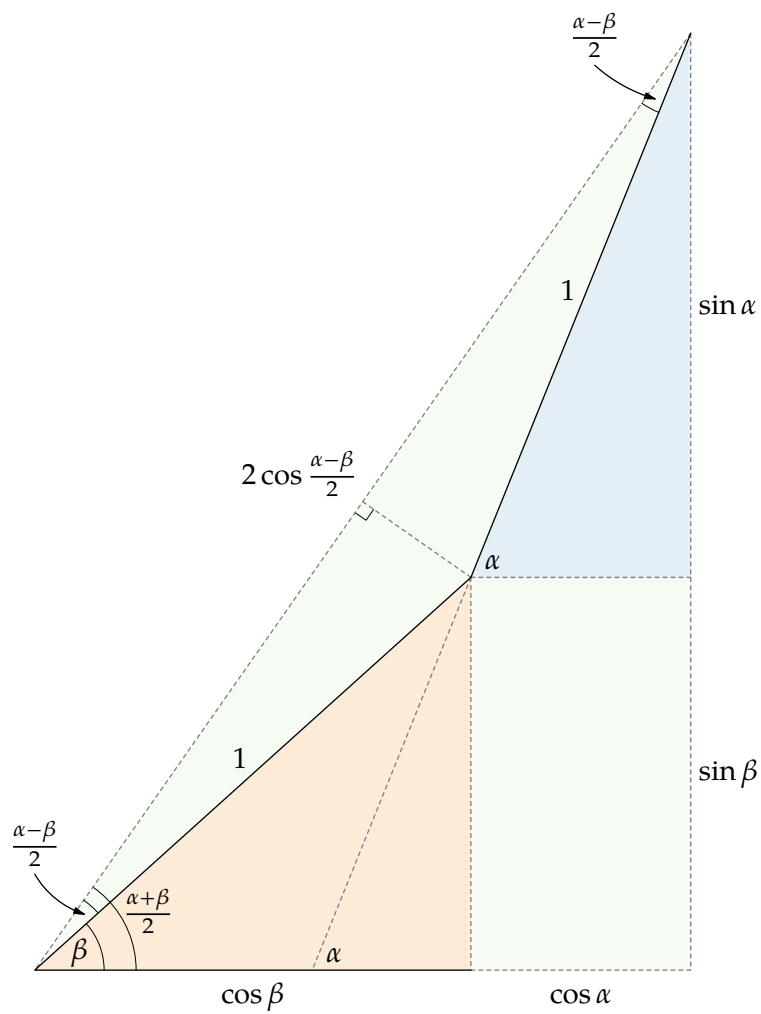
$$\theta = \frac{\alpha - \beta}{2}, \quad \gamma = \frac{\alpha + \beta}{2}$$

$$\sin \alpha - \sin \beta = v = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$\cos \beta - \cos \alpha = u = 2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

— Sidney H. Kung

The sum-to-product identities II

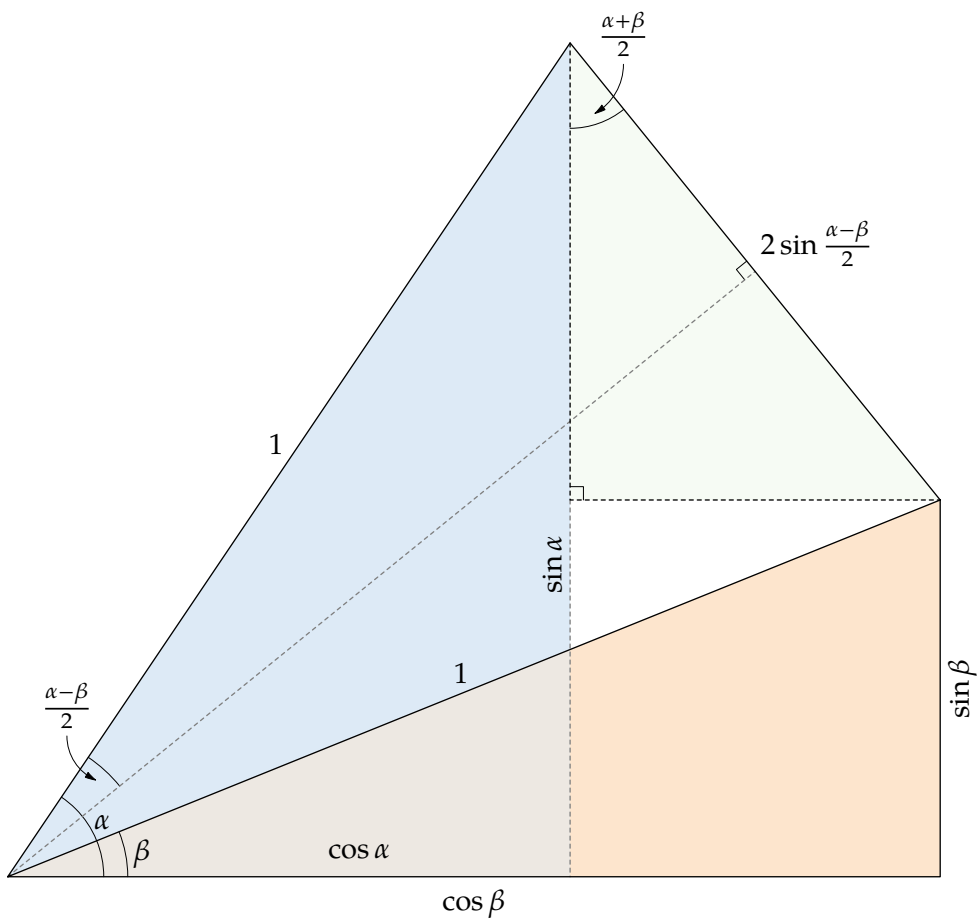


$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$\sin \alpha + \sin \beta = 2 \cos \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

— Yukio Kobayashi

The difference-to-product identities II



$$\cos \beta - \cos \alpha = 2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

— Yukio Kobayashi

Adding like sines



$$R_\phi = \sqrt{A^2 + B^2 + 2AB \cos \phi}, \quad \tan \theta = \frac{B \sin \phi}{A + B \cos \phi}$$

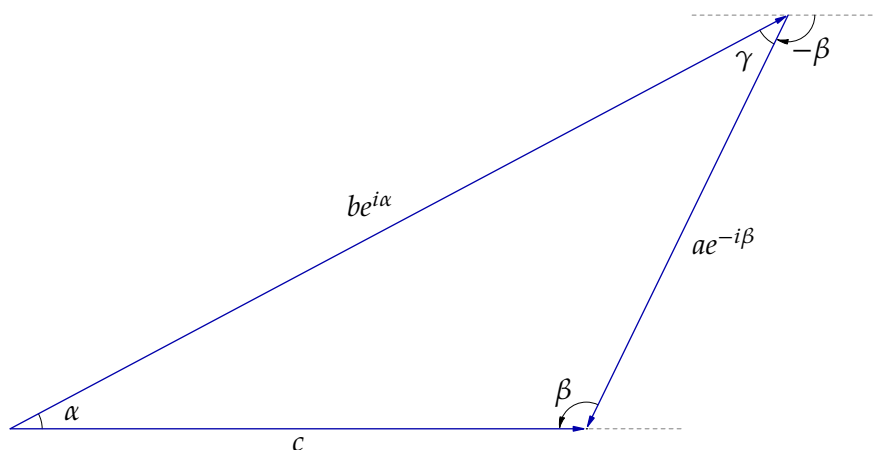
$$A \sin x + B \sin(x + \phi) = R_\phi \sin(x + \theta)$$

$$\phi = \pi/2 \Rightarrow \tan \theta = B/A$$

$$\therefore A \sin x + B \cos x = \sqrt{A^2 + B^2} \sin(x + \theta)$$

— Rick Mabry and Paul Deiermann

A complex approach to the laws of sines and cosines



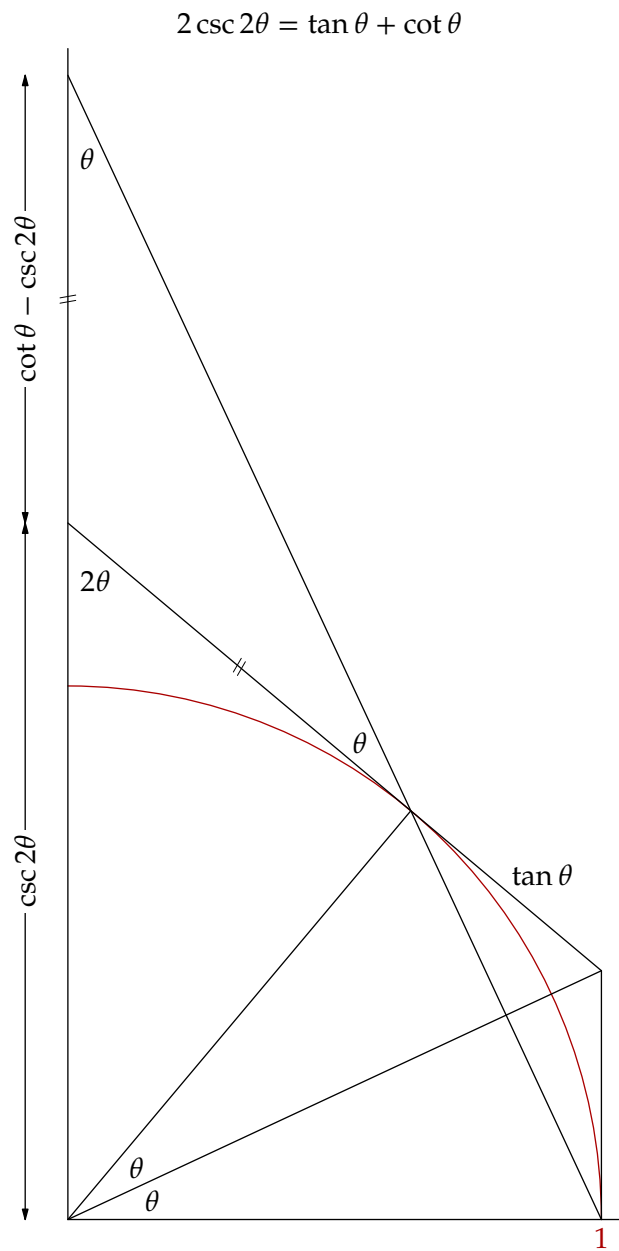
$$c = be^{i\alpha} + ae^{-i\beta} = (b \cos \alpha + a \cos \beta) + i(b \sin \alpha - a \sin \beta)$$

if c is real, then $b \sin \alpha - a \sin \beta = 0$, hence $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$

$$\begin{aligned} c^2 = |c|^2 &= (b \cos \alpha + a \cos \beta)^2 + (b \sin \alpha - a \sin \beta)^2 \\ &= a^2 + b^2 + 2ab \cos(\alpha + \beta) \\ &= a^2 + b^2 - 2ab \cos \gamma \end{aligned}$$

— William V. Grounds

Eisenstein's duplication formula



G. Eisenstein, *Mathematische Werke*, Chelsea, NY. 1975, p.411

A familiar limit for e

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$



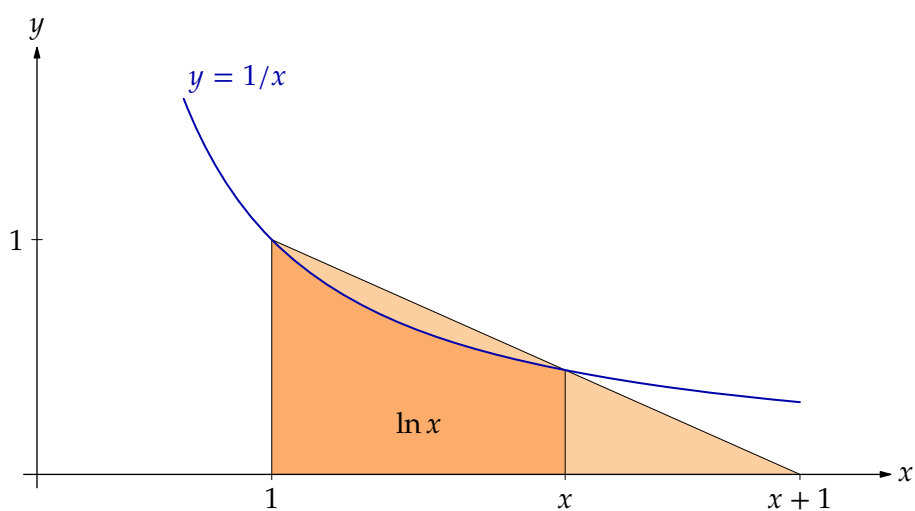
$$\frac{1}{n} \cdot \frac{n}{n+1} \leq \ln \left(1 + \frac{1}{n}\right) \leq \frac{1}{n} \cdot 1$$

$$\frac{n}{n+1} \leq n \cdot \ln \left(1 + \frac{1}{n}\right) \leq 1$$

$$\therefore \lim_{n \rightarrow \infty} \ln \left(\left(1 + \frac{1}{n}\right)^n \right) = 1$$

A common limit

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = 0$$



$$\ln x < \frac{1}{2}x$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^{x-\ln x}} = 0$$

— Alan H. Stein and Dennis McGavran

Geometric evaluation of a limit

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}} = 2$$



— Guanshen Ren

The derivative of the inverse sine



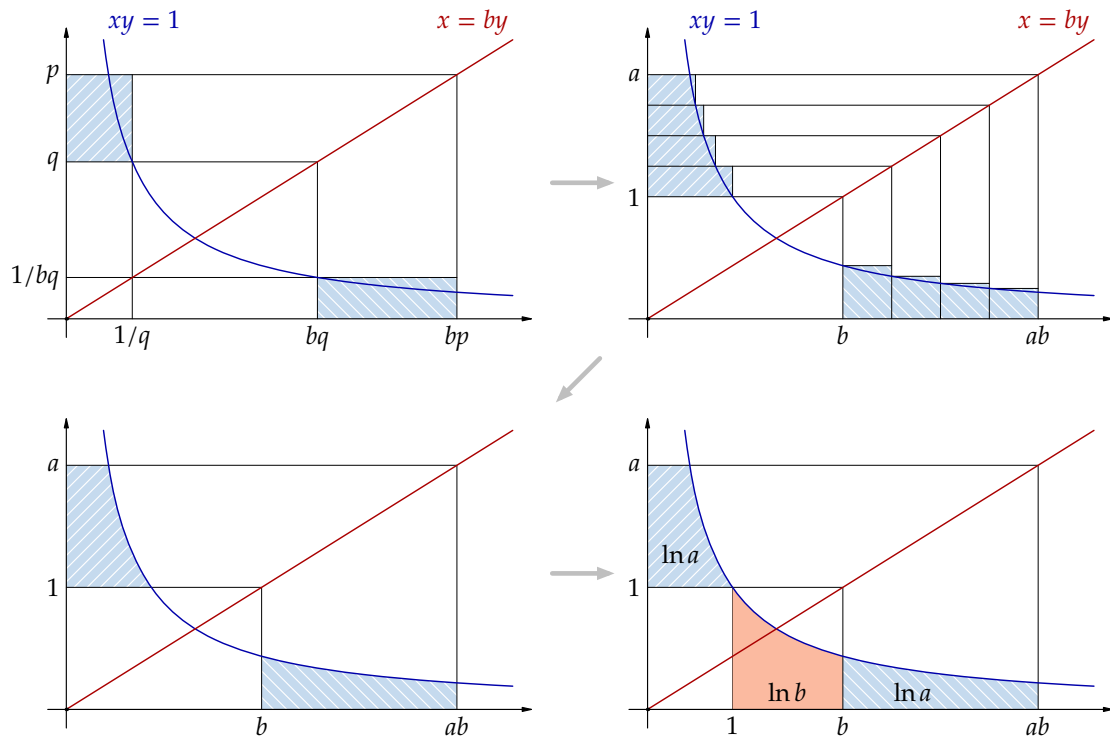
$$L = \sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

$$\therefore \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

— Craig Johnson

The logarithm of a product

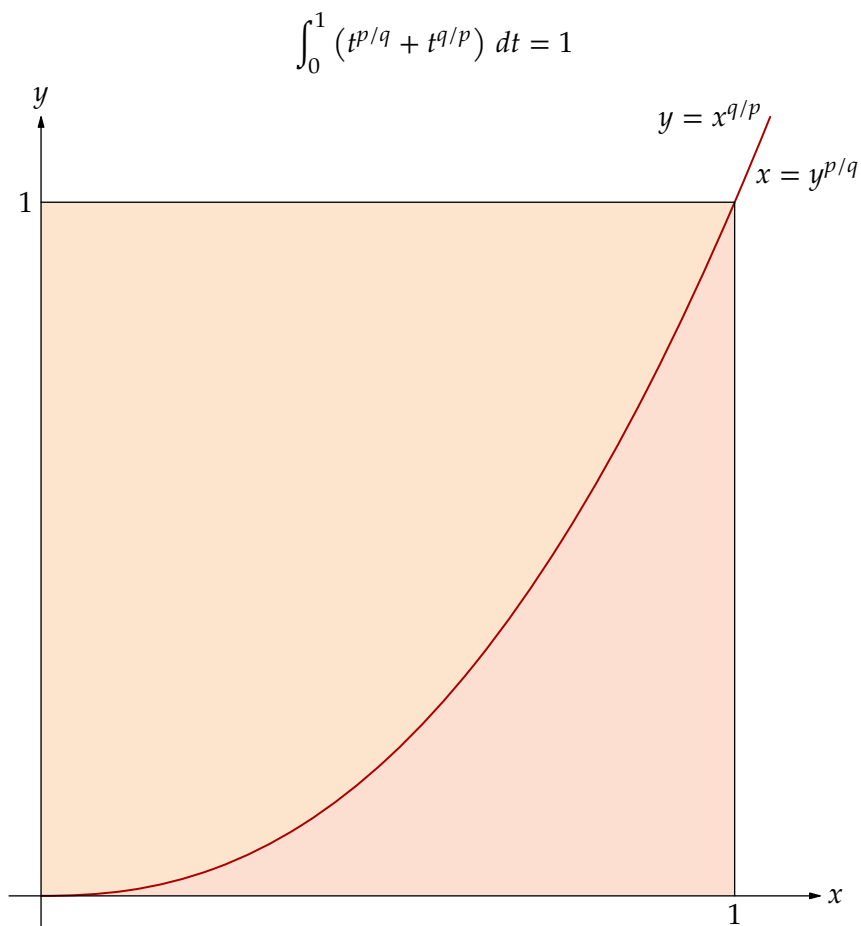
$$\ln ab = \ln a + \ln b$$



$$\text{Area}(\text{shaded}) = \text{Area}(\text{shaded})$$

— Jeffery Ely

An integral of a sum of reciprocal powers



— Peter R. Newbury

The arctangent integral



— Aage Bondesen

The method of last resort — Weierstrass substitution



$$u = \tan \frac{\theta}{2}, \quad DE = 2 \sin \frac{\theta}{2} = \frac{2u}{\sqrt{1+u^2}}$$

$$\frac{CE}{DE} = \frac{OA}{BA} \implies \sin \theta = \frac{2u}{1+u^2}$$

$$\frac{CD}{DE} = \frac{OB}{BA} \implies \cos \theta = \frac{1-u^2}{1+u^2}$$

— Paul Deiermann

The trapezoidal rule — for increasing functions

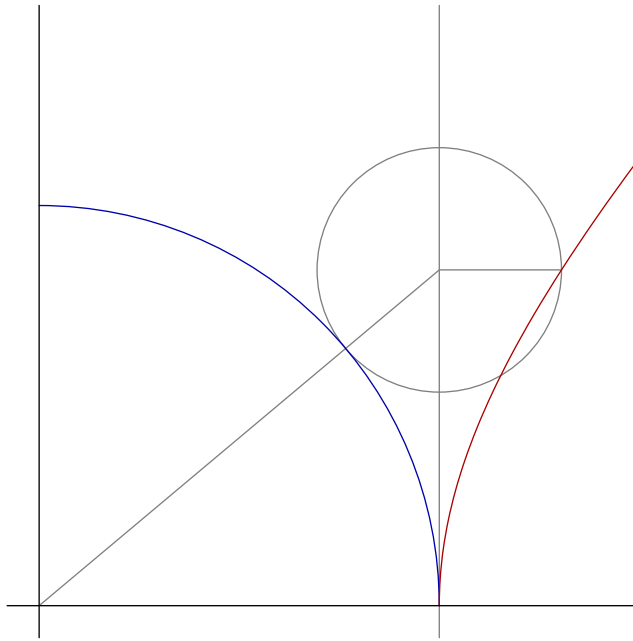


$$\int_a^b f(x) dx = \sum_{i=0}^{n-1} f(x_i) \frac{b-a}{n} + \frac{1}{2} \left(f(x_n) - f(x_0) \right) \frac{b-a}{n}$$

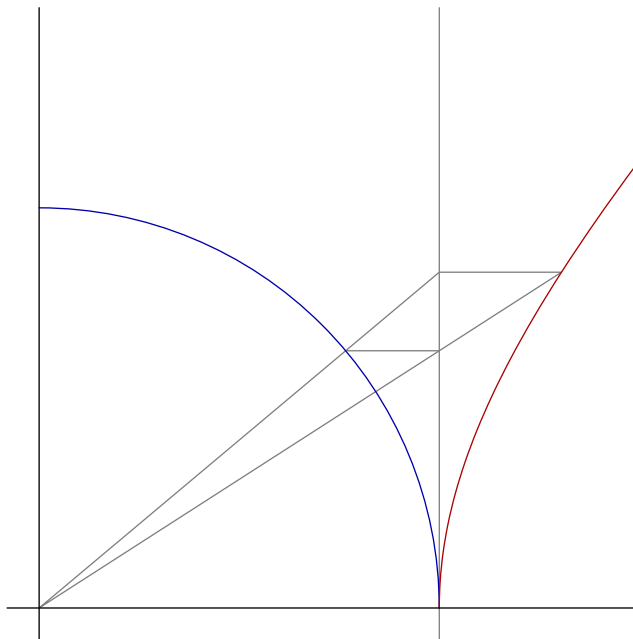
— Jesús Urías

Construction of a hyperbola

I.



II.



— Ernest J. Eckert

The focus and directrix of an ellipse

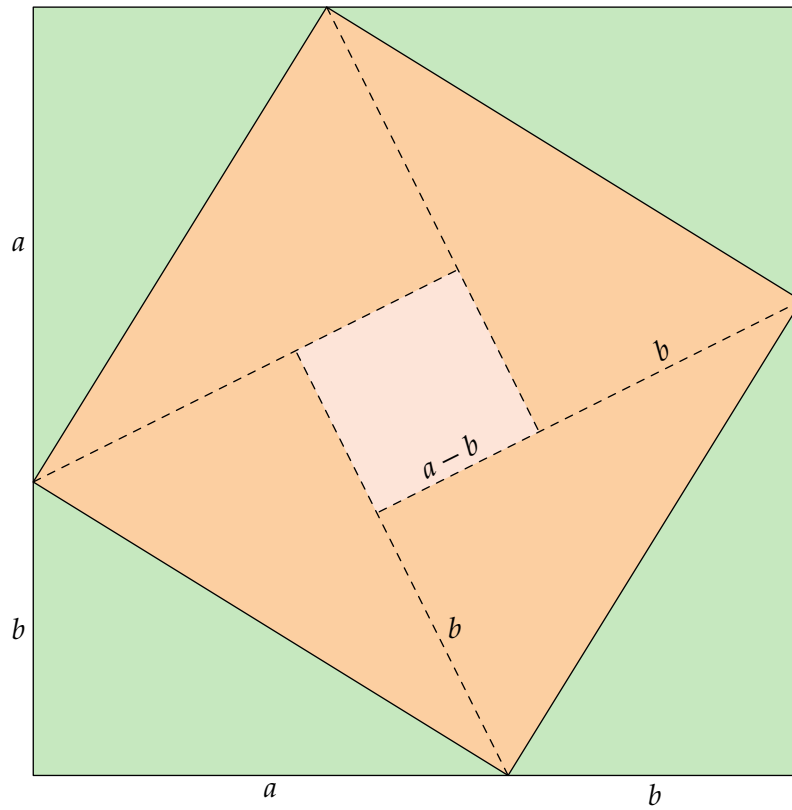


— Michel Bataille

Inequalities

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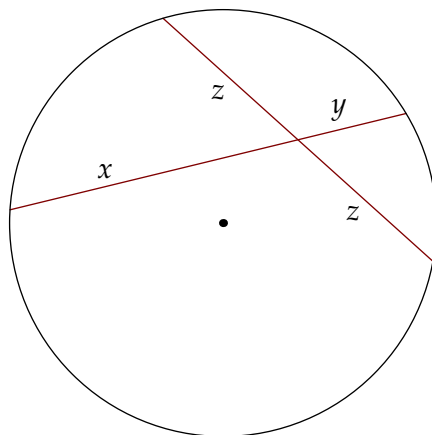
The arithmetic mean – geometric mean inequality IV



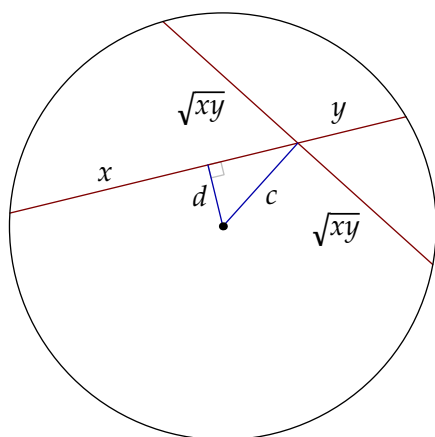
$$(a+b)^2 \geq 4ab \implies \frac{a+b}{2} \geq \sqrt{ab}$$

— Ayoub B. Ayoub

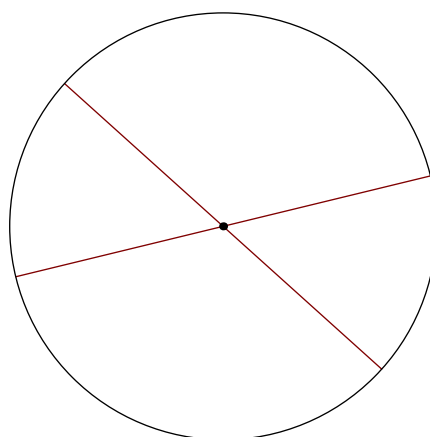
The arithmetic mean – geometric mean inequality V



$$z^2 = xy$$



$$d < c \implies x + y > 2\sqrt{xy}$$



$$d = c = 0 \implies x + y = 2\sqrt{xy}$$

— Sidney H. Kung

The arithmetic mean – geometric mean inequality VI



$$0 < a < b, \quad 0 < t < 1 \quad \Rightarrow \quad (1-t)a + tb > a^{1-t}b^t$$

$$t = \frac{1}{2} \quad \Rightarrow \quad \frac{a+b}{2} > \sqrt{ab}$$

— Michael K. Brozinsky

The arithmetic mean – geometric mean inequality for three positive numbers

LEMMA: $ab + bc + ac \leq a^2 + b^2 + c^2$



THEOREM: $3abc \leq a^3 + b^3 + c^3$



— Claudi Alsina