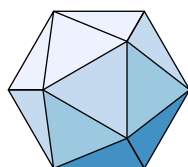


# Proofs without words II

More exercises in METAPOST

Toby Thurston

March 2020



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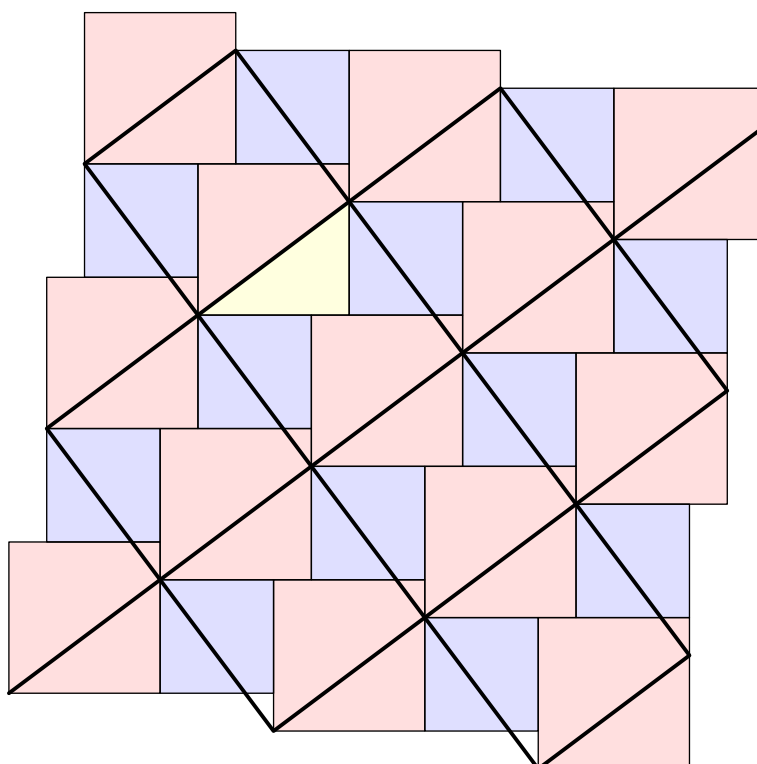
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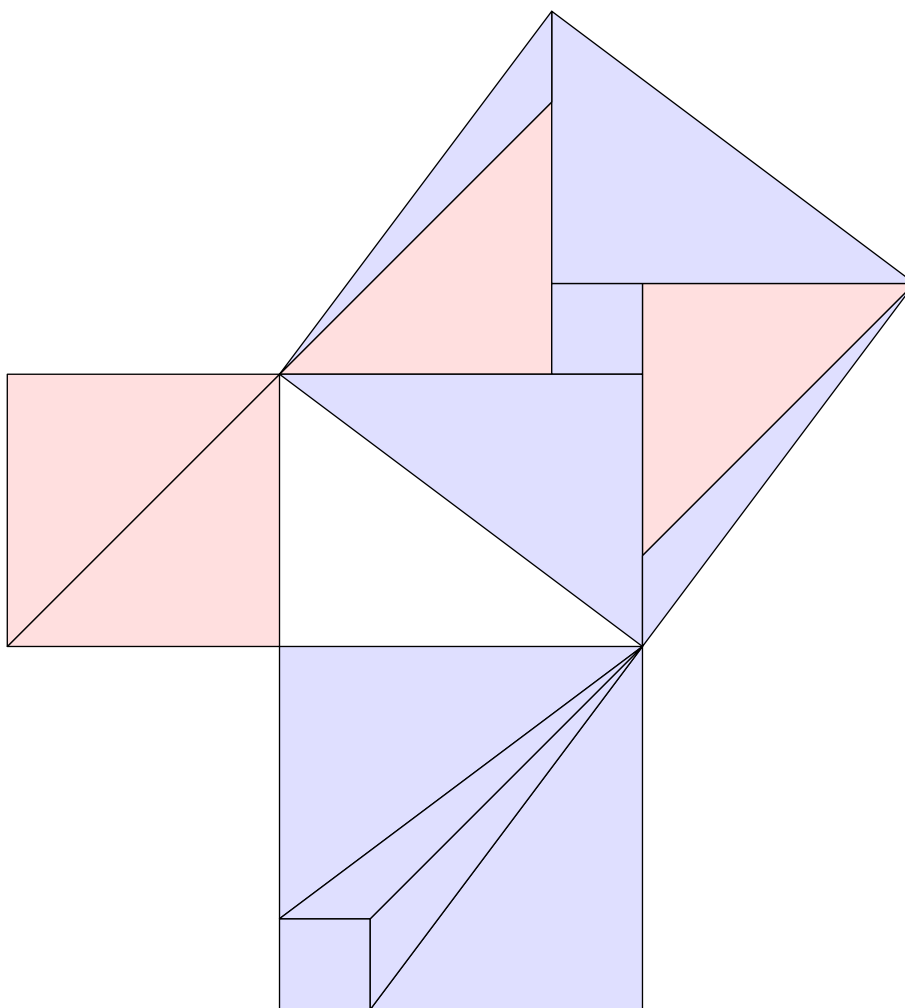
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## The Pythagorean theorem VII



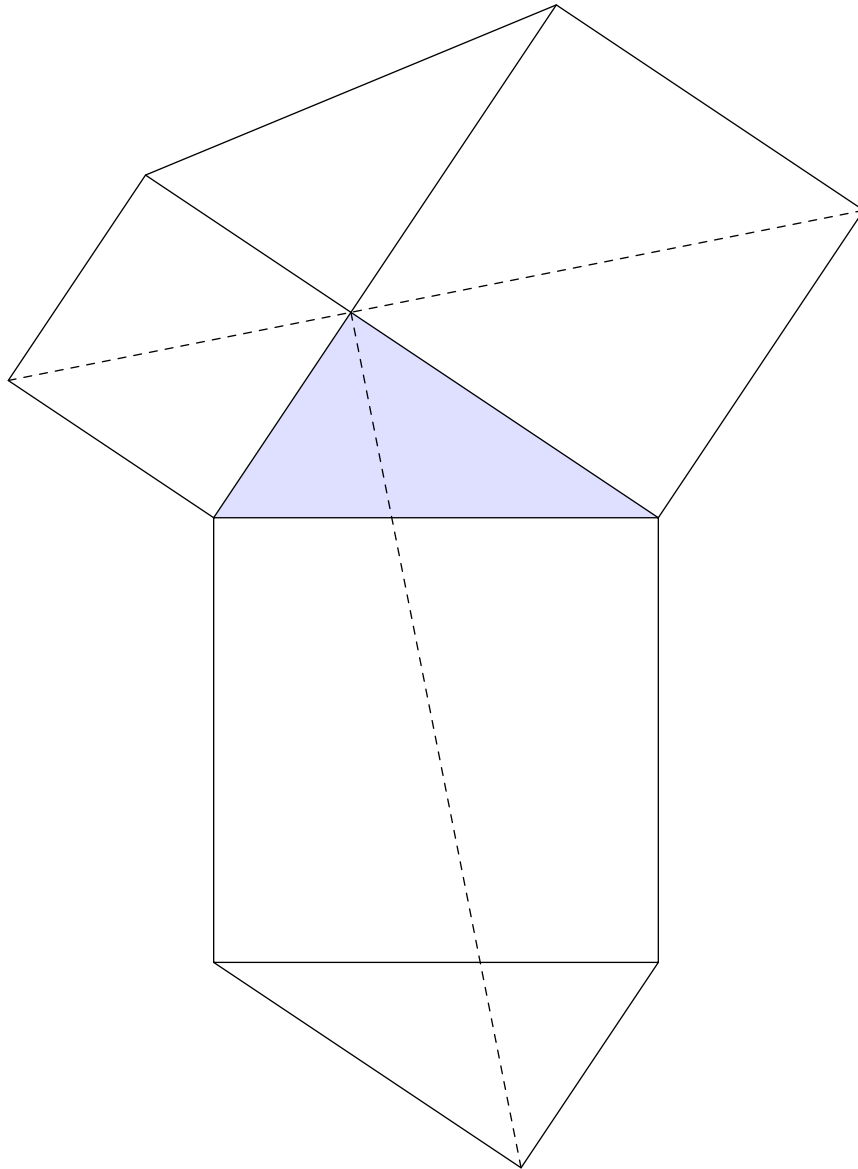
— Annairizi of Arabia (circa 900)

## The Pythagorean theorem VIII



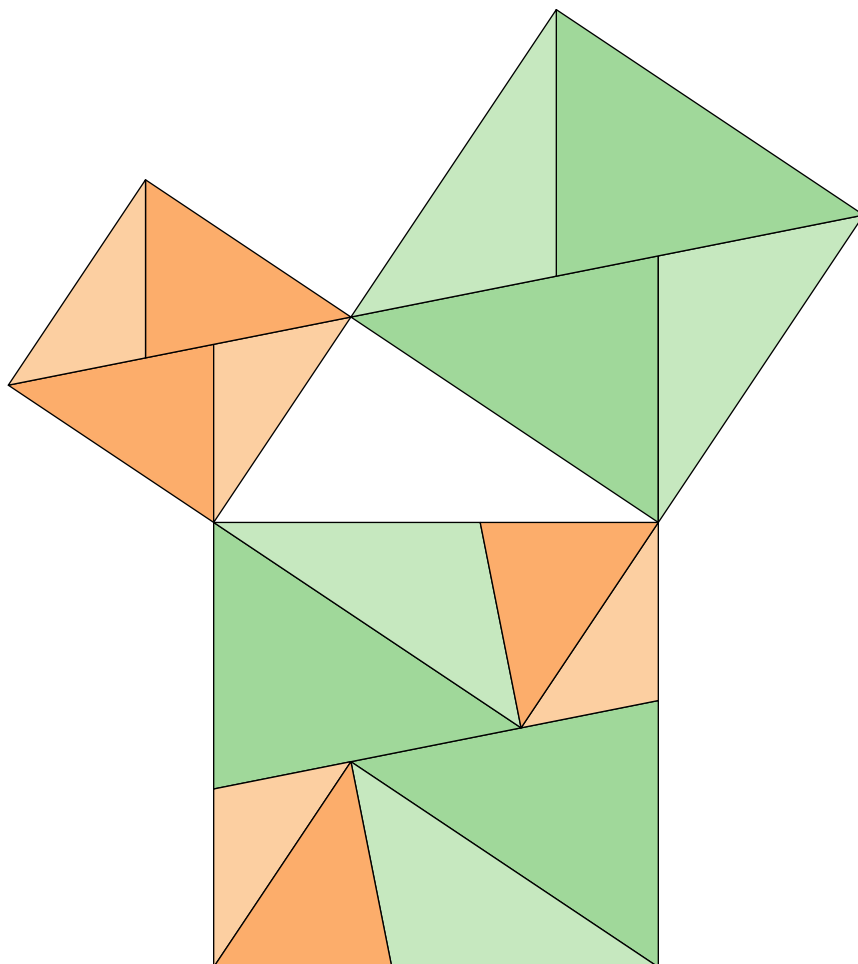
— Liu Hui (3rd century A.D.)

## The Pythagorean theorem IX



— Leonardo da Vinci (1452–1519)

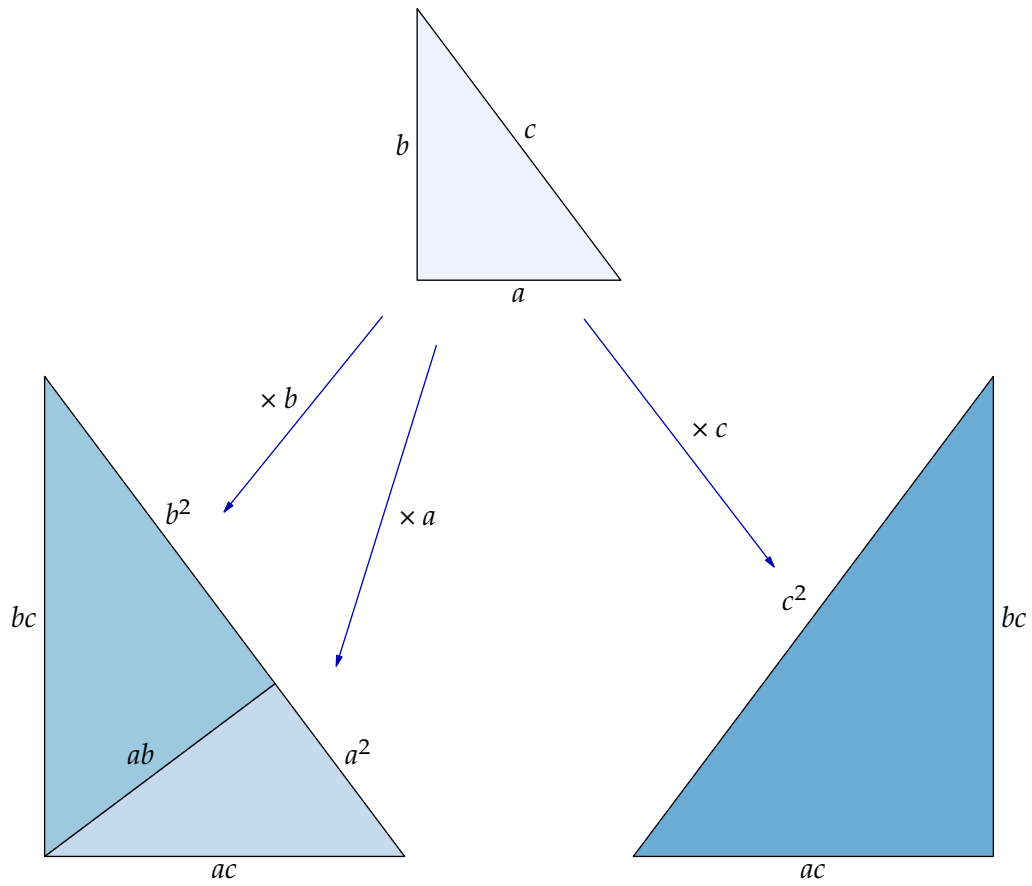
## The Pythagorean theorem X



— J. E. Böttcher

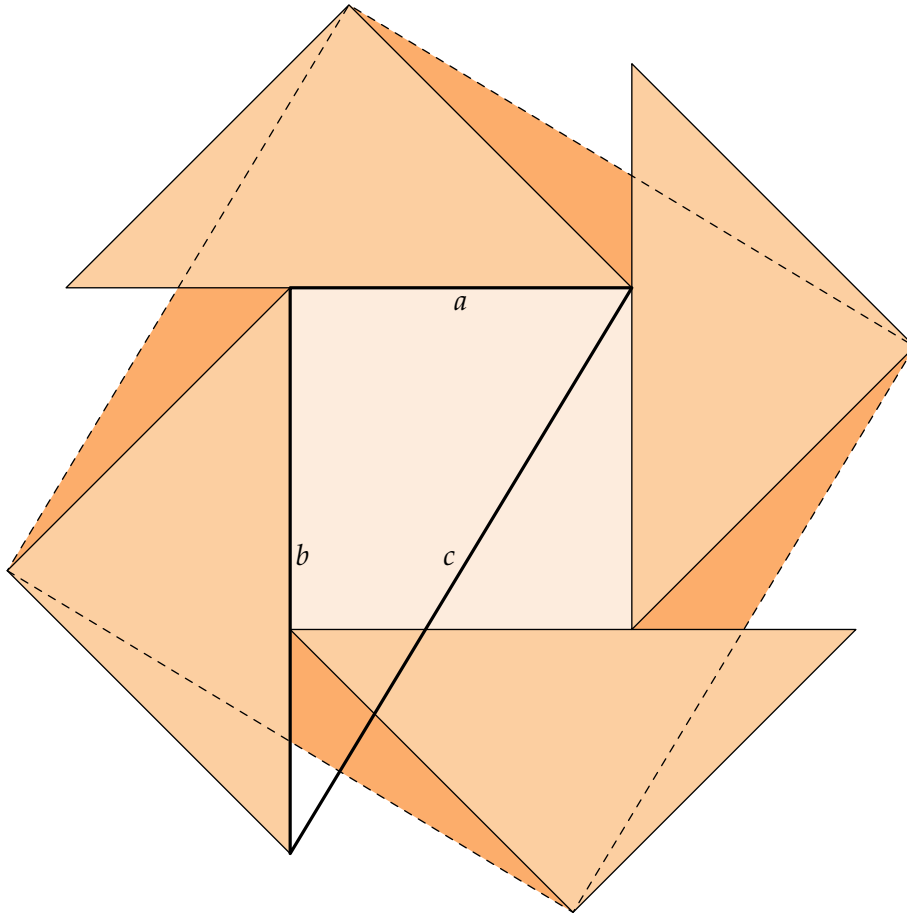


## The Pythagorean theorem XI



— Frank Burk

## The Pythagorean theorem XII

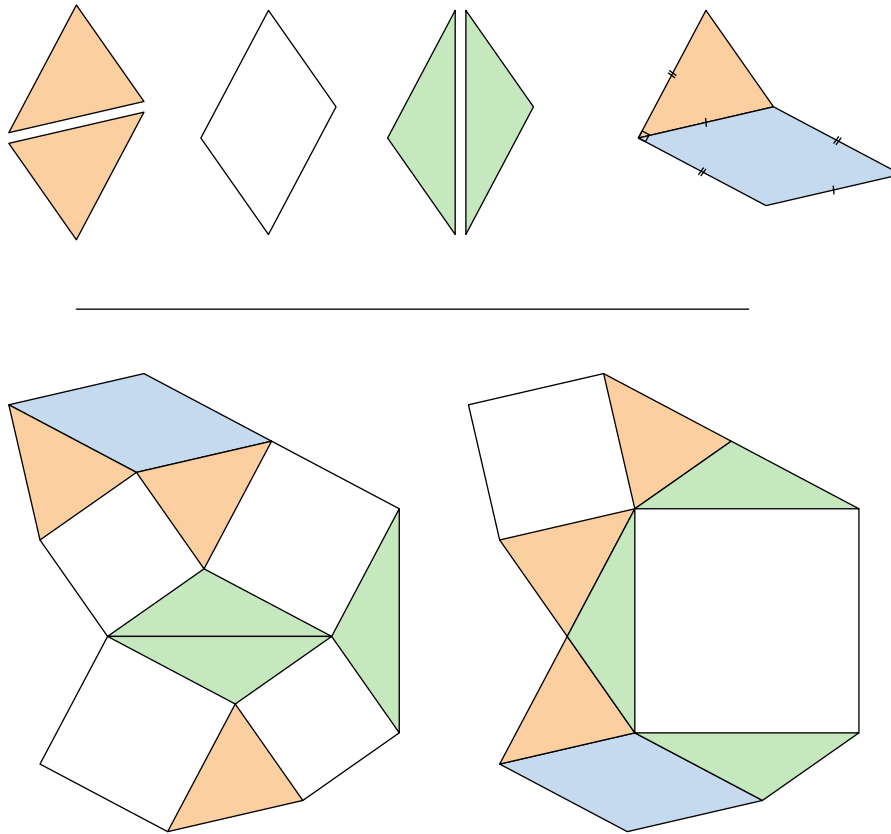


$$a^2 + b^2 = c^2$$

— Poo-Sung Park

## A generalization from Pythagoras

The sum of the area of two squares, whose sides are the lengths of two diagonals of a parallelogram, is equal to the sum of the area of four squares, whose sides are its four sides.



COROLLARY: The Pythagorean theorem (when the parallelogram is a rectangle).

— David S. Wise

## A theorem of Hippocrates of Chios (circa 440 BC)

The combined area of the lunes constructed on the legs of a given right angle triangle is equal to the area of the triangle.

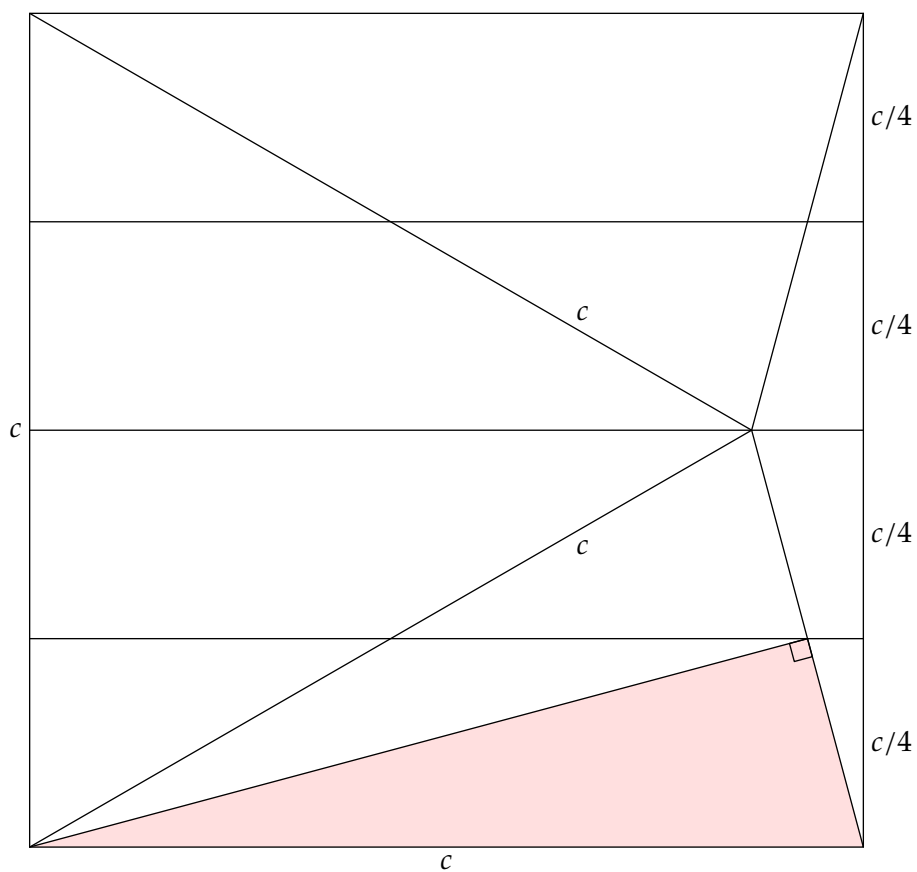


$$\begin{aligned}
 A_1 + A_2 &= A_3 \\
 (L_1 + S_1) + (L_2 + S_2) &= T + S_1 + S_2 \\
 L_1 + L_2 &= T
 \end{aligned}$$

— Eugene A. Margerum and Michael M. McDonnell

## The area of a right triangle with acute angle $\pi/12$

The area of a right triangle is  $\frac{1}{8}(\text{hypotenuse})^2$  if and only if one acute angle is  $\pi/12$ .



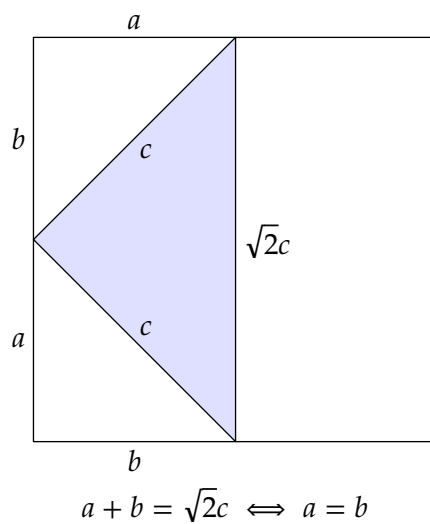
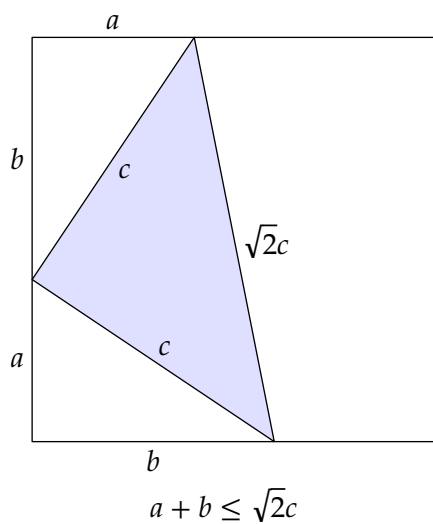
— Klara Pinter

## A right angle inequality

Let  $c$  be the hypotenuse of a right triangle whose other two sides are  $a$  and  $b$ . Prove that

$$a + b \leq \sqrt{2}c.$$

When does equality hold?



— Canadian Mathematical Olympiad 1969

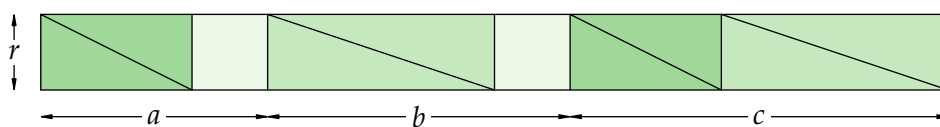
# The inradius of a right triangle



$$\text{I. } r = \frac{ab}{a + b + c}$$

$$\text{II. } r = \frac{a + b - c}{2}$$

$$\text{I. } ab = r(a + b + c)$$



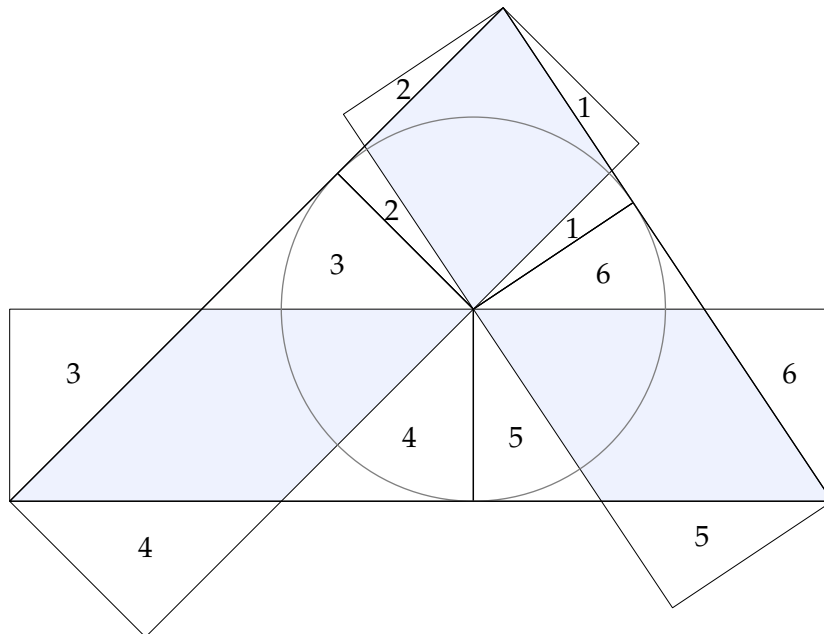
— Liu Hui (3rd century A.D.)

$$\text{II. } c = a + b - 2r$$



**The product of the perimeter of a triangle and its inradius is twice the area of the triangle**

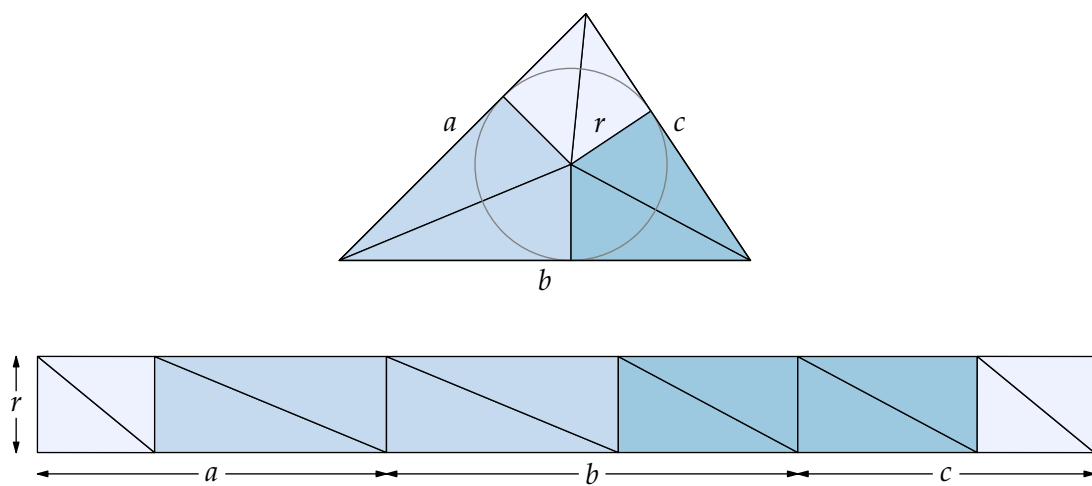
I.



NOTE: Regions bearing the same number are equal in area.

— Grace Lin

II.





## Four triangles with equal area



— Steven L. Snover

**The triangle of medians has  $3/4$  the area of the original triangle**



$$\frac{3}{4} \text{area}(\triangle abc) = \text{area}(\triangle m_a m_b m_c)$$

— Norbert Hungerbühler

## Heptasection of a triangle

If the one-third points on each side of a triangle are joined to opposite vertices, the resulting central triangle is equal in area to one-seventh that of the initial triangle.



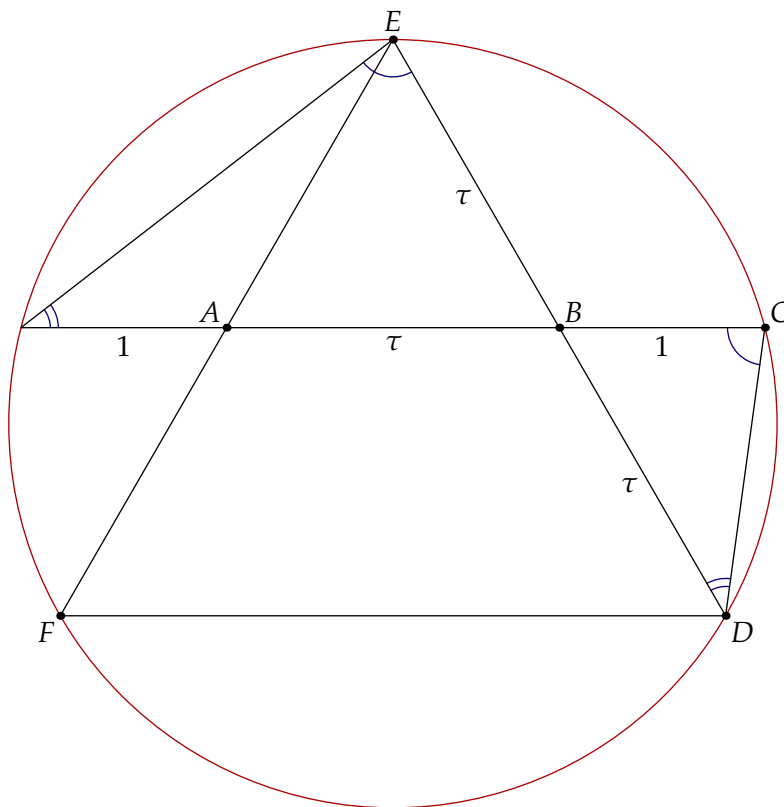
— William Johnston and Joe Kennedy

## A Golden Section problem from the *Monthly*

(Problem E3007, *American Mathematical Monthly*, 1983, p.482)

Let  $A$  and  $B$  be the midpoints of the sides  $EF$  and  $ED$  of an equilateral triangle  $DEF$ . Extend  $AB$  to meet the circumcircle (of  $DEF$ ) at  $C$ . Show that  $B$  divides  $AC$  according to the golden section.

SOLUTION:



$$\tau^2 = \tau + 1$$

— Jan van de Craats

## **Tiling with squares and parallelograms**

If squares are constructed eternally on the sides of the parallelogram, their centres form a square.



— Alfinio Flores

## The area of a quadrilateral I

The area of a quadrilateral is less than or equal to half the product of the lengths of its diagonals, with equality if and only if the diagonals are perpendicular.

### I. Convex quadrilaterals



$$\begin{aligned}\text{Area} &= \frac{1}{2} \overline{AC} \cdot (h + k) \\ &\leq \frac{1}{2} \overline{AC} \cdot \overline{BD}\end{aligned}$$

### II. Concave quadrilaterals



$$\begin{aligned}\text{Area} &= \frac{1}{2} \overline{AC} \cdot (h - k) \\ &\leq \frac{1}{2} \overline{AC} \cdot \overline{BD}\end{aligned}$$

— David B. Sher, Ronald Skurnick, and Dean C. Nataro

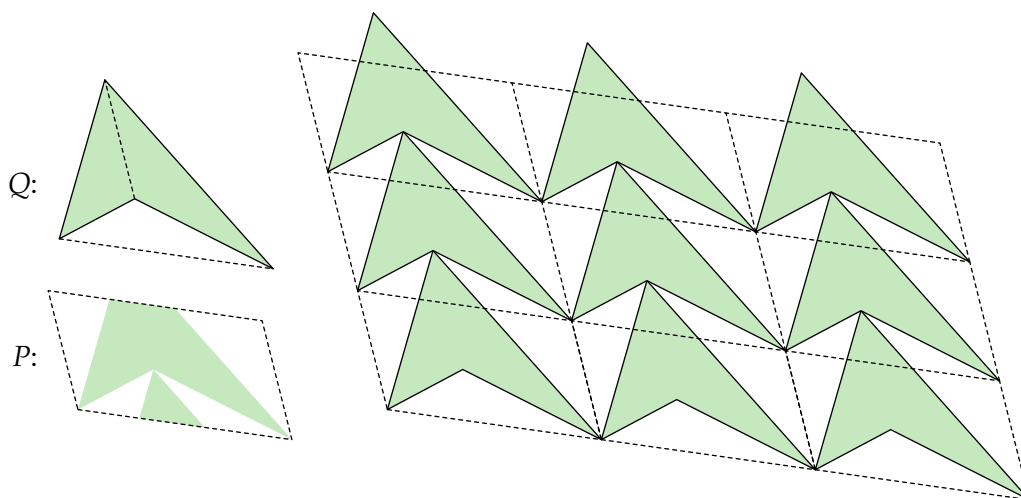
## The area of a quadrilateral II

The area of a quadrilateral  $Q$  is equal to one-half the area of a parallelogram  $P$  whose sides are parallel to and equal in length to the diagonals of  $Q$ .

I.  $Q$  convex



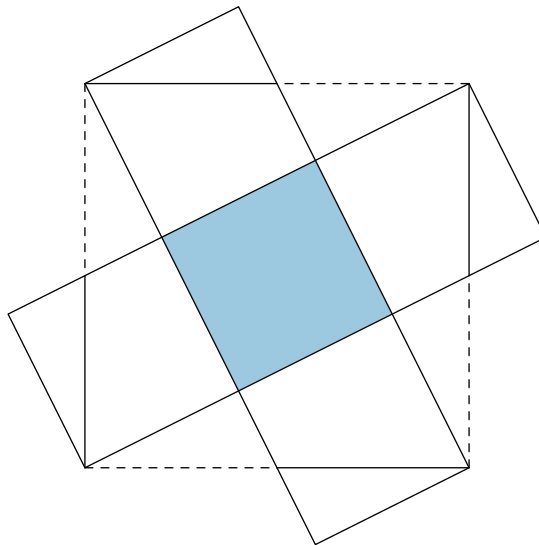
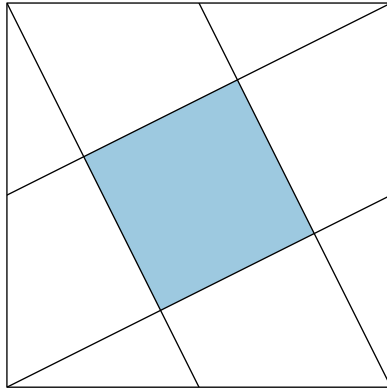
II.  $Q$  concave



$$\text{area}(Q) = \frac{1}{2} \text{area}(P)$$

### **A square within a square**

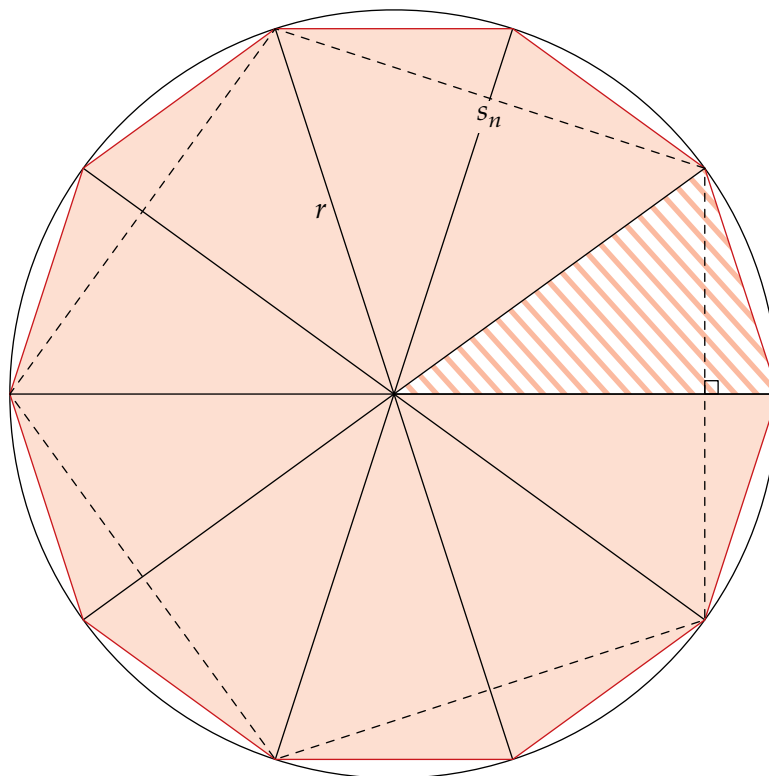
If lines from the vertices of a square are drawn to the mid-points of adjacent sides (as shown in the figure), then the area of the smaller square so produced is one-fifth that of the given square.





## Areas and perimeters of regular polygons

The area of a regular  $2n$ -gon inscribed in a circle is equal to one-half the radius of the circle times the perimeter of a regular  $n$ -gon similarly inscribed ( $n \geq 3$ ).



$$\begin{aligned}\frac{1}{2n} \text{area}(P_{2n}) &= \frac{1}{2} \cdot r \cdot \frac{1}{2}s_n \\ \text{area}(P_{2n}) &= \frac{1}{2}r \cdot ns_n \\ &= \frac{1}{2}r \cdot \text{perimeter}(P_n)\end{aligned}$$

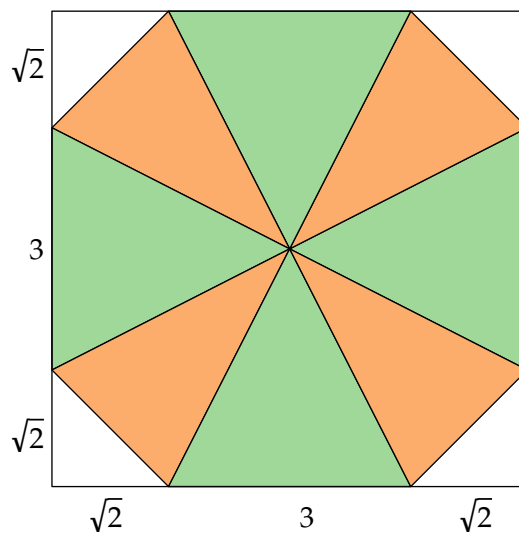
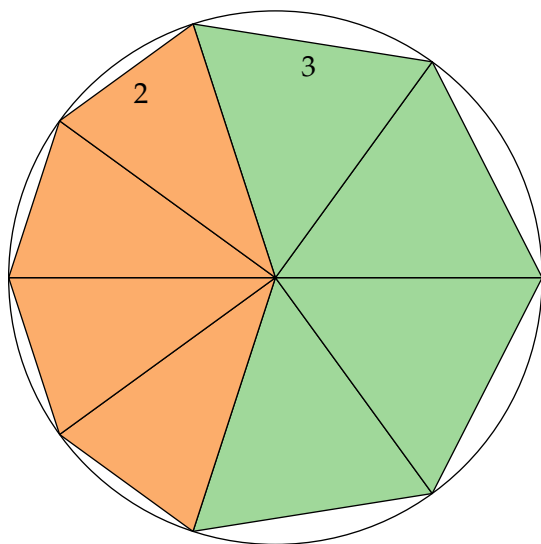
COROLLARY [Bhāskara, *Litāvatī* (India, 12th century AD)]: The area of a circle is equal to one-half the product of its radius and circumference.

## The area of a Putnam octagon

(Problem B1, 39th Annual William Lowell Putnam Mathematical Competition, 1978).

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form  $r + s\sqrt{t}$ , with  $r, s$ , and  $t$  positive integers.

SOLUTION:



$$A = (3 + 2\sqrt{2})^2 - 4 \cdot \frac{1}{2} (\sqrt{2})^2 = 9 + 6\sqrt{2} + 6\sqrt{2} + 8 - 4 = 13 + 12\sqrt{2}$$

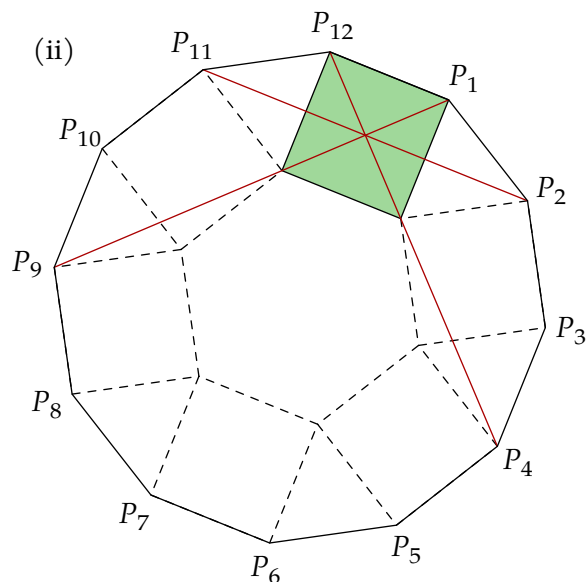
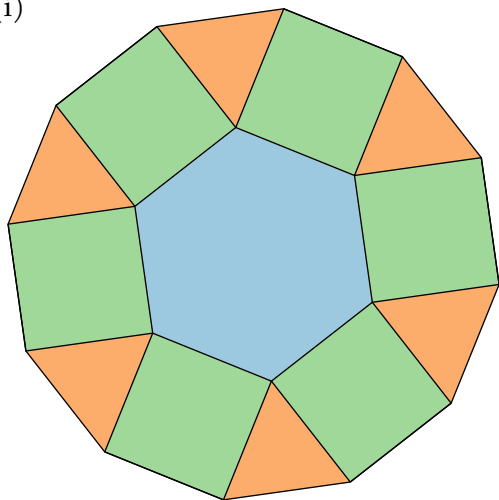
## A Putnam dodecagon

(Problem I-1, 24th Annual William Lowell Putnam Mathematical Competition, 1963)

- (i) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.
- (ii) Let  $P_1, P_2, \dots, P_{12}$  be the successive vertices of a regular dodecagon. Discuss the intersection(s) of the three diagonals  $P_1P_9$ ,  $P_2P_{11}$ , and  $P_4P_{12}$ .

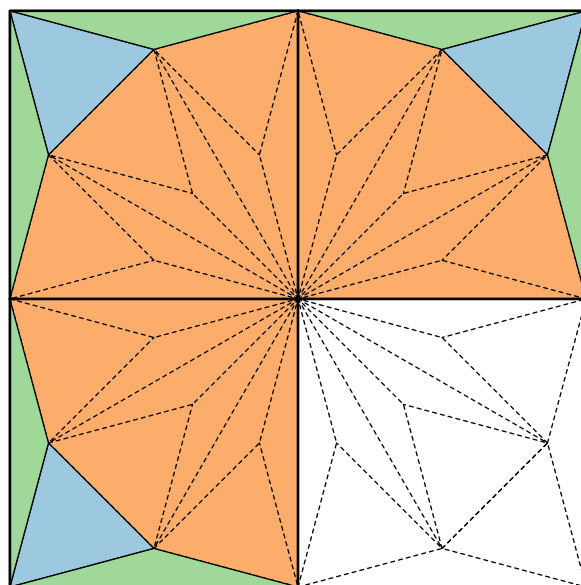
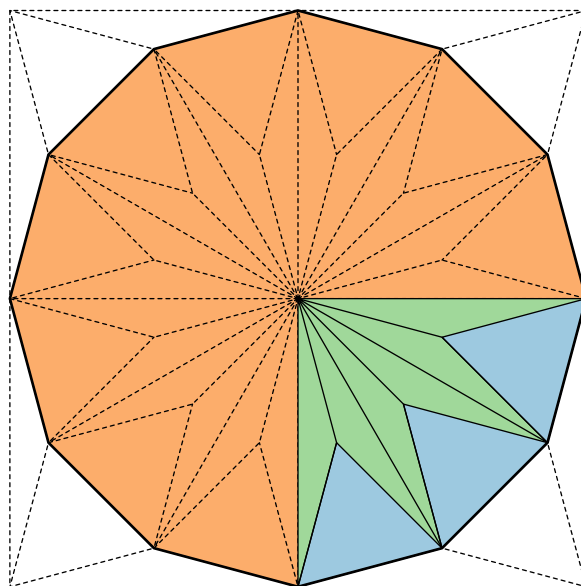
SOLUTION:

(i)



## The area of a regular dodecagon

A regular dodecagon with circumradius one has area three.



— J. Kürshák

## Fair allocation of a pizza

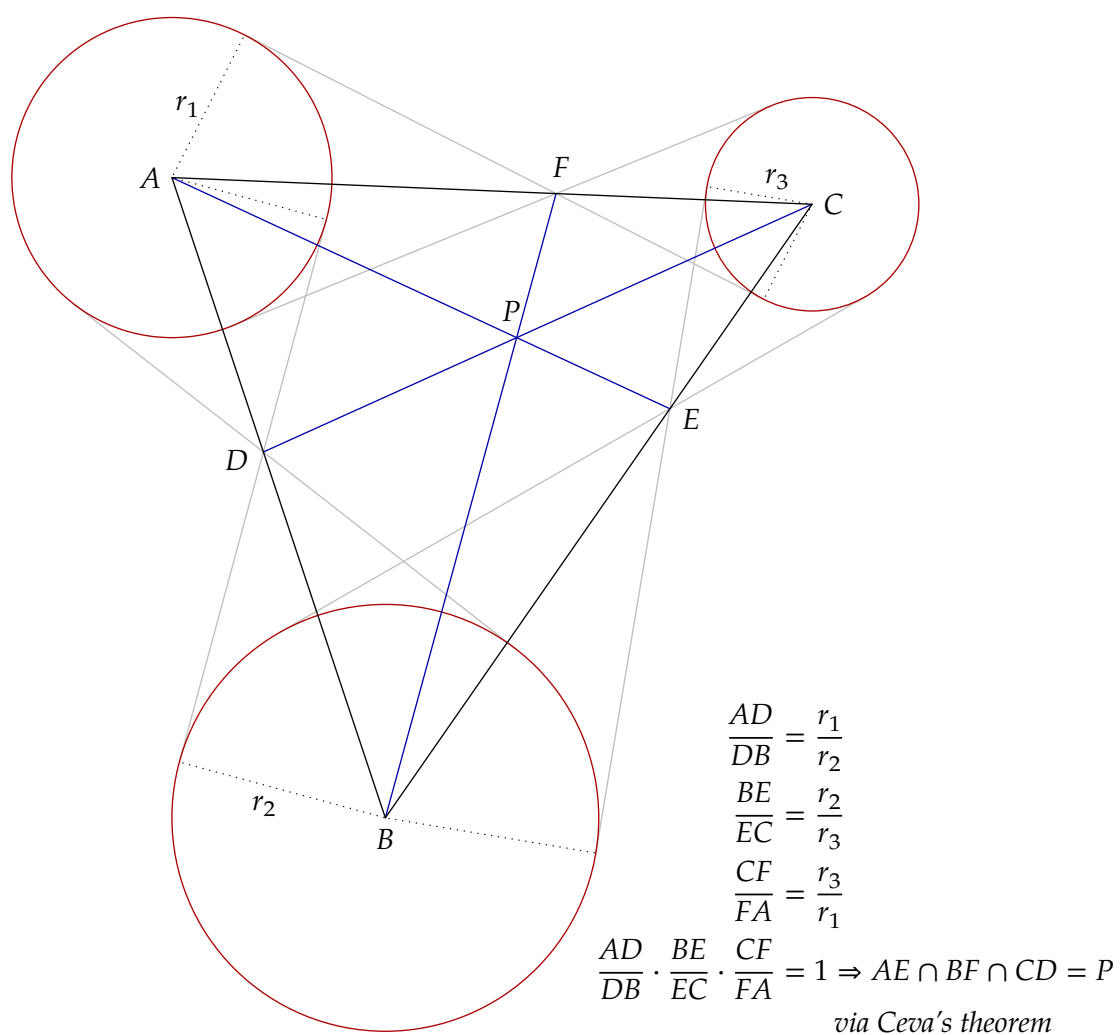
THE PIZZA THEOREM: If a pizza is divided into eight pieces by making cuts at  $45^\circ$  angles through an arbitrary point in the pizza, then the sums of the areas of alternate slices are equal.

PROOF:



## A three-circle theorem

Given three non-intersecting, mutually external circles, connect the intersection of the internal common tangents of each pair of circles with the centre of the other circle. Then the resulting three line segments are concurrent.



— R. S. Hu

## A constant chord

Suppose two circles  $Q$  and  $R$  intersect in  $A$  and  $B$ . A point  $P$  on the arc of  $Q$  which lies outside  $R$  is projected through  $A$  and  $B$  to determine chord  $CD$  of  $R$ . Prove that no matter where  $P$  is chosen on its arc, the length of chord  $CD$  is always the same.



$$\angle C'AC = \angle P'AP = \angle P'BP = \angle D'BD$$

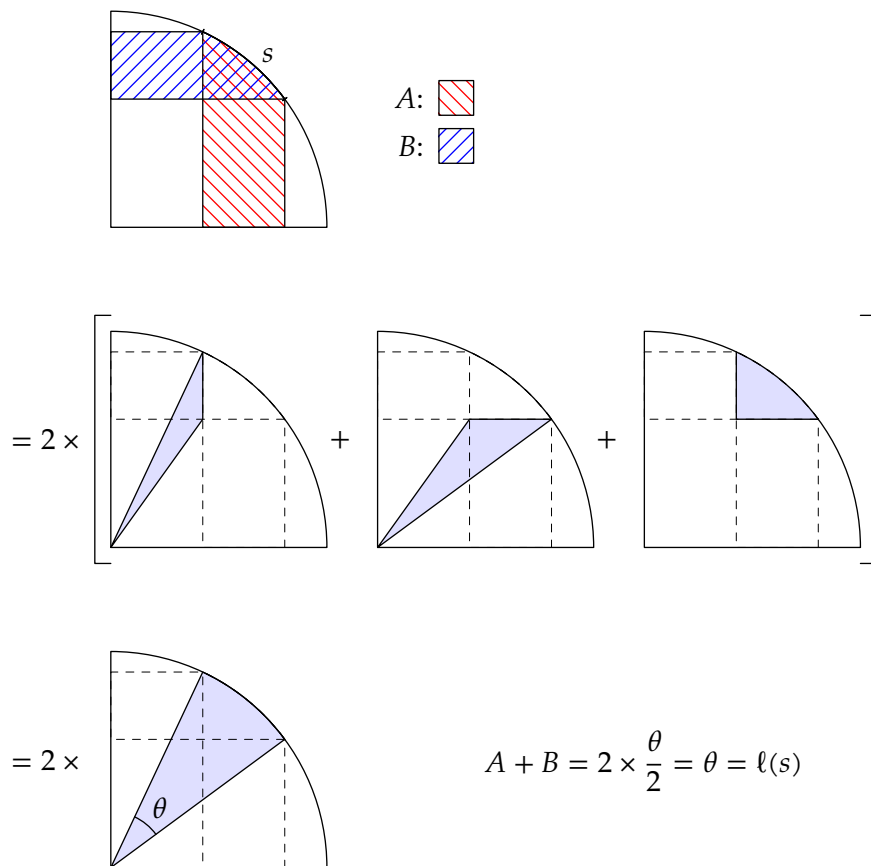
$$\widehat{C'C} = \widehat{D'D}, \quad \widehat{C'D'} = \widehat{CD}$$

$$C'D' = CD$$

## A Putnam area problem

Let  $s$  be any arc of the unit circle lying entirely in the first quadrant. Let  $A$  be the area of the region lying below  $s$  and above the  $x$ -axis, and let  $B$  be the area of the region lying to the right of the  $y$ -axis and to the left of  $s$ . Prove that  $A + B$  depends only on the arc length, and not on the position, of  $s$ .

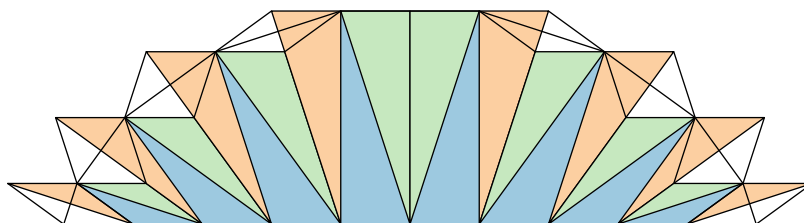
SOLUTION:





## The area under a polygonal arch

The area under a polygonal arch generated by one vertex of a regular  $n$ -gon rolling along a straight line is three times the area of the polygon.

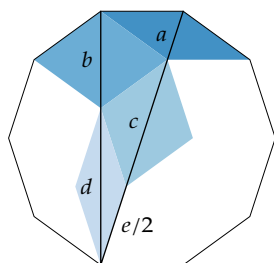
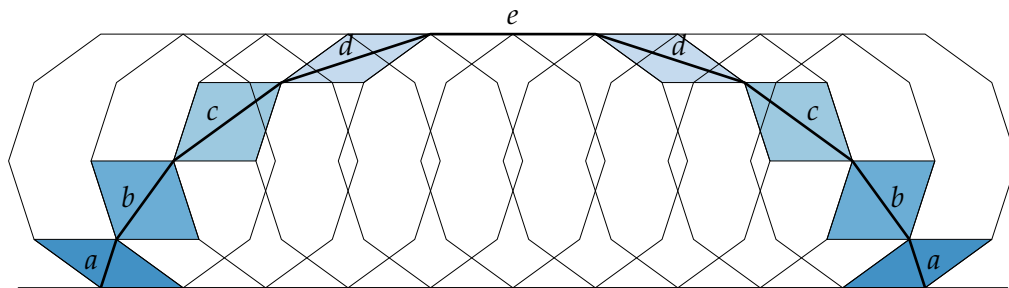


**COROLLARY:** The area under one arch of a cycloid is three times the area of the generating circle.

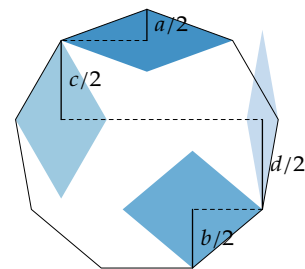
— Philip R. Mallinson

## The length of a polygonal arch

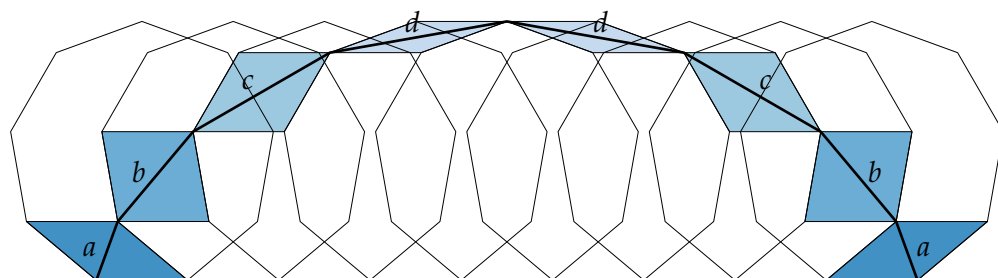
The length of the polygonal arch generated by one vertex of a regular  $n$ -gon rolling along a straight line is four times the length of the in-radius plus four times the length of the circum-radius of the  $n$ -gon.



Even  $n...$



Odd  $n...$



**COROLLARY:** The arc length of one arch of a cycloid is eight times the radius of the generating circle.

— Philip R. Mallinson

## The volume of a frustum of a square pyramid



$$P_4 = 3P_5$$

$$P_1 + P_3 = 2P_2 + 4P_4 \Rightarrow P_1 + P_2 + P_3 = 3P_2 + 12P_5 = 3(P_2 + 4P_5) = 3P$$

$$\therefore V = \frac{h}{3} (a^2 + ab + b^2)$$

— Sidney J. Kung

**The product of four (positive) numbers in arithmetic progression is always the difference of two squares**

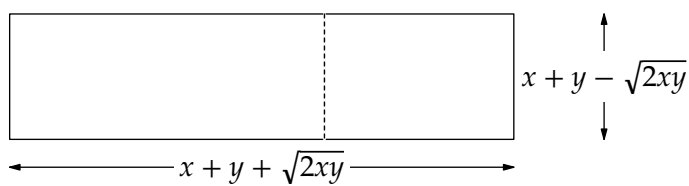
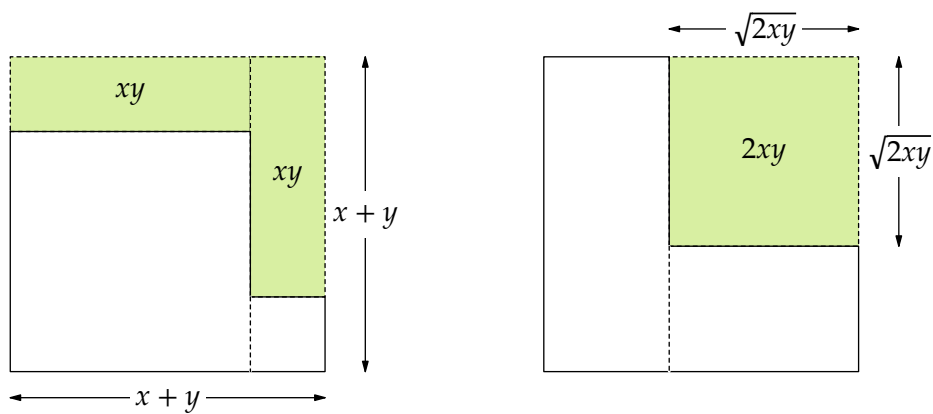
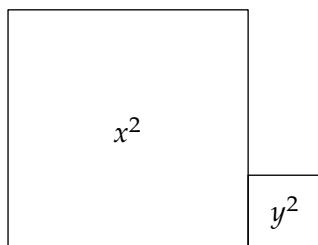


$$a(a+d)(a+2d)(a+3d) = (a^2 + 3ad + d^2)^2 - (d^2)^2$$

— RBN

### Algebraic areas III: Factoring the sum of two squares

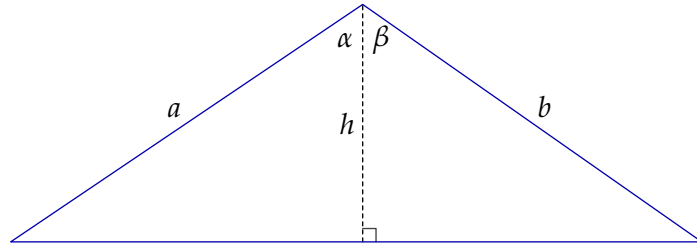
$$x^2 + y^2 = (x + \sqrt{2xy} + y)(x - \sqrt{2xy} + y)$$



# Trigonometry, Calculus, & Analytic Geometry

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## Sine of the sum - II



$$\alpha, \beta \in (0, \pi/2) \implies h = a \cos \alpha = b \cos \beta$$



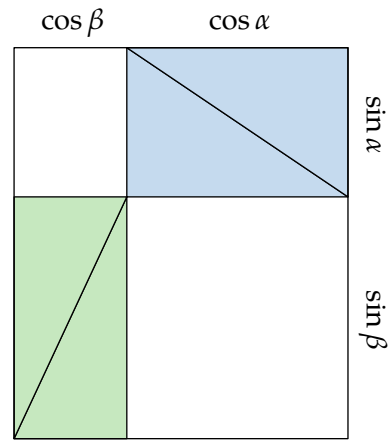
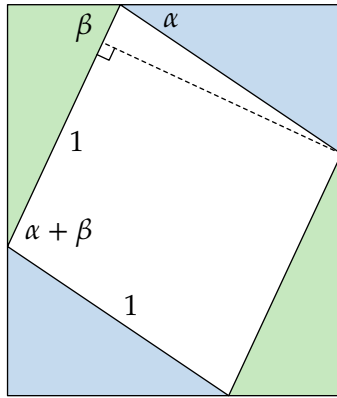
$$\begin{aligned} \frac{1}{2}ab \sin(\alpha + \beta) &= \frac{1}{2}ah \sin \alpha + \frac{1}{2}bh \sin \beta \\ &= \frac{1}{2}ab \cos \beta \sin \alpha + \frac{1}{2}ba \cos \alpha \sin \beta \\ \therefore \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{aligned}$$

— Christopher Brüningsen

### Sine of the sum – III

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

I.



II.



— Volker Priebe and Edgar A. Ramos



## Cosine of the sum

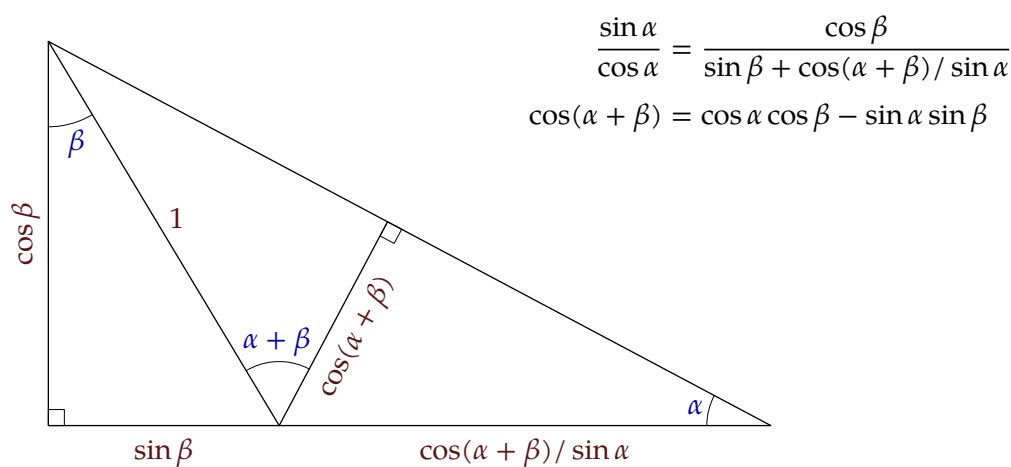


$$\frac{1}{2}ab \sin\left(\frac{\pi}{2} - (\alpha + \beta)\right) = \frac{1}{2}b \cos \alpha \cdot a \cos \beta - \frac{1}{2}b \sin \alpha \cdot a \sin \beta$$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

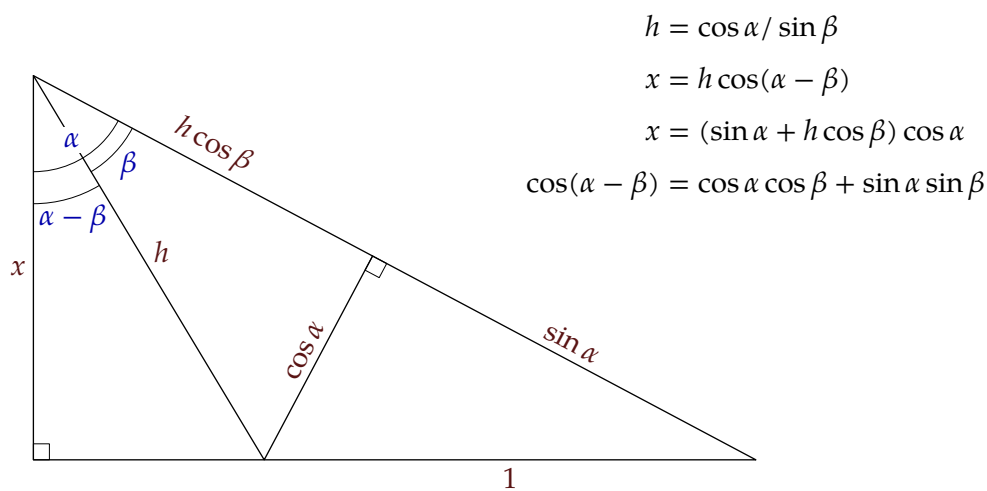
— Sidney H. Kung

## Geometry of addition formulas



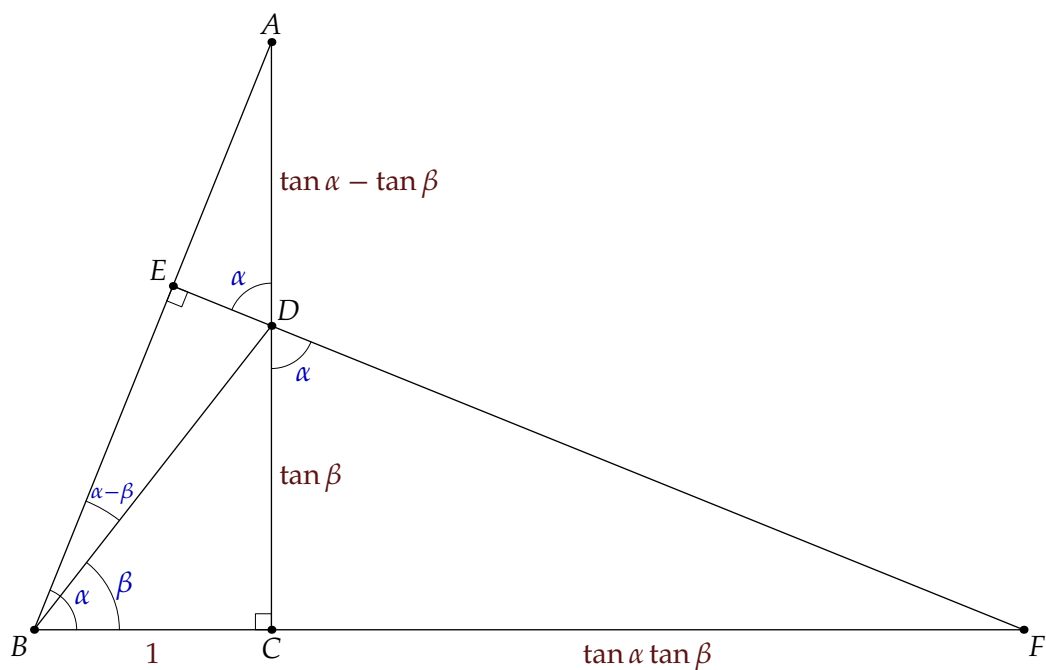
— Leonard M. Smiley

## Geometry of subtraction formulas



— Leonard M. Smiley

## The difference identity for tangents I

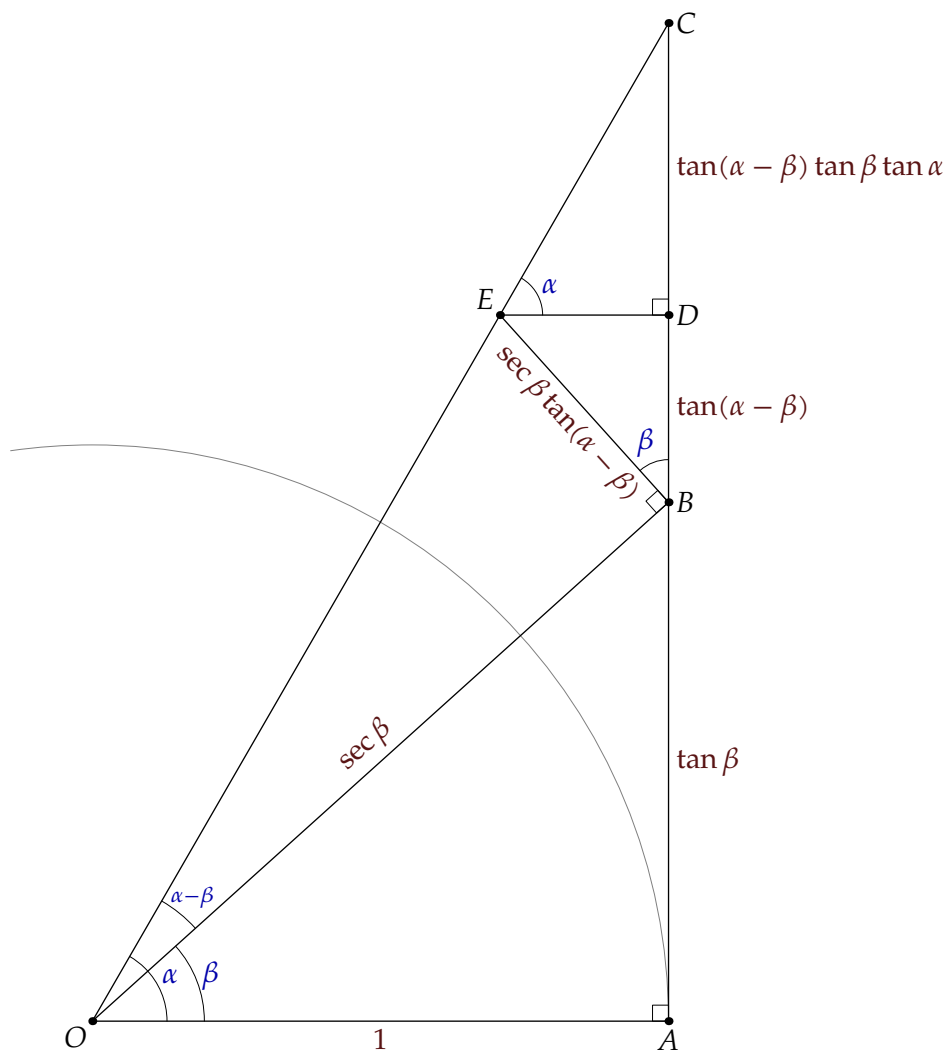


$$\frac{BF}{BE} = \frac{AD}{DE}$$

$$\therefore \tan(\alpha - \beta) = \frac{DE}{BE} = \frac{AD}{BF} = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

— Guanshen Ren

## The difference identity for tangents II



$$AC - AB = BD + DC$$

$$\therefore \tan \alpha - \tan \beta = \tan(\alpha - \beta) + \tan \alpha \tan \beta \tan(\alpha - \beta)$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

— Fukuzo Suzuki

## One figure, six identities



The figure

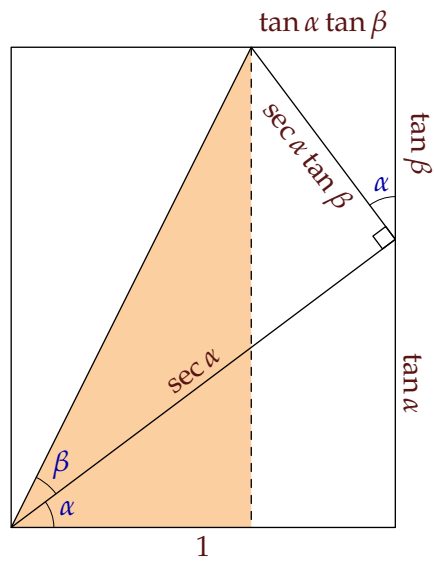
$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}$$



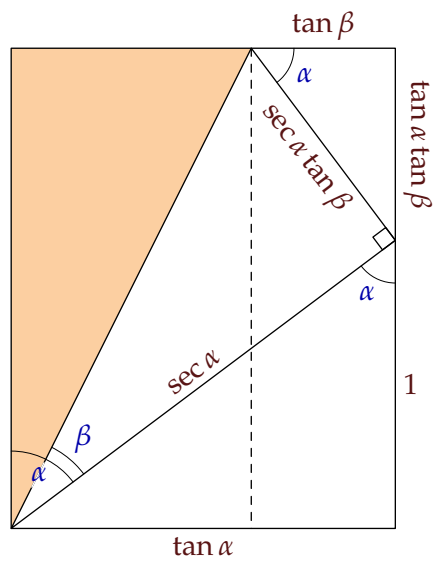
$$\begin{aligned}\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta\end{aligned}$$



$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 + \tan \alpha \tan \beta}$$



$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$



— RBN

## The double-angle formulas II



$$2 \sin \theta \cos \theta = \sin 2\theta$$



$$2 \cos^2 \theta = 1 + \cos 2\theta$$

— Yihnan David Gau



## The double-angle formulas III (via the laws of sines and cosines)

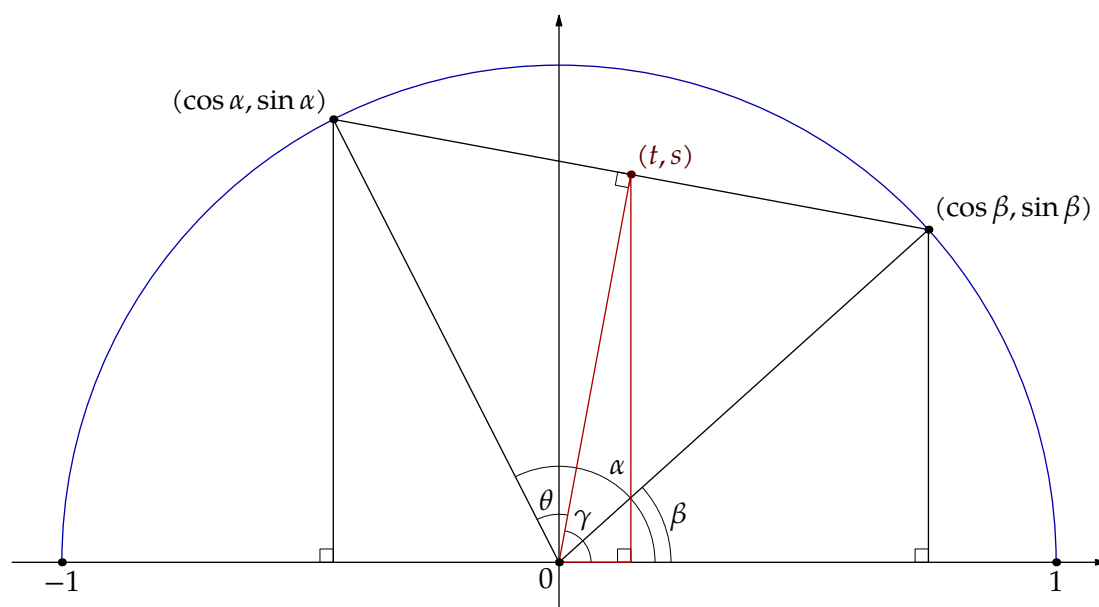


$$\frac{\sin 2\theta}{2 \sin \theta} = \frac{\sin(\pi/2 - \theta)}{1} = \cos \theta$$
$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$(2 \sin \theta)^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 2\theta$$
$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

— Sidney H. Kung

## The sum-to-product identities I



$$\theta = \frac{\alpha - \beta}{2}, \quad \gamma = \frac{\alpha + \beta}{2}$$

$$\frac{\sin \alpha + \sin \beta}{2} = s = \cos \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

$$\frac{\cos \alpha + \cos \beta}{2} = t = \cos \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

— Sidney H. Kung

## The difference-to-product identities I



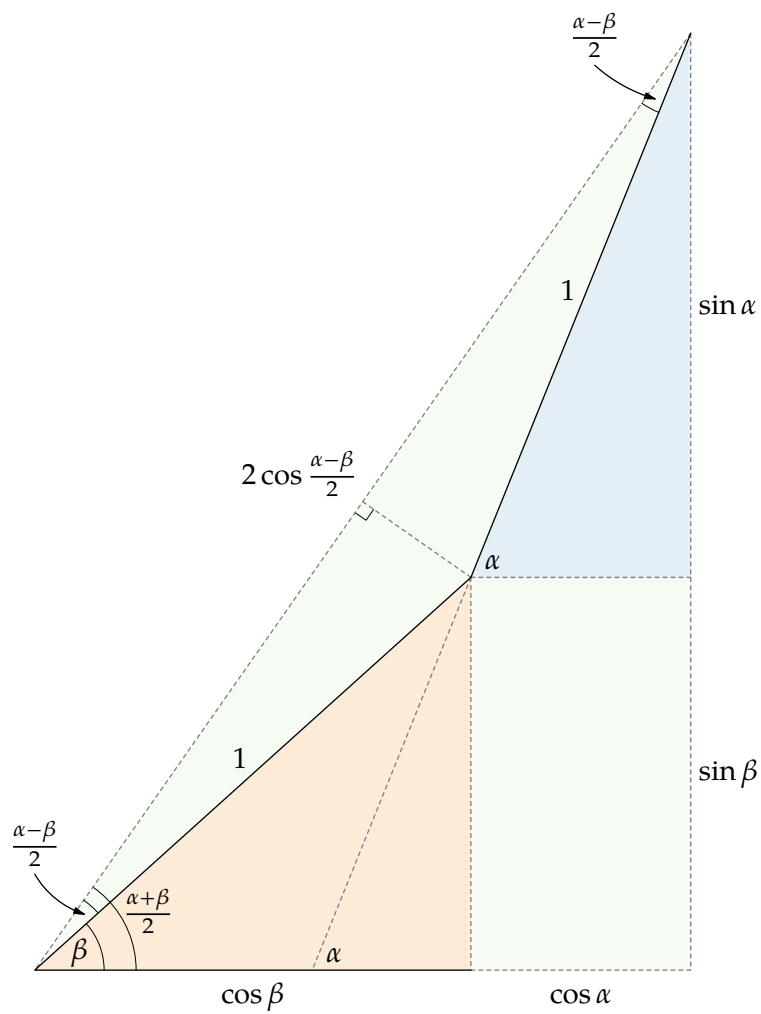
$$\theta = \frac{\alpha - \beta}{2}, \quad \gamma = \frac{\alpha + \beta}{2}$$

$$\sin \alpha - \sin \beta = v = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$\cos \beta - \cos \alpha = u = 2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

— Sidney H. Kung

## The sum-to-product identities II



$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

$$\sin \alpha + \sin \beta = 2 \cos \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

— Yukio Kobayashi

## The difference-to-product identities II



$$\cos \beta - \cos \alpha = 2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$$

— Yukio Kobayashi

## Adding like sines



$$R_\phi = \sqrt{A^2 + B^2 + 2AB \cos \phi}, \quad \tan \theta = \frac{B \sin \phi}{A + B \cos \phi}$$

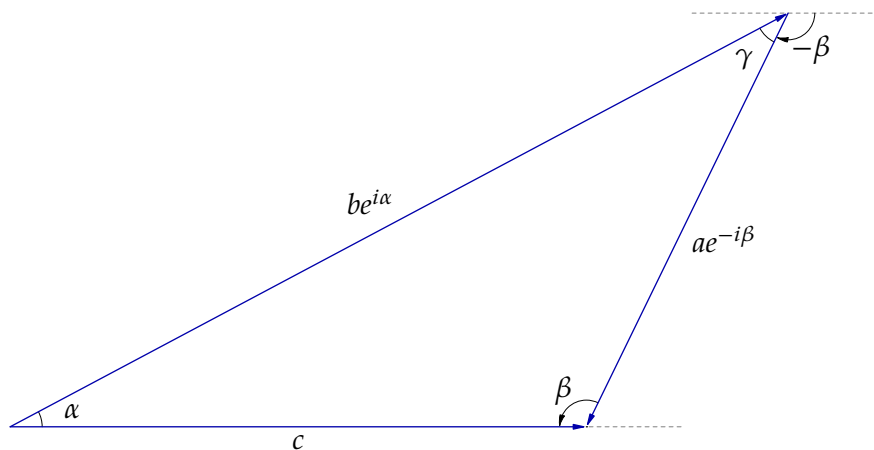
$$A \sin x + B \sin(x + \phi) = R_\phi \sin(x + \theta)$$

$$\phi = \pi/2 \Rightarrow \tan \theta = B/A$$

$$\therefore A \sin x + B \cos x = \sqrt{A^2 + B^2} \sin(x + \theta)$$

— Rick Mabry and Paul Deiermann

## A complex approach to the laws of sines and cosines



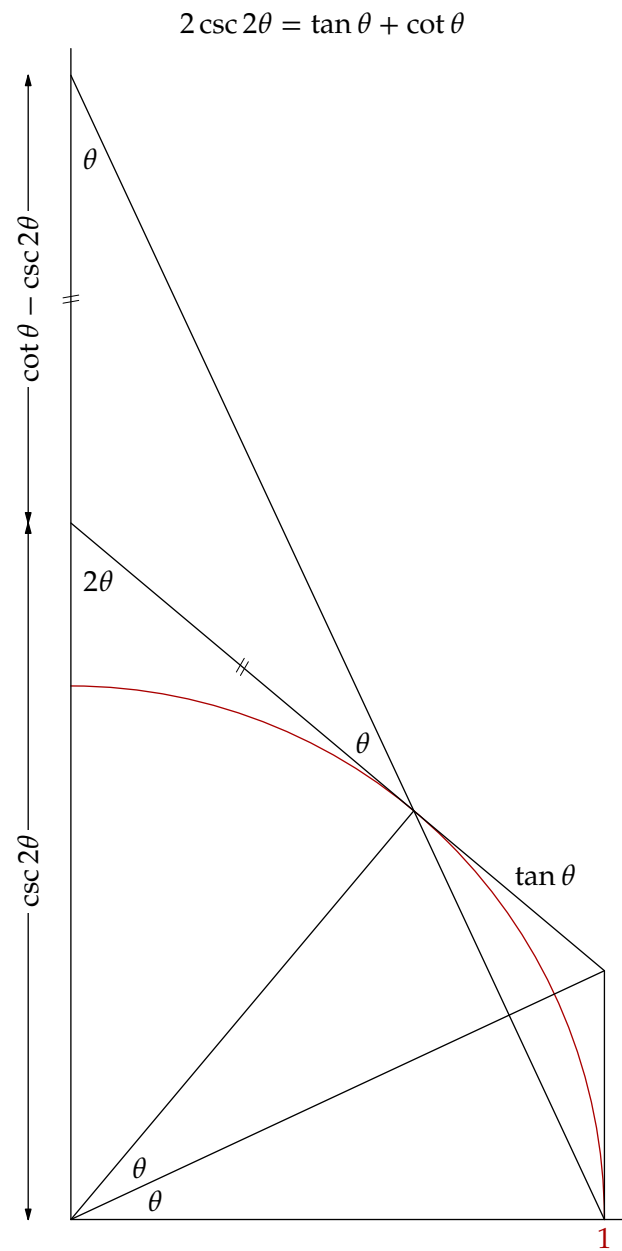
$$c = be^{i\alpha} + ae^{-i\beta} = (b \cos \alpha + a \cos \beta) + i(b \sin \alpha - a \sin \beta)$$

if  $c$  is real, then  $b \sin \alpha - a \sin \beta = 0$ , hence  $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$

$$\begin{aligned} c^2 = |c|^2 &= (b \cos \alpha + a \cos \beta)^2 + (b \sin \alpha - a \sin \beta)^2 \\ &= a^2 + b^2 + 2ab \cos(\alpha + \beta) \\ &= a^2 + b^2 - 2ab \cos \gamma \end{aligned}$$

— William V. Grounds

## Eisenstein's duplication formula



G. Eisenstein, *Mathematische Werke*, Chelsea, NY. 1975, p.411



## A familiar limit for $e$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$



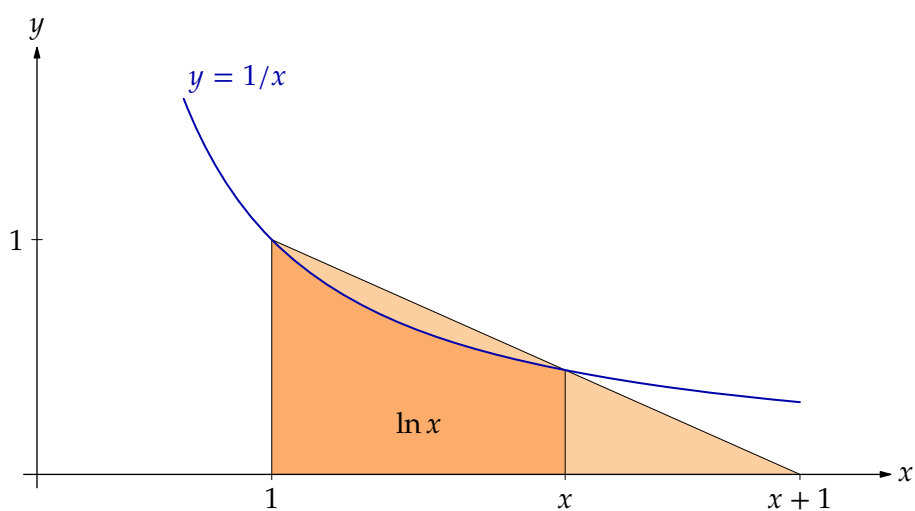
$$\frac{1}{n} \cdot \frac{n}{n+1} \leq \ln \left(1 + \frac{1}{n}\right) \leq \frac{1}{n} \cdot 1$$

$$\frac{n}{n+1} \leq n \cdot \ln \left(1 + \frac{1}{n}\right) \leq 1$$

$$\therefore \lim_{n \rightarrow \infty} \ln \left( \left(1 + \frac{1}{n}\right)^n \right) = 1$$

## A common limit

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = 0$$



$$\ln x < \frac{1}{2}x$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^{x-\ln x}} = 0$$

— Alan H. Stein and Dennis McGavran

## Geometric evaluation of a limit

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}} = 2$$



— Guanshen Ren

## The derivative of the inverse sine



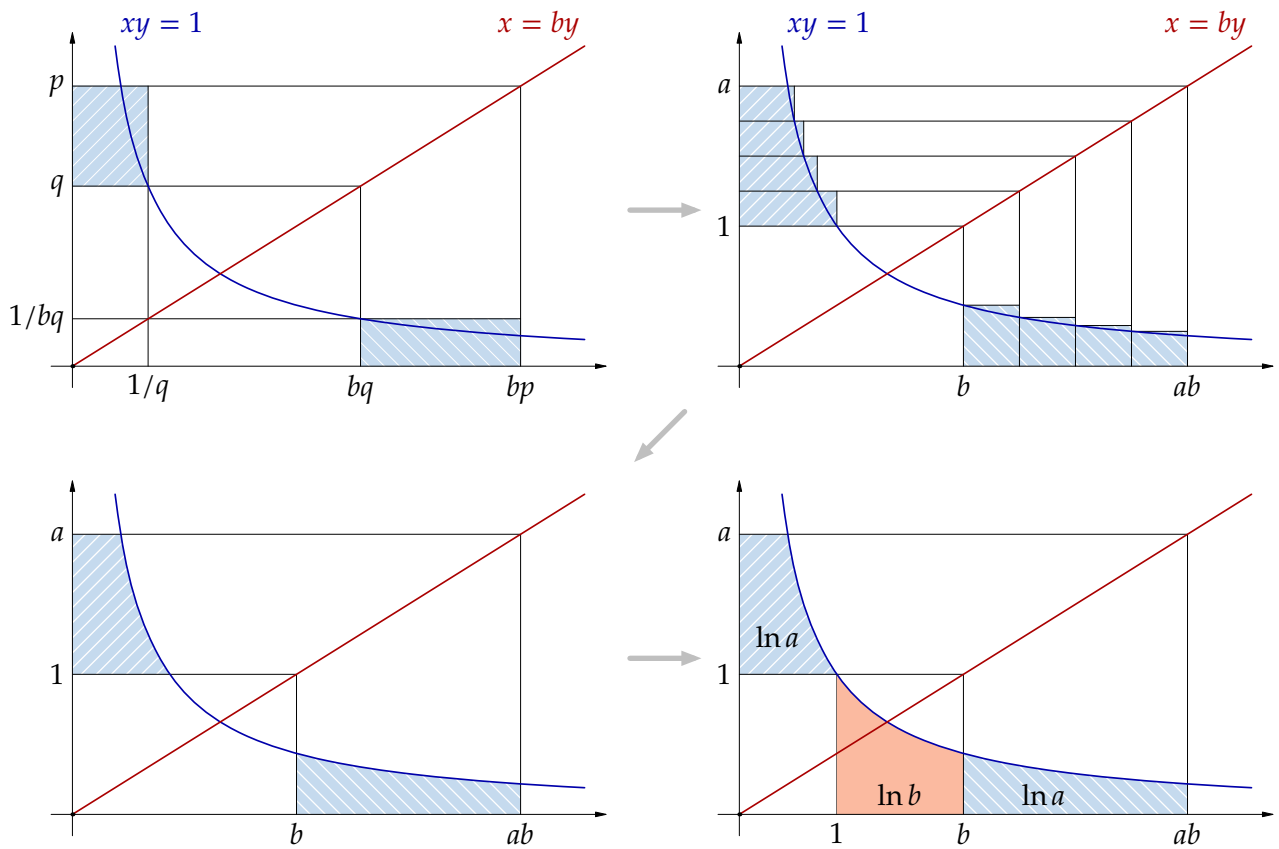
$$L = \sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

$$\therefore \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

— Craig Johnson

## The logarithm of a product

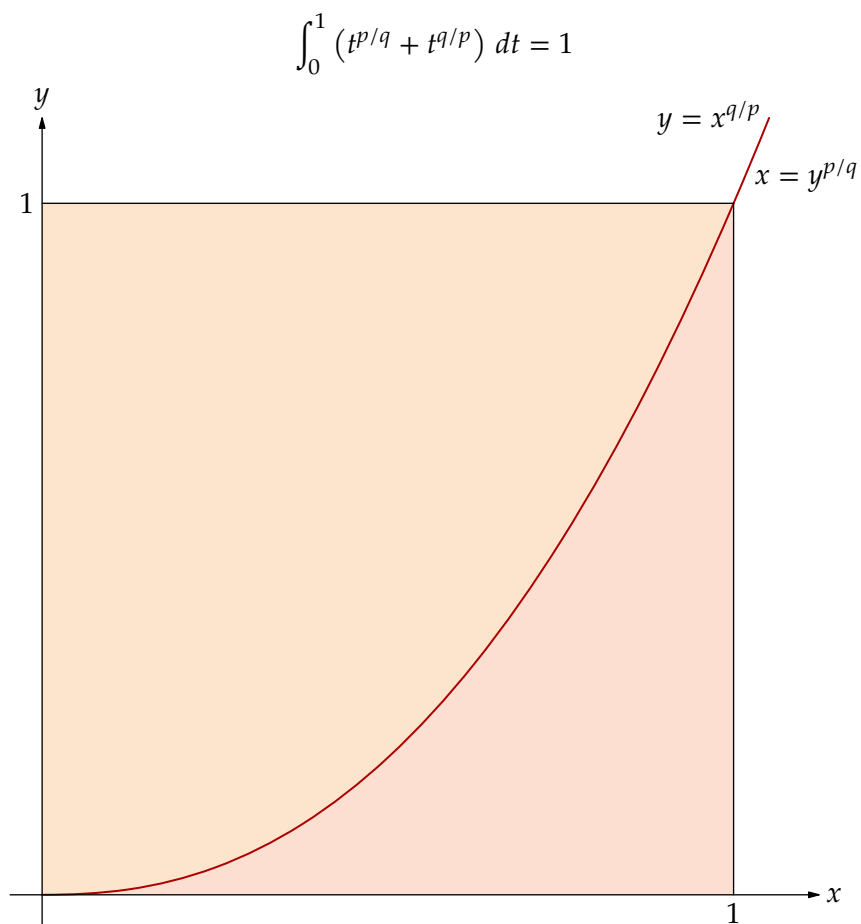
$$\ln ab = \ln a + \ln b$$



$$\text{Area}(\text{shaded area}) = \text{Area}(\text{shaded area})$$

— Jeffery Ely

## **An integral of a sum of reciprocal powers**



— Peter R. Newbury

## The arctangent integral



— Aage Bondesen

## The method of last resort — Weierstrass substitution



$$u = \tan \frac{\theta}{2}, \quad DE = 2 \sin \frac{\theta}{2} = \frac{2u}{\sqrt{1+u^2}}$$

$$\frac{CE}{DE} = \frac{OA}{BA} \implies \sin \theta = \frac{2u}{1+u^2}$$

$$\frac{CD}{DE} = \frac{OB}{BA} \implies \cos \theta = \frac{1-u^2}{1+u^2}$$

— Paul Deiermann



## The trapezoidal rule — for increasing functions

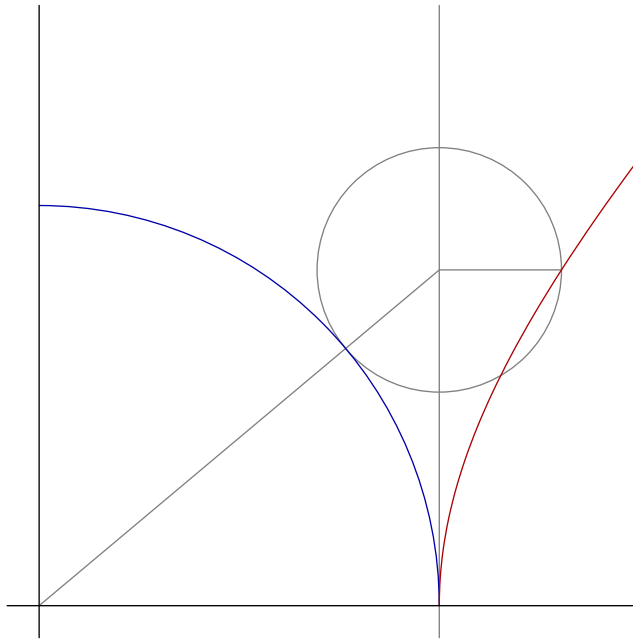


$$\int_a^b f(x) dx = \sum_{i=0}^{n-1} f(x_i) \frac{b-a}{n} + \frac{1}{2} \left( f(x_n) - f(x_0) \right) \frac{b-a}{n}$$

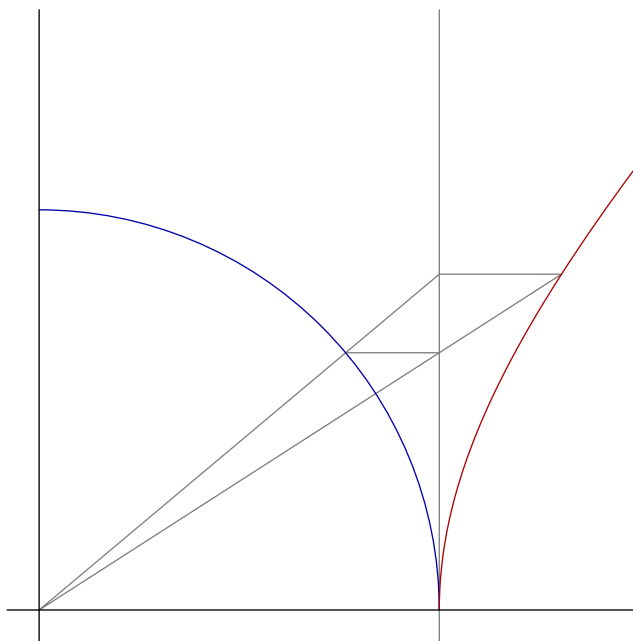
— Jesús Urías

## Construction of a hyperbola

I.



II.



— Ernest J. Eckert

## The focus and directrix of an ellipse

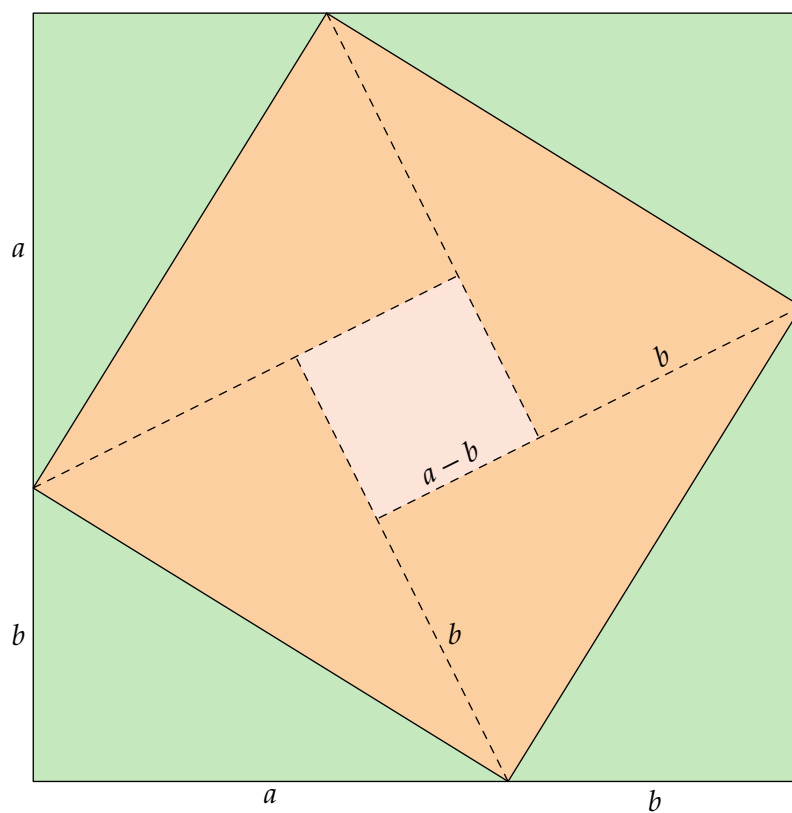


— Michel Bataille

# Inequalities

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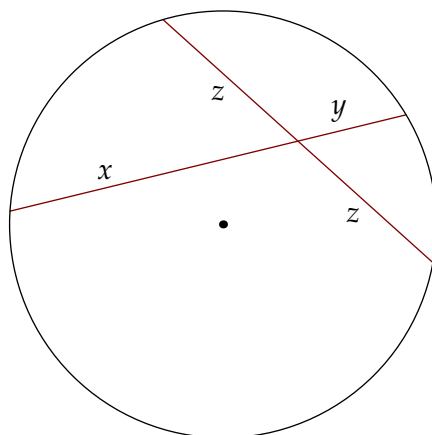
# The arithmetic mean – geometric mean inequality IV



$$(a+b)^2 \geq 4ab \implies \frac{a+b}{2} \geq \sqrt{ab}$$

— Ayoub B. Ayoub

# The arithmetic mean – geometric mean inequality V



$$z^2 = xy$$



$$d < c \implies x + y > 2\sqrt{xy}$$



$$d = c = 0 \implies x + y = 2\sqrt{xy}$$

— Sidney H. Kung

## The arithmetic mean – geometric mean inequality VI



$$0 < a < b, 0 < t < 1 \Rightarrow (1-t)a + tb > a^{1-t}b^t$$

$$t = \frac{1}{2} \Rightarrow \frac{a+b}{2} > \sqrt{ab}$$

— Michael K. Brozinsky

# The arithmetic mean – geometric mean inequality for three positive numbers

LEMMA:  $ab + bc + ac \leq a^2 + b^2 + c^2$



THEOREM:  $3abc \leq a^3 + b^3 + c^3$



— Claudi Alsina



# The arithmetic-geometric-harmonic mean inequality

$$a, b > 0 \implies \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$



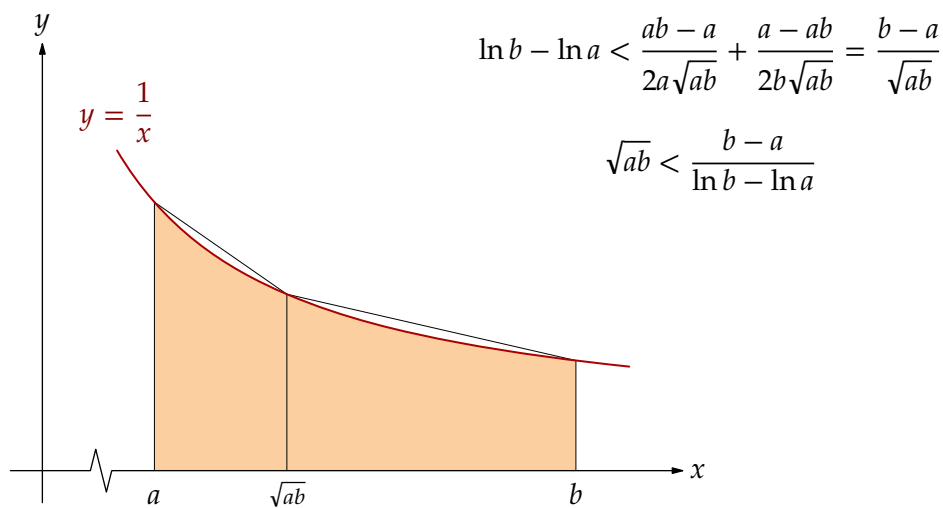
$$\overline{AM} = \frac{a+b}{2}, \quad \overline{GM} = \sqrt{ab}, \quad \overline{HM} = \frac{2ab}{a+b},$$

$$\overline{AM} \geq \overline{GM} \geq \overline{HM}.$$

— Pappus of Alexandria (circa A.D. 320)

## The arithmetic-logarithmic-geometric mean inequality

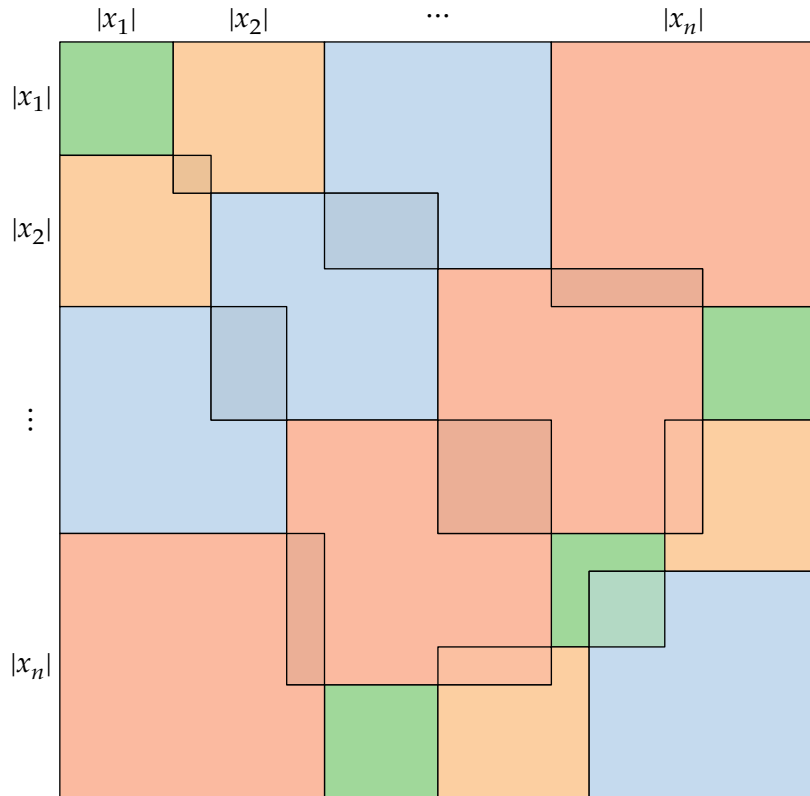
$$b > a > 0 \implies \frac{a+b}{2} > \frac{b-a}{\ln b - \ln a} > \sqrt{ab}$$



— RBN

## The mean of the squares exceeds the square of the mean

$$\frac{1}{n} \sum_{i=1}^n x_i^2 \geq \left( \frac{1}{n} \sum_{i=1}^n x_i \right)^2$$



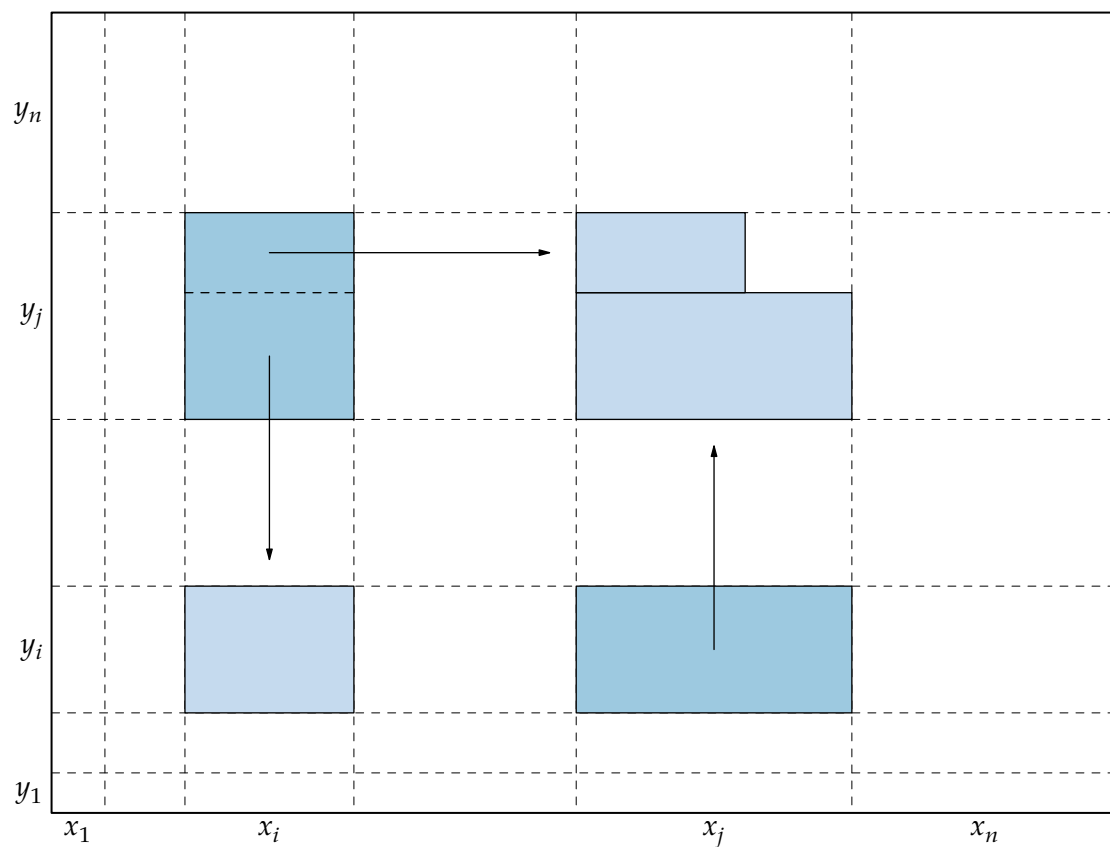
$$n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (|x_1| + |x_2| + \cdots + |x_n|)^2 \geq (x_1 + x_2 + \cdots + x_n)^2$$

$$\therefore \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \geq \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right)^2$$

— RBN

## The Chebyshev inequality for positive monotone sequences

$$\sum_{i=1}^n x_i \sum_{i=1}^n y_i \leq \sum_{i=1}^n x_i y_i$$



$$x_i < x_j \text{ \& } y_i < y_j \quad \Rightarrow \quad x_i y_j + x_j y_i \leq x_i y_i + x_j y_j$$

$$\therefore (x_1 + x_2 + \cdots + x_n) (y_1 + y_2 + \cdots + y_n) \leq n (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

— RBN

# Jordan's inequality

$$0 \leq x \leq \frac{\pi}{2} \Rightarrow \frac{2x}{\pi} \leq \sin x \leq x$$



$$\begin{aligned} OB = OM + MP &\geq OA \Rightarrow \widehat{PBQ} \geq \widehat{PAQ} \geq \overline{PQ} \\ &\Rightarrow \pi \sin x \geq 2x \geq 2 \sin x \\ &\Rightarrow \frac{2x}{\pi} \leq \sin x \leq x \end{aligned}$$

— Feng Yuefeng

## Young's inequality

W. H. Young, "On classes of summable functions and their Fourier series", *Proc. Royal Soc. (A)*, 87 (1912) 225–229.

**THEOREM:** Let  $\phi$  and  $\psi$  be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each others. Then for  $a, b \geq 0$  we have

$$ab \leq \int_0^a \phi(x) dx + \int_0^b \psi(y) dy$$

with equality if and only if  $b = \phi(a)$ .

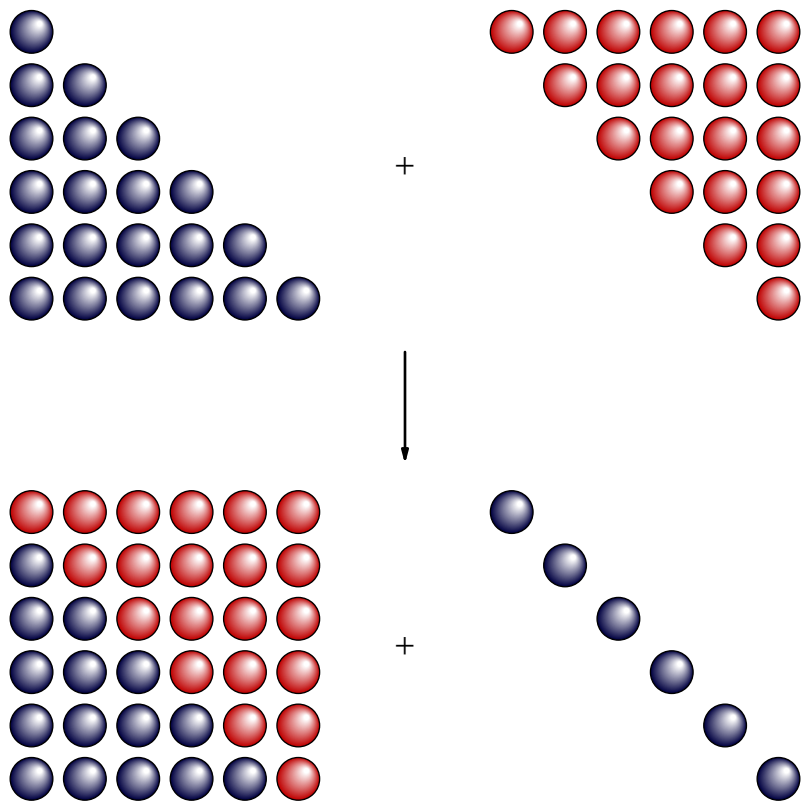
**PROOF:**



# Integer sums

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### Sums of integers III



$$1 + 2 + \cdots + n = \frac{1}{2} (n^2 + n)$$

— S. J. Barlow



## Sums of consecutive positive integers

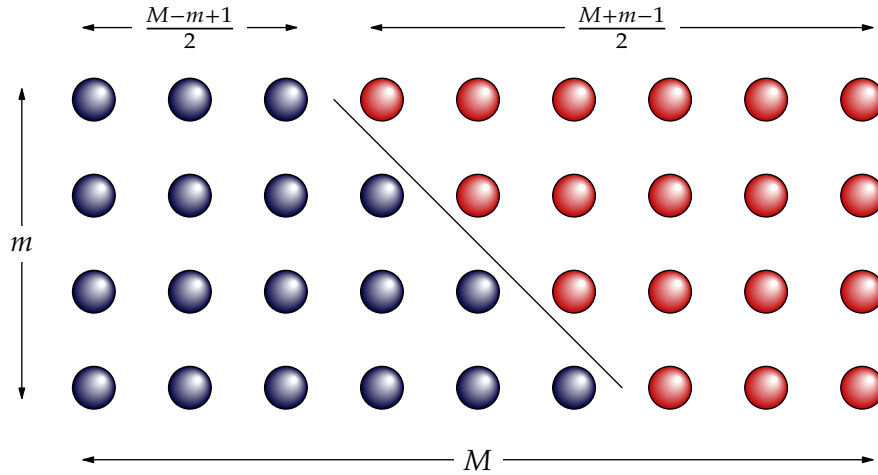
Every integer  $N > 1$ , not a power of two, can be expressed as the sum of two or more positive integers.

$$N = 2^n(2k + 1) \quad (n \geq 0, k \geq 1)$$

$$m = \min \{2^{n+1}, 2k + 1\}$$

$$M = \max \{2^{n+1}, 2k + 1\}$$

$$2N = mM$$

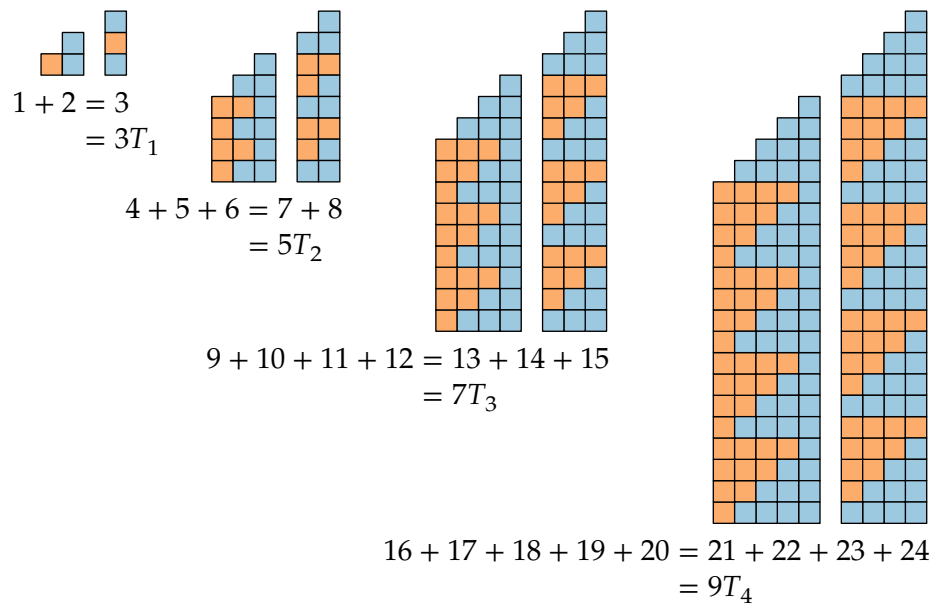


$$N = \left( \frac{M-m+1}{2} \right) + \left( \frac{M-m+1}{2} + 1 \right) + \cdots + \left( \frac{M+m-1}{2} \right)$$

— C. L. Frenzen

## Consecutive sums of consecutive integers II

$$T_k = 1 + 2 + \cdots + k \Rightarrow$$



$$\begin{aligned} n^2 + (n^2 + 1) + \cdots + (n^2 + n) &= (n^2 + n + 1) + \cdots + (n^2 + 2n) \\ &= (2n + 1)T_n \end{aligned}$$

## Sums of squares VI

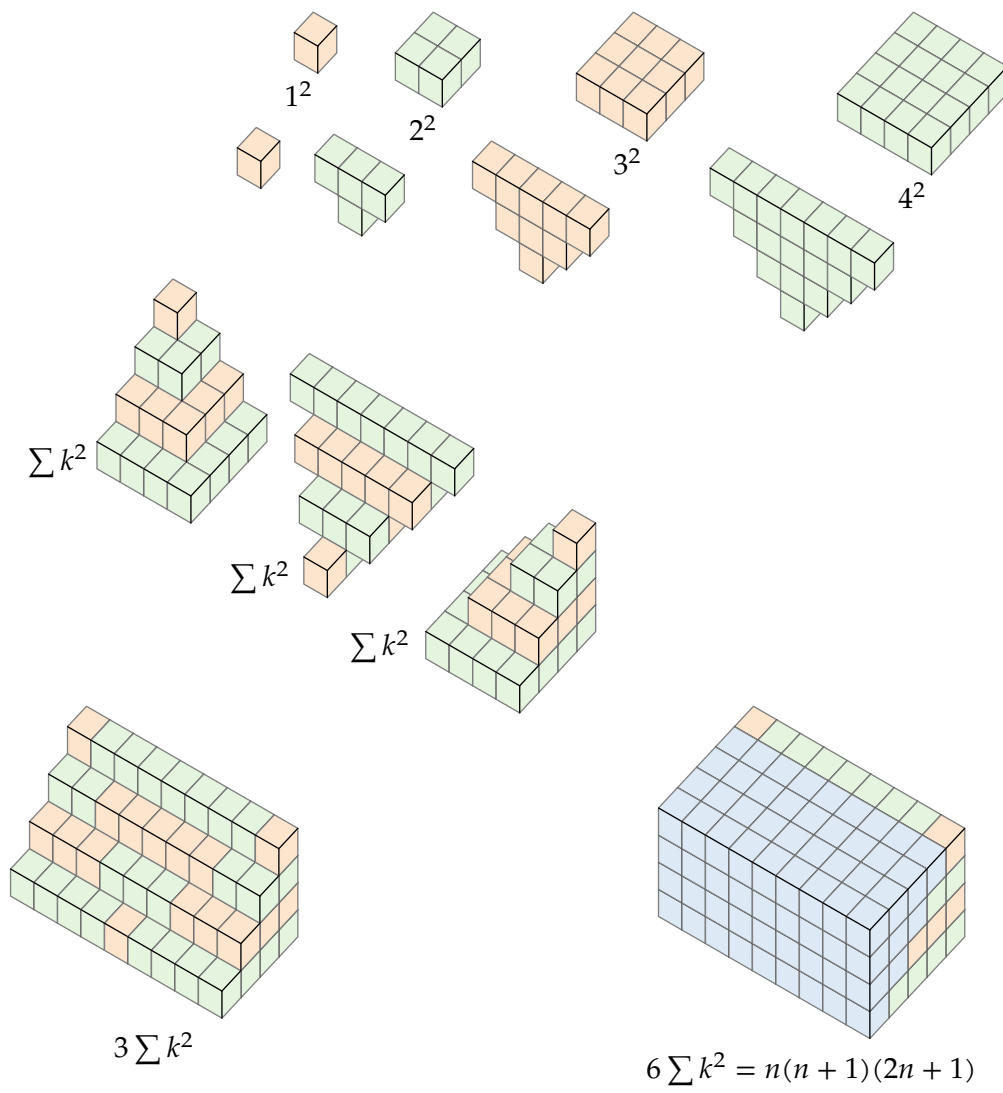


$$\begin{aligned}
 1^2 + 2^2 + \dots + n^2 &= \frac{1}{3}n^2 \times n + 4 \times \frac{n(n+1)}{2} \times \frac{1}{4} - 4 \times n \times \frac{1}{12} \\
 &= \frac{1}{6}n(n+1)(2n+1)
 \end{aligned}$$

— I. A. Sakmar

## Sums of squares VII

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$



— Nanny Wermuth and Hans-Jürgen Schuh

## Sums of squares VIII

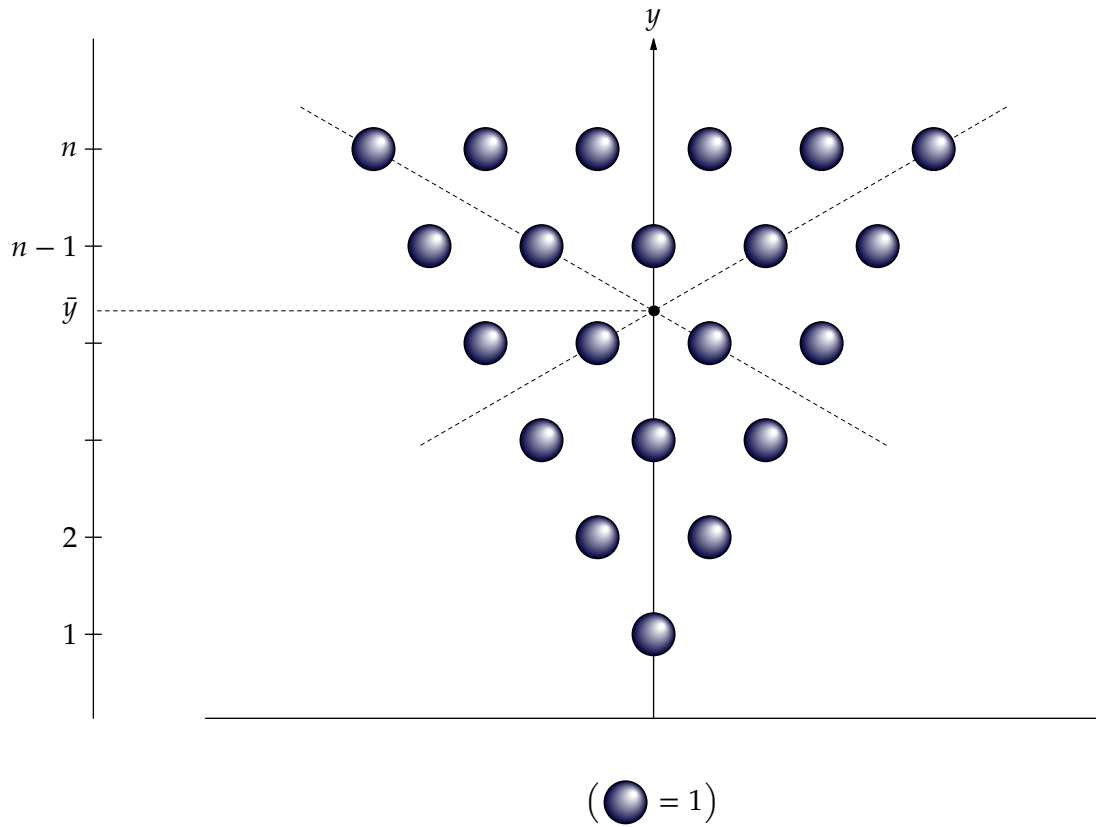
$$k^2 = 1 + 3 + \cdots + (2k - 1) \quad \Rightarrow \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\begin{array}{c}
 \begin{array}{c}
 1 \\
 3 \quad 1 \\
 5 \quad 3 \quad \vdots \\
 \vdots \quad 5 \quad \vdots \quad 1 \\
 2n-3 \quad \vdots \quad 3 \quad 1 \\
 2n-1 \quad 2n-3 \quad \cdots \quad 5 \quad 3 \quad 1
 \end{array}
 +
 \begin{array}{c}
 2n-1 \\
 2n-3 \quad 2n-3 \\
 \vdots \quad \vdots \\
 5 \quad \cdots \quad 5 \quad 5 \\
 3 \quad 3 \quad \cdots \quad 3 \quad 3 \\
 1 \quad 1 \quad 1 \quad \cdots \quad 1 \quad 1
 \end{array}
 +
 \begin{array}{c}
 1 \\
 1 \quad 3 \\
 1 \quad 3 \quad 5 \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 1 \quad 3 \quad 5 \quad \cdots \quad 2n-3 \quad 2n-1
 \end{array}
 \\
 \\
 =
 \begin{array}{c}
 2n+1 \\
 2n+1 \quad 2n+1 \\
 2n+1 \quad 2n+1 \quad 2n+1 \\
 \vdots \quad \vdots \\
 2n+1 \quad 2n+1 \quad 2n+1 \quad \cdots \quad 2n+1 \\
 2n+1 \quad 2n+1 \quad 2n+1 \quad \cdots \quad 2n+1 \quad 2n+1
 \end{array}
 \end{array}$$

$$3(1^2 + 2^2 + \cdots + n^2) = (2n+1)(1 + 2 + \cdots + n)$$

$$\therefore 1^2 + 2^2 + \cdots + n^2 = \frac{2n+1}{3} \cdot \frac{n(n+1)}{2}$$

## Sums of squares IX (via centroids)



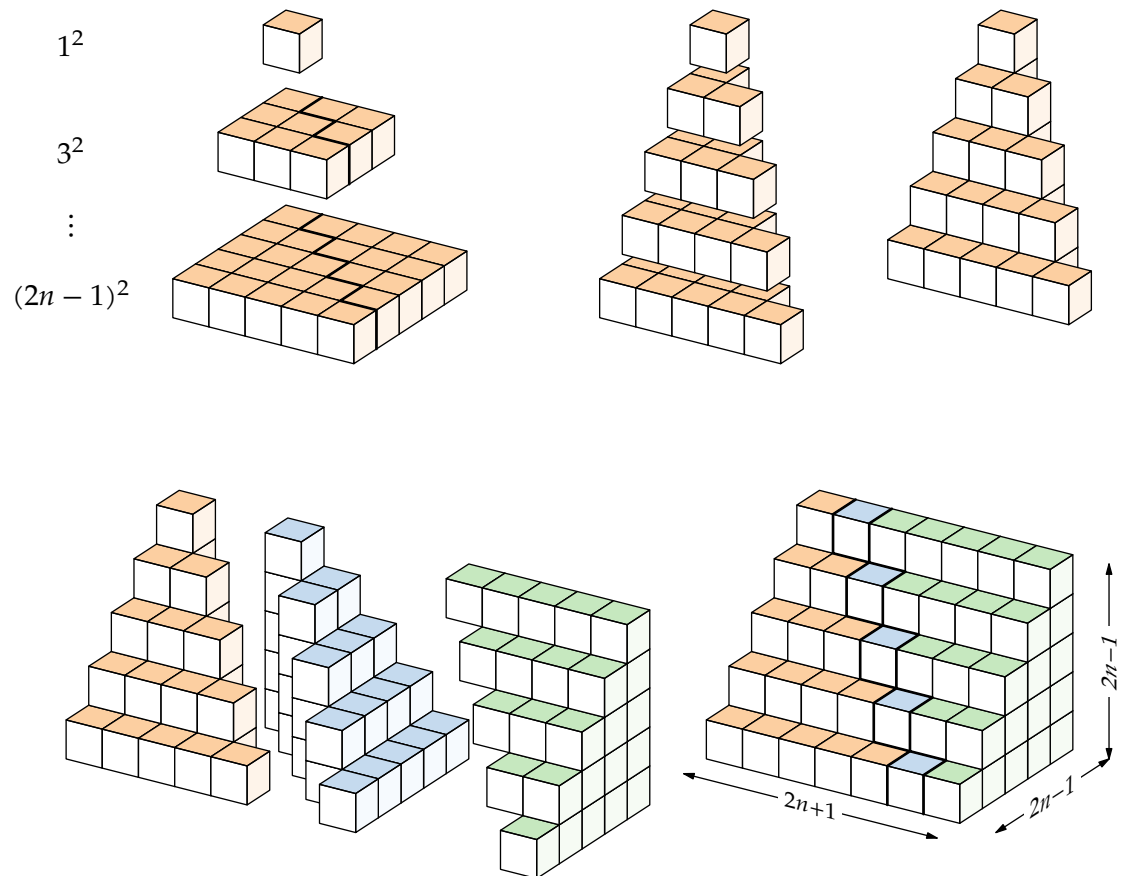
$$\bar{y} = 1 + \frac{2}{3}(n-1) = \frac{1 \cdot 1 + 2 \cdot 2 + \cdots + n \cdot n}{1 + 2 + \cdots + n}$$

$$\therefore 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{2} \cdot \frac{1}{3}(2n+1) = \frac{1}{6}n(n+1)(2n+1)$$

— Sidney H. Kung

## Sums of odd squares

$$1^2 + 2^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

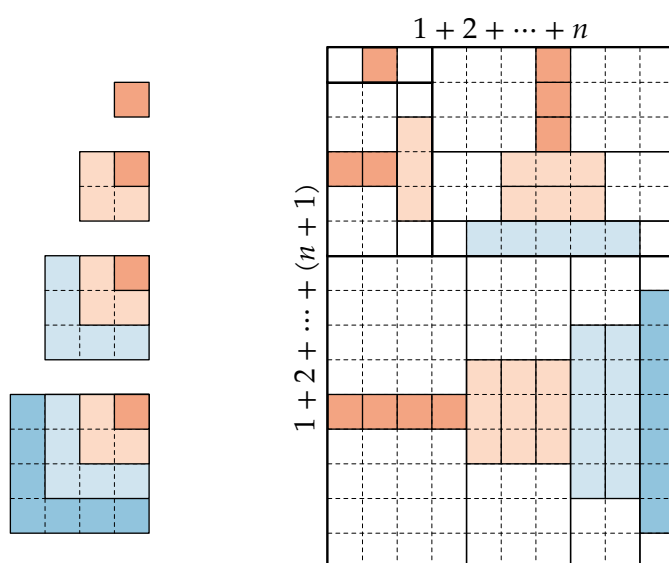


$$\begin{aligned} 3 \times (1^2 + 3^2 + \cdots + (2n-1)^2) &= (1 + 2 + \cdots + (2n-1)) \times (2n+1) \\ &= \frac{(2n-1)(2n)(2n+1)}{2} = n(2n-1)(2n+1) \end{aligned}$$

— RBN

## Sums of sums of squares

$$\sum_{k=1}^n \sum_{i=1}^k i^2 = \frac{1}{3} \binom{n+1}{2} \binom{n+2}{2}$$



$$3(1^2) + 3(1^2 + 2^2) + 3(1^2 + 2^2 + 3^2) + \dots + 3(1^2 + 2^2 + \dots + n^2) = \binom{n+1}{2} \binom{n+2}{2}$$

— C. G. Wastun



## Pythagorean runs

$$3^2 + 4^2 = 5^2$$

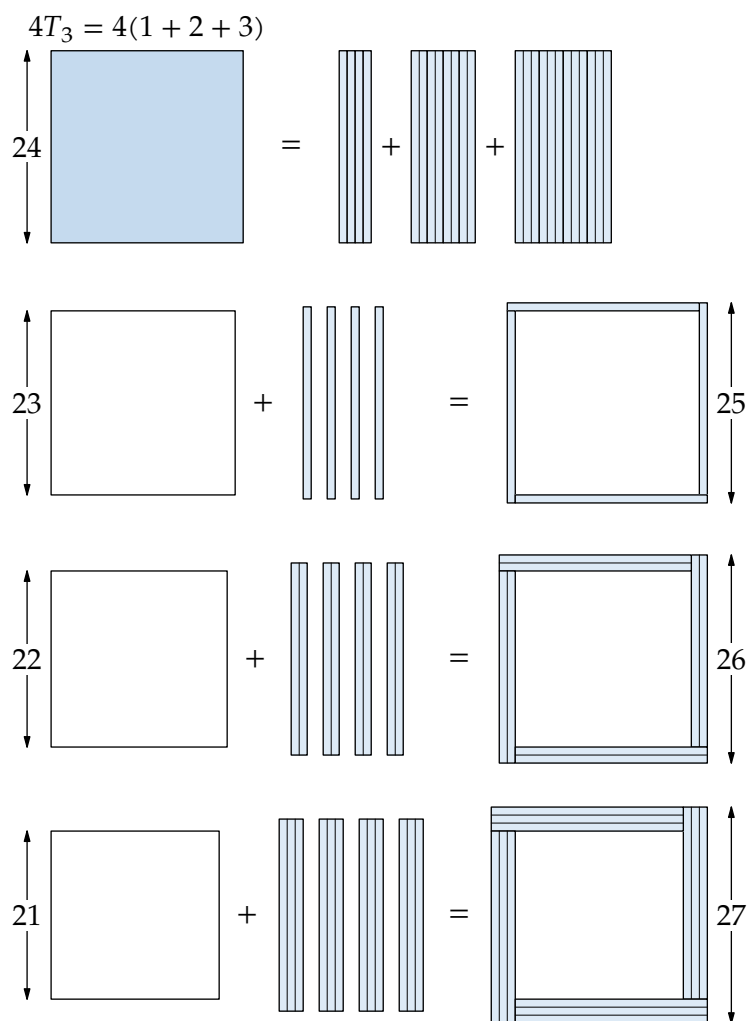
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

$\vdots$

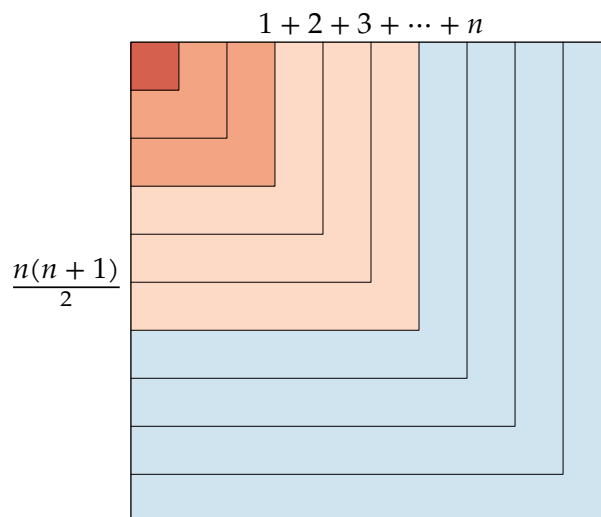
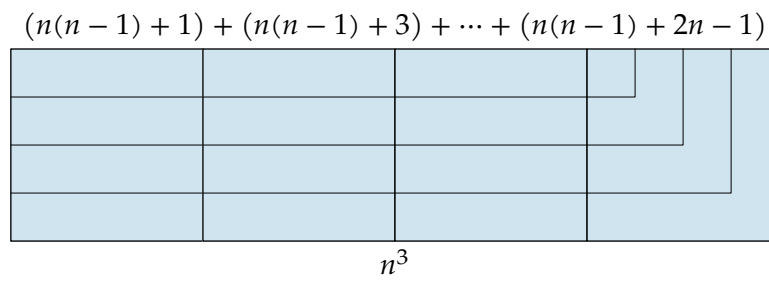
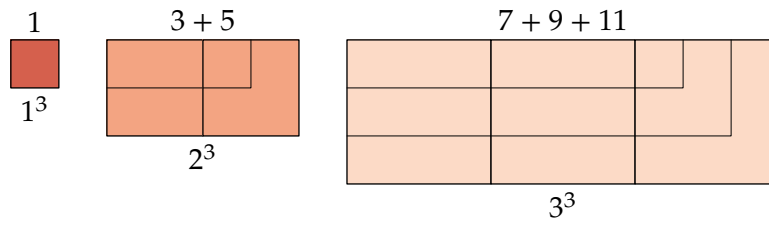
$$T_n = 1 + 2 + \cdots + n \Rightarrow (4T_n - n)^2 + \cdots + (4T_n)^2 = (4T_n + 1)^2 + \cdots + (4T_n + n)^2$$

e.g.,  $n = 3$ :



— Michael Boardman

## Sums of cubes VII



$$1^3 + 2^3 + \cdots + n^3 = 1 + 3 + 5 + \cdots + 2\frac{n(n-1)}{2} - 1 = \left(\frac{n(n-1)}{2}\right)^2$$

— Alfinio Flores

## Sums of integers as sums of cubes

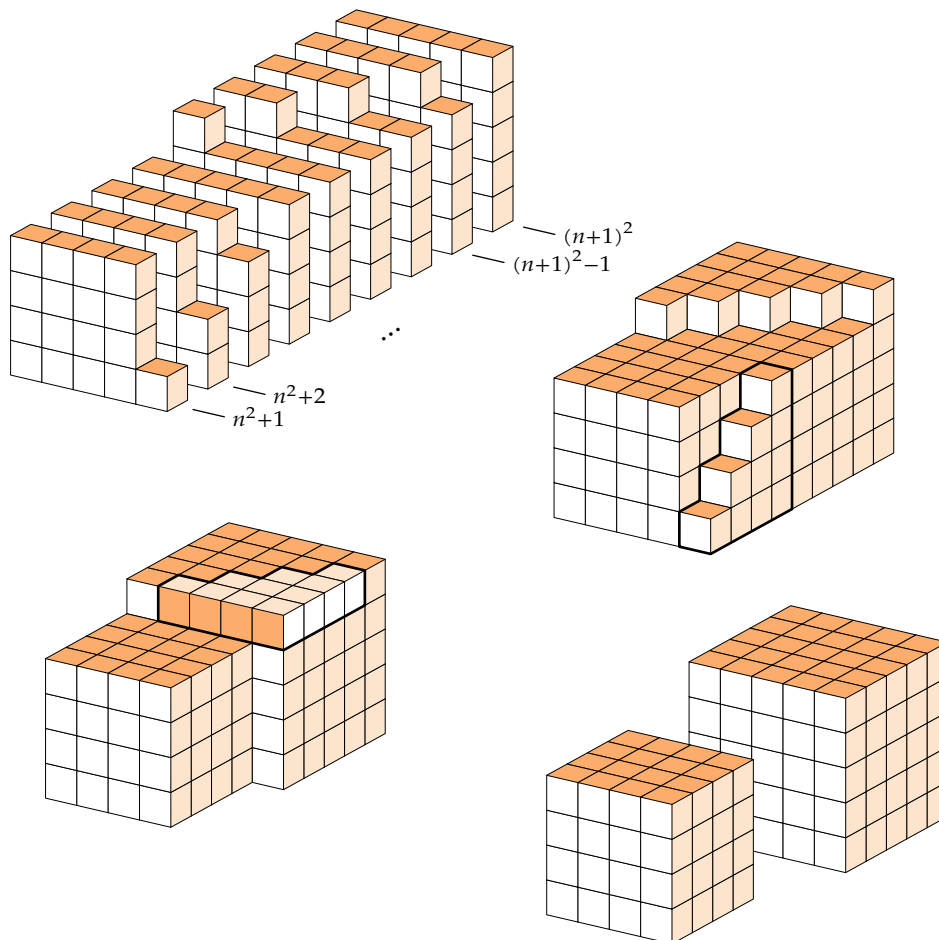
$$2 + 3 + 4 = 1 + 8$$

$$5 + 6 + 7 + 8 + 9 = 8 + 27$$

$$10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$$

$\vdots$

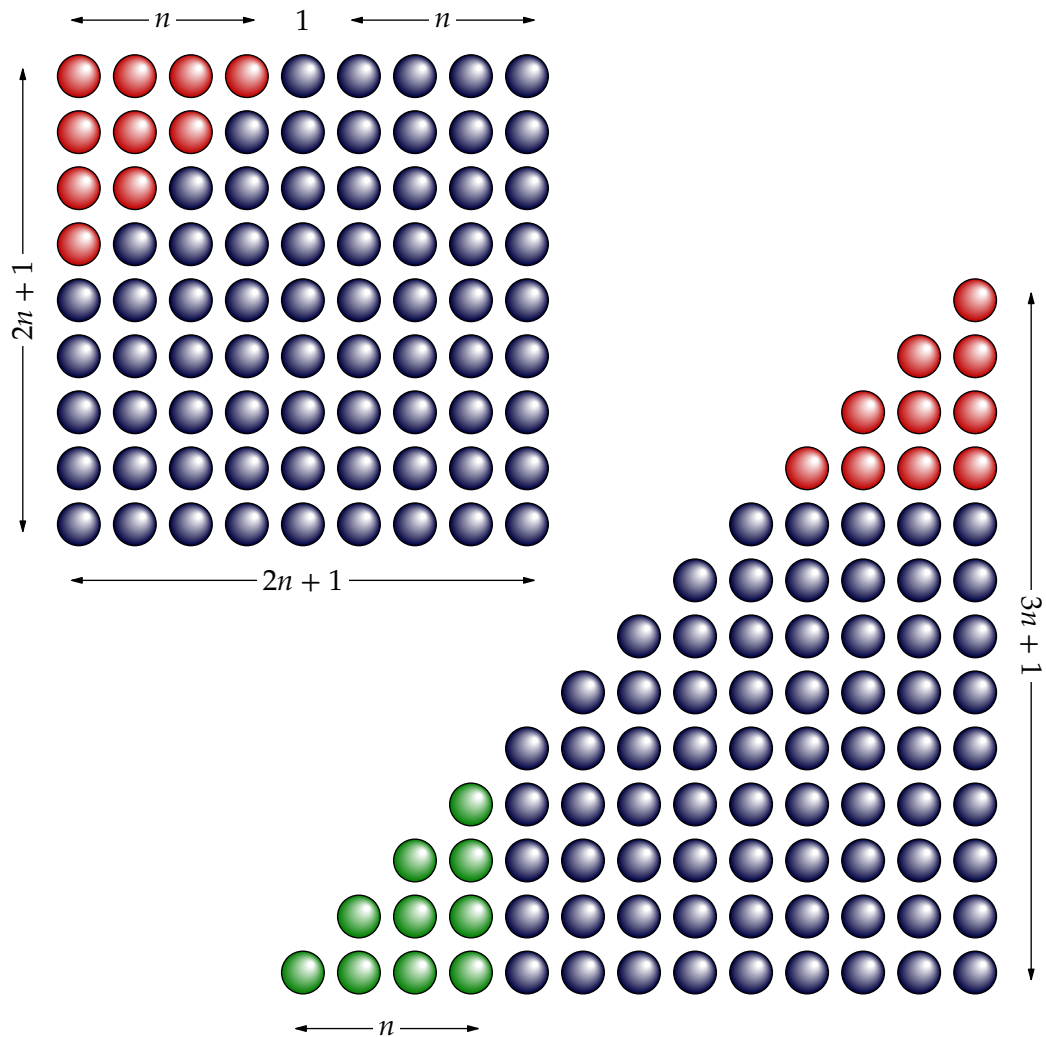
$$(n^2 + 1) + (n^2 + 2) + \cdots + (n + 1)^2 = n^3 + (n + 1)^3$$



— RBN

**The square of any odd number is the difference between two triangular numbers**

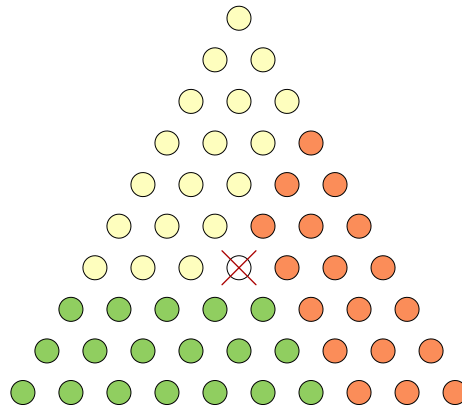
$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad (2n + 1)^2 = T_{3n+1} - T_n$$



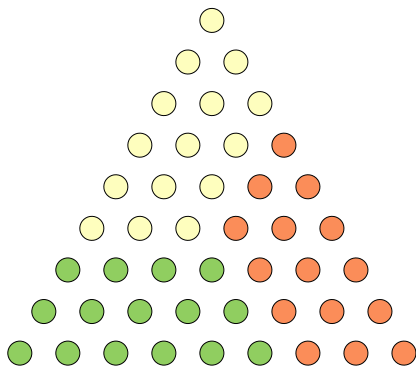
— RBN

## Triangular numbers mod 3

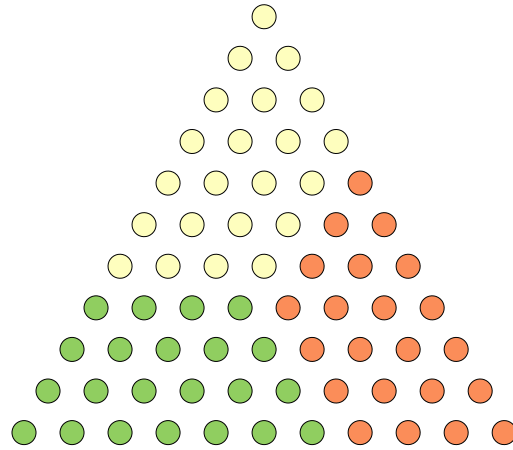
$$1 + 2 + \cdots + n = T_n \Rightarrow \begin{cases} T_n \equiv 1 \pmod{3}, & n \equiv 1 \pmod{3} \\ T_n \equiv 0 \pmod{3}, & n \not\equiv 1 \pmod{3} \end{cases}$$



$$T_{3k+1} = 1 + 3(T_{2k+1} - T_{k+1})$$



$$T_{3k} = 3(T_{2k} - T_k)$$



$$T_{3k+2} = 3(T_{2k+1} - T_k)$$

# Counting triangular numbers IV: Counting cannonballs

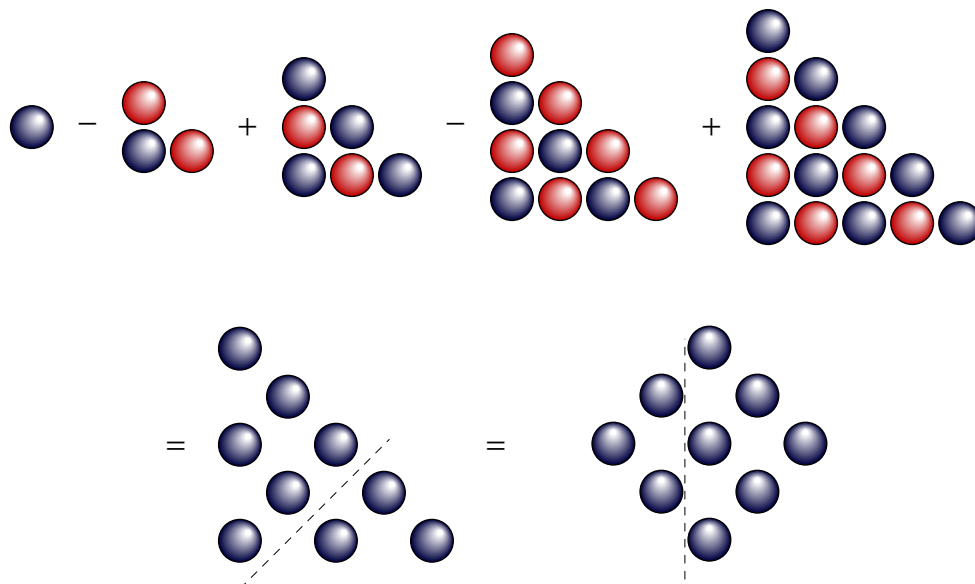
$$1 + 2 + \cdots + k = T_k \Rightarrow \sum_{k=1}^n T_k = \sum_{k=1}^n k(n - k + 1)$$



— Deanna B. Haunsperger and Stephen F. Kennedy

## Alternating sums of triangular numbers

$$1 + 2 + \cdots + k = T_k \Rightarrow \sum_{k=1}^{2n-1} (-1)^{k+1} T_k = n^2$$



— RBN

# The sum of the squares of consecutive triangular numbers is triangular

$$1 + 2 + \cdots + n = T_n \Rightarrow T_{n-1}^2 + T_n^2 = T_{n^2}$$



NOTE: This is a companion result to the more familiar  $T_{n-1} + T_n = n^2 \rightarrow$

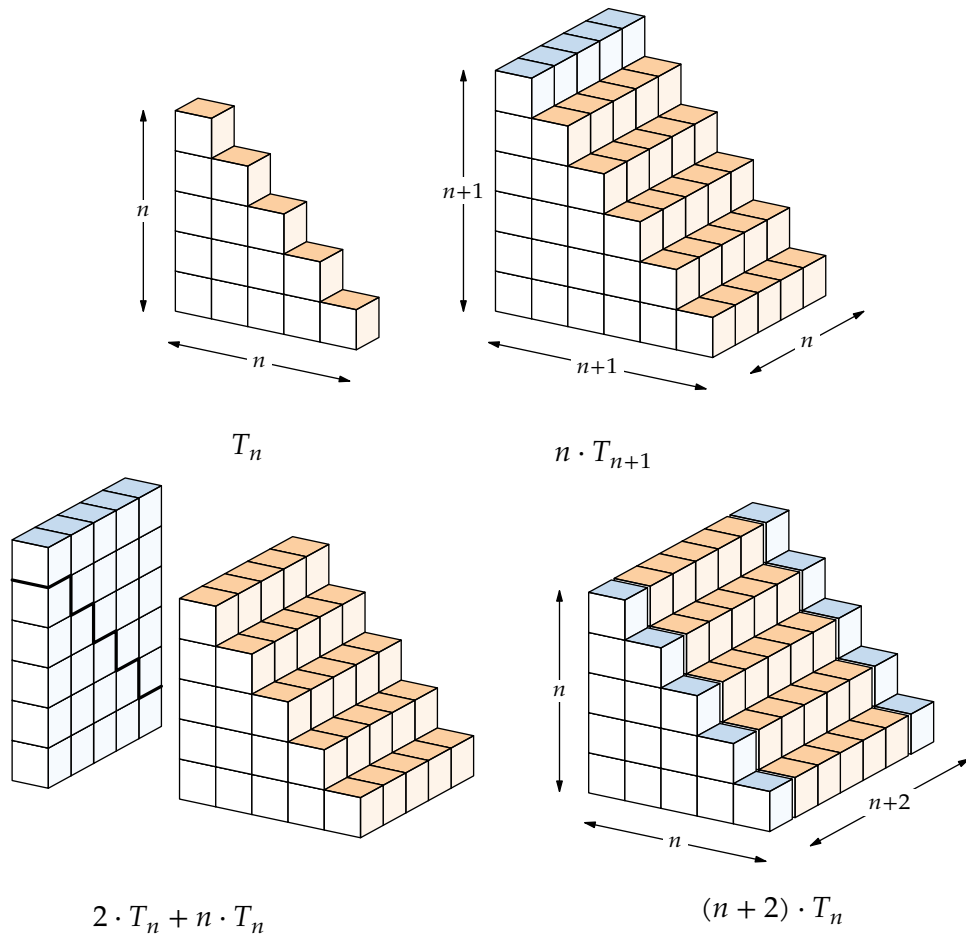


— RBN



## Recursion for triangular numbers

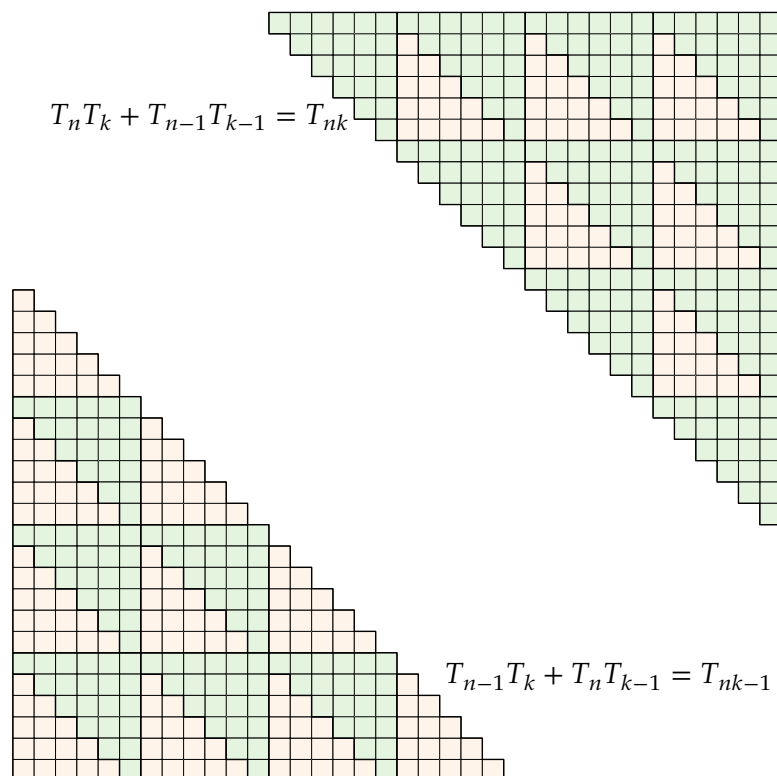
$$1 + 2 + \cdots + n = T_n \Rightarrow T_{n+1} = \frac{n+2}{n} T_n$$



$$n \cdot T_{n+1} = (n+2) \cdot T_n \Rightarrow T_{n+1} = \frac{n+2}{n} T_n$$

## Identities for triangular numbers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



— RBN

## More identities for triangular numbers

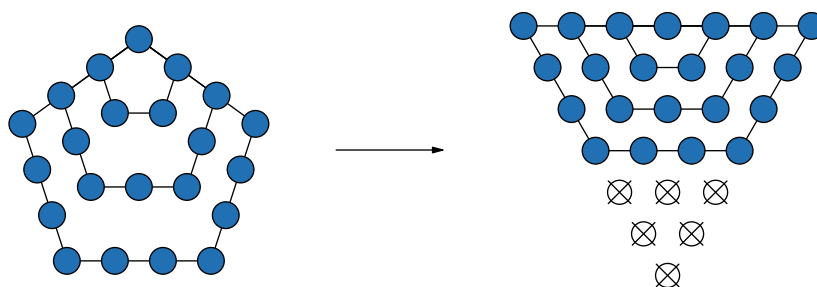
$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



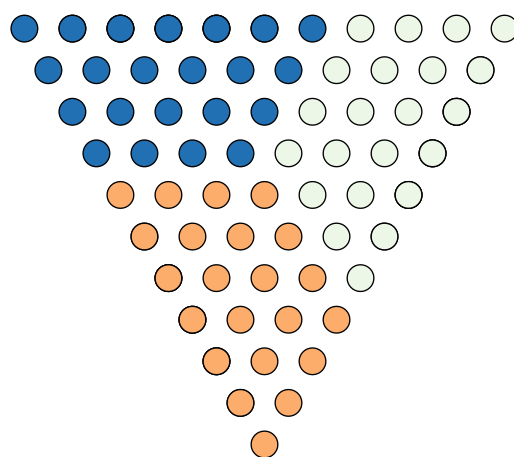
— James O. Chilaka

# Identities for pentagonal numbers

$$\left. \begin{array}{l} P_n = 1 + 4 + 7 + \cdots + (3n - 2) \\ T_n = 1 + 2 + 3 + \cdots + n \end{array} \right\} \Rightarrow$$

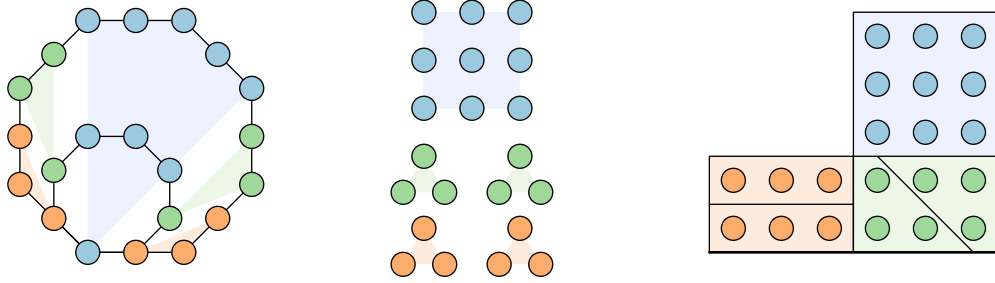


$$P_n = T_{2n-1} - T_{n-1}$$

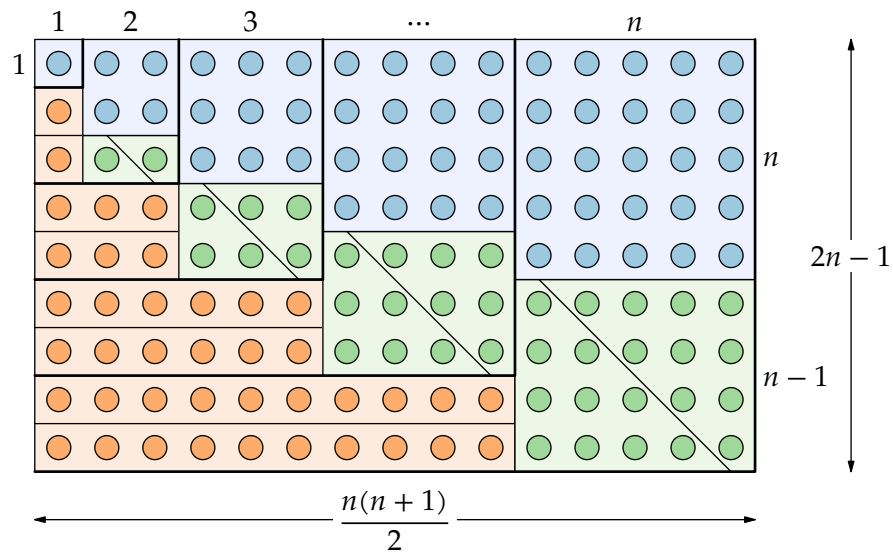


$$3P_n = T_{3n-1}$$

## Sums of octagonal numbers



$$T_k = 1 + 2 + \cdots + k \Rightarrow O_k = k^2 + 4T_{k-1}$$

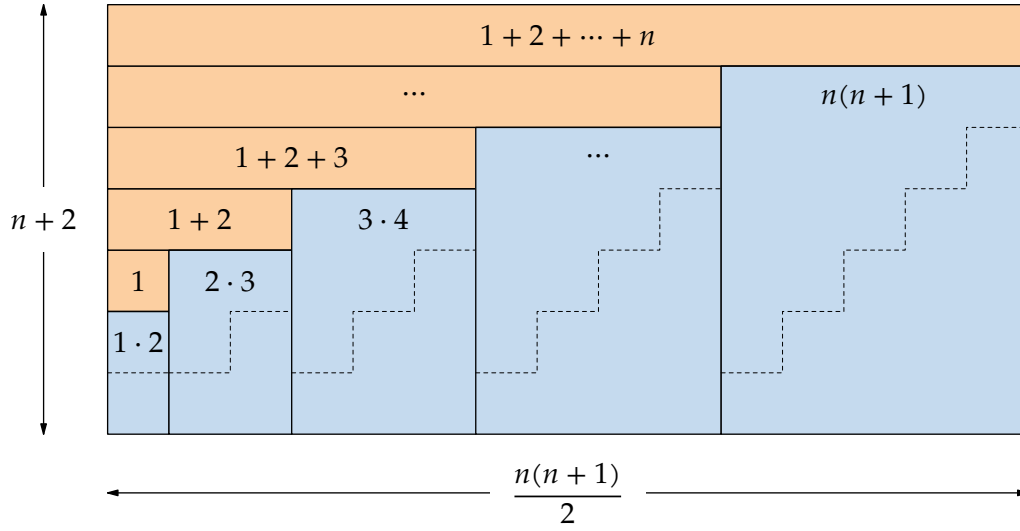


$$\sum_{k=1}^n O_k = 1 + 8 + 21 + 40 + \cdots + (n^2 + 4T_{n-1}) = \frac{n(n+1)(2n-1)}{2}$$

— James O. Chilaka

## Sums of products of consecutive integers I

$$\sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}$$



$$T_k = 1 + 2 + \dots + k \Rightarrow$$

$$1 \cdot 2 + 2 \cdot 3 + \dots + n(n+1) + (T_1 + T_2 + \dots + T_n) = \frac{n(n+1)(n+2)}{2},$$

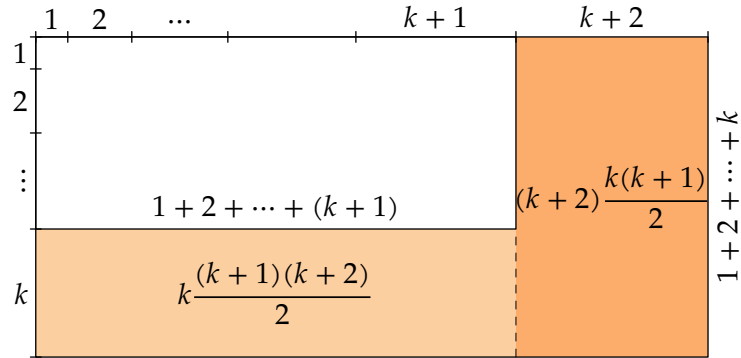
$$(T_1 + T_2 + \dots + T_n) = \frac{1}{2} (1 \cdot 2 + 2 \cdot 3 + \dots + n(n+1)),$$

$$\therefore \frac{3}{2} (1 \cdot 2 + 2 \cdot 3 + \dots + n(n+1)) = \frac{n(n+1)(n+2)}{2}.$$

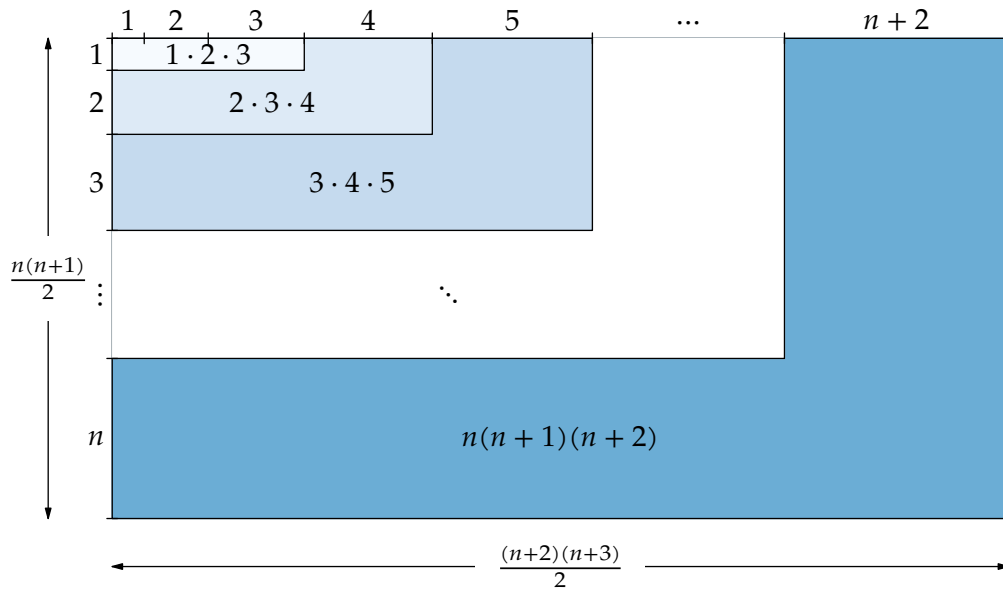
— James O. Chilaka

## Sums of products of consecutive integers II

$$\sum_{k=1}^n k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$



$$k \frac{(k+1)(k+2)}{2} + (k+2) \frac{k(k+1)}{2} = k(k+1)(k+2)$$

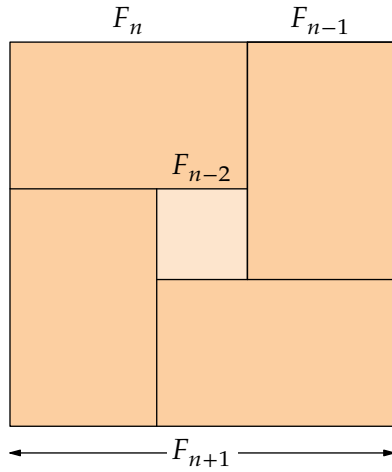


$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + n(n+1)(n+2) = \frac{n(n+1)}{2} \times \frac{(n+2)(n+3)}{2} = \frac{n(n+1)(n+2)(n+3)}{4}$$

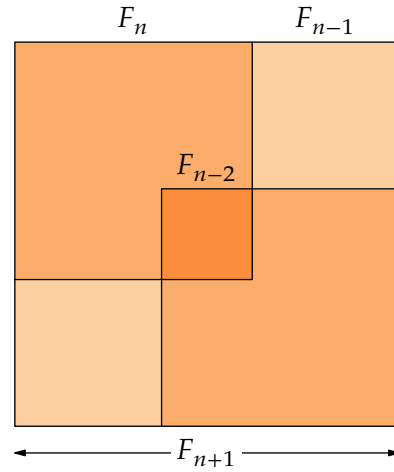
— James O. Chilaka

## Fibonacci identities

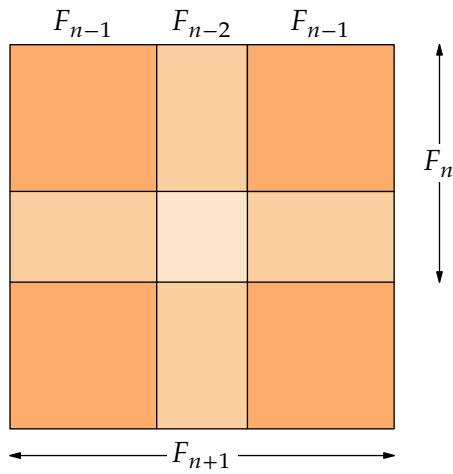
$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \Rightarrow$$



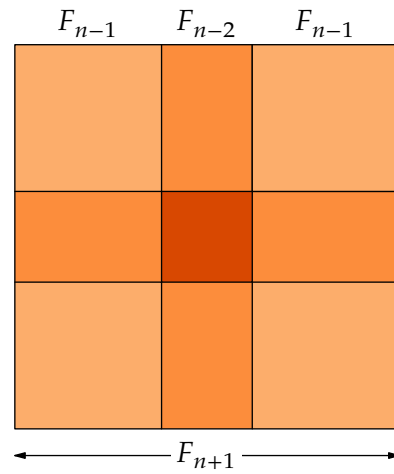
$$F_{n+1}^2 = 4F_n F_{n-1} + F_{n-2}^2$$



$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2$$



$$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-1}F_{n-2} + F_{n-2}^2$$



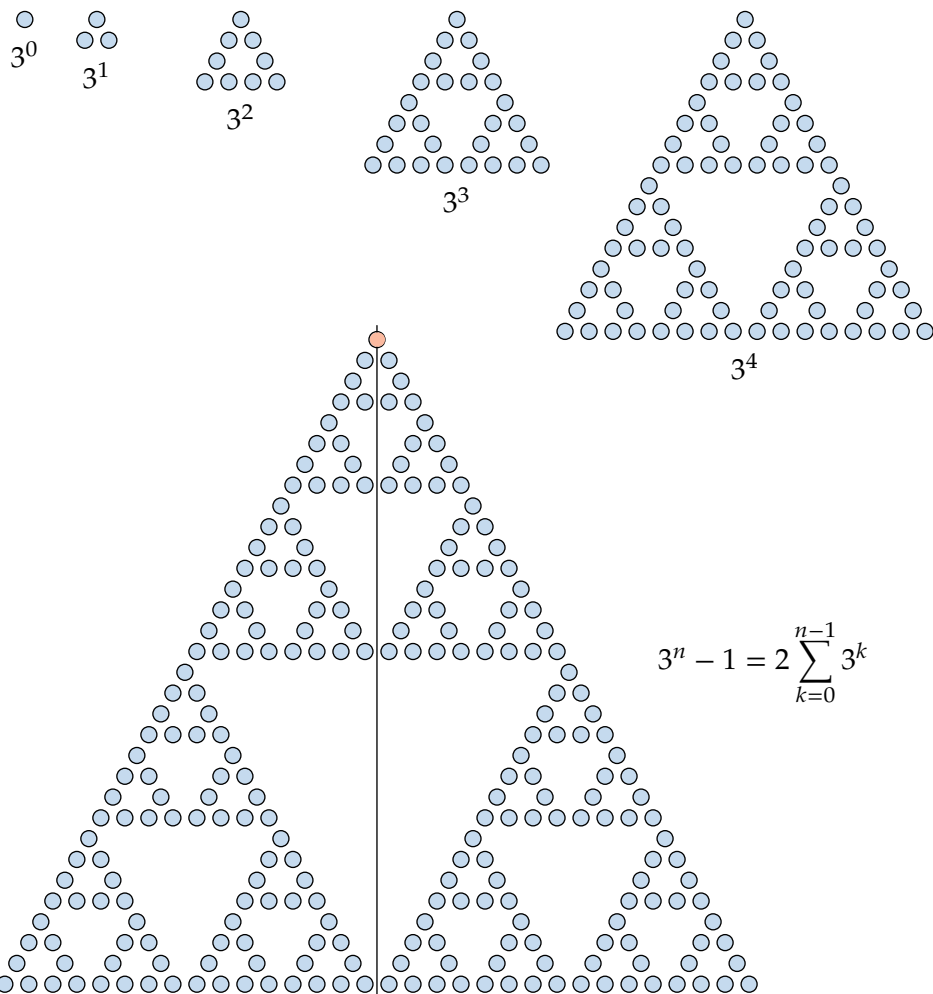
$$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$$

— Alfred Brousseau



## Sums of powers of three

$$\sum_{k=0}^{n-1} 3^k = \frac{3^n - 1}{2}$$



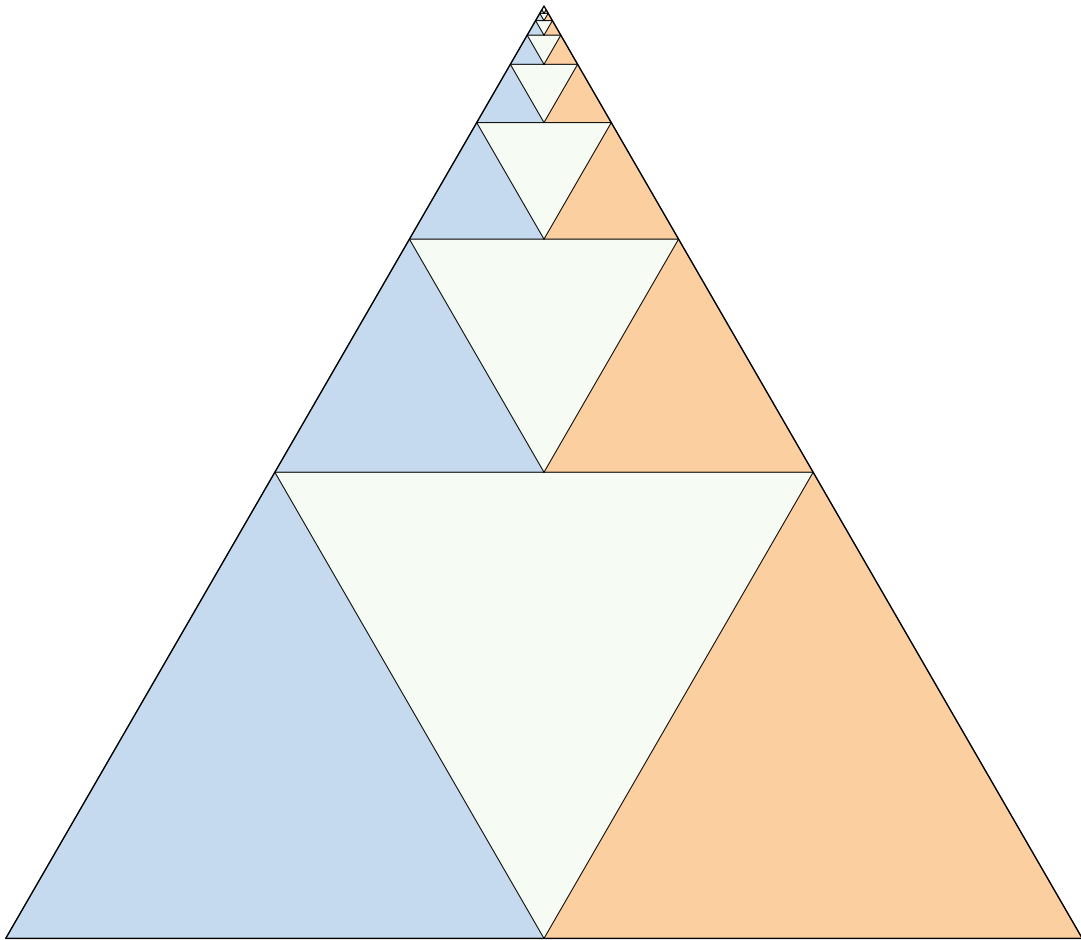
— David B. Sher

## Infinite series, linear algebra, & other topics

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## A geometric series

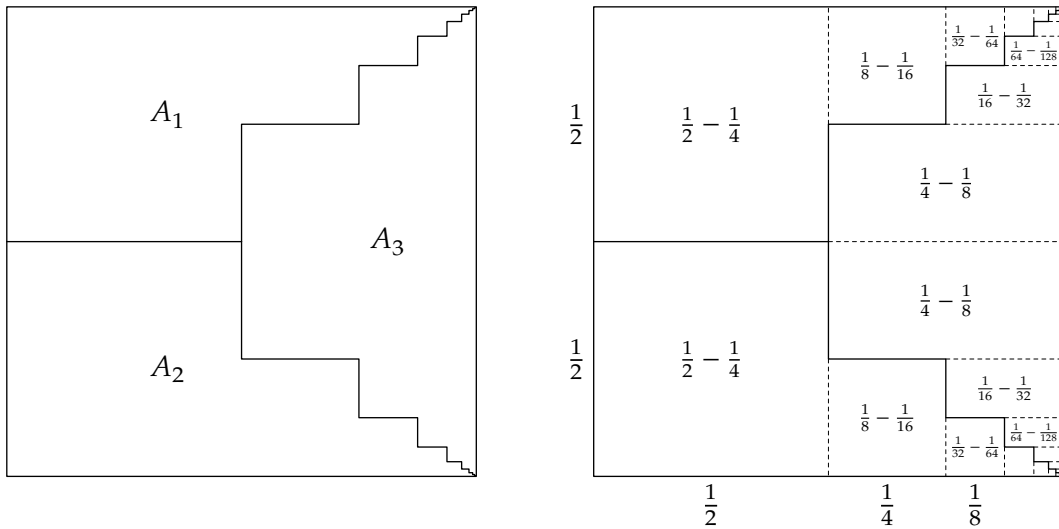
$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$$



— Rick Mabry

## An alternating series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots = \frac{1}{3}$$



$$A_1 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots,$$

$$A_1 = A_2 = A_3, \quad A_1 + A_2 + A_3 = 1,$$

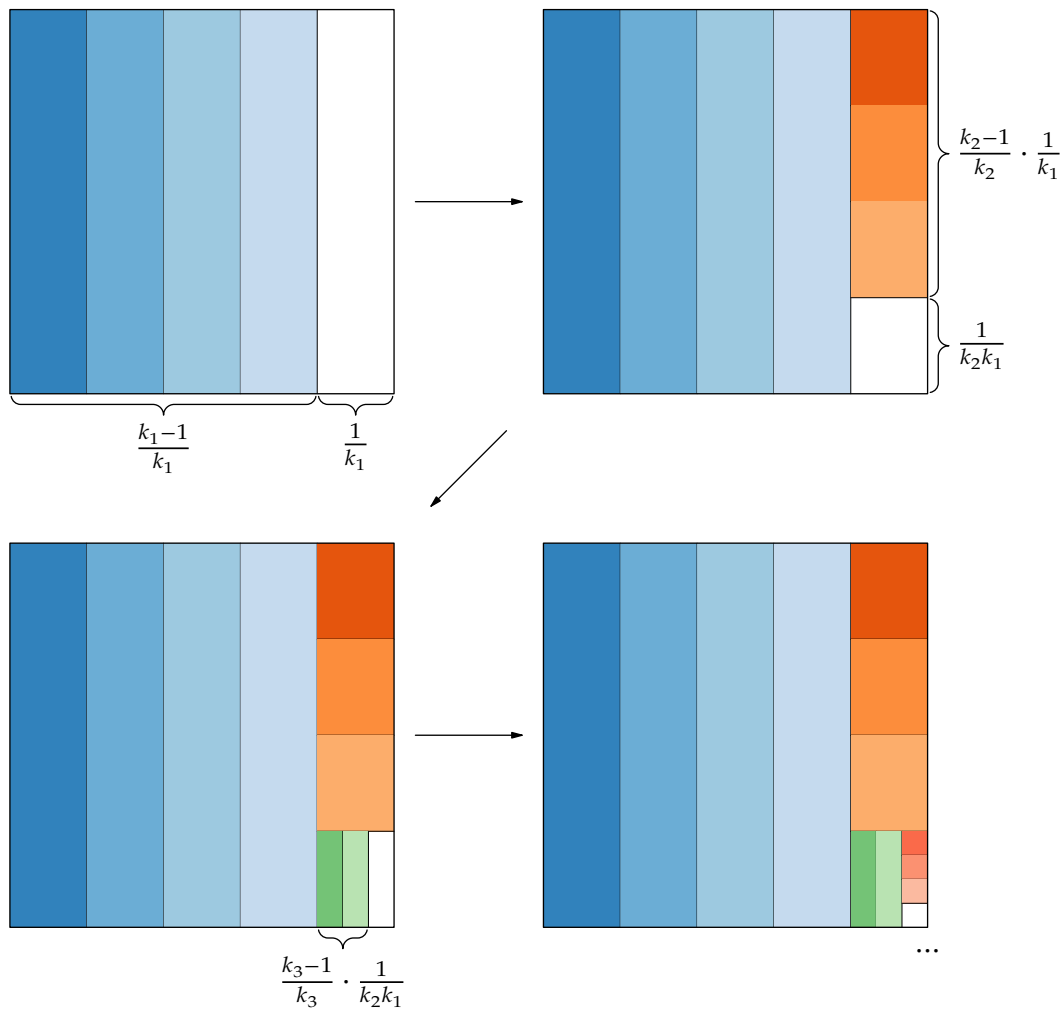
$$\therefore A_1 = \frac{1}{3}.$$

— James O. Chilaka

## A generalized geometric series

Let  $\{k_1, k_2, k_3\}$  be a sequence of integers, each of which is at least 2. Then

$$\frac{k_1 - 1}{k_1} + \frac{k_2 - 1}{k_2 k_1} + \frac{k_3 - 1}{k_3 k_2 k_1} + \dots = 1$$



— John Mason