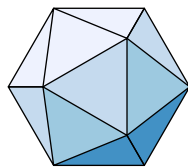


Proofs without words I

Exercises in METAPOST

Toby Thurston

March 2021 — September 2022



Contents

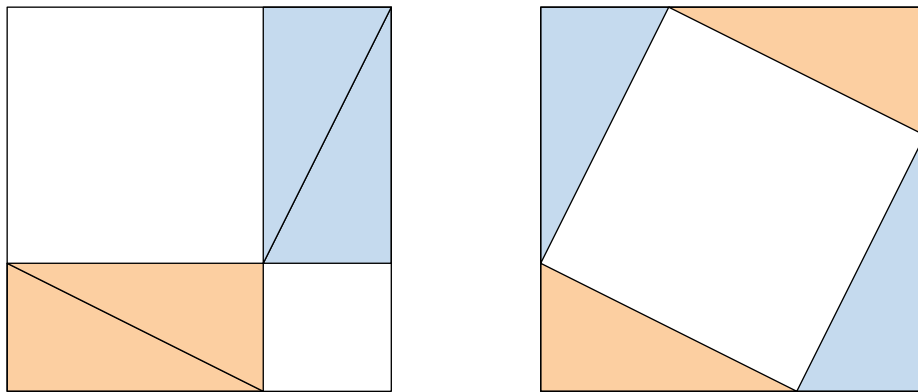
Geometry and Algebra	3
Trigonometry, Calculus, & Analytic Geometry	53
Inequalities	89
Integer sums	127
Sequences and series	207
Miscellaneous	229

Geometry and Algebra

The Pythagorean theorem I	5
The Pythagorean theorem II	7
The Pythagorean theorem III	9
The Pythagorean theorem IV	11
The Pythagorean theorem V	13
The Pythagorean theorem VI	15
A Pythagorean theorem: $aa' = bb' + cc'$	17
The rolling circle squares itself	19
On trisecting an angle	21
Trisection in an infinite number of steps	23
Trisection of a line segment	25
The vertex angles of a star sum to 180°	27
Viviani's theorem I	29
Viviani's theorem II	31
A theorem about right angles	33
Area and the projection theorem of a right triangle	35
Chords and tangents of equal length	37
Completing the square	39
Algebraic areas I	41
Algebraic areas II	43
Sum of squares identity	45
Polygonal numbers	47
The volume of a frustrum of a square pyramid	49
The volume of a hemisphere via Cavalieri's Principle	51

```
path s, t;
s = unitsquare shifted -(1/2, 1/2) scaled 144;
t = point 0 of s -- point 2/3 of s -- point -1/3 of s -- cycle;
picture P[];
P2 = image(
  for i=0 upto 3:
    fill t rotated 90i withcolor if odd i: Blues 7 2 else: Oranges 7 2 fi;
    draw t rotated 90i;
  endfor
  draw s;
);
P1 = image(
  fill t withcolor Oranges 7 2; draw t;
  t := t rotatedabout(point 3/2 of t, 180);
  fill t withcolor Oranges 7 2; draw t;
  t := t shifted (point 0 of t - point 2 of t);
  t := t rotatedabout(point 2 of t, -90);
  fill t withcolor Blues 7 2; draw t;
  t := t rotatedabout(point 3/2 of t, 180);
  fill t withcolor Blues 7 2; draw t;
  draw s;
);
draw P1;
draw P2 shifted 200 right;
```

The Pythagorean theorem I



— adapted from the *Chou pei san ching*

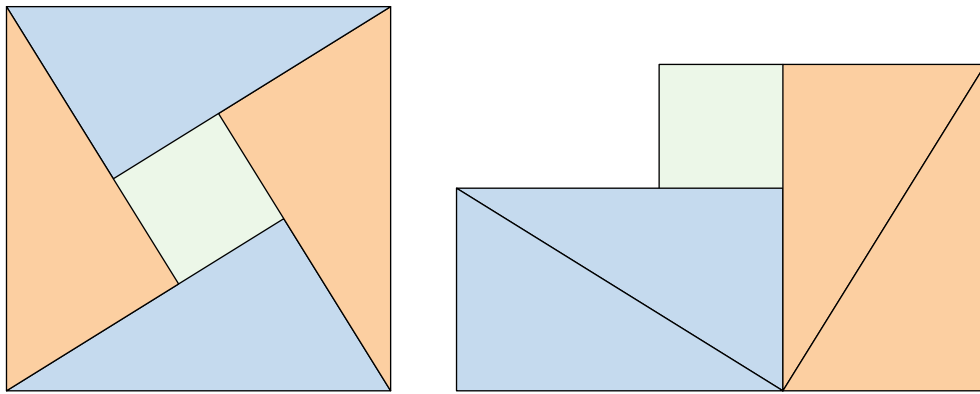
```

path s, t;
s = fullcircle scaled 144;
t = (point 4 of s -- point 0 of s -- point sqrt(2) of s -- cycle) shifted point 6 of s;
s := for i=0 upto 3: point 2 of t rotated 90i -- endfor cycle;
picture P[];
P1 = image(
    fill s withcolor Greens 7 1;
    for i=0 upto 3:
        fill t rotated 90i withcolor if odd i: Oranges 7 2 else: Blues 7 2 fi;
        draw t rotated 90i;
    endfor
);
numeric theta; theta = angle (point 2 of t - point 0 of t);
s := s rotated -theta;
t := t rotated -theta;
P2 = image(
    fill s withcolor Greens 7 1; draw subpath (1, 3) of s;
    fill t withcolor Blues 7 2; draw t;
    t := t rotatedabout(point 1/2 of t, 180);
    fill t withcolor Blues 7 2; draw t;
    t := t rotatedabout(point 0 of t, -90);
    fill t withcolor Oranges 7 2; draw t;
    t := t rotatedabout(point 1/2 of t, 180);
    fill t withcolor Oranges 7 2; draw t;
);

label.ulft(P1, 10 left);
label.urt(P2, 10 right);
label.bot("\textit{Behold!}", point 1/2 of bbox currentpicture shifted 36 down);

```

The Pythagorean theorem II



Behold!

— Bhāskara (12th century)

```

path s, t, a, b, c;
s = fullcircle scaled 72;
t = (point 4 of s -- point 0 of s -- point sqrt(6) of s -- cycle) shifted point 6 of s;
a = unitsquare zscaled (point 2 of t - point 0 of t) shifted point 0 of t;
b = unitsquare zscaled (point 1 of t - point 2 of t) shifted point 2 of t;
c = unitsquare zscaled (point 0 of t - point 1 of t) shifted point 1 of t;

color v, w; v = Oranges 7 1; w = Greens 7 1;

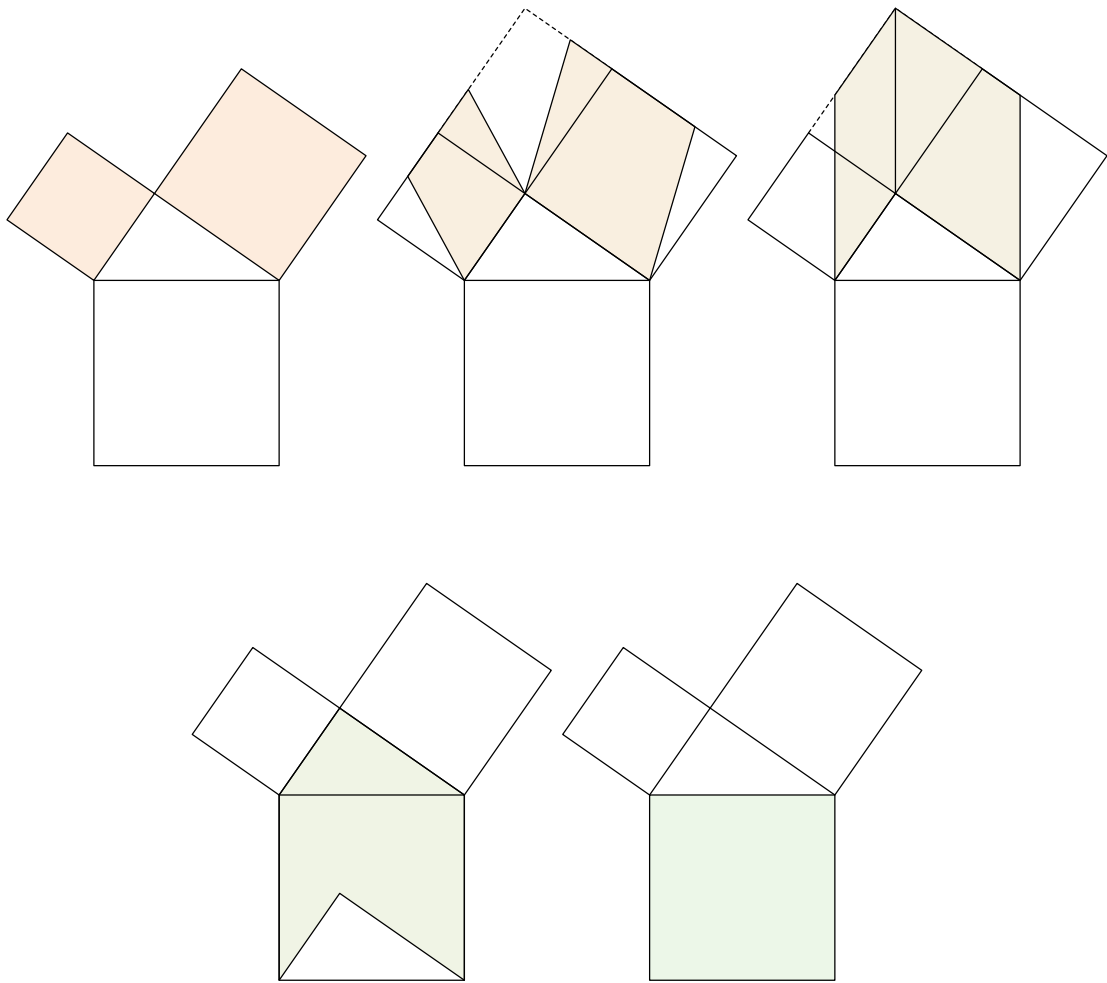
picture P[];
P0 = image(draw a; draw b; draw c);
P1 = image(fill a withcolor v; fill b withcolor v; draw P0);
z0 = whatever[point 2 of a, point 3 of a] = whatever[point 2 of b, point 3 of b];
z1 = whatever[z0, point 3 of a]; x1 = xpart point 0 of a;
z2 = whatever[z0, point 2 of b]; x2 = xpart point 1 of b;
path wedge; wedge = subpath (0,1) of a -- subpath (0, 1) of b -- z2 -- z0 -- z1 -- cycle;

P2 = image(
    draw point 2 of a -- z0 -- point 3 of b dashed evenly scaled 1/2;
    path a', b'; numeric t, u;
    t = angle (point 1 of a - point 0 of a);
    u = angle (point 1 of b - point 0 of b);
    a' = a shifted - point 0 of a rotated -t slanted 1/2 rotated t shifted point 0 of a;
    b' = b shifted - point 0 of b rotated -u slanted -1/3 rotated u shifted point 0 of b;
    fill a' withcolor 1/4[v,w]; draw a';
    fill b' withcolor 1/4[v,w]; draw b';
    draw P0
);
P3 = image(
    draw point 2 of a -- z0 -- point 3 of b dashed evenly scaled 1/2;
    fill wedge withcolor 1/2[v,w]; draw wedge; draw point 1 of a -- z0;
    draw P0
);
P4 = image(
    fill wedge shifted (point 0 of a - z1) withcolor 3/4[v,w];
    draw wedge shifted (point 0 of a - z1);
    draw P0
);
P5 = image(fill c withcolor w; draw P0);

draw P1;
draw P2 shifted (144,0);
draw P3 shifted (288,0);
draw P4 shifted (72, -200);
draw P5 shifted (216, -200);

```


The Pythagorean theorem III



— based on Euclid's proof

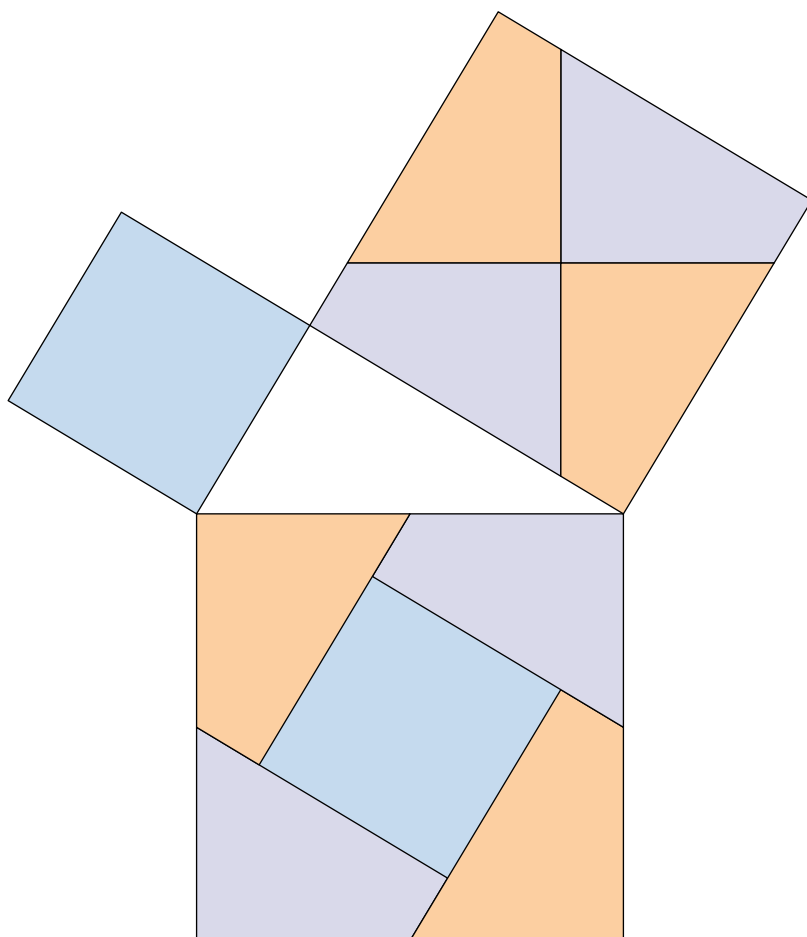
```
path c, a, a', bq, bq'; numeric r; r = 59;
c = unitsquare shifted -(1/2, 1/2) scaled 160;
a = c scaled cosd(r) rotated r;
pair p, q;
p = whatever[point 0 of a, point 1 of a] = whatever[point 0 of c, point 1 of c];
q = whatever[point 0 of a, point 3 of a] = whatever[point 0 of c, point 3 of c];
bq = point 0 of c -- p -- point 0 of a -- q -- cycle;

fill a withcolor Blues 7 2;
for i=0 upto 3:
    fill bq rotated 90i withcolor if odd i: Oranges 7 2 else: Purples 7 2 fi;
    draw bq rotated 90i;
endfor

a' = a shifted (point 3 of c - point 0 of a);
fill a' withcolor Blues 7 2;
draw a';

bq' = bq rotated 180 shifted (point 1 of a' - point 2 of (bq rotated 180));
pair o; o = point 0 of bq';
for i=0 upto 3:
    fill bq' rotatedabout(o, 90i) withcolor if odd i: Oranges 7 2 else: Purples 7 2 fi;
    draw bq' rotatedabout(o, 90i);
endfor
```

The Pythagorean theorem IV



— H. E. Dudeney (1917)

```

path t, t';
t = (origin -- 377 right -- 144 up -- cycle) scaled 3/4;
t' = t rotated -90 shifted (point 2 of t + point 1 of t rotated 90);

draw unitsquare scaled 8 withcolor 1/2;
draw unitsquare scaled 8 rotated -90 shifted point 0 of t' withcolor 1/2;
draw unitsquare scaled 8 rotated angle (point 1 of t - point 2 of t)
    shifted point 2 of t withcolor 1/2;

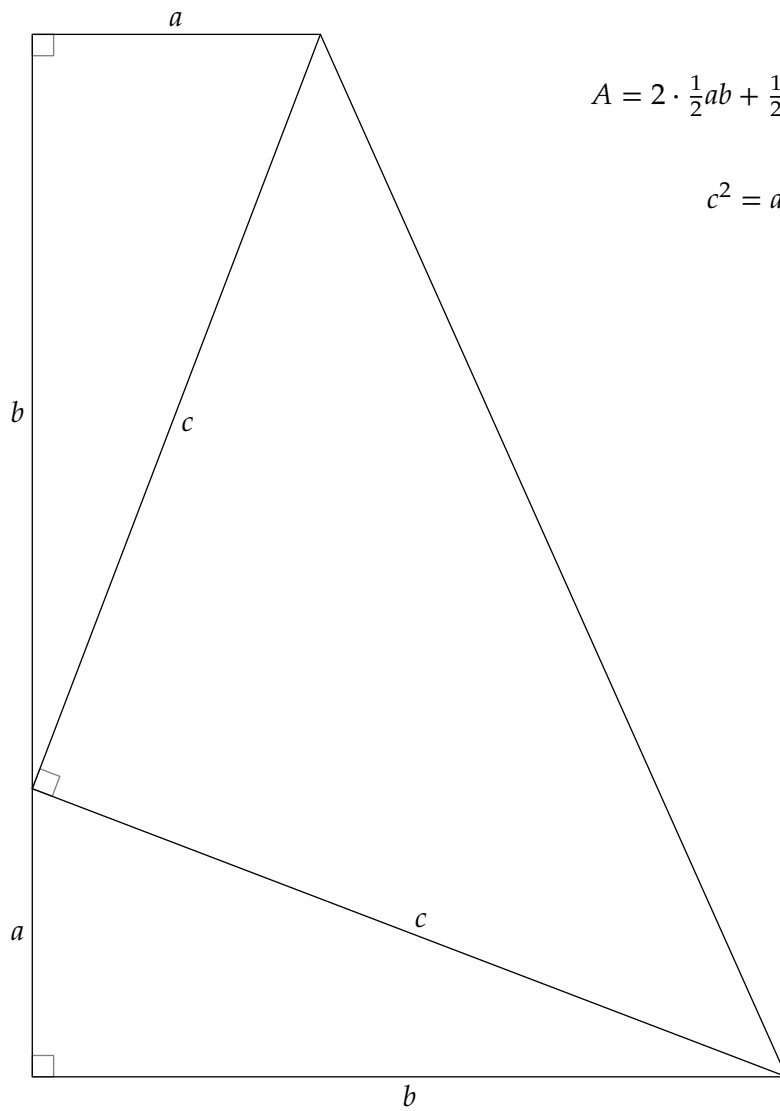
draw t;
draw t';
draw point 1 of t -- point 2 of t';

label.lft("$a$", point -1/2 of t);
label.bot("$b$", point 1/2 of t);
label.urt("$c$", point 3/2 of t);
label.top("$a$", point -1/2 of t');
label.lft("$b$", point 1/2 of t');
label.lrt("$c$", point 3/2 of t');

label.bot(btex \vbox{\openup 24pt\halign{\hfil $$$ \hfil\cr
A = 2 \cdot \frac{1}{2} ab + \frac{1}{2} c^2 = \frac{1}{2}\left(a+b\right)^2\cr
c^2 = a^2 + b^2\cr}} etex, (xpart point 1 of t, ypart point 2 of t' - 12));

```

The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a+b)^2$$

$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)

```

numeric r;
r = 144;  z1 = r * dir 66;

draw unitsquare scaled 8 rotated 90 shifted (x1, 0) withcolor 1/2;
draw (left--right) scaled r withcolor Blues 7 7;
draw origin -- z1 -- (x1, 0) withcolor Blues 7 7;
draw fullcircle scaled 2r withcolor Reds 7 7;

label.top("$a$", (1/2 x1, 0));
label.rt("$b$", (x1, 1/2 y1));
label.ulft("$c$", 1/2 z1);
label.top("$c$", (-1/2 r, 0));
label.top("$c-a$", (1/2(r+x1), 0));

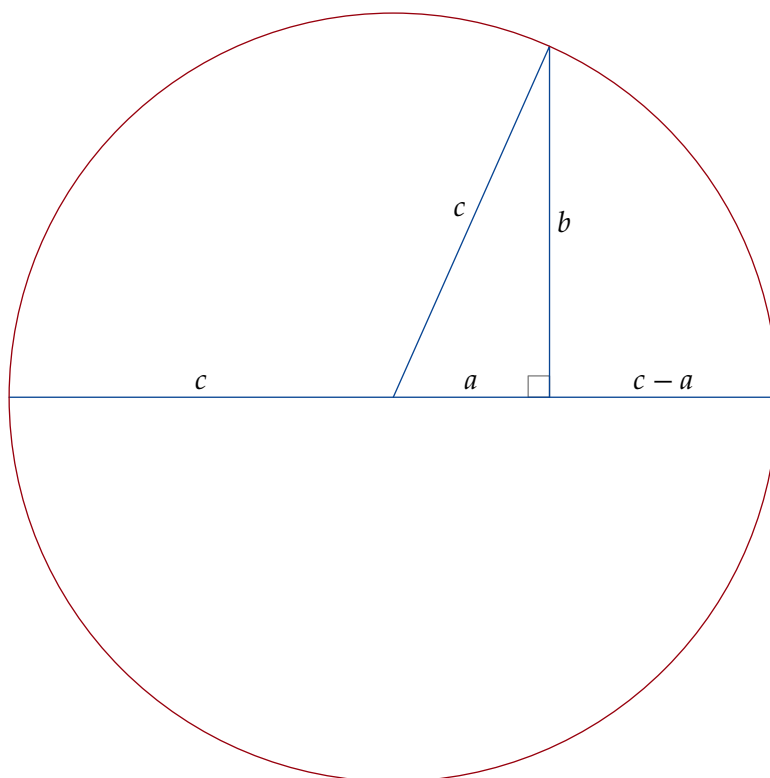
label.lft(btex \vbox{\openup 24pt\halign{\hfil $\displaystyle$\ \hfil\cr
\frac{c+a}{b} = \frac{b}{c-a} \cr
a^2 + b^2 = c^2\cr}} etex, point -1/2 of bbox currentpicture + 16 left);

```

The Pythagorean theorem VI

$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2 + b^2 = c^2$$



— Michael Hardy

```

path c; c = fullcircle scaled 377;
z0 = point 4 of c; z1 = point 0 of c; z2 = point 2.828 of c;
z3 = 5/16[z0, z1];
z4 = whatever [z1, z2];
z4 - z3 = whatever * (z2 - z0);
picture P;
P = image(
    draw unitsquare scaled 6 rotated angle (z0 - z2) shifted z2 withcolor 1/2;
    draw unitsquare scaled 6 rotated angle (z3 - z4) shifted z4 withcolor 1/2;
    draw z3 -- z1 -- z2 -- z0 -- z3 -- z4;

    label.bot ("a$", 1/2[z0, z1] shifted 10 down); label.top("$a'", 7/16[z3, z1]);
    label.ulft("$b$", 1/2[z0, z2]); label.ulft("$b'", 1/2[z3, z4]);
    label.urt ("c$", 1/2[z1, z2]); label.llft("$c'", 9/16[z1, z4]);

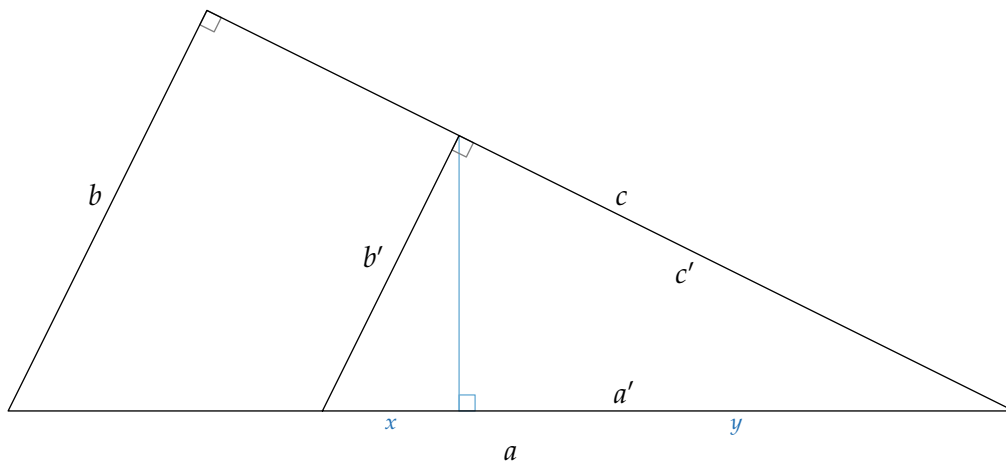
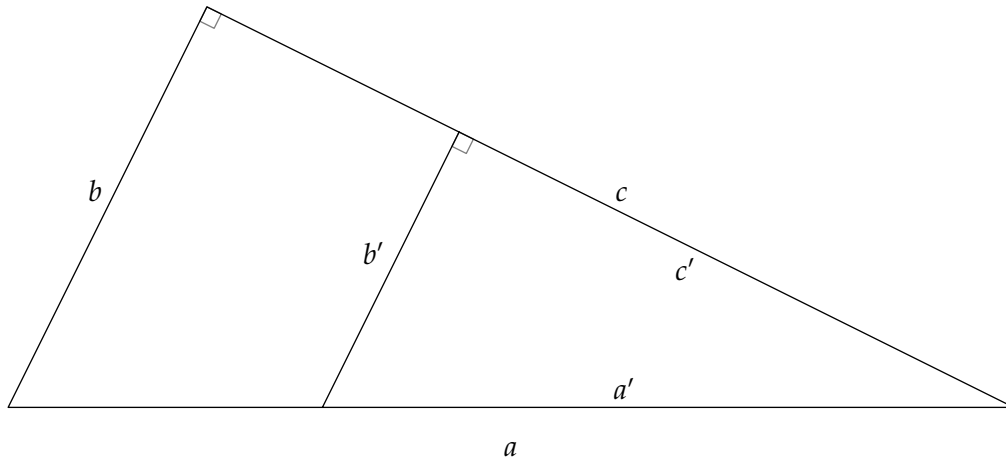
);

draw P shifted 200 up;
x5 = x4; y5 = 0;
draw unitsquare scaled 6 shifted z5 withcolor Blues 7 4;
draw z4--z5 withcolor Blues 7 4;
draw P;
label.bot("$\scriptstyle x$", 1/2[z3, z5]) withcolor Blues 7 6;
label.bot("$\scriptstyle y$", 1/2[z1, z5]) withcolor Blues 7 6;

label.bot(btex \vbox{\openup 8pt\halign{\hfil $\displaystyle # $\hfil\cr
\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';\cr
\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';\cr
\therefore\quad aa' = a\left(x+y\right) = bb' + cc'.\cr
}} etex, point 1/2 of bbox currentpicture shifted 24 down);

```


A Pythagorean theorem: $aa' = bb' + cc'$



$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$

$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$

$$\therefore aa' = a(x + y) = bb' + cc'.$$

— Enzo R. Gentile

```

numeric r; r = 64;
numeric pi; pi = 3.141592653589793;
path base, h, c, c', s;

base = (left--right) scaled 7/2r;
h = halfcircle rotated 180 scaled (pi * r + r);
c = fullcircle scaled 2r rotated 90 shifted point 0 of h shifted (0, r);
c' = fullcircle scaled 2r rotated 270 shifted point 4 of h shifted (-r, r);
s = unitssquare scaled (sqrt(pi) * r) rotated -90 shifted point 0 of c';

fill c withcolor Blues 7 1;
fill s withcolor Blues 7 1;

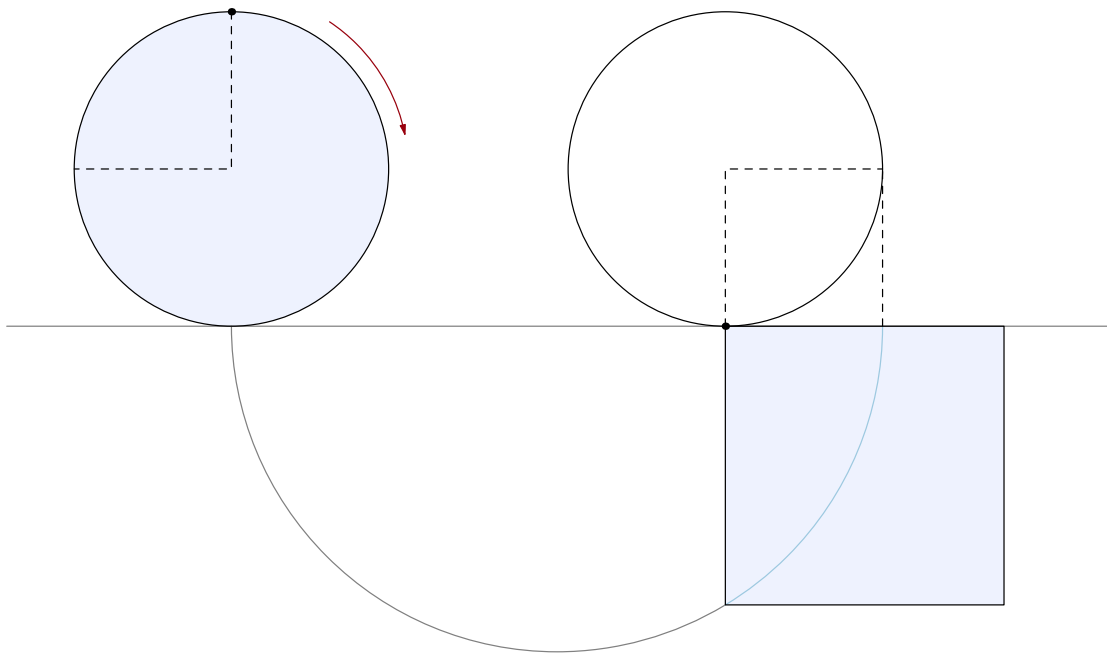
draw base withcolor 1/2;
draw subpath (0, 4 + 1/45 angle point 1 of s) of h withcolor 1/2;
draw subpath (4 + 1/45 angle point 1 of s, 4) of h withcolor Blues 7 3;
draw s;
draw point infinity of h -- point 2 of c' dashed evenly;

forsuffixes $=c, c':
    draw point 0 of $ -- center $ -- point 2 of $ dashed evenly;
    draw $; drawdot point 0 of $ withpen pencircle scaled dotlabeldiam;
endfor

drawarrow subpath (5/4, 1/4) of fullcircle scaled (2r + 16)
    shifted center c withcolor Reds 7 7;

```

The rolling circle squares itself



— Thomas Elsner

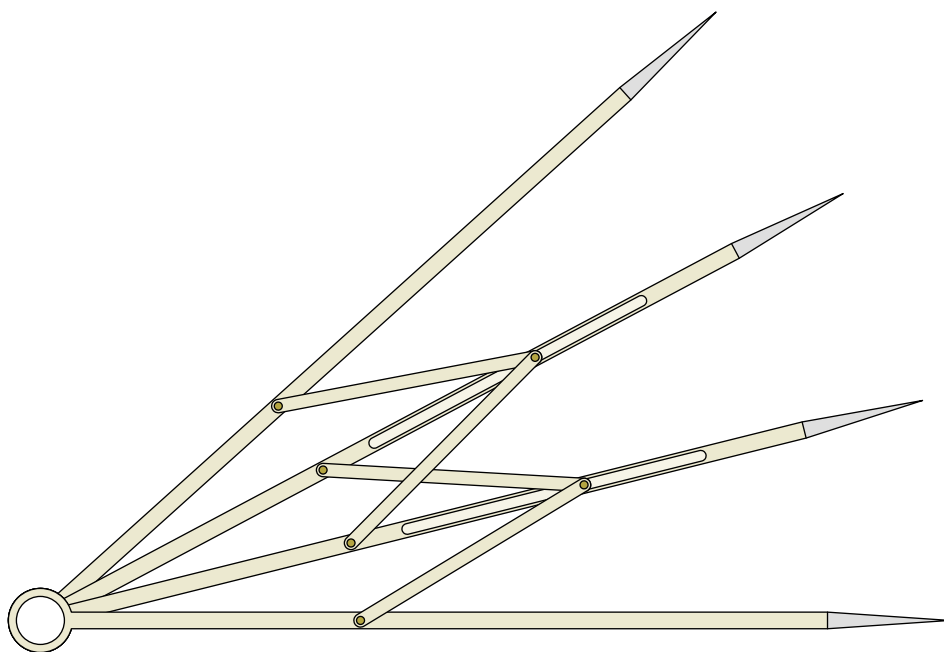
```

picture link, pointer, pointer_groove;
color metal, light_metal;
metal = 1/256 (181, 166, 66);
light_metal = 3/4[metal, white];

link = image(
  path a, b, a', b', c;
  a = fullcircle scaled 3; a' = a shifted (98,0);
  b = fullcircle scaled 5; b' = b shifted center a';
  c = subpath(2,6) of b -- subpath(-2,2) of b' -- cycle;
  fill c withcolor light_metal; draw c;
  fill a withcolor metal; draw a;
  fill a' withcolor metal; draw a';
);
pointer = image(
  path a, b, c; numeric r;
  a = fullcircle scaled 18;
  b = fullcircle scaled 24;
  r = 1/3;
  c = subpath(r,8-r) of b --
    point 8-r of b shifted (10cm,0) --
    point 0 of b shifted (116mm,0) --
    point 8+r of b shifted (10cm,0) -- cycle;
  fill c withcolor light_metal;
  fill subpath (9,11) of c -- cycle withcolor 7/8 white;
  draw point 9 of c -- point 11 of c;
  draw c;
  fill a withcolor white; draw a;
);
pointer_groove = image(
  draw pointer;
  path g;
  g = (halfcircle scaled 4 rotated 90 --
    halfcircle scaled 4 rotated 270 shifted (4cm,0) --
    cycle) shifted (5cm,0);
  fill g withcolor 7/8[metal,white]; draw g;
);
draw pointer rotated 42;
draw pointer_groove rotated 28;
draw pointer_groove rotated 14;
draw pointer rotated 0;
z0 = 210 right rotated 14;
z1 = 120 right;
numeric t; t = angle (z0-z1);
draw link rotated t shifted z1 rotatedabout(z0,-34.5);
draw link rotated t shifted z1 rotatedabout(z0,-34.5) rotated 14;
draw link rotated t shifted z1;
draw link rotated t shifted z1 rotated 14;

```

On trisecting an angle



— Rufus Isaacs

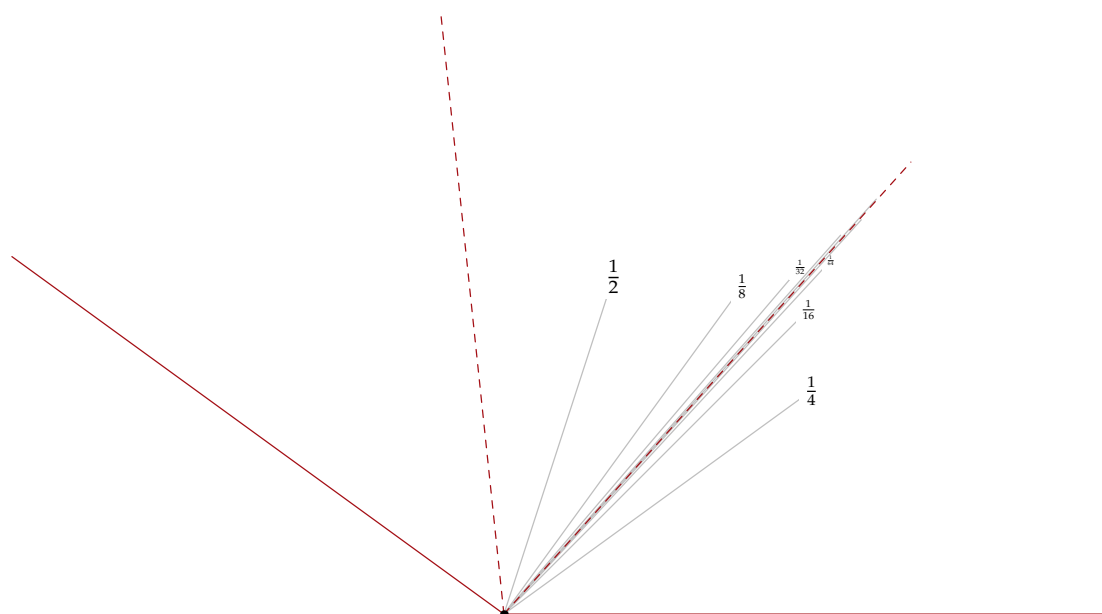
```

numeric alpha, beta;
alpha = 144;
beta = 0;
for i=1 upto 9:
    beta := beta if odd i: + else: - fi alpha * (2 ** -i);
    path ray;
    ray = origin -- (130 + 10i) * dir beta;
    draw ray withcolor 3/4;
    if i < 7:
        picture t;
        t = thelabel("$\frac{1{" & decimal (2**i) & "}$", origin)
            scaled (1 - i/8) shifted point 1 of ray;
        unfill bbox t; draw t;
    fi
endfor

for i = 0 upto 3:
    draw origin -- 240 dir (i * alpha/3)
        if i mod 3 > 0: dashed evenly fi
        withcolor Reds 6 6;
endfor
draw origin withpen pencircle scaled dotlabeldiam;
label.bot("$\displaystyle \frac{13}{16}=\frac{12}{16}-\frac{14}{16}+\frac{18}{16}-\frac{16}{16}+\cdots$",
    point 1/2 of bbox currentpicture shifted 24 down);

```

Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

— Eric Kincanon

Geometry and Algebra

```

picture P[];
pair A, B, C, D, E, F;
A = origin; B = 80 right; C = B rotated 60; D = C rotated 180;
E = 1/2 [B, D]; F = p[A,B] = q[E, C];

path ca, cb;
ca = fullcircle scaled 2 abs (A-B);
cb = ca rotated 180 shifted B;

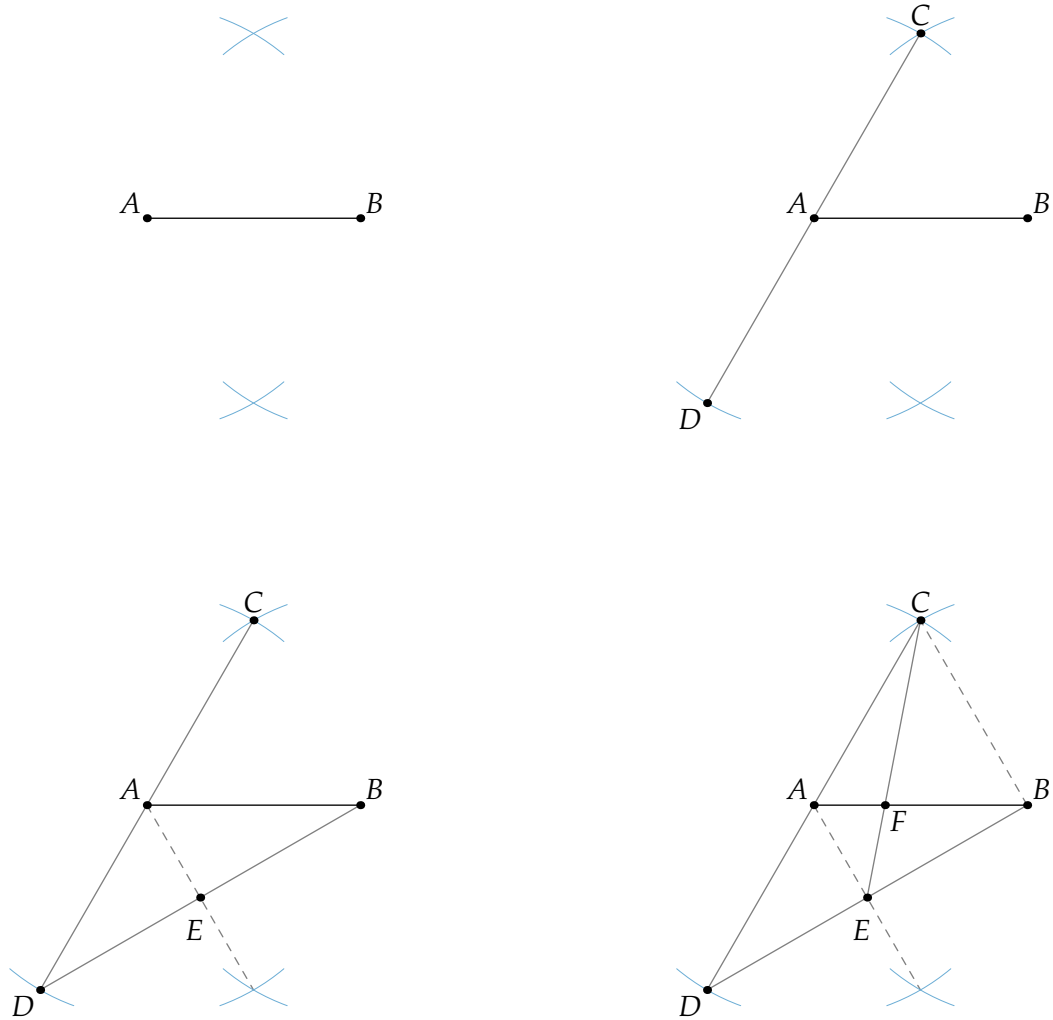
P0 = image(
    drawoptions(withpen pencircle scaled 1/4 withcolor Blues 7 4);
    draw subpath 1/45(50, 70) of ca; draw subpath 1/45(50, 70) of cb;
    draw subpath -1/45(50, 70) of ca; draw subpath -1/45(50, 70) of cb;
    drawoptions();
);
P9 = image(draw A -- B; dotlabel.ulft("$A$", A); dotlabel.urc("$B$", B));
P1 = image(draw P0; draw P9);
P2 = image(
    draw P0;
    drawoptions(withpen pencircle scaled 1/4 withcolor Blues 7 4);
    draw subpath 1/45(230, 250) of ca;
    drawoptions();
    draw C -- D withcolor 1/2;
    dotlabel.top("$C$", C);
    dotlabel.llft("$D$", D);
    draw P9;
);
P3 = image(
    draw B--D withcolor 1/2;
    draw A -- C reflectedabout(A,B) dashed evenly withcolor 1/2;
    draw E withpen pencircle scaled dotlabeldiam;
    label("$E$", E-(2,12));
    draw P2;
);
P4 = image(
    draw B--C dashed evenly withcolor 1/2;
    draw C--E withcolor 1/2;
    draw P3;
    dotlabel.lrt("$F$", F);
);

draw P1;
draw P2 shifted (250, 0);
draw P3 shifted (0, -220);
draw P4 shifted (250, -220);

label.bot("$\overline{AF} = \frac{1}{3}\cdot\overline{AB}$",
    point 1/2 of bbox currentpicture shifted 36 down);

```


Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

```

z3 = -z5 = 120 left;
z1 = 180 dir 81;
z2 = 250 dir 130;
z4 = 90 dir -100;

z6 = z5 + 72 right;
z7 = whatever [z2, z5] = whatever [z1, z4];
z8 = whatever [z3, z5] = whatever [z1, z4];
y9 = y1; z9 - z5 = whatever * (z1 - z4);

path star;
star = z3 -- z5 -- z2 -- z4 -- z1 -- cycle;
draw star withcolor Blues 7 7;
draw z6 -- z5 -- z9 dashed evenly withcolor Blues 7 5;

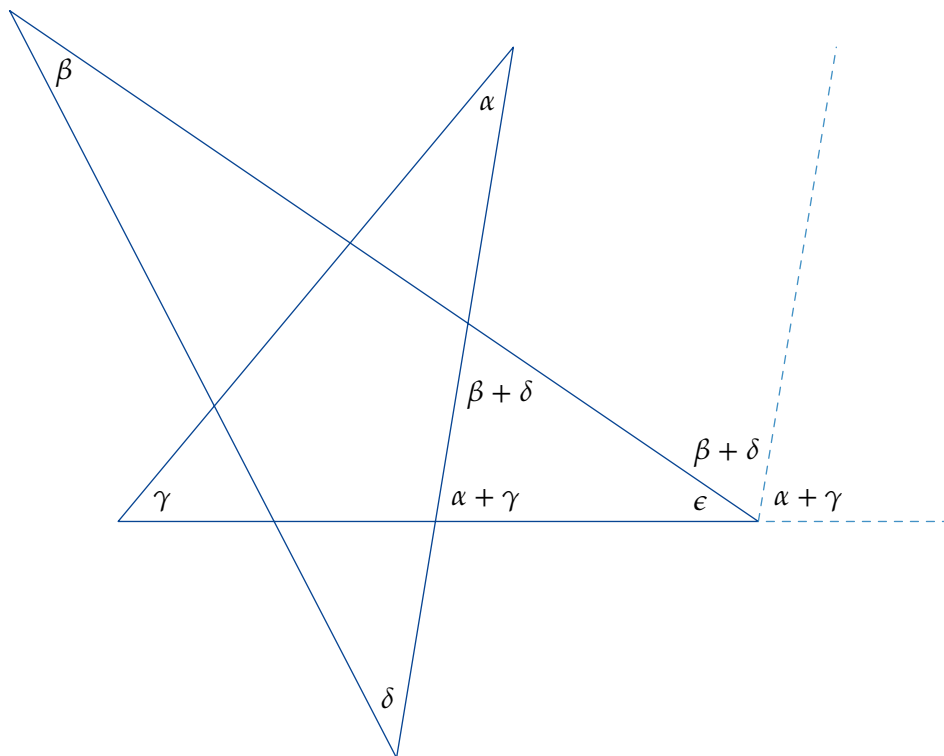
def angle_point(expr a, b, c, r) =
    b + r * (unitvector(a-b) + unitvector(c-b))
enddef;

label("$\alpha$", angle_point(z3, z1, z4, 12));
label("$\beta$", angle_point(z5, z2, z4, 16));
label("$\gamma$", angle_point(z5, z3, z1, 10));
label("$\delta$", angle_point(z1, z4, z2, 12));
label("$\epsilon$", angle_point(z2, z5, z3, 12));

label("$\alpha+\gamma$", angle_point(z1, z8, z5, 16) + 8 down);
label("$\alpha+\gamma$", angle_point(z9, z5, z6, 16) + 8 down);
label("$\beta+\delta$", angle_point(z5, z7, z4, 18) + 1 up);
label("$\beta+\delta$", angle_point(z2, z5, z9, 18) + 2 down);

```

The vertex angles of a star sum to 180°



— Fouad Nakhli

```

pair A', B', C', A, B, C, D, E, F, G, P, Q, R;
A' = origin; B' = 300 right; C' = B' rotated 60; P = 3/8[B', C'];
xpart Q = xpart C';
ypart Q = ypart E = ypart P;
D = whatever[A', B']; xpart D = xpart P;
E = whatever[A', C'];
R = whatever[A', C']; R - P = whatever * (A' - C') rotated 90;
G = whatever[C', Q] = whatever [R, P];
A = whatever[A', B']; G-A = whatever * (C' - A');
B - B' = A - A' = C - C';
F = whatever[B, C]; F-P = whatever * (B-C) rotated 90;

def right_angle_mark(expr a, b, s) =
  subpath (1,3) of unitsquare scaled s rotated angle(b-a) shifted a
enddef;

drawoptions(withcolor 1/2);
draw right_angle_mark(D, B, 6);
draw right_angle_mark(F, P, 6);
draw right_angle_mark(G, P, 6);
draw right_angle_mark(Q, P, 6);
draw right_angle_mark(R, A', 6);
draw E--P;
draw A'--B'--C'--cycle;
drawoptions();

draw P--F withcolor Reds 7 7;
draw R--G--Q withcolor 1/2[Reds 7 7, white];
draw G--P dashed evenly scaled 3/4 withcolor Greens 7 7;
draw G--C' dashed evenly scaled 3/4 withcolor 1/2[Greens 7 7, white];

draw P--D dashed withdots scaled 1/4 withcolor Blues 7 7;
draw A--B--C--cycle;

drawdot P withpen pencircle scaled dotlabeldiam;

forsuffixes $=A, A', B, B', D, Q: label.bot("\strut$" & str $ & "$", $); endfor
forsuffixes $=C, C': label.top("$" & str $ & "$", $); endfor
label.urc("$F$", F);
label("$G$", G + 10 dir 192);

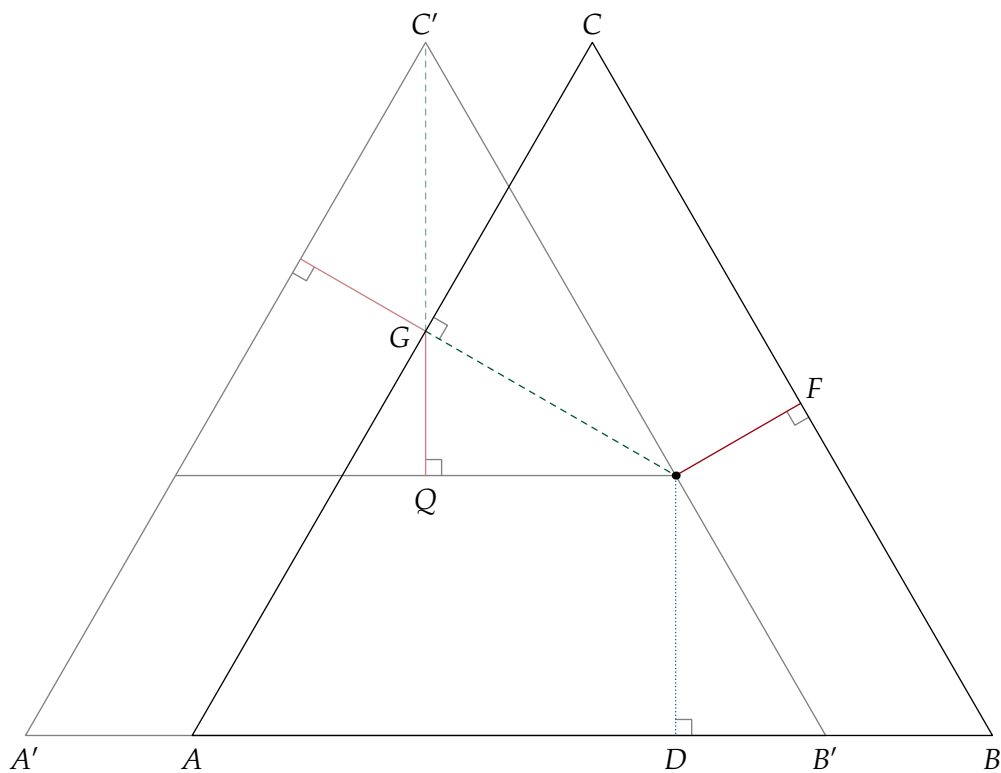
label.top(btex \vbox{\halign{\hss #\hss\cr
The perpendiculars to the sides from a point on\cr
the boundary or within an equilateral triangle\cr
add up to the height of the triangle.\cr
}} etex, point 5/2 of bbox currentpicture shifted 42 up);

label(btex \textit{This shows a particular example, with $C'GQ$ collinear, rather
than the general case} etex, point 1/2 of bbox currentpicture shifted 42 down);

```

Viviani's theorem I

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



This shows a particular example, with $C'GQ$ collinear, rather than the general case

— Samuel Wolf

Geometry and Algebra

```

def distance(expr a, b, c) = abs ypart ((a-b) rotated -angle (c-b)) enddef;

pair a, b, c, p;
a = 89 up; b = a rotated 120; c = b rotated 120; p = 21 dir 42;

numeric h[];
h0 = distance(a, b, c); h1 = distance(p, a, b);
h2 = distance(p, b, c); h3 = distance(p, c, a);

path t[];
t0 = a--b--c--cycle;
t1 = t0 rotated -120 shifted -point 2 of t0 scaled (h1/h0) shifted p;
t2 = t0 shifted -point 0 of t0 scaled (h2/h0) shifted p;
t3 = t0 rotated +120 shifted -point 1 of t0 scaled (h3/h0) shifted p;

z0 = 1/3[1/2[point 2 of t1, point 1 of t3], point 0 of t0];
z1 = 2/3[point 0 of t1, point 3/2 of t1];

color s[];
s1 = Reds 7 2; s2 = Oranges 7 2; s3 = Blues 7 2;
picture p[];
forsuffixes $=1,2,3: p$ = image(fill t$ withcolor s$; draw t$--point 3/2 of t$); endfor

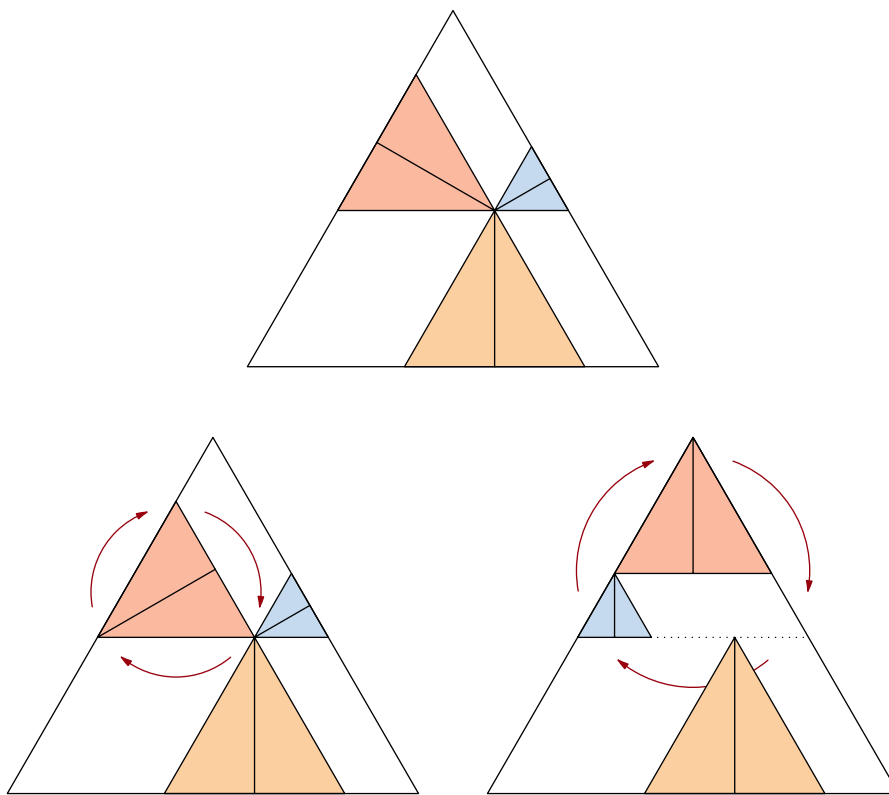
picture P[];
P1 = image(draw p1; draw p2; draw p3; draw t0;);
P2 = image(
  path cor;
  cor = reverse fullcircle rotated 90 scaled 4/3 h1 scaled 15/16 shifted z1;
  drawarrow subpath 1/45(20, 100) of cor withcolor Reds 7 7;
  drawarrow subpath 1/45(140, 220) of cor withcolor Reds 7 7;
  drawarrow subpath 1/45(260, 340) of cor withcolor Reds 7 7;
  draw p1 rotatedabout(z1, -120); draw p2; draw p3; draw t0);
P3 = image(
  path cor;
  cor = reverse fullcircle rotated 90 scaled 4/3 (h1+h3) scaled 7/8 shifted z0;
  drawarrow subpath 1/45(20, 100) of cor withcolor Reds 7 7;
  drawarrow subpath 1/45(140, 220) of cor withcolor Reds 7 7;
  drawarrow subpath 1/45(260, 340) of cor withcolor Reds 7 7;
  draw point 2 of t1 -- point 1 of t3 dashed withdots scaled 1/2;
  draw p2; draw p1 rotatedabout(z1, -120) rotatedabout(z0, -120);
  draw p3 rotatedabout(z0, -120); draw t0);

draw P1 shifted 160 up;
draw P2 shifted 90 left;
draw P3 shifted 90 right;
label.top(btex \vbox{\halign{\hss #\hss\cr
The perpendiculars to the sides from a point on\cr
the boundary or within an equilateral triangle\cr
add up to the height of the triangle.\cr
}} etex, point 5/2 of bbox currentpicture shifted 42 up);

```

Viviani's theorem II

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

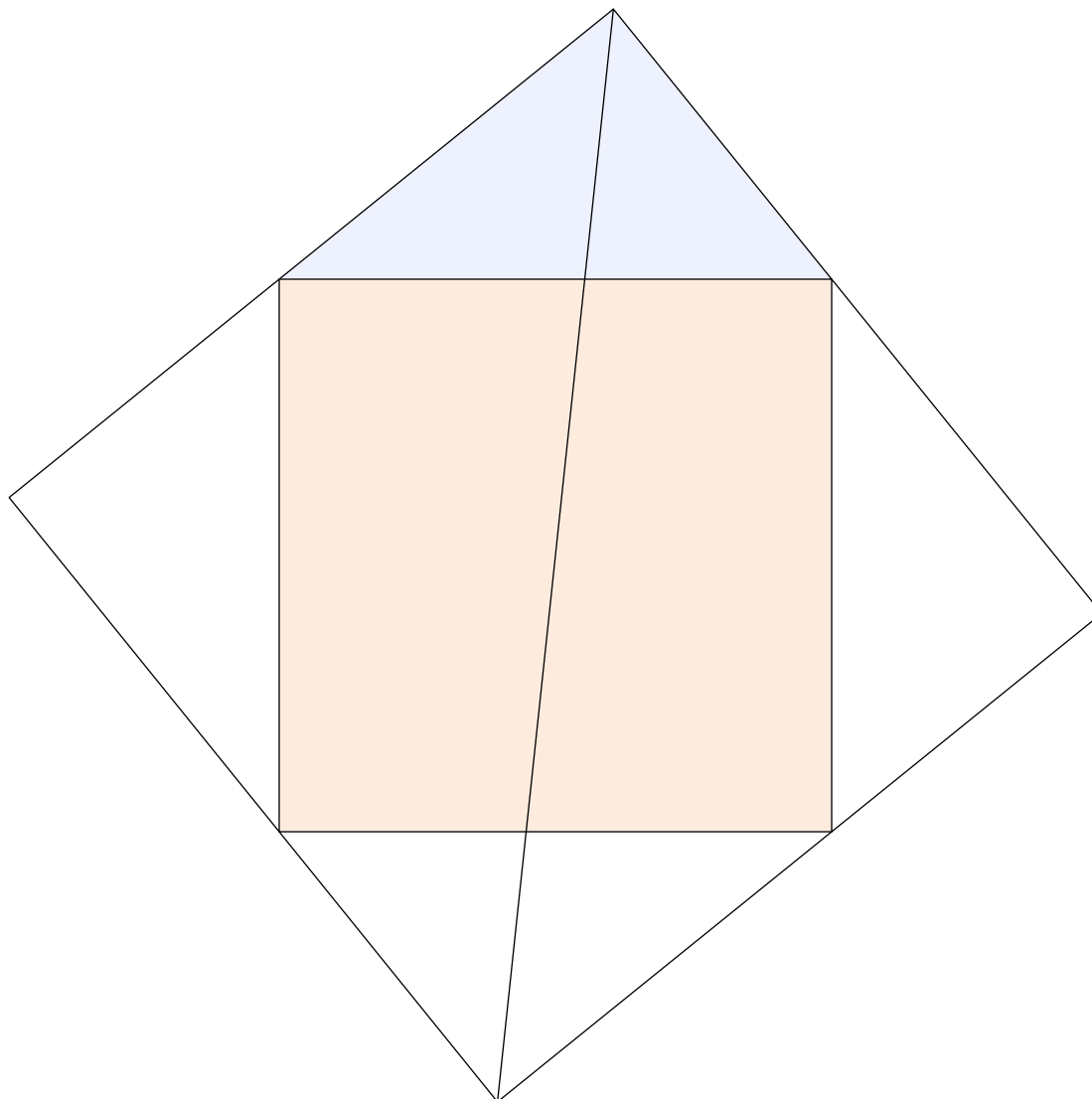
```
path s, t;
s = unitsquare shifted -(1/2,1/2) scaled 210;
t = subpath (3, 2) of s -- point 1.732 of fullcircle scaled 210
  shifted point 5/2 of s -- cycle;

fill t withcolor Blues 7 1;
fill s withcolor Oranges 7 1;
for i=0 upto 3: draw t rotated 90i; endfor;
draw point 2 of t -- point 2 of t rotated 180;

label.top(btex \vbox{\halign{\hss #\hss\cr
The internal bisector of the right angle of a right\cr
triangle bisects the square on the hypotenuse\cr
}} etex, point 5/2 of bbox currentpicture shifted 42 up);
```


A theorem about right angles

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse



— Roland H. Eddy

```

def angle_arc(expr a, o, b, r) =
    fullcircle scaled 2r rotated angle (a-o) shifted o cutafter (o--b)
enddef;
Geometry and Algebra

path c; c = fullcircle scaled 180; pair A, B, C, D, E, F;
A = point 4 of c; B = point 0 of c; C = point 1.7 of c; D = (xpart C, ypart A);
E = C rotatedabout(D, -90); F = B rotatedabout(D, -90);

color r, b, g; r = Reds 7 1; g = Greens 7 1; b = Blues 7 1; picture P[];
P1 = image(
    fill A--B--C--cycle withcolor r;
    draw unitsquare scaled 6 rotated angle (A-C) shifted C withcolor 3/4 r;
    draw unitsquare scaled 6 rotated angle (C-D) shifted D withcolor 3/4 r;
    draw angle_arc(D, A, C, 12) withpen pencircle scaled 1 withcolor Reds 7 5;
    draw angle_arc(D, C, B, 12) withpen pencircle scaled 1 withcolor Reds 7 5;
    draw D--C--B--A--C;
    label.bot("$A$", A); label.bot("$B$", B); label.top("$C$", C); label.bot("$D$", D));

P2 = image(
    fill A--D--C--cycle withcolor r; fill A--D--F--cycle withcolor g;
    fill F--D--E--cycle withcolor r; fill C--D--E--cycle withcolor b;
    draw unitsquare scaled 6 rotated angle (C-D) shifted D withcolor 3/4 r;
    draw angle_arc(D, A, C, 12) withpen pencircle scaled 1 withcolor Reds 7 5;
    draw angle_arc(D, E, F, 12) withpen pencircle scaled 1 withcolor Reds 7 5;
    drawarrow subpath (7/4, 1/4) of quartercircle scaled 42 shifted D withcolor Blues 6 6;
    draw A--F--E--C--A--E; draw C--F;
    label.lft ("A$", A); label.top ("C$", C); label.llft("$D$", D);
    label.rft ("E$", E); label.bot ("F$", F));

P3 = image(
    fill A--D--C--cycle withcolor r; fill C--D--E--cycle withcolor b;
    z3 = whatever[A,C]; z3 - E = whatever * (A-C) rotated 90;
    begingroup; interim aangle := 180;
    drawarrow E--z3 dashed evenly scaled 3/4 withpen pencircle scaled 1/4;
    label.urt("$h$", 1/4[z3, E]);
    endgroup;
    draw A--E--C--A; draw C--D;
    label.bot("$A$", A); label.top("$C$", C); label.bot("$D$", D); label.bot("$E$", E));

P4 = image(
    fill A--D--C--cycle withcolor r; fill A--D--F--cycle withcolor g;
    z4 = whatever[A,C]; z4 - F = whatever * (A-C) rotated 90;
    draw z4--F dashed evenly scaled 3/4 withpen pencircle scaled 1/4;
    label.urt("$h$", 1/4[z4, F]);
    draw A--C--F--A--D;
    label.lft ("A$", A); label.urt ("C$", C); label.rft ("D$", D); label.lrt ("F$", F));

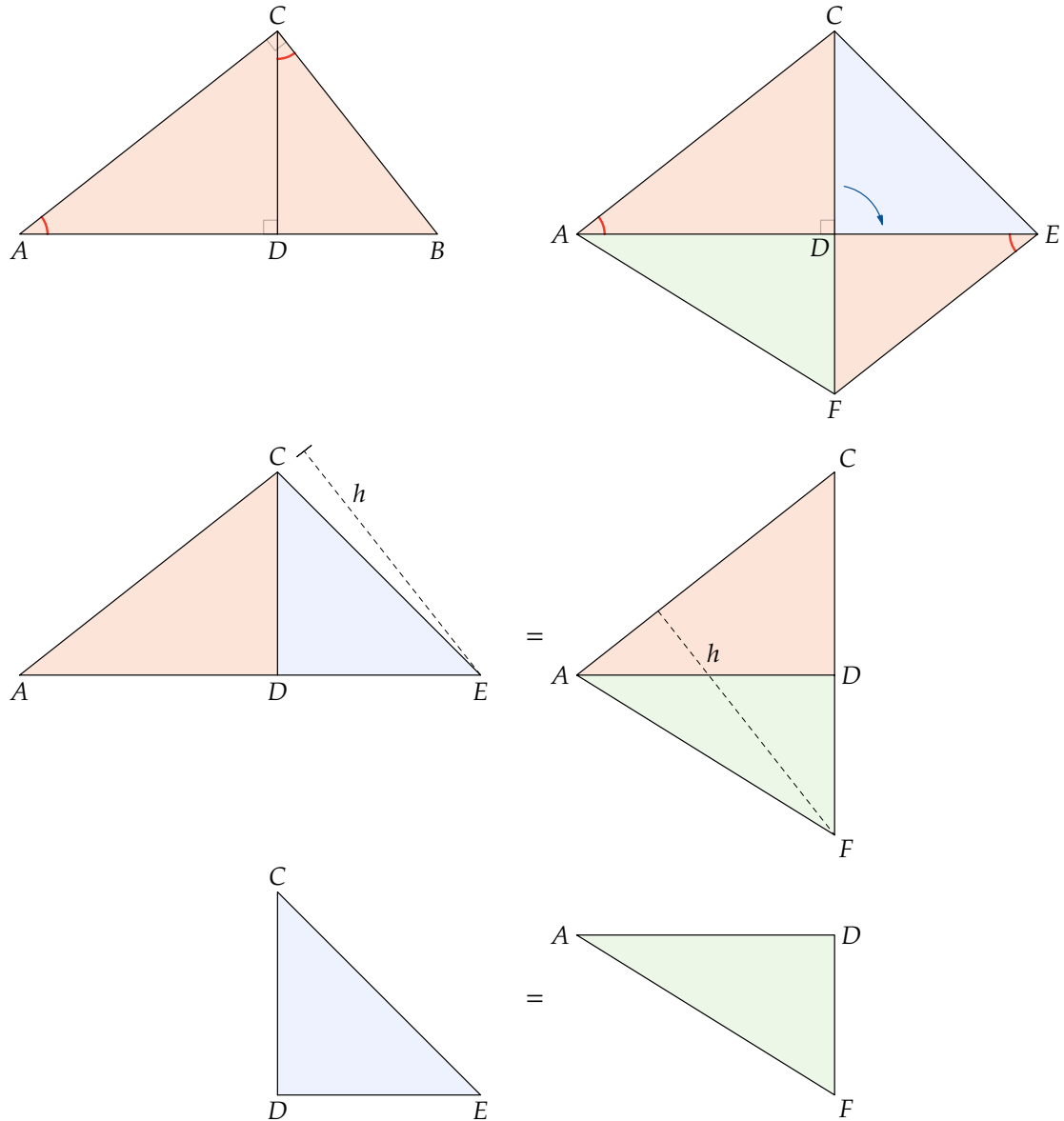
P5 = image(fill C--D--E--cycle withcolor b; draw D--E--C--D;
    label.top("$C$", C); label.bot("$D$", D); label.bot("$E$", E));

P6 = image(fill A--D--F--cycle withcolor g; draw A--F--D--A;
    label.lft("$A$", A); label.rft("$D$", D); label.lrt("$F$", F));

draw P1 shifted 120 left; draw P2 shifted 120 right;
numeric y; y = -190;
draw P3 shifted (-120, y); label("$\{=\}$", (12, y+16)); draw P4 shifted (+120, y);
y := y - 112;
draw P5 shifted (-120, y-abs(D-B)); label("$\{=\}$", (12, y-28)); draw P6 shifted (+120, y);
label("$CD^2 = AD\cdot DB$", point 1/2 of bbox currentpicture shifted 42 down);

```

Area and the projection theorem of a right triangle



$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

```

def angle_arc(expr a, o, b, r) =
    fullcircle scaled 2r rotated angle (a-o) shifted o cutoafter (o--b)
enddef;

path C[]; pair O, P, Q, R;
C1 = fullcircle scaled 280; O = point 0 of C1;
C2 = fullcircle scaled 200 shifted O;

numeric t, u;
(t, u) = C1 intersectiontimes C2;
P = point t of C1;
Q = point 8-t of C1;
z0 = whatever[P, P + direction t of C1]; y0 = ypart point 6 of C1;
R = C2 intersectionpoint (z0--P);

draw center C1 -- P -- point 4 of C1 -- O withcolor 7/8;

forsuffixes $=P, Q, R:
    draw O -- $ withcolor Blues 7 6;
endfor
draw 5/4[Q, P] -- 5/4[P, Q] withcolor Reds 7 6;
draw 5/4[P, R] -- 5/4[R, P] withcolor Reds 7 6;

draw angle_arc(O, Q, P, 30);
draw angle_arc(O, Q, P, 28);
draw angle_arc(O, P, R, 30);
draw angle_arc(O, P, R, 28);

draw C1 withcolor 1/2;
draw C2 withcolor 1/2;;

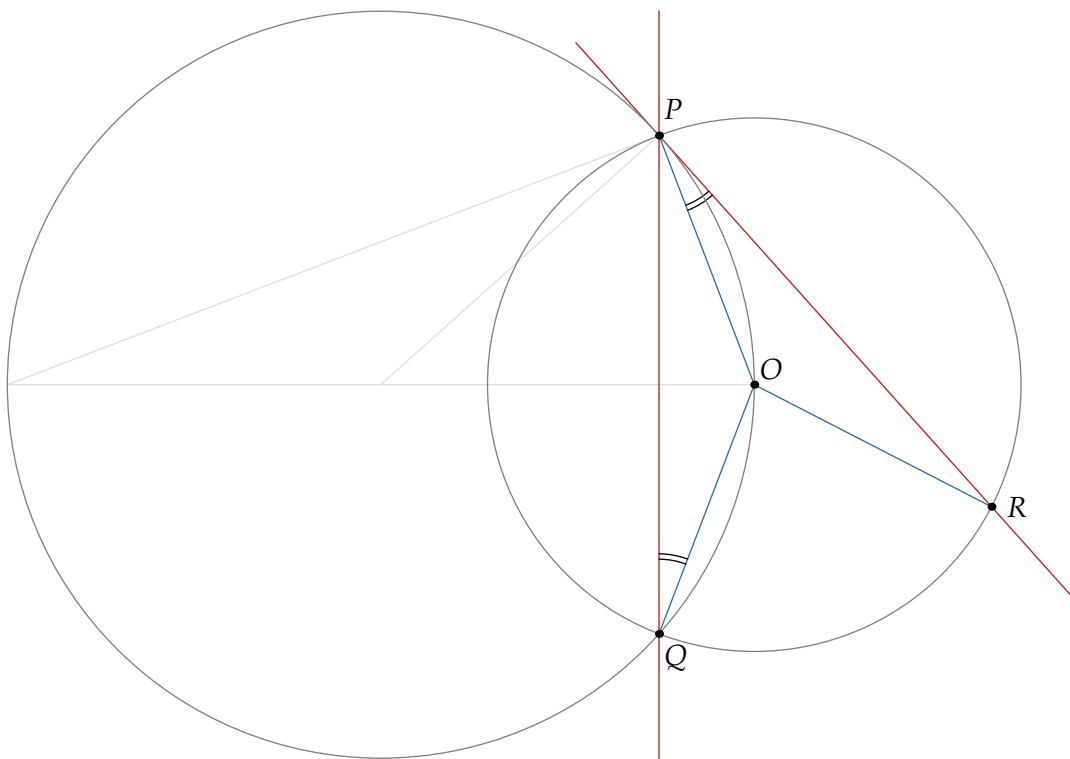
dotlabel.urt("$O$", O);
dotlabel.urt("\strut $P$", P);
dotlabel.lrt("\strut $Q$", Q);
dotlabel.rt("$\;R$", R);

label.top(btex \vbox{\openup6pt\halign{\hss #\hss\cr
If circle $C_1$ passes through the center $O$ of circle $C_2$, the length\cr
of the common chord $\overline{PQ}$ is equal to the tangent segment $\overline{PR}$.\cr
}} etex, point 5/2 of bbox currentpicture shifted 42 up);

```

Chords and tangents of equal length

If circle C_1 passes through the center O of circle C_2 , the length of the common chord \overline{PQ} is equal to the tangent segment \overline{PR} .



— Roland H. Eddy

```

path xx, ax, hax, haha;
numeric x, a;
x = 89; a = 34;
xx = unitsquare shifted 1/2 left scaled x shifted 12 up;
ax = unitsquare shifted 1/2 left xscaled x yscaled -a shifted 12 down;
hax = unitsquare shifted 1/2 left xscaled x yscaled -1/2 a shifted 12 down;
haha = unitsquare scaled 1/2 a rotated -90 shifted point 1 of xx shifted (8, -8);

picture P[];
P1 = image(
  fill xx withcolor Oranges 7 1; draw xx;
  label.top("$x$", point 5/2 of xx);
  label.lft("$x$", point 7/2 of xx);
  label("$\{ \} + \{ \} \$", origin);
  fill ax withcolor Blues 7 2; draw ax;
  label.lft("$a$", point 7/2 of ax);
);

P2 = image(
  fill xx withcolor Oranges 7 1; draw xx;
  label("$\{ \} + \{ \} \$", origin);
  for i=0, 1:
    fill hax shifted (0, -24i) withcolor Blues 7 2;
    draw hax shifted (0, -24i);
  endfor
);

P3 = image(
  fill xx withcolor Oranges 7 1; draw xx;
  hax := hax shifted (point 0 of xx - point 0 of hax);
  fill hax withcolor Blues 7 2; draw hax;
  hax := hax shifted - point 0 of hax rotated 90 shifted point 1 of xx;
  fill hax withcolor Blues 7 2; draw hax;

  fill haha withcolor Blues 7 1;
  draw haha dashed withdots scaled 1/4;
);

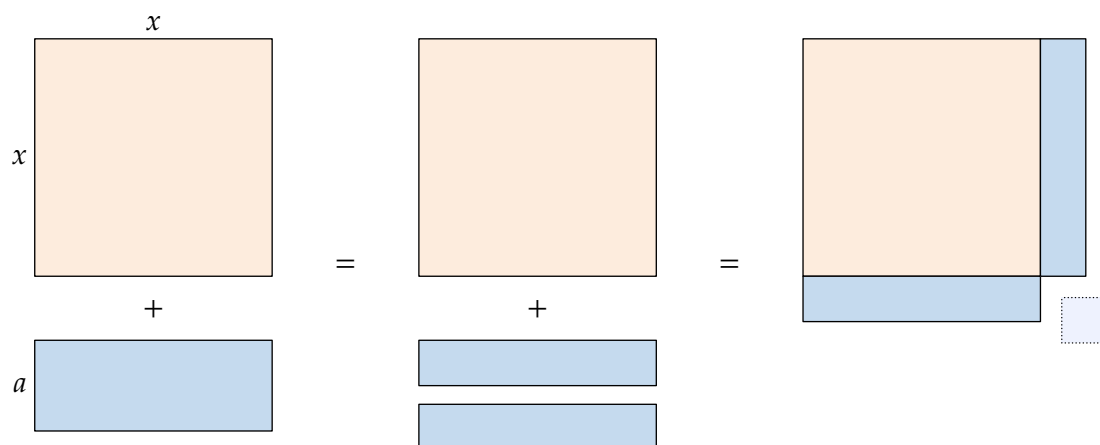
draw P1 shifted 144 left;
label("$=$", (-72, 16));
draw P2;
label("$=$", (72, 16));
draw P3 shifted 144 right;

label.top("$x^2 + ax = \left(x + a/2\right)^2 - \left(a/2\right)^2$",
point 5/2 of bbox currentpicture shifted 42 up);

```

Completing the square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

```

numeric a, b; a = 89; b = 21; picture P[];
P1 = image(
  fill unitsquare xscaled a yscaled b shifted (0, a) withcolor Greens 7 1;
  fill unitsquare xscaled b yscaled a shifted (a, 0) withcolor Greens 7 1;
  draw (a, 0) -- (a, a+b) dashed withdots scaled 1/4;
  draw (0, a) -- (a+b, a) dashed withdots scaled 1/4;
  draw (a-b, a) -- (a-b, a+b) dashed withdots scaled 1/4;
  draw (a, a-b) -- (a+b, a-b) dashed withdots scaled 1/4;
  draw unitsquare scaled (a+b);
  label.bot("\strut $a$", (1/2a, 0));
  label.bot("\strut $b$", (a+1/2b, 0));
  label.lft("$a$", (0, 1/2a));
  label.lft("$b$", (0, a+1/2b));
);
P2 = image( draw unitsquare scaled (a-b); label.bot("\strut $a-b$", 1/2(a-b, 0)));
P3 = image(
  draw unitsquare scaled a;
  draw unitsquare scaled b shifted (a,a);
  label.bot("\strut $a$", (1/2a, 0));
  label("$b$", (a + 1/2b, a + 1/2b));
);
P4 = image(
  fill unitsquare scaled a withcolor Greens 7 1;
  fill unitsquare scaled b shifted (a,a) withcolor Greens 7 1;
  fill unitsquare scaled (a-b) withcolor background;
  draw (a-b, a) -- (a-b, a-b) -- (a, a-b) dashed withdots scaled 1/4;
  draw (0, a-b) -- (a-b, a-b) -- (a-b, 0);
  draw unitsquare scaled a;
  draw unitsquare scaled b shifted (a,a);
  label.bot("\strut $a-b$", 1/2(a-b, 0));
  label.bot("\strut $b$", (a-1/2b, 0));
  label("$b$", (a + 1/2b, a + 1/2b));
);

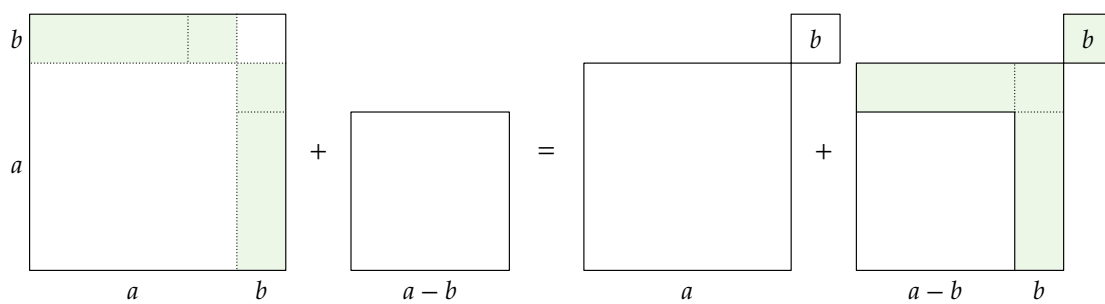
draw P1;
numeric x, y; y = 3/4 (a-b);
x := a + b + 14; label("$+$", (x,y)); x := x + 14; draw P2 shifted (x,0);
x := x + a - b + 16; label("$=$", (x,y)); x := x + 16; draw P3 shifted (x,0);
x := x + a + 14; label("$+$", (x,y)); x := x + 14; draw P4 shifted (x,0);

label.top("$\left(a+b\right)^2 + \left(a-b\right)^2 = 2\left(a^2 + b^2\right)$",
point 5/2 of bbox currentpicture shifted 42 up);

```


Algebraic areas I

$$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$$



— Shirley Wakin

input arrow_label

Geometry and Algebra

numeric a, b, c; a = 80; 2b = a; 2c = b;

def make_box(expr p, shade) = image(fill p withcolor shade; draw p) enddef;

```
path s[]; picture t[];
s1 = unitsquare scaled (a-b-c);          t1 = make_box(s1, Reds 7 2);
s2 = unitsquare scaled (2c);              t2 = make_box(s2, Oranges 7 2);
s3 = unitsquare scaled (a-b+c);           t3 = make_box(s3, YlGn 7 2);
s4 = unitsquare scaled (2b);              t4 = make_box(s4, Greens 7 2);
s5 = unitsquare scaled (a+b-c);           t5 = make_box(s5, Blues 7 2);
s6 = unitsquare xscaled (a+b-c) yscaled (a-b+c); t6 = make_box(s6, Purples 7 2);
s7 = unitsquare xscaled (a-b+c) yscaled (a+b-c); t7 = make_box(s7, Purples 7 2);
```

```
picture P[];
P1 = image(
  draw t4;
  draw t7 shifted point 1 of s4; draw t6 shifted point 3 of s4;
  draw t2 shifted ((1,1) scaled (a+b-c));

  draw t5 shifted (a + b + c + 20, 0);
  draw t3 shifted (2a + 2b + 40, 0);
  draw t1 shifted (3a + b + c + 60, 0);

  arrow_label(origin, 2b * up, "$2b$", -12);
  arrow_label((0, a+b+c), (a+b+c, a+b+c), "\strut$a+b+c$", -12);
  label.rt("$2c$", (a+b+c, a+b));
  label.top("$a+b-c$", (3/2a + 3/2b + 1/2c + 20, a + b - c + 4));
  label.top("$a-b+c$", (5/2a + 3/2b + 1/2c + 40, a - b + c + 4));
  label.top("$a-b-c$", (7/2a + 1/2b + 1/2c + 60, a - b - c + 4));
);
P2 = image(
  draw t5;
  draw t7 shifted point 1 of s5;
  draw t1 shifted (point 2 of s5 - point 3 of s1);
  draw t3 shifted point 2 of s5;
  draw t6 shifted point 3 of s5;

  draw t4 shifted (2a + 30, 0);
  draw t2 shifted (2a + 30 + 2b + 30, 0);

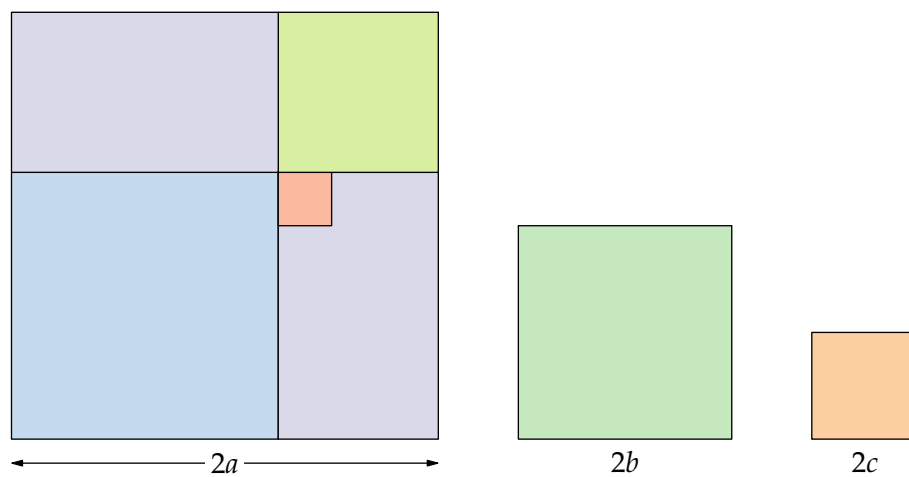
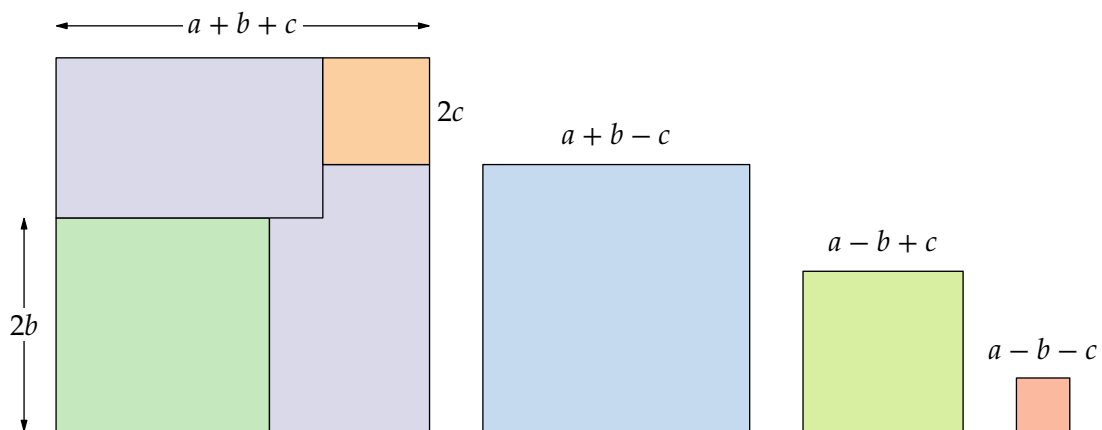
  arrow_label(origin, 2a * right, "$2a$", 9);
  label.bot("\strut$2b$", (2a + 30 + b, 0));
  label.bot("\strut$2c$", (2a + 30 + 2b + 30 + c, 0));
);
```

```
label.top(P1, (0, 2a+2b));
label.top(P2, origin);
```

```
label.top(btex $\left(a+b+c\right)^2 + \left(a+b-c\right)^2
+ \left(a-b+c\right)^2 + \left(a-b-c\right)^2
= \left(2a\right)^2 + \left(2b\right)^2 + \left(2c\right)^2$ etex,
point 5/2 of bbox currentpicture shifted 42 up);
```

Algebraic areas II

$$(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$



— Sam Pooley and K. Ann Drude

```

input arrow_label
picture P[];
numeric a,b,c,d;
a = sqrt(90); 1.732a = 1.414b; c + d = 3/2a; 1.414c = d;

P1 = image(
  path s[];
  s1 = unitsquare xscaled -(a*a) yscaled -(d*d);
  s2 = unitsquare xscaled (b*b) yscaled -(d*d);
  s3 = unitsquare xscaled (b*b) yscaled (c*c);
  s4 = unitsquare xscaled -(a*a) yscaled (c*c);

  fill s1 withcolor Reds 7 1; draw s1;
  fill s2 withcolor Greens 7 1; draw s2;
  fill s3 withcolor Oranges 7 1; draw s3;
  fill s4 withcolor Blues 7 1; draw s4;

  label("$a^2d^2$", center s1);
  label("$b^2d^2$", center s2);
  label("$b^2c^2$", center s3);
  label("$a^2c^2$", center s4);

  arrow_label(point 3 of s4, point 2 of s4, "$a^2$", 8);
  arrow_label(point 2 of s4, point 1 of s4, "$c^2$", 8);
  arrow_label(point 2 of s3, point 3 of s3, "$b^2$", 8);
  arrow_label(point 1 of s1, point 2 of s1, "$d^2$", 8);

);
% ... and so on for P2, P3, P4, P5, and P6.
draw P1;
draw P2;
draw P3;
draw P4;
draw P5;
draw P6;

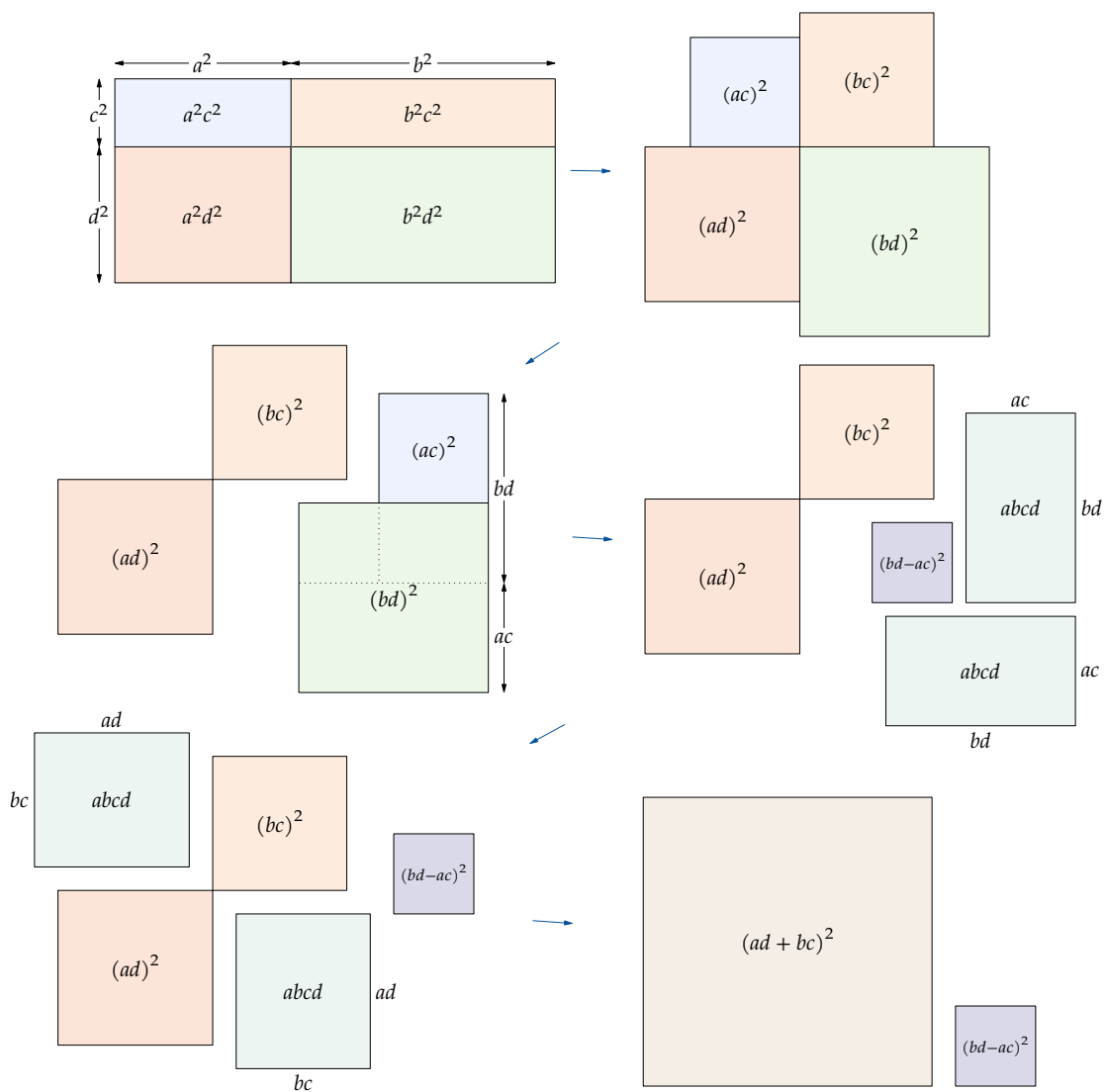
def connect_with_arrow(expr a, b) =
  drawarrow (left-- 4 right) scaled 4
    rotated angle (b-a) shifted 1/2[a,b]
    withcolor Blues 5 5;
enddef;
connect_with_arrow(center P1 + 10 right, center P2);
connect_with_arrow(center P2, center P3);
connect_with_arrow(center P3, center P4);
connect_with_arrow(center P4, center P5);
connect_with_arrow(center P5, center P6);

label.top(btex $\left(a^2+b^2\right)\left(c^2+d^2\right)$
  = $\left(ab + bc\right)^2$
  + $\left(bd-ac\right)^2$ $ etex,
point 5/2 of bbox currentpicture shifted 42 up);

```

Sum of squares identity

$$(a^2 + b^2)(c^2 + d^2) = (ab + bc)^2 + (bd - ac)^2$$



— Diophantus of Alexandria

```

vardef around(expr p, r) =
  if pair p:                                fullcircle scaled 2r shifted p
  elseif path p and (length p = 0): fullcircle scaled 2r shifted point 0 of p
  elseif path p:
    for i = 1 upto length p:
      subpath (i-1, i) of p
      shifted (r * unitvector(direction i-1/2 of p rotated -90)) ..
    endfor
    if not cycle p:
      for i = length p downto 1:
        subpath (i, i-1) of p
        shifted (r * unitvector(direction i-1/2 of p rotated 90)) ..
      endfor
    fi cycle
  fi
enddef;
% k-th n-gonal number...
numeric k, n; k = 6; n = 6;
path gon[]; for i=2 upto k:
  gon[i] = (origin for j=1 upto n-1: -- dir (240/n*j) endfor -- cycle) scaled 50(i-1);
endfor
numeric r; r = 8;
path a; a = around(origin, r);
fill a withcolor Blues 7 3; draw a;
for i=1 upto n-1:
  a := around(point i of gon2 -- point i of gon[k], r);
  fill a withcolor Oranges 7 1; draw a;
endfor
for i=1 upto n-2:
  a := around(point i+1/2 of gon[3] -- point i+1/(k-1) of gon[k] --
    point i+1-1/(k-1) of gon[k] -- cycle, r);
  fill a withcolor Greens 7 1; draw a;
endfor
for i=2 upto n-2:
  draw origin -- point i of gon[k] dashed evenly;
endfor

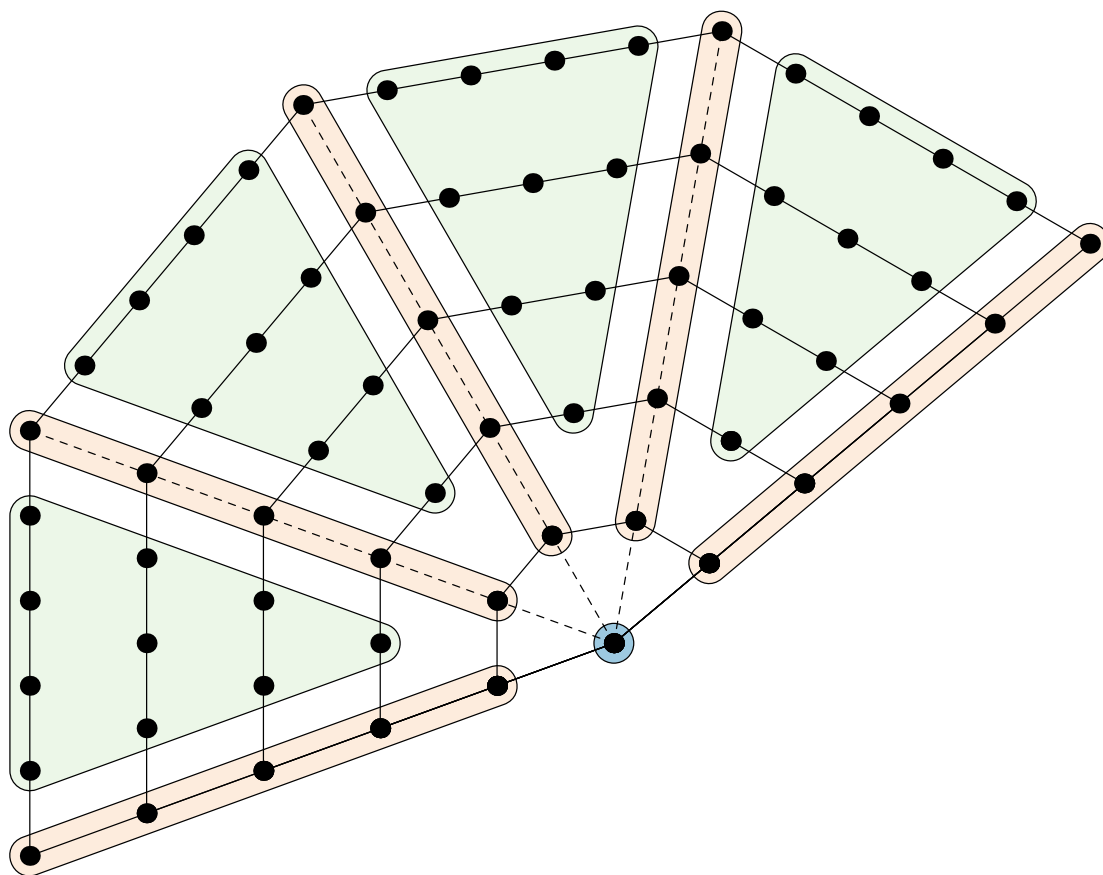
for i=2 upto k:
  draw gon[i];
  for j = i-1 upto (n-1)*i:
    draw point j/(i-1) of gon[i] withpen pencircle scaled r;
  endfor
endfor
draw origin withpen pencircle scaled r;

label.top(btex The  $k^{\text{th}}$   $n$ -gonal number is  $1 + \frac{k-1}{2} \frac{(k-1)(n-1)}{2} + \frac{1}{2} \frac{(k-2)(k-1)(n-1)}{2}$  etex,
  point 5/2 of bbox currentpicture shifted 42 up);

```

Polygonal numbers

The k^{th} n -gonal number is $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$



— Dave Logothetti

```

input isometric_projection
set_projection(22, -34);

path base, hlid, mlid;
numeric h, a, b; h = 6; b = 7; a = 3;
base = p(0, 0, 0) -- p(0, 0, b) -- p(-b, 0, b) -- p(-b, 0, 0) -- cycle;
hlid = p(0, h, 0) -- p(0, h, a) -- p(-a, h, a) -- p(-a, h, 0) -- cycle;
mlid = p(0, b-a, 0) -- p(0, b-a, a) -- p(-a, b-a, a) -- p(-a, b-a, 0) -- cycle;

picture P[];
P1 = image(
    path lid; lid = hlid;
    fill subpath (0, 1) of base -- subpath (1, 0) of lid -- cycle withcolor Blues 8 1;
    fill lid withcolor Blues 8 2;
    drawoptions(dashed withdots scaled 1/2 withcolor 1/2);
    draw subpath (1, 3) of base;
    draw point 2 of base -- point 2 of lid;

    drawoptions(withcolor 1/2);
    numeric t; t = 1/2;
    draw p(-t, 0, 0) -- p(-t, t, 0) -- p(0, t, 0) -- p(0, t, t) -- p(0, 0, t);
    draw p(-t, h, 0) -- p(-t, h-t, 0) -- p(0, h-t, 0) -- p(0, h-t, t) -- p(0, h, t);

    drawoptions();
    draw lid -- point 0 of base;
    draw point 3 of lid -- subpath (-1, 1) of base -- point 1 of lid;

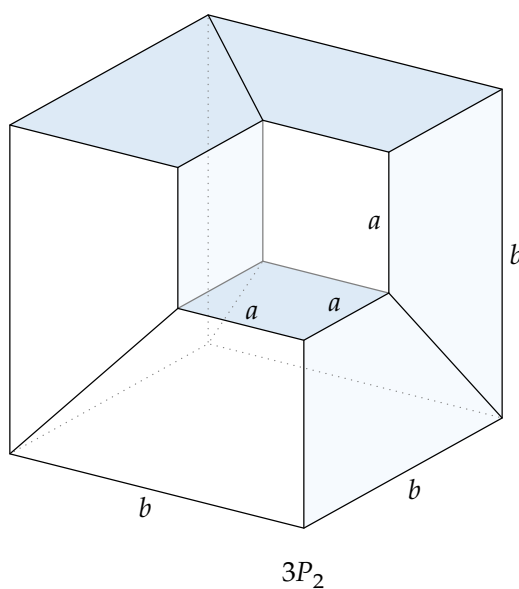
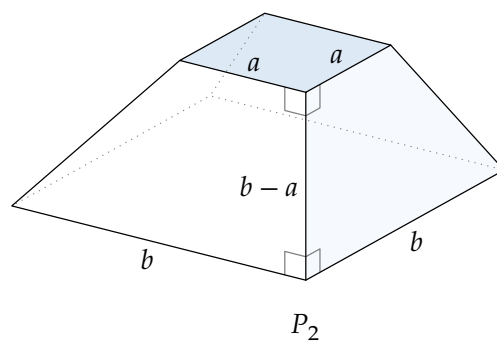
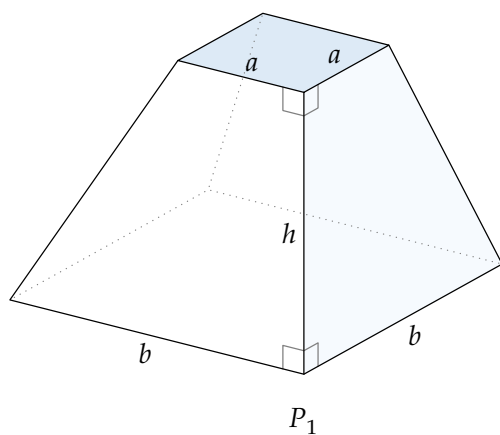
    label.lft("$h$", p(0, 1/2 h, 0));
    label.urc("$a$", point 7/2 of lid);
    label.ulft("$a$", point 1/2 of lid);
    label.lrt("$b$", point 1/2 of base);
    label.llft("$b$", point 7/2 of base);

    label("$P_1$", p(0, -1, 0));
);
% more of the same for P2 and P3...
draw P1 shifted 120 left;
draw P2 shifted 120 right;
draw P3 shifted 240 down;

label.top(btex $\displaystyle
V_{\frac{P_1}{}} = \{h\}\{b-a\}\cdot V_{\frac{P_2}{}} = \{h\}\{b-a\}\cdot\frac{1}{3}\left(b^3-a^3\right)
= \frac{h}{3}\left(a^2+ab+b^2\right)$ etex,
point 1/2 of bbox currentpicture shifted 42 down);

```


The volume of a frustum of a square pyramid



$$\underline{\underline{V_{P_1}}}hb - a \cdot \underline{\underline{V_{P_2}}}hb - a \cdot \frac{1}{3}(b^3 - a^3) = \frac{h}{3}(a^2 + ab + b^2)$$

— *The Moscow Papyrus*, c. 1850 BCE

```

input arrow_label
input isometric_projection
set_projection(18, -32);

```

Geometry and Algebra

```

numeric r, h, s, tau;
tau = 6.283185307179586;
r * tau = 400 / ipscale;
h = 3/4 r;
s = r +-+ h;

z0 = p(0,0,0); z1 = p(0,0,r); z2 = p(0,r,r); z3 = p(0,r,0); z4 = p(tau * r, 0, 0);

z5 = p(tau * (r-h), h, 0); z6 = p(tau * (r-h), h, h); z7 = p(0, h, r); z8 = p(0, h, 0);

z9 = p(0, 0, 5r); z10 = p(0, r, 5r); z11 = p(0, h, 5r);

z12 = z9 shifted p(2r, 0, 0);
z13 = z10 shifted p(r, 0, 0);
z14 = 1/2[z9, z12];
z15 = z14 shifted p(0, 0, -r);
z16 = z14 shifted p(0, 0, +r);

z17 = z14 + p(-s, h, 0);
z18 = z14 + p(0, h, -s);
z19 = z14 + p(+s, h, 0);
z20 = z14 + p(0, h, +s);

path disc, base, arc, arch;
base = for i=0 upto 11: z14 + p(r*cosd(30i), 0, r*sind(30i)) .. endfor cycle;
disc = for i=0 upto 11: z14 + p(s*cosd(30i), h, s*sind(30i)) .. endfor cycle;
numeric a, b;
a = directiontime down of base;
b = directiontime up of base;

arc = point a of base .. point a of disc .. z13 .. point b of disc .. point b of base;

drawoptions(dashed evenly withpen pencircle scaled 1/4 withcolor 1/2);
draw z1--z9--z12; draw z2--z10--z13; draw z7--z11;
draw z9--z10;

drawoptions(dashed withdots scaled 1/4 withcolor 1/2);
draw z0--z1--z4; draw z1--z2;
draw z14 -- center disc;
fill disc withcolor Blues 7 1;
draw z13 -- center disc -- z19 -- z14;
draw subpath (b, a) of base;
draw subpath (b, a) of disc;

fill z5--z6--z7--z8--cycle withcolor Oranges 7 1;
draw z6--z7--z8;

drawoptions(dashed evenly withpen pencircle scaled 1/4 withcolor 1/2);
draw z11 -- center disc;

drawoptions();
draw z0--z4--z2--z3--z0; draw z3--z4; draw z8--z5--z6;
draw arc;
draw subpath (a, 12 + b) of base;
draw subpath (a, 12 + b) of disc;

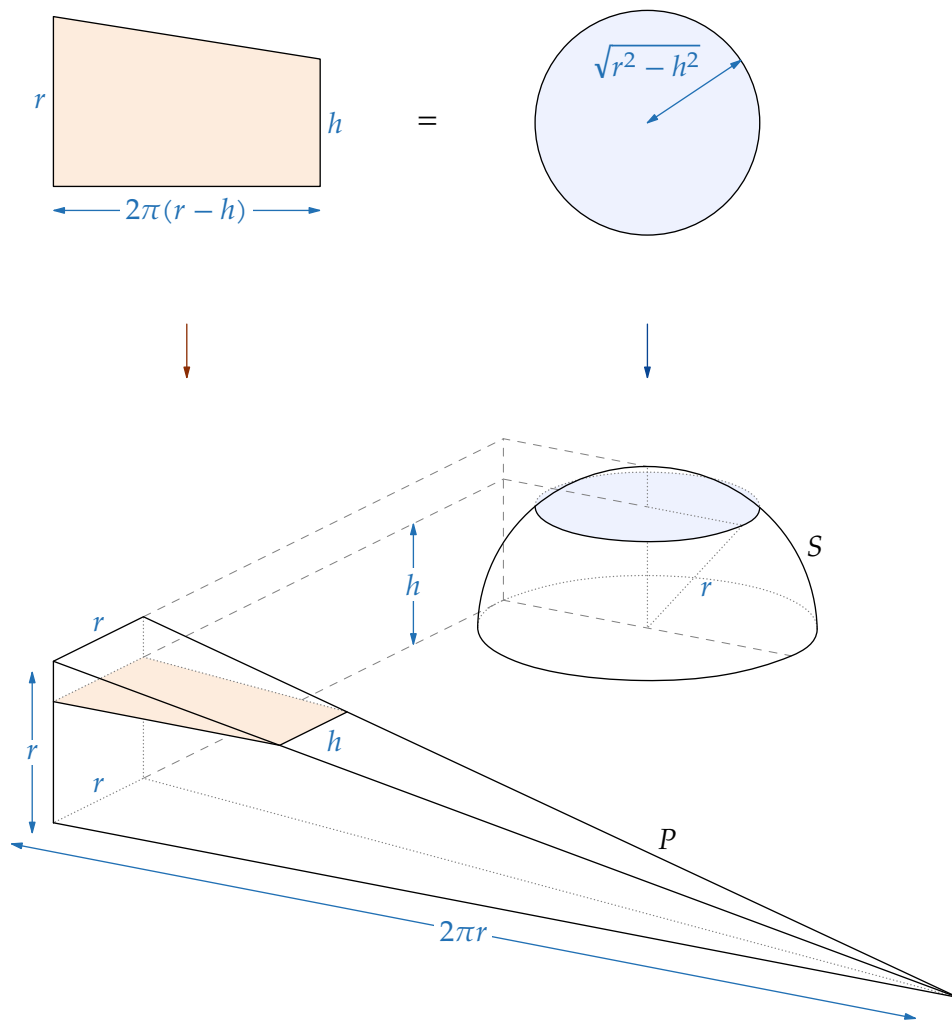
```

```

% ... you will need to browse the source for the rest of this
% drawing, which is stretching the limits of `isometric_projection`

```

The volume of a hemisphere via Cavalieri's Principle



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

— Sidney H. Kung

Trigonometry, Calculus, & Analytic Geometry

Sine of the sum	55
Area and difference formulas	57
The law of cosines I	59
The law of cosines II	61
The law of cosines III (via Ptolemy's theorem)	63
The double-angle formulae	65
The half-angle tangent formulae	67
Mollweide's equation	69
Tangent, cotangent, secant, and cosecant	71
Substitution to make a rational function of sine and cosine	73
Sums of arctangents	75
The distance between a point and a line	77
The midpoint rule is better than the trapezoidal rule for concave functions	79
Integration by parts	81
The graphs of f and f^{-1} are reflections about the line $y = x$	83
The reflection property of the parabola	85
Area under an arch of the cycloid	87

```

numeric x, y, z, r;
pair A, B, C, P;

x = 75; y = 46; x + y + z = 180;
r = 180;
A = r * dir (270 - z);
B = r * dir (270 + z);
C - A = whatever * dir x;
C - B = whatever * dir (180-y);
P = whatever[A,B]; C - P = whatever * up;

path am[];
am1 = fullcircle scaled 42 rotated angle (B-A) shifted A cutafter (A--C);
am2 = fullcircle scaled 42 rotated angle (C-B) shifted B cutafter (B--A);
am3 = fullcircle scaled 42 rotated angle (A-C) shifted C cutafter (C--B);
am4 = fullcircle scaled 42 rotated angle A cutafter (origin -- 1/2[A,B]);
forsuffixes $=1,2: draw am$ withcolor 3/4; endfor
forsuffixes $=3,4: draw am$ withcolor Reds 6 5; endfor

draw subpath (1,3) of unitsquare scaled 6 shifted 1/2[A,B] withcolor 3/4;
draw subpath (1,3) of unitsquare scaled 6 shifted P withcolor 3/4;

draw fullcircle scaled 2r withcolor Blues 7 6;
fill fullcircle scaled 2 withcolor Blues 7 6;

draw C--P dashed evenly withcolor 3/4;
draw A--B--C--A--origin--1/2[A,B];

label.urc("$\alpha$", point arctime 3/4 arclength am1 of am1 of am1);
label.lft("$\beta$", point arctime 1/2 arclength am2 of am2 of am2);
label.lrt("$\gamma$", point arctime 1/2 arclength am3 of am3 of am3);
label.llft("$\gamma$", point arctime 1/2 arclength am4 of am4 of am4);
label.urc("$a$", 1/2[B,C]);
label.ulft("$b$", 1/2[C, A]);
label.bot("$c$", 1/2[A, B]);
label.top("$c/2$", 5/16[A, B]);
label.top("$r$", 1/2 A);

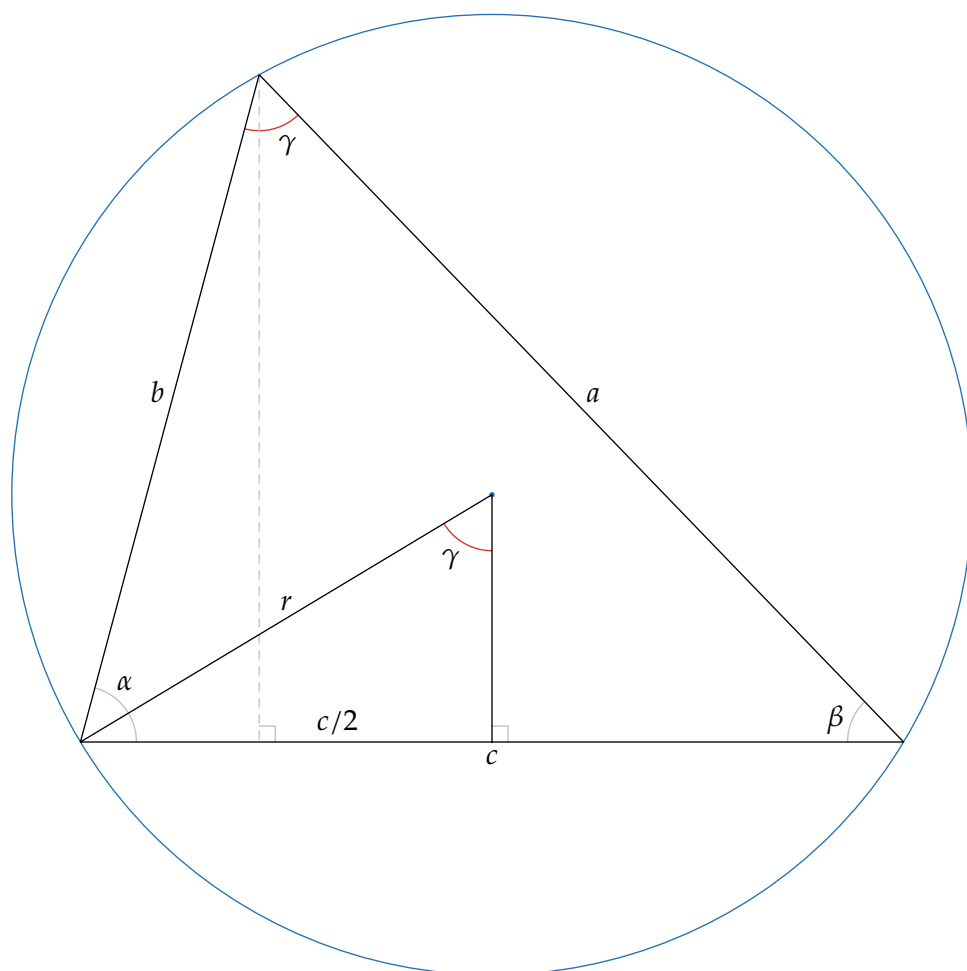
label.top("$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$"
& "\enspace for \ $\alpha+\beta < \pi$",
point 5/2 of bbox currentpicture shifted 42 up);

label.bot(btex \vbox{\openup 6pt\halign{\hss # \hss\cr
$c = a \cos\beta + b \cos\alpha$\cr
$r=1/2$ \quad $\Longrightarrow$ \quad
$\sin\gamma = \frac{c/2}{1/2} = c$, \enspace $\sin\alpha=a$, \enspace $\sin\beta=b$\cr
$\sin\bigl(\alpha+\beta\bigr) = \sin\bigl(\pi - (\alpha+\beta)\bigr) =
\sin\gamma = \sin\alpha\cos\beta + \sin\beta\cos\alpha$\cr}} etex,
point 1/2 of bbox currentpicture shifted 12 down);

```

Sine of the sum

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \text{ for } \alpha + \beta < \pi$$



$$c = a \cos \beta + b \cos \alpha$$

$$r = 1/2 \implies \sin \gamma = \frac{c/2}{1/2} = c, \sin \alpha = a, \sin \beta = b$$

$$\sin(\alpha + \beta) = \sin(\pi - (\alpha + \beta)) = \sin \gamma = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

— Sidney H. Kung

```

numeric x, y, a, b;
x = 56; y = 42; a = 120 / cosd(x); a * cosd(x) = b * cosd(y);

path t[];
t1 = orange -- b * dir y -- a * dir x -- cycle;
t2 = orange -- (xpart point 1 of t1, 0) -- point 1 of t1 -- cycle;

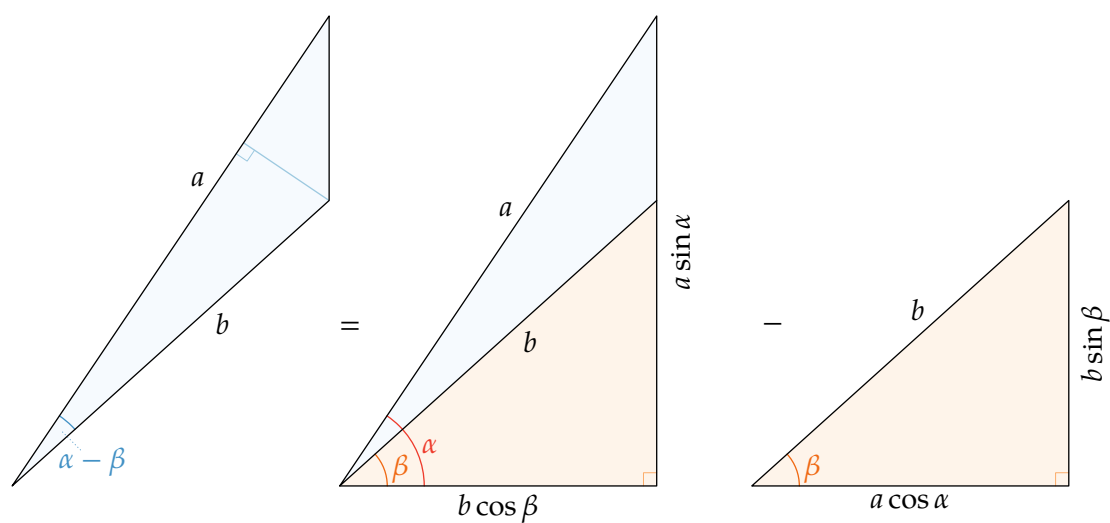
path a[];
a1 = fullcircle scaled 64 cutafter subpath (2,3) of t1;
a2 = fullcircle scaled 36 cutafter subpath (2,3) of t2;
a3 = fullcircle scaled 64 cutbefore subpath (2,3) of t2 cutafter subpath (2,3) of t1;

picture P[];
P1 = image(
    fill t1 withcolor Blues 8 1;
    pair p; p = whatever[point 2 of t1, point 3 of t1];
    p - point 1 of t1 = whatever * ((point 3 of t1 - point 2 of t1) rotated 90);
    drawoptions(withcolor Blues 7 3);
    draw subpath (1,3) of unitsquare scaled 5
        rotated angle (point 3 of t1 - point 2 of t1)
        shifted p withpen pencircle scaled 1/4;
    draw p -- point 1 of t1;
    drawoptions(withcolor Blues 7 5);
    pair q, r;
    q = 7/8 point arctime 1/2 arclength a3 of a3 of a3; r = q + (8, -8);
    draw r .. q dashed withdots scaled 1/4 withpen pencircle scaled 1/4;
    draw a3; label("$\alpha$-$\beta$", r+(4,-4));
    drawoptions();
    draw t1;
    label.ulft("$a$", point -5/8 of t1);
    label.lrt("$b$", point 5/8 of t1);
);
% .. and so on for P2 and P3
draw P1;
draw P2 shifted 124 right;
draw P3 shifted 280 right;
label("$=$", (128, 60));
label("$-$", (288, 60));

label.bot(btex \vbox{\openup 6pt\halign{\hfil $$$\{\}=#$\hfil\cr
\frac{1}{2}\cdot a\cdot b\sin\bigl(\alpha-\beta\bigr)&\frac{1}{2}\cdot a\sin\alpha\cdot
b\cos\beta - \frac{1}{2}\cdot a\cos\alpha\cdot b\sin\beta\cr
&\sin\bigl(\alpha-\beta\bigr)&\sin\alpha\cos\beta -
&\cos\alpha\sin\beta\cr}} etex,
point 1/2 of bbox currentpicture shifted 13 down);
% Browse the source for the lower drawing

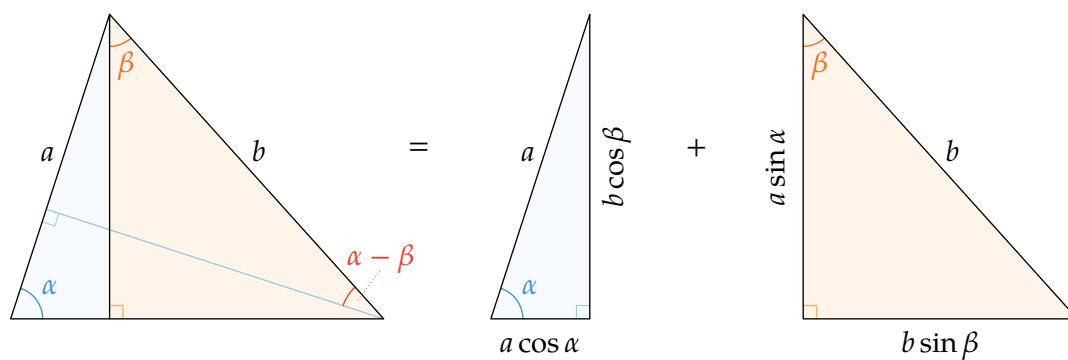
```


Area and difference formulas



$$\frac{1}{2} \cdot a \cdot b \sin(\alpha - \beta) = \frac{1}{2} \cdot a \sin \alpha \cdot b \cos \beta - \frac{1}{2} \cdot a \cos \alpha \cdot b \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\frac{1}{2} \cdot a \cdot b \cos(\alpha - \beta) = \frac{1}{2} \cdot a \cos \alpha \cdot b \cos \beta + \frac{1}{2} \cdot a \sin \alpha \cdot b \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

— Sidney H. Kung

numeric a, b, theta; a = 136; b = 9/16 a; theta = 40;

path A, B, C, arc, Am, Bm; *Trigonometry, Calculus, & Analytic Geometry*

A = unitsquare scaled a rotated -90;
 B = unitsquare scaled b rotated theta;
 C = point 3 of A
 -- point 1 of B rotated about (point 3 of A, -90)
 -- point 3 of A rotated about (point 1 of B, +90)
 -- point 1 of B -- cycle;

z0 = whatever [point 0 of A, point 3 of A]; point 1 of B - z0 = whatever * up;
 arc = quartercircle rotated 180 scaled 2 abs (point 1 of B - z0)
 shifted point 1 of B
 cut before subpath (0,1) of B;
 Am = unitsquare scaled -abs (z0 - point 3 of A) shifted point 3 of A;
 Bm = unitsquare scaled abs (point 0 of arc - point 1 of B) rotated theta shifted point 0 of arc;

picture P[];

P1 = image(
 draw subpath (1,3) of unitsquare scaled 6 shifted z0 withcolor 1/2;
 draw z0 -- point 1 of B dashed evenly scaled 1/2;

 draw arc dashed withdots scaled 1/4;

 fill A withcolor Oranges 7 1;
 fill B withcolor Oranges 7 1;
 fill C withcolor Blues 7 1;

 fill Am withcolor Blues 7 1;
 fill Bm withcolor Blues 7 1;

 draw A; draw B; draw C; draw Am; draw Bm;

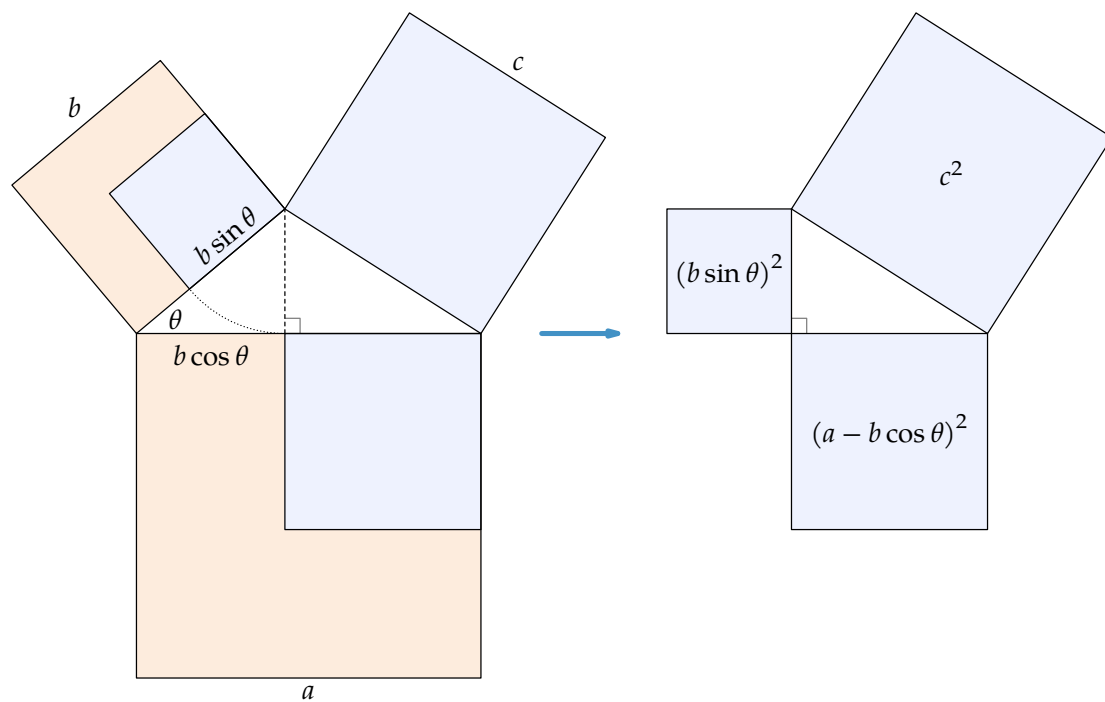
 label.bot ("a", point 3/2 of A);
 label.ulft ("b", point 5/2 of B);
 label.urc ("c", point 3/2 of C);

 label("\$\theta\$", 16 dir 1/2 theta);
 label.bot ("strut b cos \theta", 1/2 z0);
 draw thelabel.top ("b sin \theta", origin) rotated theta shifted point 1/2 of Bm;
);

P2 = image(
 Bm := Bm rotated about (point 1 of B, 90-theta);
 draw subpath (1,3) of unitsquare scaled 6 shifted z0 withcolor 1/2;
 for suffixes \$=Am, Bm, C: fill \$ withcolor Blues 7 1; draw \$; endfor
 label("\$\left(a - b \cos \theta\right)^2\$", center Am);
 label("\$\left(b \sin \theta\right)^2\$", center Bm);
 label("\$c^2\$", center C);
);

draw P1; draw P2 shifted 200 right;
 drawarrow 160 right -- 190 right withpen pencircle scaled 2 withcolor Blues 7 5;
 label.bot (btex \vbox{\openup 8pt\halign{\hfil \$\$\$\{=\}\hfil\cr
 c^2 \&\left(b \sin \theta\right)^2 + \left(a - b \cos \theta\right)^2\cr
 &b^2 \sin^2 \theta + a^2 - 2ab \cos \theta + b^2 \cos^2 \theta\cr
 &a^2 + b^2 \left(\sin^2 \theta + \cos^2 \theta\right) - 2ab \cos \theta\cr
 &a^2 + b^2 - 2ab \cos \theta\cr}} etex,
 point 1/2 of bbox currentpicture shifted 32⁵⁸ down);

The law of cosines I



$$\begin{aligned}
 c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\
 &= b^2 \sin^2 \theta + a^2 - 2ab \cos \theta + b^2 \cos^2 \theta \\
 &= a^2 + b^2 (\sin^2 \theta + \cos^2 \theta) - 2ab \cos \theta \\
 &= a^2 + b^2 - 2ab \cos \theta
 \end{aligned}$$

— Timothy A. Sipka

```

numeric a, b;  a = 0.98; b = 2.718;
path c; c = fullcircle scaled 421;
z0 = whatever[point a of c, point a+4 of c] = whatever[point 0 of c, point b of c];

fill center c -- point 0 of c -- z0 -- cycle withcolor Greens 7 1;

% mark the angles
draw unitsquare scaled 8 rotated angle (point 4 of c - point b of c)
    shifted point b of c withcolor 3/4;
draw halfcircle scaled 64 shifted point 0 of c
    cutbefore (point 0 of c -- point b of c) withcolor Reds 7 7;
label("$\theta$", 26 dir (180 - 1/4(180 - 45b)) shifted point 0 of c);

draw point a of c -- point a + 4 of c;
draw point 4 of c -- point 0 of c -- point b of c -- cycle;
draw c withcolor Blues 7 7;
draw center c withpen pencircle scaled dotlabeldiam withcolor Blues 7 7;

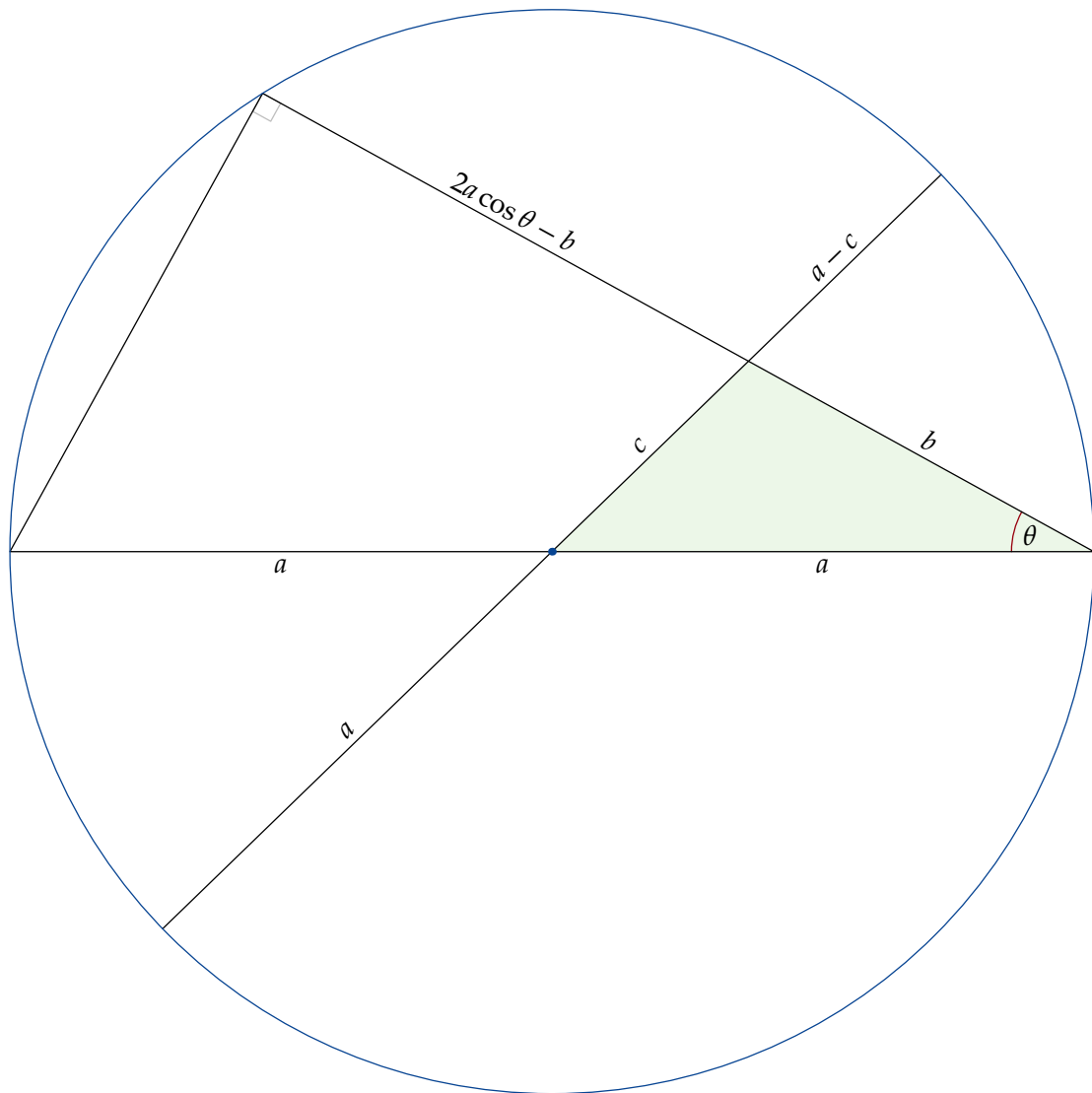
% add some labels with a one-off macro
vardef midlabel@#(expr t, a, b) =
    draw thelabel@#(t, origin) rotated angle (b-a) shifted 1/2[a,b]
enddef;

midlabel.top("$a$", point 4+a of c, origin);
midlabel.bot("$a$", point 4 of c, origin);
midlabel.bot("$a$", origin, point 0 of c);
midlabel.top("$b$", z0, point 0 of c);
midlabel.top("$c$", origin, z0);
midlabel.top("$a-c$", z0, point a of c);
midlabel.top("$2a\cos\theta-b$", point b of c, z0);

label.bot("$\bigl(2a\cos\theta - b\bigr) \cdot b"
    & " = \bigl(a - c\bigr) \cdot \bigl(a + c\bigr)$",
    point 1/2 of bbox currentpicture shifted 42 down);
label.bot("$c^2 = a^2 + b^2 - 2ab\cos\theta$",
    point 1/2 of bbox currentpicture shifted 12 down);

```

The law of cosines II



$$(2a \cos \theta - b) \cdot b = (a - c) \cdot (a + c)$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

— Sidney H. Kung

```

path c; numeric a, b;
c = fullcircle scaled 421;
a = 1/4; b = -7/8;

z0 = point a of c; z1 = point 4-a of c;
z3 = point b of c; z2 = point 4-b of c;

draw fullcircle scaled 42 rotated angle (z0-z3) shifted z3
  cutafter (z2--z3) withcolor Reds 7 6;
draw fullcircle scaled 42 rotated angle (z3-z2) shifted z2
  cutafter (z1--z2) withcolor Reds 7 6;

draw z0--z2--z3--z0--z1;
draw z2--z1--z3 dashed evenly;

draw c withcolor Blues 7 7;

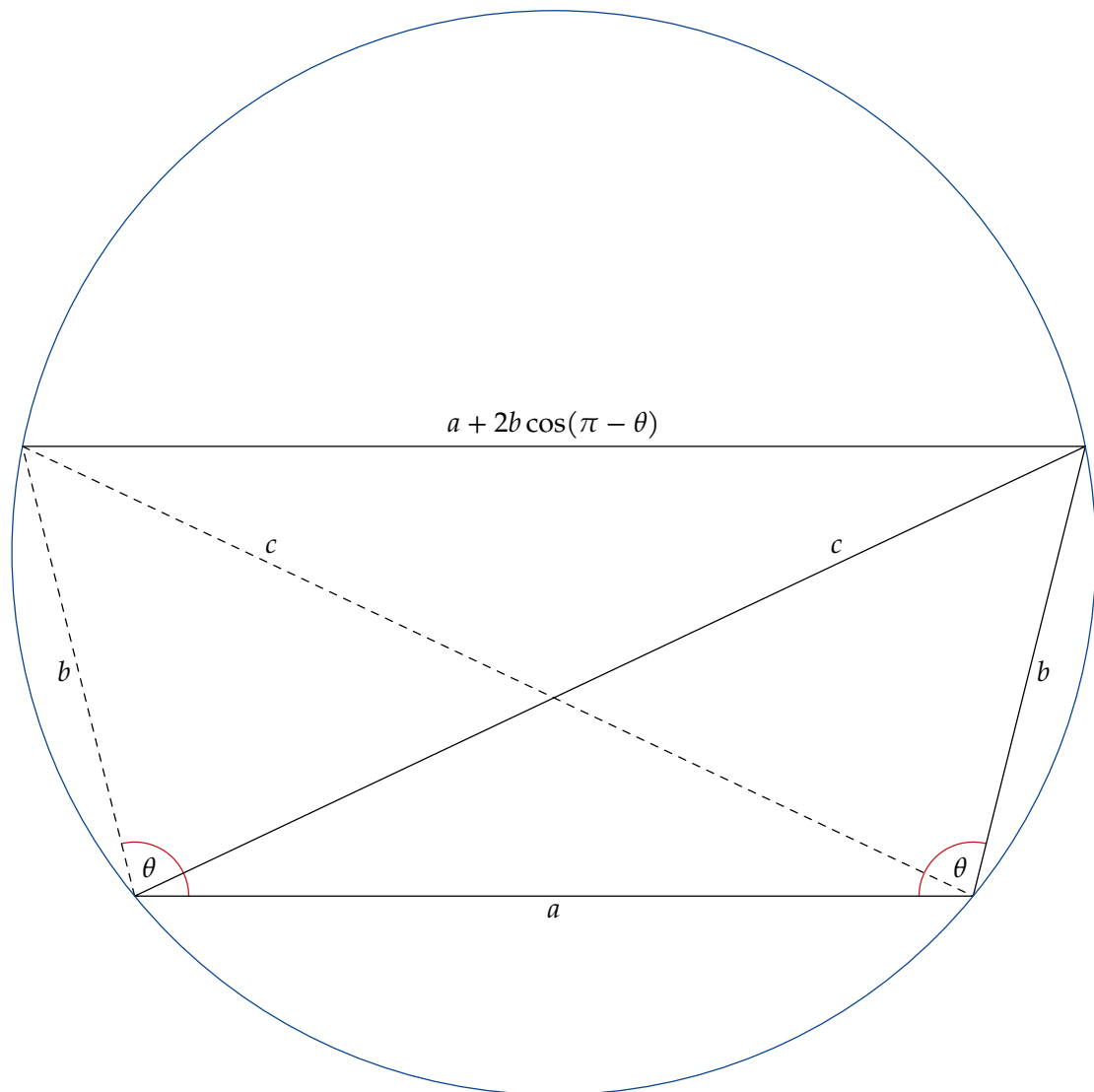
label.top("$a+2b\cos\bigl(\pi-\theta\bigr)$", 1/2[z0, z1]);
label.bot("$a$", 1/2[z2, z3]);
label.lft("$b$", 1/2[z1, z2]);
label.rt("$b$", 1/2[z3, z0]);
label.ulft("$c$", 3/4[z2, z0]);
label.urft("$c$", 3/4[z3, z1]);

label("$\theta$", z3 + 8 (unitvector(z0-z3)+unitvector(z1-z3)));
label("$\theta$", z2 + 8 (unitvector(z0-z2)+unitvector(z1-z2)));

label.bot(btex \vbox{\openup 8pt\halign{\hfil $$$ \hfil\cr
  c \cdot c = b \cdot b + \Bigl(a + 2b \cos\bigl(\pi-\theta\bigr)\Bigr) \cdot a\cr
  c^2 = a^2 + b^2 - 2ab\cos\theta\cr}} etex,
  point 1/2 of bbox currentpicture shifted 42 down);

```

The law of cosines III (via Ptolemy's theorem)



$$c \cdot c = b \cdot b + (a + 2b \cos(\pi - \theta)) \cdot a$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

— Sidney H. Kung

```

path h; pair A, B, C, D, O; numeric theta;

h = halfcircle scaled 420;

O = origin;
A = point 4 of h;
B = point 0 of h;
C = point 5/4 of h;
D = (xpart C, ypart A);

2 theta = angle C;

draw unitsquare scaled 8 rotated angle (C-D) shifted D withcolor 3/4;
draw unitsquare scaled 8 rotated angle (A-C) shifted C withcolor 3/4;

draw A--C--B withcolor Reds 7 7;
draw O--C--D withcolor Reds 7 7;

drawoptions(withcolor Blues 7 6);
draw h;
label.ulft("$x^2 + y^2 = 1$", point 3 of h);
drawoptions();

primarydef o through p =
  (1+o/arclength(p))[point 1 of p, point 0 of p] --
  (1+o/arclength(p))[point 0 of p, point 1 of p]
enddef;
drawarrow 16 through (A--B);
drawarrow 16 through (O--point 2 of h);

dotlabel.bot("$A$", A);
dotlabel.bot("$B$", B);
dotlabel.urt("$C$ \smash{\;\bigl(\cos^2\theta,\;\sin^2\theta\bigr)}$", C);
dotlabel.bot("$D$", D);
dotlabel.llft("$O$", O);

label("$\theta$", 28 dir 1/2 theta shifted A);
label("$2\theta$", 20 dir theta);

label("$x$", B shifted 24 right);
label("$y$", point 2 of h shifted 24 up);

vardef midlabel@#(expr t, a, b) =
  draw thelabel@#(t, origin) rotated angle (b-a) shifted 1/2[a,b]
enddef;
midlabel.top("$2\cos\theta$", A, C);
midlabel.top("$2\sin\theta$", C, B);

label.bot("$\triangle ACD \sim \triangle ABC$",
  point 1/2 of bbox currentpicture shifted 42 down);

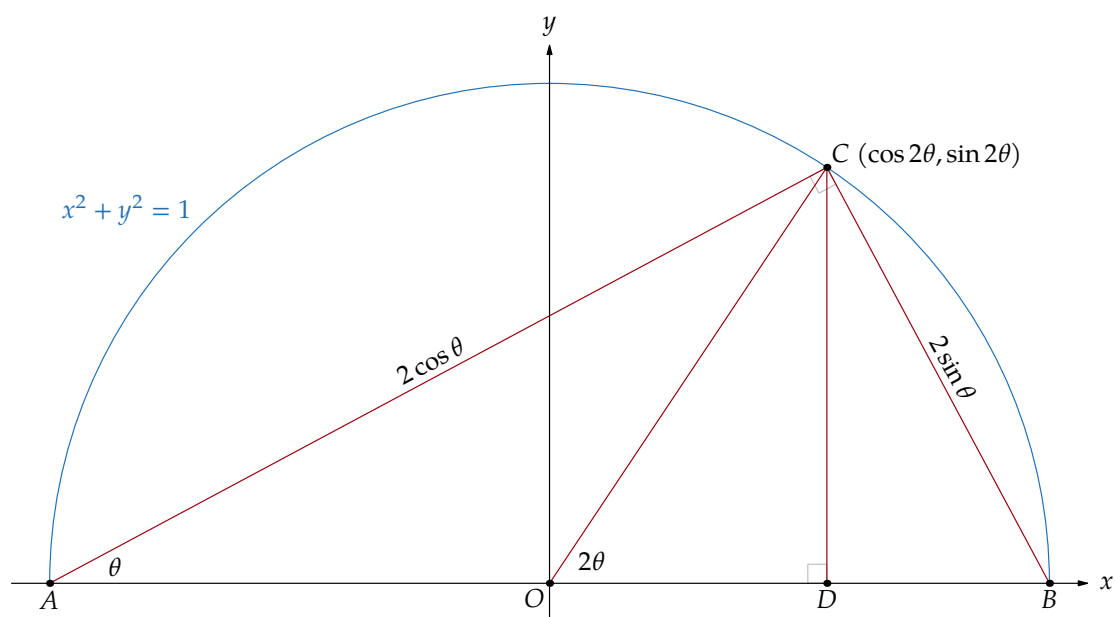
% fix bbox path in order to draw labels side by side
path p; p = bbox currentpicture shifted 20 down;

label.bot(btex \vbox{\openup 8pt\halign{\hfil $$$\hfil\cr
  CD \Big/ AC = BC \Big/ AB\cr
  \sin 2\theta \big/ 2 \cos\theta = 2 \sin\theta \big/ 2\cr
  \sin 2\theta = 2\sin\theta \cos\theta\cr}} etex, point 1/4 of p);

label.bot(btex \vbox{\openup 8pt\halign{\hfil $$$\hfil\cr
  AD \Big/ AC = AC \Big/ AB\cr
  \bigl(1 + \cos 2\theta \bigr) \big/ 2 \cos\theta = 2 \cos\theta \big/ 2\cr
  \cos 2\theta = 2\cos^2\theta - 1\cr}} etex, point 3/4 of p);

```


The double-angle formulae



$$\triangle ACD \sim \triangle ABC$$

$$CD/AC = BC/AB$$

$$\sin 2\theta / 2 \cos \theta = 2 \sin \theta / 2$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$AD/AC = AC/AB$$

$$(1 + \cos 2\theta) / 2 \cos \theta = 2 \cos \theta / 2$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

— Roger B. Nelsen

Trigonometry, Calculus, & Analytic Geometry

```

path h; pair A, B, C, D, O; numeric theta;

h = halfcircle scaled 420;

O = origin;
A = point 4 of h;
B = point 0 of h;
C = point 5/4 of h;
D = (xpart C, ypart A);

theta = angle C;

draw unitsquare scaled 8 rotated angle (C-D) shifted D withcolor 3/4;
draw unitsquare scaled 8 rotated angle (A-C) shifted C withcolor 3/4;

drawoptions(withcolor Reds 7 7);
draw A--C--B;
draw O--C--D;

drawoptions();
label("$\theta/2$", 38 dir 1/4 theta shifted A);
label("$\theta/2$", 42 dir (270 + 1/4 theta) shifted C);
label("$\theta$", 20 dir 1/2 theta);

drawoptions(withcolor Blues 7 6);
draw h;
label.ulft("$x^2 + y^2 = 1$", point 3 of h);

drawoptions();
primarydef o through p =
    (1+o/arclength(p))[point 1 of p, point 0 of p] --
    (1+o/arclength(p))[point 0 of p, point 1 of p]
enddef;
drawarrow 16 through (A--B);
drawarrow 16 through (O--point 2 of h);

label("$x$", B shifted 24 right);
label("$y$", point 2 of h shifted 24 up);

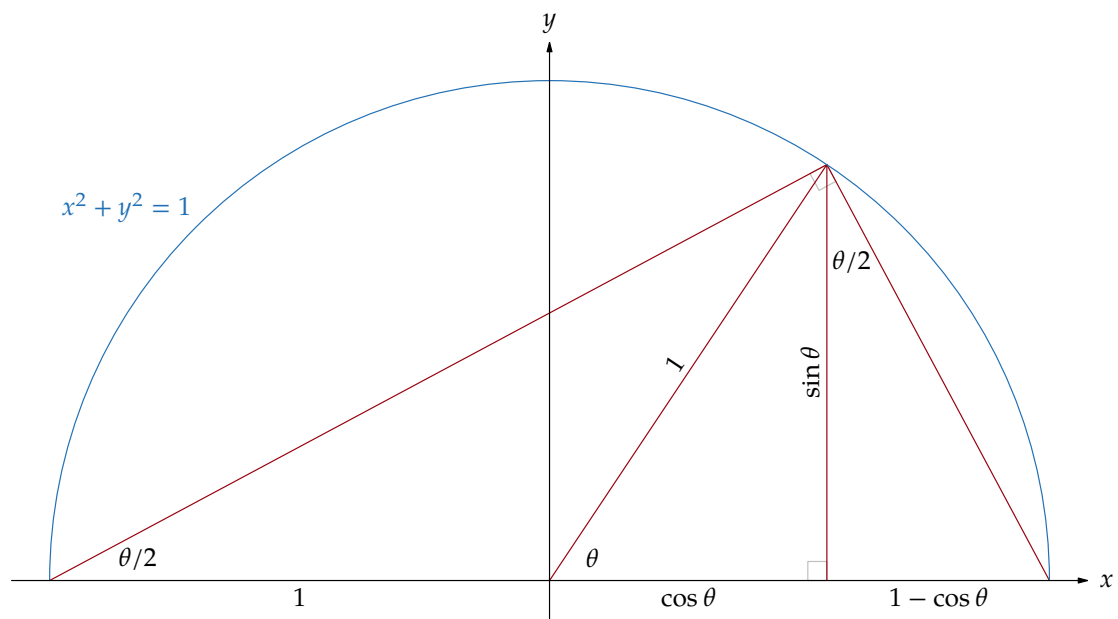
vardef midlabel@#(expr t, a, b) =
    draw thelabel@#(t, origin) rotated angle (b-a) shifted 1/2[a,b]
enddef;

midlabel.bot("$1$", A, origin);
midlabel.top("$1$", origin, C);
midlabel.bot("$\cos\theta$", origin, D);
midlabel.bot("$1-\cos\theta$", D, B);
midlabel.top("$\sin\theta$", D, C);

label.bot(btex $\displaystyle
\tan \frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}
$ etex, point 1/2 of bbox currentpicture shifted 42 down);

```

The half-angle tangent formulae



$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

— R. J. Walker

```

z1 = 210 left; z2 = 210 right; z3 = 180 dir 113;
Trigonometry, Calculus, & Analytic Geometry
pair t; t = unitvector(z1-z3) + unitvector(z2-z3);
z4 - z3 = whatever * t;
z4 - z1 = whatever * t rotated 90;

z5 = whatever[z1,z4] = whatever[z2,z3];
z6 = whatever[z1,z4]; z6 - z2 = whatever * (z3 - z4);
z7 = whatever[z1,z4]; z7 - z2 = whatever * (z3 - z1);

draw subpath (1,3) of unitsquare scaled 8 rotated angle (z4-z1) shifted z4 withcolor 3/4;
draw subpath (1,3) of unitsquare scaled 8 rotated angle (z1-z6) shifted z6 withcolor 3/4;

draw z3--z4 dashed withdots scaled 1/2;
draw z2--z6 dashed withdots scaled 1/2;

drawoptions(withcolor Blues 8 7);
draw halfcircle scaled 48 shifted z1 cutafter (z1--z3);
label("$\alpha$", z1 + 32 dir 1/2 angle (z3-z1));

drawoptions(withcolor Greens 8 7);
draw reverse halfcircle scaled 48 shifted z2 cutafter (z2--z3);
label("$\beta$", z2 + 32 dir (90 + 1/2 angle (z3-z2)));

drawoptions(withcolor Oranges 8 7);
draw halfcircle scaled 48 rotated angle (z4-z3) shifted z3 cutafter (z2--z3);
draw halfcircle scaled 48 rotated angle (z7-z2) shifted z2 cutafter (z2--z6);
label("$\frac{\gamma}{2}$", z3 + 20 (unitvector(z4-z3) + unitvector(z2-z3)));

drawoptions(withcolor Reds 8 7);
picture a; a = image(
    for s=48,52:
        draw halfcircle scaled s rotated angle (z3-z2) shifted z5 cutafter (z5--z1);
    endfor
);
draw a;
draw a rotatedabout(z5, 180);
draw a rotatedabout(z5, 180) reflectedabout(z2,z6);
label("$\frac{\alpha+\beta}{2}$", z5 + 22 (unitvector(z1-z5) + unitvector(z3-z5)));

drawoptions(withcolor Purples 8 7);
draw halfcircle scaled 108 shifted z1 cutafter (z1--z4);
pair s, t; s = z1 + 58 dir 1/2 angle (z4-z1); t = s + (32, -18);
label.bot("$\frac{\alpha-\beta}{2}$", t);
drawarrow t {up} .. {left} s withpen pencircle scaled 1/4;

drawoptions();
draw z1--z2--z3--cycle;
draw z1--z7--z2;

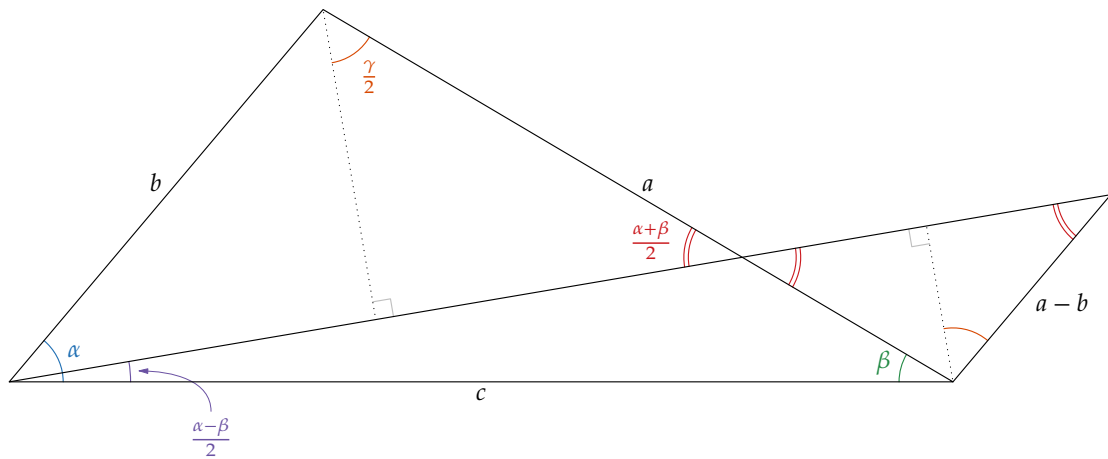
label.urt ("a", 1/2[z2, z3]);
label.ulft ("b", 1/2[z3, z1]);
label.bot ("c", 1/2[z1, z2]);
label.lrt ("a-b", 1/2[z2, z7]);

label.top(btex $\displaystyle
(a-b)\cos\frac{\gamma}{2} = c \sin\left(\frac{\alpha-\beta}{2}\right)
$ etex, point 5/2 of bbox currentpicture shifted 42 up);

```

Mollweide's equation

$$(a - b) \cos \frac{\gamma}{2} = c \sin \left(\frac{\alpha - \beta}{2} \right)$$



— H. Arthur DeKleine

```

numeric s, theta; theta = 28; s = 120;
path c; c = fullcircle scaled 2s;

z0 = whatever * dir theta;
z1 = whatever * dir theta;
z2 = (x1, y0) = (xpart point 0 of c, ypart point 6 of c);

drawoptions(withcolor 3/4);
draw unitsquare scaled 6 rotated 0 shifted point 6 of c;
draw unitsquare scaled 6 rotated 90 shifted z2;
draw unitsquare scaled 6 rotated 180 shifted point 0 of c;
draw unitsquare scaled 6 rotated 270 shifted center c;

drawoptions(withcolor Blues 8 8);
draw c;

drawoptions();
draw z0--z1--z2--cycle;
draw point 0 of c -- center c -- point 6 of c;

drawoptions(withcolor Blues 8 8);
draw center c withpen pencircle scaled dotlabeldiam;

drawoptions(withcolor Reds 8 7);
label("$\theta$", z0 + 24 dir 1/2 theta);
label("$\theta$", center c + 24 dir 1/2 theta);

drawoptions();
label.top("$1$", 1/2[point 0 of c, center c]);
label.lft("$1$", 1/2[point 6 of c, center c]);
label.bot("$1$", 1/2[point 6 of c, z2]);
label.rt("$1$", 1/2[point 0 of c, z2]);
label.rt("$\tan\theta$", 1/2[point 0 of c, z1]);
label.bot("$\cot\theta$", 1/2[point 6 of c, z0]);

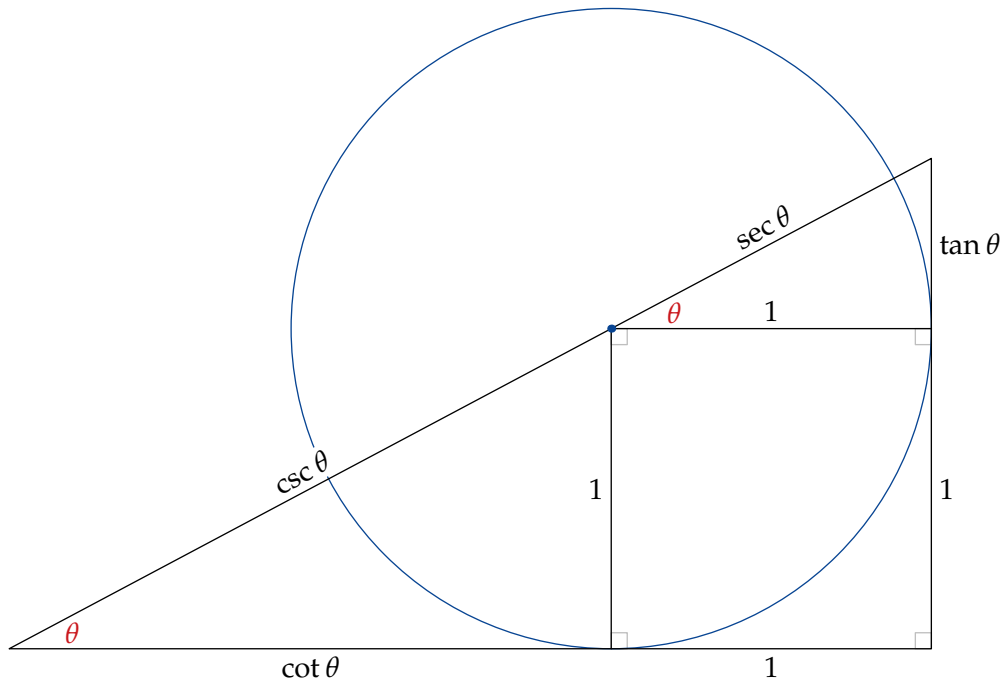
picture p;
p = thelabel.top("$\csc\theta$", origin);
unfill bbox p rotated theta shifted 1/2[z0, center c];
draw p rotated theta shifted 1/2[z0, center c];
draw thelabel.top("$\sec\theta$", origin) rotated theta shifted 1/2[z1, center c];

label.bot(btex \vbox{\openup 8pt\halign{\hfil $$$ \hfil\cr
\tan^2\theta + 1 = \sec^2 \theta\cr
\cot^2\theta + 1 = \csc^2 \theta\cr
\left(\tan\theta + 1\right)^2 +
\left(\cot\theta + 1\right)^2 =
\left(\sec\theta + \csc\theta\right)^2\cr}} etex,
point 1/2 of bbox currentpicture shifted 32 down);

label.bot(btex also \quad $\displaystyle \tan\theta =
\frac{\tan\theta+1}{\cot\theta + 1}$ etex,
point 1/2 of bbox currentpicture shifted 24 down);

```

Tangent, cotangent, secant, and cosecant



$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$(\tan \theta + 1)^2 + (\cot \theta + 1)^2 = (\sec \theta + \csc \theta)^2$$

$$\text{also } \tan \theta = \frac{\tan \theta + 1}{\cot \theta + 1}$$

— William Romaine

```

numeric theta, u; theta = 60; u = 144;
z0 = u * left;
z1 = origin;
x2 = x1; z2 = whatever * dir 1/2 theta shifted z0;
y3 = y1; z2 - z3 = whatever * (z2-z0) rotated 90;
z4 = z1 shifted (z2-z3);
z5 = z2 shifted (z2-z3);
x6 = x5; y6 = y1;
picture P[];
P0 = image(
    draw z0--z1--z2--cycle withpen pencircle scaled 1;
    label.bot("$1$", 1/2[z0, z1]);
    label.rt("$z$", 1/2[z1, z2]);
    label("$\theta/2$", 32 dir 1/4 theta shifted z0 shifted 2 down)
    withcolor Reds 8 7;
);

P1 = image(
    draw unitsquare scaled 6 rotated 90 withcolor 1/2;
    draw P0;
);
% ... and so on, with more complication, for P2, P3, P4

P2 := P2 shifted (7/4u, 0);
P3 := P3 shifted (0, -7/4u);
P4 := P4 shifted (7/4u, -7/4u);

draw P1; draw P2; draw P3; draw P4;

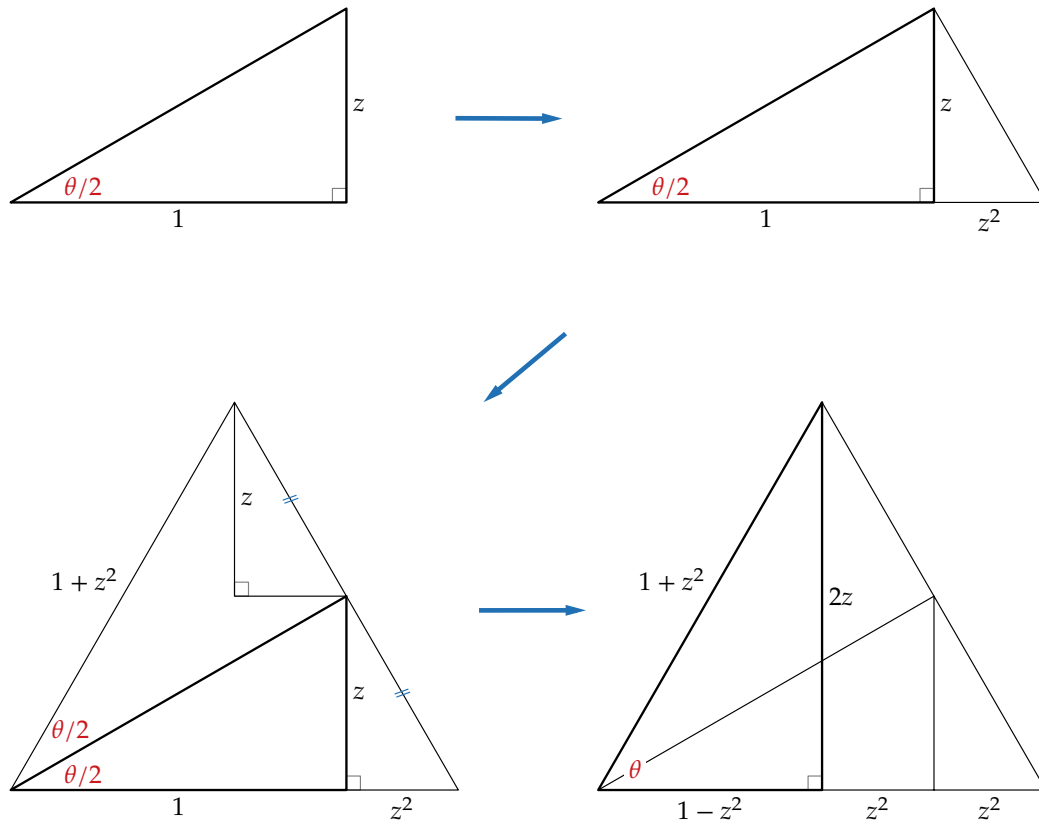
drawoptions(withpen pencircle scaled 2 withcolor Blues 8 7);
interim linecap := butt;
interim linejoin := mitered;
interim bboxmargin := 16;
picture a; a = image(drawarrow (left--right) scaled 21);
drawoptions();

for i=1 upto 3:
    draw a rotated angle (center P[i+1] - center P[i])
        shifted 1/2[center P[i], center P[i+1]];
endfor

label.bot(btex $\displaystyle
    z = \tan\frac{\theta}{2} \quad \Longrightarrow \quad
    \sin\theta = \frac{2z}{1+z^2} \quad \hbox{and} \quad
    \cos\theta = \frac{1-z^2}{1+z^2}
    $ etex, point 1/2 of bbox currentpicture shifted 42 down);

```


Substitution to make a rational function of sine and cosine



$$z = \tan \frac{\theta}{2} \implies \sin \theta = \frac{2z}{1 + z^2} \quad \text{and} \quad \cos \theta = \frac{1 - z^2}{1 + z^2}$$

— Roger B. Nelsen

```

numeric u; u = 42;
picture P[];
P1 = image(
  path t[];
  t1 = origin -- (2u, 0) -- (2u, u) -- cycle;
  t2 = origin -- (3u, -u) -- (3u, 0) -- cycle;

  fill t1 withcolor Oranges 7 1;
  fill t2 withcolor Blues 7 1;

  for x= 0 upto 3: draw (down -- up) shifted (x, 0) scaled u withcolor 3/4; endfor
  for y=-1 upto 1: draw (origin -- 3 right) shifted (0, y) scaled u withcolor 3/4; endfor

  draw fullcircle scaled 3/2 u
    rotated angle point 1 of t2
    cutafter subpath (2, 3) of t1
    withcolor Reds 7 6;

  draw t1 -- subpath (0, 2) of t2 -- point 1 of t1 withpen pencircle scaled 1;
  draw point 1 of t2 -- point 2 of t1 withpen pencircle scaled 1;

  label.bot("$\displaystyle \arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$",
    point 1/2 of bbox currentpicture shifted 21 down);
);
P2 = image(
  path t[];
  t1 = origin -- (-u, 0) -- (-u, -u) -- cycle;
  t2 = origin -- (-u, 2u) -- (-u, 0) -- cycle;
  t3 = origin -- (5u, 5u) -- (-u, 2u) -- cycle;

  fill t1 withcolor Greens 7 1;
  fill t2 withcolor Oranges 7 1;
  fill t3 withcolor Blues 7 1;

  numeric y; y = -2;
  for ss = (-1, 1), (-1, 1), (-1, 2), (-1, 3), (-1, 5), (1, 5), (3, 5):
    draw ((xpart ss, incr y) -- (ypart ss, y)) scaled u withcolor 3/4;
  endfor
  numeric x; x = -2;
  for ss = (-1, 3), (-1, 3), (-1, 4), (1, 4), (2, 5), (3, 5), (3, 5):
    draw ((incr x, xpart ss) -- (x, ypart ss)) scaled u withcolor 3/4;
  endfor

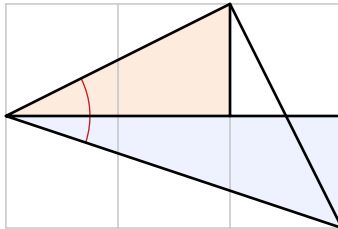
  draw halfcircle scaled 3/2 u
    rotated angle point 1 of t3
    withcolor Reds 7 6;

  draw t1 withpen pencircle scaled 1;
  draw subpath (0,2) of t2 withpen pencircle scaled 1;
  draw subpath (0,2) of t3 withpen pencircle scaled 1;

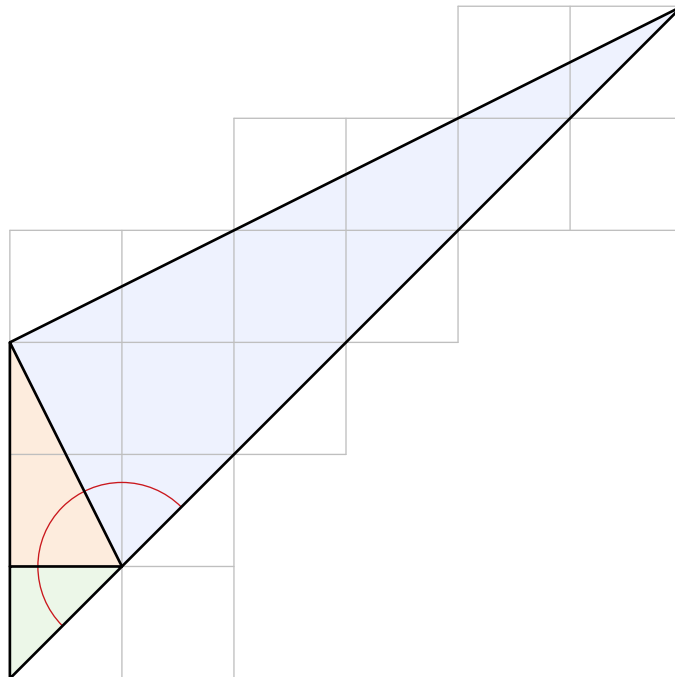
  label.bot("$\displaystyle \arctan 1 + \arctan 2 + \arctan 3 = \pi$",
    point 1/2 of bbox currentpicture shifted 21 down);
);
label.top(P1, 21 up); label.bot(P2, 21 down);

```

Sums of arctangents



$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

— Edward M. Harris

```

numeric u; u = 33;
path xx, yy;
xx = (left -- 8 right) scaled u;
yy = (down -- 11 up) scaled u;

numeric a, b, c, m;
m = 1.9;
c = -2;
a = 6.5;
b = 1;

z0 = (0, c) scaled u;
z1 = (a, a*m + c) scaled u;
z2 = (a, b) scaled u;
z3 = whatever[z0, z1]; z2 - z3 = whatever * (z0 - z1) rotated 90;

path p, t;
p = ((1/2, 1/2 m + c) -- (a+1/2, (a+1/2)*m + c)) scaled u;
t = (origin -- right -- (1, m)) scaled u shifted 1/4[z3, z1];

drawoptions(withpen pencircle scaled 1/4 withcolor Reds 8 6);
draw subpath (1,3) of unitsquare scaled 6 rotated 90 shifted point 1 of t;
draw subpath (1,3) of unitsquare scaled 6 rotated angle (z2-z3) shifted z3;

drawoptions(withcolor Reds 8 8);
draw t;
draw z1 -- z2 -- z3;
draw p;
draw thelabel.top("$y=mx + c$", origin) rotated angle (z1-z0) shifted point 1/5 of p;

drawoptions(withcolor Blues 8 7);
label.llft("$d$", 1/2[z2, z3]);
label.rt("$\big| ma + c - b \big|$", 1/2[z2, z1]);
label.bot("$1$", point 1/2 of t);
label.rt("$m$", point 3/2 of t);
label.ulft("$\sqrt{1+m^2}$", 1/2[point 0 of t, point 2 of t]);

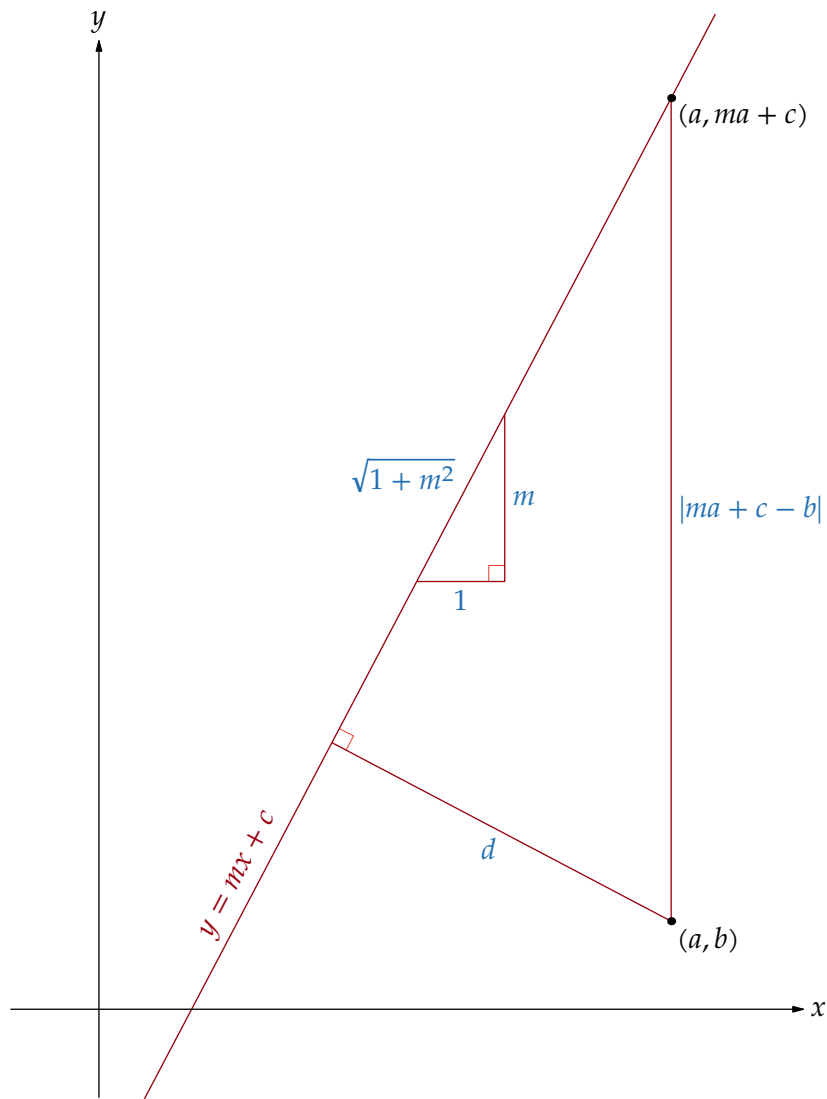
drawoptions();
dotlabel.lrt ("$(a, ma+c)$", z1);
dotlabel.lrt ("$(a, b)$", z2);

drawarrow xx;
drawarrow yy;
label.rt ("$x$", point 1 of xx);
label.top ("$y$", point 1 of yy);

label.bot(btex $\displaystyle
\frac{d}{1} = \frac{\left|ma+c-b\right|}{\sqrt{1+m^2}}$ etex,
point 1/2 of bbox currentpicture shifted 42 down);

```

The distance between a point and a line



$$\frac{d}{1} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

— R. L. Eisenman

```

numeric u; u = 12;
pair a, b, fa, fb, ta, tb, m, fm, am, bm;
xpart a = xpart fa = xpart ta = -4u = -xpart b = - xpart fb = - xpart tb;
ypart a = ypart b = 0;
ypart ta = 8u;
ypart tb = 11u;
ypart fa = 2u;
ypart fb = 9u;
m = 1/2[a, b];
fm = 1/2[ta, tb];
xpart am = xpart bm = xpart m;
am - fa = whatever * (tb - ta);
bm - fb = whatever * (tb - ta);

path base, lid, curve;
base = fa -- a -- b -- fb;
lid = fa -- ta -- tb -- fb;
curve = fa {dir 85} .. fm {tb-ta} .. fb;

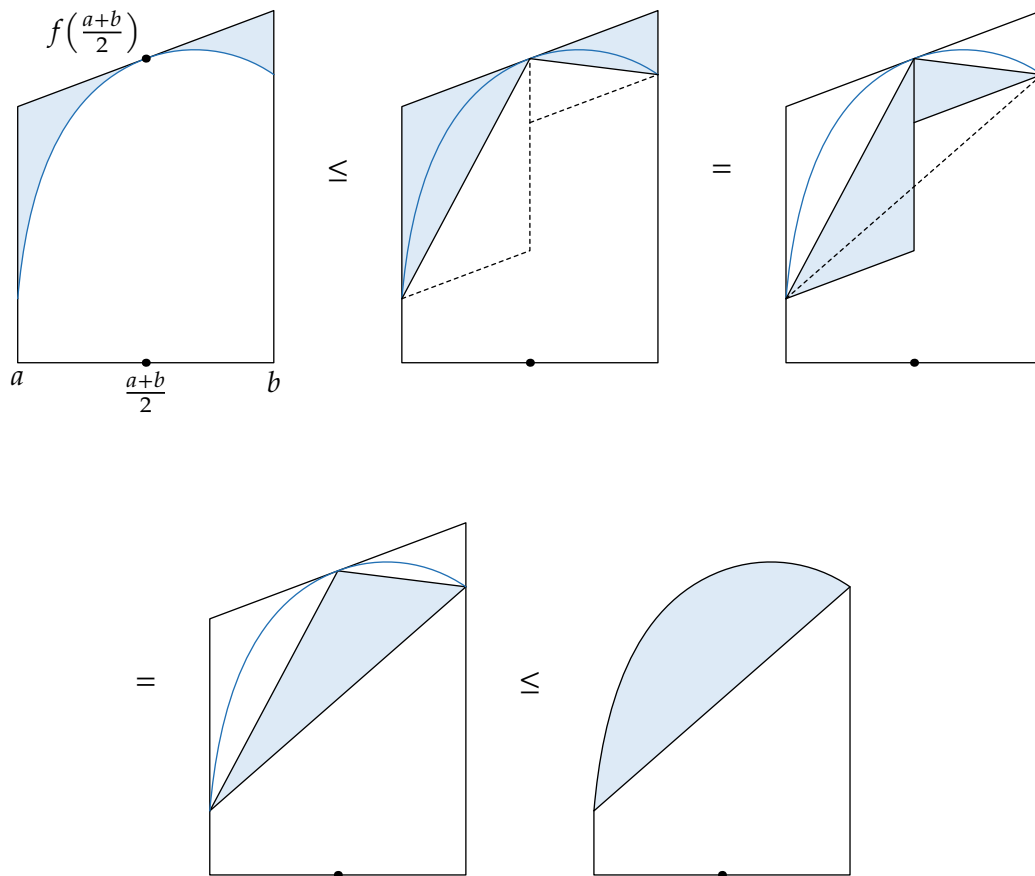
picture P[];
P1 = image(
    fill lid & reverse curve & cycle withcolor Blues 8 2;
    draw base;
    draw lid;
    draw curve withcolor Blues 8 7;
    label.bot("$a$", a);
    label.bot("$b$", b);
    dotlabel.bot("$\frac{a+b}{2}$", m);
    dotlabel.ulft("$f\left(\frac{a+b}{2}\right)$", fm);
);
% ... and so on for P2 .. P5

draw P1; draw P2 shifted (12u, 0); draw P3 shifted (24u, 0);
label("$\le$", (6u, 6u)); label("$=$", (18u, 6u));

draw P4 shifted (6u, -16u); draw P5 shifted (18u, -16u);
label("$=$", (0u, -10u)); label("$\le$", (12u, -10u));

```

The midpoint rule is better than the trapezoidal rule for concave functions



— Frank Burk

```

path ff, xx, yy;
pair p, q, r, s;

ff = (80, 55) {dir 10} .. (333, 233);
xx = 5 left -- 13 right + (xpart point 1 of ff, 0);
yy = 5 down -- 13 up + (0, ypart point 1 of ff);

numeric t; t = 1/12;

p = (xpart point t of ff, 0);
q = (xpart point 1-t of ff, 0);
r = (0, ypart point t of ff);
s = (0, ypart point 1-t of ff);

fill p -- subpath(t, 1-t) of ff -- q -- cycle withcolor Oranges 7 1;
fill r -- subpath(t, 1-t) of ff -- s -- cycle withcolor Blues 7 1;

drawoptions(withpen pencircle scaled 1/4);
draw p -- point t of ff -- r;
draw q -- point 1-t of ff -- s;

drawarrow xx;
drawarrow yy;
draw ff withpen pencircle scaled 3/4 withcolor Reds 8 7;

label.rt("$\left\{\begin{array}{c} \text{vcenter}\{\halign{\hfil\cr u=f(x)\cr v=g(x)\cr}\}\right.$",
    point 1 of ff);

label.rt("$u$", point 1 of xx);
label.top("$v$", point 1 of yy);

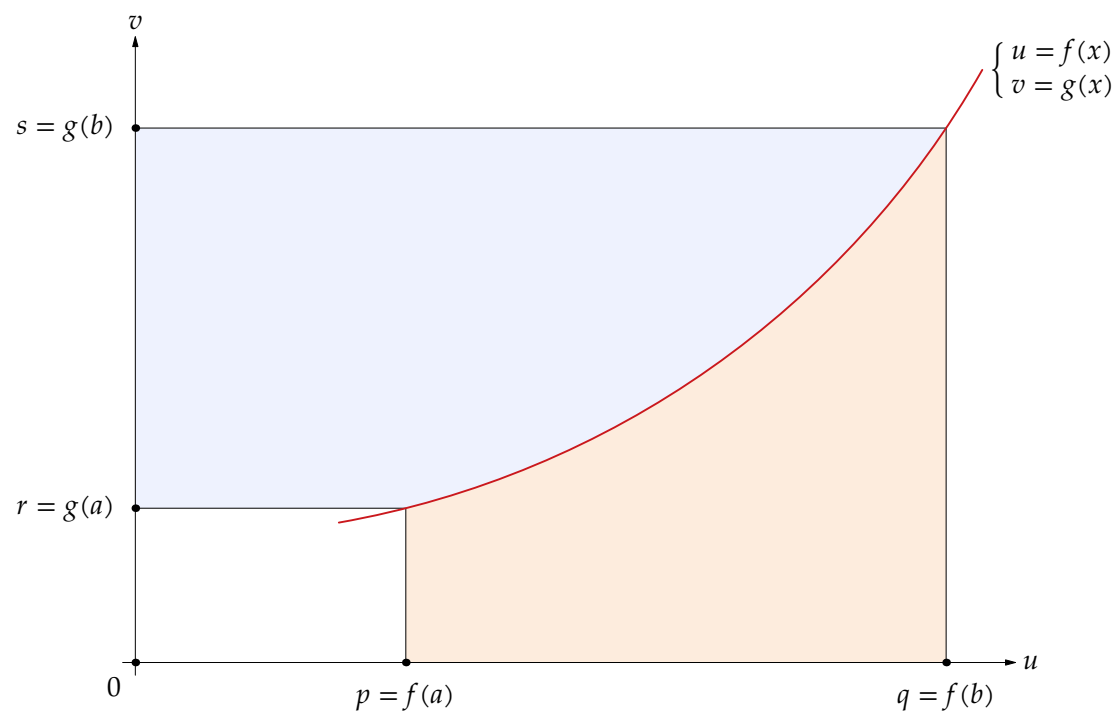
interim labeloffset := 8;
dotlabel.lft("$s=g(b)$", s);
dotlabel.lft("$r=g(a)$", r);
dotlabel.bot("$p=f(a)$", p);
dotlabel.bot("$q=f(b)$", q);
dotlabel.llft("$0$", origin);

def box(expr s) =
    "\pdfliteral{" &
    decimal redpart s & " " & decimal greenpart s & " " & decimal bluepart s &
    " rg}\vrule height 5mm width 8mm depth 2mm\pdfliteral{0 g}"
enddef;

label.bot("\vbox{\openup 16pt\halign{\hfil $\displaystyle # $\hfil\cr" &
    "\hbox{Area\ " & box(Blues 7 1) & "}" + \hbox{Area\ " & box(Oranges 7 1) & "}=qs-pr\cr" &
    "\int_r^s u:dv + \int_p^q v: du = uv \: \bigg|_{(p, r)}^{(q, s)}\cr" &
    "\int_a^b f(x) g'(x) \: dx = f(x) g(x) \: \Big|_a^b - \int_a^b g(x) f'(x) \: dx\cr}}",
    point 1/2 of bbox currentpicture shifted 21 down);

```


Integration by parts



$$\text{Area } \text{blue} + \text{Area } \text{orange} = qs - pr$$

$$\int_r^s u \, dv + \int_p^q v \, du = uv \Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) \, dx$$

— Richard Courant

```

z0 = -8(1,1); z1 = 221(1,1);
path xx, yy, xy;
xx = (x0, 0) -- (x1, 0); yy = xx rotated 90; xy = z0 -- z1;

picture P[];
P0 = image(
    drawarrow xx;
    drawarrow yy;
    draw xy withcolor Blues 8 7;
    draw thelabel.ulft("$y=x$", origin) rotated 45
        shifted point 1 of xy withcolor Blues 8 7;
);

P1 = image(
    path ff, ff';
    ff = point 1/3 of xx shifted 12 up .. {dir 5} point 1 of xx shifted 72 up;
    ff' = ff reflectedabout(z0, z1);
    numeric a, b; (a, b) = point 3/8 of ff;
    drawoptions(dashed evenly withpen pencircle scaled 1/4);
    draw (a,b) -- (b, a) -- (b, b) -- (a,b) -- (a, a) -- (b, a);
    drawoptions(withcolor Reds 8 7);
    draw ff; label.rt("$y=f(x)$", point 1 of ff);
    draw ff' dashed withdots scaled 1/4;
    label.top("$x=f(y)$, $y=f^{-1}(x)$", point 1 of ff');
    drawoptions();
    draw P0;

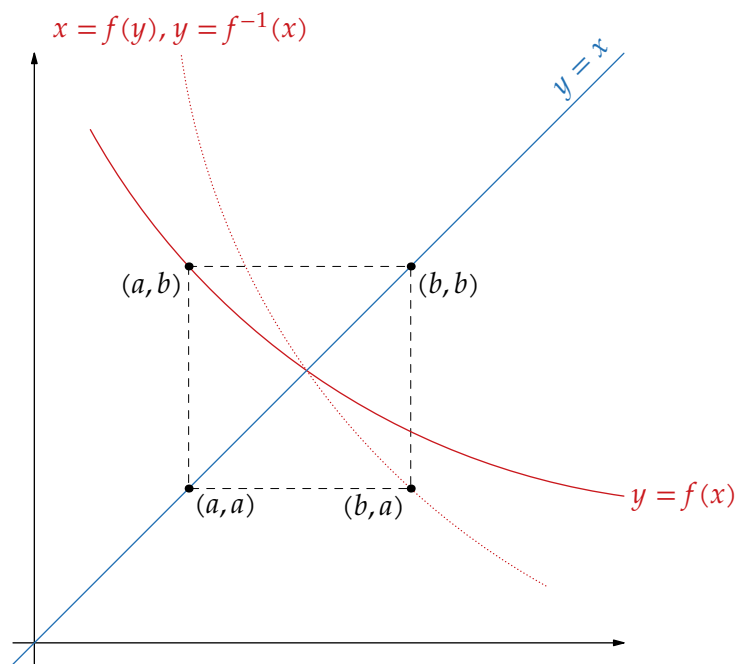
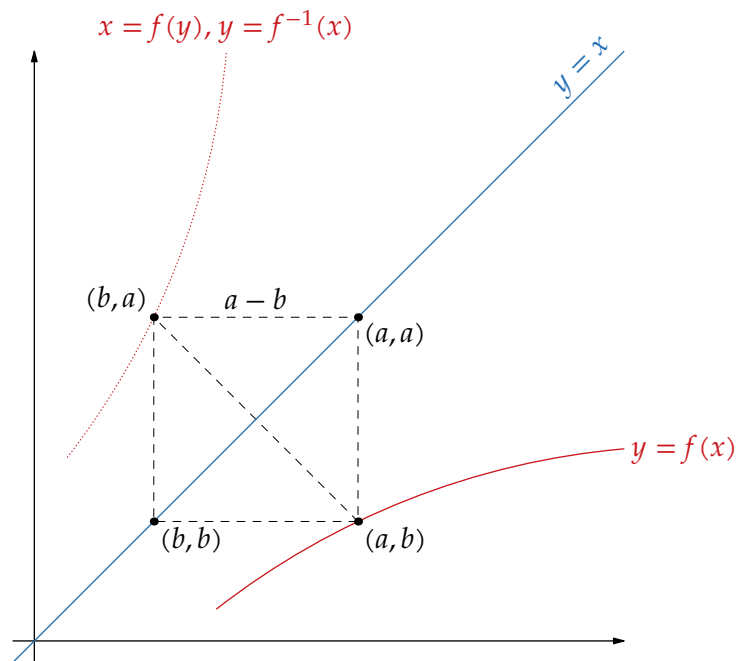
    dotlabel.lrt("(a, b)", (a, b));
    dotlabel.lrt("(a, a)", (a, a));
    dotlabel.lrt("(b, b)", (b, b));
    dotlabel.ulft("(b, a)", (b, a));
    label.top("$a-b$", 1/2(a+b, 2a));
);

P2 = image(
    path ff, ff';
    ff = point 7/8 of yy shifted 21 right .. {dir -8} point 1 of xx shifted 55 up;
    ff' = ff reflectedabout(z0, z1);
    numeric a, b; (a, b) = point 1/4 of ff;
    drawoptions(dashed evenly withpen pencircle scaled 1/4);
    draw (a,b) -- (b, b) -- (b, a) -- (a, a) -- cycle;
    drawoptions(withcolor Reds 8 7);
    draw ff; label.rt("$y=f(x)$", point 1 of ff);
    draw ff' dashed withdots scaled 1/4;
    label.top("$x=f(y)$, $y=f^{-1}(x)$", point 1 of ff');
    drawoptions();
    draw P0;
    dotlabel.llft("(a, b)", (a, b));
    dotlabel.lrt("(a, a)", (a, a));
    dotlabel.lrt("(b, b)", (b, b));
    dotlabel.llft("(b, a)", (b, a));
);

label.top(P1, origin); label.bot(P2, 21 down);

```

The graphs of f and f^{-1} are reflections about the line $y = x$



— Ayoub B. Ayoub

```

path para, dirx, xx, yy;
numeric p, s, t, u;
p = 3/2; s = 1/8; u = 28; t = 12/s;
para = ((49/4/p, -7) for y=s - 7 step s until 7: -- (y * y / 4p, y) endfor) scaled u;
dirx = (-p*u, ypart point 0 of para) -- (-p*u, ypart point infinity of para);
xx = (-u-p*u, 0) -- (p*u + xpart point 0 of para, 0);
yy = dirx shifted (p*u, 0);

z0 = p * u * right;
z1 = point t of para;
z2 = z0 reflectedabout(z1, direction t of para rotated 90 shifted z1);
z3 = z2 rotatedabout(z1, 180);
z4 = 1/2[z0, z3];
z5 = z4 rotatedabout(z1, 180);

drawoptions(withcolor 3/4);
draw subpath (3.8, 4.2 + 1/45 angle (z1-z0)) of fullcircle scaled 2 abs(z0-z1) shifted z1;
draw subpath (1, 3) of unitssquare scaled 6 rotated angle (z0-z3) shifted z4;

drawoptions(withcolor Blues 8 4);
draw z4--z5;
draw z3--z1;
draw 1.2[z0,z3] -- 1.6[z3,z0];
drawoptions(withcolor Blues 8 6);
draw thelabel.ulft("$m_1 = y' = 2p/y$", origin) rotated angle (z5-z4) shifted z5;
draw thelabel.ulft("$m_2 = -y/2p$", origin) rotated angle (z0-z3) shifted 1.5[z3, z0];
numeric a; a = 1/2 angle (z1-z0);
label("$\alpha$", z1 + 36 dir (180 + 3/2 a));
label("$\beta$", z1 + 36 dir (180 + 1/2 a));
label("$\gamma$", z1 + 36 dir (1/2 a));

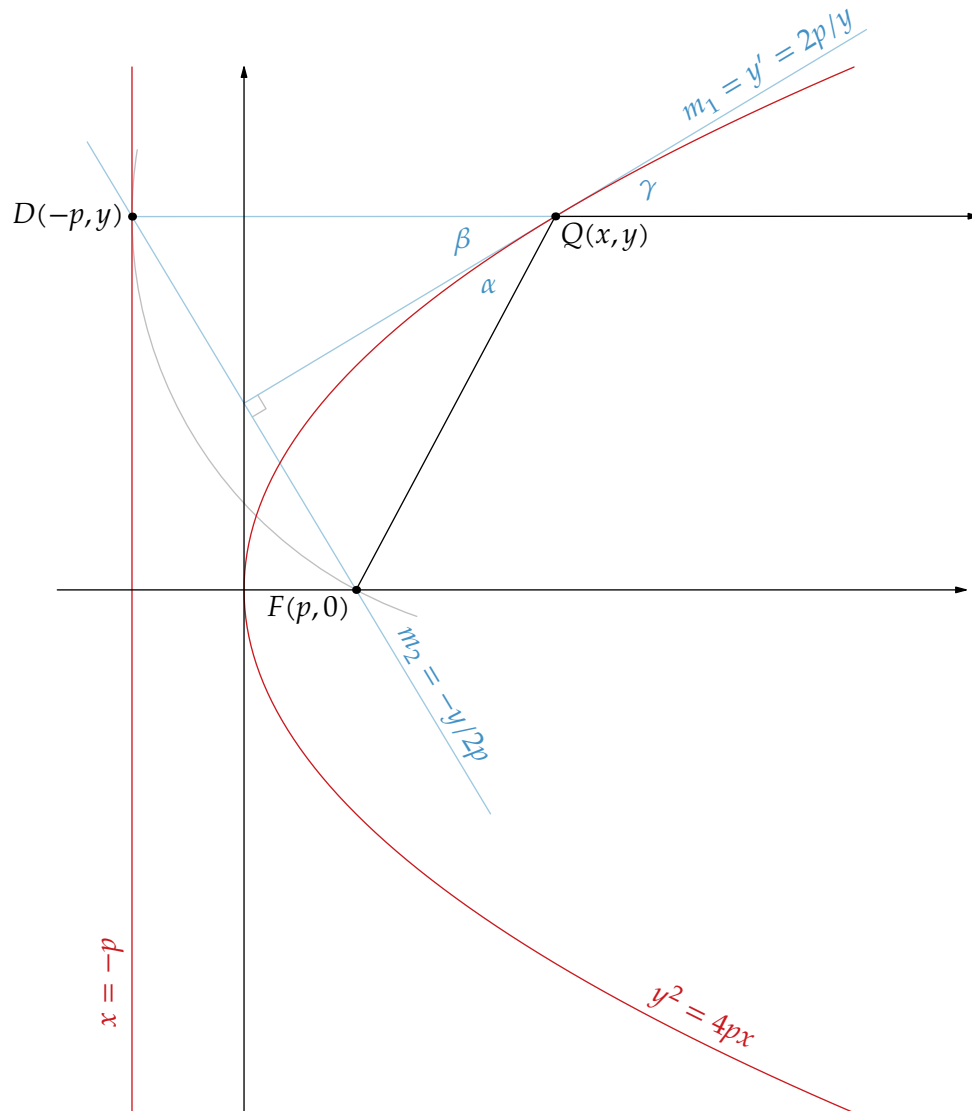
drawoptions(withcolor Reds 8 7);
draw para; draw dirx;
draw thelabel.top("$y^2=4px$", origin)
    rotated (180 + angle direction 8 of para) shifted point 8 of para;
draw thelabel.top("$x=-p$", origin)
    rotated 90 shifted point 1/8 of dirx;

drawoptions();
drawarrow z0 -- z1 -- z2;
drawarrow xx;
drawarrow yy;

dotlabel.llft("$F(p,0)$", z0);
dotlabel.lrt("$Q(x,y)$", z1);
dotlabel.lft("$D(-p,y)$", z3);
label.bot("\mathsurround 6pt" &
    "$QF=QD$ and $m_1 \cdot m_2 = -1$, therefore $\alpha=\beta=\gamma$",
    point 1/2 of bbox currentpicture shifted 42 down);

```

The reflection property of the parabola



$$QF = QD \text{ and } m_1 \cdot m_2 = -1, \text{ therefore } \alpha = \beta = \gamma$$

— Ayoub B. Ayoub

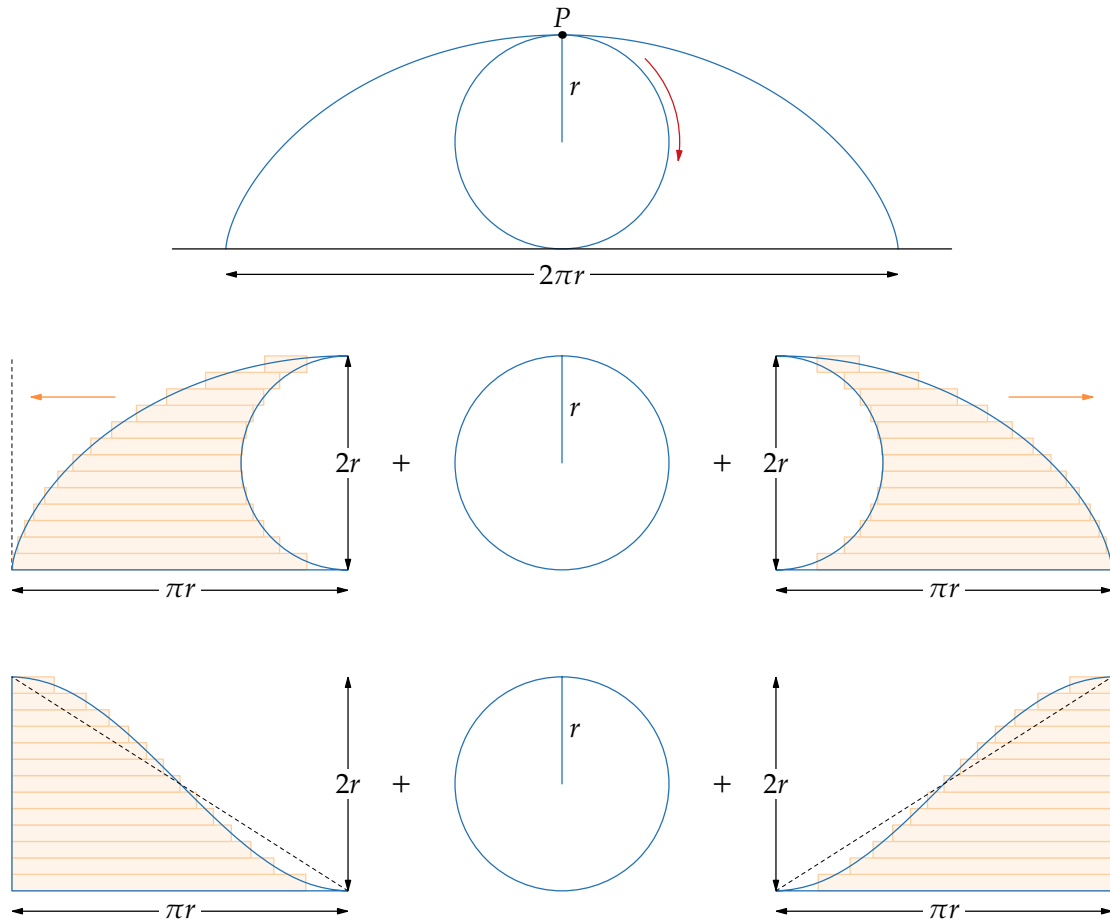
```

input arrow_label
picture P[];
path c, cycloid, base;
numeric pi, r, s; r = 42; s = 1; pi = 3.141592653589793;
c = fullcircle scaled 2r rotated 90;
cycloid = point 0 of c rotated -180 shifted (-pi * r, 0) for t = s-180 step s until 180:
    -- point 0 of c rotated -t shifted (t / 180 * pi * r, 0) endfor;
base = point 0 of cycloid shifted 21 left -- point infinity of cycloid shifted 21 right;
P0 = image(
    draw center c -- c withcolor Blues 8 7;
    label.rt("$r$", 1/2 point 0 of c);
);
P1 = image(
    draw P0;
    draw cycloid withcolor Blues 8 7;
    draw base;
    drawarrow subpath (7, 5.8) of c scaled 1.1 withcolor Reds 8 7;
    dotlabel.top("$P$", point 0 of c);
    arrow_label(point 0 of cycloid, point infinity of cycloid, "$2\pi r$", 10);
);
P2 = image(
    draw P0;
    label("$+$", (-3/2r, 0));
    label("$+$", (+3/2r, 0));
    % see source for the rest of this part ...
);
P3 = image(
    draw P0;
    label("$+$", (-3/2r, 0));
    label("$+$", (+3/2r, 0));
    % see source for the rest of this part ...
);
draw P1;
draw P2 shifted (0, -3r);
draw P3 shifted (0, -6r);

label.bot(btex \vbox{\openup 12pt\halign{\hfil $\displaystyle \#\hfil\cr
\frac{1}{2}\pi r \cdot 2r \quad\quad + \quad\quad\quad
\pi r^2 \quad\quad\quad
\frac{1}{2}\pi r \cdot 2r \cr
\hbox to 0pt{\hss\small therefore\quad}A = 3\pi r^2\cr}} etex,
point 1/2 of bbox currentpicture shifted 42 down);

```

Area under an arch of the cycloid



$$\frac{1}{2}\pi r \cdot 2r \quad + \quad \pi r^2 \quad + \quad \frac{1}{2}\pi r \cdot 2r$$

therefore $A = 3\pi r^2$

— Richard M. Beekman

Inequalities

The arithmetic mean – geometric mean inequality I	91
The arithmetic mean – geometric mean inequality II	93
The arithmetic mean – geometric mean inequality III	95
Two extremum problems	97
The HM–GM–AM–QM inequalities I	99
The HM–GM–AM–QM inequalities II	101
The HM–GM–AM–QM inequalities III	103
Five means — and their means	105
$e^\pi > \pi^e$	107
$A^B > B^A$ for $e \leq A < B$	109
The mediant property	111
Regle des nombres moyens – I	113
Regle des nombres moyens – II	115
The sum of a positive number and its reciprocal is at least two	117
Aristarchus' inequalities	119
The Cauchy-Schwartz inequality	121
Bernoulli's inequality	123
Napier's inequality	125

```

input arrow_label
path h; h = halfcircle scaled .95 \mpdim{\textwidth};
z0 = point 2.7818 of h;
x1 = x0; y1 = 0;

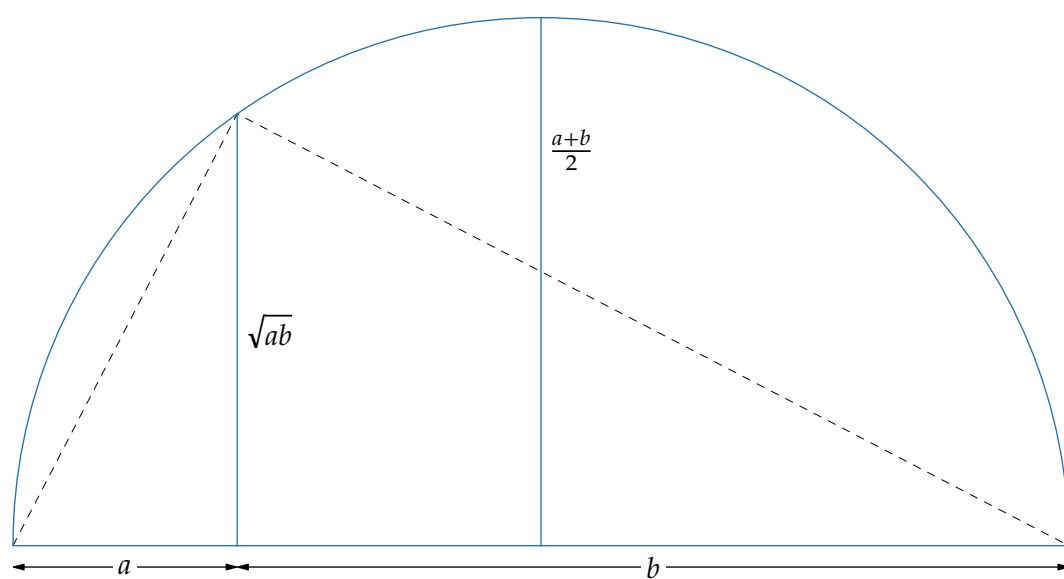
draw point 4 of h -- z0 -- point 0 of h dashed evenly withpen pencircle scaled 1/4;
drawoptions(withcolor Blues 8 7);
draw h -- cycle; draw origin -- point 2 of h; draw z0--z1;

drawoptions();
arrow_label(point 4 of h, z1, "$a$", 8);
arrow_label(z1, point 0 of h, "$b$", 8);

label.rt("$\sqrt{ab}$", 1/2[z0, z1]);
label.rt("$\frac{a+b}{2}$", 3/4 point 2 of h);
label.bot("$\displaystyle \sqrt{\frac{ab}{2}} \leq \sqrt{\frac{a+b}{2}}$",
    point 1/2 of bbox currentpicture shifted 42 down);

```

The arithmetic mean – geometric mean inequality I



$$\sqrt{\frac{ab}{2}} \leq a + b$$

— Charles D. Gallant

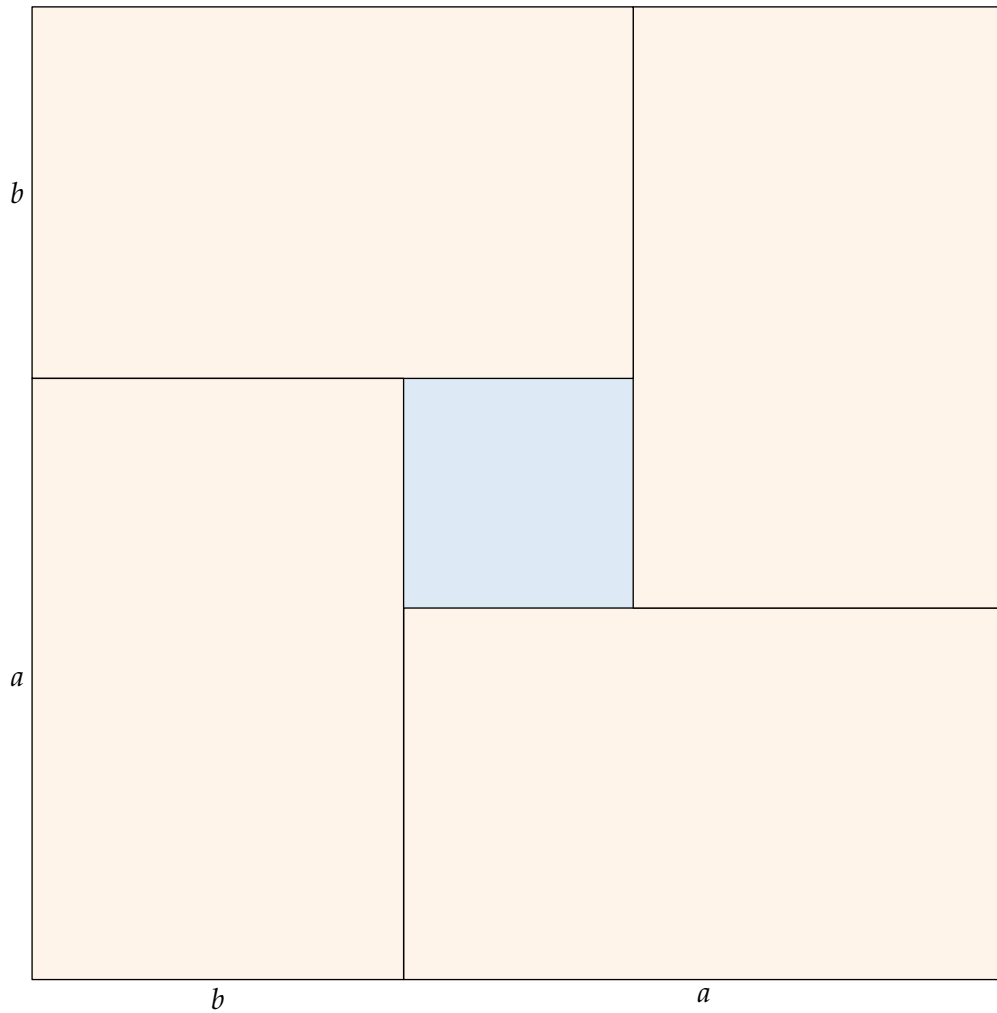
```

numeric a, b; a + b = 7/8 \mpdim{\textwidth}; a = 610/377 b;
path r, s;
s = unitsquare shifted -(1/2, 1/2) scaled (a-b);
r = unitsquare xscaled a yscaled -b shifted point 0 of s;
fill s withcolor Blues 8 2;
for t=0 upto 3:
    fill r rotated 90t withcolor Oranges 8 1; draw r rotated 90t;
endfor
label.bot("$a$", point 5/2 of r);
label.lft("$a$", point 5/2 of r rotated 270);
label.lft("$b$", point 3/2 of r rotated 180);
label.bot("$b$", point 3/2 of r rotated 270);

label.bot("$(a+b)^2 - (a-b)^2 = 4ab$",
    point 1/2 of bbox currentpicture shifted 21 down);
label.bot("$\displaystyle \frac{a+b}{2} \ge \sqrt{ab}$",
    point 1/2 of bbox currentpicture shifted 13 down);

```

The arithmetic mean – geometric mean inequality II



$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

— Doris Schattschneider

```

input arrow_label
numeric a, b;
5b = 3a; a + b = \mpdim{\textwidth};
z1 = (1/2(a+b) +-+ 1/2(a-b), 1/2(b-a));
path C, c;
C = fullcircle scaled a rotated 90;
c = fullcircle scaled b rotated 90 shifted z1;
drawoptions(withcolor Blues 8 5);
draw origin -- z1;
draw (0, y1) -- z1 dashed evenly scaled 1/2;
draw C -- point 4 of C -- point 4 of c -- c;
drawoptions();
arrow_label(point 0 of C, point 4 of C, "$a$", 10);
arrow_label(point 0 of c, point 4 of c, "$b$", -10);
arrow_label(point 4 of C, point 4 of c, "$\sqrt{ab}$", 10);
path aa; aa = (center C -- center c) shifted 12 up;
drawdbllarrow aa;
picture t; t = thelabel("$\frac{a+b}{2}$", point 1/2 of aa);
    unfill bbox t shifted 2.7 up; draw t;

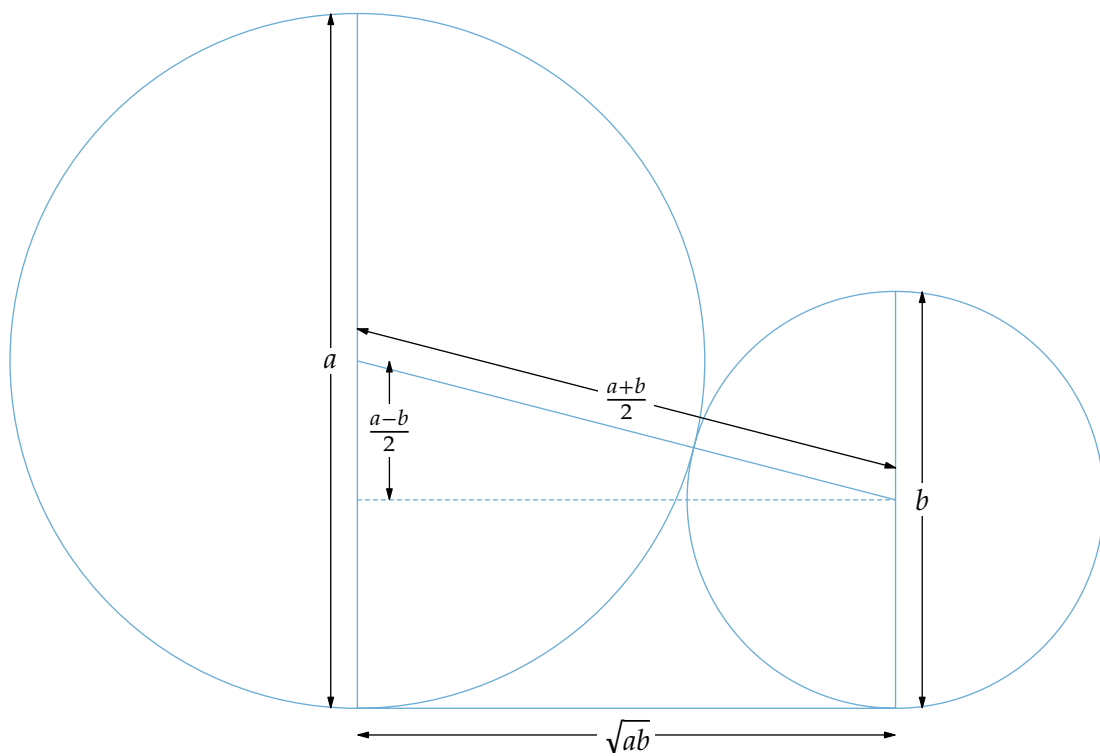
arrow_label((0, y1), center C, "$\frac{a-b}{2}$", 12);

label.top("\mathsurround 6pt"
    & "$\displaystyle \frac{a+b}{2} \ge \sqrt{ab}$,"
    & "with equality iff $a=b$",
    point 5 /2 of bbox currentpicture shifted 42 up);

```

The arithmetic mean – geometric mean inequality III

$$\frac{a+b}{2} \geq \sqrt{ab}, \text{ with equality iff } a = b$$



— Roland H. Eddy

```

numeric P, s, minx, maxx; s = 2; P = 1440; minx = 4s; maxx = P/minx;
path pp, xx, yy;
pp = (minx, P/minx) for x=minx+s step s until maxx+eps: -- (x, P/x) endfor;
xx = (-minx, 0) -- (maxx+minx, 0); yy = xx rotated 90;

z0 = point 2/3 length pp of pp;
z1 = (x0+y0, 0); z2 = (0, x0+y0); z3 = (2 sqrt(P), 0); z4 = (0, 2 sqrt(P));

picture T[];

T0 = image(
  undraw (left--right) scaled 1/2 \mpdim{\textwidth} shifted point 1/2 of xx;
  draw origin -- point 1 of xx rotated 45 dashed evenly withpen pencircle scaled 1/4;
  drawarrow xx; drawarrow yy;
  draw z1 -- z2 withcolor Blues 8 7;
  label.rt("$x$", point 1 of xx);
  label.top("$y$", point 1 of yy);
  dotlabel.urt("$\bigl(x, y\bigr)$", z0);
  dotlabel.bot("$\bigl(S, 0\bigr)$", z1);
  dotlabel.lft("$\bigl(0, S\bigr)$", z2));
T1 = image(
  draw T0;
  draw pp withcolor Reds 8 7;
  label.rt("$xy=P$", point 1/2 of pp) withcolor Reds 8 7;
  draw z3 -- z4 withcolor Blues 8 7;
  dotlabel.urt("$\bigl(\sqrt{\scriptstyle P}, \sqrt{\scriptstyle P}\bigr)$", 1/2(x3, y4));
  dotlabel.bot("$\bigl(2\sqrt{\scriptstyle P}, 0\bigr)$", z3);
  dotlabel.lft("$\bigl(0, 2\sqrt{\scriptstyle P}\bigr)$", z4);

  label.top(btex \vbox{\hsize 3.7in\centering For a given product,
    the sum of two positive numbers is minimal when the numbers are
    equal.} etex, point 5 /2 of bbox currentpicture shifted 13 up));
T2 = image(
  fill unitsquare xscaled x0 yscaled y0 withcolor Reds 7 1;
  fill unitsquare scaled 1/2(x0+y0) withcolor Blues 7 1;
  fill unitsquare xscaled 1/2(x0+y0) yscaled y0 withcolor 1/2[Reds 7 1, Blues 7 1];
  draw unitsquare xscaled x0 yscaled y0 withpen pencircle scaled 1/4;
  draw unitsquare scaled 1/2(x0+y0) withpen pencircle scaled 1/4;

  picture eq; eq = image(
    for t=-1/2, 1/2:
      draw (up--down) rotated -5 scaled 2 shifted (t, 0) withpen pencircle scaled 1/4;
    endfor
  );
  draw eq shifted (1/4x0+3/4y0, y0);
  draw eq shifted (3/4x0+1/4y0, y0);
  draw eq rotated 90 shifted (1/2x0+1/2y0, 1/4x0+3/4y0);

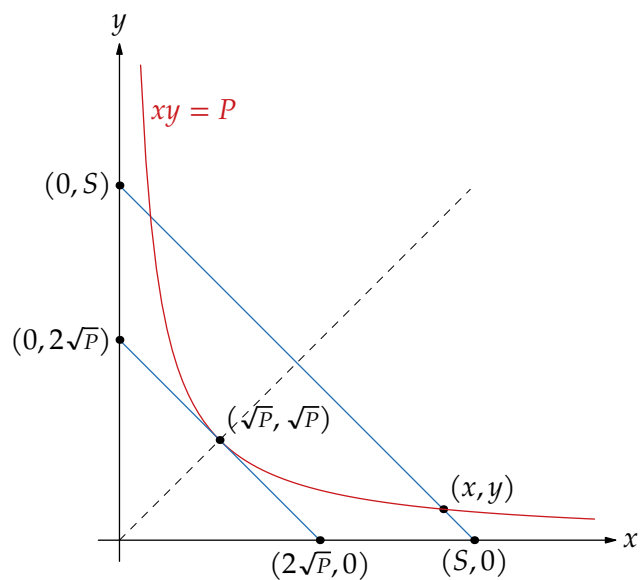
  draw T0;
  dotlabel.urt("$\frac{1}{2}\bigl(S, S\bigr)$", 1/2[z1,z2]);
  label.top(btex \vbox{\hsize 3.7in\centering For a given sum,
    the product of two positive numbers is maximal when the numbers are
    equal.} etex, point 5 /2 of bbox currentpicture shifted 13 up));

label.top(T1, 9 up); label.bot(T2, 9 down);

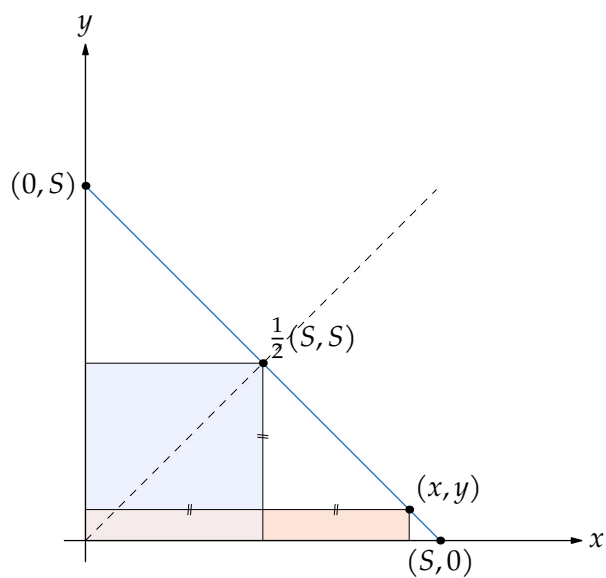
```


Two extremum problems

For a given product, the sum of two positive numbers is minimal when the numbers are equal.



For a given sum, the product of two positive numbers is maximal when the numbers are equal.



— Paulo Montuchi and Warren Page

Inequalities

```

path c; c = fullcircle scaled 233;
pair A, G, H, M, P, Q, R;
A = center c;
P = point 4 of c;
Q = point 6 of c;
R = point 8 of c;
M - P = (tw-18, 0);
G = c intersectionpoint halfcircle scaled abs(M-A) shifted 1/2[A, M];
H = (xpart G, ypart A);

drawoptions(withcolor 3/4);
draw unitsquare scaled 8 shifted H;
draw unitsquare scaled 8 rotated angle (Q-A) shifted A;
draw unitsquare scaled 8 rotated angle (A-G) shifted G;

drawoptions(withcolor Blues 8 4);
draw c;

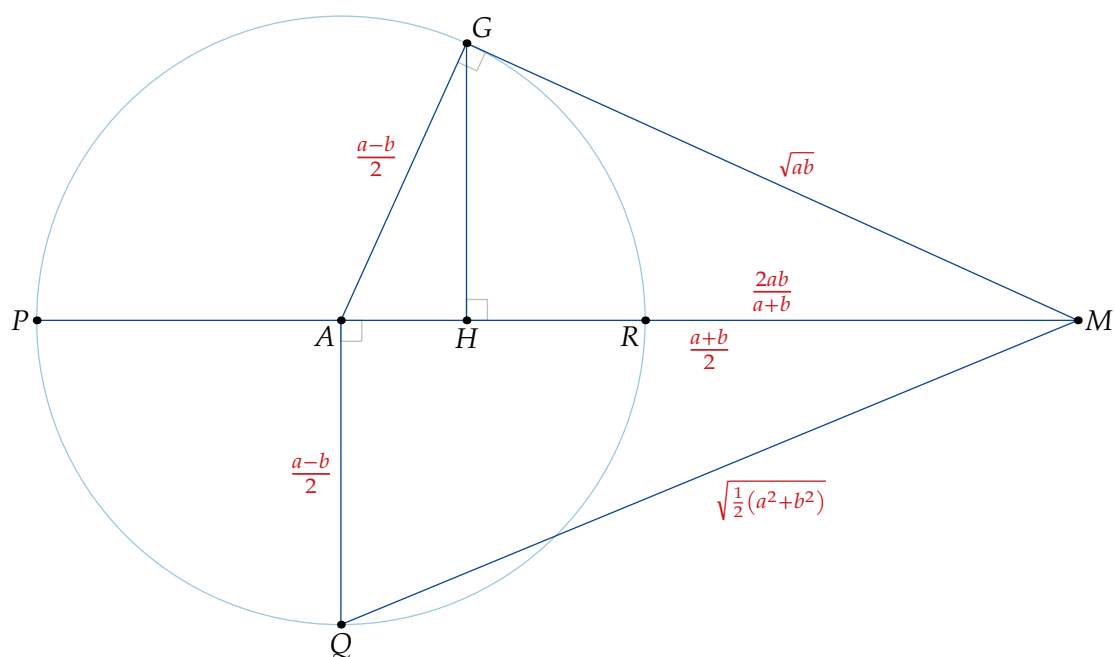
drawoptions(withcolor Blues 8 8);
draw P -- (M -- G -- A -- Q -- cycle);
draw G -- H;

drawoptions(withcolor Reds 8 7);
label.ulft("$\frac{a-b}{2}$", 1/2[A, G]);
label.lft("$\frac{a-b}{2}$", 1/2[A, Q]);
label.bot("$\frac{a+b}{2}$", 1/2[A, M]);
label.top("$\frac{2ab}{a+b}$", 1/2[H, M]);
label.urt("$\scriptstyle\sqrt{ab}$", 1/2[G, M]);
label.lrt("$\scriptstyle\sqrt{\frac{1}{2}\left(a^2+b^2\right)}$", 1/2[Q, M]);
drawoptions();
dotlabel.llft("$A$", A);
dotlabel.llft("$R$", R);
dotlabel.lft("$P$", P);
dotlabel.rt("$M$", M);
dotlabel.bot("$Q$", Q);
dotlabel.bot("$H$", H);
dotlabel.urt("$G$", G);

label.bot("$PM=a$, \quad $RM=b$, \quad $a>b>0$",
point 1/2 of bbox currentpicture shifted 21 down);
label.bot("$HM < GM < AM < QM$",
point 1/2 of bbox currentpicture shifted 13 down);
label.bot("$\displaystyle\frac{2ab}{a+b}<\sqrt{ab}$" &
"<\frac{a+b}{2}<\sqrt{\frac{1}{2}\left(a^2+b^2\right)}$",
point 1/2 of bbox currentpicture shifted 13 down);

```

The HM–GM–AM–QM inequalities I



$$PM = a, \quad RM = b, \quad a > b > 0$$

$$HM < GM < AM < QM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{1}{2}(a^2+b^2)}$$

— Roger B. Nelsen

Inequalities

```

pair A, B, C, D, E, F, G;
D = origin; A = - C;
C - A = (tw, 0);
B = 7/26[A, C];
E = (B -- B shifted 400 up) intersectionpoint halfcircle scaled abs(A-C);
F - B = whatever * (E - D);
F - E = whatever * (E - D) rotated 90;
G = E - B rotated angle (E - F);

draw halfcircle scaled abs (A-C) withcolor 7/8;

drawoptions(withcolor Blues 8 4);
draw subpath (1, 3) of unitssquare scaled 8 rotated angle (D-B) shifted B;
draw subpath (1, 3) of unitssquare scaled 8 rotated angle (B-F) shifted F;
draw subpath (1, 3) of unitssquare scaled 8 rotated angle (D-E) shifted E;

drawoptions(withcolor Blues 8 7);
draw A -- C;
draw B -- E -- D;
draw B -- F -- G -- D;

drawoptions(withcolor Reds 8 7);

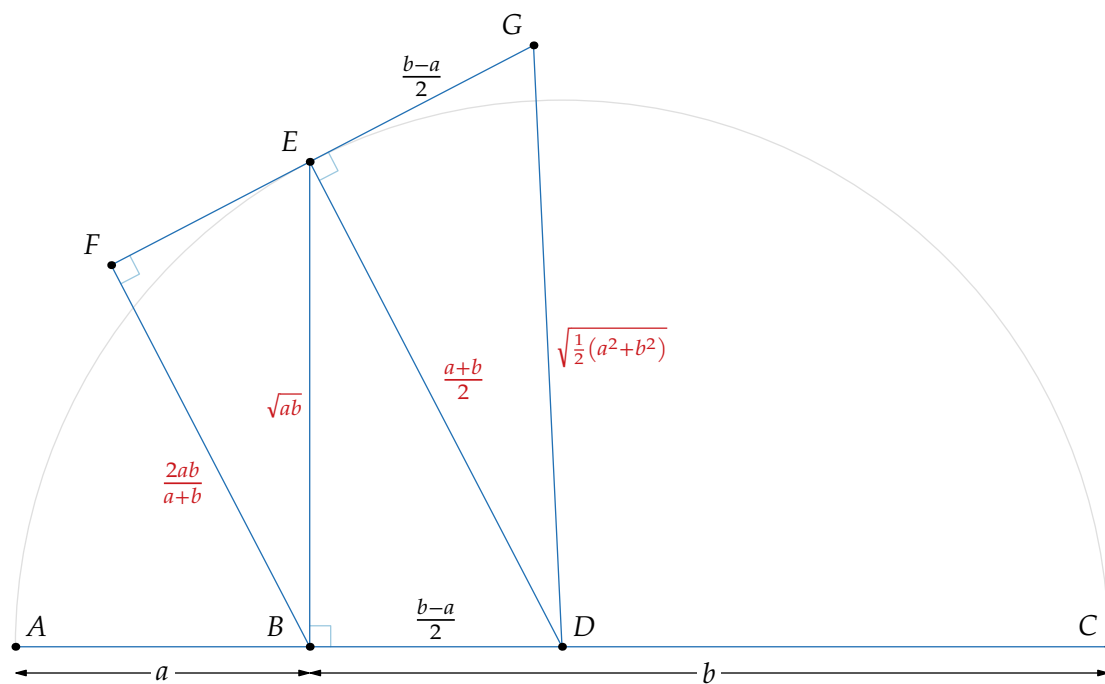
label.llft("$\frac{2ab}{a+b}$", 1/2[B, F]);
label.lft("$\scriptstyle\sqrt{ab}$", 1/2[B, E]);
label.urt("$\frac{a+b}{2}$", 1/2[D, E]);
label.rt ("$\scriptstyle\sqrt{\frac{1}{2}\left(a^2+b^2\right)}$", 1/2[D, G]);

drawoptions();
input arrow_label;
arrow_label(A, B, "$a$", 10);
arrow_label(B, C, "$b$", 10);
label.top("$\frac{b-a}{2}$", 1/2[B, D]);
label.top("$\frac{b-a}{2}$", 1/2[E, G]);
interim labeloffset := 6;
dotlabel.urt("$A$", A);
dotlabel.ulft("$B$\enspace", B);
dotlabel.ulft("$C$", C);
dotlabel.urt("$D$", D);
dotlabel.ulft("$E$", E);
dotlabel.ulft("$F$", F);
dotlabel.ulft("$G$", G);

label.bot("$AB=a$, \quad $BC=b$, \quad $AD=DC=\frac{a+b}{2}$",
point 1/2 of bbox currentpicture shifted 42 down);
label.bot("$BE \perp AB$, \quad $DE=AD$",
point 1/2 of bbox currentpicture shifted 13 down);
label.bot("$FE \perp ED$, \quad $FB \parallel ED$, \quad $EG=BD=\frac{b-a}{2}$",
point 1/2 of bbox currentpicture shifted 13 down);

```

The HM–GM–AM–QM inequalities II



$$AB = a, \quad BC = b, \quad AD = DC = \frac{a+b}{2}$$

$$BE \perp AB, \quad DE = AD$$

$$FE \perp ED, \quad FB \parallel ED, \quad EG = BD = \frac{b-a}{2}$$

— Sidney H. Kung

```

numeric u; u = 20;
path U, A, B;
U = unitsquare shifted -(1/2, 1/2) scaled u;
A = unitsquare scaled 3u shifted point 2 of U rotated 90;
B = unitsquare scaled 4u shifted point 0 of U;

picture P[];
P1 = image(
    for t=0, 180:
        fill A rotated t withcolor Greens 8 1;
        fill B rotated t withcolor Reds 8 1;
    endfor
    fill U withcolor Reds 8 2;
    for t=0, 180:
        draw subpath (1, 3) of A rotated t;
        draw B rotated t;
    endfor

    label.top("$a$", point 3/2 of A);
    label.lft("$a$", point 5/2 of A);
    label.top("$b$", point 5/2 of B);
    label.lft("$b$", point 3/2 of B rotated 180);

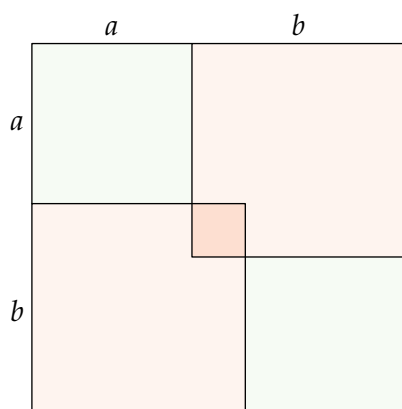
    label.rt(btex
        \vbox{\openup 12pt\halign{\hbox to 64pt{\hfil$#$}&${}\ge #$\hfil\cr
        2a^2 + 2b^2 & \left(a+b\right)^2\cr
        \sqrt{\frac{12\left(a^2+b^2\right)}{}} & \displaystyle \frac{a+b}{2}\cr}}
        etex, point 3/2 of bbox currentpicture shifted 34 right);
);

% ... and similar for P2, P3

draw P1;
draw P2 shifted (9u * down);
draw P3 shifted (18u * down);

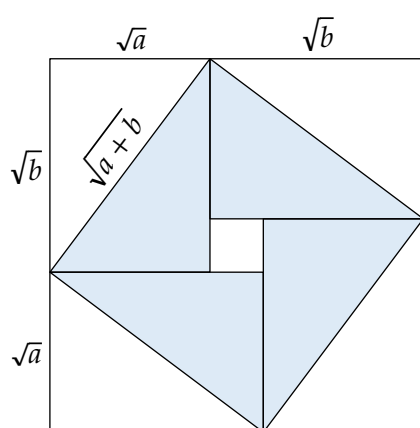
```

The HM–GM–AM–QM inequalities III



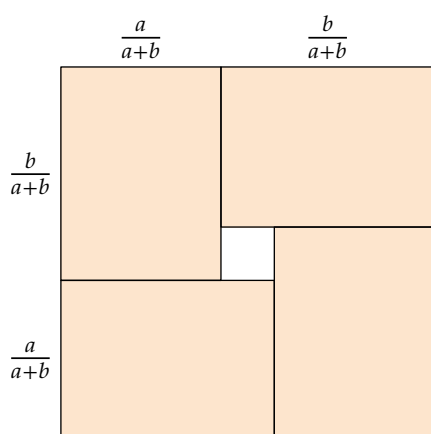
$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\frac{1}{2}(a^2 + b^2)} \geq \frac{a+b}{2}$$



$$(\sqrt{a+b})^2 \geq 4 \cdot \frac{1}{2} \sqrt{a} \sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$



$$1 \geq 4 \cdot \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

— Roger B. Nelsen

```

numeric a, b, am, gm, rms, hm, ch; a + b + 32 = tw; b = 4a;
am = 1/2(a+b); gm = sqrt(a*b); rms = 1/2 sqrt 2 * (a++b);
hm = 2a / (a+b) * b; ch = a / (a+b) * a + b / (a+b) * b;
path xx, yy, qq, ss, pp, arc, harc, hpp;
xx = 10 left -- (a+b+20) * right; qq = xx rotated 45; yy = xx rotated 90;
ss = (-10, a + b + 10) -- (a + b + 10, -10);
arc = quartercircle scaled 2 (a++b);
harc = subpath(1/2,3/2) of quartercircle scaled (sqrt(2) * (a+b));

numeric s, ix; ix = sqrt(a*b); s = 4;
pp = (ix, ix) for x=s+ix step s until a+b-8: -- (x, b/x*a) endfor;
pp := reverse pp reflectedabout(origin, point 1 of qq) & pp;
numeric s, ix; ix = 1/2 sqrt(2) * (a++b); s = 4;
hpp = (ix, ix) for x=s+ix step s until b+eps: -- (x, a / 2x * a + b / 2x * b) endfor;
hpp := reverse hpp reflectedabout(origin, point 1 of qq) & hpp;

drawoptions(withcolor Greens 8 7);
draw pp; label.rt("$xy=ab$", point infinity of pp);
draw hpp; label.rt("$2xy=a^2+b^2$", point infinity of hpp);

drawoptions(withcolor Oranges 8 7);
draw arc; draw harc; begingroup; interim ahlength := 2;
z0 = 1/3[point infinity of pp, point infinity of hpp];
z1 = 2/3[point infinity of pp, point infinity of hpp];
path aa, bb;
aa = (z0 -- point 0 of harc) cutafter fullcircle scaled 4 shifted point 0 of harc;
numeric t, u; (t, u) = arc intersectiontimes (aa shifted (z1-z0));
bb = (z1 -- point t of arc) cutafter fullcircle scaled 4 shifted point t of arc;

drawarrow aa withpen pencircle scaled 1/4 withcolor Oranges 8 5;
drawarrow bb withpen pencircle scaled 1/4 withcolor Oranges 8 5;

label.rt("$2\left(x^2 + y^2\right) = \left(a+b\right)^2$", z0);
label.rt("$x^2 + y^2 = a^2 + b^2$", z1);
endgroup;

drawoptions(withcolor Blues 8 7);
draw qq; draw thelabel.top("$x=y$", origin) rotated 45 shifted point 0.9 of qq;
draw ss; draw thelabel.top("$x+y=a+b$", origin) rotated -45 shifted point 0.9 of ss;

drawoptions(dashed evenly scaled 1/2);
draw (hm, ch) -- (am, ch) -- (am, am) -- (ch, am) -- (ch, hm);
draw (gm, rms) -- (gm, gm) -- (rms, gm);

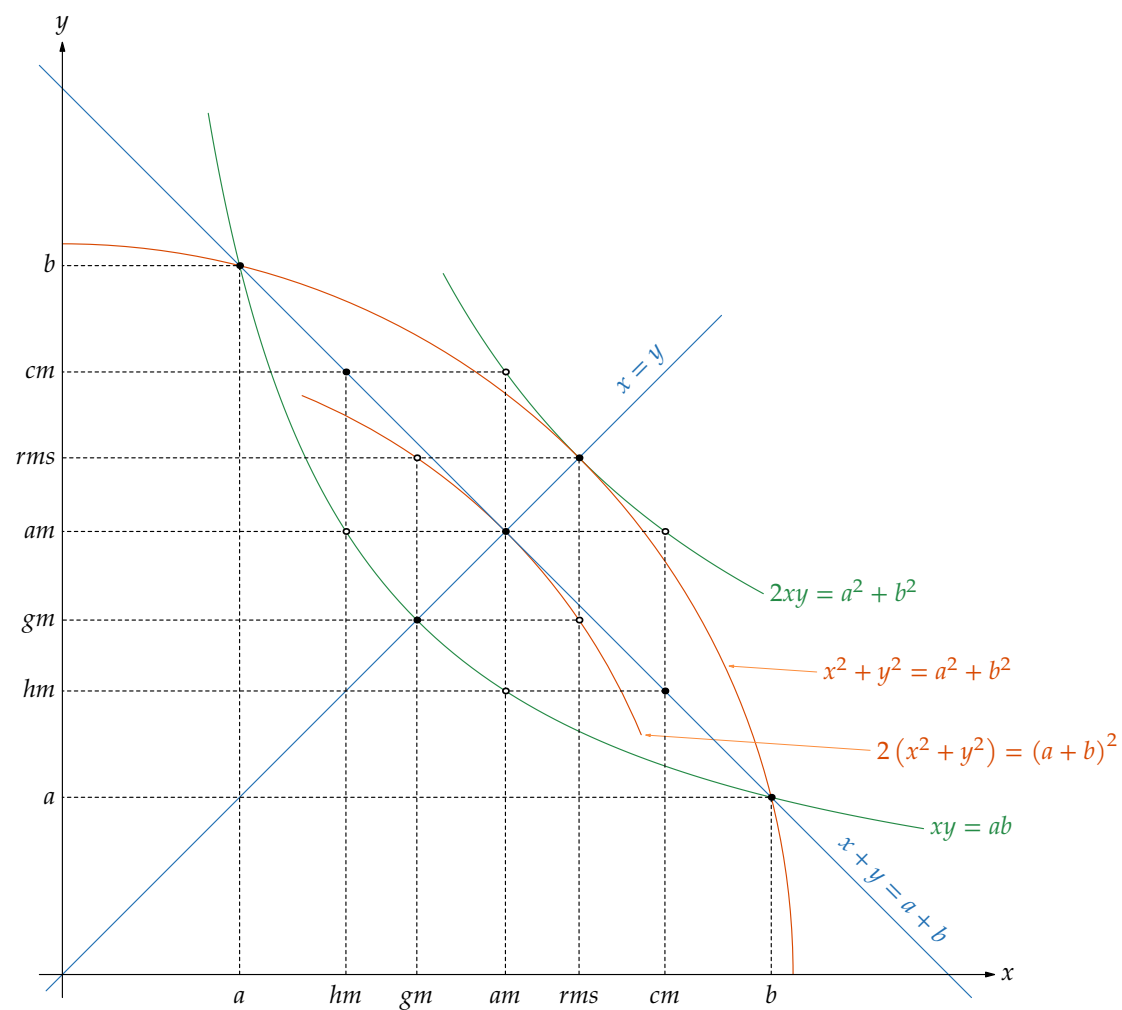
def connect(expr p, q, P, Q) =
  draw (p, 0) -- (p, q) -- (0, q) dashed evenly scaled 1/2;
  label.bot("\strut" & P, (p, 0)); label.lft("\strut" & Q, (0, q));
  draw (p, q) withpen pencircle scaled dotlabeldiam;
enddef;

drawoptions();
connect(a, b, "$a$", "$b$"); connect(am, am, "$am$", "$am$");
connect(b, a, "$b$", "$a$"); connect(gm, gm, "$gm$", "$gm$");
connect(rms, rms, "$rms$", "$rms$");
connect(ch, hm, "$cm$", "$hm$"); connect(hm, ch, "$hm$", "$cm$");

for p = (hm, am), (gm, rms), (am, ch), (am, hm), (rms, gm), (ch, am):
  draw p withpen pencircle scaled dotlabeldiam;
  undraw p withpen pencircle scaled 1/2 dotlabeldiam;
endfor

```


Five means — and their means



— Roger B. Nelsen

```

path ff, xx, yy;
def f(expr x) = 1/256 mlog(x) / x enddef;
numeric minx, maxx, s, u, v;
minx = 13/8; s = 1/16; maxx = 19/4;
u = 89;
v = 3328-256;
ff = ((minx, f(minx)) for x=minx+s step s until maxx:
    .. (x, f(x))
endfor) xscaled u yscaled v;
xx = (point 0 of ff -- (xpart point infinity of ff, ypart point 0 of ff)) shifted 30 down;
yy = (point 0 of ff -- point 0 of ff shifted (0, 0.1v)) shifted 30 left;

numeric pi, e, fpi, fe;
pi = 3.141592653589793 u; fpi = f(3.141592653589793) * v;
e = 2.718281828459045 u; fe = f(2.718281828459045) * v;

path ee, pp;
ee = (e, ypart point 0 of xx) -- (e, fe) -- (xpart point 0 of yy, fe);
pp = (pi, ypart point 0 of xx) -- (pi, fpi) -- (xpart point 0 of yy, fpi);

draw ee withcolor Reds 8 4;
draw pp withcolor Oranges 8 4;
draw ff withcolor Blues 8 7;

drawarrow xx;
drawarrow yy;

for x=2 upto 4:
    draw (down--up) scaled 2 shifted (x * u, ypart point 0 of xx);
    label.bot("$" & decimal x & "$", (x * u, ypart point 0 of xx - 2));
endfor
for y=32, 35, 38,:
    draw (left--right) scaled 2 shifted (xpart point 0 of yy, y/100 * v);
    label.lft("$" & decimal (y/100) & "$", (xpart point 0 of yy - 2, y/100 * v));
endfor

drawoptions(withcolor Reds 8 7);
label.bot("$e$", point 0 of ee shifted 4 down);
label.lft("$1/e$", point 2 of ee shifted 2 left);

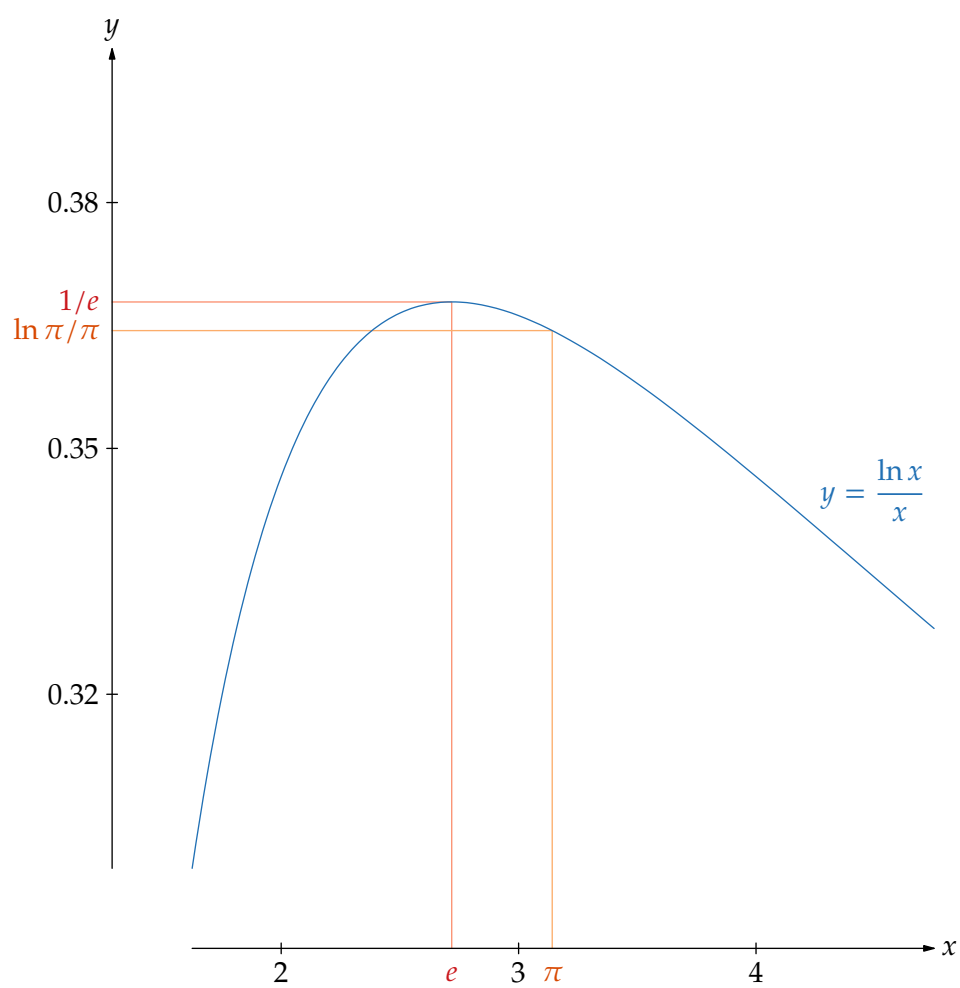
drawoptions(withcolor Oranges 8 7);
label.bot("$\pi$", point 0 of pp shifted 4 down);
label.lft("$\ln\pi/\pi$", point 2 of pp shifted 2 left);

drawoptions(withcolor Blues 8 7);
label.urt("$\displaystyle y=\frac{\ln x}{x}$", point 42 of ff);

drawoptions();
label.rt("$x$", point 1 of xx);
label.top("$y$", point 1 of yy);

```

$$e^\pi > \pi^e$$



— Fouad Nakhli

Inequalities

```

path ff, xx, yy;
def f(expr x) = 1/256 mlog(x) enddef;
numeric minx, maxx, s, u, v, A, B, e;
minx = 1/4; s = 1/4; maxx = 12;
u = (tw-40)/maxx;
v = 89;
ff = ((minx, f(minx)) for x=minx+s step s until maxx:
    .. (x, f(x))
    endfor) xscaled u yscaled v;
xx = 24 left -- (xpart point infinity of ff + 10, 0);
yy = (0, ypart point 0 of ff) -- (0, ypart point infinity of ff + 10);
A = 1/2 maxx;
B = 3/4 maxx;
e = 2.718281828459;
primarydef o through p =
    (1+o/arclength(p))[point 1 of p, point 0 of p] --
    (1+o/arclength(p))[point 0 of p, point 1 of p]
enddef;

forsuffixes $=e, A, B:
    z$ = ($ * u, f($) * v);
    draw ($ * u, 0) -- z$;
    path p; p = 32 through (origin -- z$);
    draw p withcolor Reds 8 7;
    if not (str $ = "e"):
        draw thelabel.rtl("$y=m_" & str $ & "x$", origin)
            rotated angle z$ shifted point 1 of p withcolor Reds 8 8;
    fi
endfor

drawoptions(withcolor Blues 8 7);
draw ff withpen pencircle scaled 3/4;
label.rtl("$y=\ln x$", point 1/2 of ff shifted 4 right);

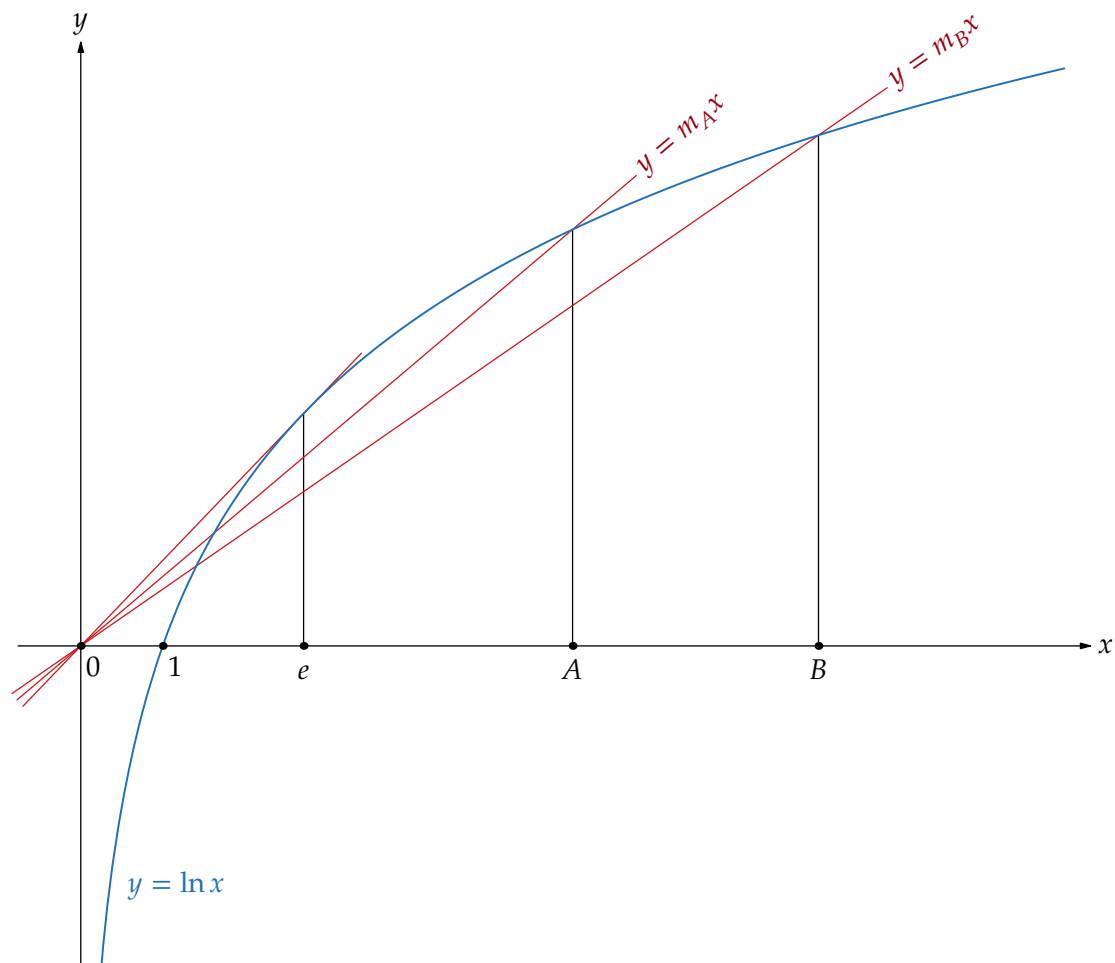
drawoptions();
drawarrow xx;
drawarrow yy;
label.rtl("$x$", point 1 of xx);
label.top("$y$", point 1 of yy);

dotlabel.lrt("\strut $0$", (0, 0));
dotlabel.lrt("\strut $1$", (u, 0));
dotlabel.bot("\strut $e$", (e*u, 0));
dotlabel.bot("\strut $A$", (A*u, 0));
dotlabel.bot("\strut $B$", (B*u, 0));

label.bot(btex \vbox{\openup 12pt\halign{\hfil $$$${\}\quad
    \mathbin{\Longrightarrow}\quad #$\cr
    e \le A < B \& m_A > m_B \cr
    & \frac{\ln A}{A} > \frac{\ln B}{B}\cr
    & A^B > B^A\cr
}} etex, point 1/2 of bbox currentpicture shifted 42 down);

```

$A^B > B^A$ **for** $e \leq A < B$



$$\begin{aligned}
 e \leq A < B &\implies m_A > m_B \\
 &\implies \frac{\ln A}{A} > \frac{\ln B}{B} \\
 &\implies A^B > B^A
 \end{aligned}$$

— Charles D. Gallant

```

numeric a, b, c, d;
b + d + 16 = tw;
b = 2a;
a = 3/4 c = d;

path t[];
t1 = origin -- (b, 0) -- (b, a) -- cycle;
t2 = (origin -- (d, 0) -- (d, c) -- cycle) shifted point 2 of t1;
t3 = origin -- (b+d, 0) -- (b+d, a+c) -- cycle;

fill t1 withcolor Blues 8 1;
fill t2 withcolor Oranges 8 1;

forsuffixes $=1,2,3:
    draw subpath (1,3) of unitsquare scaled 6 rotated 90
        shifted point 1 of t$ withcolor 3/4;
endfor

draw subpath(1, 3) of t1;
draw subpath(-1, 1) of t2;
draw t3;

label.rt("$a$", point 3/2 of t1);
label.rt("$c$", point 3/2 of t2);
label.rt("$a$", 1/2[point 1 of t3, point 1 of t2]);

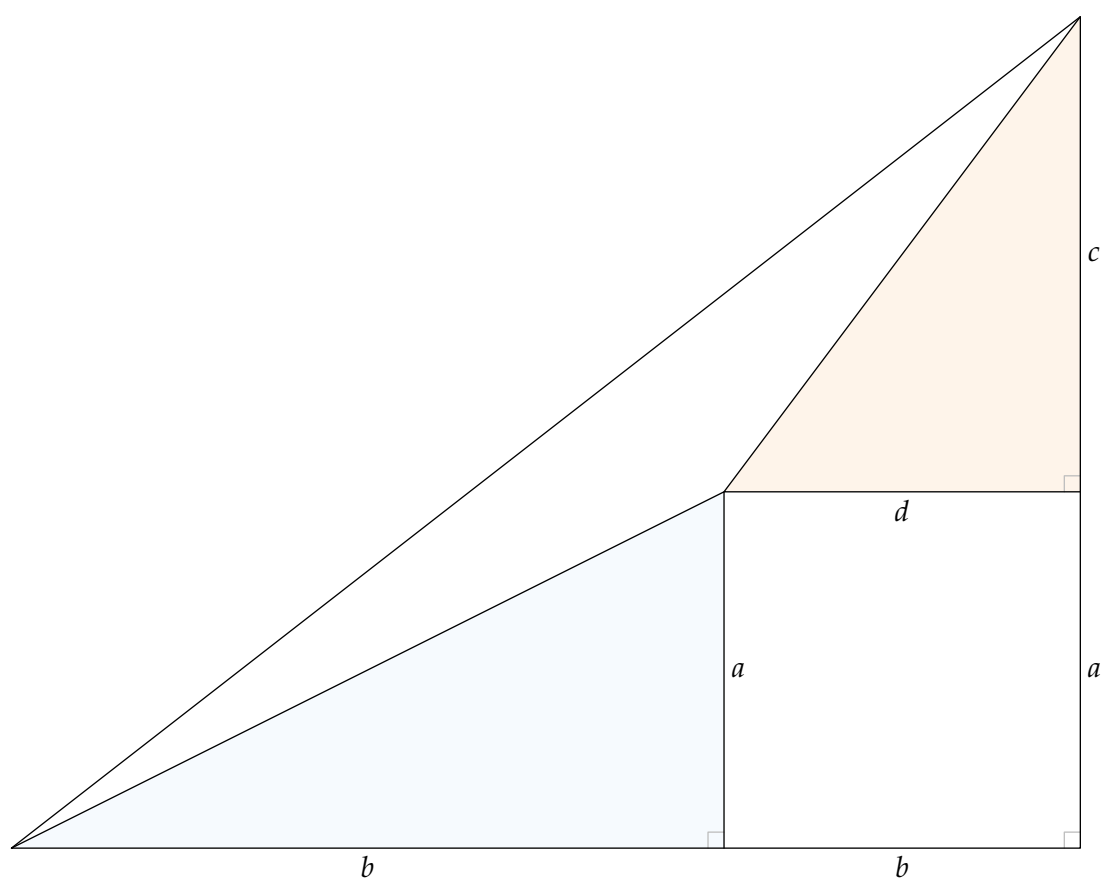
label.bot("$b$", point 1/2 of t1);
label.bot("$d$", point 1/2 of t2);
label.bot("$b$", 1/2[point 1 of t3, point 1 of t1]);

label.top(btex $\displaystyle \frac{a}{b} < \frac{c}{d} \quad \Rightarrow \quad
\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ etex,
point 5/2 of bbox currentpicture shifted 42 up);

```

The mediant property

$$\frac{a}{b} < \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$



— Richard A. Gibbs

```

path xx, yy;
xx = 12 left -- 233 right;
yy = 12 down -- 144 up;

z1 = .95(1/4[point 1 of xx, point 1 of yy]);
z3 = .95(3/4[point 1 of xx, point 1 of yy]);
z2 = 1/2[z1, z3];

drawoptions(withcolor Blues 8 7);
draw origin -- z1 -- z3 -- cycle;
draw origin -- z2;
picture m; m = image(draw (up--down) scaled 2 rotated -5 shifted 1/2 left;
                     draw (up--down) scaled 2 rotated -5 shifted 1/2 right);
draw m rotated angle (z3 - z1) shifted 1/2[z1, z2];
draw m rotated angle (z3 - z1) shifted 1/2[z2, z3];

draw thelabel.top("$y=m_1 x$", origin) rotated angle z1 shifted 3/4 z1;
draw thelabel.top("$y=m_2 x$", origin) rotated angle z2 shifted 3/4 z2;
draw thelabel.top("$y=m_3 x$", origin) rotated angle z3 shifted 3/4 z3;

drawoptions();
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);

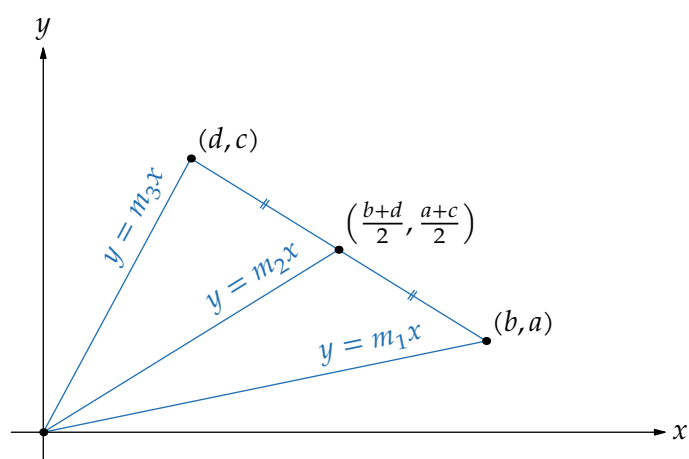
dotlabel.urt("$ (b,a) $", z1);
dotlabel.urt("$\left(\frac{b+d}{2}, \frac{a+c}{2}\right) $", z2);
dotlabel.urt("$ (d,c) $", z3);
drawdot origin withpen pencircle scaled dotlabeldiam;

label.top(btex $\displaystyle
a, b, c, d > 0; \quad \frac{a}{b} < \frac{c}{d}
\quad \Longrightarrow \quad
\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} $
etex, point 5/2 of bbox currentpicture shifted 13 up);
label.bot(btex $m_1 < m_3 \quad \Longrightarrow \quad m_1 < m_2 < m_3 $
etex, point 1/2 of bbox currentpicture shifted 13 down);

```


Regle des nombres moyens – I

$$a, b, c, d > 0; \quad \frac{a}{b} < \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$



$$m_1 < m_3 \quad \Rightarrow \quad m_1 < m_2 < m_3$$

— Li Changming

```

numeric a, b, c, d, u;
u = 89;
55a = 34b; b = d = 1;
64c = 61d;

path A, B, C, D, A', B';

A = unitsquare scaled u yscaled (a/b);
A' = unitsquare scaled u yscaled (c/d);
B = unitsquare scaled u yscaled (a/b) xscaled (b/(b+d))
    shifted point 1 of A shifted 42 right;
B' = A' shifted point 0 of B;
C = unitsquare scaled u yscaled (c/d) xscaled (d/(b+d))
    shifted point 1 of B;
D = A' shifted point 1 of C shifted 42 right;

draw A' dashed evenly scaled 1/2;
draw B' dashed evenly scaled 1/2;

fill A withcolor Oranges 8 2;
fill B withcolor Oranges 8 2;
fill C withcolor Blues 8 2;
fill D withcolor Blues 8 2;

draw A; draw B; draw C; draw D;

vardef superlabel(expr t, z) =
    interim bboxmargin := 6;
    save P; picture P; P = thelabel(t, origin);
    save s; path s; s = superellipse(point 3/2 of bbox P, point 5/2 of bbox P,
        point 7/2 of bbox P, point 1/2 of bbox P, 0.78);
    unfill s shifted z; draw P shifted z;
enddef;

superlabel("$\frac{a}{b}$", center A);
superlabel("$\frac{a}{b+d}$", center B);
superlabel("$\frac{c}{b+d}$", center C);
superlabel("$\frac{c}{d}$", center D);

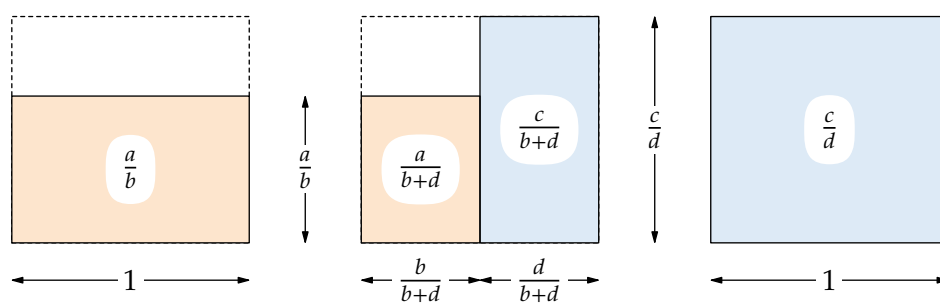
input arrow_label
arrow_label(point 0 of A, point 1 of A, "$1$", 14);
arrow_label(point 0 of B, point 1 of B, "$\frac{b}{b+d}$", 14);
arrow_label(point 0 of C, point 1 of C, "$\frac{d}{b+d}$", 14);
arrow_label(point 0 of D, point 1 of D, "$1$", 14);

arrow_label(1/2[point 1 of A, point 0 of B], 1/2[point 2 of A, point 3 of B],
    "$\frac{a}{b}$", 0);
arrow_label(1/2[point 1 of C, point 0 of D], 1/2[point 2 of C, point 3 of D],
    "$\frac{c}{d}$", 0);

label.bot(btex $\displaystyle
    \frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}
    $ etex, point 1/2 of bbox currentpicture shifted 21 down);

```

Regle des nombres moyens – II



$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

— Roger B. Nelsen

input superlabel

Inequalities

picture P[];

```

P1 = image(numeric x; x = 89/72; path box[];
  box1 = unitsquare scaled 72 xscaled x yscaled (1/x);
  box2 = box1 rotated 90; box2 := box2 shifted (point 1 of box1 - point 3 of box2);
  box3 = box1 rotated 180; box3 := box3 shifted (point 1 of box2 - point 3 of box3);
  box4 = box1 rotated 270; box4 := box4 shifted (point 1 of box3 - point 3 of box4);

  forsuffices $=1,2,3,4:
    fill box$ withcolor if odd $: Oranges else: Blues fi 8 2;
    superlabel("$1$", center box$);
    draw subpath (-2, 1) of box$;
  endfor
  label.bot("\strut$x$", point 1/2 of box1); label.bot("\strut$\frac{1}{x}$", point -1/2 of box2);
  label.lft("$x$", point 1/2 of box4); label.lft("$\frac{1}{x}$", point -1/2 of box1);
  label.top("I.", point 3 of bbox currentpicture shifted 3.25 up));
P2 = image(path xx, yy; xx = 8 left -- 150 right; yy = xx rotated 90;
  numeric u; u = 100/3;
  for i=1 upto 4:
    draw (left--right) shifted (0, i*u);
    draw (up--down) shifted (i*u, 0);
  endfor
  drawarrow xx; label.rt("$x$", point 1 of xx);
  drawarrow yy; label.top("$y$", point 1 of yy);
  path a, b;
  a = ((-24/100, 224/100) -- (224/100, -24/100)) scaled u;
  b = ((1,1) for x=1+1/8 step 1/8 until 4: -- (x, 1/x) endfor) scaled u;
  b := reverse b reflectedabout(origin, point 0 of b) & b;

  draw b withcolor Oranges 8 7;
  draw a withcolor Blues 8 7;
  label.urt("$y=\frac{1}{x}$", point 40 of b) withcolor Oranges 8 7;
  label.lrt("$y=2-x$", point 1 of a shifted 3 up) withcolor Blues 8 7;
  label.llft("II.", point 3 of bbox currentpicture));
numeric x, u; x = 7/4; u = 80;
z1 = (0, x - 1/x) scaled u; z2 = (-2, 0) scaled u; z3 = -z2;
z4 = origin rotatedabout(z1, 90); z5 = whatever[z1, z3]; x5 = x4;

P3 = image(draw unitsquare scaled 8 rotated 90 withcolor 1/2;
  draw origin -- z1 -- z2 -- cycle;
  label.bot("$2$", 1/2 z2); label.rt("$x-\frac{1}{x}$", 1/2z1);
  label.ulft("$x+\frac{1}{x}$", 1/2[z1, z2]);
  label("III.", point 3 of bbox currentpicture));
P4 = image( fill origin -- z1 -- z3 -- cycle withcolor Oranges 8 2;
  fill z4 -- z1 -- z5 -- cycle withcolor Blues 8 2;
  draw z1 -- origin -- z3 -- z1 -- z4 -- z5;
  draw z5 -- (x5, 0) dashed withdots scaled 1/2;

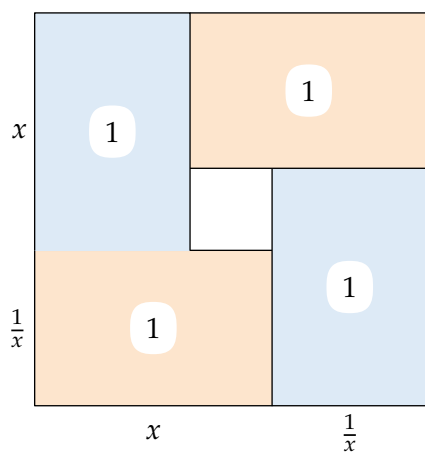
  label.bot("$x$", 1/2 z3); label.top("$1$", 1/2[z1, z4]);
  label.lft("$1$", 1/2 z1); label.rt("$\frac{1}{x}$", 1/2[z4, z5]);
  label("IV.", point 3 of bbox currentpicture shifted 4 left));

draw P1 shifted 180 left; draw P2 shifted 32 right;
draw P3 shifted (-32, -180); draw P4 shifted (+32, -180);

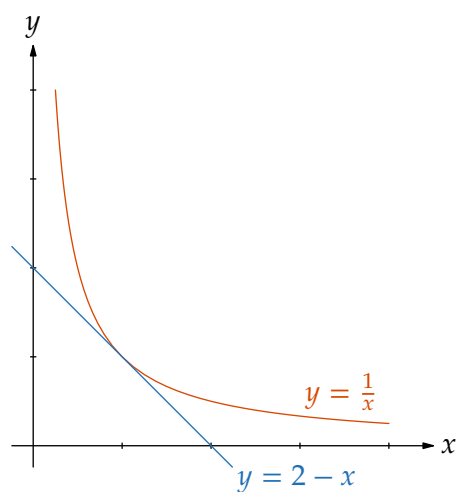
```

The sum of a positive number and its reciprocal is at least two

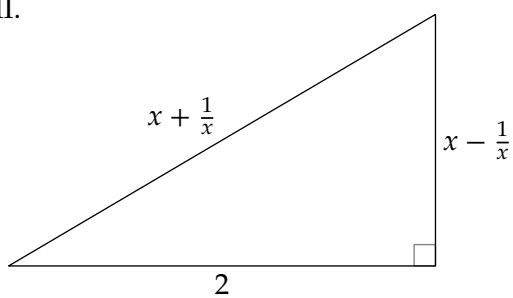
I.



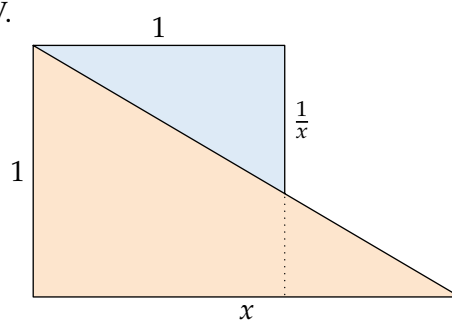
II.



III.



IV.



— Roger B. Nelsen

```

numeric u, v, a, b, halfpi; u = 160; v = 89; a = 1.2; b = 0.8a; halfpi = 1.570796;

vardef sin(expr x) = sind(57.2957795 x) enddef;
vardef cos(expr x) = cosd(57.2957795 x) enddef;
vardef tan(expr x) = sin(x) / cos(x) enddef;

numeric ma; ma = a + 1/8;
path ss, tt, sbb, tbb;
ss = origin for x=1/32 step 1/32 until ma+eps: -- (x * u, sin(x) * v) endfor;
tt = origin for x=1/32 step 1/32 until a+1/32: -- (x * u, tan(x) * v) endfor;
sbb = origin -- (ma * u, sin(b) / b * ma * v);
tbb = origin -- (ma * u, tan(b) / b * ma * v);

draw ss withcolor Blues 8 7; draw sbb withcolor Blues 8 5;
draw tt withcolor Oranges 8 7; draw tbb withcolor Oranges 8 5;

for $=a,b:
  draw ($*u, 0) -- ($*u, tan($)*v) dashed evenly scaled 1/2 withpen pencircle scaled 1/4;
  draw ($*u, sin($)*v) withpen pencircle scaled dotlabeldiam;
  draw ($*u, tan($)*v) withpen pencircle scaled dotlabeldiam;
endfor
draw (a*u, sin(b)/b*a*v) withpen pencircle scaled dotlabeldiam;
draw (a*u, tan(b)/b*a*v) withpen pencircle scaled dotlabeldiam;

path xx, yy; xx = 12 left -- 12 right shifted (halfpi * u, 0); yy = xx rotated 90;
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);

label.rt("$y=\sin x$", point infinity of ss shifted down);
label.top("$y=\tan x$", point infinity of tt);

label.urt("$y=\frac{\sin \beta}{\beta}x$", point infinity of sbb shifted 4 down);
label.urt("$y=\frac{\tan \beta}{\beta}x$", point infinity of tbb shifted 3 down);

vardef hbarlabel@#(expr t, z) =
  draw (left--right) scaled 3/2 shifted z;
  interim labeloffset := 5; label@#(t, z);
enddef;
vardef vbarlabel@#(expr t, z) =
  draw (down--up) scaled 3/2 shifted z;
  interim labeloffset := 5; label@#(t, z);
enddef;

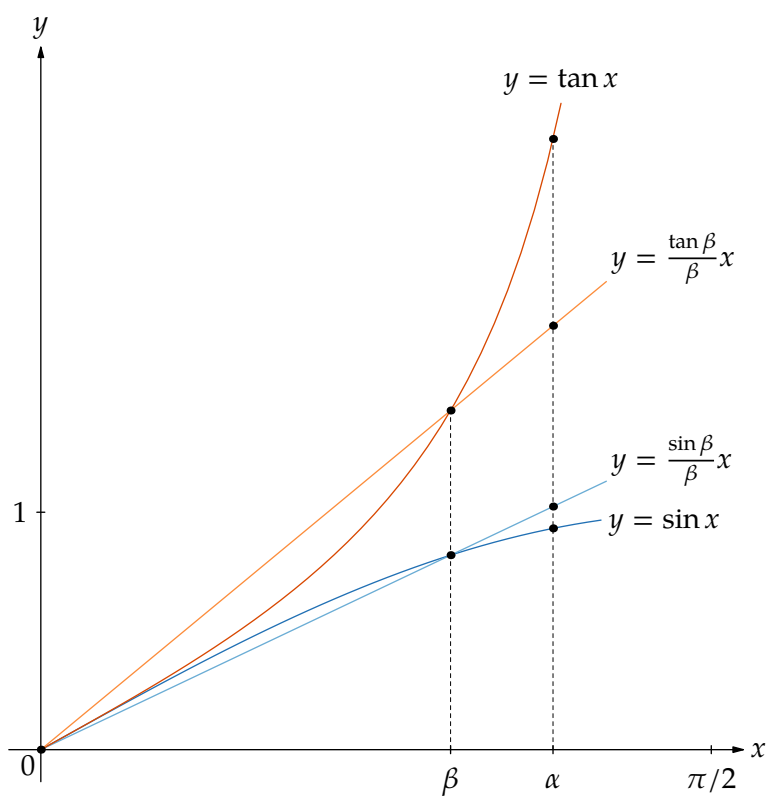
hbarlabel.lft("$1$", (0, v));
dotlabel.llft("$0$", origin);
vbarlabel.bot("\strut $\beta$", (b * u, 0));
vbarlabel.bot("\strut $\alpha$", (a * u, 0));
vbarlabel.bot("\strut $\pi/2$", (halfpi * u, 0));

label.top(btex $\displaystyle
0 < \beta < \alpha < \frac{\pi}{2} \implies
\frac{\sin \alpha}{\alpha} < \frac{\sin \beta}{\beta} < \frac{\alpha}{\tan \alpha} < \frac{\tan \alpha}{\alpha} < \frac{\tan \beta}{\beta}
$ etex, point 5/2 of bbox currentpicture shifted 42 up);
label.bot(btex $\displaystyle
\sin \alpha < \frac{\sin \beta}{\beta} \alpha, \quad
\tan \alpha > \frac{\tan \beta}{\beta} \alpha $ etex,
point 1/2 of bbox currentpicture shifted 42 down);
label.bot(btex $\displaystyle \therefore \quad
\frac{\sin \alpha}{\alpha} < \frac{\sin \beta}{\beta} < \frac{\alpha}{\tan \alpha} < \frac{\tan \alpha}{\alpha} < \frac{\tan \beta}{\beta}
$ etex, point 1/2 of bbox currentpicture shifted 8 down);

```

Aristarchus' inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \Rightarrow \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$



$$\sin \alpha < \frac{\sin \beta}{\beta} \alpha, \quad \tan \alpha > \frac{\tan \beta}{\beta} \alpha$$

$$\therefore \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

— Roger B. Nelsen

```

path t[]; z0 = 89 dir 280; z1 = 75 dir 200;
t1 = (x1, y0) -- z0 -- z1 -- cycle; Inequalities
t2 = (xpart point 2 of (t1 rotated 180), ypart point 1 of t1)
    -- point 2 of (t1 rotated 180) -- point 1 of t1 -- cycle;

picture P[];

for i=1,2:
  if i = 2:
    t1 := t1 rotated - angle (point 1 of t1 - point 2 of t1);
    t1 := t1 shifted - point 3/2 of t1;
    t1 := t1 shifted - (0, 1/2 abs (point 1 of t2 - point 2 of t2));

    t2 := t2 rotated (90 - angle (point 1 of t2 - point 2 of t2));
    t2 := t2 shifted (point 1 of t1 - point 2 of t2);
  fi

  P[i] = image(
    if i = 2:
      draw unitsquare scaled 6 shifted point 2 of t1;
      label.top("$\sqrt{a^2+b^2}$", point 3/2 of t1);
      label.lft("$\sqrt{x^2+y^2}$", point 3/2 of t2);
    fi
    forsuffices r=0, 180:
      fill t1 rotated r withcolor Blues 8 2;
      fill t2 rotated r withcolor Oranges 8 2;
      draw unitsquare scaled 6 rotated angle (point 1 of t1 - point 0 of t1)
          shifted point 0 of t1 rotated r;
      draw unitsquare scaled 6 rotated angle (point 1 of t2 - point 0 of t2)
          shifted point 0 of t2 rotated r;
      draw t1 rotated r; draw t2 rotated r;
      if i=1:
        label("$|a|$", point 3/7 of t1 shifted 8 down rotated r);
        label("$|b|$", point -3/7 of t1 shifted 8 left rotated r);
        label("$|x|$", point 3/7 of t2 shifted 8 right rotated r);
        label("$|y|$", point -3/7 of t2 shifted 8 down rotated r);
      fi
    endfor
  );
endfor

label.lft(P1, 12 left); label("$\le$", origin); label.rt(P2, 12 right);

label.top(btex $
\left| \langle a,b \rangle \cdot \langle x,y \rangle \right| \le
\left| \langle a,b \rangle \right| \left| \langle x,y \rangle \right|
$ etex, point 5/2 of bbox currentpicture shifted 42 up);

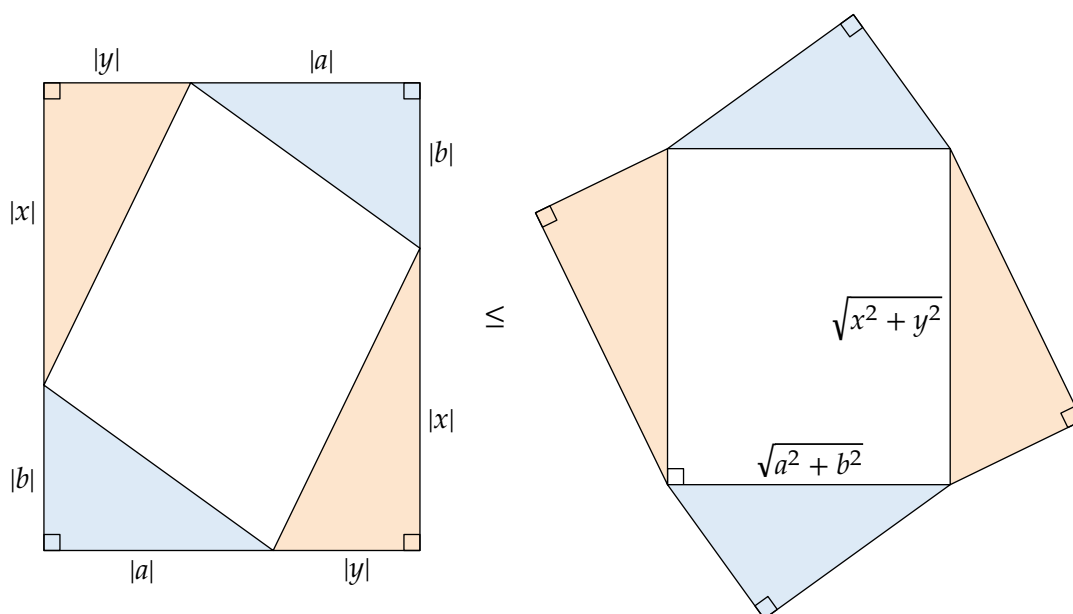
label.bot(btex $
\left( |a|+|y| \right) \left( |b|+|x| \right) \le
2 \left( \frac{1}{2} |a||b| + \frac{1}{2} |x||y| \right) + \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}
$ etex, point 1/2 of bbox currentpicture shifted 21 down);

label.bot(btex $\therefore \quad
\left| ax+by \right| \le |a||x|+|b||y| \le \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}
$ etex, point 1/2 of bbox currentpicture shifted 16 down);

```


The Cauchy-Schwartz inequality

$$|\langle a, b \rangle \cdot \langle x, y \rangle| \leq \|\langle a, b \rangle\| \|\langle x, y \rangle\|$$



$$(|a| + |y|) (|b| + |x|) \leq 2 \left(\frac{1}{2} |a| |b| + \frac{1}{2} |x| |y| \right) + \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

$$\therefore |ax + by| \leq |a| |x| + |b| |y| \leq \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

— Roger B. Nelsen

```

numeric u; u = 53;
path xx, yy; xx = (-12, 0) -- (12 + 2.5u, 0); yy = (0, -12-u) -- (0, 12 + 2.5u);
path ff, dff; numeric s; s = 1/8;
vardef f(expr x) = x**1.9 - 1 enddef;
ff = ((0, f(0)) for x = s step s until 2: .. (x, f(x)) endfor) scaled u;
dff = (3/2 left -- 2 right) scaled u
    rotated angle direction 1/s of ff shifted point 1/s of ff;

picture P[];
P1 = image(
    drawoptions(withcolor Blues 7 5);
    draw dff; label.rt("$y=r(x-1)$", point infinity of dff);

    drawoptions(withcolor Blues 7 7);
    draw ff; label.rt("$y=x^r - 1$", point infinity of ff);

    drawoptions();
    drawarrow xx; label.rt("$x$", point 1 of xx);
    drawarrow yy; label.top("$y$", point 1 of yy);

    dotlabel.llft("$0$", origin); dotlabel.lft("$-1$", u * down);
    dotlabel.lrt("$1$", u * right);
    unfill fullcircle scaled 3/4 dotlabeldiam shifted (u * right);

    label.top("\hbox to \textwidth{I. First semester calculus\hss}",
        point 5/2 of bbox currentpicture shifted 8 up);
);
% P2 is made up of several subpictures ...
% ... see the document source for details.

P2 = image(
    label.lft(P21, 12 left); label.rt(P22, 12 right);
    label.top("\hbox to \textwidth{II. Second semester calculus\hss}",
        point 5/2 of bbox currentpicture shifted 8 up);
);

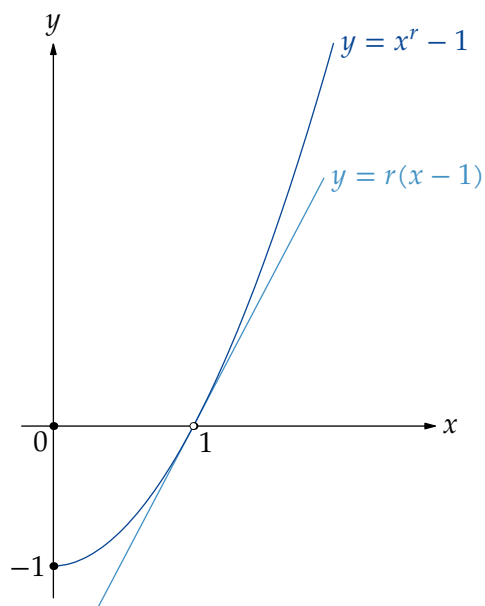
label.top(P1, 7 up); label.bot(P2, 7 down);
label.top("$x>0$, $x \ne 1$, $r>1$: $x^r - 1 > r(x-1)$",
    point 5/2 of bbox currentpicture shifted 13 up);

```

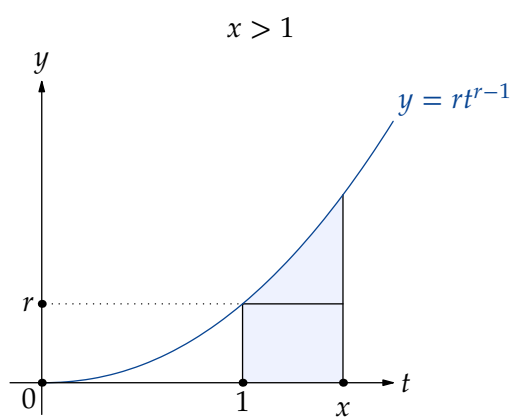
Bernoulli's inequality

$$x > 0, x \neq 1, r > 1: x^r - 1 > r(x - 1)$$

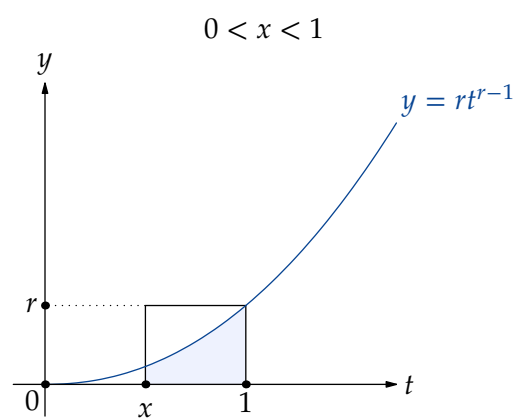
I. First semester calculus



II. Second semester calculus



$$x^r - 1 = \int_1^x rt^{r-1} dt > r(x - 1)$$



$$1 - x^r = \int_x^1 rt^{r-1} dt < r(1 - x)$$

— Roger B. Nelsen

```

numeric u, v, a, b; u = 64; v = 42; a = 3; b = 21;

vardef f(expr x) = (x * u, 1/256 mlog(x) * v) enddef;
path ff; ff = f(1/8) for t = 1/4 step 1/8 until 4: .. f(t) endfor;
path yy; yy = (0, ypart point 0 of ff 12);
path xx; xx = 12 left -- 12 right + 4u * right; picture P[];
P1 = image(path L[];
  L1 = (left--right) scaled u rotated angle direction b of ff shifted point b of ff;
  L3 = (left--right) scaled u rotated angle direction a of ff shifted point a of ff;
  L2 = point a of ff -- point b of ff;

drawoptions(dashed withdots scaled 1/4 withcolor Blues 7 6);
  forsuffices $=1,2,3: draw L$; endfor
  label.ulft("$L_1$", point 7/8 of L1); label.ulft("$L_3$", point 7/8 of L3);
  label.lrt("$L_2$", point 1/4 of L2);

drawoptions(withcolor Reds 7 7);
  draw ff; label.rt("\rlap{$y=\ln x}$", point infinity of ff);

drawoptions();
forsuffices $=a, b:
  draw (up--down) shifted (xpart point $ of ff, 0);
  label.top("$" & str $ & "$", (xpart point $ of ff, 0));

  fill fullcircle scaled dotlabeldiam shifted point $ of ff;
  unfill fullcircle scaled 2/3 dotlabeldiam shifted point $ of ff;

endfor

drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);
label.bot("$\hbox{slope of $L_1$}<\hbox{slope of $L_2$}<\hbox{slope of $L_3$}}$",
  point 1/2 of bbox currentpicture);
label.top("\hbox to \textwidth{I. First semester calculus\hss}",
  point 5/2 of bbox currentpicture shifted 8 up));
path ff; ff = (1/4 u, 4 v) for x=3/8 step 1/8 until 4: .. (x * u, v / x) endfor;
path yy; yy = 12 down -- (0, 12 + ypart point 0 of ff);

P2 = image(path A, B, C;
  z0 = point a of ff; z1 = point b of ff; x2=x0; x3=x1; y2=y3=0;
  A = unitsquare xscaled (x1-x0) yscaled y0 shifted z2;
  B = unitsquare xscaled (x1-x0) yscaled y1 shifted z2;
  C = z3 -- z2 -- subpath(a,b) of ff -- cycle;
  fill C withcolor Blues 7 1; fill B withcolor Blues 7 2;
  draw A withcolor 1/2; draw B withcolor 1/2;

  label.bot("\strut$a$", z2); label.bot("\strut$b$", z3);

drawoptions(withcolor Reds 7 7);
draw ff; label.top("\rlap{$y=\frac{1}{x}$", point infinity of ff);
drawoptions();
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);
label.bot("$\frac{1}{b-a} < \int_a^b \frac{1}{x}\,dx < \frac{1}{a(b-a)}$",
  point 1/2 of bbox currentpicture);
label.top("\hbox to \textwidth{II. Second semester calculus\hss}",
  point 5/2 of bbox currentpicture shifted 8 up));

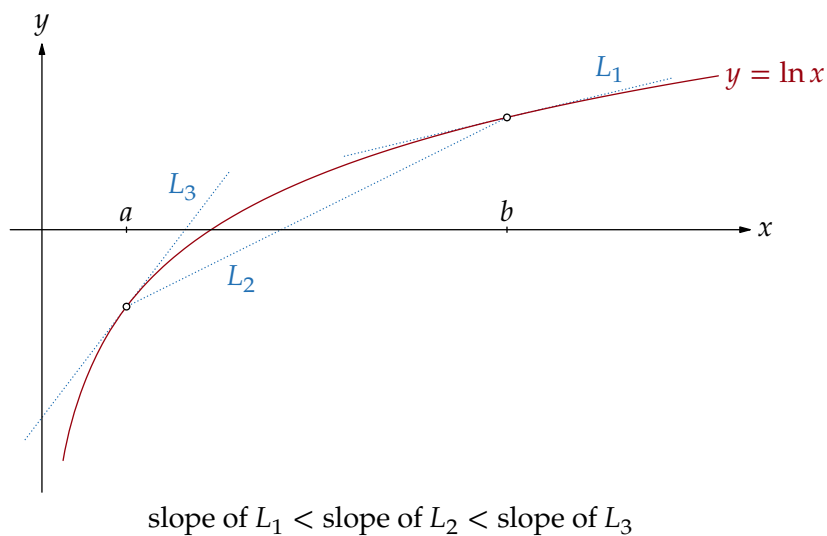
label.top(P1, 7 up); label.bot(P2, 7 down);
label.top("$b>a>0$ implies $\displaystyle\frac{1}{b}<\frac{\ln b-\ln a}{b-a}<\frac{1}{a}$",
  point 5/2 of bbox currentpicture shifted 13 up);

```

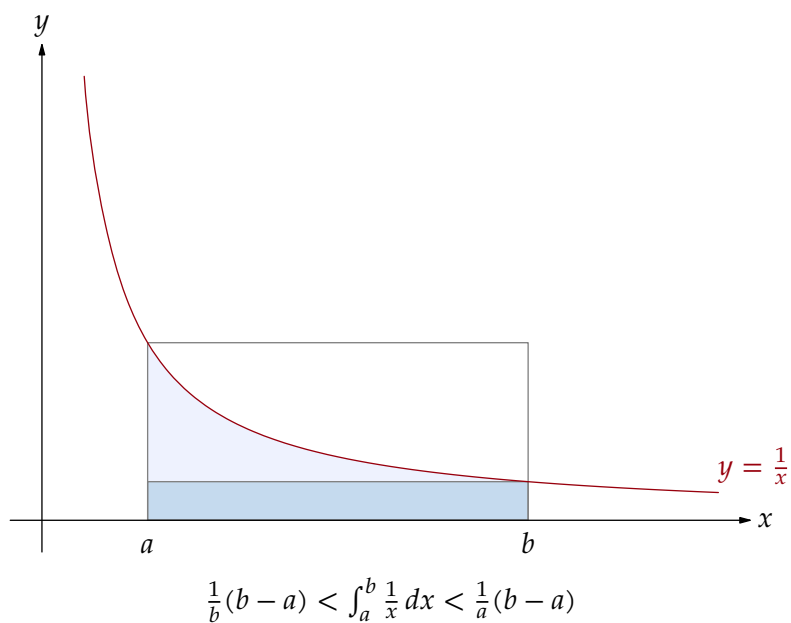
Napier's inequality

$$b > a > 0 \text{ implies } \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

I. First semester calculus



II. Second semester calculus



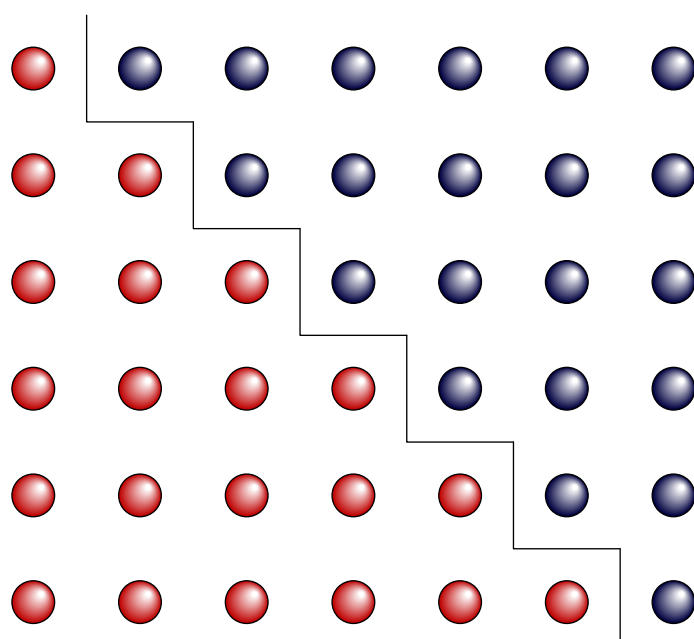
— Roger B. Nelsen

Integer sums

Sums of integers I	129
Sums of integers II	131
Sums of odd integers I	133
Sums of odd integers II	135
Sums of odd integers III	137
Squares and sums of integers I	139
Squares and sums of integers II	141
Arithmetic progressions with sum equal to square of number of terms	143
Sums of squares I	145
Sums of squares II	147
Sums of squares IV	149
Sums of squares V	151
Alternating sums of squares	153
Sums of squares of Fibonacci numbers	155
Sums of cubes I	157
Sums of cubes II	159
Sums of cubes III	161
Sums of cubes IV	163
Sums of cubes V	165
Sums of cubes VI	167
Sums of integers and sums of cubes	169
Sums of odd cubes are triangular numbers	171
Sums of fourth powers	173
k -th powers as sums of consecutive odd numbers	175
Sums of triangular numbers I	177
Sums of triangular numbers II	179
Sums of triangular numbers III	181
Sums of oblong numbers I	183
Sums of oblong numbers II	185
Sums of pentagonal numbers	187
On squares of positive integers	189
Consecutive sums of consecutive integers	191
Count the dots	193
Identities for triangular numbers	195
A triangular identity	197
Every hexagonal number is a triangular number	199
One domino = two squares : concentric squares	201
Sums of consecutive powers of 9 are sums of consecutive integers	203
Sums of hex numbers are cubes	205

```
input paintball
numeric n; n = 7;
for i=1 upto n-1:
  for j=1 upto n:
    draw if j > i: bball else: rball fi shifted ((j, -i) scaled 280/n);
    if i=j:
      draw (up--origin--right) shifted (j+1/2, -i-1/2) scaled (280/n);
    fi
  endfor
endfor
% remove the extra parts of the stepped line
clip currentpicture to bbox currentpicture scaled 0.975;
label.bot("$1+2+\cdots + n = \frac{1}{2} n (n+1)$",
  point 1/2 of bbox currentpicture shifted 21 down);
```


Sums of integers I



$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$$

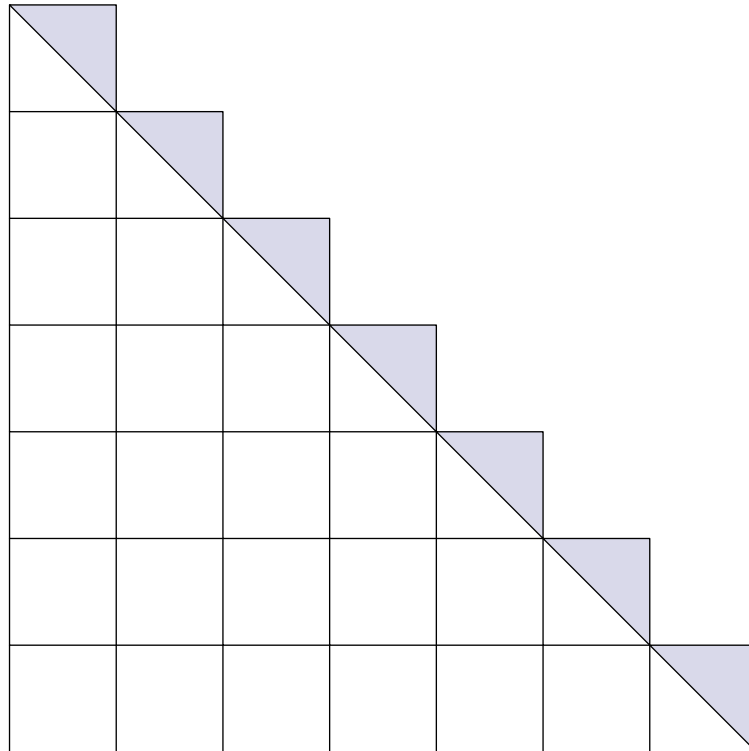
— Ancient Greek

```

numeric n; n = 7;
path t; t = (origin -- right -- up -- cycle) rotatedabout(1/2[right, up], 180);
for i=1 upto n:
  for j=0 upto n-i:
    draw (up--origin--right) shifted (i, j) scaled (280/n);
  endfor
  fill t shifted (i, n-i) scaled (280/n) withcolor Purples 8 3;
  draw t shifted (i, n-i) scaled (280/n);
endfor
label.bot("$1+2+\cdots + n = \frac{n^2}{2} + \frac{n}{2}$",
  point 1/2 of bbox currentpicture shifted 21 down);

```

Sums of integers II



$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

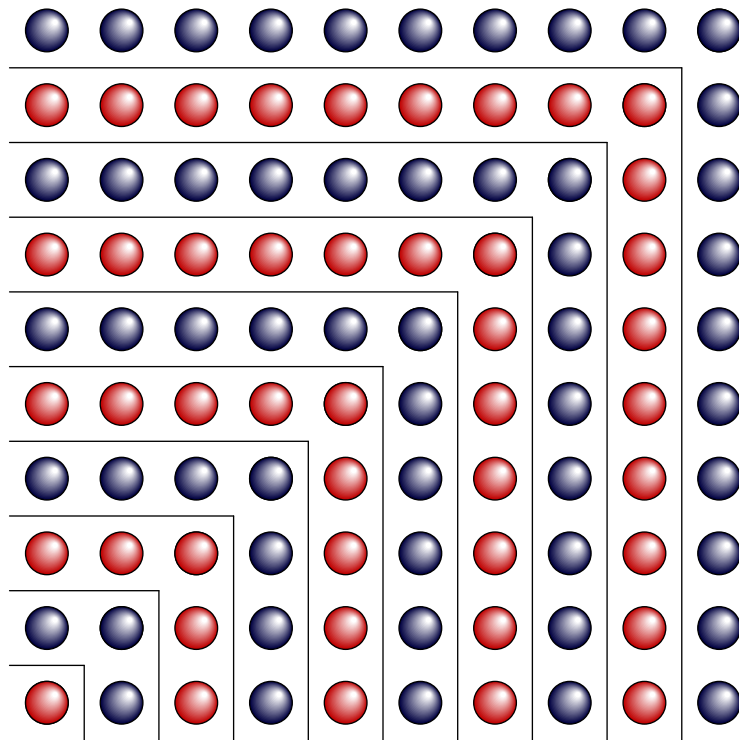
— Ian Richards

Integer sums

```
input paintball
numeric n; n = 10;
picture half;
half = image(
  for i=1 upto n:
    for j=1 upto i:
      draw if odd i: rball else: bball fi shifted ((i, j) scaled (280/n));
    endfor
    if i < n:
      draw (origin -- i * up) shifted (i+1/2, 1/2) scaled (280/n);
    fi
  endfor
);
draw half; draw half reflectedabout(origin, (1, 1));

label.bot("$1+3+5+\cdots + (2n-1) = n^2$",
  point 1/2 of bbox currentpicture shifted 21 down);
```

Sums of odd integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

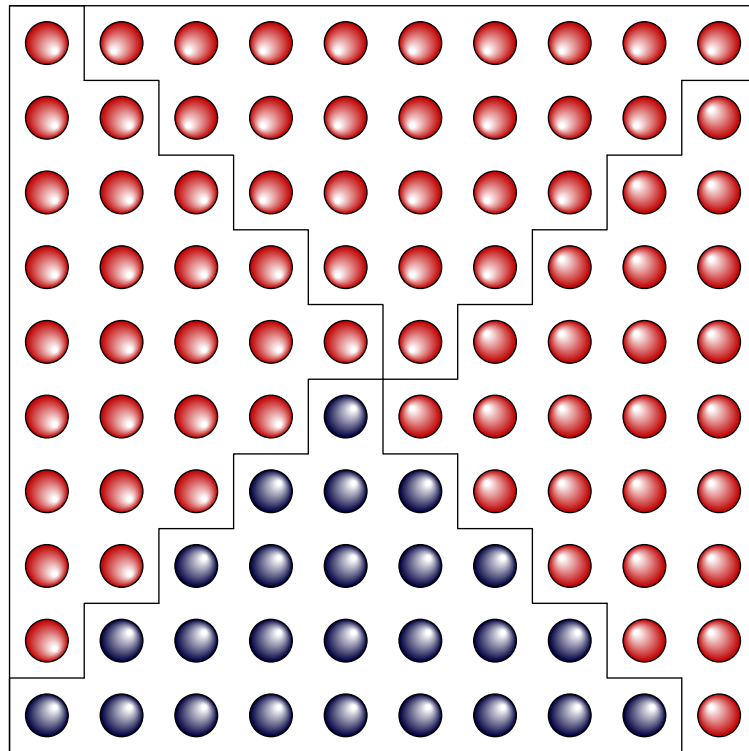
— Nichomachus of Gerasa

```

input paintball
numeric n, u; n = 10; u = 280/n;
for i=0 upto 3:
  for j=1 upto floor (n/2):
    for k=j upto n-j:
      draw if i=0: bball else: rball fi
      shifted ((k, j) scaled u)
      rotatedabout((n/2+1/2,n/2+1/2) scaled u, 90i);
      if k=j:
        draw (down--origin--right)
        shifted (k-1/2, j+1/2) scaled u
        rotatedabout((n/2+1/2,n/2+1/2) scaled u, 90i);
      fi
    endfor
  endfor
endfor
interim bboxmargin := -1/4;
draw bbox currentpicture;
label.bot("$1+3+\cdots + (2n-1) = \frac{1}{4}\left(2n\right)^2 = n^2$",
  point 1/2 of bbox currentpicture shifted 24 down);

```

Sums of odd integers II



$$1 + 3 + \cdots + (2n - 1) = \frac{1}{4} (2n)^2 = n^2$$

— Roger B. Nelsen

Integer sums

```

picture P[]; path t; numeric n; n = 6;
t = for i=0 upto 2: 16 down rotated 120i -- endfor cycle;

pair u, v;
u = point 2 of t - point 0 of t;
v = point 1 of t - point 2 of t;

P1 = image(
  fill t withcolor Blues 8 2; label("$\Delta$", origin);
  for i=0 upto n-1:
    for j=0 upto i:
      draw t shifted (j * v) shifted (i * u);
    endfor
    label.lrt(
      if i=n-1: "$2n-1$"
      elseif i=n-2: "\rotatebox{105}{$\ddots}$"
      else: "$" & decimal (2i+1) & "$" fi,
      point 2/3 of t shifted (i * v) shifted (i * u));
    endfor
  );
P2 = image(
  path s, s';
  s = t shifted -(xpart point 1 of t, ypart point 0 of t);
  s' = s scaled n;

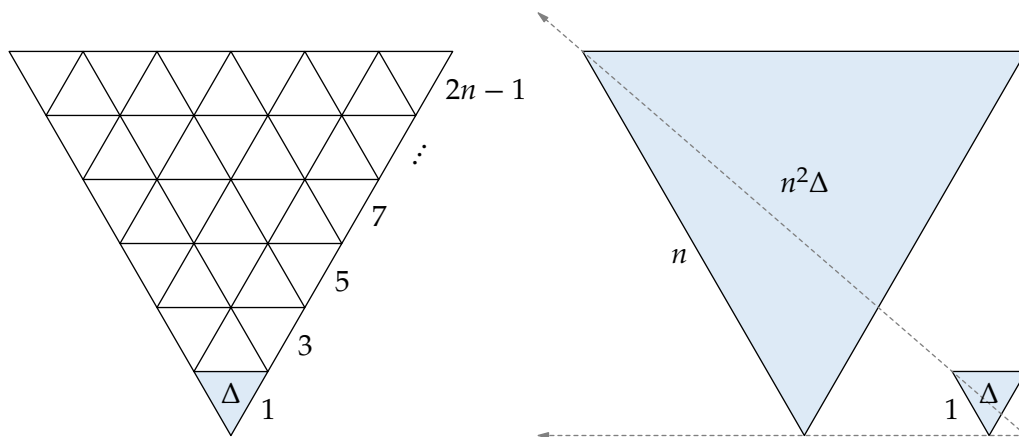
  forsuffices $=s, s':
    fill $ withcolor Blues 8 2; draw $;
  endfor
  z1 = point 2 of s' scaled 1.1;
  drawoptions(dashed evenly scaled 1/2 withcolor 1/2);
  drawarrow origin -- z1;
  drawarrow origin -- (x1, 0);
  drawarrow origin -- (0, y1);
  drawoptions();
  label("$\Delta$", 1/3[point 3/2 of s, point 0 of s]);
  label("$n^2\Delta$", 1/3[point 3/2 of s', point 0 of s']);
  label.llft("$1$", point -2/3 of s);
  label.llft("$n$", point -1/2 of s');
);

interim bboxmargin := 12;
label.ulft(P1, origin);
label.urt(P2, origin shifted down);

label.bot("$\Delta+3\cdot\Delta+\cdots + (2n-1)\cdot\Delta = A = n^2\cdot\Delta$",
  point 1/2 of bbox currentpicture shifted 24 down);
label.bot("$\displaystyle\sum_{i=1}^n \left(2i - 1\right) = n^2$",
  point 1/2 of bbox currentpicture shifted 4 down);

```


Sums of odd integers III



$$\Delta + 3 \cdot \Delta + \cdots + (2n - 1) \cdot \Delta = A = n^2 \cdot \Delta$$

$$\sum_{i=1}^n (2i - 1) = n^2$$

— Jenő Lehel

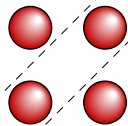
```

input paintball
for n=2 upto 4:
  numeric y; y = - 20 n * n;
  picture p; p = image(
    for i=1 upto n:
      for j=1 upto n:
        draw rball shifted 28(i, j);
      endfor
    endfor
    path b; b = bbox currentpicture;
    drawoptions(dashed evenly withpen pencircle scaled 1/4);
    for i=1 upto n-1:
      for j=-1, 1:
        draw (left-- 2 right) scaled 40n rotated 45 shifted ((28i-14)*j, 0);
      endfor
    endfor
    drawoptions();
    clip currentpicture to b;
  );
  label(p, (-80, y));
  label("$"
    for i=1 upto n: & decimal i & "+" endfor
    for i=n-1 downto 2: & decimal i & "+" endfor
    & "1 =" & decimal n & "^2$",
    (80, y));
endfor

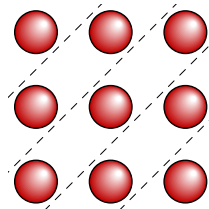
label.bot("$1+2+\cdots+(n-1) + n + (n-1) + \cdots + 2 + 1 = n^2$",
  point 1/2 of bbox currentpicture shifted 42 down);

```

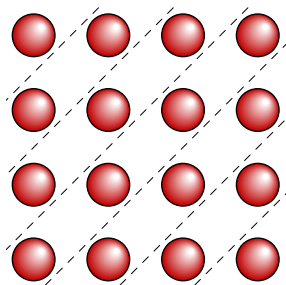
Squares and sums of integers I



$$1 + 2 + 1 = 2^2$$



$$1 + 2 + 3 + 2 + 1 = 3^2$$



$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

$$1 + 2 + \cdots + (n - 1) + n + (n - 1) + \cdots + 2 + 1 = n^2$$

— Ancient Greek

```

input paintball
for n=2 upto 4:
  numeric y; y = - 24 n * n;
  picture p; p = image(
    for i=1 upto n:
      for j=1 upto n:
        draw rball shifted 28(i, j);
      endfor
    endfor
  );
  picture q; q = image(
    for i=1 upto n-1:
      for j=1 upto n-1:
        draw bball shifted 28(i+1/2, j+1/2);
      endfor
    endfor
  );
  picture r; r = image(
    drawoptions(dashed evenly withpen pencircle scaled 1/4);
    for i=1 upto n-1:
      for j=-1, 1:
        draw (left-- 2 right) scaled 40n rotated 45 shifted ((28i-14)*j, 0);
      endfor
    endfor
    drawoptions();
    clip currentpicture to bbox p;
  );
  label(p, (-144, y + 42));
  label(q, (-144, y + 42));
  label(r, (-144, y + 42));
  label("$=$", (-64, y+42));
  label(p, (144, y + 42));
  label("$+$", (64, y+42));
  label(q, (0, y + 42));

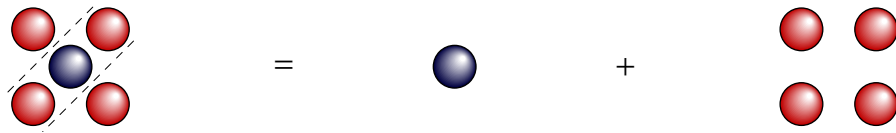
  label.bot("$"
    for i=1 step 2 until 2n-1: & decimal i & "+" endfor
    for i=2n-3 step -2 until 3: & decimal i & "+" endfor
    & "1 =" & decimal (n-1) & "^2 + " & decimal n & "^2$",
    (0, y - 3n));
endfor

label.bot("$\vdots$", point 1/2 of bbox currentpicture shifted 21 down);

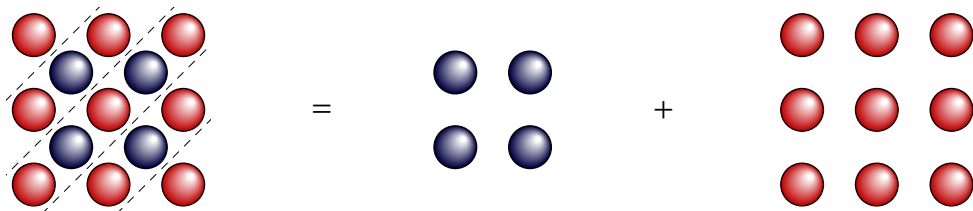
label.bot("$1+3+\cdots+(2n-1) + (2n+1) + (2n-1) + \cdots + 3 + 1 = n^2 + (n+1)^2$",
  point 1/2 of bbox currentpicture shifted 21 down);

```

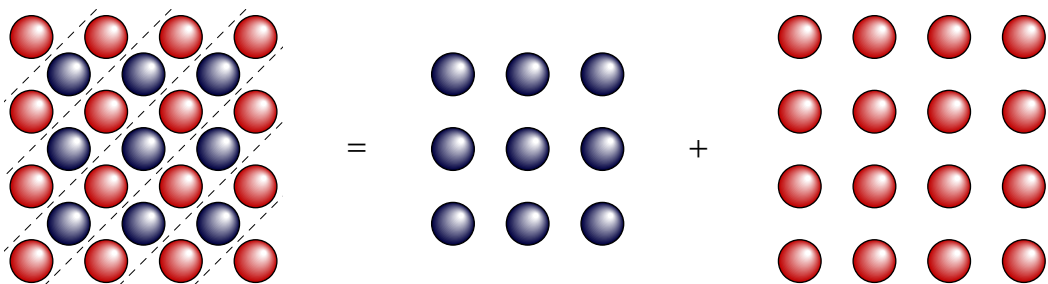
Squares and sums of integers II



$$1 + 3 + 1 = 1^2 + 2^2$$



$$1 + 3 + 5 + 3 + 1 = 2^2 + 3^2$$



$$1 + 3 + 5 + 7 + 5 + 3 + 1 = 3^2 + 4^2$$

⋮

$$1 + 3 + \cdots + (2n - 1) + (2n + 1) + (2n - 1) + \cdots + 3 + 1 = n^2 + (n + 1)^2$$

— Hee Sik Kim

```

numeric u; u = 21;
vardef folded_bar(expr n, k) =
  if k < 2:
    unitsquare xscaled (n + k)
  else:
    numeric p, q;
    if odd k:
      p = n + 1/2 (k - 1);
      q = n + k - p - 1;
      origin -- (p, 0) -- (p, -q) -- (p+1, -q) -- (p+1, 1)
    else:
      p = n + 1/2 k;
      q = n + k - p + 1;
      origin -- (p, 0) -- (p, q) -- (p-1, q) -- (p-1, 1)
    fi -- up -- cycle
  fi
enddef;

numeric y; y = 0;
for n = 1 upto 4:
  y := y - 2(n-1) * u - u;
  for k = 0 upto 2n - 2:
    path s; s = folded_bar(n, k) scaled u
      shifted (0, y + 1/2 u * (if odd k: k+1 else: -k fi));
    fill s withcolor
      if k = 0: Purples 8 2
      elseif odd floor(k/2): Blues 8 2
      else: Oranges 8 2
      fi;
  endfor

  drawoptions(dashed withdots scaled 1/2);
  for i = 1 upto 2n - 2:
    draw ((n-1) * down -- n * up) shifted (i, 0) scaled u shifted (0, y);
    draw (origin -- (2n-1) * right) shifted (0, i-n+1) scaled u shifted (0, y);
  endfor
  drawoptions();

  for k = 0 upto 2n - 2:
    path s; s = folded_bar(n, k) scaled u
      shifted (0, y + 1/2 u * (if odd k: k+1 else: -k fi));
    draw s;
  endfor
endfor

label.rt(btex \vbox{\openup 6pt \halign{##$\hss\cr
  n = 4\cr 4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2\cr}} etex,
  (8u, y + 1/2 u));

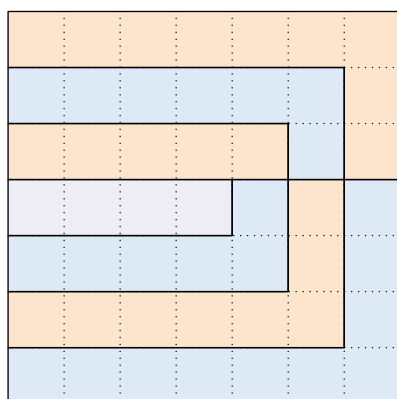
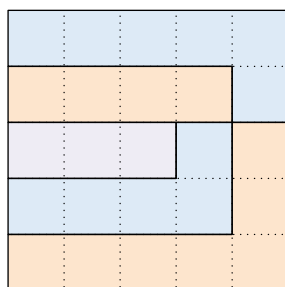
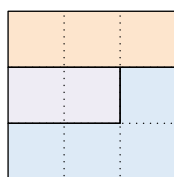
label.llft(btex
  $\displaystyle \sum_{k=n}^{\{3n-2\}} k = \left(2n-1\right)^2$;\quad
  $n=1,2,3,\dots$
etex, urcorner currentpicture);

```

Arithmetic progressions with sum equal to square of number of terms



$$\sum_{k=n}^{3n-2} k = (2n-1)^2; \quad n = 1, 2, 3, \dots$$



$$n = 4$$

$$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$$

— James O. Chilaka

```

input isometric_projection
set_projection(22, -34);
picture P[], b_cube, b_semicube, r_semicube;
b_cube = cube(Blues 8 4, Blues 8 2, background);
b_semicube = semicube(Blues 8 4, Blues 8 2, background);
r_semicube = semicube(Reds 8 6, Reds 8 4, Reds 8 2);

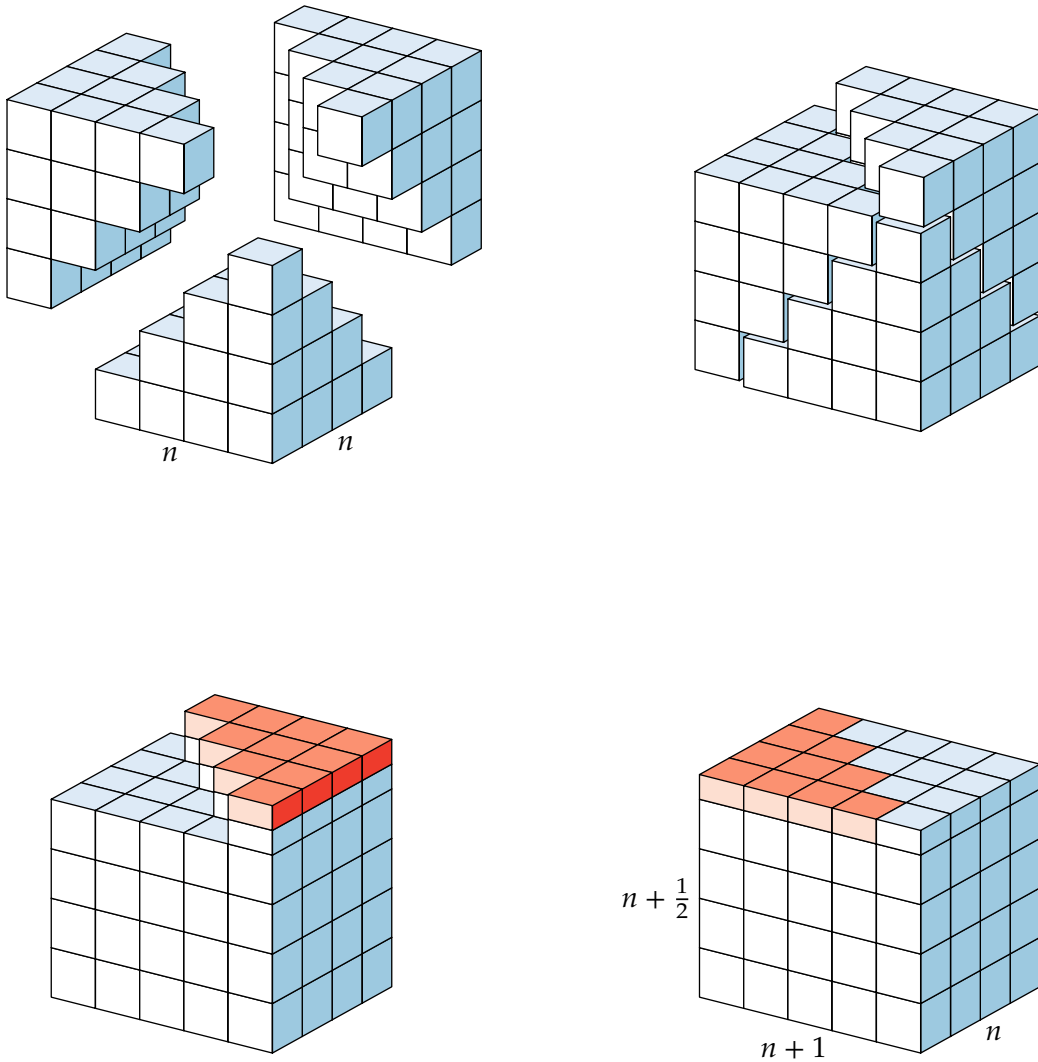
P1 = image(
  pair a, b; a = p(0, 2, 3); b = p(-5, 2, 0);
  for k= 0 upto 3:
    for j = k upto 3:
      for i = k upto 3:
        draw b_cube shifted p(i-3, k, 3-j);
        draw b_cube shifted a shifted p(i-3, j, 3-k);
        draw b_cube shifted b shifted p(k, i, 3-j);
      endfor
    endfor
  endfor
  label.lrt("$n$", p(0, 0, 2));
  label.llft("$n$", p(-2, 0, 0));
);
P2 = image(
  % .. as P1 but with a and b rather smaller
);
P3 = image(
  for i=-4 upto 0:
    for j = 0 upto 3:
      for k = 3 downto 0:
        draw b_cube shifted p(i, j, k);
      endfor
    endfor
  endfor
  for k = 3 downto 0:
    for i = -k upto 0:
      draw b_semicube shifted p(i, 4, k);
      draw r_semicube shifted p(i, 4.5, k);
    endfor
  endfor
);
P4 = image(
  % .. as P3 except for top layer, and labels
);
draw P1 shifted 12 down; draw P2 shifted 243 right;
draw P3 shifted 233 down; draw P4 shifted 233 down shifted 243 right;

label.top(btex $
  1^2 + 2^2 + \cdots + n^2 =
  \frac{1}{3} n \left(n + 1\right)\left(n + \frac{1}{2}\right)
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);

```


Sums of squares I

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)\left(n + \frac{1}{2}\right)$$



— Man-Keung Siu

```

numeric u, N; u = 14.4; N = 6;
picture P[];
P1 = image(
  for n = 1 upto N:
    for m = n downto 1:
      path s; s = unitsquare scaled (-m * u) shifted (0, - n / 2 * n * u);
      fill s withcolor Spectral[7][m+1];
      for k = 1 upto m - 1:
        draw subpath (2, 3) of s shifted (0, k*u);
        draw subpath (1, 2) of s shifted (k*u, 0);
      endfor
      draw s withpen pencircle scaled 3/2;
    endfor
  endfor
);
P2 = image(
  for n = 1 upto N:
    path s; s = unitsquare xscaled (2n - 1) yscaled (N - n + 1) scaled u
    shifted ((N + 1 - n, -2 - N * n + n*(n-1)/2) scaled u);
    fill s withcolor Spectral[7][1+n];
    for k = 0 upto N - n:
      draw (left--right) scaled (N+1/2) scaled u
      shifted point 1/2 of s shifted (0, k*u)
      if k=0: withpen pencircle scaled 3/2 fi;
    endfor
    draw s withpen pencircle scaled 3/2;
  endfor
  for k=0 upto 2N + 1:
    draw ((k, -2) -- (k, -2 - N / 2 * (N + 1))) scaled u
    if (k=0) or (k=2N+1): withpen pencircle scaled 3/2 fi;
  endfor
  draw ((0, -2) -- (2N+1, -2)) scaled u withpen pencircle scaled 3/2;
  label.bot("$2n+1$", point 1/2 of bbox currentpicture);
  label.lft(TEX("$1+2+\cdots+n$") rotated 90, point -1/2 of bbox currentpicture);
);

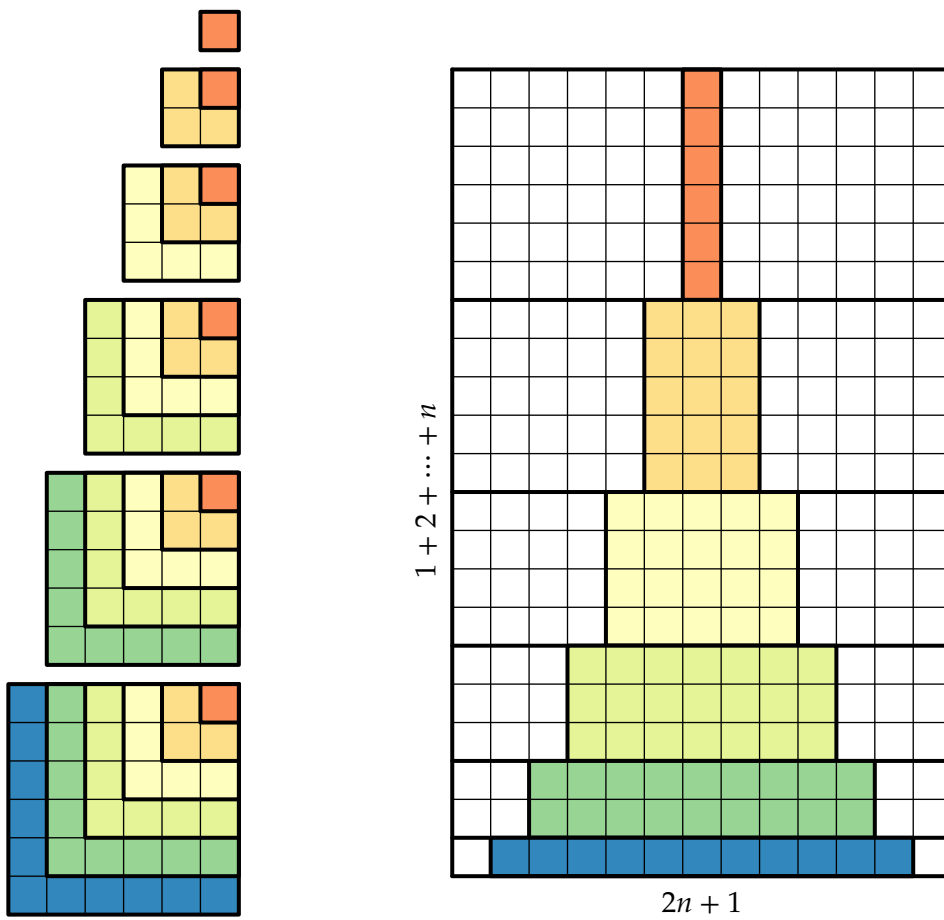
draw P1;
draw P2 shifted 80 right;

label.top(btex
$3\left(1^2 + 2^2 + \cdots + n^2\right) =
\left(2n+1\right)\left(1 + 2 + \cdots + n\right)$ etex,
point 5/2 of bbox currentpicture shifted 21 up);

```

Sums of squares II

$$3(1^2 + 2^2 + \cdots + n^2) = (2n + 1)(1 + 2 + \cdots + n)$$



— Dan Kalman

```

numeric u, N; u = 14.4; N = 6;

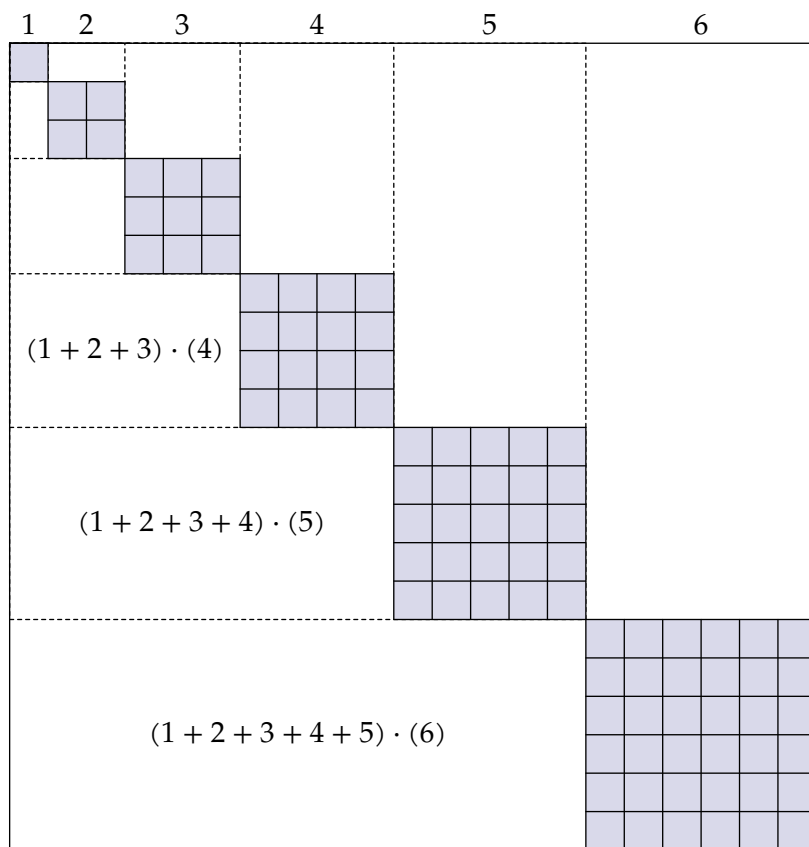
numeric x, y; x = 0; y = -1;
for n = 1 upto N:
  label("$" & decimal n & "$", (x + n / 2, 1/2) scaled u);
  draw unitsquare scaled 1/2 (n * n + n) scaled u rotated -90
    if n < N: dashed evenly scaled 1/2 fi;
  fill unitsquare scaled n shifted (x, y) scaled u withcolor Purples 8 3;
  for m = 0 upto n:
    draw (origin -- n * right) shifted (x, y + m) scaled u;
    draw (origin -- n * up)    shifted (x + m, y) scaled u;
  endfor
  if n > 3:
    label("$\left(1" for i=2 upto n-1: & "+" & decimal i endfor & "\right)\cdot(" & decimal n & ")\right)$",
      1/2[(0, y+n) scaled u, (x,y) scaled u]);
  fi
  x := x + n;
  y := y - (n+1);
endfor

label.top(btex $\displaystyle
  \sum_{k=1}^n k^2 = \left(\sum_{k=1}^n k \right)^2
  - 2 \sum_{k=1}^{n-1} \left( \left(k+1\right) \sum_{i=1}^k i \right)
$ etex, point 5/2 of bbox currentpicture shifted 21 up);

```

Sums of squares IV

$$\sum_{k=1}^n k^2 = \left(\sum_{k=1}^n k \right)^2 - 2 \sum_{k=1}^{n-1} \left((k+1) \sum_{i=1}^k i \right)$$



— James O.Chilaka

```

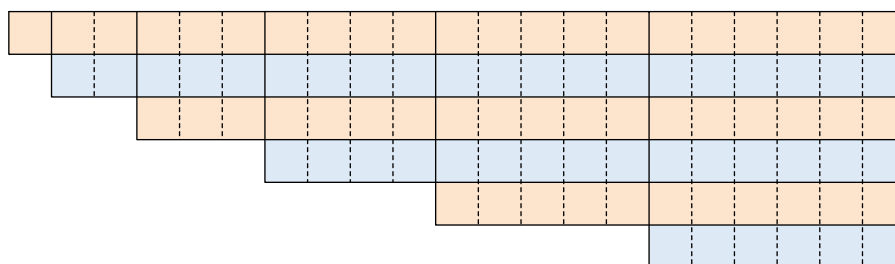
numeric u, N, m; u = 16; N = 6; 2m = N * N + N;
path s[];
for n=1 upto N:
    s[n] = unitsquare xscaled -m shifted (0, -n) scaled u;
    fill s[n] withcolor if odd n: Oranges else: Blues fi 8 2;
    m := m - n;
endfor
numeric m; 2m = N * N + N;
for n=1 upto N:
    for k=1 upto n-1:
        draw ((k - m, 0) -- (k - m, -n)) scaled u dashed evenly scaled 1/2;
    endfor
    draw (-m * u, 0) -- subpath (1, 0) of s[n]
        if n=N: -- subpath (3, 2) of s1 fi;
    m := m - n;
endfor

label.top(btex $\displaystyle
    \sum_{i=1}^n \sum_{j=i}^n j = \sum_{i=1}^n i^2
$ etex, point 5/2 of bbox currentpicture shifted 21 up);

```

Sums of squares V

$$\sum_{i=1}^n \sum_{j=i}^n j = \sum_{i=1}^n i^2$$



— Pi-Chun Chuang

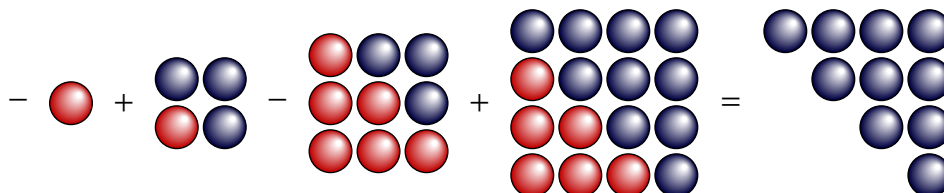
```

input paintball
picture P[];
P1 = image(draw rball);
P2 = image(for i=0 upto 1:
  for j=0 upto 1:
    draw if i+j=0: rball else: bball fi shifted (18i, 18j);
  endfor
endfor);
P3 = image(for i=0 upto 2:
  for j=0 upto 2:
    draw if i+j<3: rball else: bball fi shifted (18i, 18j);
  endfor
endfor);
P4 = image(for i=0 upto 3:
  for j=0 upto 3:
    draw if i+j<3: rball else: bball fi shifted (18i, 18j);
  endfor
endfor);
P5 = image(for i=0 upto 3:
  for j=0 upto 3:
    if i+j >= 3: draw bball shifted (18i, 18j) fi;
  endfor
endfor);
label("${}-{}$", origin);
label.rt(P1, point 3/2 of bbox currentpicture);
label.rt("${}+{}$", point 3/2 of bbox currentpicture);
label.rt(P2, point 3/2 of bbox currentpicture);
label.rt("${}-{}$", point 3/2 of bbox currentpicture);
label.rt(P3, point 3/2 of bbox currentpicture);
label.rt("${}+{}$", point 3/2 of bbox currentpicture);
label.rt(P4, point 3/2 of bbox currentpicture);
label.rt("${}={}$", point 3/2 of bbox currentpicture);
label.rt(P5, point 3/2 of bbox currentpicture);
label.bot(btex $\displaystyle \sum_{k=1}^n \left(-1\right)^k k^2
= \left(-1\right)^n T_n = \left(-1\right)^n \frac{n(n+1)}{2}$ etex,
point 1/2 of bbox currentpicture shifted 13 down);

```


Alternating sums of squares

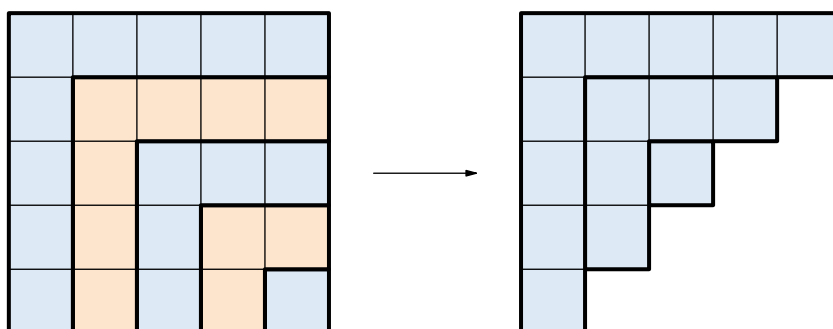
I.



$$\sum_{k=1}^n (-1)^k k^2 = (-1)^n T_n = (-1)^n \frac{n(n+1)}{2}$$

— Dave Logothetti

II.



$$n^2 - (n-1)^2 + \cdots + (-1)^{n-1} (1)^2 = \sum_{k=0}^n (-1)^k (n-k)^2 = \frac{n(n+1)}{2}$$

— Steven L. Snover

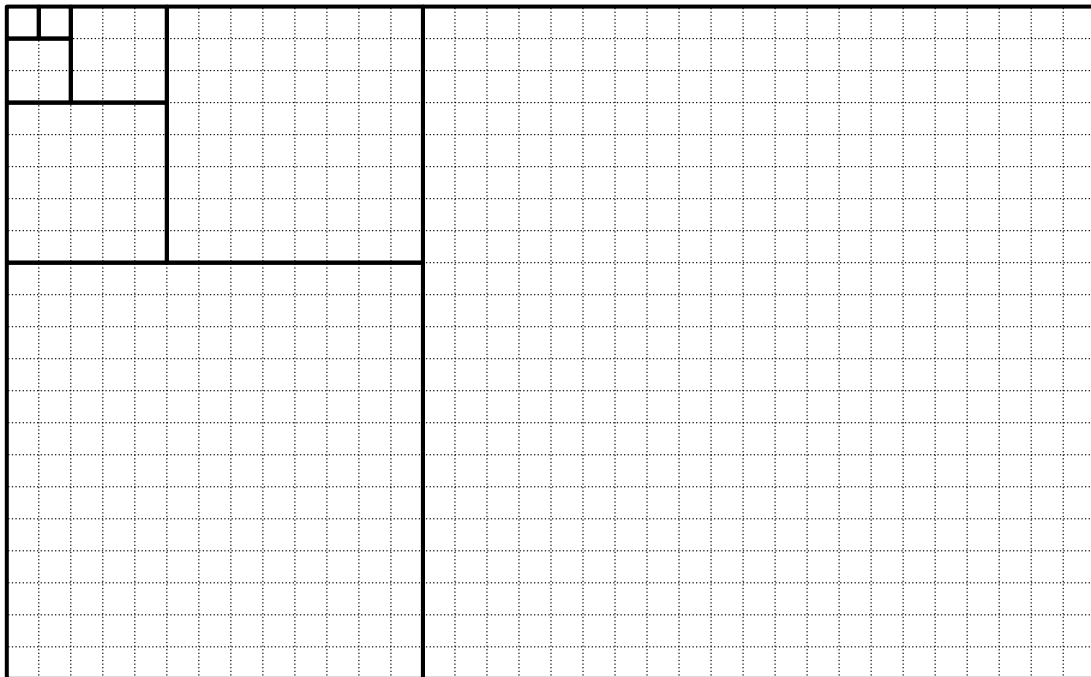
```

numeric u; u = 12;
for i=1 upto 34:
    draw ((i, 0) -- (i, -21)) scaled u dashed withdots scaled 1/4;
endfor
for i=1 upto 20:
    draw ((0, -i) -- (34, -i)) scaled u dashed withdots scaled 1/4;
endfor
numeric y; y = 0;
for i=1, 2, 5, 13:
    y := y + i * u;
    draw unitsquare scaled (u * i) shifted (0, -y) withpen pencircle scaled 3/2;
    draw unitsquare scaled y shifted (u * i, -y) withpen pencircle scaled 3/2;
endfor

label.bot(btex $F_1=F_2=1$; $F_{n+2}=F_{n+1} + F_n$
    \quad hence\quad
    $F_1^2+F_2^2+\cdots+F_n^2=F_nF_{n+1}$
    etex, point 1/2 of bbox currentpicture shifted 13 down);

```

Sums of squares of Fibonacci numbers



$$F_1 = F_2 = 1; F_{n+2} = F_{n+1} + F_n \quad \text{hence} \quad F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

— Alfred Brousseau

Integer sums

```

numeric u, n, x, y;
u = 9;
n = 8;
x = y = 0;
for i=1 upto n:
  y := y - (i*u);
  for j=0 upto floor (i/2) - 1:
    path s; s = unitsquare scaled (i*u) shifted (i*u*j, y);
    fill s withcolor if odd i: Blues else: Oranges fi 8 2;
  endfor
  for j=1 upto ceiling (i/2):
    path s; s = unitsquare scaled (i*u) shifted (x, -i*u*j);
    fill s withcolor if odd i: Blues else: Oranges fi 8 2;
    if 2j = i:
      fill center s -- subpath (-1/2, 1/2) of s -- cycle
        withcolor Oranges 8 3;
    fi
  endfor
  x := x + (i*u);
endfor
numeric x, y;
x = y = 0;
for i=1 upto n:
  y := y - (i*u);
  for j=0 upto floor (i/2) - 1:
    path s; s = unitsquare scaled (i*u) shifted (i*u*j, y);
    draw s;
  endfor
  for j=1 upto ceiling (i/2):
    path s; s = unitsquare scaled (i*u) shifted (x, -i*u*j);
    draw s;
  endfor
  x := x + (i*u);
endfor

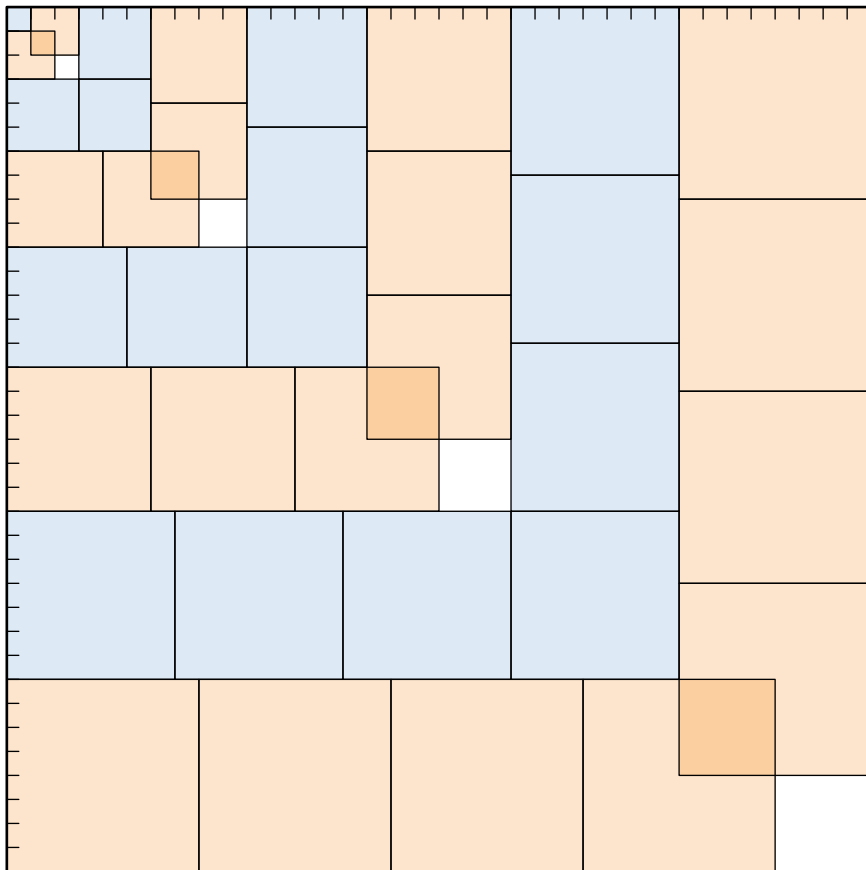
numeric N; N = 1/2 n * (n + 1);
for i=1 upto N-1:
  draw (i*u, 0) -- (i*u, -1/2 u);
  draw (0, -i*u) -- (1/2u, -i*u);
endfor
draw unitsquare scaled u xscaled N yscaled -N withpen pencircle scaled 1;

label.top("$1^3+2^3+3^3+\cdots+n^3 = \left(1+2+3+\cdots+n\right)^2$",
point 5/2 of bbox currentpicture shifted 13 up);

```

Sums of cubes I

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— Solomon W. Golomb

```

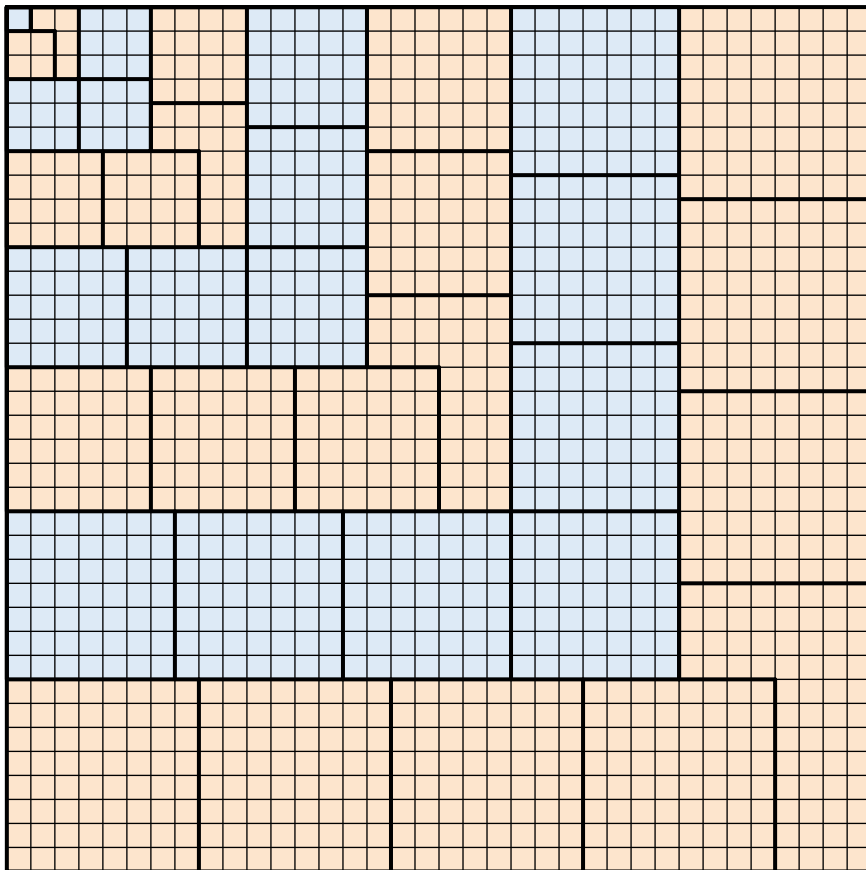
numeric u, N;
u = 9;
N = 8;
path s; s = unitsquare scaled u yscaled -1;
for n = N downto 1:
    numeric t; t = n / 2 * (n + 1);
    fill s scaled t withcolor if odd n: Blues else: Oranges fi 8 2;
endfor
numeric t; t = N / 2 * (N + 1);
for i = 1 upto t - 1:
    draw (i*u, 0) -- (i*u, -t*u);
    draw (0, -i*u) -- (t*u, -i*u);
endfor
for n = N downto 1:
    numeric t; t = n / 2 * (n + 1);
    draw s scaled t withpen pencircle scaled 3/2;
    for i=n step n until t:
        draw ((i, -t) -- (i, n-t)) scaled u withpen pencircle scaled 3/2;
    endfor
    for i=n step n until t-eps:
        numeric a; a = if t - i < n: n/2 else: 0 fi;
        draw ((t-n, a-i) -- (t-a, a-i)) scaled u withpen pencircle scaled 3/2;
    endfor
endfor

label.top("$1^3+2^3+3^3+\cdots+n^3 = \left(1+2+3+\cdots+n\right)^2$",
point 5/2 of bbox currentpicture shifted 13 up);

```

Sums of cubes II

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— J. Barry Love

```

input isometric_projection
set_projection(100/3, -45);
picture P[], b_cube, o_cube, r_cube;
b_cube = cube(Blues 8 4, Blues 8 2, Blues 8 1);
o_cube = cube(Oranges 8 4, Oranges 8 2, Oranges 8 1);
r_cube = cube(Reds 8 4, Reds 8 2, Reds 8 1);

P1 = image(
for n=1, 2, 3:
  for i=1 upto n:
    for j=1 upto n:
      for k=1 upto n:
        draw o_cube shifted p(i - 3/4 n * n, n * n / 2 + j, n - k);
      endfor
    endfor
  endfor
endfor);
P2 = image(
for n=1, 2, 3:
  for i=1 upto n:
    for j=1 upto n:
      for k=1 upto n:
        draw if k = 1: o_cube elseif k = 2: b_cube else: r_cube fi
          shifted p(i - 3/4 n * n, n * n / 2 + j, n - 1.4k);
        endfor
      endfor
    endfor
  endfor
endfor);
P3 = image(
draw o_cube;
for i=1 upto 3:
  for j=1 upto 3:
    if (i*j) > 1:
      draw if (i>1) and (j>1): o_cube else: b_cube fi shifted p(i, j, .5);
    fi
  endfor
endfor
for i=1 upto 6:
  for j=1 upto 6:
    if (i>3) or (j>3):
      draw if i<=3: b_cube elseif j<=3: r_cube else: o_cube fi shifted p(i, j, 2);
    fi
  endfor
endfor);
P4 = image(
for i=1 upto 6:
  for j=1 upto 6:
    draw o_cube shifted p(i, j, 1);
  endfor
endfor);

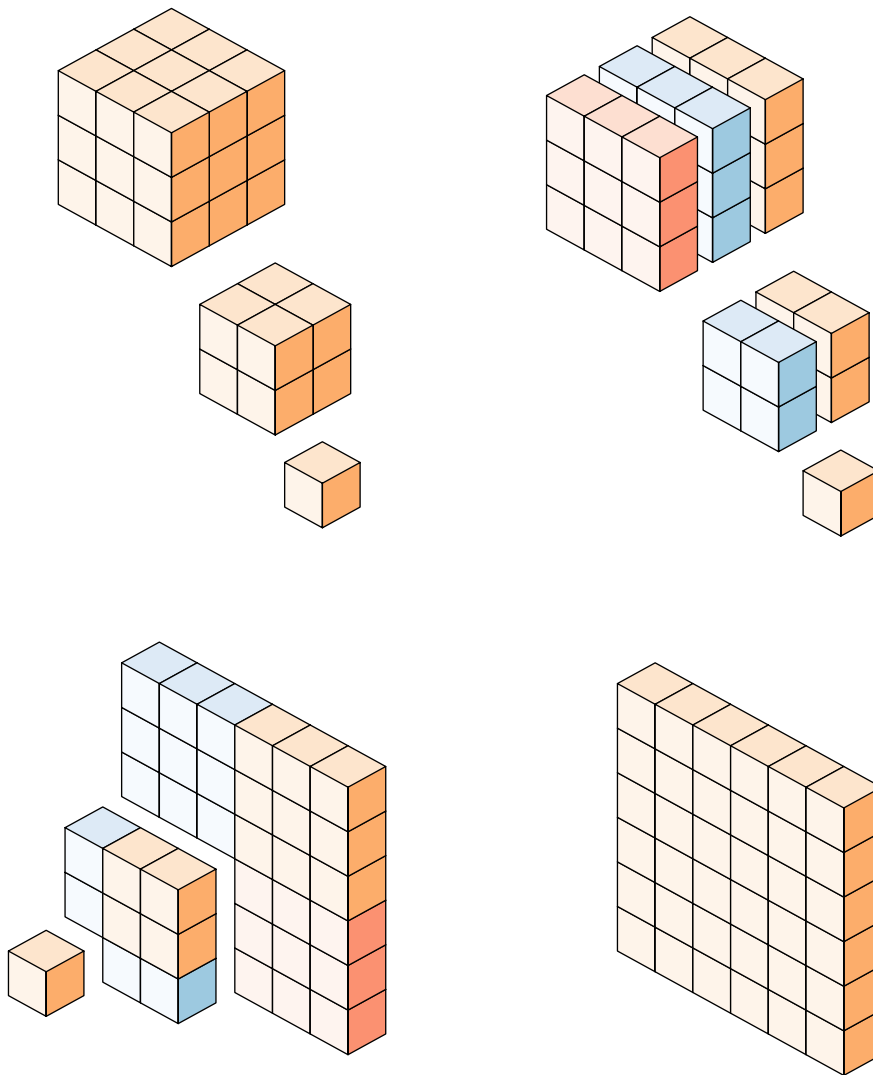
draw P1; draw P2 shifted (200, 0);
draw P3 shifted (-100, -160); draw P4 shifted (100, -160);

label.top("$1^3+2^3+3^3+\cdots+n^3 = \left(1+2+3+\cdots+n\right)^2$",
point 5/2 of bbox currentpicture shifted 21 up);

```


Sums of cubes III

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— Alan L. Fry

```

color s[]; s1 = Reds 8 2; s2 = Oranges 8 2; s3 = Blues 8 2;
numeric u; 36 u = tw;
for i = 1 upto 5:
    numeric o; o = 1/2 i * (i + 1);
    path b; b = unitsquare scaled i shifted -(o, o) scaled u;
    for r=0 upto 3:
        for j = 0 upto i-1:
            if known s[r]:
                fill b shifted (j*i*u, 0) rotated 90r withcolor s[r];
            else:
                drawoptions(dashed evenly scaled 1/2 withpen pencircle scaled 1/4);
                for k=1 upto i-1:
                    draw subpath (3,4) of b shifted (j*i*u+k*u, 0);
                    draw subpath (0,1) of b shifted (j*i*u, k*u);
                endfor
                drawoptions();
            fi
        draw b shifted (j*i*u, 0) rotated 90r;
    endfor
endfor
input arrow_label

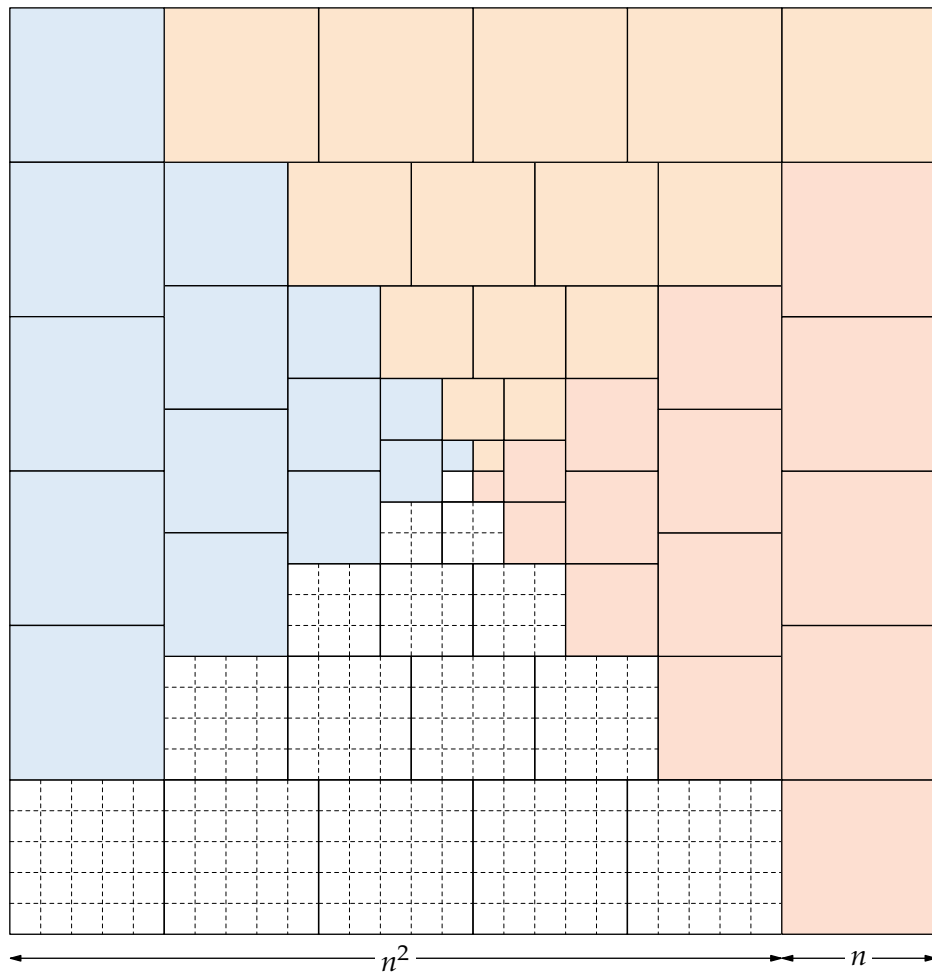
arrow_label((-15u, -15u), (10u, -15u), "$n^2$", 9);
arrow_label((10u, -15u), (15u, -15u), "$n$", 9);

label.top("$1^3+2^3+3^3+\cdots+n^3 = \frac{1}{4} \left(n(n+1)\right)^2$",
point 5/2 of bbox currentpicture shifted 21 up);

```

Sums of cubes IV

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} (n(n+1))^2$$



— Antonella Cupillari

Integer sums

```

input isometric_projection
set_projection(80/3, -30); ipscale := 16;
picture P[], Cube[]; numeric N; N = 4;
Cube0 = cube(background, background, background);
Cube1 = cube(Reds 8 4, Reds 8 2, Reds 8 1);
Cube2 = cube(Oranges 8 4, Oranges 8 2, Oranges 8 1);
Cube3 = cube(Purples 8 4, Purples 8 2, Purples 8 1);
Cube4 = cube(Blues 8 4, Blues 8 2, Blues 8 1);
P1 = image(for n = N downto 1:
    numeric a; a = 1/2 n * n - 1/2;
    for m = 1 upto n-1:
        numeric b; b = 1/2 m * m - 1/2;
        for z = 1 upto n:
            for x = 0 upto m-1:
                draw Cube[if n < N: 0 else: 4 fi] shifted p(b + x, 0, a + n - z);
            endfor
        endfor
    endfor
    for x = 0 upto n-1:
        for z = 1 upto n:
            draw Cube[if n < N: 0 else: 4 fi] shifted p(a + x, 0, a + n - z);
        endfor
    endfor
endfor
for n = 1 upto N:
    numeric a; a = 1/2 n * n - 1/2;
    for m = n-1 downto 1:
        numeric b; b = 1/2 m * m - 1/2;
        for x = 0 upto n-1:
            for z = 1 upto m:
                draw Cube[if n < N: 0 else: 4 fi] shifted p(a + x, 0, b + m - z);
            endfor
        endfor
    endfor
endfor);
% ... and so on to assemble all the other pictures

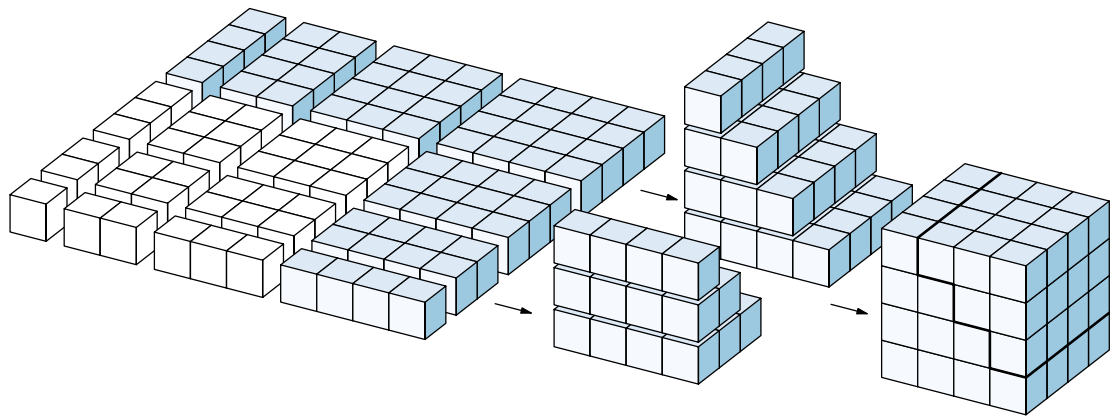
P7 = image(
    draw P1; draw P2 shifted p(12.5, 0, 8);
    draw P3 shifted p(13.5, 0, 0); draw P4 shifted p(22, 0, 1);
    drawarrow p(11, 0, 9.5) -- p(12, 0, 9.5);
    drawarrow p(11, 0, 2.5) -- p(12, 0, 2.5);
    drawarrow p(18, 0, 6.5) -- p(19, 0, 6.5);
    label.bot("$t_n = 1 + 2 + \cdots + n$ \quad \Rightarrow \quad $t_n^2 - t_{n-1}^2 = n^3$",
        point 1/2 of bbox currentpicture));

P8 = image(
    draw P5; draw P6; drawarrow p(11, 0, 2.5) -- p(12, 0, 2.5);
    label.bot("$t_n^2 = \left(1 + 2 + \cdots + n\right)^2 = 1^3 + 2^3 + \cdots + n^3$",
        point 1/2 of bbox currentpicture));

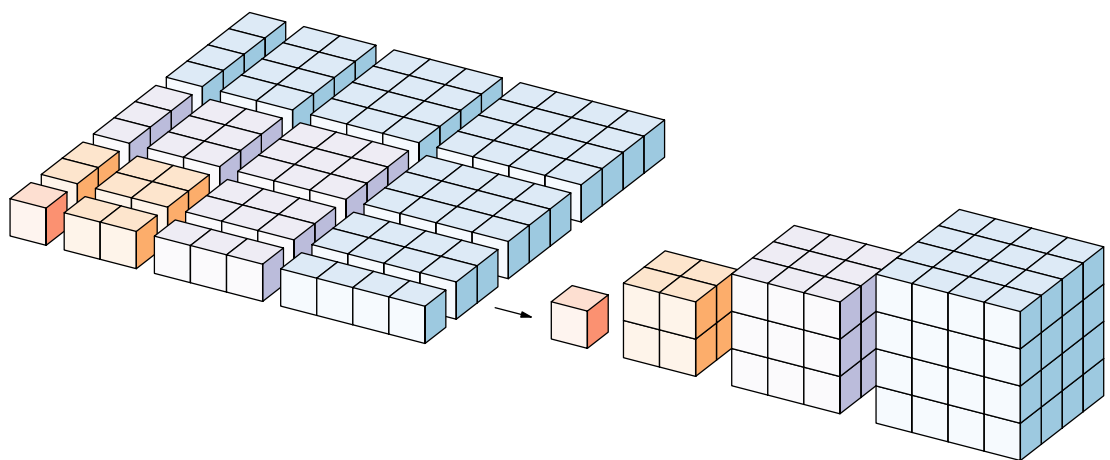
draw P7; draw P8 shifted 240 down;

```

Sums of cubes V



$$t_n = 1 + 2 + \dots + n \Rightarrow t_n^2 - t_{n-1}^2 = n^3$$



$$t_n^2 = (1 + 2 + \dots + n)^2 = 1^3 + 2^3 + \dots + n^3$$

— Roger Nelsen

```

vardef cartouche(expr w, d, r) = save p; path p; p =
  quartercircle rotated 180 shifted ( 1/2, 1/2) scaled r shifted (0-d, -d) --
  quartercircle rotated 270 shifted (-1/2, 1/2) scaled r shifted (w+d, -d) --
  quartercircle rotated 0 shifted (-1/2, -1/2) scaled r shifted (w+d, +d) --
  quartercircle rotated 90 shifted ( 1/2, -1/2) scaled r shifted (0-d, +d) --
  cycle; image(fill p withcolor Oranges 8 2; draw p;)
enddef;
vardef boomer(expr n, w, h, d, r) = save p; path p; p =
  quartercircle rotated 180 shifted ( 1/2, 1/2) scaled r shifted ( 0-d, -h*n-d) --
  quartercircle rotated 270 shifted (-1/2, 1/2) scaled r shifted (w*n+d, -h*n-d) --
  quartercircle rotated 0 shifted (-1/2, -1/2) scaled r shifted (w*n+d, 0+d) --
  if n > 0:
    quartercircle rotated 90 shifted (+1/2, -1/2) scaled r shifted (w*n-d, 0+d) --
    reverse
    quartercircle rotated 270 shifted (-1/2, 1/2) scaled r shifted (w*n-d, -h*n+d) --
  fi
  quartercircle rotated 90 shifted (+1/2, -1/2) scaled r shifted ( 0-d, -h*n+d) --
  cycle; image(fill p withcolor Blues 8 2; draw p)
enddef;

picture P[]; P0 = image(
  label.lrt(btex \vbox{\openup 16pt\halign{\hss ${}\#{}\&\&\hbox to 36pt{\hss$\$ $\hss}\cr
    & 1 & 2 & 3 & \cdots & n\cr
+ & 2 & 4 & 6 & \cdots & 2n\cr
+ & 3 & 6 & 9 & \cdots & 3n\cr
+ & \vdots & \vdots & \vdots & \ddots & \vdots \cr
+ & n & 2n & 3n & \cdots & n^2\cr
}} etex, origin));

P1 = image(
  picture c; c = cartouche(144, 8, 4);
  for i=0, 1, 2, 4: draw c shifted (34, -29.4i - 6); endfor
  draw P0;
  label.lrt("${}=\quad \sum_{i=1}^n i + 2\sum_{i=1}^n i + \cdots + n\sum_{i=1}^n i$",
    point 0 of bbox currentpicture shifted 16 down);
  label.lrt("${}=\quad \left(\sum_{i=1}^n i\right)^2$",
    point 0 of bbox currentpicture shifted 10 down);
);
P2 = image(
  numeric u, v; u = 36; v = 29.4;
  for i = 0, 1, 2, 4: draw boomer(i, u, v, 8, 4) shifted (33, -6); endfor
  draw P0;
  label.lrt("${}=\quad 1 + 2 \cdot 2^2 + 3 \cdot 3^2 + \cdots + n \cdot n^2$",
    point 0 of bbox currentpicture shifted 16 down);
  label.lrt("${}=\quad \sum_{i=1}^n i^3$",
    point 0 of bbox currentpicture shifted 16 down);
);

draw P1 shifted 112 left; draw P2 shifted 112 right;

```

Sums of cubes VI

$$\begin{array}{rcl}
& \boxed{1 \quad 2 \quad 3 \quad \dots \quad n} \\
+ & \boxed{2 \quad 4 \quad 6 \quad \dots \quad 2n} \\
+ & \boxed{3 \quad 6 \quad 9 \quad \dots \quad 3n} \\
+ & \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
+ & \boxed{n \quad 2n \quad 3n \quad \dots \quad n^2} \\
= & \sum_{i=1}^n i + 2 \sum_{i=1}^n i + \dots + n \sum_{i=1}^n i \\
= & \left(\sum_{i=1}^n i \right)^2
\end{array}$$

$$\begin{array}{rcl}
& \boxed{1} \quad \boxed{2} \quad \boxed{3} \quad \dots \quad \boxed{n} \\
+ & \boxed{2} \quad \boxed{4} \quad \boxed{6} \quad \dots \quad \boxed{2n} \\
+ & \boxed{3} \quad \boxed{6} \quad \boxed{9} \quad \dots \quad \boxed{3n} \\
+ & \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
+ & \boxed{n} \quad \boxed{2n} \quad \boxed{3n} \quad \dots \quad \boxed{n^2} \\
= & 1 + 2 \cdot 2^2 + 3 \cdot 3^2 + \dots + n \cdot n^2 \\
= & \sum_{i=1}^n i^3
\end{array}$$

— Farhood Pouryoussefi

```

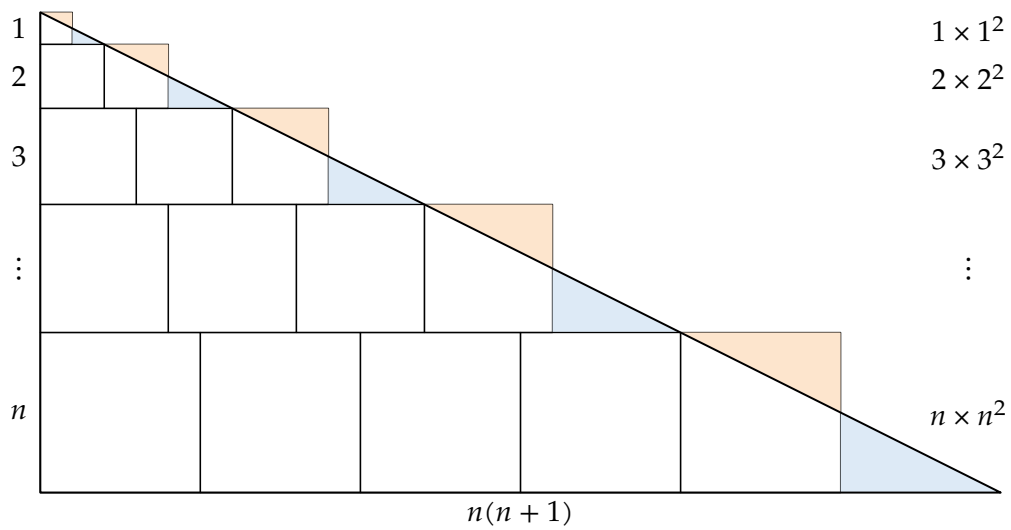
numeric N, u, y;
N = 5; y = 0; u = 12;
for n = 1 upto N:
  numeric w; w = n * u;
  path s, t;
  s = unitsquare scaled w;
  y := y - w;
  label("$" &
    if n = N: "n"
    elseif n = N - 1: "\vdots"
    else: decimal n
    fi & "$", (-8, y + 1/2 w));
  label("$" &
    if n = N: "n \times n^2"
    elseif n = N - 1: "\vdots"
    else: decimal n & "\times" & decimal n & "^2"
    fi & "$", (N * N * u + 4 u, y + 1/2 w));
  for x = 0 upto n - 1:
    draw s shifted (x * w, y);
  endfor
  t = subpath (3/2, 3) of s shifted ((n - 1) * w, y) -- cycle;
  fill t withcolor Oranges 8 2;
  fill t rotatedabout(point 0 of t, 180) withcolor Blues 8 2;
endfor
draw origin -- (0, y) -- (N * (N + 1) * u, y) -- cycle withpen pencircle scaled 3/4;
label.bot("$n(n+1)$", (1/2 N * (N + 1) * u, y));
label.top(btex
  \vbox{\openup 12pt\halign{\hss $$$${}=#$ \hss \cr
    1 + 2 + \cdots + n & \frac{1}{2} n(n+1)\cr
    1^3 + 2^3 + \cdots + n^3 & \left(\frac{1}{2} n(n+1)\right)^2\cr
  }} etex, point 5/2 of bbox currentpicture shifted 42 up);

```


Sums of integers and sums of cubes

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$$



— Georg Schrage

Integer sums

```

numeric N, u, y;
N = 4; y = 0; u = 6;

% see document source for definitions of sq, sqmark, and sqclip

picture P[];
P1 = image(
  for n = 1 upto N:
    string s, S;
    s = if n = N: "\left(2n-1\right)" else: decimal (2n - 1) fi;
    S = "$" & s & "^3 = " & s & "\times" & s & "^2 = {}$";
    numeric y; y = (n * n + n + n) * -u;
    label.lft(S, (80, y));
    for i = 1 upto 2n - 1:
      draw if (i=1) or (i=2n-1): sqmark(2n - 1) else: sq(2n-1) fi
        shifted (80, y + 1/2 u)
        shifted (if i > n: (n-1, i - n - 1/2) else: (i-1, -1/2) fi * (2n+1) * u);
    endfor
  endfor);

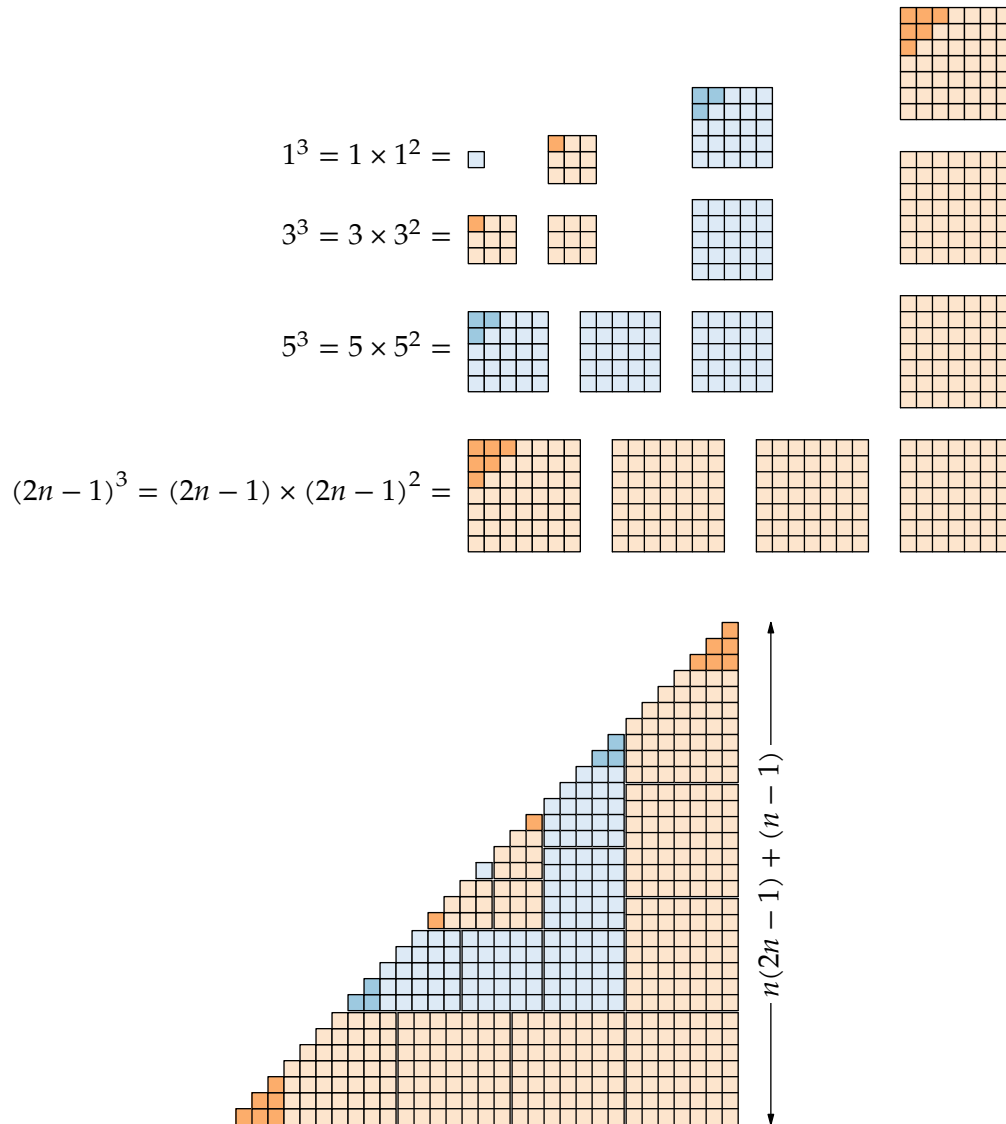
input arrow_label
P2 = image(
  numeric x, y; x = y = 0;
  for n = 1 upto N:
    picture C, Up, Left; C = sq(2n - 1); Left = sqclip(2n - 1);
    Up = Left rotated -90 reflectedabout(up, down) shifted ((2n-1, 3n-2) * u);
    draw C shifted ((x, y) * u);
    for i = 2 upto n:
      draw if i < n: C else: Up fi
        shifted ((x, y + ((i-1) * (2n - 7/8))) * u);
      draw if i < n: C else: Left shifted ((1-n)*u,0) fi
        shifted ((x - ((i-1) * (2n - 7/8)), y) * u);
    endfor
    x := x + 2n - 1 + 1/8;
    y := y - 2n - 1 - 1/8;
  endfor
  arrow_label(lrcorner currentpicture, urcorner currentpicture,
    TEX("$n(2n-1) + (n-1)$") rotated 90, 12));

label.top(P1, 10 up);
label.bot(P2, 10 down);

label.bot(btex $
  1^3 + 3^3 + 5^3 + \cdots + \left( 2n - 1 \right)^3
  = 1 + 2^2 + 3^3 + \cdots + \left( 2n^2 - 1 \right)
  = n^2 \left( 2n^2 - 1 \right)
$ etex, point 1/2 of bbox currentpicture shifted 34 down);

```

Sums of odd cubes are triangular numbers



$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = 1 + 2 + 3 + \cdots + (2n^2 - 1) = n^2 (2n^2 - 1)$$

— Monte J. Zerger

```

numeric u, o; u = 12; o = 0;
for n = 1 upto 4:
  path s; s = unitsquare rotated -90 scaled n scaled n shifted (o, -o) scaled u;
  fill s withcolor Greens 8 2;
  if n=1:
    label("$\scriptstyle 1^4$", center s);
  else:
    path t, tt;
    t = subpath (4, 3) of s -- subpath (3, 4) of s shifted (0, o*u) -- cycle;
    tt = t reflectedabout(origin, dir -45);
    fill t withcolor Purples 8 2;
    fill tt withcolor Purples 8 2;
    draw subpath (-1, 1) of t;
    draw subpath (-1, 1) of tt;
    label("$" if n=2: & "\scriptstyle" fi & decimal n & "^2\left(1^2"
      for i=2 upto n-1:
        & "+" & decimal i & "^2"
      endfor & "\right)$", center t);
    fill (superellipse(right, up, left, down, 0.78)) scaled u shifted center s
      withcolor Greens 9 1;
    label("$" & decimal n & "^4$", center s);
  fi
  o := o + n * n;
endfor
path s; s = unitsquare xscaled o yscaled -o scaled u; draw s;
for i=1 upto o-1:
  draw (origin -- 2 up) shifted ((i, -o)*u);
  if i > 5: draw (origin -- 2 down) shifted ((i, 0)*u); fi
  draw (origin -- 2 left) shifted ((o, -i)*u);
  draw (origin -- 2 right) shifted ((0, -i)*u);
endfor

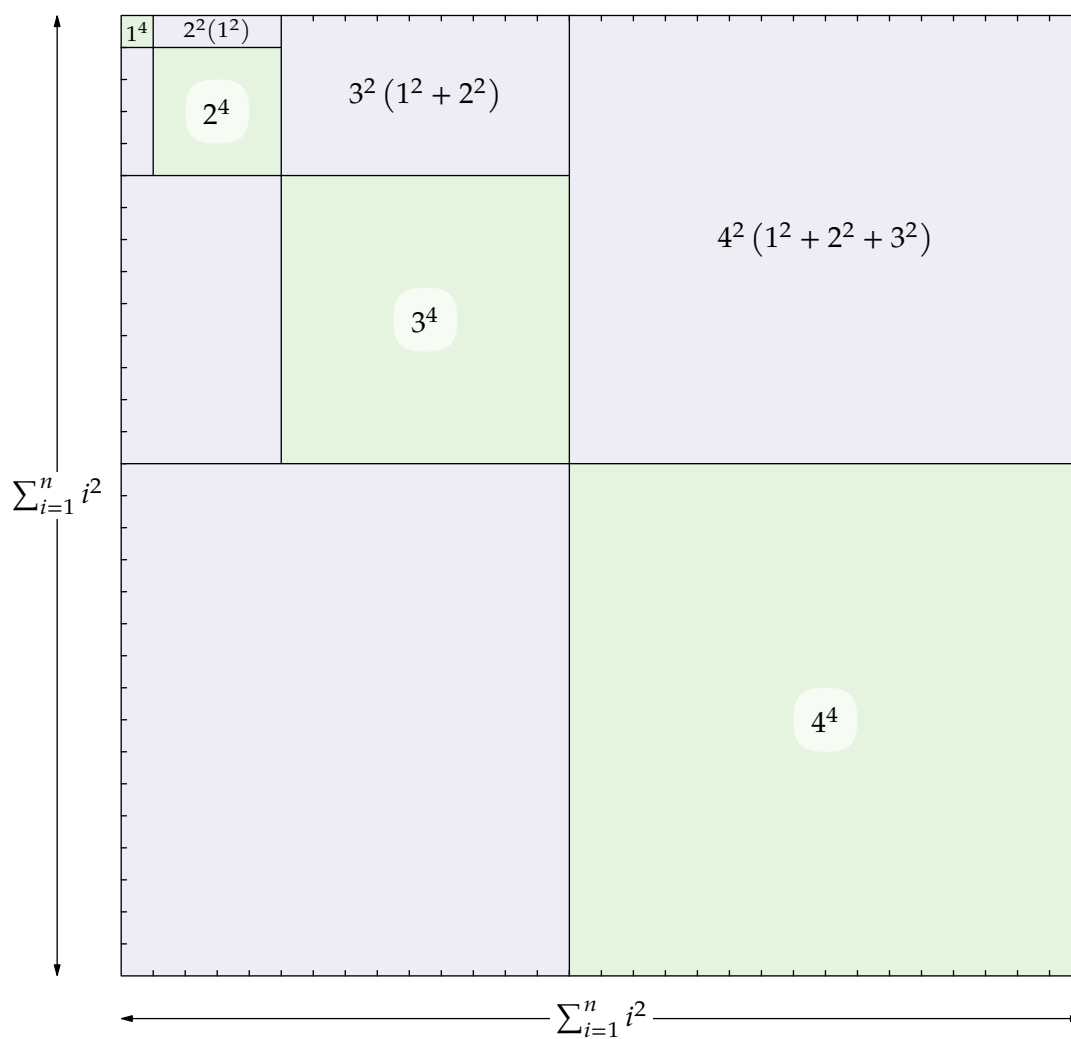
input arrow_label
arrow_label(point 4 of s, point 3 of s, "$\sum_{i=1}^n i^2$", 24);
arrow_label(point 3 of s, point 2 of s, "$\sum_{i=1}^n i^2$", 16);

label.top(btex $
\sum_{i=1}^n i^4
= \left(\sum_{i=1}^n i^2 \right)^2
- 2 \left(\sum_{k=2}^n \left(k^2 \sum_{i=1}^{k-1} i^2 \right)\right)$
etex, point 5/2 of bbox currentpicture shifted 34 up);

```

Sums of fourth powers

$$\sum_{i=1}^n i^4 = \left(\sum_{i=1}^n i^2\right)^2 - 2\left(\sum_{k=2}^n \left(k^2 \sum_{i=1}^{k-1} i^2\right)\right)$$



— Elizabeth M. Markham

```

numeric u, N;
u = 16; N = 5;

for n = N downto 1:
    path s; s = unitsquare scaled u xscaled (N * N - N + n) yscaled n;
    fill s withcolor if odd n: Blues else: Oranges fi 8 2;
endfor

for i=1 upto N * N - N - 1:
    draw (origin -- up * N * u) shifted (i*u, 0) dashed evenly scaled 1/2;
endfor

z0 = ((N * N - N) * u, 0);
draw (origin -- up * N * u) shifted z0;

for i = 1 upto N - 1:
    draw (x0 + i * u, N * u) -- (x0 + i * u, y0 + i * u) -- (N * N * u, y0 + i * u)
        dashed evenly scaled 1/2;
endfor

for n = N downto 1:
    path s; s = unitsquare scaled u xscaled (N * N - N + n) yscaled n;
    draw s;
endfor
input arrow_label

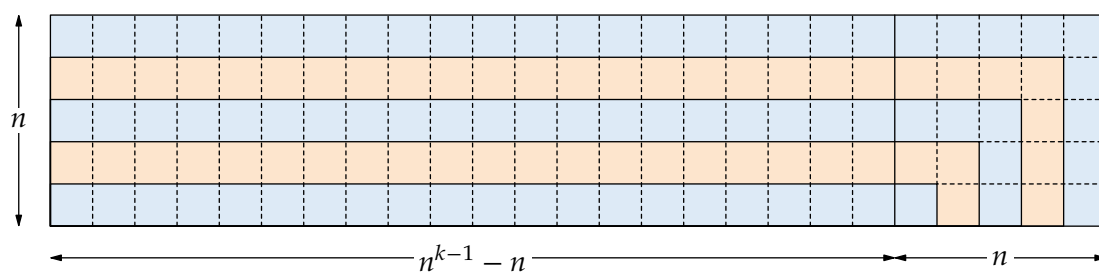
arrow_label(origin, z0, "$n^{k-1}-n$", 12);
arrow_label(z0, (N*N*u, 0), "$n$", 12);
arrow_label(origin, (0, N*u), "$n$", -12);

label.top(btex $
n^k = \left(n^{k-1} - n + 1\right)
      + \left(n^{k-1} - n + 3\right) + \cdots
      + \left(n^{k-1} - n + 2n - 1\right)$ for $k=2, 3, \dots$
etex, point 5/2 of bbox currentpicture shifted 34 up);

```

k -th powers as sums of consecutive odd numbers

$$n^k = (n^{k-1} - n + 1) + (n^{k-1} - n + 3) + \cdots + (n^{k-1} - n + 2n - 1) \text{ for } k = 2, 3, \dots$$



— N. Gopalakrishnan Nair

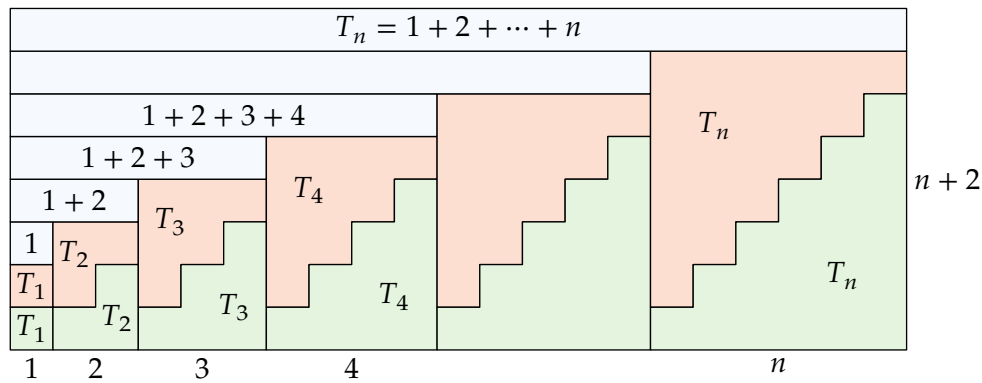
```

numeric t, u, N; t = 0; u = 16; N = 6;
for n = 1 upto N:
  path W, M, B;
  W = (origin -- (n, 0) -- (n, n) for i = 1 upto n:
    -- (n-i, n-i+1) -- (n-i, n-i)
  endfor -- cycle) shifted (t, 0) scaled u;
  M = W rotatedabout(point n + 3/2 of W, 180);
  fill M withcolor Reds 8 2; draw M;
  fill W withcolor Greens 8 2; draw W;
  t := t + n;
  B = unitsquare xscaled -t scaled u shifted point 0 of M;
  fill B withcolor Blues 8 1; draw B;
  if n = 1:
    label("$1$", center B);
    label("$T_1$", center W);
    label("$T_1$", center M);
    label.bot("$1$", point 1/2 of W);
  elseif n < N - 1:
    label("$1$ for i=2 upto n: & '+' & decimal i endfor & '$', center B);
    label("$T_-$ & decimal n & '$', 1/2[point 1 of W, point n + 3/2 of W]);
    label("$T_-$ & decimal n & '$', 1/2[point 1 of M, point n + 3/2 of W]);
    label.bot("$" & decimal n & "$", point 1/2 of W);
  elseif n = N:
    label("$T_n = 1 + 2 + \cdots + n$", center B);
    label("$T_n$", 1/2[point 1 of W, point n + 3/2 of W]);
    label("$T_n$", 1/2[point 1 of M, point n + 3/2 of W]);
    label.bot("$n$", point 1/2 of W);
    label.rt("$n+2$", point 3/2 of W shifted (0, u));
  fi
endfor
label.top(btex $T_n = 1 + 2 + \cdots + n$ \quad implies\quad
  $\displaystyle T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$
  etex, point 5/2 of bbox currentpicture shifted 34 up);
label.bot(btex $\left(T_1+T_2+\cdots+T_n\right) = (n+2) \cdot T_n $
  etex, point 1/2 of bbox currentpicture shifted 34 down);
label.bot(btex $\displaystyle T_1+T_2+\cdots+T_n =
  \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$
  etex, point 1/2 of bbox currentpicture shifted 21 down);

```


Sums of triangular numbers I

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



$$3(T_1 + T_2 + \cdots + T_n) = (n+2) \cdot T_n$$

$$T_1 + T_2 + \cdots + T_n = \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

— Monte J. Zerger

```

numeric t, N; t = 0; N = 5;
input arrow_label
input isometric_projection
ipscale := 14; set_projection(25, -30);
picture P[], r_cube;
r_cube = cube(Reds 8 4, Reds 8 2, background);
P7 = image(
  for n=1 upto N:
    for i = n downto 1:
      for j = 1 upto i:
        draw r_cube shifted p(2n, j, i);
      endfor
    endfor
  endfor
);
P1 = image(
for n=1 upto N:
  for i = n downto 1:
    for j = 1 upto i:
      draw r_cube shifted p(n, j, i);
    endfor
  endfor
endfor
);
P8 = image(
  draw P1;
  arrow_label(p(N + 1/2, 1, N + 1), p(N + 1/2, N + 1, N + 1), "$n$", 0);
);

path a; a = (left--right) scaled 18;
drawarrow a;
label.lft(P7, point 0 of a shifted 16 left);
label.rt(P8, point 1 of a);
% more juggling of the drawing order for other pictures...

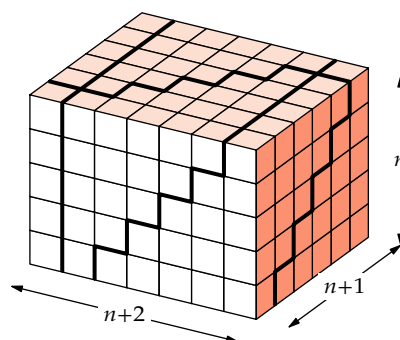
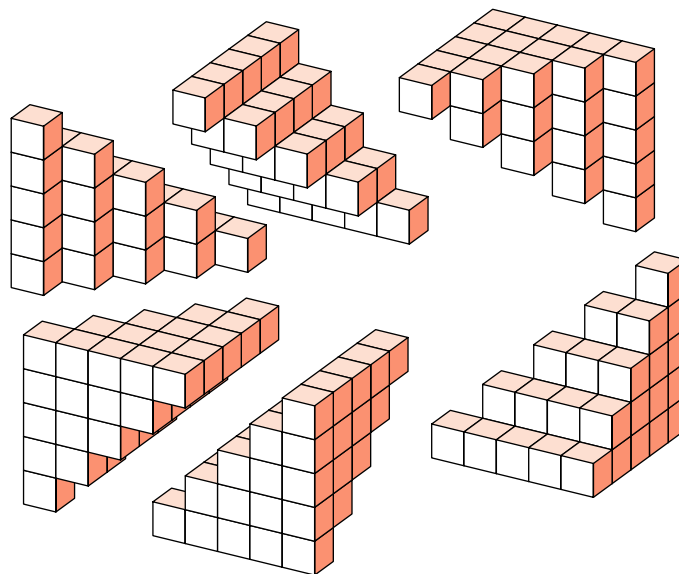
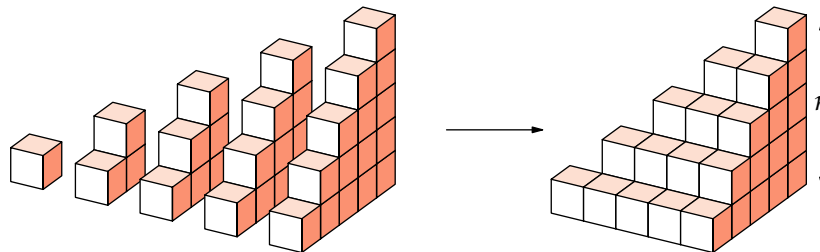
label.bot(P9, point 1/2 of bbox currentpicture shifted 21 down);
label.bot(P10, point 1/2 of bbox currentpicture shifted 21 down);

label.top(btex $T_n = 1 + 2 + \cdots + n$ \quad implies\quad
  $\displaystyle T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$
etex, point 5/2 of bbox currentpicture shifted 34 up);

```

Sums of triangular numbers II

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



— Roger B. Nelsen

```

path t; t = for i=1 upto 3: 85up rotated 120i -- endfor cycle;
string s[];
s0 = "$\scriptstyle n$";
s1 = "$\scriptstyle n-1$";
s2 = "$\cdot$";
s3 = "$\cdot$";
s4 = "3";
s5 = "2";
s6 = "1";
s7 = "$\scriptstyle n+2$";
numeric N; N = 6;
for p=0 upto 2:
  t := t rotated 120;
  picture P;
  P = image(
    label(s0, point 0 of t);
    for n = 1 upto N:
      for i = 0 upto n:
        label(s[n], (i/n)[point -n/N of t, point n/N of t]);
      endfor
    endfor
  );
  draw P shifted (180p, 0);
endfor
label("$+$", (90, 30));
label("$+$", (270, 30));
label("$=$", (90, -150));
picture P;
P = image(
  label(s7, point 0 of t);
  for n = 1 upto N:
    for i = 0 upto n:
      label(s7, (i/n)[point -n/N of t, point n/N of t]);
    endfor
  endfor
);
draw P shifted (180, -180);

label.top(btex $T_n = 1 + 2 + \cdots + n$ \quad implies\quad
$T_1 + T_2 + \cdots + T_n = \frac{1}{6} n(n+1)(n+2)$
etex, point 5/2 of bbox currentpicture shifted 34 up);

label.bot(btex $3\left(T_1 + T_2 + \cdots + T_n\right) = T_n \cdot (n+2)$
etex, point 1/2 of bbox currentpicture shifted 34 down);

```

Sums of triangular numbers III

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{1}{6}n(n+1)(n+2)$$

$$\begin{array}{ccccccc}
 & & 1 & & & n & & & 1 \\
 & & 1 & 2 & & n-1 & n-1 & & 2 & 1 \\
 & & 1 & 2 & 3 & & \cdot & \cdot & \cdot & & 3 & 2 & 1 \\
 & & 1 & 2 & 3 & \cdot & & & \cdot & \cdot & \cdot & \cdot & & \cdot & 3 & 2 & 1 \\
 & & 1 & 2 & 3 & \cdot & \cdot & & 3 & 3 & 3 & 3 & 3 & & \cdot & \cdot & \cdot & 3 & 2 & 1 \\
 & & 1 & 2 & 3 & \cdot & \cdot & n-1 & & 2 & 2 & 2 & 2 & 2 & 2 & & n-1 & \cdot & \cdot & \cdot & 3 & 2 & 1 \\
 & & 1 & 2 & 3 & \cdot & \cdot & n-1 & n & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & n & n-1 & \cdot & \cdot & 3 & 2 & 1
 \end{array}$$

$$\begin{array}{c}
 n+2 \\
 n+2 \quad n+2 \\
 n+2 \quad n+2 \quad n+2 \\
 = \\
 n+2 \quad n+2 \quad n+2 \quad n+2 \\
 n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \\
 n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \\
 n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \\
 n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2 \quad n+2
 \end{array}$$

$$3(T_1 + T_2 + \cdots + T_n) = T_n \cdot (n+2)$$

```

input isometric_projection Integer sums
ipscale := 24; set_projection(25, -30);
picture cb[], P[];
cb1 = cuboid(-1, 2, 1, Blues 8 4, Blues 8 2, background);
cb2 = cuboid(-1, 4, 3, Blues 8 4, Blues 8 2, background);
cb3 = cuboid(-2, 4, 3, Blues 8 4, Blues 8 2, background);
P1 = image(
  draw cb1;
  for i = -1 upto 1:
    draw cb1 shifted p(i, 0, 0) shifted 144 right;
  endfor
  drawoptions(withcolor Blues 8 7);
  label.bot("$\scriptstyle 1$", p(-1/2, 0, 0));
  label.lrt("$\scriptstyle 1$", p(0, 0, 1/2));
  label.rt("$\scriptstyle 2$", p(0, 1, 1));
  for i = -1 upto 1:
    label.bot("$\scriptstyle 1$", p(i-1/2, 0, 0) shifted 144 right);
  endfor
  label.lrt("$\scriptstyle 1$", p(1, 0, 1/2) shifted 144 right);
  label.rt("$\scriptstyle 2$", p(1, 1, 1) shifted 144 right);
  drawoptions();
  label("$= \quad \frac{13}{3}$", (64, 21));
);
P2 = image(
  draw cb3 shifted 20 left;

  draw cb2 shifted 80 right;
  draw cb3 shifted p(-1, 0, 0) shifted 240 right;
  for i = 0 upto 2:
    draw cb2 shifted p(i, 0, 0) shifted 240 right;
  endfor
  label("$= \quad \frac{13}{3}$", (150, 64));
  label("$\frac{13}{3}$", (-78, 64));
  label("$+$", (38, 64));
  drawoptions(withcolor Blues 8 7);
  label.llft("$\scriptstyle n-1$", p(-3/4,0,0)) shifted 20 left;
  label.lrt("$\scriptstyle n$", p(0,0,3/2)) shifted 20 left;
  label.rt("$\scriptstyle n+1$", p(0,5/2,3)) shifted 20 left;

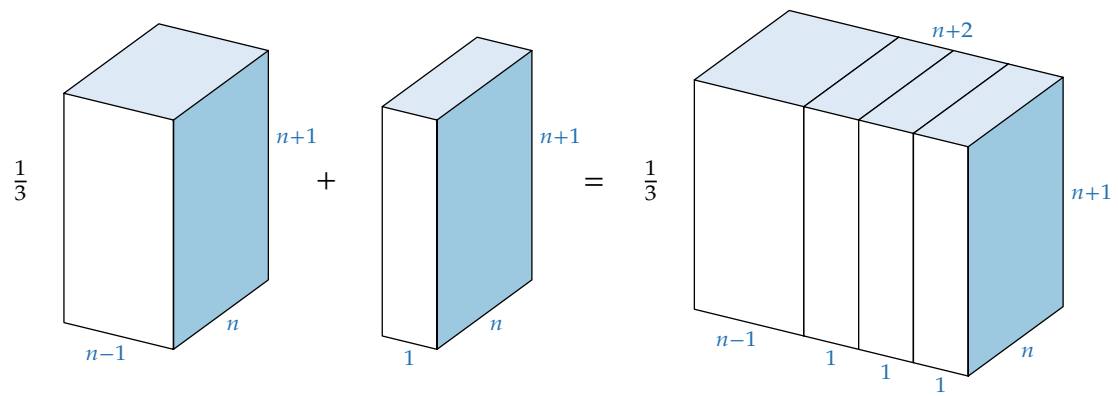
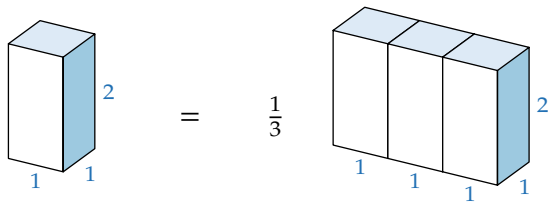
  label.bot("$\scriptstyle 1$", p(-1/2,0,0)) shifted 80 right;
  label.lrt("$\scriptstyle n$", p(0,0,3/2)) shifted 80 right;
  label.rt("$\scriptstyle n+1$", p(0,5/2,3)) shifted 80 right;

  label.llft("$\scriptstyle n-1$", p(-7/4,0,0)) shifted 240 right;
  label.lrt("$\scriptstyle n$", p(2,0,3/2)) shifted 240 right;
  label.rt("$\scriptstyle n+1$", p(2,2,3)) shifted 240 right;
  label.urt("$\scriptstyle n+2$", p(-1/2,4,3)) shifted 240 right;
  label.bot("$\scriptstyle 1$", p(-1/2,0,0)) shifted 240 right;
  label.bot("$\scriptstyle 1$", p(1/2,0,0)) shifted 240 right;
  label.bot("$\scriptstyle 1$", p(3/2,0,0)) shifted 240 right;
  drawoptions();
);
draw P1; draw P2 shifted 240 down;
label.top("$ (1\times 2)+(2\times 3)+(3\times 4) + \cdots + (n-1)n = \frac{13}{3} (n-1) n (n+1)$",
  point 5/2 of bbox currentpicture shifted 34 up);

```

Sums of oblong numbers I

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + \cdots + (n-1)n = \frac{1}{3}(n-1)n(n+1)$$



— T. C. Wu

```

input isometric_projection
ipscale := 24; set_projection(28, -28);
numeric i; i = 0;
picture c[];
forsuffixes s = Reds, Greens, Oranges, Blues:
  c[incr i] = cube(s 8 4, s 8 3, s 8 2);
endfor

picture P[];
def make_boxes(expr n, X, Y, Z, Sx, Sy, Sz) =
  P[n] = image(
    for x = 1 upto X:
      for y = 0 upto Y-1:
        for z = Z-1 downto 0:
          draw c[n] shifted p(x, y, z);
        endfor
      endfor
    endfor
  );
  P[11n] = image(draw P[n];
    label.lft(Sy, p(0, 1/2Y, 0));
    label.bot(Sx, p(1/2X, 0, 0));
    label.lrt(Sz, p(X, 0, 1/2Z));
  ) enddef;

make_boxes(1, 2, 1, 3, "2", "1", "3");
make_boxes(2, 2, 3, 3, "2", "3", "3");
make_boxes(3, 3, 4, 3, "3", "4", "3");
make_boxes(4, 5, 4, 3, "$n+1$\strut", "$n$", "3");

P5 = image(
  draw P4 shifted p(0, 0, 3);
  draw P2; draw P1 shifted p(0, 3, 0);
  draw P3 shifted p(2,0,0);
);

numeric s; s = 80;
label(P11, (-2s,5/2s)); label("$+$", (-s,5/2s));
label(P22, (0,5/2s)); label("$+$", (+s,5/2s));
label(P33, (+2s,5/2s));

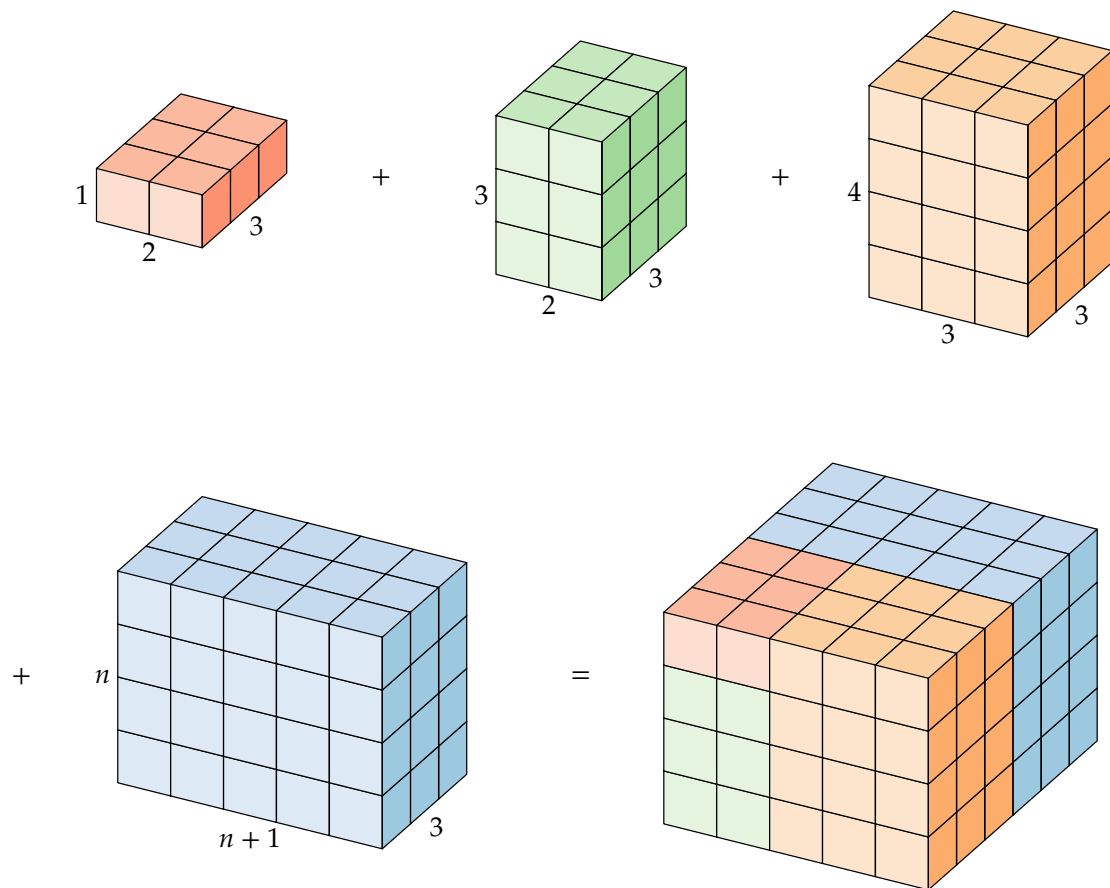
label("$+$", (-2.8s, 0));
label(P44, (-3/2s, 0));
label("$=$", (0, 0));
label(P5, (3/2s, 0));

label.top(btex $3 \bigl(1\times2 + 2\times3 + 3\times4 + \cdots + n(n+1) \bigr)
  = n (n+1) (n+2)$ etex, point 5/2 of bbox currentpicture shifted 34 up);

```


Sums of oblong numbers II

$$3(1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1)) = n(n+1)(n+2)$$



— Sidney H. Kung

```

input isometric_projection Integer sums
ipscale := 16; set_projection(24, -28);
picture P[], c[]; numeric i; i = 0;
forsuffixes s = Reds, Greens, Oranges, Blues:
  c[incr i] = cube(s 8 4, s 8 3, s 8 2);
endfor
P1 = image(for i=1 upto 4:
  numeric z; z = 3i * sqrt(i);
  for j = i downto 1:
    for k = 1 upto i:
      draw c[i] shifted p(-j, k, z);
    endfor
  endfor
  for k=1 upto i-1:
    for j=i-k downto 1:
      draw c[i] shifted p(-j - 1/2 k, i + k, z);
    endfor
  endfor
endfor);
P2 = image(for i=1 upto 4:
  numeric z; z = 3i * sqrt(i);
  for j = i downto 1:
    for k = 1 upto i:
      draw c[i] shifted p(-j, k, z);
    endfor
  endfor
  for k=1 upto i-1:
    for j=i-k downto 1:
      draw c[i] shifted p(-j, i, z -k);
    endfor
  endfor
endfor);
P3 = image(path base, lid;
  base = p(-1,1,1) -- p(-6, 1, 1) -- p(-6, 1, 5) -- p(-1, 1, 5) -- cycle;
  lid = base shifted p(0,4,0);
  drawoptions(dashed withdots scaled 1/2);
  draw base; for i=0 upto 3: draw point i of base -- point i of lid; endfor
  drawoptions();
  for z = 4 downto 1:
    for x = z downto 1:
      for y = 1 upto 4:
        draw c[max(y,z)] shifted p(-x, y, z);
      endfor
    endfor
  endfor
  draw lid dashed withdots scaled 1/2;
  label.lrt("$n$", point -1/2 of base);
  label.rt("$n$", 1/2[point -1 of base, point -1 of lid]);
  label.urt("$n+1$", point 5/2 of lid));

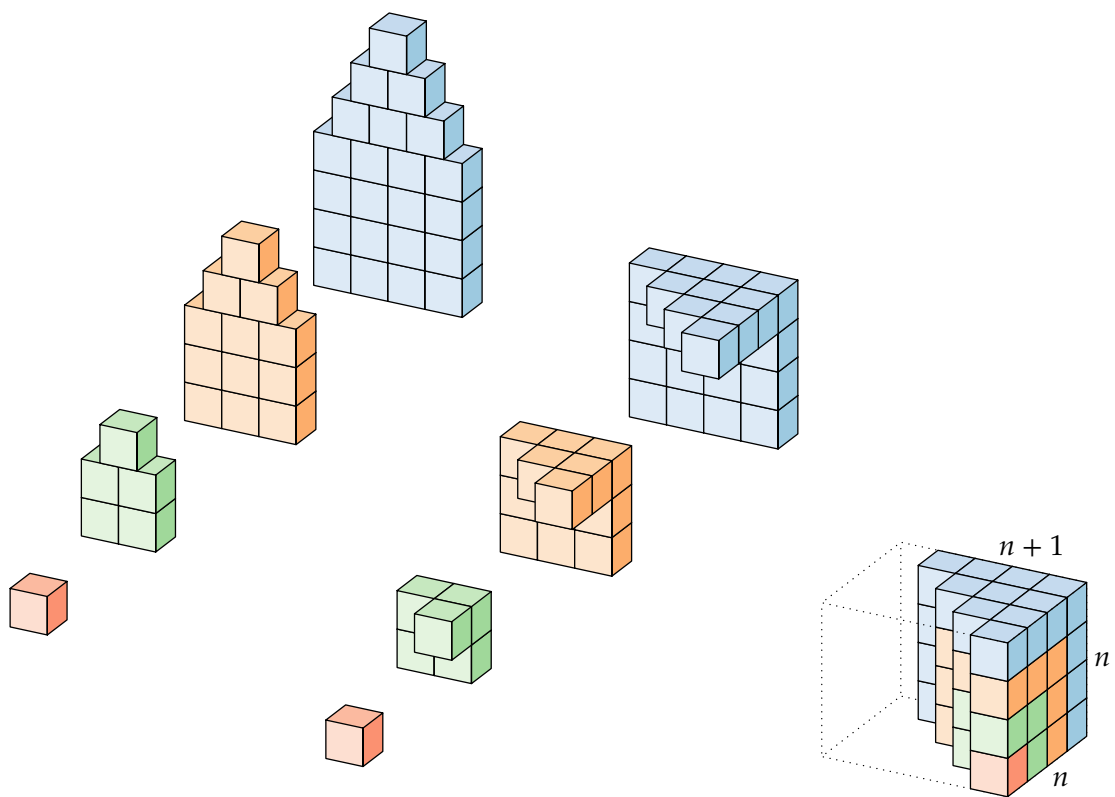
draw P1; draw P2 shifted (120, -50); draw P3 shifted (380, -50);

label.top(btex $\displaystyle
\frac{1\cdot 2}{2} + \frac{2\cdot 5}{2} + \frac{3\cdot 8}{2} + \cdots + \frac{n(3n-1)}{2}
= \frac{n^2(n+1)}{2}$ etex, point 5/2 of bbox currentpicture shifted 34 up);

```

Sums of pentagonal numbers

$$\frac{1 \cdot 2}{2} + \frac{2 \cdot 5}{2} + \frac{3 \cdot 8}{2} + \cdots + \frac{n(3n-1)}{2} = \frac{n^2(n+1)}{2}$$



— William A. Miller

```

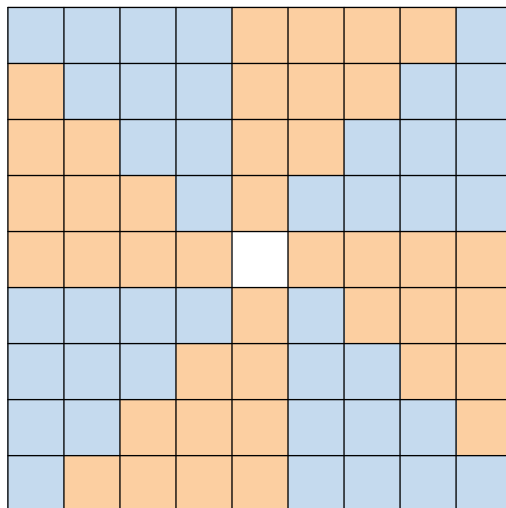
numeric u, n; u = 21; n = 4;
picture P[];
P1 = image(
  fill unitsquare shifted -(1/2, 1/2) scaled (2n+1) scaled u withcolor Oranges 8 3;
  unfill unitsquare shifted -(1/2, 1/2) scaled u;
  for k=0 upto 3:
    for i=0 upto n-1:
      for j=i upto n-1:
        fill unitsquare shifted (j + 1/2, i + 1/2)
          scaled u rotated 90k withcolor Blues 8 3;
      endfor
    endfor
  endfor
  for i=-n upto n+1:
    draw (left--right) scaled (n+1/2) shifted (0, i-1/2) scaled u;
    draw (down--up) scaled (n+1/2) shifted (i-1/2, 0) scaled u;
  endfor
  label.bot("$\left(2n+1\right)^2 = 8T_n + 1$",
    point 1/2 of bbox currentpicture shifted 8 down);
);
P2 = image(
  for k=0 upto 3:
    for i=1 upto n-1:
      for j=i upto n-1:
        fill unitsquare shifted (j, i - 1)
          scaled u rotated 90k withcolor Blues 8 3;
        fill unitsquare shifted (i - 1, j)
          scaled u rotated 90k withcolor Oranges 8 3;
      endfor
    endfor
  endfor
  for i=-n upto n:
    draw (left--right) scaled n shifted (0, i) scaled u;
    draw (down--up) scaled n shifted (i, 0) scaled u;
  endfor

  label.bot("$\left(2n\right)^2 = 8T_{n-1} + 4n$",
    point 1/2 of bbox currentpicture shifted 8 down);
);
interim labeloffset := 13;
label.top(P1, origin); label.bot(P2, origin);
label.top("$T_n = 1 + 2 + \cdots + n$ \quad \rightarrow",
  point 5/2 of bbox currentpicture shifted 21 up);

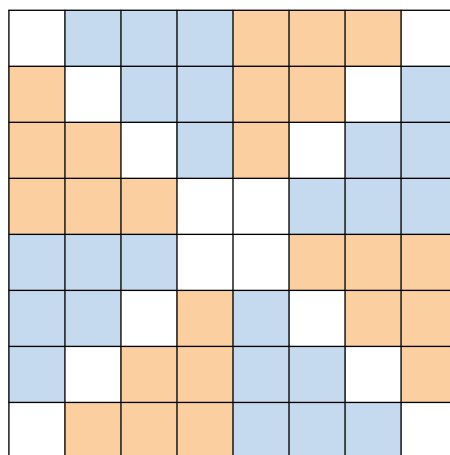
```

On squares of positive integers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



$$(2n + 1)^2 = 8T_n + 1$$



$$(2n)^2 = 8T_{n-1} + 4n$$

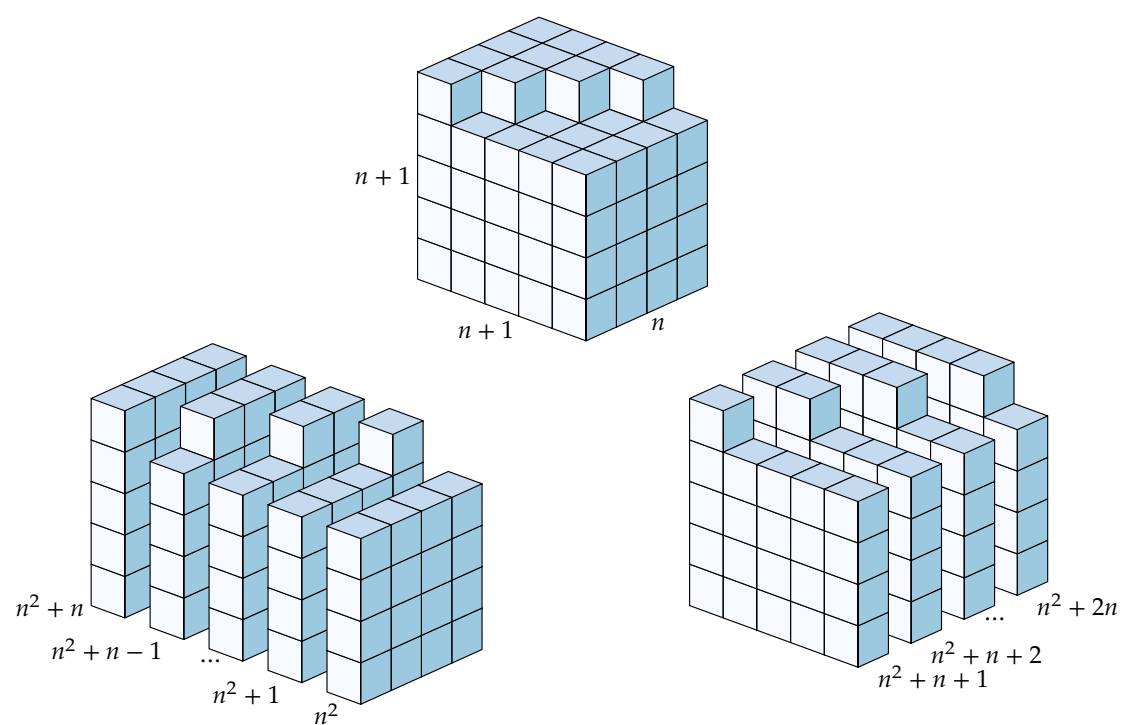
— Edwin G. Landauer

```

input isometric_projection
ipscale := 20;
set_projection(24, -42);
picture P[];
picture c;
c = cube(Blues 8 4, Blues 8 3, Blues 8 1);
for n=1 upto 3:
  P[n] = image(
    for x=0 upto 4:
      for z=3 downto 0:
        for y=0 upto 4:
          if (y=4) and (z < x): else:
            draw c shifted p(x if n=2: + 3/4 x fi, y, z if n=3: + 3/4 z fi);
          fi
        endfor
      endfor
    endfor
  if n=1:
    label.lrt("$n$", p(4,0,2));
    label.llft("$n+1$", p(2,0,0));
    label.lft("$n+1$", p(-1,5/2,0));
  elseif n=2:
    label.llft("$n^2$", p(4 + 12/4 - 1/2, 0,0));
    label.llft("$n^2+1$", p(3 + 9/4 - 1/2, 0,0));
    label.llft("$\cdots$", p(2 + 6/4 - 1/2, 0,0));
    label.llft("$n^2+n-1$", p(1 + 3/4 - 1/2, 0,0));
    label.lft("$n^2+n$", p(-1,0,0));
  elseif n=3:
    label.lrt("$n^2+n+1$", p(4, 0, 1/2));
    label.lrt("$n^2+n+2$", p(4, 0, 1/2 + 7/4));
    label.lrt("$\cdots$", p(4, 0, 1/2 + 14/4));
    label.lrt("$n^2+2n$", p(4, 0, 1/2 + 21/4));
  fi
);
endfor
draw P1 shifted 144 up;
draw P2 shifted 144 left;
draw P3 shifted 120 right;
label.bot(btex\vbox{\openup 6pt \halign{\hfill $$$${}#{}$$$$\hfill\cr
1+2&=&3\cr
4+5+6&=&7+8\cr
9+10+11+12&=&13+14+15\cr
16+17+18+19+20&=&21+22+23+24\cr
&\vdots&\cr
n^2+(n^2+1)+\cdots+(n^2+n)&=&(n^2+n+1)+\cdots+(n^2+2n)\cr}} etex,
point 1/2 of bbox currentpicture shifted 13 down);

```

Consecutive sums of consecutive integers



$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$

$$\vdots$$

$$n^2 + (n^2 + 1) + \cdots + (n^2 + n) = (n^2 + n + 1) + \cdots + (n^2 + 2n)$$

— Roger B. Nelsen

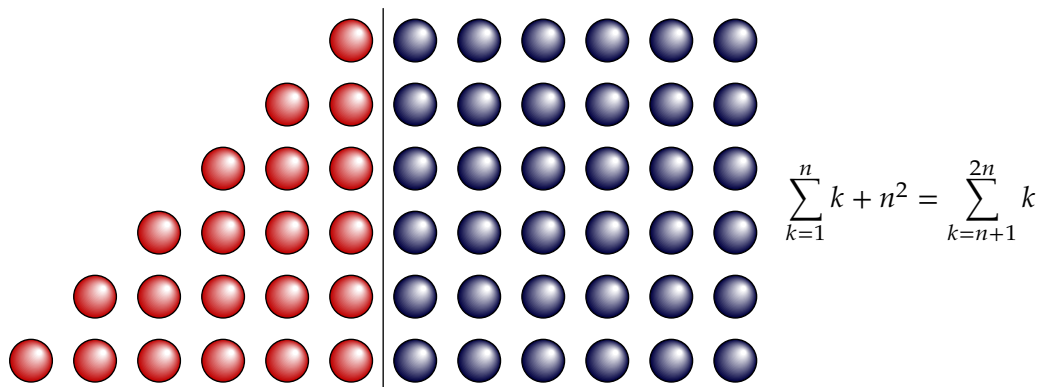
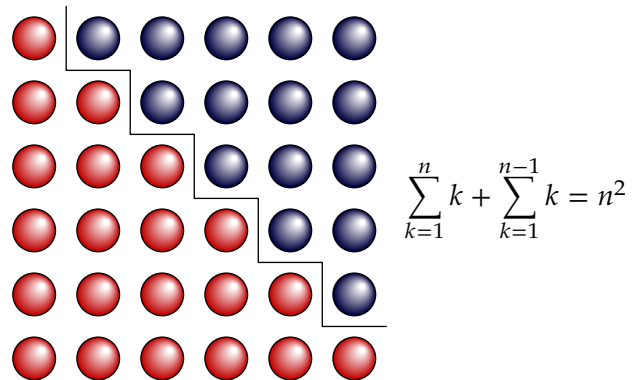
```

input paintball
numeric n; n = 6;
picture P[];
P1 = image(
  for i=1 upto n:
    for j=1 upto n:
      draw if j <= n+1-i: rball else: bball fi shifted (24i, 24j);
    endfor
  endfor
  draw (for i=1 upto n-1: (i, n+1-i) -- (i, n-i) -- endfor (n, 1))
    shifted (1/2, 1/2) scaled 24;

);
P2 = image(
  for i=1-n upto n:
    for j=1 upto n:
      if j < i+n+1:
        draw if i<1: rball else: bball fi shifted (24i, 24j);
      fi
    endfor
  endfor
  draw (origin -- (0, n)) shifted (1/2, 1/2) scaled 24;
);
draw P1 shifted 90 up;
draw P2 shifted 90 down;
label.rt("$\displaystyle \sum_{k=1}^n k + \sum_{k=1}^{n-1} k = n^2$", (160, 174));
label.rt("$\displaystyle \sum_{k=1}^n k + n^2 = \sum_{k=n+1}^{2n} k$", (160, -6));

```


Count the dots



Integer sums

```

vardef trig(expr n, shade, edge) = image(
  for y = 1 upto n:
    for x = 1 upto n + 1 - y:
      path c; c = fullcircle shifted (2x-1, 2y-1) scaled 8;
      fill c withcolor shade;
      draw subpath (1/2, 3/2) of c
        shifted - center c scaled 3/4 shifted center c
        withcolor 3/4[shade, white];
      draw c;
    endfor
  endfor
  if edge:
    draw origin -- (16n, 0)
    for i = 1 upto n:
      -- 16(n + 1 - i, i) -- 16(n - i, i)
    endfor -- cycle;
  fi
)
enddef;

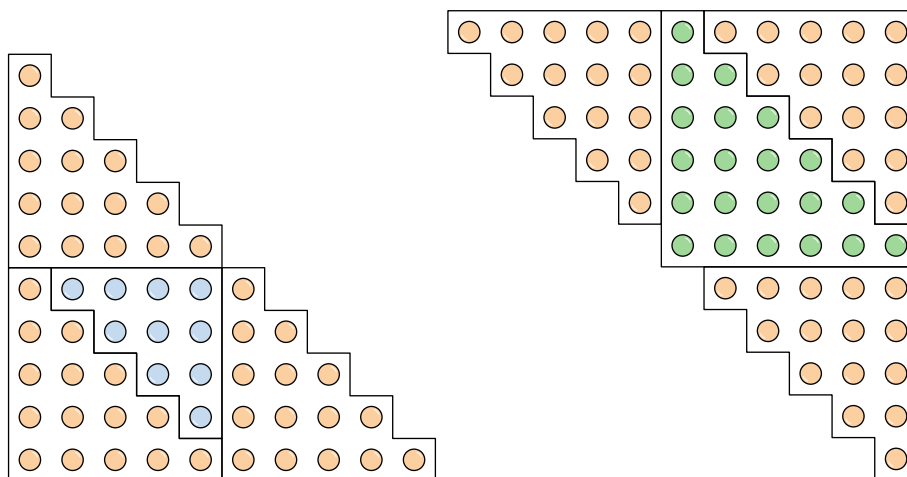
picture t; t = trig(5, Oranges 8 3, true);
picture P[];
P1 = image(draw t; draw t shifted 80 up; draw t shifted 80 right;
  draw trig(4, Blues 8 3, true) rotated 180 shifted (80, 80);
  label.bot("$3T_n + T_{n-1} = T_{2n}$",
    point 1/2 of bbox currentpicture shifted 8 down);
);
P2 = image(
  draw t rotated 180;
  draw t rotated 180 shifted 96 left;
  draw t rotated 180 shifted 96 down;
  draw trig(6, Greens 8 4, true) shifted -(96, 96);
  label.bot("$3T_n + T_{n+1} = T_{2n+1}$",
    point 1/2 of bbox currentpicture shifted 8 down);
);
P3 = image(draw t; draw t shifted 80 up; draw t shifted 80 right;
  draw trig(4, Blues 8 3, true) rotated 180 shifted (80, 80);
  draw t rotated 180 shifted (176, 176);
  draw t rotated 180 shifted ( 80, 176);
  draw t rotated 180 shifted (176, 80);
  draw trig(6, Greens 8 4, true) shifted (80, 80);
  label.bot("$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$",
    point 1/2 of bbox currentpicture shifted 8 down);
);

label.ulft(P1, origin); label.urt (P2, origin); label.bot (P3, 21 down);
label.top("$T_n = 1 + 2 + \cdots + n$ \Longrightarrow",
  point 5/2 of bbox currentpicture shifted 13 up);

```

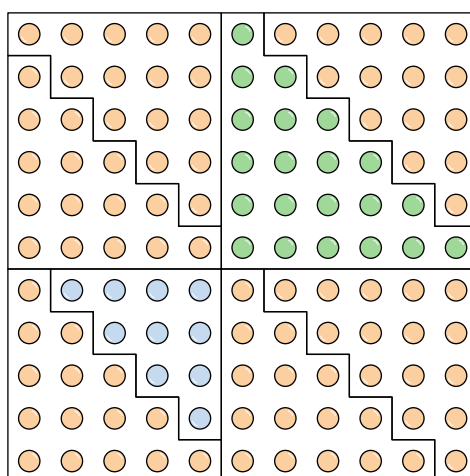
Identities for triangular numbers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



$$3T_n + T_{n-1} = T_{2n}$$

$$3T_n + T_{n+1} = T_{2n+1}$$



$$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$$

```

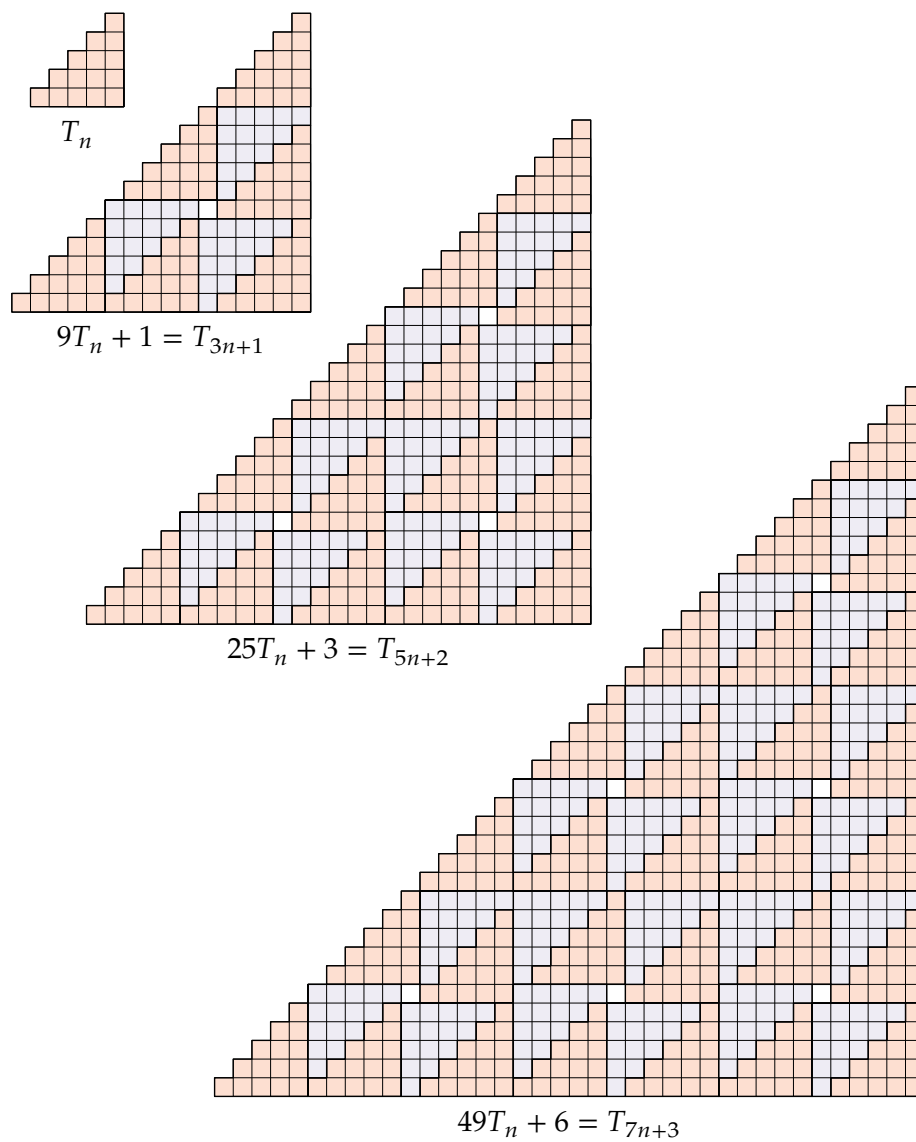
numeric n, s; n = 5; s = 7;
path edge; edge = (origin for i=0 upto n-1:
  -- (-i, n-i) -- (-i-1, n-i)
endfor -- (-n, 0) -- cycle) scaled s;

def make_trig(expr shade) = image(
  fill edge withcolor shade;
  for i=1 upto n:
    draw unitsquare xscaled (i-n-1) yscaled i scaled s withpen pencircle scaled 1/4;
  endfor
  draw edge;
) enddef;
picture t, u;
t = make_trig(Reds 8 2);
u = make_trig(Purples 8 2) rotated 180;
picture P[];
for k=1 step 2 until 7:
  P[k] = image(
    for i = 1 upto k:
      for j = 1 upto i:
        pair z; z = (1-j, 1-i) scaled n shifted (-floor(j/2), 0)
        if odd j: shifted (0, - floor (i/2))
        else: shifted (if odd i: 0 else: 1 fi, -floor((i-1)/2))
        fi scaled s;
        draw t shifted z;
        if odd j:
          if j > 1:
            draw u shifted z shifted (0, n*s + s)
          fi
        else:
          draw u shifted z shifted (0, n*s)
        fi;
      endfor
    endfor
    label.bot(
      if k=1: "$T_n$"
      else: "$" & decimal (k*k) & "T_n + " & decimal floor(k*k/8) &
        "=T_{" & decimal k & "n+" & decimal floor(k/2) & "}"$"
      fi,
      point 1/2 of bbox currentpicture);
  );
endfor

draw P1; draw P3 shifted (70, 0); draw P5 shifted (175, -40); draw P7 shifted (300, -140);
label.bot("$\left(2k+1\right)^2T_n + T_k = T_{(2k+1)n+k}$",
  point 1/2 of bbox currentpicture shifted 21 down);

```

A triangular identity



$$(2k + 1)^2 T_n + T_k = T_{(2k+1)n+k}$$

— Roger B. Nelsen

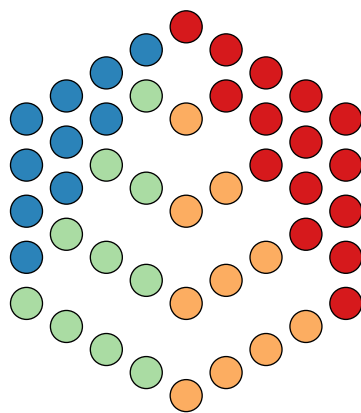
```

numeric u; u = 12;
path ring; ring = fullcircle scaled u; picture ball[];
ball1 = image(fill ring withcolor Spectral 4 1; draw ring);
ball2 = image(fill ring withcolor Spectral 4 2; draw ring);
ball3 = image(fill ring withcolor Spectral 4 3; draw ring);
ball4 = image(fill ring withcolor Spectral 4 4; draw ring);
picture P[]; interim bboxmargin := 10;
P1 = image(path H;
  H = (for i=0 upto 5: up rotated -60i -- endfor cycle) shifted down;
  draw ball1 shifted point 0 of H;
  for i=2 upto 5:
    path h; h = H scaled ((i-1) * 1.44 u);
    draw ball1 shifted point 1 of h;
    for j = 1 upto 4:
      for k = 1 upto i - 1:
        draw ball[j] shifted point j + k / (i-1) of h;
      endfor
    endfor
  endfor
  label.bot("$H_5$", point 1/2 of bbox currentpicture));
P2 = image(
  for i = 1 upto 9:
    for j = 1 upto i:
      draw ball[if i < 5: 4
        elseif (i > 5) and (j < i-4): 3
        elseif (i > 5) and (j > 5): 2
        else: 1 fi] shifted ((j - 1/2 i, -0.866025 i) * 1.44u);
    endfor
  endfor
  label.bot("$T_9$", point 1/2 of bbox currentpicture);
);
P3 = image(
  for i=1 upto 9:
    for j=1 upto 5:
      draw ball[if i < 6: if i + j < 7: 1 else: 2 fi
        else: if i + j < 11: 3 else: 4 fi
        fi] shifted ((i, j) * 1.44u);
    endfor
  endfor
  label.bot("$5\times 9$", point 1/2 of bbox currentpicture);
);
interim labeloffset := 32;
label.ulft(P1, origin); label.urt(P2, origin); label.bot(P3, origin);
label.top(btex $
  \left.\vcenter{\openup 6pt\halign{##$\hss&${}=#$\hss\cr
  H_{n+5+\cdots+(4n-3)}\cr
  T_{n+2+\cdots+n}\cr}}\right\}
  \Longrightarrow H_n = 3T_{n-1} + T_n = T_{2n-1} = n(2n-1)
  $ etex, point 5/2 of bbox currentpicture shifted 34 up);

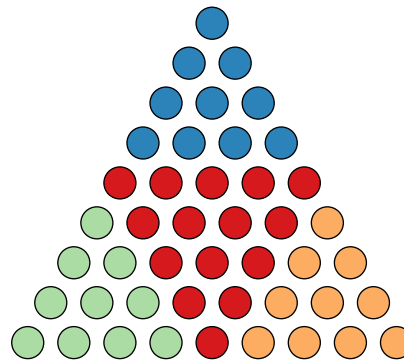
```

Every hexagonal number is a triangular number

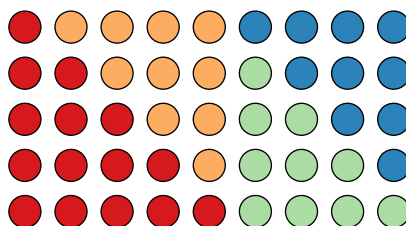
$$\left. \begin{array}{l} H_n = 1 + 5 + \cdots + (4n - 3) \\ T_n = 1 + 2 + \cdots + n \end{array} \right\} \Rightarrow H_n = 3T_{n-1} + T_n = T_{2n-1} = n(2n - 1)$$



H_5



T_9



5×9

```

numeric u; u = 16;
path dom; dom = unitsquare xscaled 2 scaled 2u;
numeric c; c = 0;
for n=1 upto 4:
  for k = 0 upto 3:
    for x = 0 upto n - 1:
      path d; d = dom shifted ((4x - 2n + 1, 2n-1) * u) rotated 90k;
      fill d withcolor if odd incr c: Blues else: Oranges fi [9][if odd n: 1 else: 2 fi];
      draw d;
    endfor
  endfor
endfor

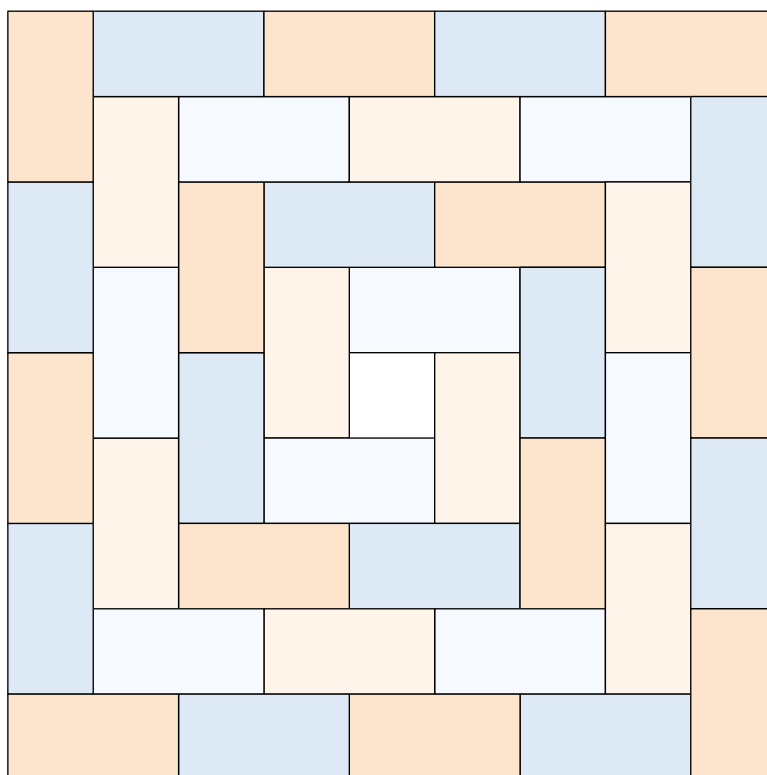
label.top(btex
  \vbox{\openup 12pt\halign{\hss$\displaystyle #$\hss\cr
    1 + 4\times 2 + 8 \times 2 + 12 \times 2 + 16 \times 2 = 9^2\cr
    1 + 2 \sum_{k=1}^n \, , \, 4k = \left(2n+1\right)^2\cr}}
  etex, point 5/2 of bbox currentpicture shifted 34 up);

```


One domino = two squares : concentric squares

$$1 + 4 \times 2 + 8 \times 2 + 12 \times 2 + 16 \times 2 = 9^2$$

$$1 + 2 \sum_{k=1}^n 4k = (2n + 1)^2$$



— Shirley A. Wakin

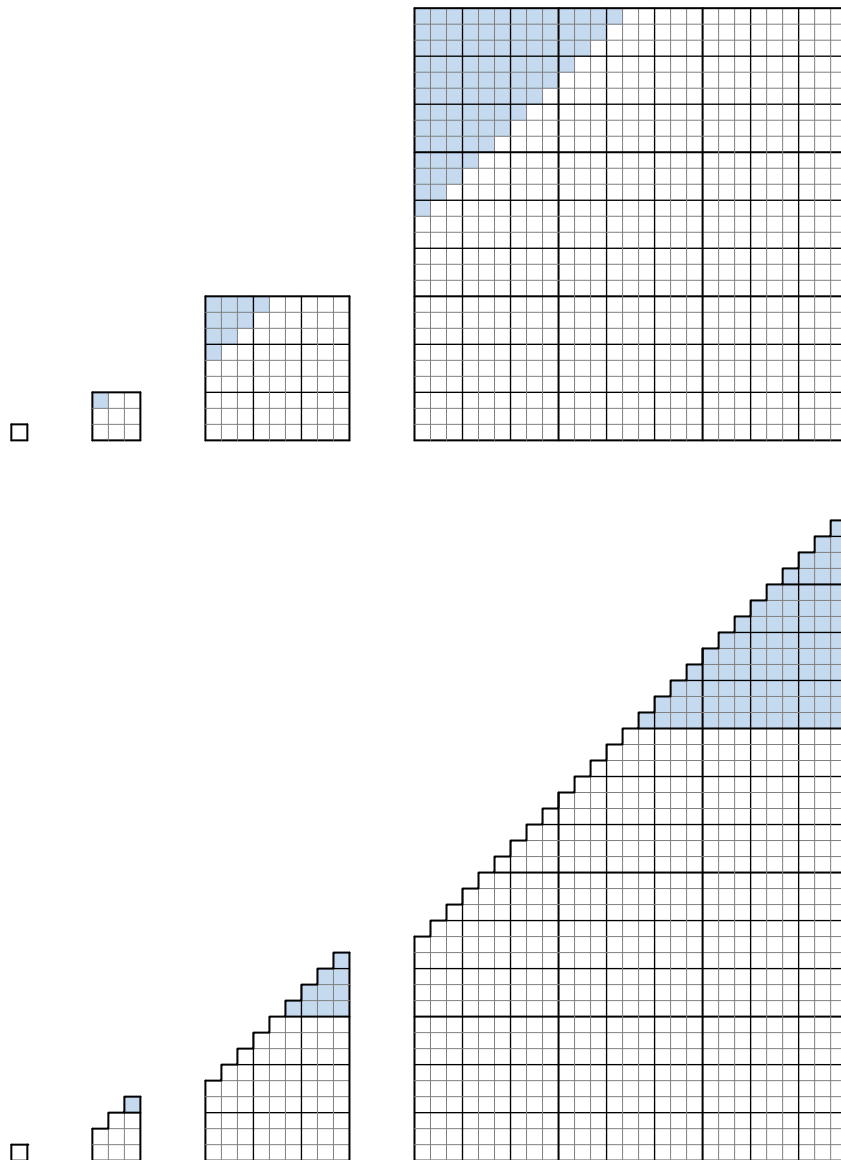
```

numeric u; u = 6;
def marked(expr i, N) =
  if (i = N) or (i mod 9 = 0): withpen pencircle scaled 3/4
  elseif i mod 3 = 0: withpen pencircle scaled 1/2
  else: withpen pencircle scaled 1/4 withcolor 1/2
  fi
enddef;
vardef frame(expr N, Closed) = image(
  for i = 0 upto N-1:
    for j = 0 upto N-1:
      path s; s = unitsquare shifted (i, j) scaled u;
      if j > i + floor(N / 2):
        if not Closed:
          s := s rotatedabout((N/2, N) scaled u, 180);
        fi
        fill s withcolor Blues 9 3;
      fi
    endfor
  endfor

  for i = 0 upto N:
    numeric minx; minx = if Closed or (i < floor(N / 2) + 1): 0 else: i - floor(N / 2) fi;
    numeric maxy; maxy = if Closed: N else: i + floor(N / 2) if i < N: + 1 fi fi;
    draw ((minx, 0) -- (N, 0)) shifted (0, i) scaled u marked(i, N);
    draw ((0, 0) -- (0, maxy)) shifted (i, 0) scaled u marked(i, N);
  endfor
  if not Closed:
    numeric m; m = floor(N / 2) + 1;
    draw ((0, m) -- (1, m)
      for i = 1 upto N - 1:
        -- (i, m+i) -- (i+1, m+i)
        hide(if (m + i) > N: draw ((i+1, m+i) -- (N, m+i)) scaled u marked(m+i, N); fi)
      endfor) scaled u withpen pencircle scaled 3/4;
  fi)
enddef;
for y = 0, -45u:
  numeric x; x = 0;
  for n=0 upto 3:
    picture f; f = frame(3*n, y=0); draw f shifted (x, y);
    x := x + 24 + xpart lrcorner f;
  endfor
endfor
label.bot("$1+9+\cdots+9^n = 1 + 2 + 3 + \cdots + (1+3+\cdots+3^n)$",
  point 1/2 of bbox currentpicture shifted 13 down);

```

Sums of consecutive powers of 9 are sums of consecutive integers



$$1 + 9 + \cdots + 9^n = 1 + 2 + 3 + \cdots + (1 + 3 + \cdots + 3^n)$$

— Roger B. Nelsen

```

numeric u; u = 16; path disc; disc = fullcircle scaled u;
vardef hexagon(expr N, Label) = image(
  numeric t; t = 1; fill disc withcolor Oranges 9 1; draw disc;
  for n = 1 upto N - 1:
    for r = 0 upto 5:
      for k = 1 upto n:
        path d; d = disc shifted (k/n)[(0, n*u) rotated 60r, (0, n*u) rotated (60r + 60)];
        fill d withcolor Oranges[8][n+1]; draw d; t := t + 1;
      endfor
    endfor
  endfor
  if Label:
    label.bot("$h_{\{ " & decimal N & " \} = " & decimal t & "$",
      point 1/2 of bbox currentpicture shifted 13 down);
  fi
) enddef;
picture P[];
P1 = image(
  draw hexagon(1, true);
  draw hexagon(2, true) shifted (5u, 1u);
  draw hexagon(3, true) shifted (12u, 2u);
  draw hexagon(4, true) shifted (21u, 3u);
);
input isometric_projection
set_projection(20, -45); ipscale := u;
picture ocube, halfbox; ocube = cube(Oranges 9 3, Oranges 9 1, Oranges 9 2);
halfbox = image(
  for i=1 upto 5: for j=1 upto 5:
    draw ocube shifted p(0, i, 5-j);
  endfor endfor
  for i=1 upto 4: for j=1 upto 5:
    draw ocube shifted p(i, 1, 5-j);
  endfor endfor
  for i=1 upto 4: for j=1 upto 4:
    draw ocube shifted p(i, j+1, 4);
  endfor endfor);

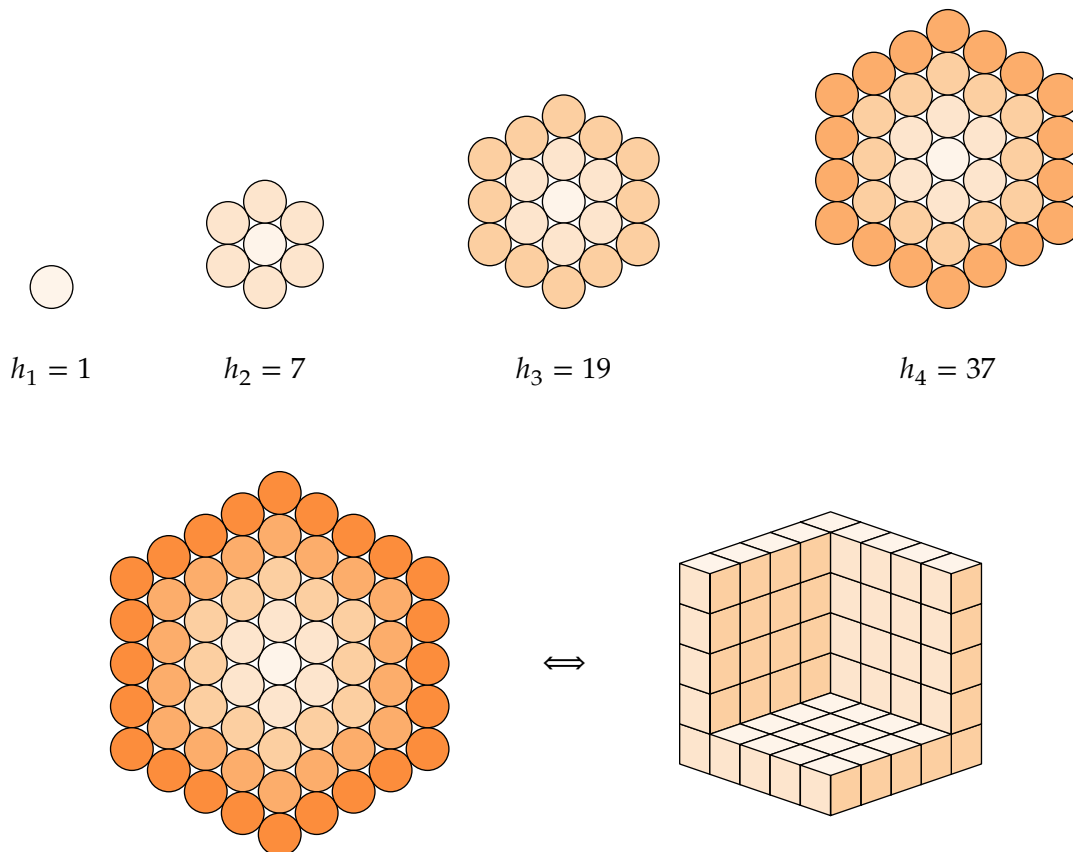
P2 = image(
  label.lft(hexagon(5, false), 40 left);
  label("\large $\iff$", origin);
  label.rt(halfbox, 40 right);
);

label.top(P1, 13 up); label.bot(P2, 13 down);

label.bot("$h_n = n^3 - (n-1)^3$", point 1/2 of bbox currentpicture shifted 13 down);
label.bot("$\therefore \quad h_1 + h_2 + \cdots + h_n = n^3$",
  point 1/2 of bbox currentpicture shifted 13 down);

```

Sums of hex numbers are cubes



$$h_n = n^3 - (n - 1)^3$$

$$\therefore h_1 + h_2 + \cdots + h_n = n^3$$

Sequences and series

A monotone sequence bounded by e	209
A recursively defined sequence for e	211
Geometric sums	213
Geometric series I	215
Geometric series II	217
Geometric series III	219
Sum of reciprocals of successive integer products	221
Sum of reciprocals of triangular numbers	223
Alternating harmonic series	225
Sum of sines	227

Sequences and series

```

vardef f(expr x) = (x, 1/256 mlog(1 + x)) enddef;
vardef fp(expr x) = if x=-1: up else: (1, 1/(1 + x)) fi enddef;

path ff;
ff = (f(-1/2){fp(-1/2)} for x=0 step 1/2 until 3: .. f(x){fp(x)} endfor) scaled 120;

interim bboxmargin := 0;
path xx, yy;
xx = subpath(0, 1) of bbox ff shifted (0, -ypart llcorner ff);
yy = subpath(0, -1) of bbox ff shifted (-xpart llcorner ff, 0);

path m[];
z0 = (xpart point 0 of xx, xpart point 0 of xx);
m0 = z0 -- -2z0;

z1 = point 5.5 of ff;
z2 = point 4 of ff;

m1 = xx scaled 2 rotated angle z1
  cutbefore subpath (-1, 1) of bbox ff cutafter subpath (1,3) of bbox ff;
m2 = xx scaled 2 rotated angle z2
  cutbefore subpath (-1, 1) of bbox ff cutafter (point 1 of m0 -- point 1 of m1);

draw m0 dashed evenly withcolor 1/2;
draw m1 withcolor 1/2;
draw m2 withcolor 1/2;
draw ff withcolor 2/3 red;
drawarrow xx;
drawarrow yy;

draw z1 -- (x1, 0) dashed withdots scaled 1/4;
draw z2 -- (x2, 0) dashed withdots scaled 1/4;

forsuffixes @=origin, z1, z2:
  draw @ withpen pencircle scaled dotlabeldiam;
  undraw @ withpen pencircle scaled 3/4 dotlabeldiam;
endfor

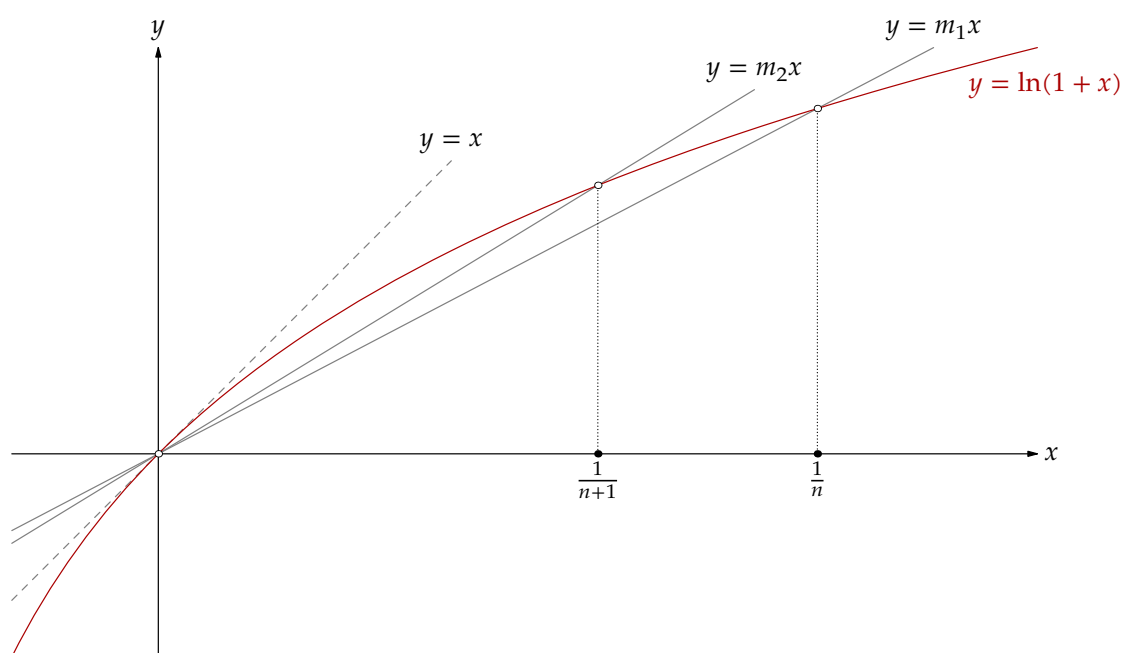
dotlabel.bot("$\frac{1}{n}$", (x1, 0));
dotlabel.bot("$\frac{1}{n+1}$", (x2, 0));

label.top("$y=x$", point 1 of m0);
label.top("$y=m_1x$", point 1 of m1);
label.top("$y=m_2x$", point 1 of m2);
label.lrt("$y=\ln(1+x)$", point 6.5 of ff) withcolor 2/3 red;
label.rt("$x$", point 1 of xx);
label.top("$y$", point 1 of yy);

label.bot(btex
\vbox{\openup 12pt\halign{\hfill $$$\quad\Longrightarrow\quad $\displaystyle # $\hfill\cr
n \ge 1 \& m_1 < m_2 < 1\cr
&\frac{\ln(1+1/n)}{1/n} < \frac{\ln(1+1/(n+1))}{1/(n+1)} < 1\cr
&\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < e\cr}} etex,
point 1/2 of bbox currentpicture shifted 34 down);

```


A monotone sequence bounded by e



$$n \geq 1 \implies m_1 < m_2 < 1$$

$$\implies \frac{\ln(1 + 1/n)}{1/n} < \frac{\ln(1 + 1/(n+1))}{1/(n+1)} < 1$$

$$\implies \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e$$

— Roger B. Nelsen

```

numeric u; u = 32;
vardef f(expr x) = 256 x / mlog(x) enddef;
path ff, xy, xx, yy;
numeric minx, s; minx = 39/32; s = 1/8;
ff = ((minx, f(minx)) for x = minx + s step s until 8: .. (x, f(x)) endfor) scaled u;
xx = 5 left -- (xpart point infinity of ff, 0);
yy = 5 down -- (0, ypart point 0 of ff);
xy = origin -- 5(u, u);

numeric e; e = 2.718281828459;
numeric x[]; x0 = 7; for i=1 upto 4: x[i] = f(x[i-1]); endfor
drawoptions(dashed evenly scaled 1/2 withpen pencircle scaled 1/4 withcolor 1/4);
draw (u, 0) -- (u, ypart point 0 of ff);
draw ((e, 0) -- (e, e) -- (0, e)) scaled u;
draw ((x0, 0) -- (x0, f(x0))) scaled u;
draw ((x1, 0) -- (x1, f(x1))) scaled u;
draw ((x2, 0) -- (x2, f(x2))) scaled u;
drawoptions(dashed withdots scaled 1/4 withpen pencircle scaled 1/2 withcolor 1/2 blue);
draw ((x0, f(x0)) for i=1 upto 3: -- (x[i], x[i]) -- (x[i], f(x[i])) endfor) scaled u;
drawoptions(withcolor Reds 8 7);
draw ff; label.top("$y=\frac{x}{\ln(x)}$", point infinity of ff);
drawoptions(withcolor Oranges 8 7);
draw xy; label.top("$y=x$", point infinity of xy);
drawoptions();
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);

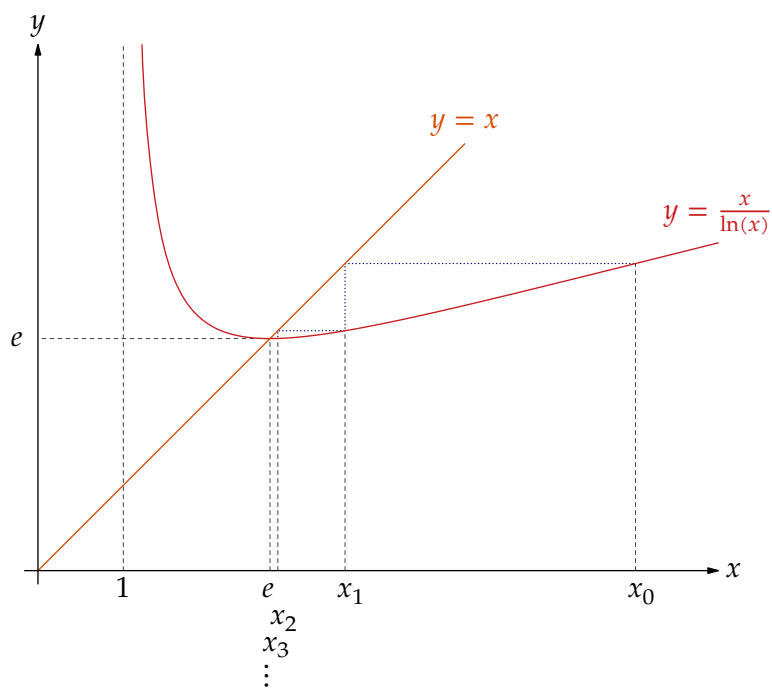
numeric dx, dy; dx = -2.8; dy = -10;
draw TEX("$1$") shifted (1 * u + dx, dy);
draw TEX("$e$") shifted (e * u + dx, dy);
draw TEX("$x_0$") shifted (x0 * u + dx, dy);
draw TEX("$x_1$") shifted (x1 * u + dx, dy);
draw TEX("$x_2$") shifted (x2 * u + dx, 2dy);
draw TEX("$x_3$") shifted (x3 * u + dx, 3dy);
draw TEX("$\vdots$") shifted (x4 * u + dx, 4.2dy);

draw TEX("$e$") shifted (dy, e * u + dx);

label.bot(btex $\displaystyle
x_0 > 1 \enspace \mathbin{\&} \enspace x_{n+1} = \frac{x_n}{\ln(x_n)}
\quad \mathbin{\&} \quad \lim_{x_n \rightarrow e}
$ etex, point 1/2 of bbox currentpicture shifted 34 down);

```

A recursively defined sequence for e



$$x_0 > 1 \ \& \ x_{n+1} = \frac{x_n}{\ln(x_n)} \implies \lim x_n = e$$

— Thomas P. Dence

```

vardef do_boxes(expr u, r, n, t) = image(
  save a, b; path a, b;
  a = unitsquare xscaled r   yscaled 1 scaled u shifted -(u,u);
  b = unitsquare xscaled (1-r) yscaled r scaled u shifted point 1 of a;
  for i=1 upto n:
    draw a withpen pencircle scaled 1/4;
    draw b withpen pencircle scaled 1/4;
    numeric sf; sf = sqrt(r**(i-1));
    if sf > 1/4: if r = 1/2:
      label(TEX("$\frac{1}{2}^i" & decimal 2i & "$") scaled sf, center b);
      label(TEX("$\frac{1}{2}^i" if i>1: & "^" & decimal (2i-1) fi & "$") scaled sf, center a);
    else:
      label(TEX("$r(1-r)^i" if i>1: & "^" & decimal (2i-1) fi & "$") scaled sf, center b);
      label(TEX("$r^i" if i>1: & "(1-r)^(i-1)" & decimal (2i-2) fi & "$") scaled sf, center a);
    fi fi
    a := a scaled (1-r);
    b := b scaled (1-r);
  endfor
  draw unitsquare scaled -u;
  label.bot(t, point 1/2 of bbox currentpicture shifted 13 down);
)
enddef;

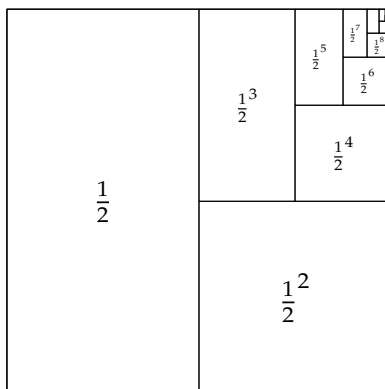
picture P[];

P1 = do_boxes(144, 1/2, 6, "$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$");
P2 = do_boxes(160, 1/3, 9, "$r + r(1-r) + r(1-r)^2 + \cdots = 1$");

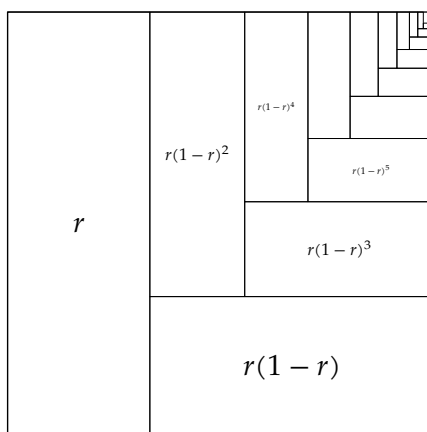
label.top(P1, 7 up);
label.bot(P2, 7 down);

```

Geometric sums



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$



$$r + r(1-r) + r(1-r)^2 + \dots = 1$$

```

input arrow_label
picture P[];
numeric a, r, u;
2a = u = 160; r = 0.81;
pair x;
x = origin;
path b[];
for i=0 upto 21:
    b[i] = unitsquare xscaled a yscaled u scaled (r**i) shifted x;
    x := point 1 of b[i];
endfor
x1 = 0; z1 = whatever[point 2 of b0, point 2 of b1];
y2 = 0; z2 = whatever[point 2 of b0, point 2 of b1];
path t; t = subpath (3, 2) of b1 -- point 2 of b0 -- cycle;

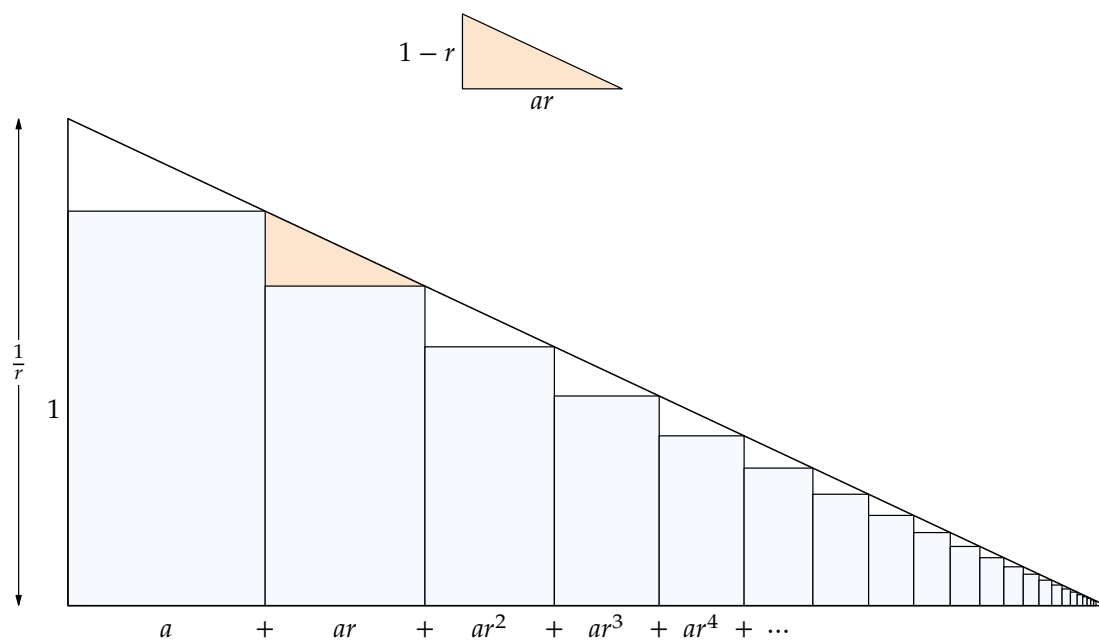
P1 = image(
    fill t withcolor Oranges 8 2;
    for i = 0 upto 21:
        fill b[i] withcolor Blues 8 1;
        %draw subpath (1, 3) of
        draw b[i];
        if i < 6:
            label.top("$" &
                if i=0: "a" &
                elseif i=1: "ar" &
                elseif (i>1) and (i<5): "ar^" & decimal i &
                elseif i=5: "\cdots" &
                fi "$", point 1/2 of b[i] shifted 16 down);
        fi
        if i < 5:
            label.top("$+$", point 1 of b[i] shifted 16 down);
        fi
    endfor
    label.lft("$1$", point -1/2 of b0);
    draw origin -- z1 -- z2 -- cycle withpen pencircle scaled 5/8;
    arrow_label(origin, z1, "$\frac{1}{r}$", -20);
);
P2 = image(
    fill t withcolor Oranges 8 2;
    draw t;
    label.bot("$ar$", point 1/2 of t);
    label.lft("$1-r$", point -1/2 of t);
);

draw P1;
draw P2 shifted (a, a);

label.bot(btex \vbox{\openup13pt\halign{\hss$\displaystyle #&$$\displaystyle {}=##$\hss\cr
\frac{a + ar + ar^2 + ar^3 + ar^4 + \cdots}{1-r}&\frac{ar}{1-r}\cr
a + ar + ar^2 + ar^3 + ar^4 + \cdots &\frac{a}{1-r}\cr
\sum_{n=0}^{\infty} ar^n&\frac{a}{1-r}\cr
}} etex, point 1/2 of bbox currentpicture shifted 55 down);

```

Geometric series I



$$\frac{a + ar + ar^2 + ar^3 + ar^4 + \dots}{1/r} = \frac{ar}{1-r}$$

$$a + ar + ar^2 + ar^3 + ar^4 + \dots = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

— J. H. Webb

```

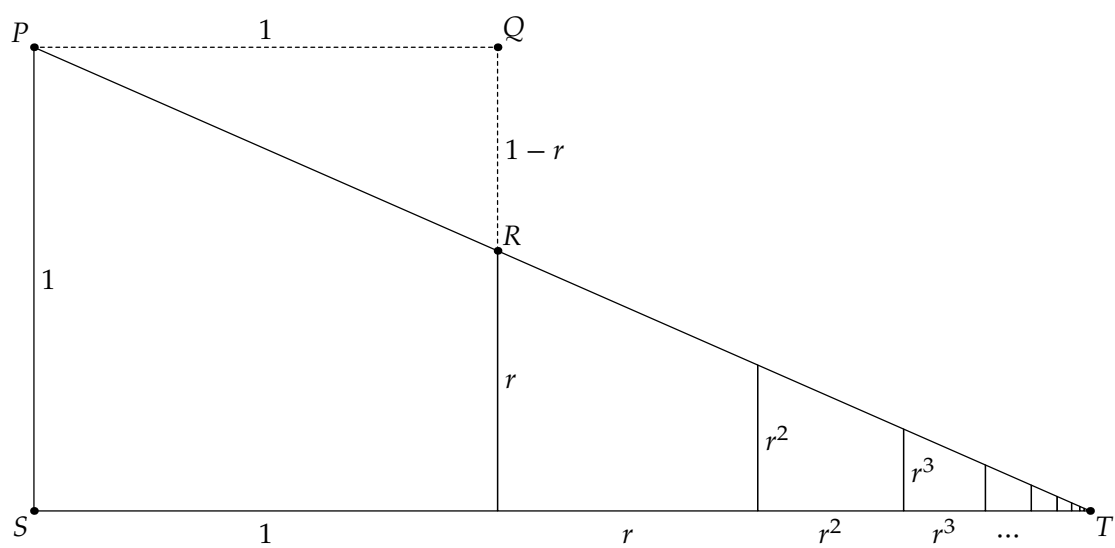
numeric r, u;
u = 180; r = 0.561;
pair P, Q, R, S, T;
S = origin; P = u * up; R = (u, u * r); Q = (u, u);
T = whatever[P, R]; ypart T = 0;
path pg; pg = origin -- (u, 0) -- (u, u*r) -- (0, u) -- cycle;
draw R -- Q -- P dashed evenly scaled 1/2;
for i=0 upto 12:
  draw pg;
  if i=0:
    label.rt("$1$", point -1/2 of pg);
    label.bot("\strut $1$", point 1/2 of pg );
  elseif i=1:
    label.rt("$r$", point -1/2 of pg);
    label.bot("\strut $r$", point 1/2 of pg);
  elseif i < 4:
    label.rt("$r^{\&decimal i}$", point -1/2 of pg);
    label.bot("\strut $r^{\&decimal i}$", point 1/2 of pg);
  elseif i = 4:
    label.bot("\strut $\cdots$", point 1/2 of pg);
  fi
  pg := pg shifted - point 0 of pg scaled r shifted point 1 of pg;
endfor

dotlabel.ulft("$P$", P);
dotlabel.urt (" $Q$", Q);
dotlabel.urt (" $R$", R);
dotlabel.llft("$S$", S);
dotlabel.lrt (" $T$", T);
label.top("$1$", 1/2[P, Q]);
label.rt (" $1-r$", 1/2[R, Q]);

label.bot(btex \vbox{\openup 13pt\halign{\hss $\displaystyle \# $\hss\cr
\triangle PQR \sim \triangle TSP\cr
\therefore \quad 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}\cr}} etex,
point 1/2 of bbox currentpicture shifted 21 down);

```


Geometric series II



$$\triangle PQR \sim \triangle TSP$$

$$\therefore 1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$$

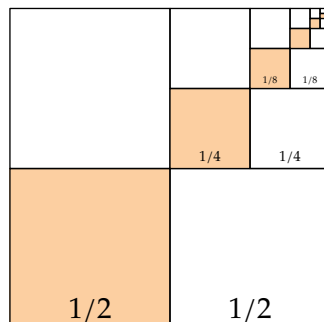
— Benjamin G. Klein and Irl C. Bivens

```

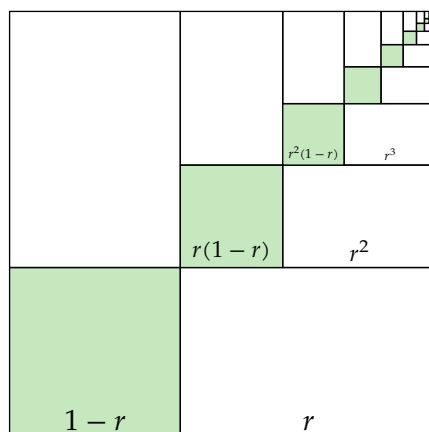
numeric u; u = 120; picture P[];
P1 = image(
  for i=1 upto 9:
    path s; s = unitsquare scaled u shifted -2(u, u) scaled (1/2 ** i);
    fill s withcolor Oranges 8 3;
    draw s;
    draw s shifted (point 1 of s - point 0 of s);
    draw s shifted (point 3 of s - point 0 of s);
    if i < 4:
      picture t; t = TEX("$1/" & decimal (2 ** i) & "$") scaled (1/i);
      label.top(t, point 1/2 of s - (0, 1/2i));
      label.top(t, point 1/2 of s + (1/2 ** i * u, - 1/2 i));
    fi
  endfor
  label.bot("$\displaystyle \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2}$",
    point 1/2 of bbox currentpicture shifted 13 down);
);
P2 = image(numeric u, r; u = 160; r = .6;
  for i = 0 upto 9:
    path s, t;
    s = unitsquare scaled (u * (1-r)) shifted -(u, u) scaled (r ** i);
    t = point 1 of s -- (0, ypart point 1 of s)
      -- (0, ypart point 2 of s) -- point 2 of s -- cycle;
    fill s withcolor Greens 8 3; draw s;
    draw t; draw t rotatedabout(center s, 90);
    if i < 3:
      picture q, p;
      if i=0:
        q = TEX("$1-r$"); p = TEX("$r$");
      elseif i=1:
        q = TEX("$r(1-r)$") scaled 0.8; p = TEX("$r^2$") scaled 0.8;
      else:
        q = TEX("$r^2(1-r)$") scaled 0.45; p = TEX("$r^3$") scaled 0.45;
      fi
      label.top(q, point 1/2 of s - (0, 1/2i));
      label.top(p, point 1/2 of t - (0, 1/2i));
    fi
  endfor
  label.bot(btex
    \vbox{\openup 13pt\halign{\hss $\displaystyle # + \cdots$&$\displaystyle \{ } = \# $\hss \cr
    (1-r)^2 + r^2(1-r)^2 + r^4(1-r)^2 & \frac{(1-r)^2}{(1-r)^2 + 2\times r(1-r)} \cr
    1 + r^2 + r^4 & \frac{1}{(1-r)^2 + 2r(1-r)} = \frac{1}{1-r^2} \cr
    a + ar + ar^2 & \frac{a}{1-r} \cr}}
    etex, point 1/2 of bbox currentpicture shifted 13 down);
);
label.top(P1, 7 up); label.bot(P2, 7 down);

```

Geometric series III



$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{3}$$



$$(1 - r)^2 + r^2(1 - r)^2 + r^4(1 - r)^2 + \dots = \frac{(1 - r)^2}{(1 - r)^2 + 2 \times r(1 - r)}$$

$$1 + r^2 + r^4 + \dots = \frac{1}{(1 - r)^2 + 2r(1 - r)} = \frac{1}{1 - r^2}$$

$$a + ar + ar^2 + \dots = \frac{a}{1 - r}$$

— Sunday A. Ajose

```

numeric u, v;
u = 72; v = 305;
numeric n; n = 5;
draw (u, v) -- (u, 0) -- (0, 0) -- (0, v + 21);
draw (n * u, 0) -- (n * u, v / (n + 1));

for x = 1 upto n + 1:
  numeric y; y = (x-1)/x;
  if x <> n - 1:
    draw (0, v) -- (x * u, 0) withcolor 2/3 blue;
    draw (0, v * y) -- (u, v * y) dashed withdots scaled 1/2;
    draw (left--right) scaled 1/2 shifted (x, 0) scaled u;
  else:
    draw (left--right) scaled 1/4 shifted (x, 0) scaled u dashed withdots;
  fi
  if x < n - 1:
    label.lft("$" & decimal (x-1) & "/" & decimal x & "$", (0, v * y));
    label.bot("\strut $" & decimal x & "$", (x * u, 0));
  elseif x = n:
    label.lft("$ (n-1)/n$", (0, v * y - 2));
    label.bot("\strut $n$", (x * u, 0));
  elseif x = n + 1:
    label.lft("$n/(n+1)$", (0, v * y + 2));
    label.bot("\strut $n+1$", (x * u, 0));
  fi
endfor
label.lft("$1$", (0, v));

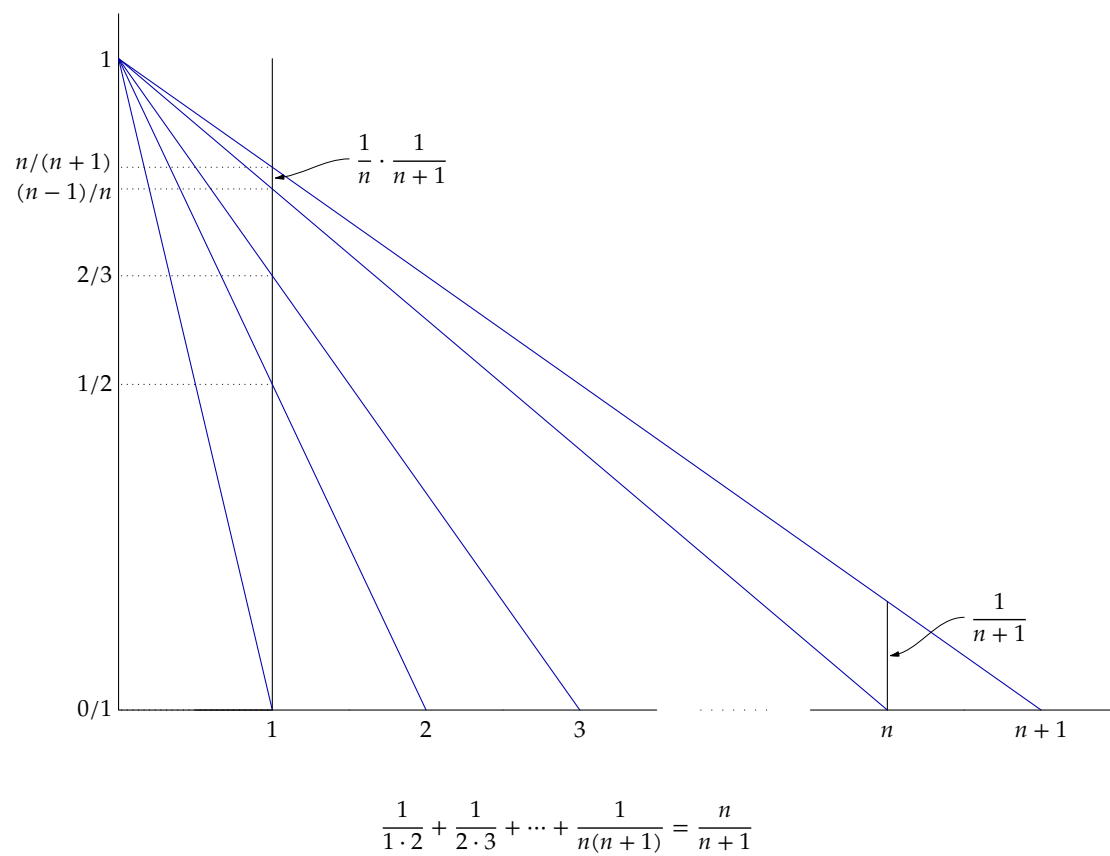
vardef label_bar(expr a, b, z, t) =
  label.rt(t, 1/2[a, b] + z);
  drawarrow (z {left} .. origin {left}
    cutafter fullcircle scaled dotlabeldiam) shifted 1/2[a, b]
    withpen pencircle scaled 1/4;
enddef;

label_bar((n * u, 0), (n * u, v / (n + 1)), (1/2u, 1/4u),
  "$\displaystyle\frac{1}{n+1}$");
label_bar((u, (n-1)*v/n), (u, n*v/(n+1)), (1/2u, 1/8u),
  "$\displaystyle\frac{1}{n}\cdot\frac{1}{n+1}$");

label.bot(btex $\displaystyle
  \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}
$ etex, point 1/2 of bbox currentpicture shifted 21 down);

```

Sum of reciprocals of successive integer products



— Roman W. Wong

Sequences and series

```

path xx, yy, ff;
numeric u, v, s, minx, maxx; s = 1/8; minx = 1; maxx = 7; u = 42; v = 102;
ff = ((minx - s, 2/(minx - s)) for x = minx step s until maxx + 2s:
    .. (x, 2/x)
    endfor) xscaled u yscaled v;
xx = origin -- (maxx + 4s, 0) scaled u;
yy = origin -- (0, 2+s) scaled v;

for x = minx upto maxx - 1:
    path a, b, c; a = unitsquare xscaled u yscaled (-2v / (x * (x + 1)));
    b = a shifted (x * u, 2v / x);
    c = a shifted (-2u, 2v/x);

    forsuffices @=b, c:
        fill @ withcolor 1/2[Oranges[8][x], white]; draw @;
    endfor
    draw point 0 of b -- (x * u, 0) dashed evenly scaled 1/2;
    draw (up--down) scaled 3/2 shifted (x * u, 0);
    if x = maxx - 1:
        label.top(btex \vbox{\openup4pt\halign{\hss $$$ \hss\cr
            1\cdot\left(\frac{2n-\frac{2}{n+1}}{2}\right)\cr
            =\frac{2n(n+1)}{2}\cr}} etex, point 1/2 of b shifted 8 up);
        label.bot("\strut $n$", (x * u, -3));
        label.bot("\strut $n+1$", (x * u + u, -3));
        draw (up--down) scaled 3/2 shifted (x * u + u, 0);
        draw point 1 of b -- (x * u + u, 0) dashed evenly scaled 1/2;
    elseif x = maxx - 2:
        label.top("$\cdots$", point 1/2 of b);
        label.bot("\strut $\cdots$", (x * u, -3));
    else:
        label.top("$\frac{1}{2}$" & decimal (1/2 * x * (x + 1)) & "$", point 1/2 of b);
        label.bot("\strut $" & decimal x & "$", (x * u, -3));
    fi
endifor
z0 = (xpart point 0 of c, 0);
z1 = (xpart point 1 of c, 0);
draw point 0 of c -- z0 dashed evenly scaled 1/2;
draw point 1 of c -- z1 dashed evenly scaled 1/2;
draw 1.1[z0, z1] -- 1.1[z1, z0];

for y=0 upto 2:
    draw (left--right) scaled 3/2 shifted (0, y * v);
    label.lft("$" & decimal y & "$", (0, y * v));
endfor

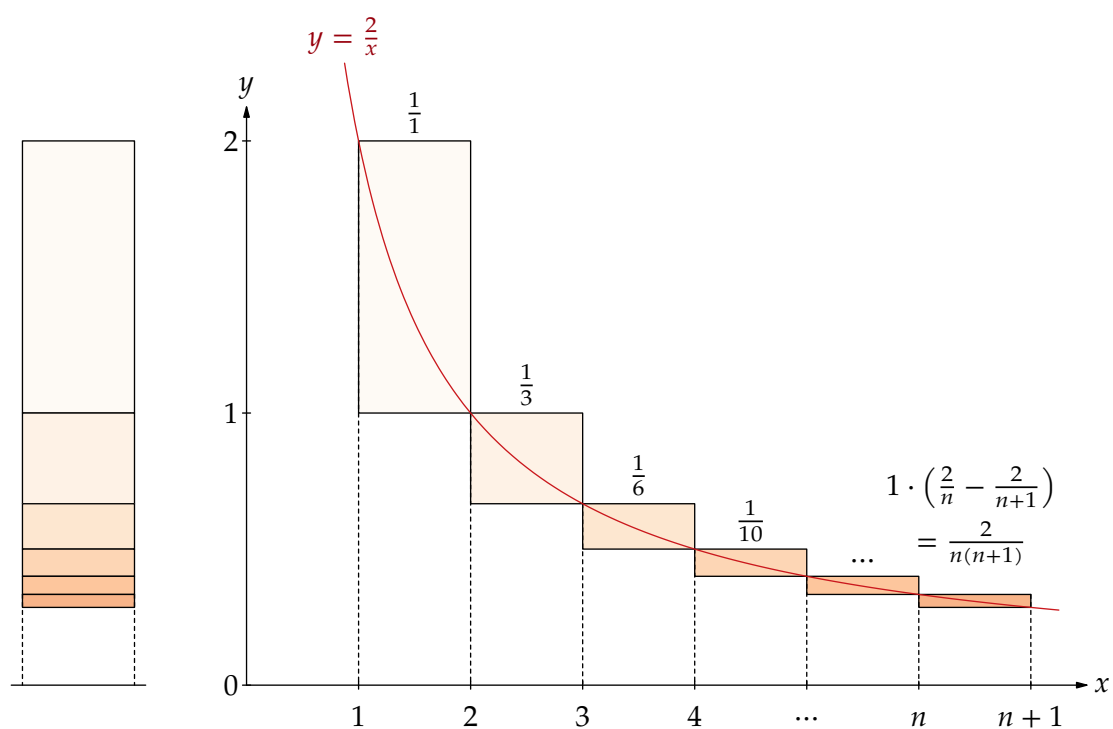
draw ff withcolor Reds 8 7; label.top("$y=\frac{2x}{2}$", point 0 of ff) withcolor Reds 8 8;
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);

label.top("$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\cdots+\frac{2}{n(n+1)}+\cdots=2$",
    point 5/2 of bbox currentpicture shifted 34 up);

```

Sum of reciprocals of triangular numbers

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{2}{n(n+1)} + \cdots = 2$$



— Roger B. Nelsen

Sequences and series

```

numeric u; u = 300;
path ff; ff = ((15/16, 16/15) for x = 1 step 1/8 until 9/4: .. (x, 1/x) endfor) scaled u;
path xx; xx = (xpart point 0 of ff, 0) -- (xpart point infinity of ff, 0);
path yy; yy = (point 0 of xx -- point 0 of ff) shifted 21 left;

vardef gcd(expr a, b) = if b = 0: a else: gcd(b, a mod b) fi enddef;
vardef reduced_fraction(expr N, D) =
  save n, d, g; numeric n, d, g; g = gcd(N, D); n = N / g; d = D / g;
  if d = 1: "$" & decimal n & "$"
  else: "$\frac{" & decimal n & "}{"}" & decimal d & "$" fi
enddef;
vardef fh(expr a) =
  save s, t; string s; numeric t;
  for n = 1 upto 8:
    t := 1/n - 1/(n+1);
    s := "$" if n>1: & "\frac{1" fi & decimal n & "-" & "\frac{1" & decimal (n+1) & "$";
    exitif abs(a-t) < eps;
  endfor s
enddef;

vardef partition(expr a, b, c, level) =
  if level > 0:
    save box; path box;
    save w, h; numeric w, h; w = abs(a-b); h = 1/b - c;
    if h > 0:
      box = unitsquare xscaled w yscaled h shifted (a, c) scaled u;
      fill box withcolor 1/4[Oranges[9][level], white]; draw box;
      if w >= 1/4: label(fh(w*h) , center box); fi
    fi
    partition(a, a + 1/2 w, c + h, level - 1);
    partition(a + 1/2 w, b, c + h, level - 1);
  fi
enddef;
partition(1, 2, 0, 6);

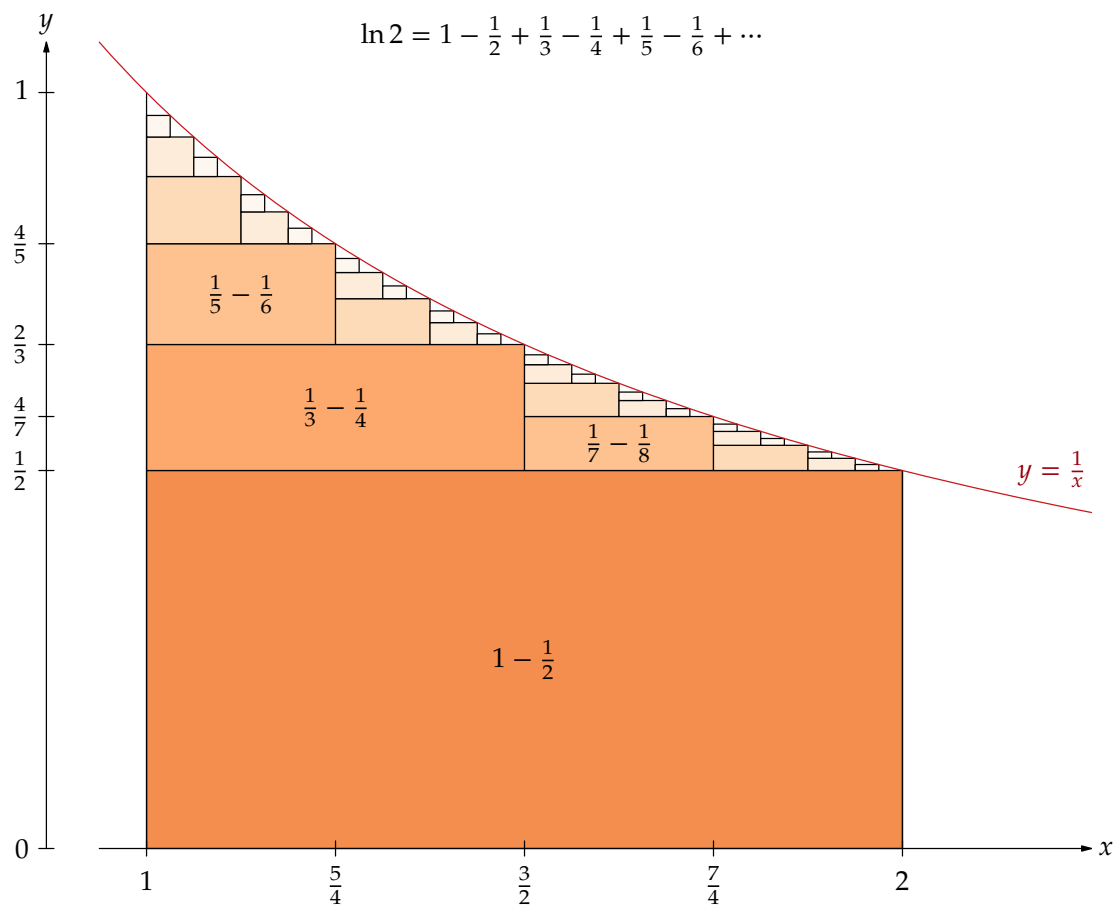
for x = 1 step 1/4 until 2:
  pair a, b; a = (x,0) scaled u; b = (xpart point 0 of yy, u / x);
  draw (up--down) scaled 3 shifted a;
  draw (left--right) scaled 3 shifted b;
  label.bot("\strut" & reduced_fraction(4x, 4), a shifted 4 down);
  label.lft("\strut" & reduced_fraction(4, 4x), b shifted 4 left);
endfor

draw ((1, 0) -- (1, 1)) scaled u; draw ((2, 0) -- (2, 1/2)) scaled u;
draw (left--right) scaled 3 shifted point 0 of yy;
label.lft("$0$", point 0 of yy shifted 4 left);

draw ff withcolor Reds 8 7;
label.ulft("$y=\frac{1}{x}$", point infinity of ff shifted 8 up) withcolor Reds 8 8 ;
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);
% ... plus the TeX labels, top and bottom

```


Alternating harmonic series



$$\frac{1}{2} \times \left(\frac{2}{3} - \frac{1}{2} \right) = \frac{1}{3} - \frac{1}{4}; \quad \frac{1}{4} \times \left(\frac{4}{5} - \frac{2}{3} \right) = \frac{1}{5} - \frac{1}{6}; \quad \frac{1}{4} \times \left(\frac{4}{7} - \frac{1}{2} \right) = \frac{1}{7} - \frac{1}{8}; \quad \dots$$

$$\ln 2 = \int_1^2 \frac{1}{x} dx = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

— Mark Finklestein

Sequences and series

```

path xx, yy, arc; numeric r;
r = 300; xx = 30 left -- (20+r) * right; yy = xx rotated 90; arc = quartercircle scaled 2r;
numeric n, theta; n = 4; theta = 8;
for i = 0 upto n:
  z[i] = point (2i+1) * theta / 45 of arc;
endfor

vardef pin@#(expr p, o, z) =
  draw z+o..z withpen pencircle scaled 1/4; label@#(p, o+z);
enddef;

for i = 0 upto n:
  draw origin -- z[i]; draw (0, y[i]) -- z[i] dashed evenly;
  path t;
  if i > 0:
    t = z[i] -- (x[i], y[i-1]) -- z[i-1] -- cycle;
    if i <> n - 1:
      pin.urt("$2\sin\theta$", 8 unitvector(direction 5/2 of t rotated -90), point 5/2 of t);
      picture p;
      p = thelabel.lft("$2\sin\theta\cos"
        & if i = n: "2n" else: decimal 2i fi
        & "\theta$", point 1/2 of t);
      unfill bbox p; draw p;
    else:
      label.lft("$\dots$", point 1/2 of t);
    fi
  else:
    t = z0 -- (x0, 0);
    pin.urt("$\sin\theta$", 8 dir 10, point 1/2 of t);
  fi
  draw t;
endfor

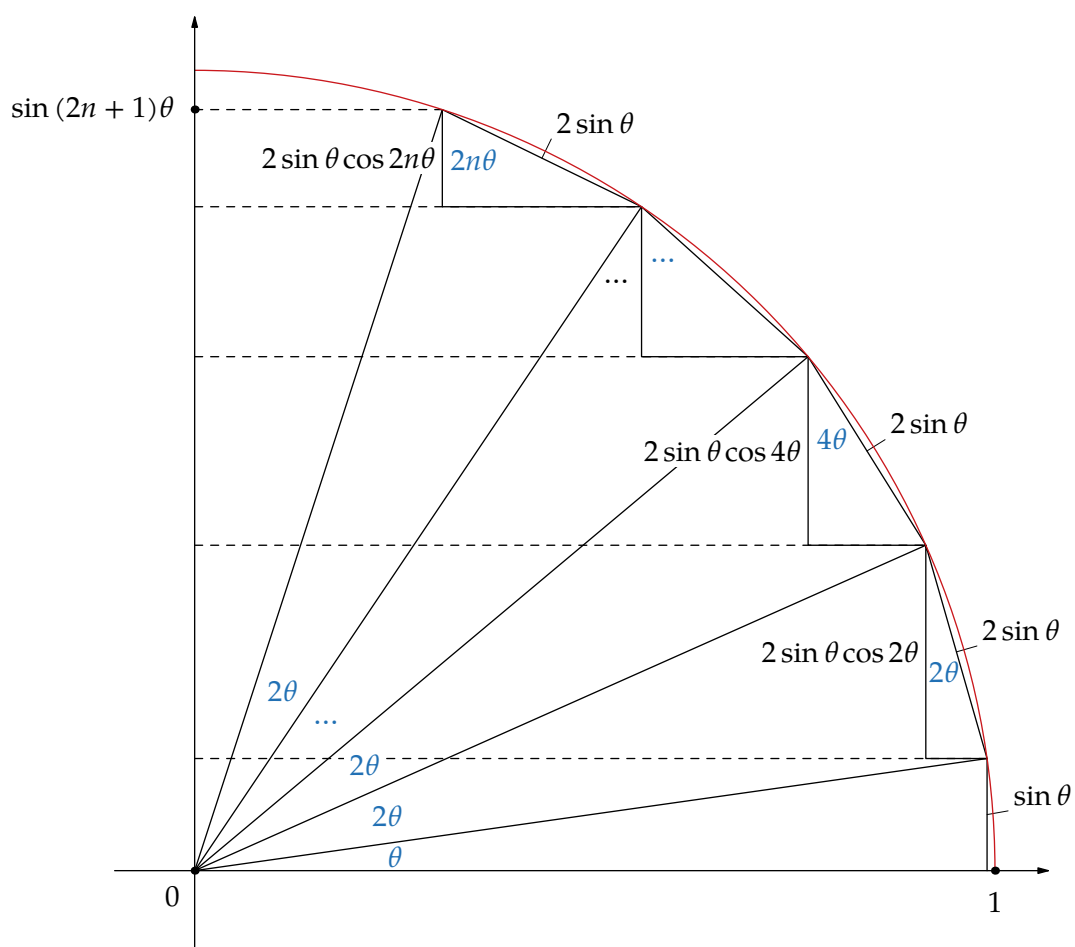
draw arc withcolor Reds 8 7; drawarrow xx; drawarrow yy;
drawoptions(withcolor Blues 8 7);
label("$\theta$", 1/4 r * dir 1/2 theta);
for i=1 upto n:
  picture t;
  t = thelabel(if i+1=n: "$\dots$" else: "$2\theta$" fi, 1/4 r * dir (2i * theta));
  unfill bbox t; draw t;
endfor
label("$2\theta$", 48 dir (270 + theta) shifted point 3/45 theta of arc);
label("$4\theta$", 32 dir (270 + 2theta) shifted point 5/45 theta of arc);
label("$\dots$", 22 dir (270 + 3theta) shifted point 7/45 theta of arc);
label("$2n\theta$", 22 dir (270 + 4theta) shifted point 9/45 theta of arc);
drawoptions();
labeloffset := 8;
dotlabel.llft("$0$", origin);
dotlabel.bot("$1$", (r, 0));
dotlabel.lft("$\sin\,(2n+1)\theta$", (0, r * sind((2n+1)*theta)));

label.top(btex $\displaystyle
  \sin\,(2n+1)\theta = \sin\theta + 2\sin\theta \sum_{k=1}^n \cos 2k\theta
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);

```

Sum of sines

$$\sin (2n + 1)\theta = \sin \theta + 2 \sin \theta \sum_{k=1}^n \cos 2k\theta$$



— J. Chris Fisher & E. L. Koh

Miscellaneous

A 2×2 determinant is the area of a parallelogram	231
Area of parallelogram	233
Gaussian quadrature as the area of either trapezoid	235
The problem of the calissons	237

```

numeric a, b, c, d; a = 233; b = 52; c = 85; d = 164;
z0 = origin; z1 = (a, 0); z2 = (a, d); z3 = (0, d); z4 = z2 + (c, 0);
z5 = (a, b); z6 = z2 + (c, b); z7 = (c, d); z8 = z2 + (0, b);

z9 = whatever[z5, z6]; y9 = y2;
z11 = whatever[z5, z6]; y11 = y0;
z10 = whatever[z6, z7]; x10 = x2;
z12 = whatever[z6, z7]; x12 = x0;

path t[];
t1 = z0 -- z11 -- z5 -- cycle; t5 = z11 -- z1 -- z5 -- cycle;
t2 = z7 -- z9 -- z6 -- cycle; t6 = z9 -- z4 -- z6 -- cycle;
t3 = z0 -- z12 -- z7 -- cycle; t7 = z12 -- z7 -- z3 -- cycle;
t4 = z5 -- z10 -- z6 -- cycle; t8 = z10 -- z6 -- z8 -- cycle;
t0 = z2 -- z9 -- z6 -- z10 -- cycle;

fill t1 withcolor Oranges 8 3; fill t7 withcolor Reds 8 3;
fill t2 withcolor Oranges 8 2; fill t8 withcolor Reds 8 3;
fill t3 withcolor Blues 8 3; fill t5 withcolor Greens 8 3;
fill t4 withcolor Blues 8 2; fill t6 withcolor Greens 8 3;
draw t1; draw t3; draw t5; draw t7;
draw t2; draw t4; draw t6; draw t8;

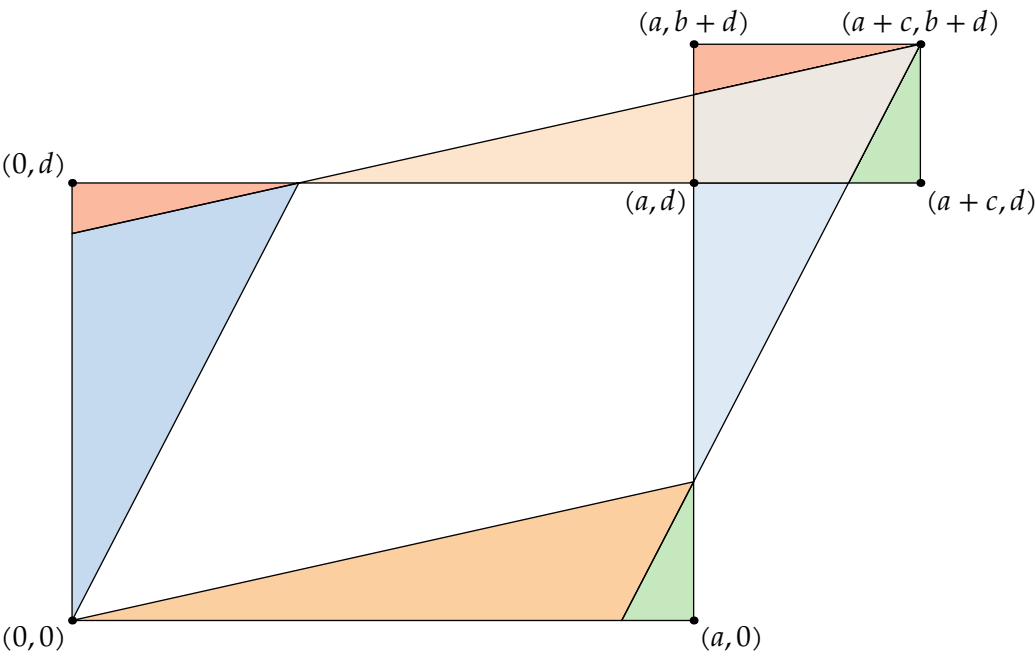
fill t0 withcolor 1/2[Oranges 8 2, Blues 8 2]; draw t0;

dotlabel.llft("$ (0, 0)$", z0);
dotlabel.lrt ("$(a, 0)$", z1);
dotlabel.llft("$ (a, d)$", z2);
dotlabel.ulft("$ (0, d)$", z3);
dotlabel.lrt ("$(a+c, d)$", z4);
dotlabel.top ("$(a+c, b+d)$", z6);
dotlabel.top ("$(a, b+d)$", z8);

numeric s; s = 1/8;
picture ad, bc, pg, t;
ad = image(draw (z0--z1--z2--z3--cycle) scaled s);
bc = image(draw (z2--z4--z6--z8--cycle) scaled s);
pg = image(draw (z0--z5--z6--z7--cycle) scaled s);
t = btex $\left|\begin{array}{c}
\text{vcenter}{\halign{\hss$#\hss&\quad\hss$#\hss\cr a\&b\cr c\&d\cr}}
\end{array}\right|=ad-bc=\{\}$ etex;
t := image(draw t; label.rt(ad, point 3/2 of bbox t));
t := image(draw t; label.rt("${}-\{\}$", point 3/2 of bbox t));
t := image(draw t; label.rt(bc, point 3/2 of bbox t));
t := image(draw t; label.rt("${}=\{\}$", point 3/2 of bbox t));
t := image(draw t; label.rt(pg, point 3/2 of bbox t));
label.bot(t, point 1/2 of bbox currentpicture shifted 55 down);

```

A 2×2 determinant is the area of a parallelogram



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \boxed{} - \boxed{} = \text{parallelogram}$$

— Solomon W. Golomb

Miscellaneous

```

numeric a, b, c, d; a = 144; d = 3/4 a; c = 1/3 a; b = 1/4 a;
path xx, yy; xx = 4 left -- (a+c) * right; yy = 4 down -- (b+d) * up;
z1 = whatever[(a,b), (a+c, b+d)]; y1 = d;
z2 = whatever[(a,b), (a+c, b+d)]; y2 = 0;
z3 = (a, 0) - z2;
picture P[];
P1 = image(
  draw (c,d) -- z1 -- z2 dashed withdots scaled 1/2;
  drawarrow xx; drawarrow yy;
  draw origin -- (a,b) -- (a+c, b+d) -- (c, d) -- cycle
    withpen pencircle scaled 3/4 withcolor Blues 8 8;
  dotlabel.rt("$ (a, b) $", (a, b));
  dotlabel.ulft("$ (c, d) $", (c, d));
);
P2 = image(
  draw z3 -- (a, 0) -- (a, d) -- (x3, d) -- cycle dashed withdots scaled 1/2;
  drawarrow xx; drawarrow yy;
  draw origin -- z2 -- z1 -- (c,d) -- cycle
    withpen pencircle scaled 3/4 withcolor Blues 8 8;
  dotlabel.rt("$ (a, b) $", (a, b));
  dotlabel.ulft("$ (c, d) $", (c, d));
);
P3 = image(
  fill unitsquare xscaled (c-x3) yscaled b shifted z3 withcolor Blues 8 2;
  drawoptions(dashed withdots scaled 1/2);
  draw (0, b) -- (a, b);
  draw (c, 0) -- (c, d) -- origin;
  drawoptions();
  drawarrow xx; drawarrow yy;
  draw z3 -- (a, 0) -- (a, d) -- (x3, d) -- cycle
    withpen pencircle scaled 3/4 withcolor Blues 8 8;
  dotlabel.rt("$ (a, b) $", (a, b));
  dotlabel.ulft("$ (c, d) $", (c, d));
);
P4 = image(
  fill unitsquare xscaled x3 yscaled (d-b) shifted (0, b) withcolor Blues 8 2;
  draw (x3, b) -- (x3, d) dashed withdots scaled 1/2;
  drawarrow xx; drawarrow yy;
  draw (0, d) -- (a, d) -- (a, 0) -- (c, 0) -- (c, b) -- (0, b) -- cycle
    withpen pencircle scaled 3/4 withcolor Blues 8 8;
  label.bot("$ a $", (a, 0)); label.lft("$ b $", (0, b));
  label.bot("$ c $", (c, 0)); label.lft("$ d $", (0, d));
);

interim labeloffset := 24;
label.ulft(P1, origin); label.urc(P2, origin);
label.llft(P3, origin); label.lrc(P4, origin);

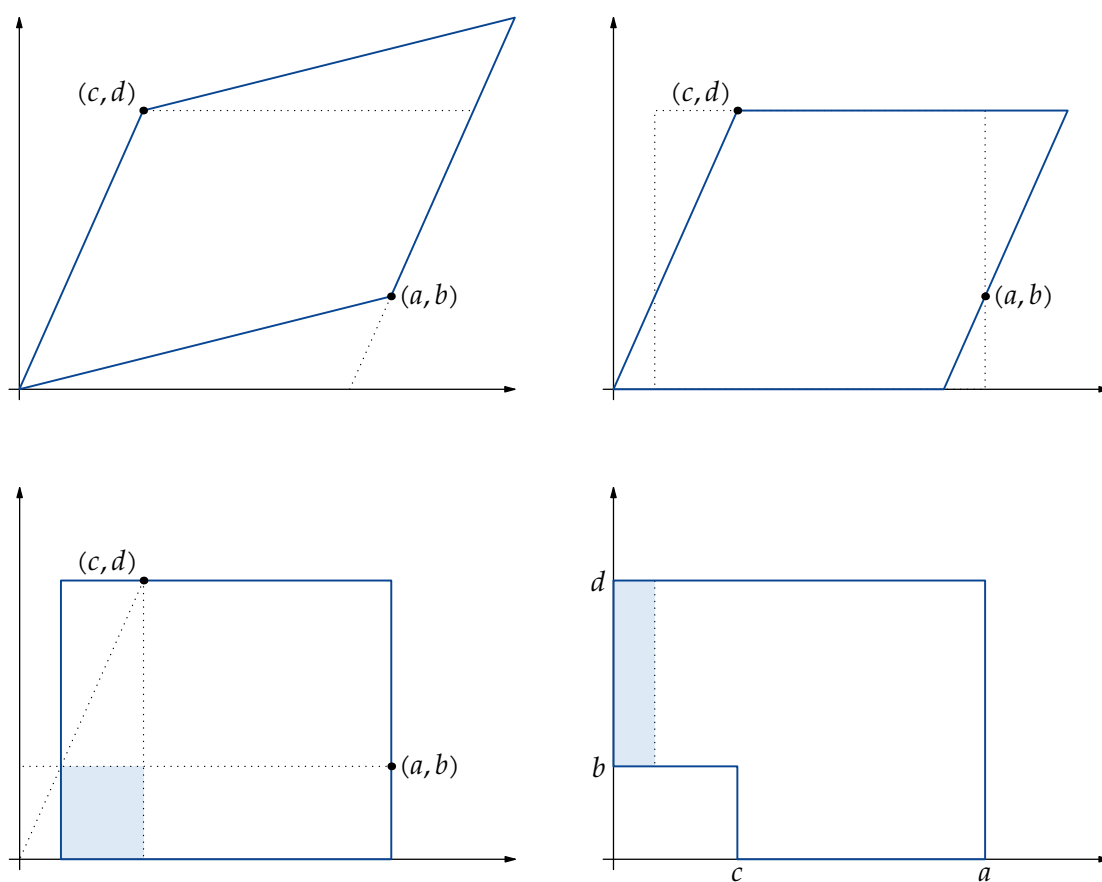
label.top(btex \vbox{\halign{\hss\vrule width 0pt depth 12pt # \hss\cr
The area of the parallelogram determined by vectors $(a,b)$ and $(c,d)$ is\cr
$\left|\, \, \, \vcenter{\halign{\hss$#\hss\quad\hss$#\hss\cr a&b\cr
c&d\cr}}\, \, \right|=\pm(ad-bc)$\cr}} etex, point 5/2 of bbox currentpicture);

```


Area of parallelogram

The area of the parallelogram determined by vectors (a, b) and (c, d) is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm(ad - bc)$$



— Yihnan David Gau

```

numeric u; u = 144;
vardef f(expr x) = 7/8 (x*x*x - x) enddef; numeric minx, maxx, s; -minx = maxx = 5/4; s = 1/8;
path ff; ff = ((minx,f(minx)) for x = minx+s step s until maxx: .. (x, f(x)) endfor) scaled u;
% move it to a convenient location, so axes can go through origin
ff := ff shifted - 1.14 point 0 of bbox ff shifted 20 up;
path xx; xx = 8 left -- right scaled 1.1 xpart (urcorner ff - llcorner ff);
path yy; yy = xx rotated 90 cutafter (left--right) shifted (0, 1.1 ypart urcorner ff);

numeric a, b, a', b'; a = 1.4; a' = 3.7; b = length ff - 1/4 - a; b' = length ff - 1/4 - a';
z0 = point a of ff; z1 = point b of ff; z2 = point a' of ff; z3 = point b' of ff;

z4 = whatever[z2, z3]; z5 = whatever[z2, z3];
x4 = xpart point 0 of ff; x5 = xpart point infinity of ff;

x0 = x6 = x8; y6 = y2; x1 = x7 = x9; y7 = y3;
z8 = whatever[z4, z5]; z9 = whatever[z4, z5];
z10 = whatever[z4, z5] = whatever[z6, z7];

% dotlabels.top(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10); % <--- in case of need

fill z2--z6--z8--cycle withcolor Oranges 8 2;
fill z3--z7--z9--cycle withcolor Blues 8 2;

fill z10 -- z2 -- z6 -- cycle withcolor Oranges 8 3;
fill z10 -- z3 -- z7 -- cycle withcolor Blues 8 3;

label.bot("\strut $a$", (x8, 0)); draw (x8, 0) -- z8;
label.bot("\strut $b$", (x7, 0)); draw (x7, 0) -- z7;
label.bot("\strut $a'$", (x2, 0)); draw (x2, 0) -- z2 dashed evenly;
label.bot("\strut $b'$", (x3, 0)); draw (x3, 0) -- z3 dashed evenly;

draw z2 -- z6 -- z7 -- z3;
drawoptions(withcolor Blues 8 7);
draw z4--z5; label.rt("$y=h(x)$", z5);
drawoptions(withcolor Oranges 8 7);
draw ff; label.top("$y=f(x)$", point infinity of ff);
drawoptions();
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);

picture mark; mark = image(forsuffixes $=left, right:
    draw (up--down) scaled 2 rotated -20 shifted 1/2 $;
endfor);
draw mark shifted (1/2(x0+x2), ypart point 0 of xx);
draw mark shifted (1/2(x1+x3), ypart point 0 of xx);

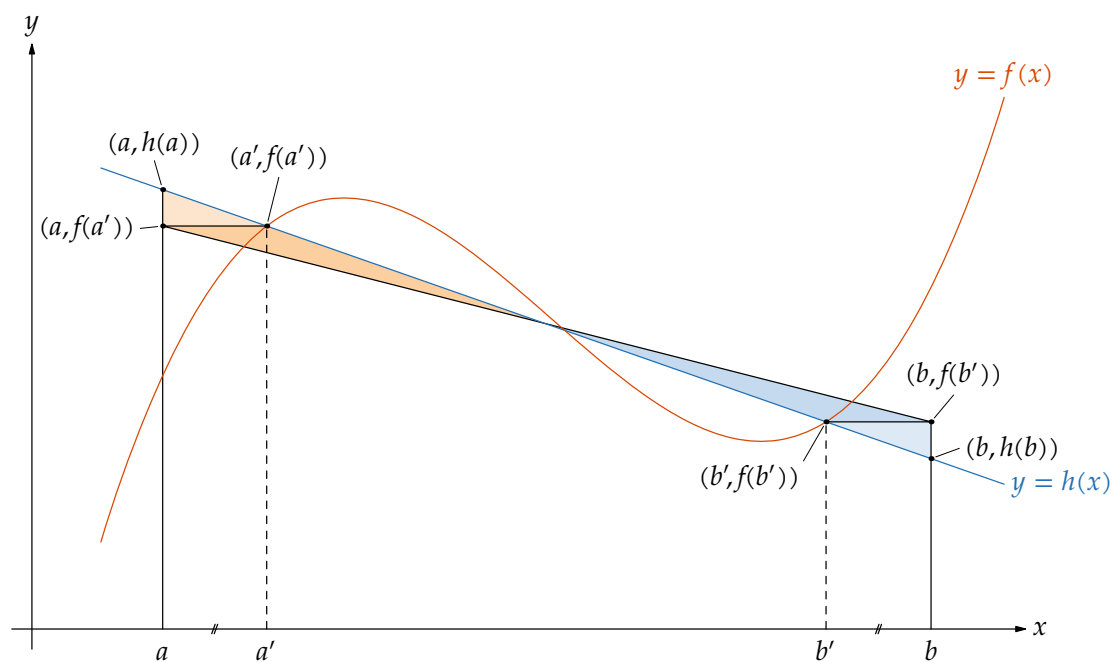
vardef pindotlabel@#(expr t, z, o) =
    draw z -- z+o cutbefore fullcircle scaled 2 dotlabeldiam shifted z
    withpen pencircle scaled 1/4;
    label@#(t, z+o); draw z withpen pencircle scaled dotlabeldiam;
enddef; interim dotlabeldiam := 2; interim labeloffset := 2;
pindotlabel.top ("$(a, h\kern-0.4pt(a))$", z8, 12 dir 106);
pindotlabel.lft ("$(a, f\kern-0.4pt(a'))$", z6, 9 dir 186);
pindotlabel.top ("$(a'\!, f\kern-0.4pt(a'))$", z2, 20 dir 76);
pindotlabel.top ("$(b, f\kern-0.4pt(b'))$", z7, 14 dir 50);
pindotlabel.rt ("$(b, h\kern-0.4pt(b))$", z9, 12 dir 20);
pindotlabel.llft ("$(b'\!, f\kern-0.4pt(b'))$", z3, 18 dir 240);

label.top(btex $\displaystyle
\frac{1}{2} \left( (b-a)\left(f(a') + f(a)\right) + \right.
\frac{1}{2} \left. \left( (b-a)\left(h(a) + h(b)\right) \right) \right) =
$ etex, point 5/2 of bbox currentpicture shifted 42 up);

```

Gaussian quadrature as the area of either trapezoid

$$\frac{1}{2} (b - a) (f(a') + f(b')) = \frac{1}{2} (b - a) (h(a) + h(b))$$



— Mike Akerman

```

input isometric_projection
set_projection(100/3, -45);
picture blue_cube; blue_cube = cube(Blues 8 4, Blues 8 2, background);
picture white_cube; white_cube = cube(background, background, background);

path case; picture nice_case;
case = p(0,1,0)--p(0,1,5)--p(0,6,5)--p(-5,6,5)--p(-5,6,0)--p(-5,1,0)--cycle;
nice_case = image(draw case withpen pencircle scaled 3/2; draw case withcolor 7/8);

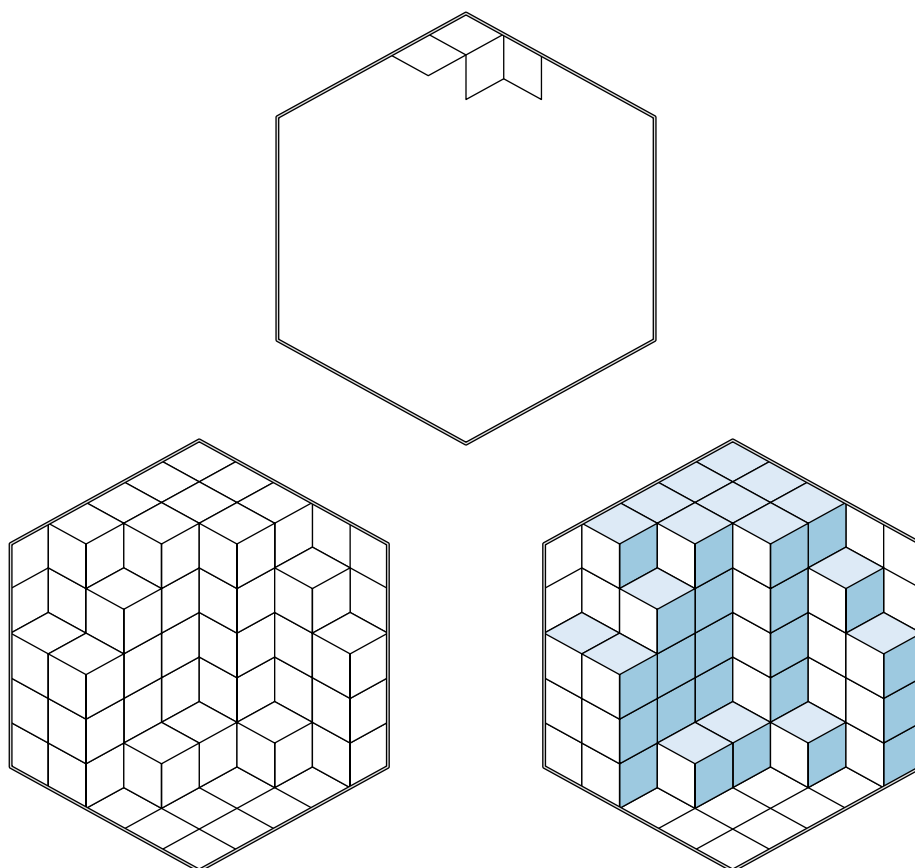
picture P[]; P0 = image(
  draw p(-5,6,3) -- p(-4,6,3) -- p(-4,6,5) -- p(-4,5,5);
  draw p(-5,6,4) -- p(-4,6,4) -- p(-4,5,4) -- p(-4,5,5) -- p(-3,5,5) -- p(-3,6,5);
  draw nice_case;
);

vardef make_pattern(expr cube) =
  % grid on "walls"
  for i = 0 upto 5:
    draw p(-i, 1, 0) -- p(-i, 1, 5) -- p(-i, 6, 5);
    draw p(0, 1, i) -- p(-5, 1, i) -- p(-5, 6, i);
    draw p(-5, i+1, 0) -- p(-5, i+1, 5) -- p(0, i+1, 5);
  endfor
  % draw the cubes
  save x, z; numeric x, z;
  x = -4; z = 4;
  for k = 5, 5, 5, 5, 3,
    5, 5, 5, 4, 3,
    5, 5, 1, 1, 0,
    4, 1, 0, 0, 0,
    3, 0, 0, 0, 0:
    for y=1 upto k: draw cube shifted p(x, y, z); endfor
    z := z - 1;
    if z < 0:
      z := 4;
      x := x + 1;
    fi
  endfor enddef;
P1 = image(make_pattern(white_cube); draw nice_case);
P2 = image(make_pattern(blue_cube); draw nice_case);

draw P0 shifted 160 up;
draw P1 shifted 100 left;
draw P2 shifted 100 right;

```

The problem of the calissons



— Guy David and Carlos Tomei