Proofs without words II

More exercises in METAPOST

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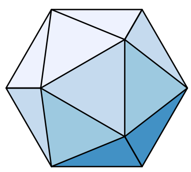
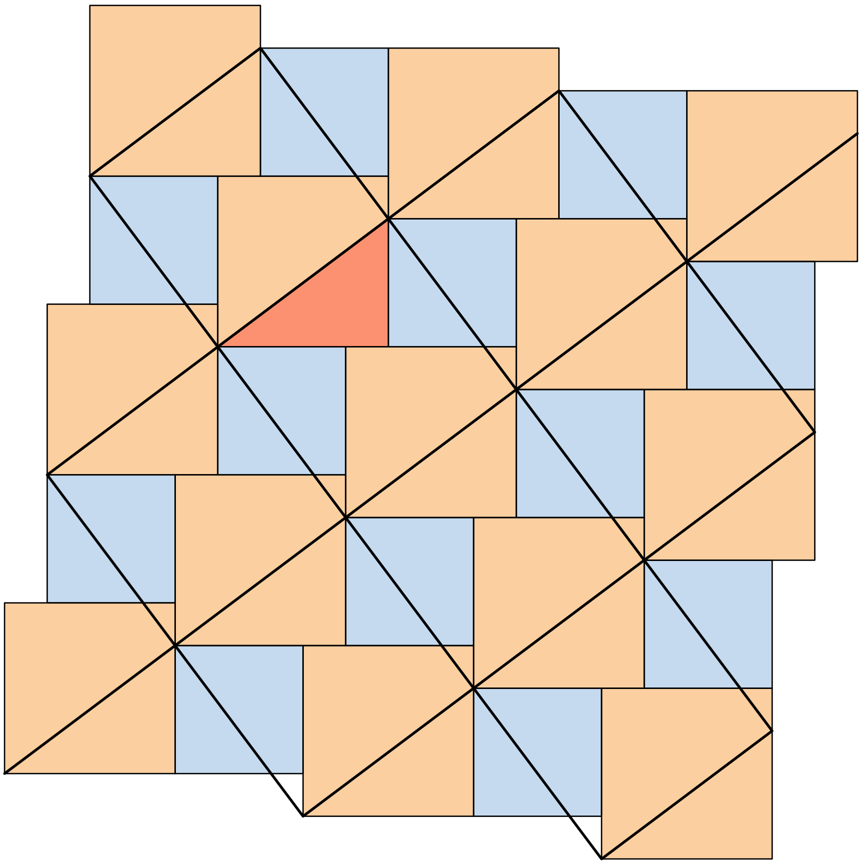


Contents

Geometry and Algebra

```
numeric u;
u = 16;
path s[];
for i=0 upto 4:
    for j=0 upto 4:
         s[5i+j] = unitsquare shifted -(1/2,1/2)
             scaled ((4-((i+j) mod 2))*u)
             shifted (1/2i*u*(7,-1))
             shifted (1/2j*u*(1,7));
         fill s[5i+j] withcolor if odd (i+j): Blues 8 3 else: Oranges 8 3 fi;
    endfor
endfor
fill subpath(0,7/4) of s[8] -- cycle withcolor Reds 8 4;
for i=0 upto 4:
    for j=0 upto 4:
        draw s[5i+j];
    endfor
endfor
drawoptions(withpen pencircle scaled 1);
draw point 0 of s[0] -- point 7/4 of s[24];
draw point 0 of s[2] -- point 7/4 of s[14]
  -- point 7/4 of s[22] -- point 0 of s[10] -- cycle;
draw point 0 of s[4] -- point 7/4 of s[ 4]
  -- point 7/4 of s[20] -- point 0 of s[20] -- cycle;
```

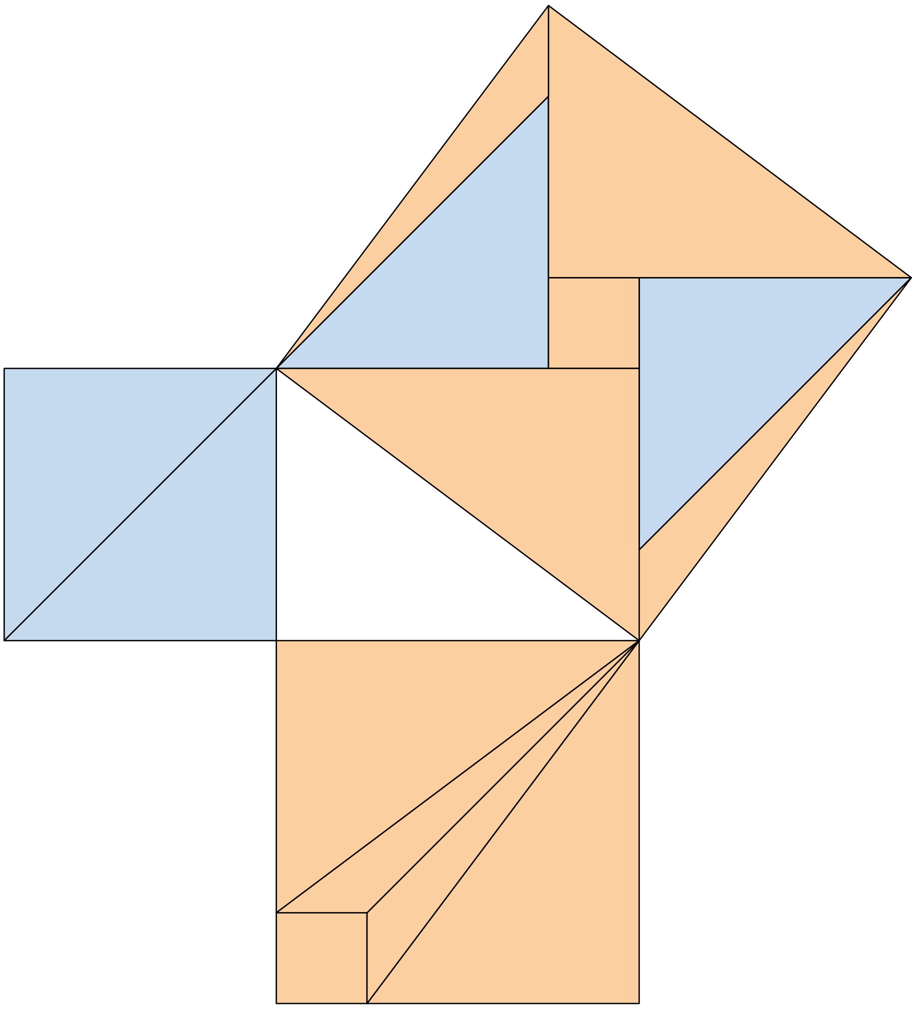
The Pythagorean theorem VII



— Annairizi of Arabia (circa 900)

```
numeric u;
u = 12mm;
path a[], b[];
a1 = ((0,0)--(1,0)--(1,1)--cycle) scaled 3u;

b0 = unitsquare scaled u;
b1 = ((1,0)--(4,0)--(4,4)--cycle) scaled u;
b2 = ((1,0)--(4,4)--(1,1)--cycle) scaled u;

a2 = a1 reflectedabout((0,0),(1,1));
b3 = b1 reflectedabout((0,0),(1,1));
b4 = b2 reflectedabout((0,0),(1,1));

picture A[], B[];

forsuffixes $=1,2:
    A$ = image(fill a$ withcolor Blues 8 3; draw a$);
    draw A$ shifted (3u * left);
endfor
forsuffixes $=0,1,2,3,4:
    B$ = image(fill b$ withcolor Oranges 8 3; draw b$);
    draw B$ shifted (4u * down);
endfor

draw A1 shifted (0u,3u);
draw A2 shifted (4u,1u);
draw B0 shifted (3u,3u);
draw B1 rotated 90 shifted (4u,-1u);
draw B2 shifted (3u,0u);
draw B3 reflectedabout(left,right) shifted (3u,8u);
draw B4 reflectedabout(down,up) rotated 90 shifted (4u,7u);
```

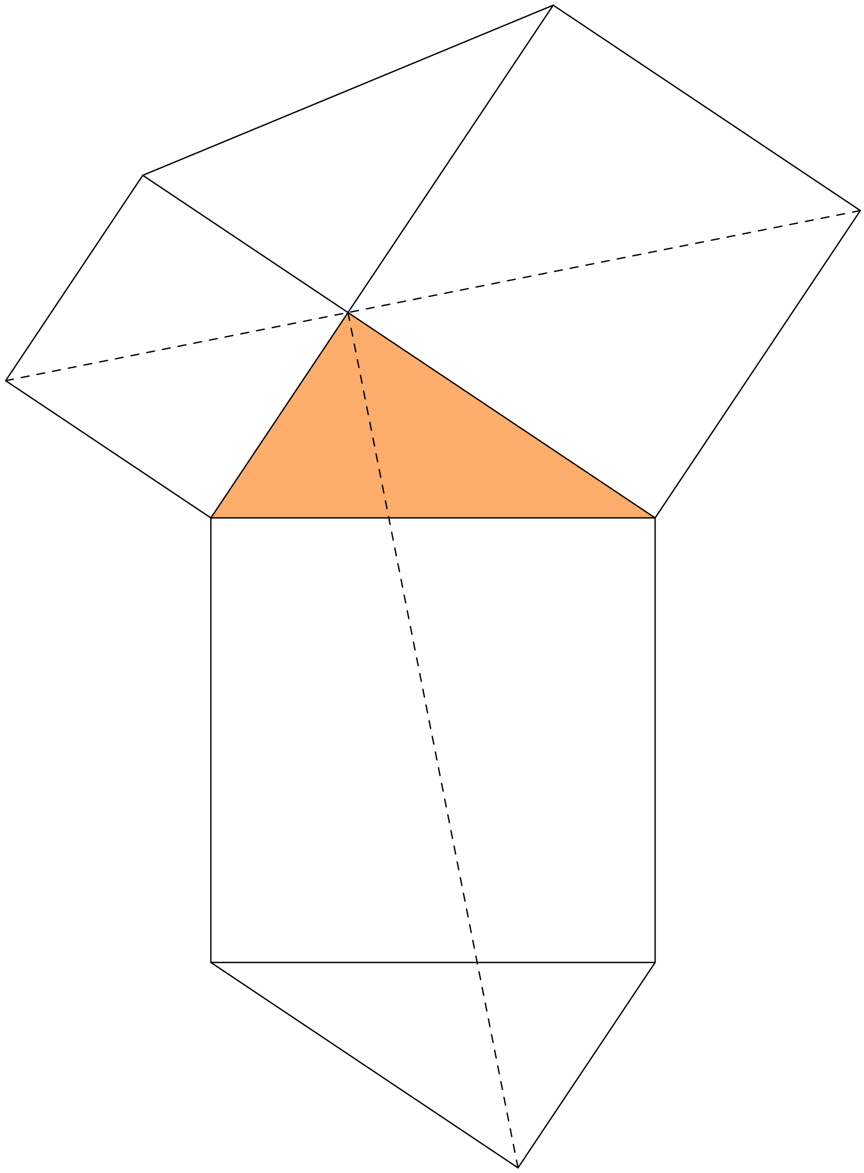
The Pythagorean theorem VIII



— Liu Hui (3rd century A.D.)

```
path c, t, s[];
c = fullcircle scaled 2/5 \mpdim\textwidth;
t = point 4 of c -- point 0 of c -- point 2.5 of c -- cycle;
fill t withcolor Oranges 7 3;
vardef square_on(expr a,b) =
 unitsquare scaled abs(a-b) rotated angle (a-b) shifted b
enddef;
for i=1 upto 3:
s[i] = square_on(point i-1 of t, point i of t);
draw s[i];
endfor
z1 = point 2 of t rotatedabout(center s1, 180);
draw point 2 of t  -- z1            dashed evenly;
draw point 2 of s1 -- z1 -- point 3 of s1;
draw point 2 of s2 -- point 3 of s3 dashed evenly;
draw point 2 of s3 -- point 3 of s2;
```

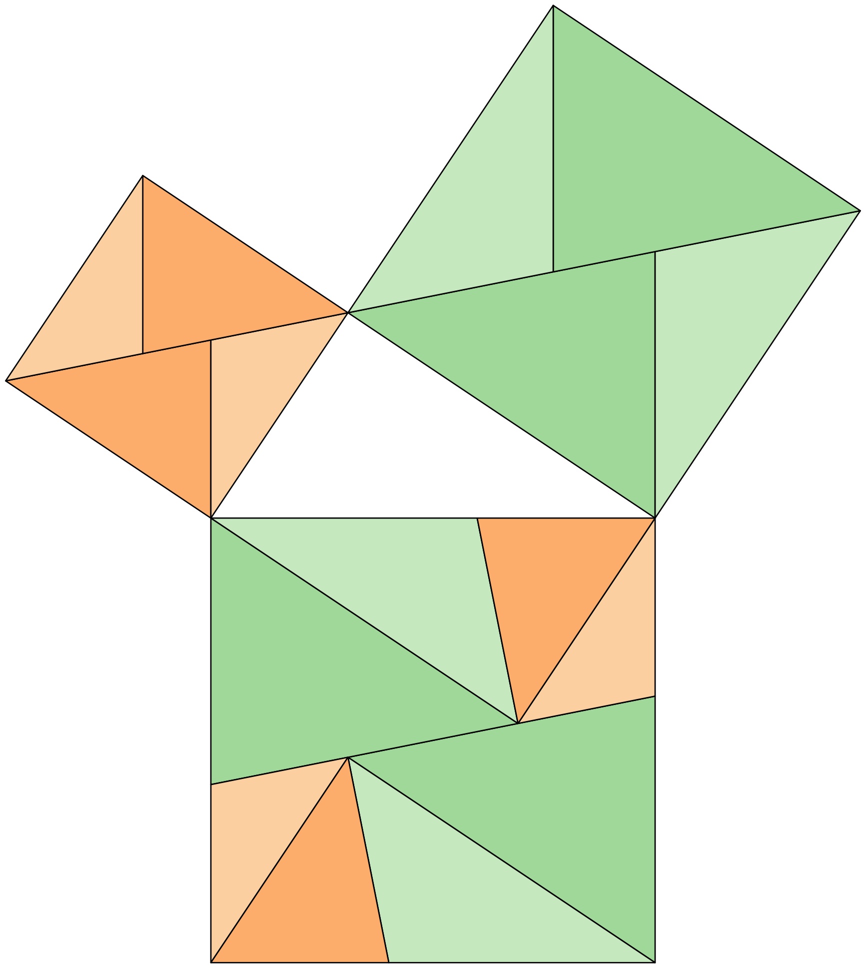
The Pythagorean theorem IX



— Leonardo da Vinci (1452–1519)

```
path c, t, s[];
c = fullcircle scaled 2/5 \mpdim\textwidth;
t = point 4 of c -- point 0 of c -- point 2.5 of c -- cycle;
vardef square_on(expr a,b) =
 unitsquare scaled abs(a-b) rotated angle (a-b) shifted b
enddef;
for i=1 upto 3:
s[i] = square_on(point i-1 of t, point i of t);
endfor
z2 = whatever [ point 3 of s3, point 2 of s2 ]
= whatever [ point 3 of s1, point 4 of s1 ] ;
z3 = whatever [ point 3 of s3, point 2 of s2 ]
= whatever [ point 1 of s1, point 2 of s1 ] ;
path p[];
p21 = point 0 of s2 -- point 1 of s2 -- z2 -- cycle;
p22 = point 1 of s2 -- point 2 of s2 -- z2 -- cycle;
p23 = p21 rotatedabout(center s2, 180);
p24 = p22 rotatedabout(center s2, 180);
p31 = point 3 of s3 -- point 4 of s3 -- z3 -- cycle;
p32 = point 0 of s3 -- point 1 of s3 -- z3 -- cycle;
p33 = p31 rotatedabout(center s3, 180);
p34 = p32 rotatedabout(center s3, 180);

color f[];
f21 = f23 =  Greens[7][3]; f22 = f24 =  Greens[7][2];
f31 = f33 = Oranges[7][3]; f32 = f34 = Oranges[7][2];

picture m[];
forsuffixes $=21,22,23,24,31,32,33,34:
 m$ = image(fill p$ withcolor f$; draw p$);
 draw m$;
endfor

draw m21 shifted (point 3 of s1 - point 1 of s2);
draw m23 shifted (point 1 of s1 - point 3 of s2);

draw m32 shifted (point 2 of s1 - point 0 of s3);
draw m34 shifted (point 0 of s1 - point 2 of s3);

draw m22 shifted -point 1 of s2 rotated 90 shifted point 3 of s1;
draw m24 shifted -point 3 of s2 rotated 90 shifted point 1 of s1;

draw m31 shifted -point 0 of s3 rotated 90 shifted point 0 of s1;
draw m33 shifted -point 2 of s3 rotated 90 shifted point 2 of s1;
```

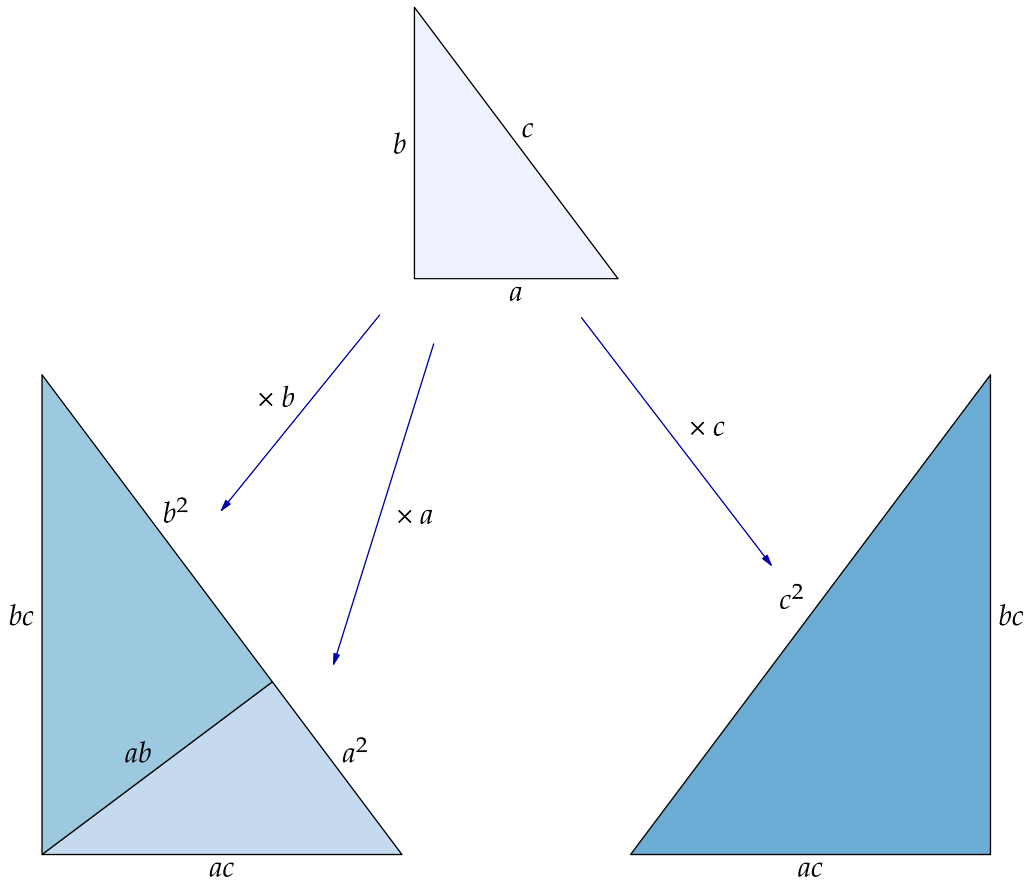
The Pythagorean theorem X



— J. E. Böttcher

```
numeric a,b,c,u;
a = 1.0605; b = 1.414; c = a++b;

u = 72;
path t[];
t0 = (origin -- (a,0) -- (0,b) -- cycle) scaled u;
t1 = t0 scaled a reflectedabout(up,down) rotated angle(b,a) rotated 90;
t2 = t0 scaled b reflectedabout(up,down) rotated angle(b,a);
t3 = t0 scaled c reflectedabout(up,down);

t0 := t0 shifted - point 1/2 of t0;
t1 := t1 shifted (1/2u*(a-6,-6)-point 2 of t1);
t2 := t2 shifted (1/2u*(a-6,-6)-point 1 of t2);
t3 := t3 shifted (point 2 of t1 reflectedabout(up,down));


fill t0 withcolor Blues[7][1]; draw t0;
fill t1 withcolor Blues[7][2]; draw t1;
fill t2 withcolor Blues[7][3]; draw t2;
fill t3 withcolor Blues[7][4]; draw t3;

label.bot("$a$", point 1/2 of t0);
label.lft("$b$", point 5/2 of t0);
label.urt("$c$", point 3/2 of t0);

label.bot ("$ac$",  point 1/2 of t3);
label.rt  ("$bc$",  point 5/2 of t3);
label.ulft("$c^2$", point 3/2 of t3);

label.urt ("$a^2$", point 1/2 of t1);
label.ulft("$ab$",  point 5/2 of t1);
label.bot ("$ac$",  point 3/2 of t1);

label.lft ("$bc$",  point 3/2 of t2);
label.urt ("$b^2$", point 5/2 of t2);

vardef centroid(expr t) = 2/3[point 2 of t, point 1/2 of t] enddef;
path a[];
for i=1 upto 3:
 a[i] = centroid(t0) -- centroid(t[i]) if i=1: shifted 20 right fi
 cutbefore fullcircle scaled 1.7u shifted centroid(t0)
 cutafter  fullcircle scaled 1.7u shifted centroid(t[i]);
 drawarrow a[i] withcolor 2/3 blue;
endfor

label.lrt ("${}\times a$", point 1/2 of a1);
label.ulft("${}\times b$", point 1/2 of a2);
label.urt ("${}\times c$", point 1/2 of a3);
```

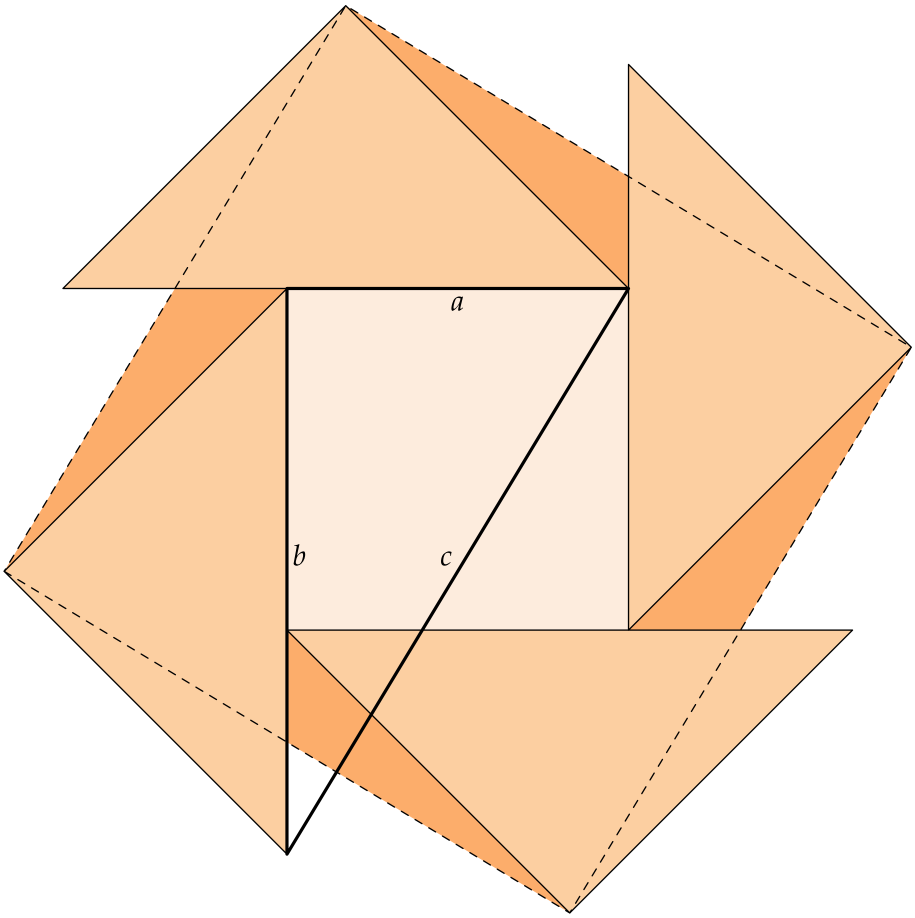
The Pythagorean theorem XI



— Frank Burk

```
numeric a,b,r,u;
u = 1cm; r = 21;
a = 3u-r;
b = 3u+r;
path t[];
for i=0 upto 3:
 t[i] = ((-a-b,a-b) -- (-a,a-2b) -- (-a,a) -- cycle) rotated 90i;
endfor
fill for i=0 upto 3: point 0 of t[i] -- endfor cycle withcolor Oranges[7][3];
fill for i=0 upto 3: point 2 of t[i] -- endfor cycle withcolor Oranges[7][1];
for i=0 upto 3:
 fill t[i] withcolor Oranges[7][2]; draw t[i];
endfor
draw for i=0 upto 3: point 0 of t[i] -- endfor cycle dashed evenly;

draw subpath(1,2) of t0 -- point 2 of t3 -- cycle withpen pencircle scaled 1.2;
label.bot("$a$", 1/2[point 2 of t0, point 2 of t3]);
label.urt("$b$", 1/2[point 2 of t0, point 1 of t0]);
label.ulft("$c$", 1/2[point 1 of t0, point 2 of t3]);
```

The Pythagorean theorem XII



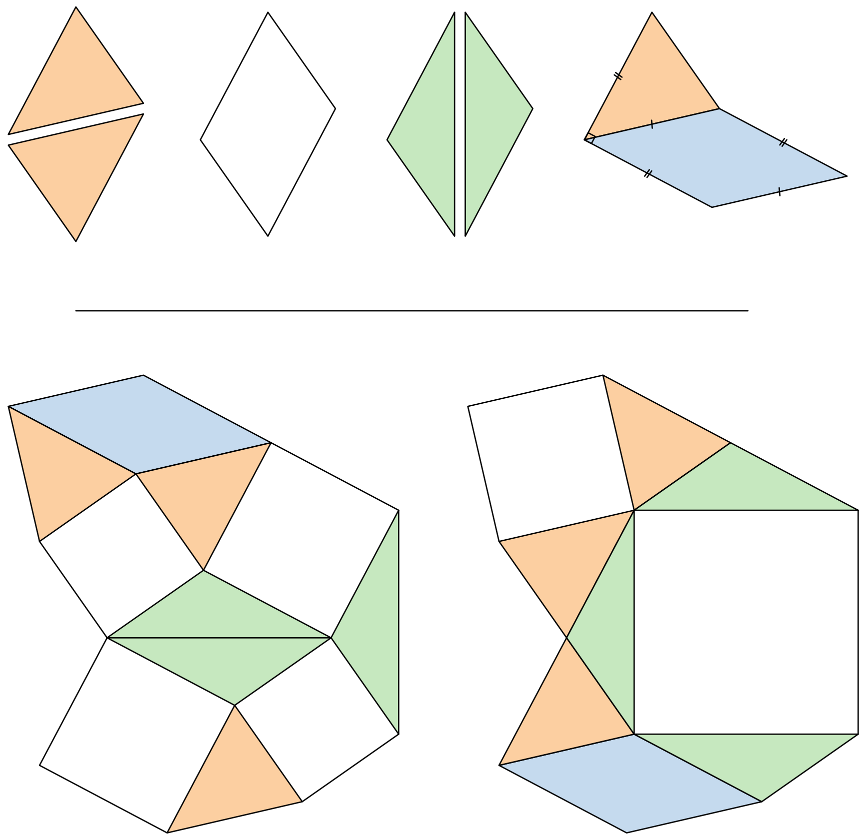
$$a^2 + b^2 = c^2$$

— Poo-Sung Park

```
numeric a,b,r; a = 42; b = 26; r = 13; -z0 = z2 = (b,0) rotated r; -z1 = z3 = (0,a);
z4 = z3 rotatedabout(z0,-90); z5 = z0 rotatedabout(1/2[z4,z2],180);
path p,h; p = z0--z1--z2--z3--cycle; h = z0--z2--z3--cycle;
path v,b; v = z0--z1--z3--cycle; b = z0--z4--z5--z2--cycle;
picture whole, cut_v, cut_h, fit, twos, fours;
whole = image(fill p withcolor background; draw p);
def fd(expr p, shade) = fill p withcolor shade; draw p enddef;
color s; s = Oranges 6 2;
cut_v = image(fd(h shifted 2 up, s); fd(h rotated 180 shifted 2 down, s));
s := Greens 6 2;
cut_h = image(fd(v shifted 2 left, s); fd(v rotated 180 shifted 2 right, s));
input mark_equal
fit = image(fd(h, Oranges[6][2]); fd(b, Blues[6][2]);
  mark_equal(z0,z2,1); mark_equal(z4,z5,1);
  mark_equal(z0,z3,2); mark_equal(z0,z4,2); mark_equal(z2,z5,2);
  draw unitsquare scaled 3 rotated angle(z4-z0) shifted z0);
path hh[], vv[];
hh1 = h rotatedabout(z0,-90); hh2 = h rotated 180 shifted 1/2[z4,z5];
vv1 = v shifted -z0 rotated -90 shifted point 2 of hh2;
vv2 = v shifted -z0 shifted point 2 of vv1;
vv3 = v shifted -z3 rotated 90 shifted point 1 of vv1;
hh3 = h shifted -z3 shifted point 0 of vv3;
fours = image(
 fd(b, Blues[6][2]);
 fd(hh1, Oranges[6][2]); fd(hh2, Oranges[6][2]); fd(hh3, Oranges[6][2]);
 fd(vv1, Greens[6][2]); fd(vv2, Greens[6][2]); fd(vv3, Greens[6][2]);
 draw point 2 of b -- point 2 of vv2;
 draw point 1 of hh3 -- point 1 of vv2;
 draw point 0 of hh3
   -- point 2 of hh3 reflectedabout(point 0 of hh3, point 2 of vv3)
   -- point 2 of vv3 -- point 1 of hh1);
path hh[], vv[];
hh1 = h; hh2 = h rotatedabout(point 2 of h, 180);
hh3 = h shifted point 1/2 of hh2 rotatedabout(point 0 of hh2,-90);
vv1 = v shifted -z0 shifted point 2 of h;
vv2 = v shifted -z1 rotated -90 shifted point 2 of vv1;
vv3 = v shifted -z3 rotated 90 shifted z2;
twos = image(
 fd(b, Blues[6][2]);
 fd(hh1, Oranges[6][2]); fd(hh2, Oranges[6][2]); fd(hh3, Oranges[6][2]);
 fd(vv1, Greens[6][2]); fd(vv2, Greens[6][2]); fd(vv3, Greens[6][2]);
 draw point 1 of vv3 -- point 2 of vv2;
 draw point 1 of hh2
   -- point 0 of hh2 reflectedabout(point 1 of hh2, point 0 of hh3)
   -- point 0 of hh3);
numeric i; i = -1;
forsuffixes $=cut_v, whole, cut_h, fit: draw $ shifted 72(incr i, 0); endfor
draw (0,-70) -- (252,-70);
numeric dy; dy = ypart (lrcorner fours - lrcorner twos);
draw fours shifted (0, -100); draw twos shifted (184,dy-100);
```

A generalization from Pythagoras

The sum of the area of two squares, whose sides are the lengths of two diagonals of a parallelogram, is equal to the sum of the area of four squares, whose sides are its four sides.



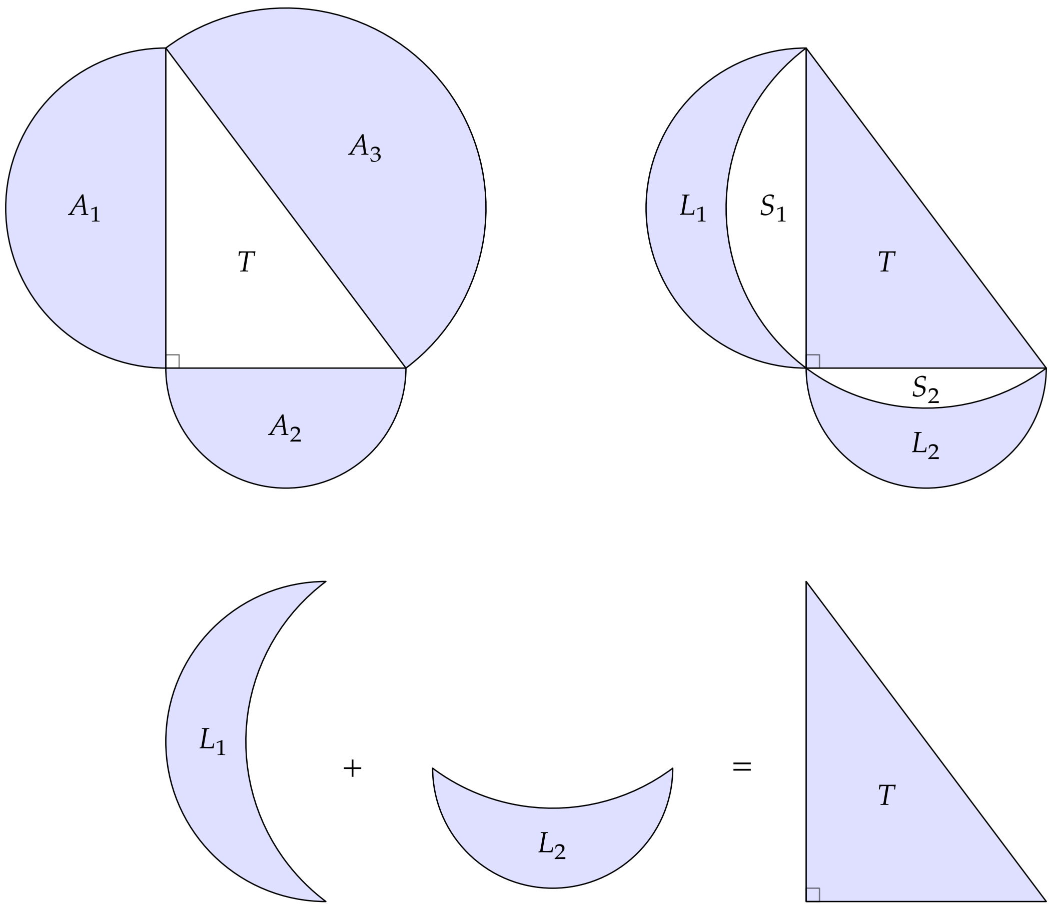
Corollary: The Pythagorean theorem (when the parallelogram is a rectangle).

— David S. Wise

```
path T; T = origin -- 90 right -- 120 up -- cycle;
path s[];
vardef semicircle(expr a,b) = halfcircle zscaled (a-b) shifted 1/2[a,b] enddef;
s0 = semicircle(point 2 of T, point 1 of T);
s1 = semicircle(point 2 of T, point 0 of T);
s2 = semicircle(point 0 of T, point 1 of T);
s3 = semicircle(point 1 of T, point 2 of T);

path A[], S[], L[];
for i=1 upto 3: A[i] = s[i] -- cycle; endfor
S1 = s0 cutafter origin -- cycle;
L1 = s0 cutafter origin .. reverse s1 .. cycle;
S2 = s0 cutbefore origin -- cycle;
L2 = s0 cutbefore origin .. reverse s2 .. cycle;

picture part[];  color f; f = 7/8[blue,white];
part1 = image(
  draw unitsquare scaled 5 withcolor 1/2 white;
  forsuffixes $=1, 2, 3:
    fill A$ withcolor f; draw A$;
    label("$A_" & str $ & "$", 1/2[point 2 of A$, point 9/2 of A$]);
  endfor
  label("$T$", 2/3 point 3/2 of T);
);
part2 = image(
 fill T withcolor f;
 draw unitsquare scaled 5 withcolor 1/2[f,black];
 draw T;
 fill L1 withcolor f; draw L1;
 fill L2 withcolor f; draw L2;
 label("$T$", 2/3 point 3/2 of T);
 label("$L_1$", point 6 of L1 shifted 18 right);
 label("$L_2$", point 5 of L2 shifted 16 up);
 label("$S_1$", point 5/2 of T shifted 12 left);
 label("$S_2$", point 1/2 of T shifted  8 down);
);
part3 = image(
 fill T withcolor f;
 draw unitsquare scaled 5 withcolor 1/2[f,black];
 draw T;
 L1 := L1 shifted (-180,0); fill L1 withcolor f; draw L1;
 L2 := L2 shifted (-140,50); fill L2 withcolor f; draw L2;
 label("$T$", 2/3 point 3/2 of T);
 label("$L_1$", point 6 of L1 shifted 18 right);
 label("$L_2$", point 5 of L2 shifted 16 up);
 label("$+$", (-170,50));
 label("$=$", (-24,50));
);

draw part1 shifted 120 left; draw part2 shifted 120 right;
draw part3 shifted (120,-200);
```

A theorem of Hippocrates of Chios (circa 440 BC)

The combined area of the lunes constructed on the legs of a given right angle triangle is equal to the area of the triangle.



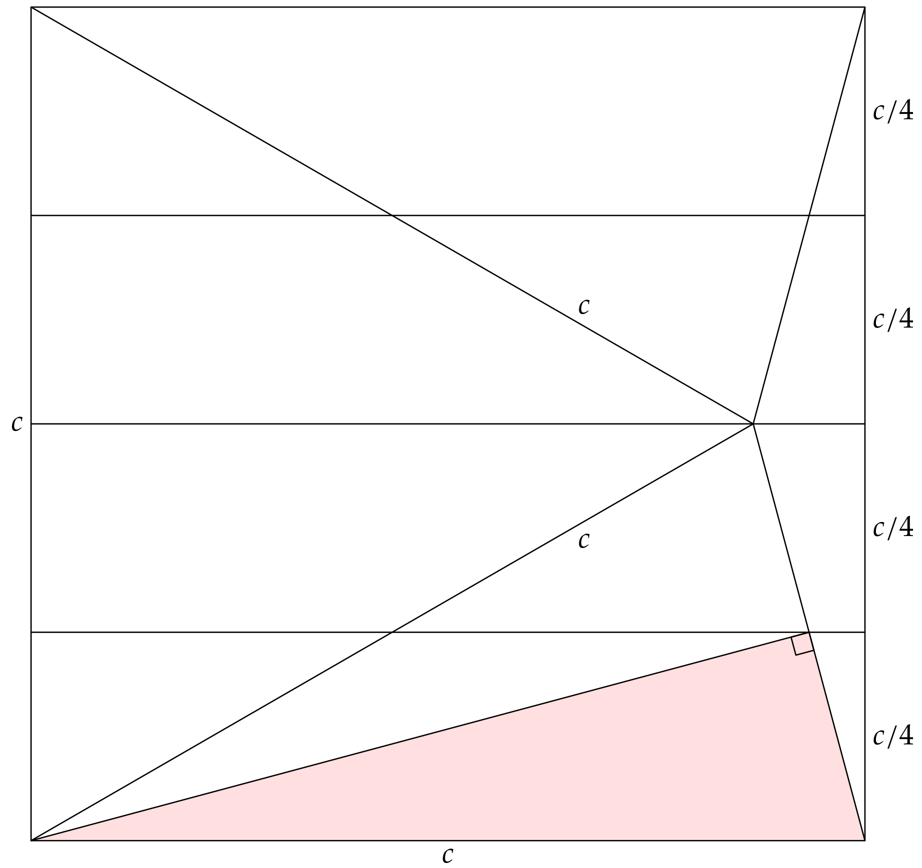
$$A_1 + A_2 = A_3$$
$$(L_1 + S_1) + (L_2 + S_2) = T + S_1 + S_2$$
$$L_1 + L_2 = T$$

— Eugene A. Margerum and Michael M. McDonnell

```
path s;
s = unitsquare scaled 3/4 \mpdim\textwidth;
z1 = point 1 of s rotated 30;
z2 = 1/2[z1,point 1 of s];
fill origin--point 1 of s--z2--cycle withcolor 7/8[red,white];
draw unitsquare scaled 7 rotated 195 shifted z2;
draw s;
draw subpath(0,1) of s shifted point -1/4 of s;
draw subpath(0,1) of s shifted point -1/2 of s;
draw subpath(0,1) of s shifted point -3/4 of s;

draw origin--z2;
for i=0 upto 3: draw z1--point i of s; endfor
label.bot("$c$", point 1/2 of s);
label.lft("$c$", point 7/2 of s);
label.lrt("$c$", 3/4 [point 0 of s, z1]);
label.urt("$c$", 3/4 [point 3 of s, z1]);
for t=9/8 step 1/4 until 2:
label.rt("$c/4$", point t of s);
endfor
```

The area of a right triangle with acute angle $\pi/12$

The area of a right triangle is $\frac{1}{8}(\text{hypotenuse})^2$ if and only if one acute angle is $\pi/12$.



— Klara Pinter

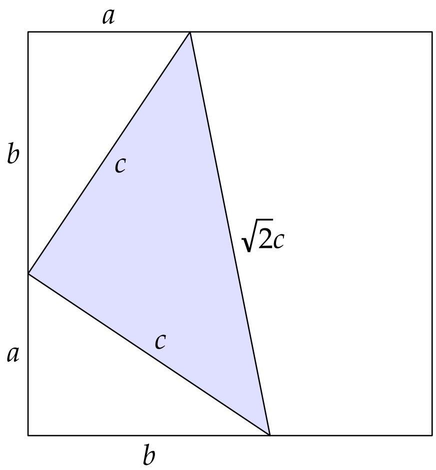
```
vardef make_fig(expr wd,r) =
    save a,b; a+b = wd; b-a=r;
    save s,t; path s,t;
    s = unitsquare scaled (a+b);
    t = (b,0) -- (a,a+b) -- (0,a) -- cycle;
    image(
        fill t withcolor 7/8 [ blue, white ];
        draw t;
        draw s;
        label.top("$a$", (1/2 a, a+b));
        label.lft("$a$", (0, 1/2 a));
        label.bot("$b$", (1/2 b, 0));
        label.lft("$b$", (0, a + 1/2 b));
        label.lrt("$c$", point 3/2 of t);
        label.urt("$c$", point 5/2 of t);
        label.rt("$\sqrt2c$", point 1/2 of t);
    )
enddef;
numeric u; u = 2/11 \mpdim\textwidth;
picture p[];
p1 = make_fig(2u,30) shifted (-3/2u,0);
p2 = make_fig(2u, 0) shifted (+3/2u,0);
draw p1; label.bot("$a+b\le\sqrt2c$",        point 1/2 of bbox p1 shifted 8 down);
draw p2; label.bot("$a+b=\sqrt2c \iff a=b$", point 1/2 of bbox p2 shifted 8 down);
```

A right angle inequality

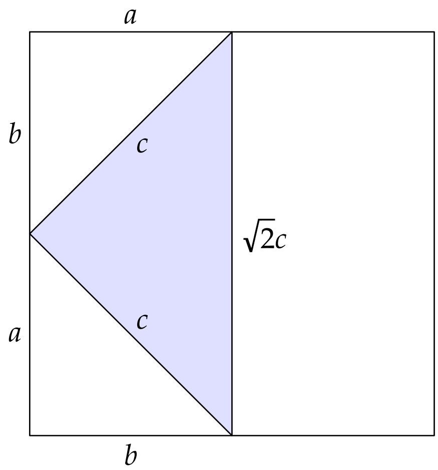
Let c be the hypotenuse of a right triangle whose other two sides are a and b. Prove that

$$a + b \leq \sqrt{2}c.$$

When does equality hold?



$a + b \leq \sqrt{2}c$

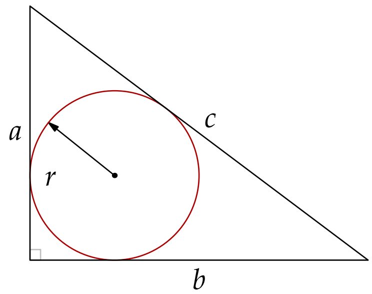


$a + b = \sqrt{2}c \iff a = b$

— Canadian Mathematical Olympiad 1969

```
input arrow_label
numeric a, b, c, r, u; u = 32; a = 3u; b = 4u; c = a++b; 2r = a + b - c;
path s; s = fullcircle scaled (a+b-c) shifted (r, r);
path t; t = origin -- (b,0) -- (0,a) -- cycle;
picture P[];
P1 = image(
  drawarrow center s -- point 3.14 of s; label.urt("$r$", 1/2 point 3.14 of s);
  drawdot center s withpen pencircle scaled 2; draw s withcolor 2/3 red;
  draw unitsquare scaled 4 withcolor 3/4; draw t;
  label.lft("$a$", point 5/2 of t);
  label.bot("$b$", point 1/2 of t);
  label.urt("$c$", point 3/2 of t));
P2 = btex \vbox to 72pt{\openup8pt\halign{#\hfil\quad&$\displaystyle #$\cr
  I.&r=\frac{ab}{a+b+c}\cr   II.&r=\frac{a+b-c}{2}\cr}\vss} etex;
P3 = btex \hbox to \textwidth{\hbox to 12pt{\hss I.}\quad $ab = r(a+b+c)$\hss} etex;
P4 = image( % this one is re-used in P5 and P8
  path p[]; p0 = unitsquare scaled r; % split the triangle into 5 parts
  p1 = (0,r)--(r,r)--(0,a)--cycle; p2 = p1 reflectedabout((r,r),(0,a));
  p3 = (r,0)--(b,0)--(r,r)--cycle; p4 = p3 reflectedabout((r,r),(b,0));
  numeric i; i = -1;
  for t=1,3,3,2,2: fill p[incr i] withcolor Greens[7][t]; draw p[i]; endfor
  draw s withcolor 1/2[2/3 red,white]);
P5 = image(
  draw P4; draw P4 rotatedabout(point 3/2 of t, 180) shifted (b/a,1);
  label.lft("$a$", point 5/2 of t); label.bot("$b$", point 1/2 of t));
P6 = image(
  fill unitsquare xscaled (a-r) yscaled r shifted (     0,0) withcolor Greens[7][3];
  fill unitsquare xscaled (r)   yscaled r shifted (   a-r,0) withcolor Greens[7][1];
  fill unitsquare xscaled (b-r) yscaled r shifted (     a,0) withcolor Greens[7][2];
  fill unitsquare xscaled (r)   yscaled r shifted ( a+b-r,0) withcolor Greens[7][1];
  fill unitsquare xscaled (a-r) yscaled r shifted (   a+b,0) withcolor Greens[7][3];
  fill unitsquare xscaled (b-r) yscaled r shifted (2a+b-r,0) withcolor Greens[7][2];

  draw (0,0)--(a+b+c,0); draw (0,r)--(a+b+c,r);
  draw   (0,0)--   (0,r)--    (a-r,0)--    (a-r,r);
  draw   (a,0)--   (a,r)-- (a+b-r,0)-- (a+b-r,r);
  draw (a+b,0)--(a+b,r)--(2a+b-r,0)--(2a+b-r,r)--(a+b+c,0)--(a+b+c,r);

  interim ahangle := 20;
  arrow_label((0,r),(0,0),"$r$", 10);
  arrow_label((0,0),(a-1/2,0),"$a$", 10);
  arrow_label((a+1/2,0),(a+b-1/2,0),"$b$", 10);
  arrow_label((a+b+1/2,0),(a+b+c,0),"$c$", 10));
%... and so on to define the other pictures
label.lft(P1, 24 left); label.rt(P2, 24 right);
label.bot(P3, point 1/2 of bbox currentpicture shifted 36 down);
label.bot(P5, point 1/2 of bbox currentpicture shifted  9 down);
label.bot(P6, point 1/2 of bbox currentpicture shifted  9 down);
label.bot(P7, point 1/2 of bbox currentpicture shifted 36 down);
label.bot(P8, point 1/2 of bbox currentpicture shifted  9 down);
```

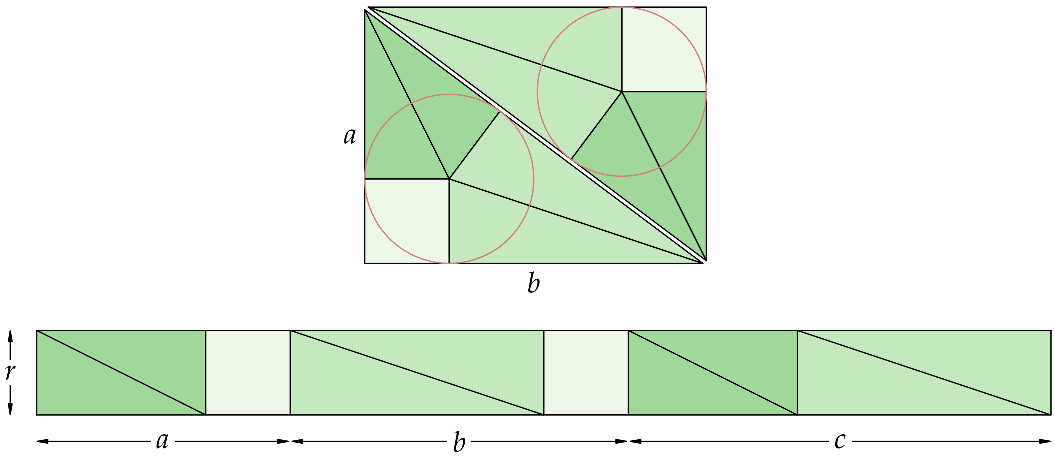
The inradius of a right triangle



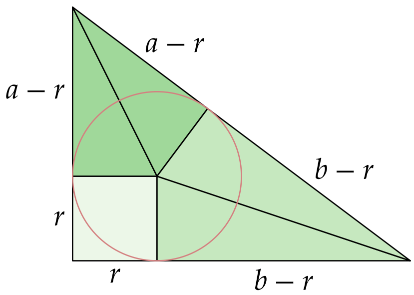
I. $\quad r = \dfrac{ab}{a+b+c}$

II. $\quad r = \dfrac{a+b-c}{2}$

I. $\quad ab = r(a+b+c)$



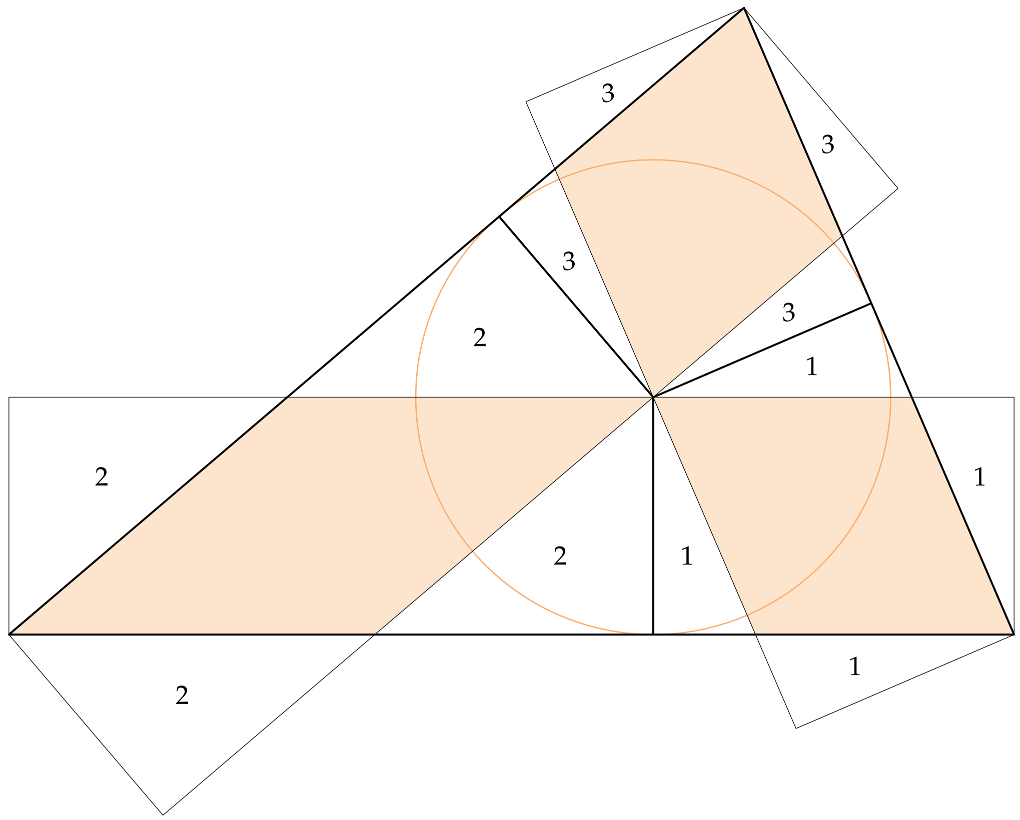
II. $\quad c = a + b - 2r$



— Liu Hui (3rd century A.D.)

```
numeric r, p[]; path c;
r = 89;
c = fullcircle scaled 2r;
p0 = 0.518; p1 = 6; p2 = 2.9;  % points round c

pair v[], t[];  % vertexes and tangents
for i = 0, 1, 2: j := (i+1) mod 3;
  t[i] = point p[i] of c;
  v[i] = whatever[precontrol p[i] of c, postcontrol p[i] of c]
       = whatever[precontrol p[j] of c, postcontrol p[j] of c];
endfor

for i = 0, 1, 2: j := (i+1) mod 3;
  pair a; a = whatever[v[i], t[i]] = whatever * (v[i]-t[j]);
  pair b; b = whatever[v[i], t[j]] = whatever * (v[i]-t[i]);
  label("\small " & decimal (i+1), 1/3(a - t[j] + 2v[i]));
  label("\small " & decimal (i+1), 1/3(b - t[i] + 2v[i]));
  label("\small " & decimal (i+1), 1/3(a + t[i]));
  label("\small " & decimal (i+1), 1/3(b + t[j]));
  fill origin -- a -- v[i] -- b -- cycle withcolor Oranges 8 2;
  draw subpath (p[i] if i=0: + 8 fi, p[j]) of c withcolor Oranges 8 4;
  draw origin -- v[i]-t[i] -- v[i] -- v[i]-t[j] -- cycle withpen pencircle scaled 1/4;
  draw origin -- t[i] -- v[i] -- t[j] withpen pencircle scaled 3/4;
endfor

label.bot(btex \small
  \textsc{Note}: \textit{Triangles marked with the same number are equal in area}.
  etex, point 1/2 of bbox currentpicture shifted 32 down);
```

The product of the perimeter of a triangle and its inradius is twice the area of the triangle I

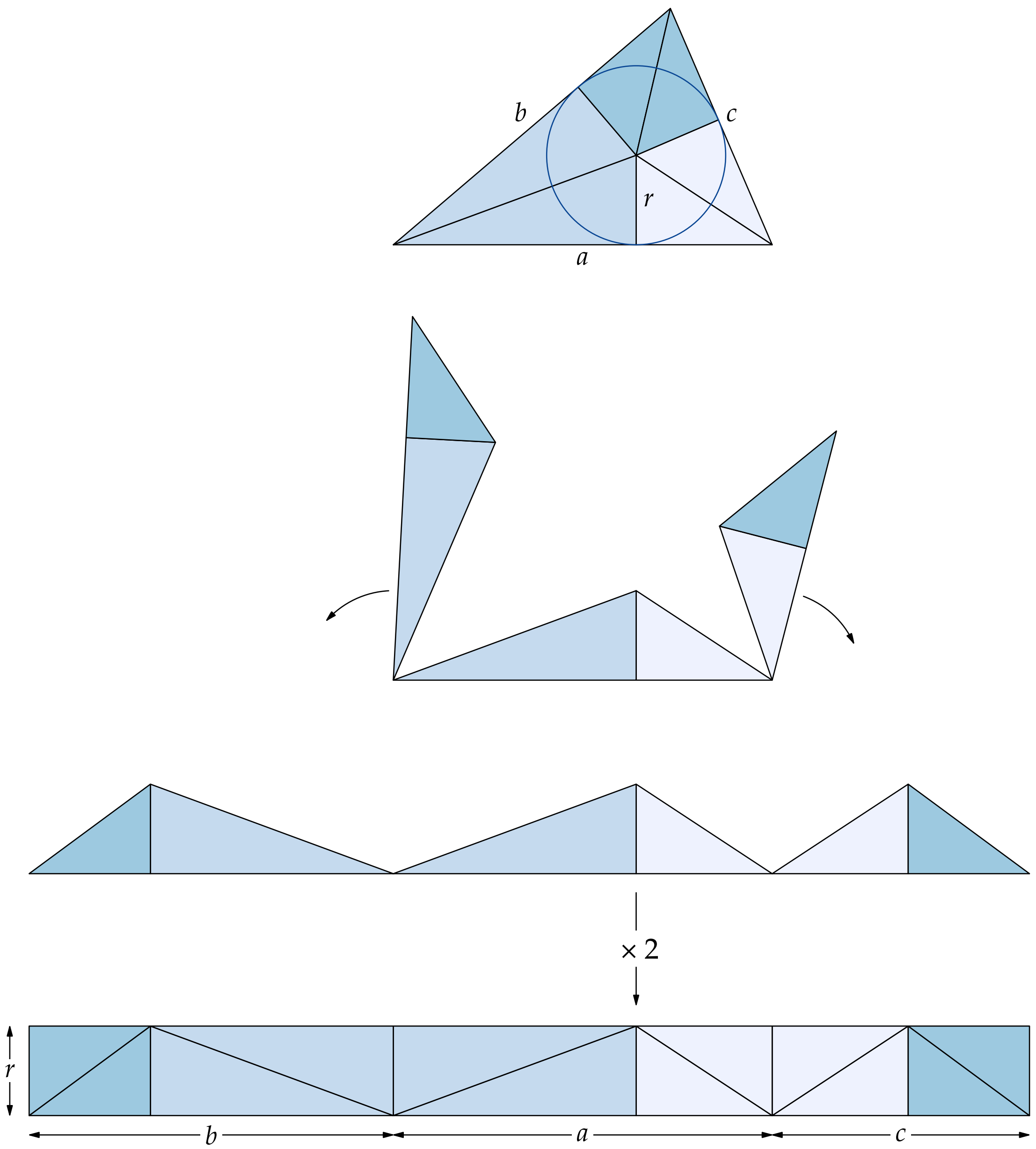


Note: *Triangles marked with the same number are equal in area.*

— Grace Lin

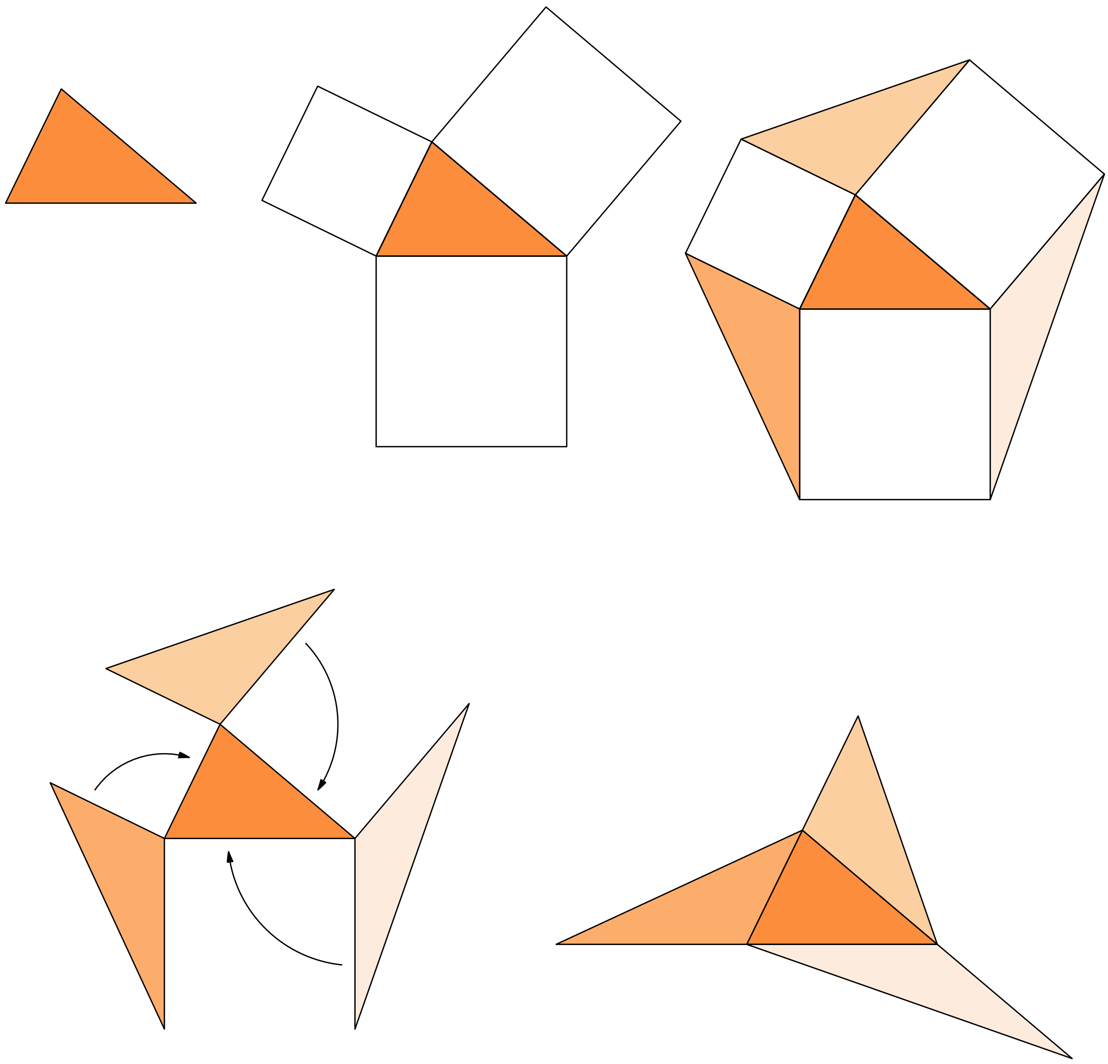
```
numeric r, a[]; path c; pair v[]; path t[];
r = 37; c = fullcircle scaled 2r; a0 = 0.518; a1 = 6; a2 = 2.9;
for i = 0, 1, 2: j := (i+1) mod 3;
v[i] = whatever[precontrol a[i] of c, postcontrol a[i] of c]
     = whatever[precontrol a[j] of c, postcontrol a[j] of c];
endfor
for i = 0, 1, 2: j := (i+1) mod 3;
  t[2i+1] = origin -- point a[i] of c -- v[i] -- cycle;
  t[2i+2] = origin -- point a[j] of c -- v[i] -- cycle;
endfor
picture P[];
P1 = image(
  for i=1 upto 6: fill t[i] withcolor Blues[7][ceiling (i/2)]; draw t[i]; endfor
  draw c withcolor Blues 7 7;
  label.bot("$a$", 1/2[v0, v1]); label.ulft("$b$", 1/2[v1, v2]);
  label.urt("$c$", 1/2[v2, v0]); label.rt("$r$", 1/2 point a1 of c);
);

numeric rotb, rotc;
3 rotb = 180 - angle(v2 - v1); 3 rotc = - angle(v2 - v0);
forsuffixes $=4,5: t$ := t$ rotatedabout(v1, rotb); endfor
forsuffixes $=6,1: t$ := t$ rotatedabout(v0, rotc); endfor
P2 = image(
  for i=1 upto 6: fill t[i] withcolor Blues[7][ceiling (i/2)]; draw t[i]; endfor
  drawarrow subpath (0,1) of fullcircle scaled 2r
      rotated (angle (point 1 of t5 - v1) + 6) shifted v1;
  drawarrow subpath (8,7) of fullcircle scaled 2r
      rotated (angle (point 1 of t6 - v0) - 6) shifted v0;
);

forsuffixes $=4,5: t$ := t$ rotatedabout(v1, 2 rotb); endfor
forsuffixes $=6,1: t$ := t$ rotatedabout(v0, 2 rotc); endfor
P3 = image(for i=1 upto 6: fill t[i] withcolor Blues[7][ceiling (i/2)]; draw t[i]; endfor);

P4 = image(drawarrow (up--down) scaled 5/8 r; unfill fullcircle scaled 15;
   label("\large ${}\times2$", origin));

input arrow_label
forsuffixes $=1,2,3,4,5,6: t$ := t$ rotatedabout(point -1/2 of t$, 180); endfor
P5 = image(
  draw P3; for i=1 upto 6: fill t[i] withcolor Blues[7][ceiling (i/2)]; draw t[i]; endfor
  arrow_label(point 1 of t5, point 0 of t5, "$r$", 8);
  arrow_label(point 0 of t5, point 0 of t4, "$b$", 8);
  arrow_label(point 0 of t3, point 0 of t2, "$a$", 8);
  arrow_label(point 0 of t1, point 0 of t6, "$c$", 8);
);
draw P1;
draw P2 shifted 180 down;
draw P3 shifted 260 down;
draw P4 shifted 328 down;
draw P5 shifted 360 down;
```

The product of the perimeter of a triangle and its inradius is twice the area of the triangle II



```
path t[], s[];
t0 = origin -- 72 right -- 48 right rotated 64 -- cycle;
for i=1 upto 3:
  s[i] = unitsquare zscaled (point i-1 of t0 - point i of t0) shifted point i of t0;
endfor
for i=1 upto 3:
  t[i] = point i of t0 -- point -1 of s[i] -- point 2 of s[i mod 3 + 1] -- cycle;
endfor
def fd(expr p, shade) = fill p withcolor shade; draw p; enddef;
def do_arrow(expr n) =
  drawarrow point 2/3 of t[n] .. point (3n-2)/3 of t0 {direction (2n-1)/2 of t0 rotated 90}
    cutbefore fullcircle scaled 10 shifted point 2/3 of t[n]
    cutafter  fullcircle scaled 10 shifted point (3n-2)/3 of t0; enddef;

picture P[];
P0 = image(fd(t0, Oranges 7 4));
P1 = image(draw P0; for i=1 upto 3: draw s[i]; endfor);
P2 = image(draw P1; for i=1 upto 3: fd(t[i], Oranges[7][i]); endfor);
P3 = image(draw P0; for i=1 upto 3: fd(t[i], Oranges[7][i]); do_arrow(i); endfor);
P4 = image(
  draw P0;
  for i=1 upto 3:
    fd(t[i] rotatedabout(point i of t0, -90), Oranges[7][i]);
  endfor
);

draw P0;
draw P1 shifted (140,-20);
draw P2 shifted (300,-40);

draw P3 shifted ( 60,-240);
draw P4 shifted (280,-280);
```

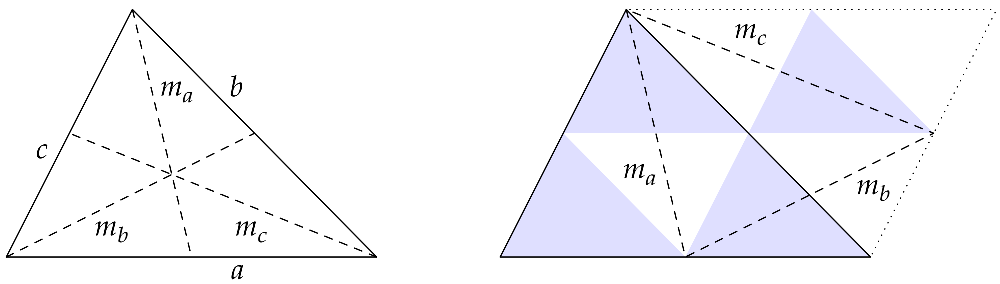
Four triangles with equal area



— Steven L. Snover

```
numeric a,b;
a = 3/4 b; b = 1/3 \mpdim\textwidth;
z0 = origin; z1 = (b,0); z2 = (0,a) rotated -27;
z3 = 1/2[z1,z2];
z4 = 1/2[z2,z0];
z5 = 1/2[z0,z1];
path t[], s[], m[];
t0 = z0--z1--z2--cycle;
t1 = z0--z4--z5--cycle;
s1 = z0--z1;
s2 = z1--z2;
s3 = z0--z2;
m1 = z0--z3;
m2 = z1--z4;
m3 = z2--z5;

picture P[];
P1 = image(
    for i=1 upto 3: draw m[i] dashed evenly; endfor
    draw t0;
    label.bot("$a$", point 5/8 of t0);
    label.urt("$b$", point 13/8 of t0);
    label.ulft("$c$", point 21/8 of t0);

    label.lrt("$m_b$", point 1/3 of m1);
    label.llft("$m_c$", point 1/3 of m2);
    label.rt("$m_a$", point 1/3 of m3);
);
P2 = image(
    forsuffixes $=0,3,4,5:
        fill t1 shifted z$ withcolor 7/8[blue,background];
    endfor
    draw s1 shifted z2 dashed withdots scaled 1/2;
    draw s3 shifted z1 dashed withdots scaled 1/2;
    draw m3 & m1 shifted z5 -- cycle dashed evenly;
    draw t0;
    label.lft("$m_a$", point 2/3 of m3);
    label.lrt("$m_b$", point 2/3 of m1 shifted z5);
    label.urt("$m_c$", 2/3[point 1 of m1 shifted z5, z2]);

);

draw P1 shifted (-2/3b,0);
draw P2 shifted (2/3b,0);
```

The triangle of medians has 3/4 the area of the original triangle



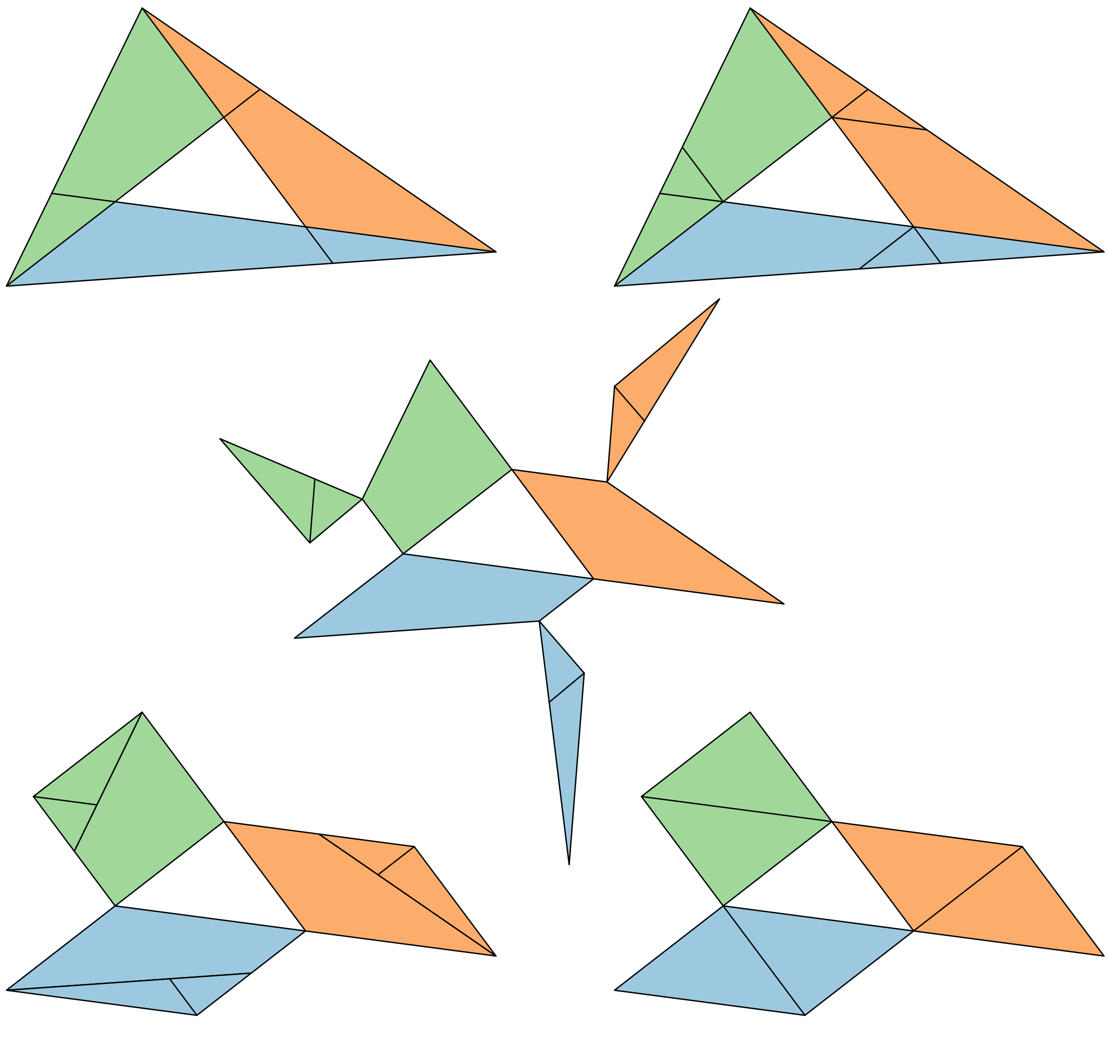
$$\frac{3}{4}\,\text{area}(\triangle abc) = \text{area}(\triangle m_a m_b m_c)$$

— Norbert Hungerbühler

```
color s[];  s1 = Greens 7 3; s2 = Blues 7 3; s3 = Oranges 7 3;
z1 = origin;
z2 = 184 right rotated 4;
z3 = 116 right rotated 64;
z12 = 2/3[z1, z2];
z23 = 2/3[z2, z3];
z31 = 2/3[z3, z1];
z4 = whatever [z1, z23] = whatever [z2, z31];
z5 = whatever [z2, z31] = whatever [z3, z12];
z6 = whatever [z3, z12] = whatever [z1, z23];
z7 = whatever [z3, z1]; z7-z4 = whatever * (z3-z6);
z8 = whatever [z1, z2]; z8-z5 = whatever * (z1-z4);
z9 = whatever [z2, z3]; z9-z6 = whatever * (z2-z5);
picture P[];
P1 = image(
  fill z1--z6--z3--cycle withcolor s1;
  fill z2--z4--z1--cycle withcolor s2;
  fill z3--z5--z2--cycle withcolor s3;
  draw z1--z2--z3--cycle;
  draw z1--z23; draw z2--z31; draw z3--z12;
);
P2 = image(
  draw P1;
  draw z4--z7; draw z5--z8; draw z6--z9;
);
P31 = image(fill z1--z4--z7--cycle withcolor s1; draw z31--z4--z7--z1--z4;);
P32 = image(fill z2--z5--z8--cycle withcolor s2; draw z12--z5--z8--z2--z5;);
P33 = image(fill z3--z6--z9--cycle withcolor s3; draw z23--z6--z9--z3--z6;);
for p = (3, -87), (4, -180): numeric n, r; (n, r) = p;
  P[n] = image(
    fill z6--z3--z7--z4--cycle withcolor s1; draw z6--z3--z7--z4--cycle;
    fill z4--z1--z8--z5--cycle withcolor s2; draw z4--z1--z8--z5--cycle;
    fill z5--z2--z9--z6--cycle withcolor s3; draw z5--z2--z9--z6--cycle;
    draw P31 rotatedabout(z7, r);
    draw P32 rotatedabout(z8, r);
    draw P33 rotatedabout(z9, r);
  );
endfor
z101 = z4 rotatedabout(z7, 180);
z102 = z5 rotatedabout(z8, 180);
z103 = z6 rotatedabout(z9, 180);
P5 = image(
  fill z4--z6--z3--z101--cycle withcolor s1; draw z101--z6--z4--z101--z3--z6;
  fill z5--z4--z1--z102--cycle withcolor s2; draw z102--z4--z5--z102--z1--z4;
  fill z6--z5--z2--z103--cycle withcolor s3; draw z103--z5--z6--z103--z2--z5;
);
draw P1 shifted 120(-.9,+1.1); draw P2 shifted 120(+1,+1.1);
draw P3;
draw P4 shifted 120(-.9,-1.1); draw P5 shifted 120(+1,-1.1);
```

Heptasection of a triangle

If the one-third points on each side of a triangle are joined to opposite vertices, the resulting central triangle is equal in area to one-seventh that of the initial triangle.



— William Johnston and Joe Kennedy

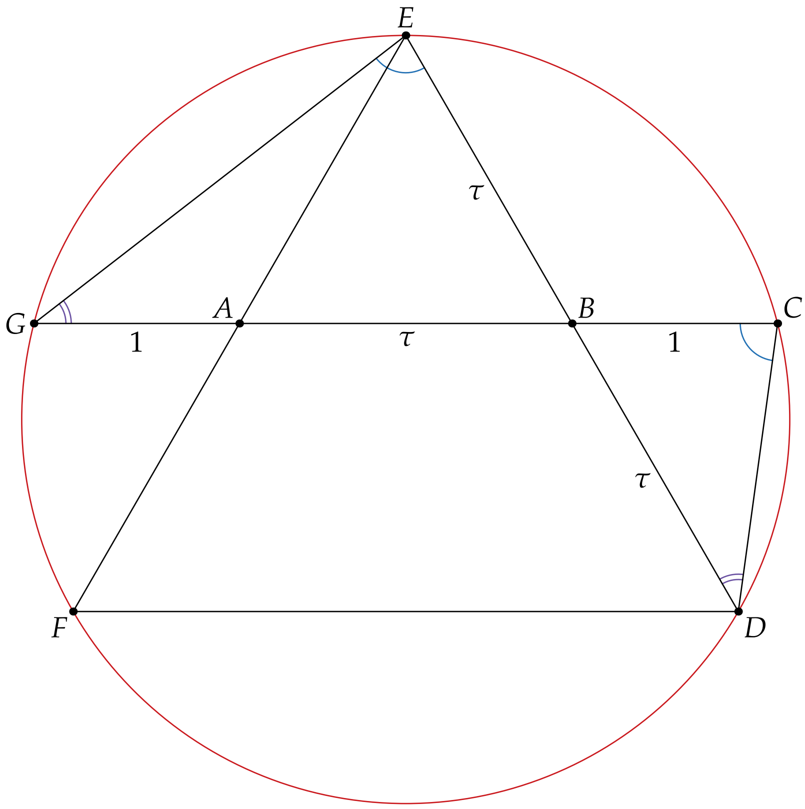
```
pair A, B, C, D, E, F, G;
E = 144 up;
F = E rotated 120;
D = F rotated 120;
A = 1/2[E, F];
B = 1/2[D, E];
path circ; circ = fullcircle scaled 2 abs E;
(whatever, t) = (A -- 2[A,B]) intersectiontimes circ;
C = point t of circ; G = point 4-t of circ;

vardef angle_mark@#(expr a, o, b) =
  fullcircle scaled @# rotated angle (a-o) shifted o cutafter (o--b)
enddef;

draw angle_mark 28 (G, C, D) withcolor Blues 8 7;
draw angle_mark 28 (G, E, D) withcolor Blues 8 7;
draw angle_mark 28 (C, D, E) withcolor Purples 8 7;
draw angle_mark 24 (C, D, E) withcolor Purples 8 7;
draw angle_mark 28 (C, G, E) withcolor Purples 8 7;
draw angle_mark 24 (C, G, E) withcolor Purples 8 7;

draw circ withcolor Reds 8 7;
draw D--E--F--cycle;  draw E--G--C--D;

dotlabel.ulft("$A$", A);
dotlabel.urt ("$B$", B);
dotlabel.urt ("$C$", C);
dotlabel.lrt ("$D$", D);
dotlabel.top ("$E$", E);
dotlabel.llft("$F$", F);
dotlabel.lft ("$G$", G);

label.llft("$\tau$", 1/2[E,B]);
label.llft("$\tau$", 1/2[D,B]);
label.bot ("$\tau$", 1/2[A,B]);
label.bot ("$1$", 1/2[A,G]);
label.bot ("$1$", 1/2[B,C]);
```

A Golden Section problem from the *Monthly*

(Problem E3007, *American Mathematical Monthly*, 1983, p.482)

Let A and B be the midpoints of the sides EF and ED of an equilateral triangle DEF. Extend AB to meet the circumcircle (of DEF) at C. Show that B divides AC according to the golden section.

SOLUTION:



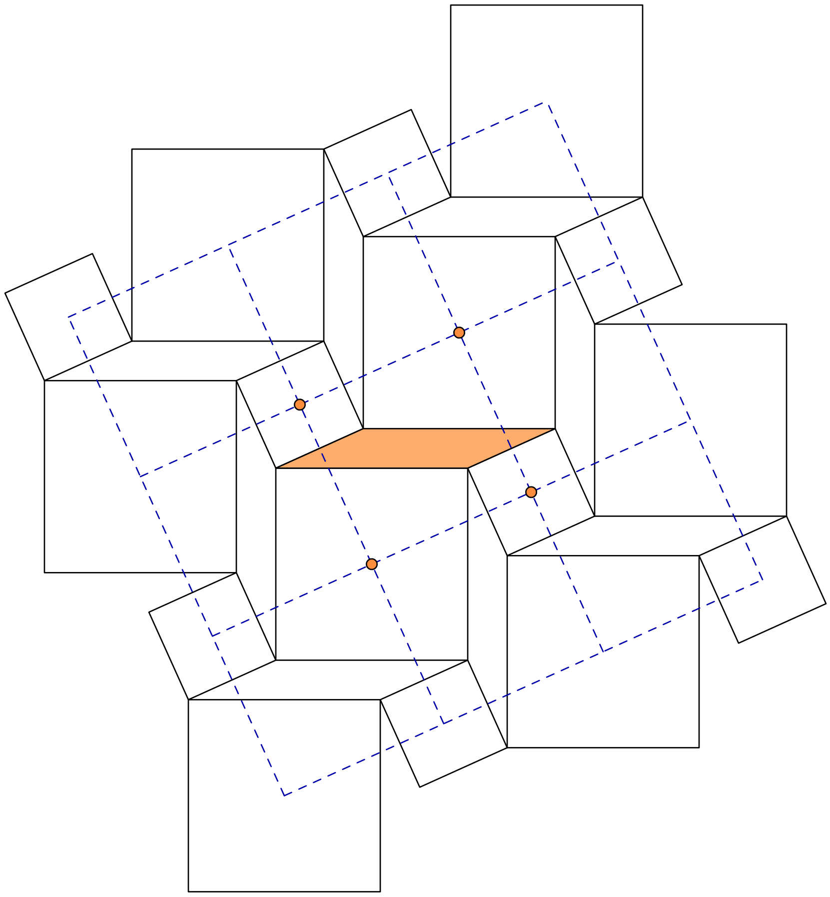
$$\tau^2 = \tau + 1$$

— Jan van de Craats

```
path s[]; s0 = unitsquare shifted (-1/2, -1/2);
numeric a, n; a = 36; n=0;
for i=1 upto 4:
    for j=1 upto 4:
        s[incr n] = s0 scaled if not odd (i+j): 2a rotated -24.29519 else: a fi
            shifted (1.822875a * (i,j));
        draw s[n];
    endfor
endfor

fill subpath (2,3) of s[6] -- subpath (0,1) of s[11] -- cycle withcolor Oranges 7 3;
draw subpath (2,3) of s[6] -- subpath (0,1) of s[11] -- cycle;

for i=0 upto 3:
    draw center s[4i+1] -- center s[4i+4] dashed evenly withcolor 2/3 blue;
    draw center s[i+1] -- center s[i+13] dashed evenly withcolor 2/3 blue;
endfor
forsuffixes @=6,7,10,11:
    fill fullcircle scaled 4 shifted center s@ withcolor Oranges 7 4;
    draw fullcircle scaled 4 shifted center s@;
endfor
currentpicture := currentpicture rotated 24.29519;
```

Tiling with squares and parallelograms

If squares are constructed eternally on the sides of the parallelogram, their centres form a square.



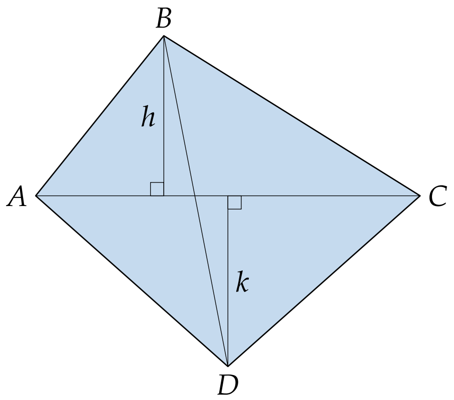
— Alfinio Flores

```
%-- Convex picture ----------------------------------
z0 = origin; z1 = (48, 60); z2 = (144, 0); z3 = (72, -64);

fill z0--z1--z2--z3--cycle withcolor Blues 6 2;
drawoptions(withpen pencircle scaled 1/4);
draw unitsquare scaled 5 rotated  90 shifted (x1,0);
draw unitsquare scaled 5 rotated -90 shifted (x3,0);
draw z0--z2;
draw (x1,0) -- z1 -- z3 -- (x3,0);
drawoptions();
draw z0--z1--z2--z3--cycle;

label.lft("$A$", z0);
label.top("$B$", z1);
label.rt ("$C$", z2);
label.bot("$D$", z3);
label.lft("$h$", (x1, 1/2 y1));
label.rt ("$k$", (x3, 1/2 y3));

label(btex \vbox{\openup3pt\halign{\hss#&${}#$\hfil\cr
  Area&  ={1\over2}\overline{AC}\cdot(h+k)\cr
      &\le{1\over2}\overline{AC}\cdot\overline{BD}\cr
  }} etex, z2 shifted 100 right);

%-- Concave picture ---------------------------------
z0 = origin; z1 = (48, 120); z2 = (144, 0); z3 = (76, 42);

fill z0--z1--z2--z3--cycle withcolor Blues 6 2;
drawoptions(withpen pencircle scaled 1/4);
draw unitsquare scaled 5 shifted (x1,0);
draw unitsquare scaled 5 shifted (x3,0);
draw z0--z2;
draw (x1,0) -- z1 -- z3 -- (x3,0);
draw z3--(x1,y3) dashed withdots scaled 1/4;
drawoptions();
draw z0--z1--z2--z3--cycle;

label.lft("$A$", z0);
label.top("$B$", z1);
label.rt ("$C$", z2);
label.urt("$D$", z3);
label.lft("$h$", (x1, 1/2 y1));
label.lft("$k$", (x3, 1/2 y3));

label(btex \vbox{\openup3pt\halign{\hss#&${}#$\hfil\cr
  Area&=  {1\over2}\overline{AC}\cdot(h-k)\cr
      &\le{1\over2}\overline{AC}\cdot\overline{BD}\cr
  }} etex, z2 shifted (100, 42));
```

The area of a quadrilateral I

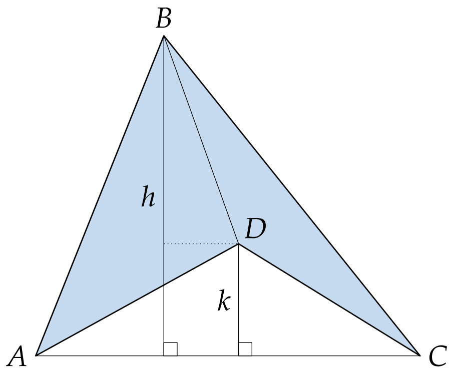
The area of a quadrilateral is less than or equal to half the product of the lengths of its diagonals, with equality if and only if the diagonals are perpendicular.

I. Convex quadrilaterals



$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\overline{AC}\cdot(h+k)\\ &\le \tfrac{1}{2}\overline{AC}\cdot\overline{BD}\end{aligned}$$

II. Concave quadrilaterals



$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\overline{AC}\cdot(h-k)\\ &\le \tfrac{1}{2}\overline{AC}\cdot\overline{BD}\end{aligned}$$

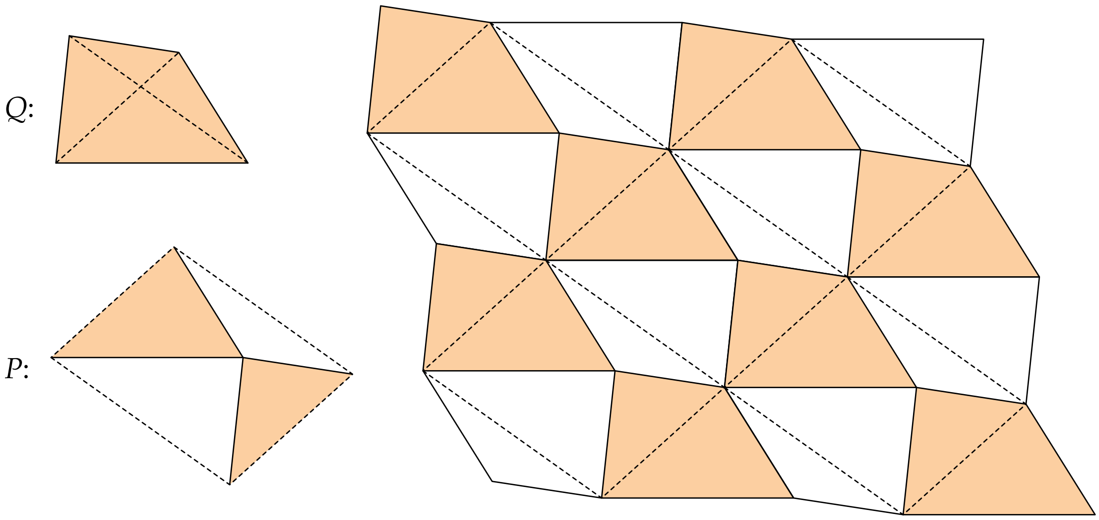
— David B. Sher, Ronald Skurnick, and Dean C. Nataro

```
%-- Convex picture ----------------------------------
path Q; Q = origin -- 72 dir 0 -- 62 dir 42 -- 48 dir 84 -- cycle;
pair s, t; s = point 1 of Q - point 3 of Q; t = point 2 of Q - point 0 of Q;
path box; box = origin -- s -- s+t -- t -- cycle;
picture qq, pp, grid;
qq = image(
  fill Q withcolor Oranges 7 2; draw Q;
  draw point 0 of Q -- point 2 of Q dashed evenly scaled 1/2;
  draw point 1 of Q -- point 3 of Q dashed evenly scaled 1/2;
  label.lft("$Q$:", (-5, 20));
);
pp = image(
  path A, B; A = Q; B = A shifted s;
  fill A withcolor Oranges 7 2; draw A;
  fill B withcolor Oranges 7 2; draw B;
  clip currentpicture to box;
  draw box dashed evenly scaled 1/2;
  label.lft("$P$:", (-5, -4));
);
grid = image(
  for i = 0 upto 1:
    for j = 0 upto 1:
      pair o; o = i * (s-t) + j * (s+t);
      path A, B; A = Q shifted o; B = A shifted s;
      fill A withcolor Oranges 7 2; draw A;
      fill B withcolor Oranges 7 2; draw B;
      draw point 0 of A -- point 1 of A shifted -t -- point 0 of B;
      draw point 2 of A -- point 1 of A shifted +t -- point 2 of B;
      draw box shifted point 0 of A dashed evenly scaled 1/2;
    endfor
  endfor
);
label.rt(qq, origin); label.rt(pp, 100 down);
label.rt(grid, point 3/2 of bbox currentpicture);
% Concave picture is essentially the same, with a simpler grid
```

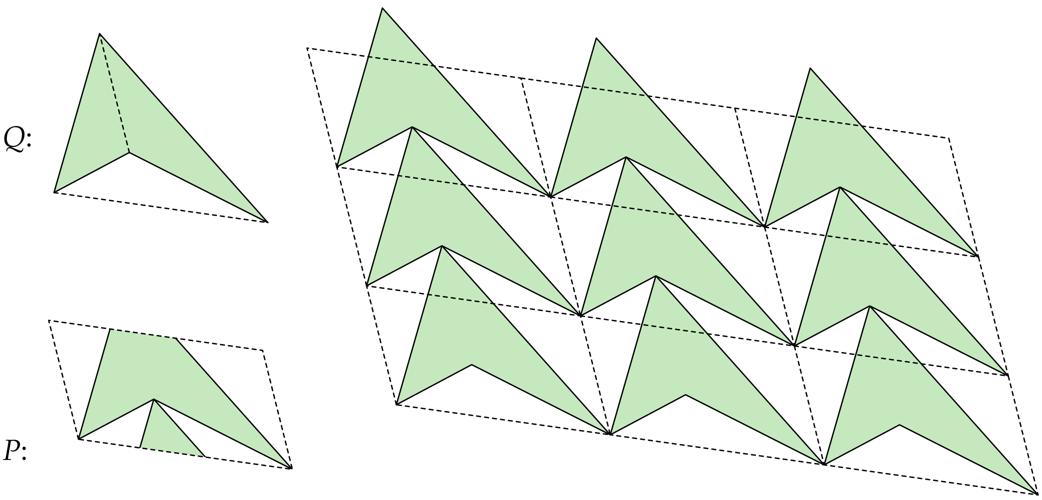
The area of a quadrilateral II

The area of a quadrilateral Q is equal to one-half the area of a parallelogram P whose sides are parallel to and equal in length to the diagonals of Q.

I. Q convex



II. Q concave

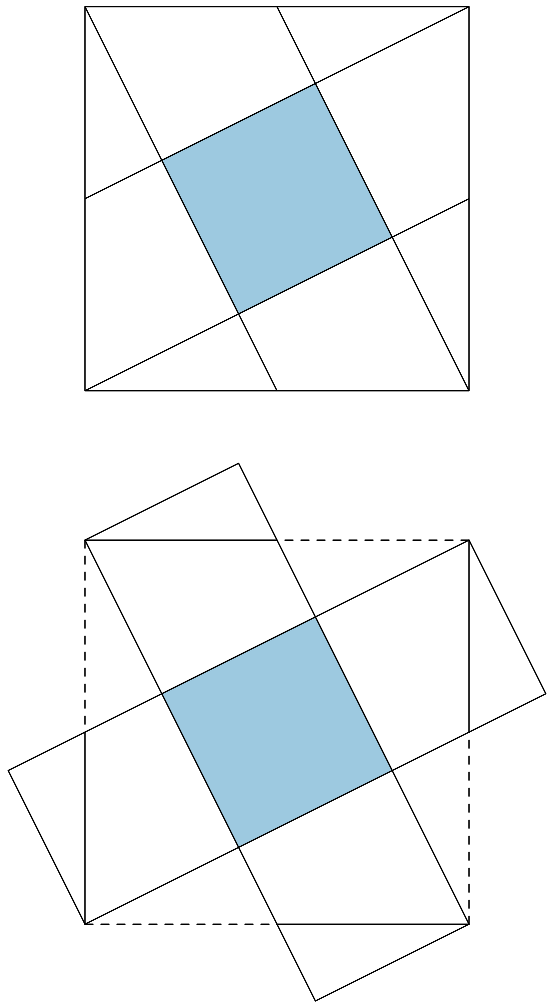


$$\text{area}(Q) = \frac{1}{2}\,\text{area}(P)$$

```
path S, r, s, t;
S = unitsquare scaled 144;
s = for i=0 upto 3: 2/5[point i of S, point i+3/2 of S] -- endfor cycle;
numeric h, w;
h = abs(point 1 of s-point 0 of s);
w = abs(point 2 of S-point 0 of S) +-+ h;
r = unitsquare xscaled w yscaled h rotated angle (point 1 of s - point 0 of s);
t = r rotatedabout(center s, 90);

picture aa, bb;
aa = image(
    fill s withcolor Blues 7 3;
    draw r; draw t;
    clip currentpicture to S;
    draw S;
);
bb = image(
    fill s withcolor Blues 7 3;
    draw r; draw t;
    for i=0 upto 3:
        draw subpath(i, i+1/2) of S dashed evenly;
        draw subpath(i+1/2, i+1) of S;
    endfor
);
draw aa;
draw bb shifted 200 down;
```

A square within a square

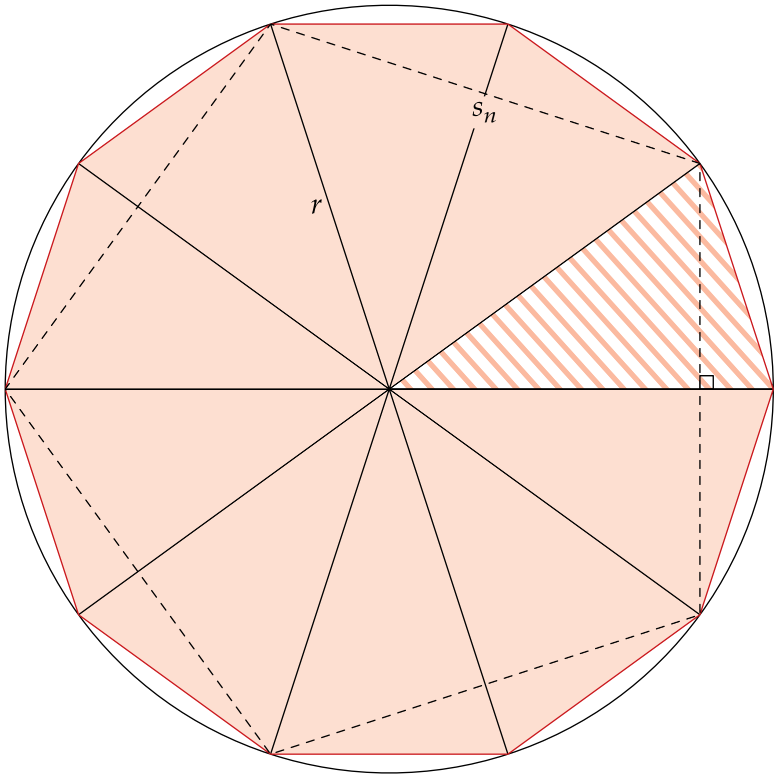
If lines from the vertices of a square are drawn to the mid-points of adjacent sides (as shown in the figure), then the area of the smaller square so produced is one-fifth that of the given square.



```
numeric n; n = 5;
path gon, gonn, circle;
circle = for t=0 upto 2n-1: 144 right rotated (180/n*t) .. endfor cycle;
gonn = for t=0 upto 2n-1: point t of circle -- endfor cycle;
gon = for t=0 upto n-1: point 1+2t of circle -- endfor cycle;

 for i=1 upto 100:
     draw (down--200 up) rotated 42 shifted (8i,0)
         withpen pencircle scaled 2
         withcolor Reds 8 3;
 endfor
 clip currentpicture to origin -- subpath(0,1) of gonn -- cycle;
 fill origin -- subpath(1,2n) of gonn -- cycle withcolor Reds 8 2;

 draw unitsquare scaled 5 shifted (xpart point 1 of gonn, 0);
 for i=0 upto 2n:
     draw origin -- point i of gonn;
 endfor
 draw circle; draw gonn withcolor Reds 8 7;

 label.lft("$r$", 1/2 point 3 of gonn);
 picture P;
 P = thelabel.bot("$s_n$", 1/2 [point 1 of gon, point 0 of gon]);
 fill bbox P withcolor Reds 8 2; draw P;
 draw gon dashed evenly;
```

Areas and perimeters of regular polygons

The area of a regular $2n$-gon inscribed in a circle is equal to one-half the radius of the circle times the perimeter of a regular n-gon similarly inscribed ($n \geq 3$).



$$\begin{aligned} \frac{1}{2n}\,\text{area}\,(P_{2n}) &= \frac{1}{2}\cdot r \cdot \frac{1}{2}s_n \\ \text{area}\,(P_{2n}) &= \frac{1}{2}r \cdot ns_n \\ &= \frac{1}{2}r \cdot \text{perimeter}\,(P_n) \end{aligned}$$

Corollary [Bhāskara, *Litāvati* (India, 12th century AD)]: The area of a circle is equal to one-half the product of its radius and circumference.

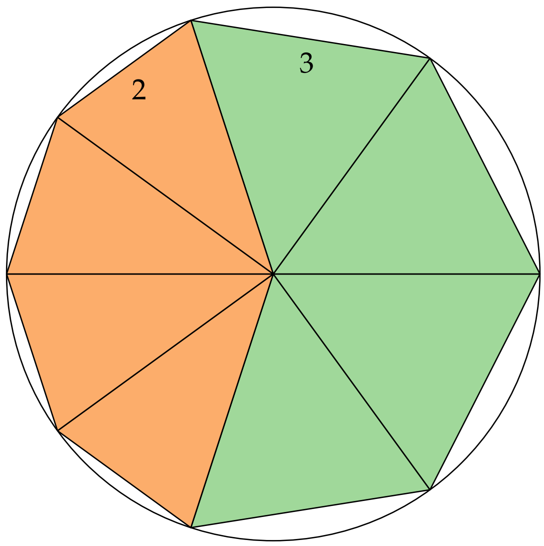
```
path tt, ttt; numeric s; s = 100;
ttt = origin -- right scaled s -- right scaled s rotated 54 -- cycle;
tt = origin -- right scaled s -- right scaled s rotated 36 -- cycle;

picture TT, TTT;
TT = image(fill tt withcolor Oranges 7 3; draw tt withpen pencircle scaled 1/2);
TTT = image(fill ttt withcolor Greens 7 3; draw ttt withpen pencircle scaled 1/2);

picture Q, A;
Q = image(
    for t=-2 upto 1:
        numeric a, b;
        a = 54 t;
        b = 144-36t;
        draw TTT rotated a;
        draw TT  rotated b;
        if t=1:
            label("3", point 3/2 of ttt scaled 0.9 rotated a);
            label("2", point 3/2 of tt scaled 0.9 rotated b);
        fi
    endfor
    draw fullcircle scaled 2s;

);

A = image(
    for t=0 upto 3:
        draw TTT rotated (90t-27);
        draw TT  rotated (90t+27);
    endfor
    path S; S = unitsquare shifted -(1/2, 1/2)
        scaled (2 abs (point 1 of ttt) +-+ abs(point 2 of ttt - point 1 of ttt));
    draw S;
    label.bot("$\sqrt2$", point 1/9 of S);
    label.bot("$\sqrt2$", point 8/9 of S);
    label.bot("$\phantom{\sqrt2}3$", point 1/2 of S);

    label.lft("$\sqrt2$", point 28/9 of S);
    label.lft("$\sqrt2$", point 35/9 of S);
    label.lft("$3$", point 7/2 of S);

);
draw Q;
draw A shifted (210, -180);
```

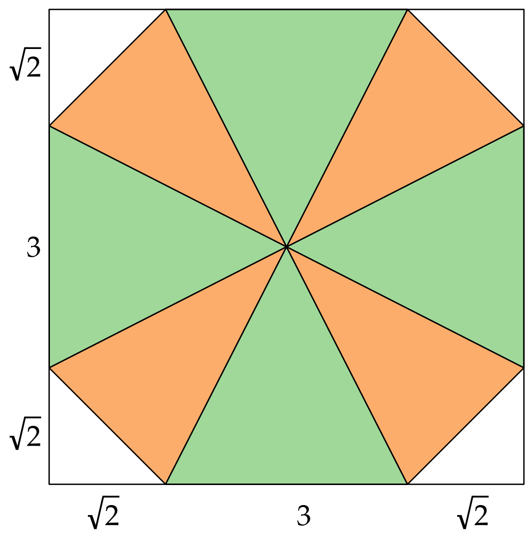
The area of a Putnam octagon

(Problem B1, 39th Annual William Lowell Putnam Mathematical Competition, 1978).

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$, with r, s, and t positive integers.

SOLUTION:





$$A = \left(3 + 2\sqrt{2}\right)^2 - 4 \cdot \frac{1}{2}\left(\sqrt{2}\right)^2 = 9 + 6\sqrt{2} + 6\sqrt{2} + 8 - 4 = 13 + 12\sqrt{2}$$

```
path S, H, D;  numeric a; a = 48;
S = unitsquare shifted -(1/2, 1/2) scaled a
               shifted (a/2*(sqrt(3)+1, 0)) rotated 68;
H = for t=0 upto 5: point 0 of S rotated -60t -- endfor cycle;
D = for t=0 upto 5: subpath (2,1) of S rotated -60t -- endfor cycle;

picture aa, bb;
aa = image(
    fill D withcolor Oranges 7 3;
    fill H withcolor Blues 7 3;
    for t=0 upto 5:
        fill S rotated 60t withcolor Greens 7 3;
        draw S rotated 60t;
    endfor
    draw D;
    label("(i)", (xpart point 9 of D, ypart point 12 of D));
);
bb = image(
    fill S withcolor Greens 7 3; draw S;
    for t=1 upto 5:
        draw S rotated 60t dashed evenly;
    endfor
    draw point 1 of D -- point  9 of D withcolor 2/3 red;
    draw point 2 of D -- point 11 of D withcolor 2/3 red;
    draw point 4 of D -- point 12 of D withcolor 2/3 red;

    draw D;
    r = abs point 0 of D;
    for t=1 upto length D:
        label("$P_{" & decimal t & "}$", point t of D scaled (1+10/r));
    endfor

    label("(ii)", (xpart point 9 of D, ypart point 12 of D));

);
draw aa shifted 120 left;
draw bb shifted (90, -120);
```

A Putnam dodecagon

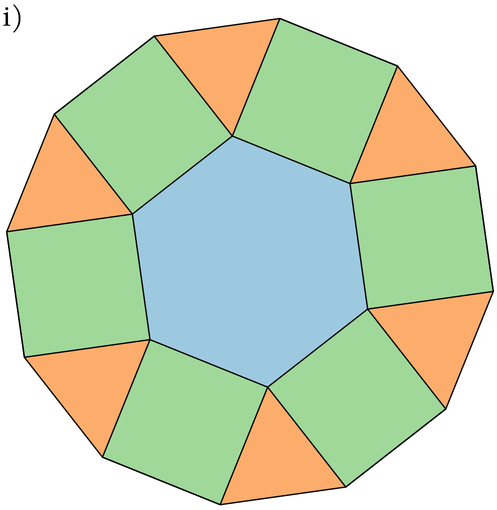
(Problem I-1, 24th Annual William Lowell Putnam Mathematical Competition, 1963)

(i) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.

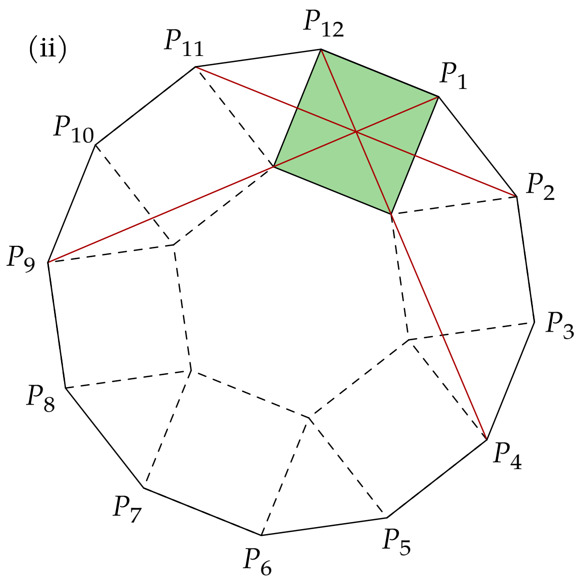
(ii) Let $P_1, P_2, \dots, P_{12}$ be the successive vertices of a regular dodecagon. Discuss the intersection(s) of the three diagonals P_1P_9, P_2P_{11}, and P_4P_{12}.

Solution:

(i)



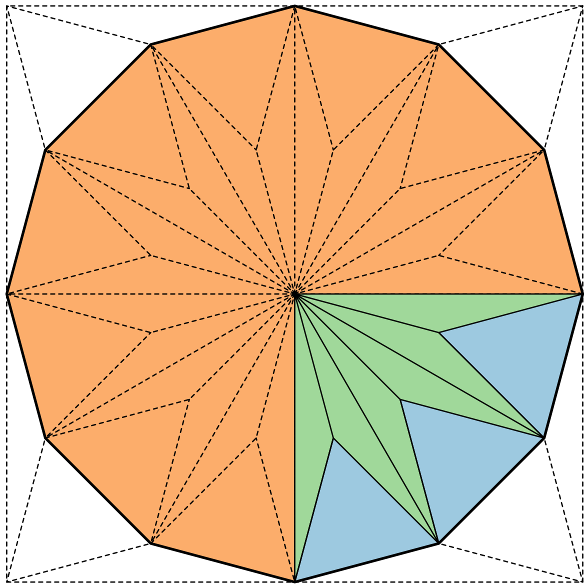
(ii)

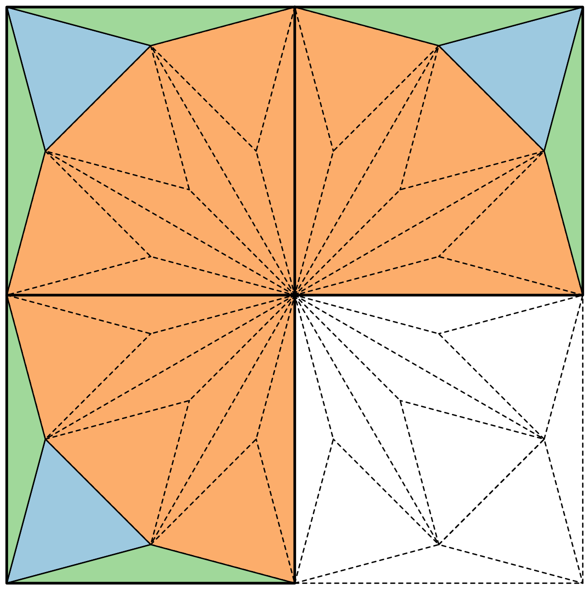


```
numeric r, s; r = 108; s = 2r * sind(15);

path D, S, T, T';
T = (r,0) -- (r,0) rotated 30 -- (s,0) rotated 15 -- cycle;
S = origin -- point 0 of T -- point 2 of T -- cycle;
D = for i=0 upto 11: point 0 of T rotated 30i -- endfor cycle;
T' = T reflectedabout(point 0 of T, point 1 of T);

picture base, first, second;
base = image(
  drawoptions(dashed evenly scaled 1/2);
  numeric n; n = 0;
  for i=0 upto 11:
    draw subpath (0,2) of T rotated 30i;
    if i mod 3 = 1:
        draw T' rotated 30i;  z[incr n] = point 2 of T' rotated 30i;
    fi
    draw S rotated 30i;
  endfor
  draw z1--z2--z3--z4--cycle;
  drawoptions();
);

first = image(
  fill origin -- subpath (0, 9) of D -- cycle withcolor Oranges 7 3;
  draw base;
  fill origin -- subpath (9, 12) of D -- cycle withcolor Greens 7 3;
  for i=9 upto 11:
    fill T rotated 30i withcolor Blues 7 3; draw T rotated 30i;
    draw S rotated 30i;
  endfor
  draw origin -- point 0 of T;
  draw D withpen pencircle scaled 1;
);
second = image(
  fill origin -- subpath (0, 9) of D -- cycle withcolor Oranges 7 3;
  draw base;
  for i=0 upto 2:
    fill T' rotated (30+90i) withcolor Blues 7 3;
    draw T' rotated (30+90i);
    fill S  rotated 90 shifted point 0 of T rotated 90i withcolor Greens 7 3;
    draw S  rotated 90 shifted point 0 of T rotated 90i;
    fill S  rotated 180 shifted z1 rotated 90i withcolor Greens 7 3;
    draw S  rotated 180 shifted z1 rotated 90i;
  endfor
  draw (x2,0) -- (x1,0) -- z1 -- z2 -- z3 -- (0, y3) -- (0,y1) withpen pencircle scaled 1;
);

draw first; draw second shifted ((2r+36) * down);
```

The area of a regular dodecagon

A regular dodecagon with circumradius one has area three.





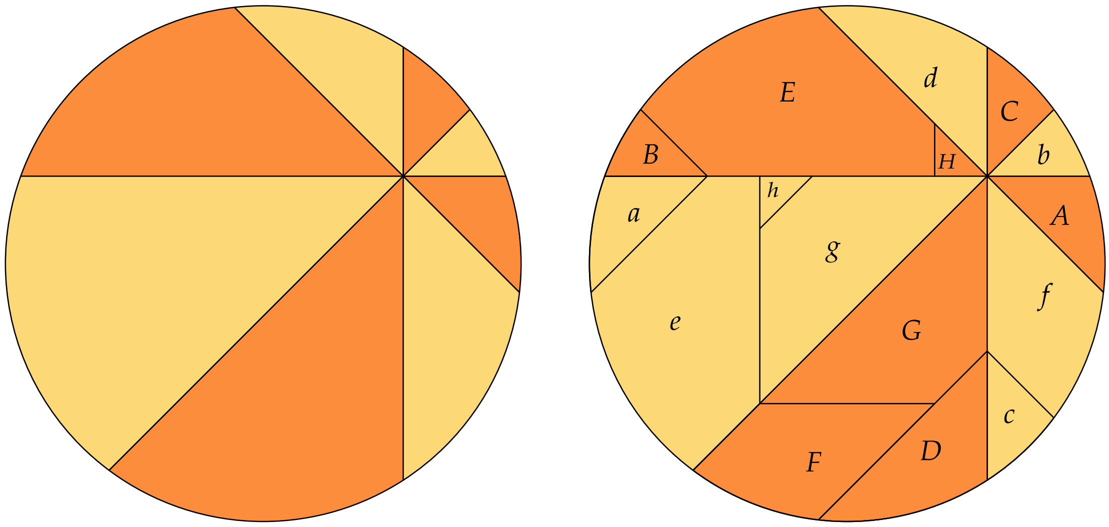
— J. Kürshák

```
path pizza; pizza = fullcircle scaled 200;
pair pp; pp = 64 right rotated 32;
numeric p[]; for i=0 upto 7:
  (p[i], whatever) = pizza intersectiontimes (origin -- 200 dir 45(i-1)) shifted pp;
endfor
path slice[]; for i=0 upto 7:
    numeric a, b; a = p[i]; b = p[(i+1) mod 8]; if b < a: b := b + 8; fi
    slice[i] = pp -- subpath (a, b) of pizza -- cycle;
    z[i] = 1/3(pp + point a of pizza + point b of pizza);
endfor

picture aa, bb;
aa = image(for i=0 upto 7:
  fill slice[i] withcolor YlOrRd 7 if odd i: 2 else: 4 fi; draw slice[i];
endfor);
bb = image(draw aa;
    label("$A$", z0); label("$a$", z0 reflectedabout(up, down));
    label("$b$", z1); label("$B$", z1 reflectedabout(up, down));
    label("$C$", z2); label("$c$", z2 reflectedabout(left, right));
    label("$d$", z3); label("$D$", z3 reflectedabout(left, right));
    draw slice[0] reflectedabout(up, down);
    draw slice[1] reflectedabout(up, down);
    draw slice[2] reflectedabout(left, right);
    draw slice[3] reflectedabout(left, right);

    for i=0 upto 3:
      z[10+i] = pp reflectedabout(dir 45i, -dir 45i);
    endfor

    path E, F, G;
    E = subpath(p[4], 4-p[2]) of pizza -- z12 -- (x11, y12) -- z11 -- cycle;
    draw subpath (-2, -1) of E; draw subpath (-3, -2) of E rotated 90;
    label("$E$", center E); label("$e$", center E rotated 90);

    F = pp -- pp reflectedabout(left, right) -- subpath(8-p[2], p[0]) of pizza -- cycle;
    draw F rotated -90;
    label("$f$", center F); label("$F$", center F rotated -90);

    G = pp -- z13 -- pp rotated -90 -- z10 -- cycle;
    draw subpath (2,3) of G reflectedabout(pp, z13);
    label("$G$", 2/3[z13, z10] shifted 15 up);
    label("$g$", 2/3[z13, z10] shifted 15 up reflectedabout(pp, z13));
    label("$\scriptstyle H$", 2/3[pp, point -3/2 of E] - (2,1) );
    label("$\scriptstyle h$", 2/3[pp, point -3/2 of E] rotated 90 + (12,-12));

);
label.lft(aa, 10 left); label.rt(bb, 10 right);
```

Fair allocation of a pizza

The Pizza Theorem: If a pizza is divided into eight pieces by making cuts at 45° angles through an arbitrary point in the pizza, then the sums of the areas of alternate slices are equal.

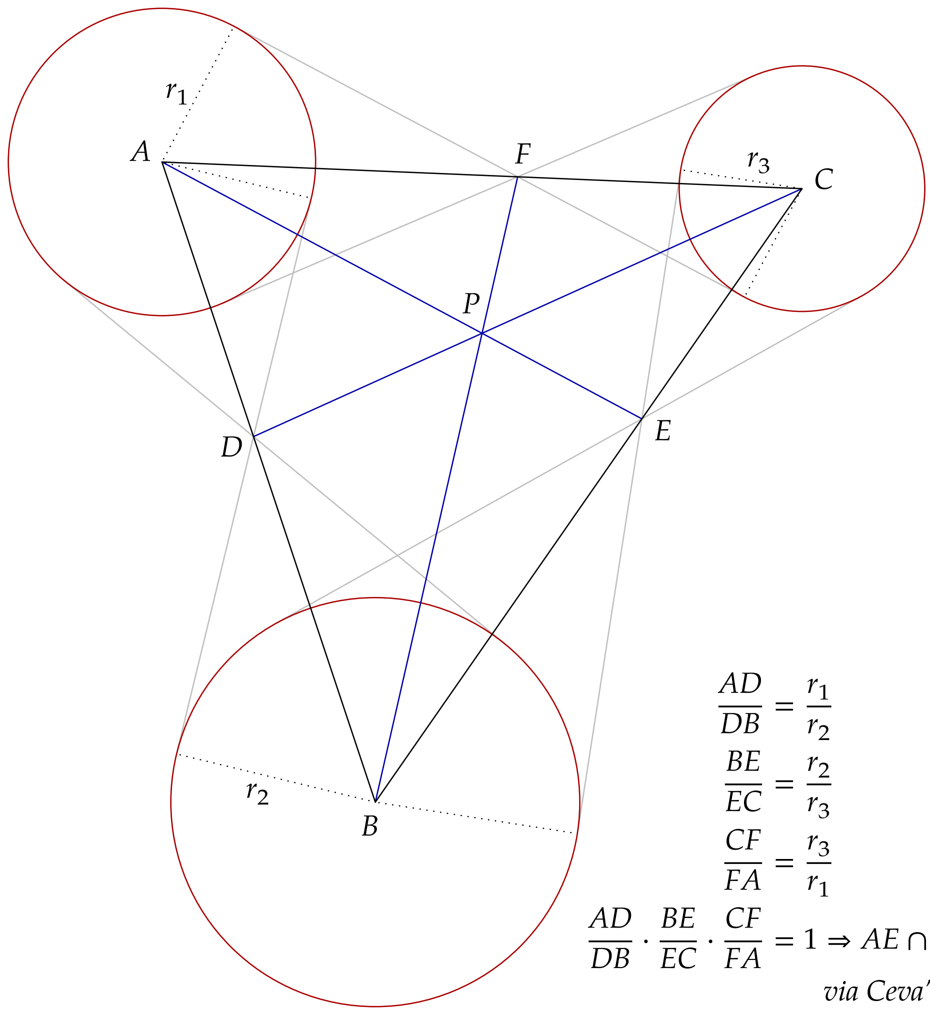
Proof:



```
pair A,B,C,D,E,F,P;
numeric r[];
A = origin;    r1 = 60 + 5 normaldeviate;
B = (80,-240); r2 = 80 + 5 normaldeviate;
C = (240,-10); r3 = 40 + 5 normaldeviate;

path c[];
c1 = fullcircle scaled 2r1 shifted A;
c2 = fullcircle scaled 2r2 shifted B;
c3 = fullcircle scaled 2r3 shifted C;

D = whatever[point 4 of c1, point 0 of c2] = whatever[point 0 of c1, point 4 of c2];
E = whatever[point 4 of c2, point 0 of c3] = whatever[point 0 of c2, point 4 of c3];
F = whatever[point 2 of c1, point 6 of c3] = whatever[point 6 of c1, point 2 of c3];
P = whatever [C,D] = whatever [A,E];

vardef tangent_point(expr c, p) =
  c intersectionpoint fullcircle scaled abs(p-center c) shifted 1/2[p,center c]
enddef;

pair t[];
t111 = tangent_point(c1,D); t113 = tangent_point(c1,F);
t121 = tangent_point(c2,D); t122 = tangent_point(c2,E);
t132 = tangent_point(c3,E); t133 = tangent_point(c3,F);
t211 = tangent_point(reverse c1,D); t213 = tangent_point(reverse c1,F);
t221 = tangent_point(reverse c2,D); t222 = tangent_point(reverse c2,E);
t232 = tangent_point(reverse c3,E); t233 = tangent_point(reverse c3,F);

drawoptions(withcolor 3/4 white);
draw t111 -- t121; draw t211 -- t221;
draw t113 -- t233; draw t213 -- t133;
draw t122 -- t232; draw t222 -- t132;
drawoptions(dashed withdots scaled 1/2);
draw A -- t113; draw t233 -- C; label.lft ("$r_1$", 1/2[A,t113]);
draw A -- t211; draw t221 -- B; label.llft("$r_2$", 1/2[B,t221]);
draw B -- t222; draw t132 -- C; label.urt ("$r_3$", 1/2[C,t132]);
drawoptions(withcolor 2/3 red); draw c1; draw c2; draw c3;
drawoptions(withcolor 2/3 blue); draw A--E; draw B--F; draw C--D;
drawoptions(); draw A--B--C--cycle;

forsuffixes s = A, B, C, D, E, F:
  label("$" & str s & "$", (1+9/abs(s-P))[P, s]);
endfor
label("$P$", P + 12 dir 110);

label.urt(btex \vbox{\openup3pt\halign{\hfill$\displaystyle #$&$\displaystyle{}=#$\hfil\cr
\frac{AD}{DB}&\frac{r_1}{r_2}\cr
\frac{BE}{EC}&\frac{r_2}{r_3}\cr
\frac{CF}{FA}&\frac{r_3}{r_1}\cr
\frac{AD}{DB}\cdot \frac{BE}{EC}\cdot \frac{CF}{FA}&1 \Rightarrow AE \cap BF \cap CD = P\cr
&\omit\qquad\textit{via Ceva's theorem}\hfil\cr
}} etex, (xpart point 0 of c2, ypart point 6 of c2));
```

A three-circle theorem

Given three non-intersecting, mutually external circles, connect the intersection of the internal common tangents of each pair of circles with the centre of the other circle. Then the resulting three line segments are concurrent.



$$\frac{AD}{DB} = \frac{r_1}{r_2}$$
$$\frac{BE}{EC} = \frac{r_2}{r_3}$$
$$\frac{CF}{FA} = \frac{r_3}{r_1}$$
$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1 \Rightarrow AE \cap BF \cap CD = P$$

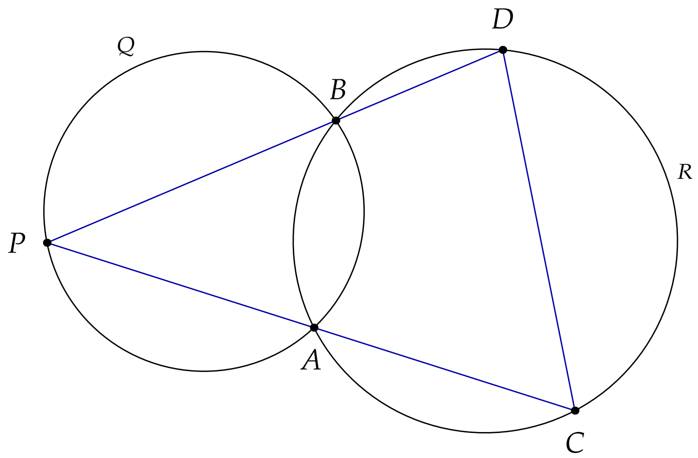
via Ceva's theorem

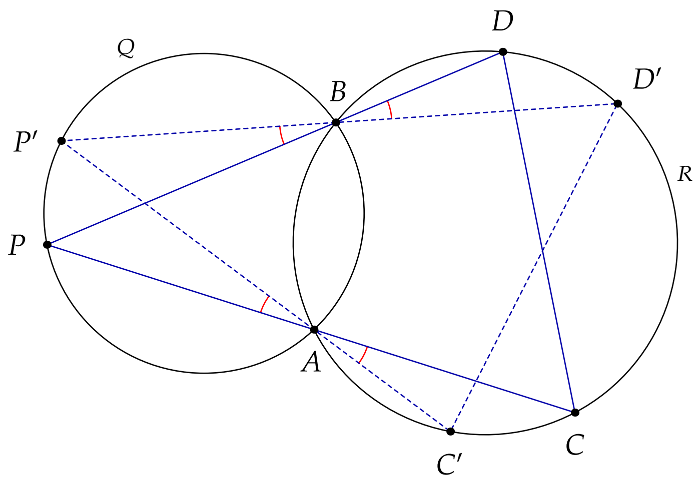
— R. S. Hu

```
path Q, R; pair A, B, C, D, P;
Q = fullcircle scaled 120 rotated 180;
R = fullcircle scaled 144 shifted 106 right rotated -6;
P = point 1/4 of Q;
A = Q intersectionpoint R;
B = reverse Q intersectionpoint R;
C = reverse R intersectionpoint (P -- 4[P, A]);
D = R intersectionpoint (P -- 4[P, B]);

picture upper;
upper = image(
    draw P--C--D--cycle withcolor 2/3 blue;
    draw Q; draw R;

    interim labeloffset := 8;
    dotlabel.bot("$A\thinspace$", A);
    dotlabel.top("$\thinspace B$", B);
    dotlabel.bot("$C$", C);
    dotlabel.top("$D$", D);
    dotlabel.lft("$P$", P);

    interim labeloffset := 4;
    label.ulft("$\scriptstyle Q$", point 6.5 of Q);
    label.urt("$\scriptstyle R$", point 0.5 of R);
);

draw upper shifted 200 up;
pair P', C', D';
P' = point 7.4 of Q;
C' = reverse R intersectionpoint (P' -- 4[P', A]);
D' = R intersectionpoint (P' -- 4[P', B]);

path ark, bark;
ark = quartercircle rotated angle (C'-A) scaled 42 shifted A cutafter (A--C);
draw ark withcolor red;
draw ark rotatedabout(A, 180) withcolor red;
bark = quartercircle rotated angle (D'-B) scaled 42 shifted B cutafter (B--D);
draw bark withcolor red;
draw bark rotatedabout(B, 180) withcolor red;

draw P'--C'--D'--cycle dashed evenly scaled 1/2 withcolor 2/3 blue;
draw upper;
interim labeloffset := 8;
dotlabel.bot("$C'$", C');
dotlabel.urt("$D'$", D');
dotlabel.lft("$P'$", P');
```

A constant chord

Suppose two circles Q and R intersect in A and B. A point P on the arc of Q which lies outside R is projected through A and B to determine chord CD of R. Prove that no matter where P is chosen on its arc, the length of chord CD is always the same.





$$\angle C'AC = \angle P'AP = \angle P'BP = \angle D'BD$$

$$\overset{\frown}{C'C} = \overset{\frown}{D'D}, \quad \overset{\frown}{C'D'} = \overset{\frown}{CD}$$

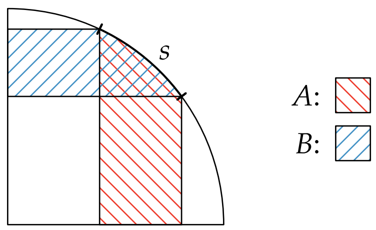
$$C'D' = CD$$

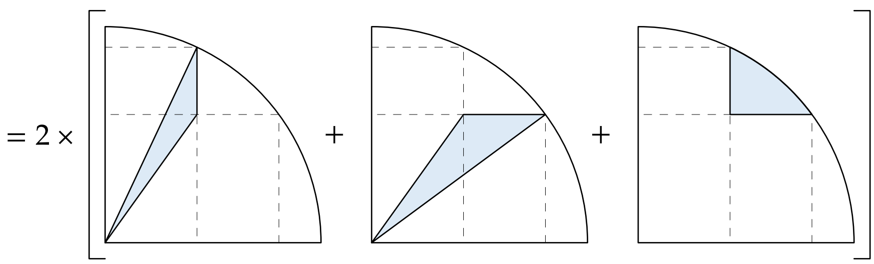
```
numeric r; r = 81;
path Q, s, A, B;
Q = (quartercircle -- origin -- cycle) scaled 2r;
s = subpath (0.81, 1.44) of Q;
z1 = point 0 of s; z2 = point infinity of s; z3 = (x2, y1);
A = s -- (x2, 0) -- (x1, 0) -- cycle;
B = s -- (0, y2) -- (0, y1) -- cycle;

color aa, bb, ff; aa = Reds 8 6; bb = Blues 8 6; ff = Blues 8 2;
picture P[];
input thatch
P1 = image(
  path a, b; a = unitsquare scaled 13 shifted (r + 42, 42);
  b = a shifted 18 down;
  thatch_angle := -45; rule A withcolor aa; rule a withcolor aa; draw a;
  thatch_angle := +45; rule B withcolor bb; rule b withcolor bb; draw b;
  draw A; draw B; draw Q;
  interim ahangle := 180; interim ahlength := 2; drawdblarrow s withpen pencircle scaled 3/4;
  label.urt("$s$", point 1.125 of s);
  label.lft("$A$:\ ", point 3.5 of a);
  label.lft("$B$:\ ", point 3.5 of b);
);
P2 = image(
  draw A dashed evenly withpen pencircle scaled 1/8;
  draw B dashed evenly withpen pencircle scaled 1/8;
  draw Q;
);
P3 = image(path p; p = origin -- z2 -- z3 -- cycle; fill p withcolor ff; draw P2; draw p);
P4 = image(path p; p = origin -- z1 -- z3 -- cycle; fill p withcolor ff; draw P2; draw p);
P5 = image(path p; p = s -- z3 -- cycle;            fill p withcolor ff; draw P2; draw p);
P6 = image(
  draw P3; draw P4 shifted 100 right; draw P5 shifted 200 right;
  numeric c; c = 6;
  path brk; brk = (0, -c) -- (-c, -c) -- (-c, r+c) -- (0, r+c);
  draw brk; draw brk reflectedabout(up, down) shifted (200 + r, 0);
  label.lft("$=2\times{}$", point 1.5 of brk);
  label("$+$", point 1.5 of brk shifted 92 right);
  label("$+$", point 1.5 of brk shifted 192 right);

);
P7 = image(
  fill s -- origin -- cycle withcolor ff; draw P2;
  path a; a = quartercircle scaled 42 rotated angle z1 cutafter (origin -- z2);
  draw a; label.urt("$\theta$", point 1/2 of a);
  draw s -- origin -- cycle;
  label.lft("$=2\times{}$", point 1.5 of brk);
);
draw P1;
draw P6 shifted 120 down;
draw P7 shifted 240 down;
label("$\displaystyle A + B = 2 \times {\theta \over 2} = \theta = \ell(s)$", (200, -200));
```

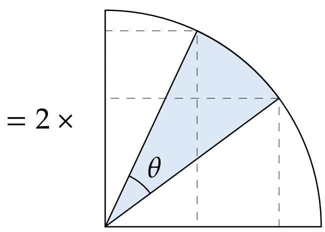
A Putnam area problem

Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis, and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

SOLUTION:





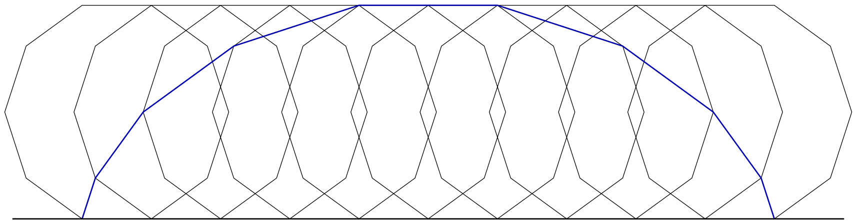


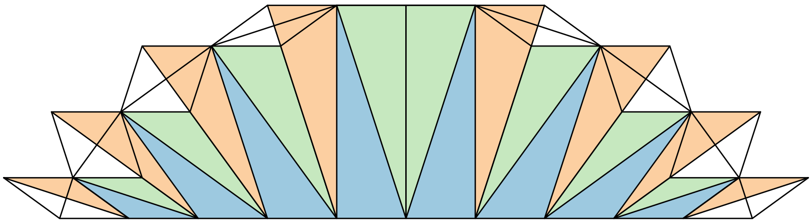
$$A + B = 2 \times \frac{\theta}{2} = \theta = \ell(s)$$

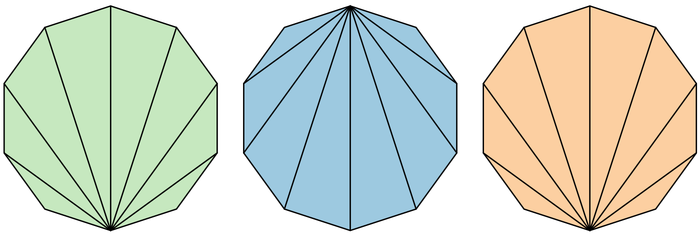
```
numeric r, s, u; r = 42; s = r * cosd 18; u = 2r * sind 18;
picture P[];
path D, side, base, arch;
D = (for t=0 upto 9: (r, 0) rotated -72 rotated 36t -- endfor cycle) shifted (1/2 u, s);
side = subpath (9, 10) of D;
base = (origin -- (12u, 0)) shifted -(u, 0);
P1 = image(
    arch =
    for i=0 upto 9:
        point -1 of D --
        hide(draw D withpen pencircle scaled 1/4; D := D rotatedabout(point i of D, -36);)
    endfor point -1 of D;
    draw arch withcolor 2/3 blue;
    draw base;
);
P2 = image(
    for i=1 upto 8:
        path part, part', part'';
        part = side shifted (i*u, 0) -- point i of arch -- cycle;
        part' = part rotatedabout(point if i > 4: 5/2 else: 3/2 fi of part, 180);
        part'' = part rotatedabout(point if i > 4: 3/2 else: 5/2 fi of part, 180);
        fill part withcolor Blues 7 3;
        fill part' withcolor Greens 7 2; draw part';
        fill part'' withcolor Oranges 7 2; draw part'';
        path m;
        m = point i if i > 4: +1 else: -1 fi of arch
         -- point   if i > 4: 0  else:  1 fi of part'';
        draw m; draw m shifted (u * if i > 4: left else: right fi);
    endfor
    draw arch -- cycle;
);
P3 = image(
    D := D shifted - center D rotated -18 shifted (s, r);
    fill D withcolor Greens 7 2;
    draw D; for i=1 upto 7: draw point 0 of D -- point 1+i of D; endfor
    D := D rotatedabout(center D, 180) shifted (5u-s, 0);
    fill D withcolor Blues 7 3;
    draw D; for i=1 upto 7: draw point 0 of D -- point 1+i of D; endfor
    D := D rotatedabout(center D, 180) shifted (5u-s, 0);
    fill D withcolor Oranges 7 2;
    draw D; for i=1 upto 7: draw point 0 of D -- point 1+i of D; endfor
);

draw P1;
draw P2 shifted (3r * down);
draw P3 shifted (6r * down);
```

The area under a polygonal arch

The area under a polygonal arch generated by one vertex of a regular n-gon rolling along a straight line is three times the area of the polygon.







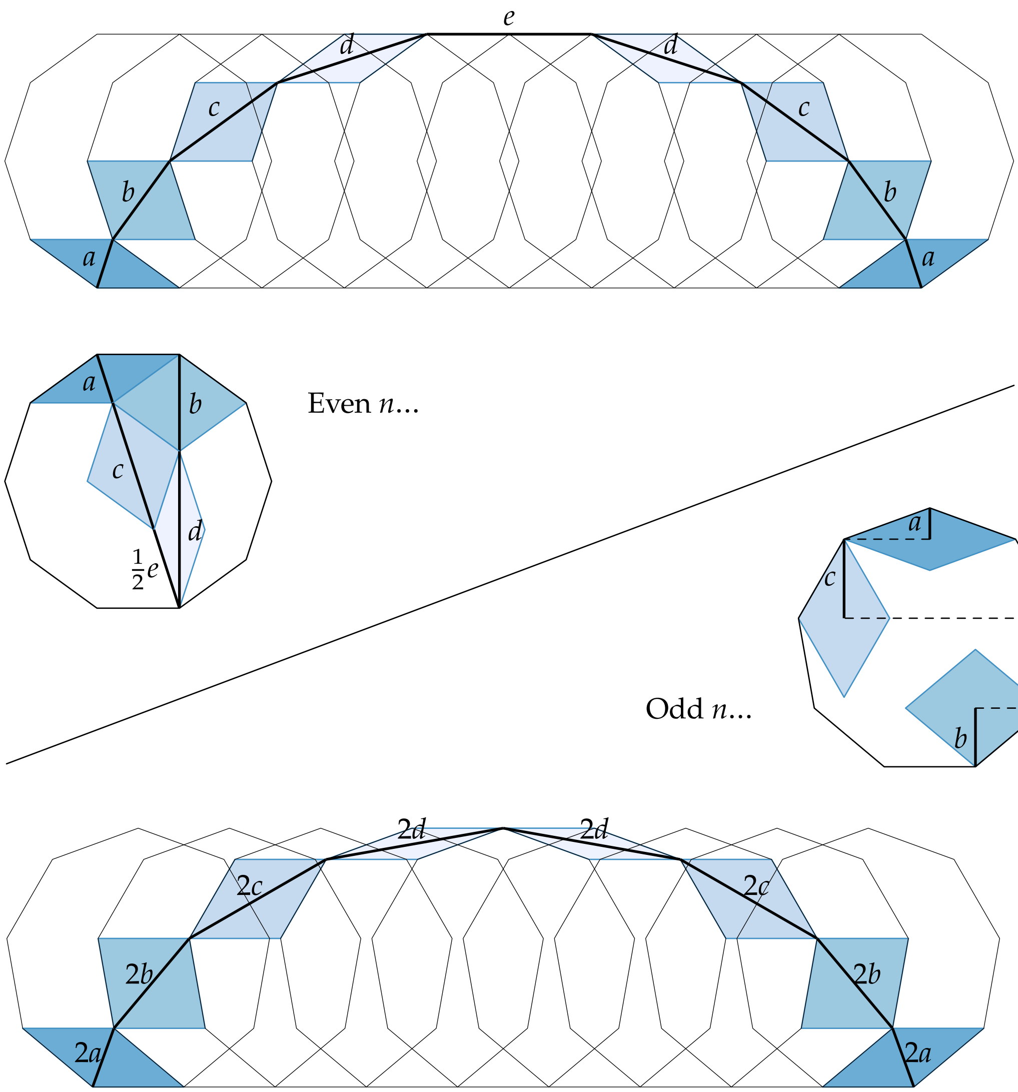
COROLLARY: The area under one arch of a cycloid is three times the area of the generating circle.

— Philip R. Mallinson

```
vardef decorate(expr rhombus, number, s) =
  fill rhombus withcolor Blues[7][s];
  draw rhombus withcolor Blues[7][5];
  path A; A = point 0 of rhombus -- point 2 of rhombus;
  cutdraw A withpen pencircle scaled 1;
  label(
    "$" & if odd number: "2" & fi char (101-s) & "$",
    point 1/2 of A + 6 unitvector(direction 1/2 of A rotated 90)
  );
enddef;
path gon[]; picture R[], P[]; numeric r; r = 50; pair side[];
for n=9, 10:
  gon[n] = for t=-1/2 upto n-1: down scaled r rotated -(360/n*t) -- endfor cycle;
  side[n] = point 0 of gon[n] - point 1 of gon[n];
  P[n] = image(
    for i=1 upto n-1:
      path R; R = subpath (i,i+1) of gon[n] shifted ((i-1)*side[n])
               -- subpath (i+1,i) of gon[n] shifted (i*side[n]) -- cycle;
      decorate(R, n, round(abs(n/2-i)));
    endfor
    for i=0 upto n-1:
        draw gon[n] shifted (i*side[n]) withpen pencircle scaled 1/4;
    endfor
  );
endfor
P90 = image(
  % See source for the internals of the nonagon.
);
P100 = image(
  z1 = point 5 of gon10 reflectedabout(point 4 of gon10, point 6 of gon10);
  z2 = point 6 of gon10 reflectedabout(point 4 of gon10, point 7 of gon10);
  z3 = z2 reflectedabout(point 0 of gon10, point 5 of gon10);
  z4 = z1 reflectedabout(z2, z3);
  z5 = z4 reflectedabout(point 0 of gon10, point 6 of gon10);
  decorate(z1 -- subpath (4,6) of gon10       -- cycle, 10, 4);
  decorate(subpath(6, 7) of gon10 -- z2 -- z1 -- cycle, 10, 3);
  decorate(z4 -- z2 -- z1 -- z3               -- cycle, 10, 2);
  decorate(z2 -- z4 -- point 0 of gon10 -- z5 -- cycle, 10, 1);
  cutdraw point 0 of gon10 -- z4 withpen pencircle scaled 1;
  label.lft("$\frac12e$", 1/2[point 0 of gon10, z4]);
  label.rt("Even $n$\dots", point 7 of gon10 shifted 20 right);
  draw gon10;
);
draw P9; draw P90 shifted (tw-2.4r , 2.4r);
draw P10 shifted (0, 6r); draw P100 shifted (0, 3.6r);
draw ulcorner P9 shifted (0, 20) -- lrcorner P10 shifted (0, 6r-36);
```

The length of a polygonal arch

The length of the polygonal arch generated by one vertex of a regular n-gon rolling along a straight line is four times the length of the in-radius plus four times the length of the circum-radius of the n-gon.

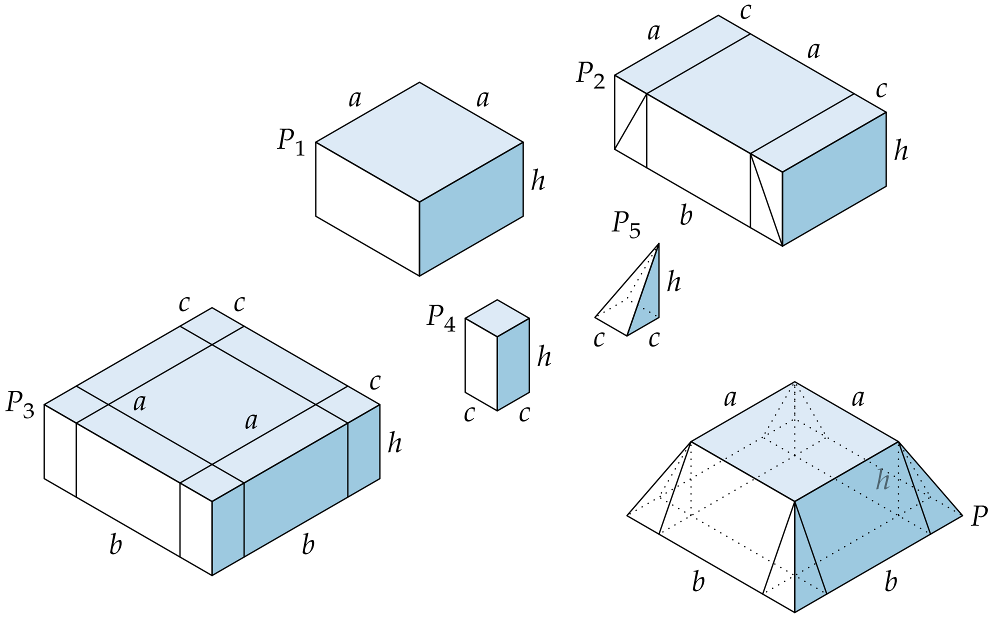


Corollary: The arc length of one arch of a cycloid is eight times the radius of the generating circle.

— Philip R. Mallinson

```
numeric a, b, c, h; 2c = b - a; a = 55; b = 89; h = 34;
% quick and dirty, fixed isometric projection
vardef p(expr x, y, z) =
    0.40824829 * (1.73205x + 1.73205z, -x + 2y + z)
enddef;
picture P[];
P1 = image(
    path f, t, s;
    f = origin -- p(a, 0, 0) -- p(a, h, 0) -- p(0, h, 0) -- cycle;
    t = p(0, h, 0) -- p(a, h, 0) -- p(a, h, a) -- p(0, h, a) -- cycle;
    s = p(a, 0, 0) -- p(a, 0, a) -- p(a, h, a) -- p(a, h, 0) -- cycle;
    fill t withcolor Blues 8 2;
    fill s withcolor Blues 8 4;
    draw f; draw s; draw t;
    label.ulft("$a$", point 7/2 of t);
    label.urt("$a$", point 5/2 of t);
    label.rt("$h$", point 3/2 of s);
    label.lft("$P_1$", point 0 of t);
);
P2 = image(
    path f, t, s;
    f = origin -- p(b, 0, 0) -- p(b, h, 0) -- p(0, h, 0) -- cycle;
    t = p(0, h, 0) -- p(b, h, 0) -- p(b, h, a) -- p(0, h, a) -- cycle;
    s = p(b, 0, 0) -- p(b, 0, a) -- p(b, h, a) -- p(b, h, 0) -- cycle;
    fill t withcolor Blues 8 2;
    fill s withcolor Blues 8 4;
    draw f; draw s; draw t;
    draw point 0 of f -- p(c, h, 0) -- p(c, h, a); draw p(c, h, 0) -- p(c, 0, 0);
    draw point 1 of f -- p(b-c, h, 0) -- p(b-c, h, a); draw p(b-c, h, 0) -- p(b-c, 0, 0);
    label.ulft("$a$", point 7/2 of t);
    label.urt("$a$", point 5/2 of t);
    label.urt("$c$", p(c/2, h, a));
    label.urt("$c$", p(b-c/2, h, a));
    label.rt("$h$", point 3/2 of s);
    label.llft("$b$", point 1/2 of f);
    label.lft("$P_2$", point 0 of t);
);
%... and so on to define four more pictures
draw P1;
draw P2 shifted p(3/2a - 2c, 0, 2a);
draw P3 shifted p(3/2a - 2c, 0, -3.5a);
draw P4 shifted p(3/2a + b - 3c, 0, -3/4a);
draw P5 shifted p(3/2a + b - 3c, 0, +1/2a);
draw P6 shifted p(4a, 0, -a);

label.bot(btex \vbox{\openup=6pt\halign{\hss # \hss\cr
$P_4 = 3P_5$\cr
$P_1+P_3 = 2P_2 + 4P_4 \quad\Rightarrow\quad
        P_1 + P_2 + P_3 = 3P_2 + 12P_5 = 3(P_2 + 4P_5) = 3P$\cr
$\therefore\quad V = \frac{h}{3}\left(a^2 + ab + b^2\right)$\cr
}} etex, point 1/2 of bbox currentpicture shifted 21 down);
```

The volume of a frustrum of a square pyramid



$$P_4 = 3P_5$$

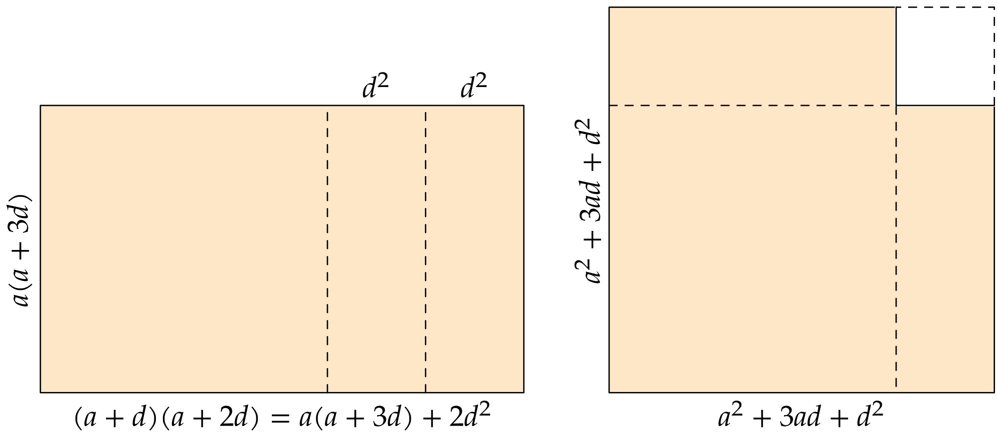
$$P_1 + P_3 = 2P_2 + 4P_4 \quad \Rightarrow \quad P_1 + P_2 + P_3 = 3P_2 + 12P_5 = 3(P_2 + 4P_5) = 3P$$

$$\therefore \quad V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

— Sidney J. Kung

```
numeric a, d; d = 6.07; a = 4.7;

path A, B, C;
A = unitsquare xscaled ((a+d)*(a+2d)) yscaled (a*(a+3d));
B = unitsquare scaled (a**2 +3a*d+d**2) shifted point 1 of A shifted 32 right;
C = unitsquare scaled (d**2) rotated 180 shifted point 2 of B;

fill A withcolor OrRd 8 2;
draw subpath(1,2) of A shifted -(d**2, 0) dashed evenly;
draw subpath(1,2) of A shifted -2(d**2, 0) dashed evenly;
draw A;

fill B withcolor OrRd 8 2;
draw B;

filldraw C withpen pencircle scaled 1 withcolor background;
draw subpath (-1, 1) of C dashed evenly;
draw subpath (1, 3) of C;

draw (xpart point 0 of B, ypart point 2 of C)
     -- point 2 of C
     -- (xpart point 2 of C, ypart point 0 of B) dashed evenly;

label.lft(textext("$a(a+3d)$") rotated 90, point -1/2 of A);
label.bot("$(a+d)(a+2d) = a(a+3d) + 2d^2$", point 1/2 of A);
label.lft(textext("$a^2+3ad+d^2$") rotated 90, point -1/2 of B);
label.bot("$a^2+3ad+d^2$", point 1/2 of B);

label.top("$d^2$", point 2 of A shifted (-1/2d*d, 0));
label.top("$d^2$", point 2 of A shifted (-3/2d*d, 0));
```

The product of four (positive) numbers in arithmetic progression is always the difference of two squares



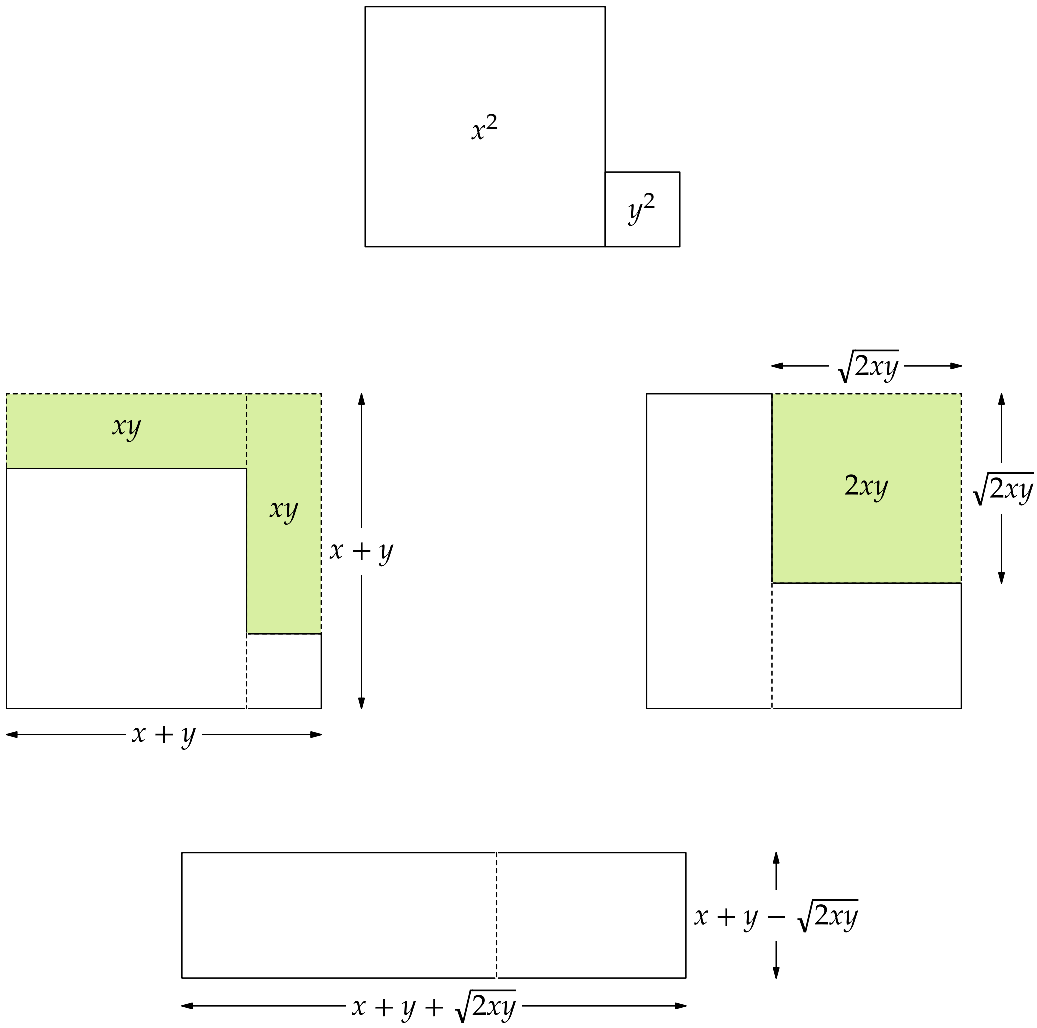
$$a(a+d)(a+2d)(a+3d) = \left(a^2 + 3ad + d^2\right)^2 - \left(d^2\right)^2$$

— RBN

```
numeric x, y; x = 90; y = 28;
path a; a = unitsquare scaled x;
path b; b = unitsquare scaled y shifted point 1 of a;
path c; c = unitsquare xscaled x yscaled y shifted point 3 of a;
path d; d = unitsquare xscaled y yscaled x shifted point 3 of b;

vardef double_arrow_label(expr t, a, b) =
  save p; picture p; p = thelabel(t, origin) rotated angle (b-a);
  save o, wd; numeric o, wd; o = labeloffset; wd = ypart (urcorner p - llcorner p);
  if wd > 2o: o := o + 1/2 wd; fi
  save arc; path arc; arc = (a--b) shifted (unitvector(a-b) rotated 90 scaled o);
  drawdblarrow arc;
  p := p rotated - angle (b-a) shifted point 1/2 of arc;
  unfill bbox p; draw p;
enddef;

picture P[];
P1 = image(draw a; draw b; label("$x^2$", center a); label("$y^2$", center b));
P2 = image(
  forsuffixes @=c, d:
    fill @ withcolor YlGn 7 2; draw @ dashed evenly scaled 1/2;
    label("$xy$", center @); endfor
  draw a; draw b;
  undraw subpath(3,4) of b withpen pencircle scaled 1;
  draw subpath(3,4) of b dashed evenly scaled 1/2;
  double_arrow_label("\strut $x+y$", point 0 of a, point 1 of b);
  double_arrow_label("\strut $x+y$", point 1 of b, point 2 of d);
);

path A; A = unitsquare xscaled (x+y-sqrt(2x*y)) yscaled (x+y);
path B; B = unitsquare yscaled (x+y-sqrt(2x*y)) xscaled sqrt(2x*y) shifted point 1 of A;
path C; C = unitsquare scaled sqrt(2x*y) shifted point 3 of B;
path D; D = A rotated 90 shifted point 0 of B;
P3 = image(
  fill C withcolor YlGn 7 2; draw C dashed evenly scaled 1/2;
  label("$2xy$", center C); draw A; draw B;
  undraw subpath(3,4) of B withpen pencircle scaled 1;
  draw subpath(3,4) of B dashed evenly scaled 1/2;
  double_arrow_label("\strut $\sqrt{2xy}$", point 1 of C, point 2 of C);
  double_arrow_label("\strut $\sqrt{2xy}$", point 2 of C, point 3 of C);
);
P4 = image(
  draw D; draw B; undraw subpath(3,4) of B withpen pencircle scaled 1;
  draw subpath(3,4) of B dashed evenly scaled 1/2;
  double_arrow_label("\strut $x+y+\sqrt{2xy}$", point 3 of D, point 1 of B);
  double_arrow_label("\strut $x+y-\sqrt{2xy}$", point 1 of B, point 2 of B);
);
draw P2 shifted 120 left; draw P3 shifted 120 right;
label.top(P1, point 5/2 of bbox currentpicture shifted 30 up);
label.bot(P4, point 1/2 of bbox currentpicture shifted 30 down);
label.top(btex $\displaystyle
x^2 + y^2 = \left(x + \sqrt {2xy} + y\right)\left(x - \sqrt {2xy} + y\right)
$ etex, point 5/2 of bbox currentpicture shifted 21 up);
```

Algebraic areas III: Factoring the sum of two squares

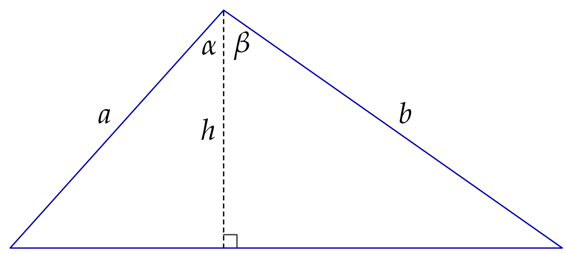
$$x^2 + y^2 = \left(x + \sqrt{2xy} + y\right)\left(x - \sqrt{2xy} + y\right)$$



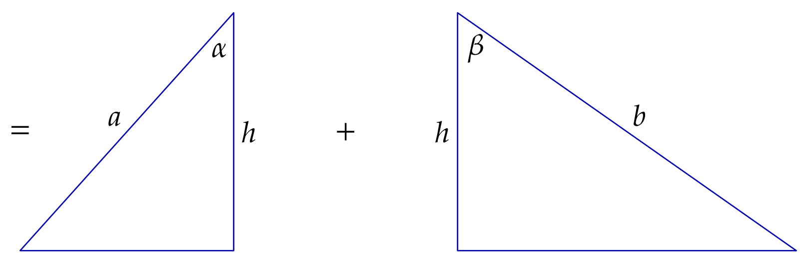
Triognometry, Calculus, & Analytic Geometry

```
numeric alpha, beta; alpha = 42; beta = 55;
numeric h; h = 89; z0 = h * up; y1 = y2 = 0;
z0 - z1 = whatever * dir (90 - alpha);
z0 - z2 = whatever * dir (90 + beta);
path t; t = z0--z1--z2--cycle;
picture P[];
P1 = image(
  draw subpath(1,3) of unitsquare scaled 5 withcolor 1/4 white;
  draw origin -- up * h dashed evenly scaled 1/2;
  draw t withcolor 2/3 blue;
  label("$\alpha$", 15 down rotated -1/2 alpha shifted point 0 of t);
  label("$\beta$", 15 down rotated 1/2 beta shifted point 0 of t);
  label.lft("$h$", (0, 1/2 h));
  label.ulft("$a$", point 1/2 of t);
  label.urt("$b$", point -1/2 of t);
  label(btex $\displaystyle \alpha, \beta \in (0, \pi/2)\quad\Longrightarrow
     \quad h=a \cos \alpha = b\cos\beta$ etex, 36 down);
);
P2 = image(
  path t', t'';
  t'  = (subpath (0, 1) of t -- (xpart point 0 of t, ypart point 1 of t) -- cycle);
  t'' = (subpath (2, 3) of t -- (xpart point 0 of t, ypart point 1 of t) -- cycle);
  t' := t' shifted 42 left;
  t'' := t'' shifted 42 right;
  draw t' withcolor 2/3 blue;
  draw t'' withcolor 2/3 blue;
  label("$\alpha$", 15 down rotated -1/2 alpha shifted point 0 of t');
  label("$\beta$", 15 down rotated 1/2 beta shifted point 1 of t'');
  label.rt("$h$", point -1/2 of t');
  label.lft("$h$", point 3/2 of t'');
  label.ulft("$a$", point 1/2 of t');
  label.urt("$b$", point 1/2 of t'');
  label("${}+{}$", 1/2[point -1/2 of t', point 3/2 of t'']);
  label("${}={}$", (xpart point 1 of t', ypart point -1/2 of t'));
);
draw P1;
draw P2 shifted 180 down;
label.bot(btex \vbox{\openup6pt\halign{\hfil $\displaystyle #$&$\displaystyle {}=#$\hfil\cr
 \frac12 ab\sin(\alpha+\beta)&\frac12 ah\sin\alpha + \frac12 bh\sin\beta\cr
&\frac12 ab\cos\beta\sin\alpha + \frac12 ba\cos\alpha\sin\beta\cr
\therefore\quad\sin(\alpha+\beta)&\sin\alpha\cos\beta + \cos\alpha\sin\beta\cr
}} etex, point 1/2 of bbox currentpicture shifted 34 down);
```

Sine of the sum - II



$$\alpha, \beta \in (0, \pi/2) \quad \Longrightarrow \quad h = a\cos\alpha = b\cos\beta$$



$$\begin{aligned}
\frac{1}{2}ab\sin(\alpha+\beta) &= \frac{1}{2}ah\sin\alpha + \frac{1}{2}bh\sin\beta \\
&= \frac{1}{2}ab\cos\beta\sin\alpha + \frac{1}{2}ba\cos\alpha\sin\beta \\
\therefore \quad \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}$$

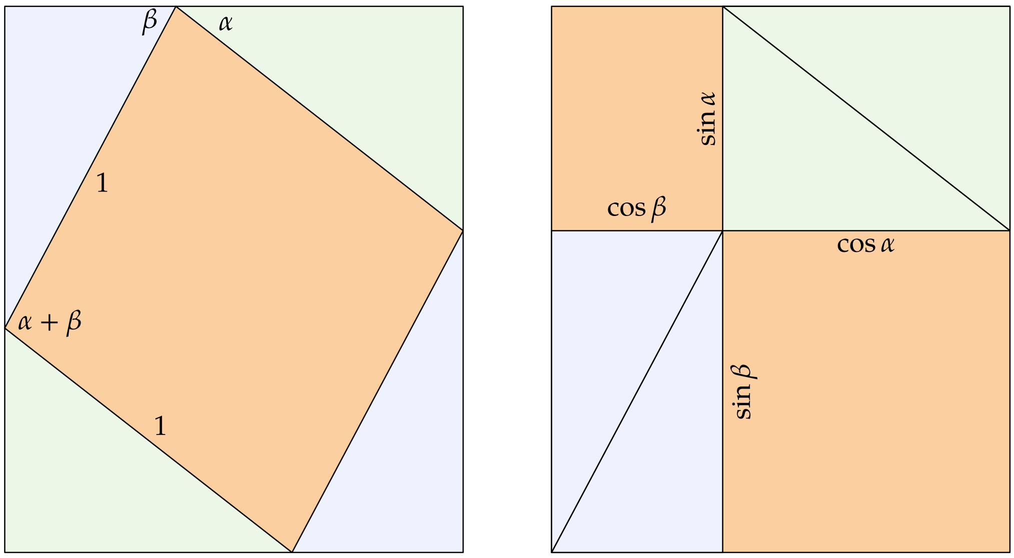
— Christopher Brüningsen

```
numeric a, b, u; a = 38; b = 62; u = 144;
numeric ca, cb, sa, sb; ca = cosd(a);  sa = sind(a); cb = cosd(b);  sb = sind(b);
path s, t[];
s = unitsquare shifted -(1/2, 1/2) xscaled (ca+cb) yscaled (sa+sb) scaled u;
t0 = (origin -- ca * right -- sa * up -- cycle) scaled u;
t1 = (origin -- cb * left  -- sb * up -- cycle) scaled u;
beginfig(1);
picture P[];
P1 = image(
  fill s withcolor Oranges 7 2;
  for i=0, 1:
    for j=0, 180:
      path p; p = t[i] shifted point i of s rotated j;
      fill p withcolor if odd i: Blues else: Greens fi 7 1; draw p;
      if i+j = 0:
        label.urt("$1$", point 3/2 of p);
        label("$\alpha+\beta$", point 2 of p + 18 dir 1/2(b-a-12));
      elseif i+j = 181:
        label.lrt("$1$", point 3/2 of p);
        label("$\alpha$", point 1 of p + 21 dir (- 1/2 a));
        label("$\beta$", point 1 of p + 12 dir (180 + 1/2 b));
      fi
    endfor
  endfor
);
P2 = image(
  fill s withcolor Oranges 7 2; draw s;
  for i = 0, 1:
    path p; p = t[i] shifted (point 0 of s - point i of t[i]) rotated 180(1-i);
    for j=0, 180:
      p := p rotatedabout(point 3/2 of p, j);
      fill p withcolor if odd i: Blues else: Greens fi 7 1; draw p;
      if i+j=180:
        label.bot("$\cos\alpha$", point 1/2 of p);
        label.lft(TEX("$\sin\alpha$") rotated 90, point -1/2 of p);
      elseif i+j=1:
        label.rt(TEX("$\sin\beta$") rotated 90, point -1/2 of p);
      elseif i+j=181:
        label.top("$\cos\beta$", point 1/2 of p);
      fi
    endfor
  endfor
);
% part II is much the same, but with smaller s
labeloffset := 20;
P5 = image(draw P1; draw P2 shifted (3/2u, 0); label.ulft("I.", ulcorner currentpicture));
P6 = image(draw P3; draw P4 shifted (3/2u, 0); label.ulft("II.", ulcorner currentpicture));
draw P5; draw P6 shifted (0, -7/4u);
label.top("$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$",
  point 5/2 of bbox currentpicture shifted 13 up);
```

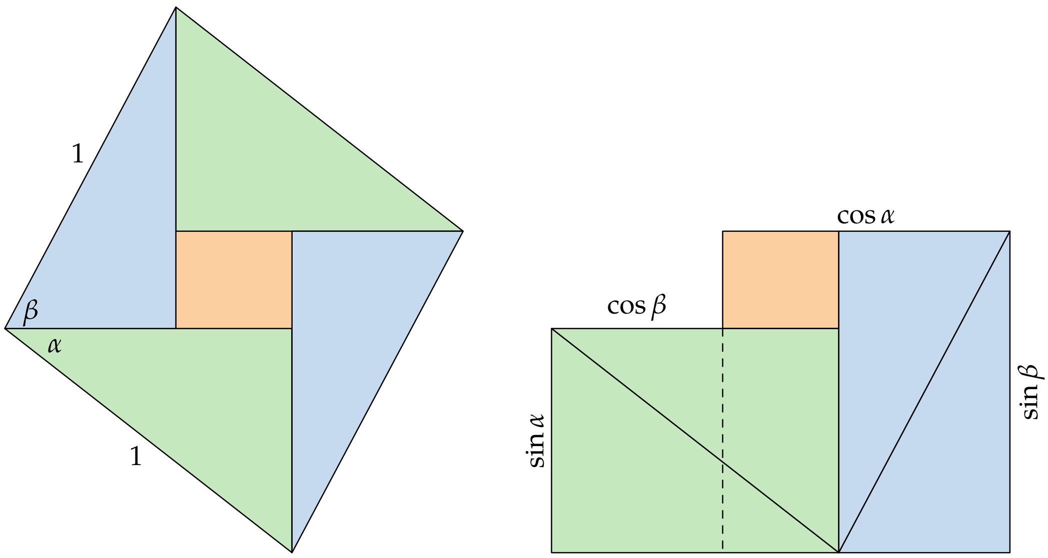
Sine of the sum – III

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

I.



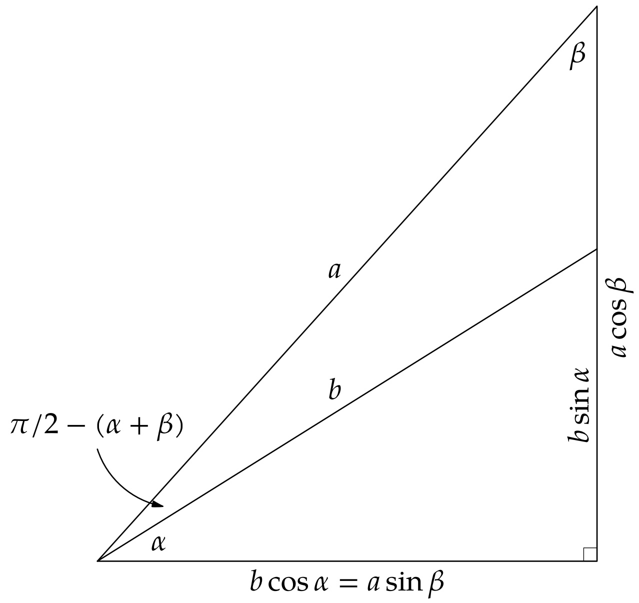
II.



— Volker Priebe and Edgar A. Ramos

```
numeric alpha, beta, a, b;
alpha = 32;
beta = 42;
a = 280;
b * cosd(alpha) = a * sind(beta);
z0 = right scaled a rotated (90 - beta);
z1 = right scaled b rotated alpha;
path t;
t = origin -- (x0, 0) -- z0 -- cycle;

draw unitsquare scaled 5 rotated 90 shifted point 1 of t withpen pencircle scaled 1/4;
draw t;
draw origin -- z1;

label.bot("$b\cos\alpha = a\sin\beta$", point 1/2 of t);
label.rt(TEX("$a\cos\beta$") rotated 90, point 1.44 of t);
label.lft(TEX("$b\sin\alpha$") rotated 90, (x0, 1/2y1));

label.ulft("$a$", 1/2 z0);
label.ulft("$b$", 1/2 z1);

label("$\alpha$", 24 dir 1/2 alpha);
label("$\beta$", 20 dir (270 - 1/2 beta) shifted point 2 of t);
z2 = 42 up;
drawarrow z2 {dir -72} .. 32 dir 1/2(90 + alpha - beta);
label.top("$\pi/2-(\alpha+\beta)$", z2);
```

Cosine of the sum

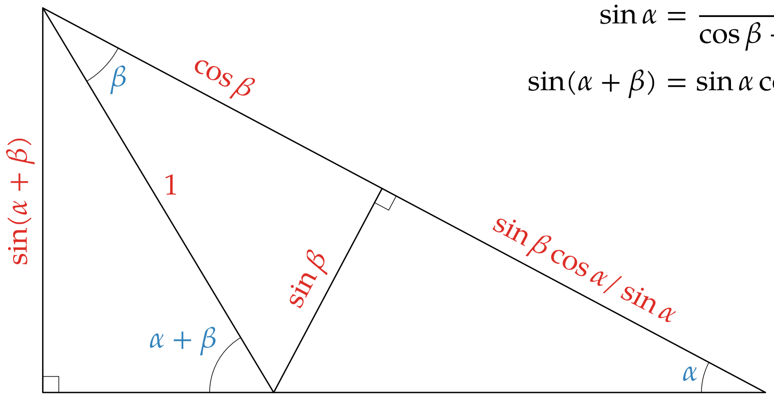


$$\frac{1}{2}ab\sin\left(\frac{\pi}{2}-(\alpha+\beta)\right) = \frac{1}{2}b\cos\alpha\cdot a\cos\beta - \frac{1}{2}b\sin\alpha\cdot a\sin\beta$$
$$\therefore\quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

— Sidney H. Kung

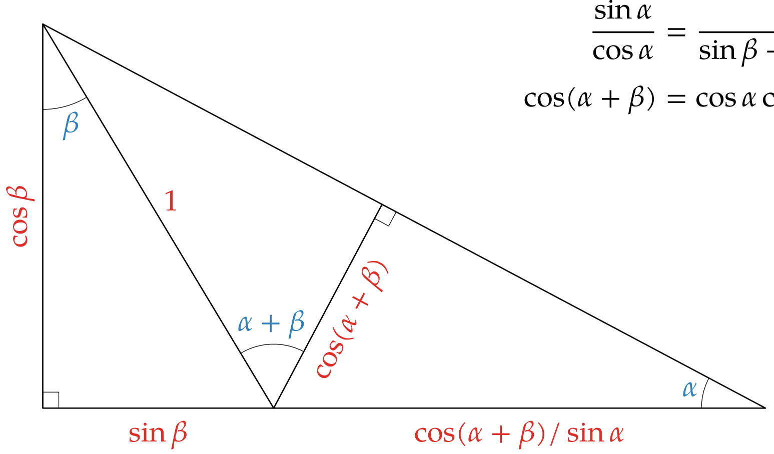
```
vardef angle_mark(expr a, o, b, r, t, shade) =
  draw fullcircle scaled 2r rotated angle (a-o) shifted o cutafter (o--b)
       withpen pencircle scaled 1/4;
  save p; picture p;
  p = thelabel(t, origin);
  save offset, alpha; numeric offset, alpha;
  alpha = 1/2 (angle (a-o) + angle (b-o));
  offset = r + arclength ((origin -- 100 dir alpha
  shifted center bbox p) cutafter bbox p);
  draw p shifted o shifted (offset * dir alpha) withcolor shade;
enddef;
numeric alpha, beta; alpha = 28; beta = 45 - 1/2 alpha;
z4 = 144 up;
z1 = whatever * right; z4-z1 = whatever * dir (180-alpha-beta);
z2 = whatever * right; z4-z2 = whatever * dir (180-alpha);
z3 = whatever[z2, z4]; z1-z3 = whatever * dir (90-alpha);
path t; t = origin -- z1 -- z2 -- z3 -- z4 -- cycle;
picture P[];
P1 = image(
  angle_mark(z3, z2, z1, 24, "$\alpha$", Blues 6 5);
  angle_mark(z1, z4, z3, 32, "$\beta$", Blues 6 5);
  angle_mark(z4, z1, origin, 24, "$\alpha+\beta$", Blues 6 5);
  draw unitsquare scaled 6 withpen pencircle scaled 1/4;
  draw unitsquare scaled 6 rotated angle (z1-z3) shifted z3 withpen pencircle scaled 1/4;
  draw z3--z1--z4; draw t;
  drawoptions(withcolor Reds 6 5);
  label.urt("$1$", 1/2[z1, z4]);
  label.lft(TEX("$\sin(\alpha+\beta)$") rotated 90, 1/2z4);
  draw thelabel.top(TEX("$\sin\beta$"), origin) rotated angle (z3-z1) shifted 1/2[z3, z1];
  draw thelabel.top(TEX("$\cos\beta$"), origin) rotated angle (z3-z4) shifted 1/2[z3, z4];
  draw thelabel.top(TEX("$\sin\beta\cos\alpha/\sin\alpha$"), origin)
  rotated angle (z2-z3) shifted 1/2[z2, z3];
  drawoptions();
  label(btex \vbox{\openup6pt\halign{\hfil$\displaystyle #$&$\displaystyle{}=#$\hfil\cr
  \sin\alpha & {\sin(\alpha+\beta) \over \cos\beta + \sin\beta\cos\alpha/\sin\alpha}\cr
  \sin(\alpha+\beta) & \sin\alpha\cos\beta + \cos\alpha\sin\beta\cr}} etex, (x2, y4-12));
);
% similarly for P2
draw P1;
draw P2 shifted 240 down;
```

Geometry of addition formulas



$$\sin\alpha = \frac{\sin(\alpha+\beta)}{\cos\beta + \sin\beta\cos\alpha/\sin\alpha}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$



$$\frac{\sin\alpha}{\cos\alpha} = \frac{\cos\beta}{\sin\beta + \cos(\alpha+\beta)/\sin\alpha}$$

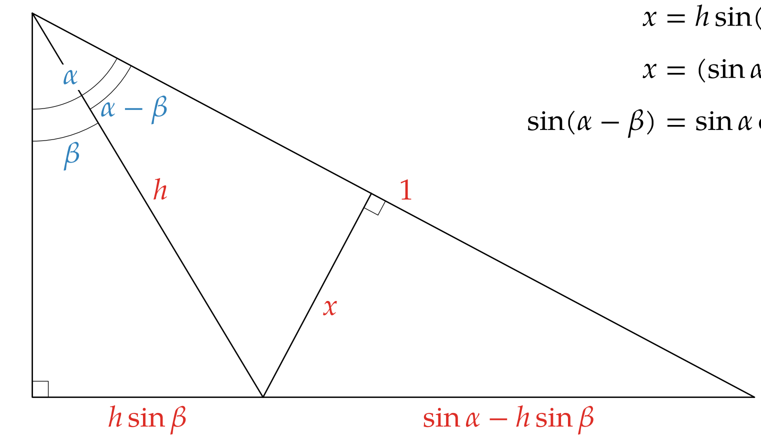
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

— Leonard M. Smiley

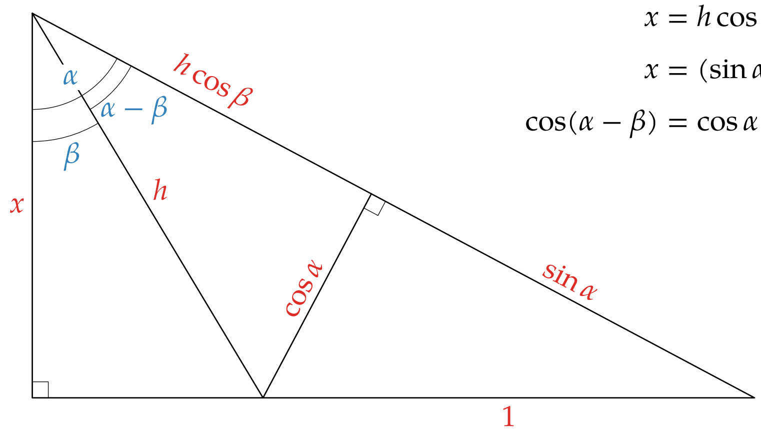
```
% Using the same angle_mark macro defined above...
numeric alpha, beta; alpha = 62; beta = 1/2 alpha;

z4 = 144 up;
z1 = whatever * right; z4-z1 = whatever * dir (90 + beta);
z2 = whatever * right; z4-z2 = whatever * dir (90 + alpha);
z3 = whatever[z2, z4]; z1-z3 = whatever * dir alpha;

picture P[];
P0 = image(   % -------------- the common parts
  % mark the right angles
  draw unitsquare scaled 6 withpen pencircle scaled 1/4;
  draw unitsquare scaled 6 rotated angle (z1-z3) shifted z3 withpen pencircle scaled 1/4;
  % mark the other angles
  angle_mark(origin, z4, z1, 48, "$\beta$", Blues 6 5);
  angle_mark(z1, z4, z2, 42, "$\alpha-\beta$", Blues 6 5);
  % do the complicated angle mark by hand...
  draw subpath (6, 6+alpha/45) of fullcircle scaled 72 shifted z4 withpen pencircle scaled 1/4;
  picture A; A = thelabel("$\alpha$", 28 down rotated 1/2 alpha shifted z4);
  % draw the triangle and the internal lines
  draw z4 -- origin -- z2 -- z4 -- z1 -- z3;
  % and now erase the background and add the alpha label
  unfill bbox A; draw A withcolor Blues 6 5;
);
P1 = image(
  draw P0;
  % labels...
  drawoptions(withcolor Reds 6 5);
    label.urt("$h$", 1/2[z1,z4]);
    label.urt("$1$", 1/2[z2,z4]);
    label.lrt("$x$", 1/2[z1,z3]);
    label.bot("$h\sin\beta$", 1/2z1);
    label.bot("$\sin\alpha-h\sin\beta$", 1/2[z1,z2]);
  drawoptions();
  label("\vbox{\openup6pt\halign{\hfil$\displaystyle #$&$\displaystyle{}=#$\hfil\cr" &
    "                  h & \cos\alpha/\cos\beta \cr" &
    "                  x & h\sin(\alpha-\beta) \cr" &
    "                  x & (\sin\alpha-h\sin\beta)\cos\alpha \cr" &
    "\sin(\alpha-\beta) & \sin\alpha\cos\beta - \cos\alpha\sin\beta \cr }}", (x2, y4-12));
);
% P2 is the same except for the labels
draw P1;
draw P2 shifted 240 down;
```

Geometry of subtraction formulas

$$h = \cos\alpha / \cos\beta$$
$$x = h\sin(\alpha - \beta)$$
$$x = (\sin\alpha - h\sin\beta)\cos\alpha$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$



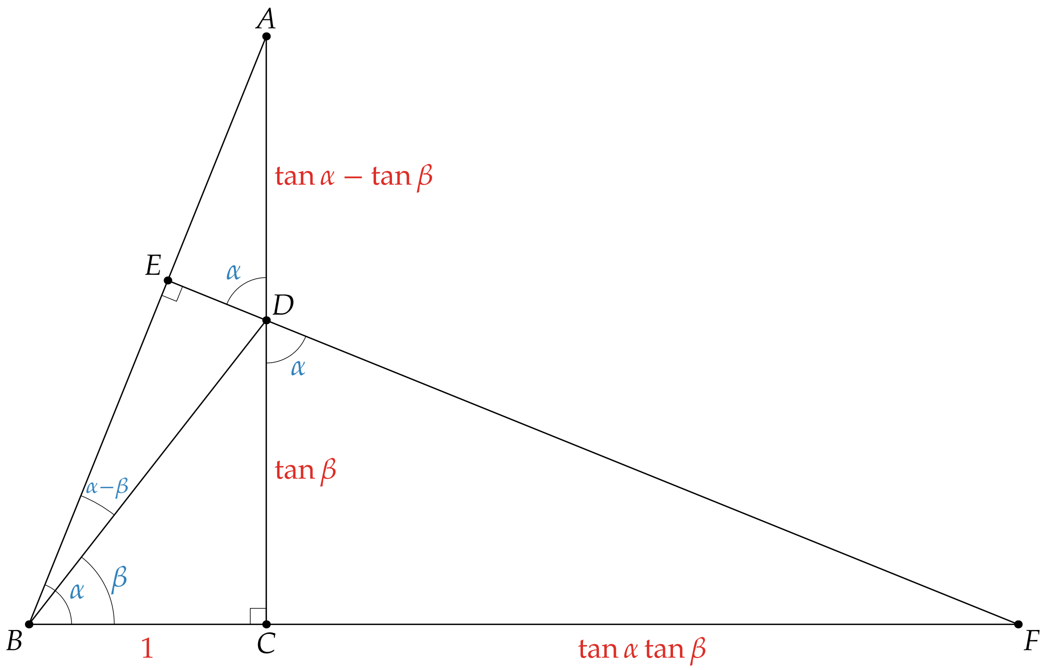
$$h = \cos\alpha / \sin\beta$$
$$x = h\cos(\alpha - \beta)$$
$$x = (\sin\alpha + h\cos\beta)\cos\alpha$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



— Leonard M. Smiley

```
numeric alpha, beta;
alpha = beta + 16; beta = 52;
z.C = origin;
z.B = 89 left;
x.A = x.D = x.C = y.F;
z.A - z.B = whatever * dir alpha;
z.D - z.B = whatever * dir beta;
z.E = whatever[z.A, z.B]; z.E - z.D = whatever * (z.A - z.B) rotated 90;
z.F = whatever[z.D, z.E];

% Using the same angle_mark macro defined above...

angle_mark(z.A, z.D, z.E, 16, "$\alpha$", Blues 6 5);
angle_mark(z.C, z.D, z.F, 16, "$\alpha$", Blues 6 5);
angle_mark(z.C, z.B, z.D, 32, "$\beta$", Blues 6 5);
angle_mark(z.C, z.B, z.A, 16, "$\alpha$", Blues 6 5);
angle_mark(z.D, z.B, z.A, 52, "$\scriptstyle\alpha-\beta$", Blues 6 5);

draw unitsquare scaled 6 rotated angle (z.B-z.E) shifted z.E withpen pencircle scaled 1/4;
draw unitsquare scaled 6 rotated angle (z.A-z.C) shifted z.C withpen pencircle scaled 1/4;

drawoptions(withcolor Reds 6 5);
label.rt("$\tan\beta$", 1/2[z.C, z.D]);
label.rt("$\tan\alpha-\tan\beta$", 1/2[z.A, z.D]);
label.bot("\strut$1$", 1/2[z.B, z.C]);
label.bot("\strut$\tan\alpha\tan\beta$", 1/2[z.C, z.F]);
drawoptions();

draw z.C--z.A--z.B--z.F--z.E; draw z.B--z.D;
dotlabel.top ("$A$", z.A);
dotlabel.llft("$B$", z.B);
dotlabel.bot ("$C$", z.C);
dotlabel.urt ("$D$", z.D);
dotlabel.ulft("$E$", z.E);
dotlabel.lrt ("$F$", z.F);

label.bot(btex \vbox{\openup8pt\halign{\hfil$\displaystyle#$&$\displaystyle{}=#$\hfil\cr
{BF\over BE}&{AD\over DE}\cr
\therefore\quad\tan(\alpha-\beta)={DE\over BE}&
{AD\over BF} ={\tan\alpha-\tan\beta\over1+\tan\alpha\tan\beta}\cr
}} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

The difference identity for tangents I



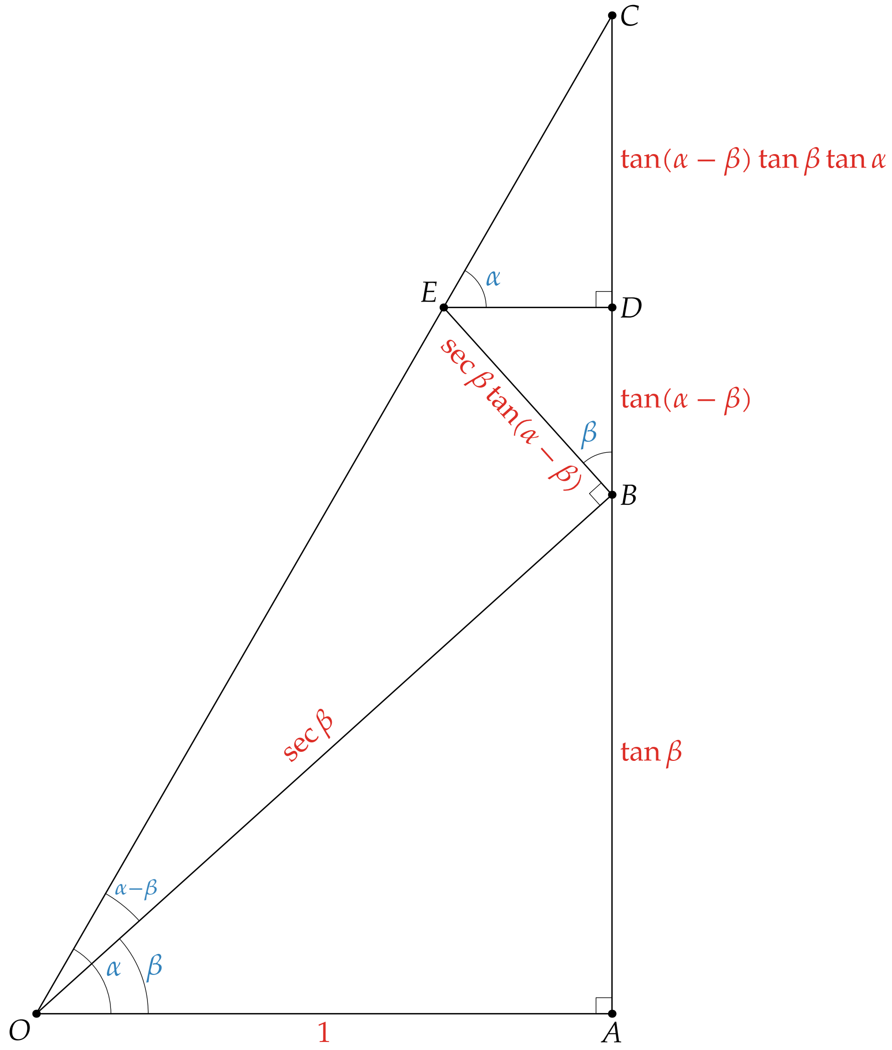
$$\frac{BF}{BE} = \frac{AD}{DE}$$

$$\therefore \quad \tan(\alpha-\beta) = \frac{DE}{BE} = \frac{AD}{BF} = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

— Guanshen Ren

```
numeric alpha, beta; alpha = beta + 18; beta = 42;
z.A = 216 right;
x.A = x.B = x.C = x.D;
z.B = whatever * dir beta;
z.C = whatever * dir alpha;
z.E = whatever [origin, z.C];
z.E - z.B = whatever * z.B rotated 90;
y.E = y.D;

% Using the same angle_mark macro defined above...

angle_mark(z.A, origin, z.C, 28, "$\alpha$", Blues 6 5);
angle_mark(z.A, origin, z.B, 42, "$\beta$", Blues 6 5);
angle_mark(z.B, origin, z.C, 52, "$\scriptstyle\alpha-\beta$", Blues 6 5);
angle_mark(z.D, z.B, z.E, 16, "$\beta$", Blues 6 5);
angle_mark(z.D, z.E, z.C, 16, "$\alpha$", Blues 6 5);

draw unitsquare scaled 6 rotated 90        shifted z.A withpen pencircle scaled 1/4;
draw unitsquare scaled 6 rotated (90+beta) shifted z.B withpen pencircle scaled 1/4;
draw unitsquare scaled 6 rotated 90        shifted z.D withpen pencircle scaled 1/4;

drawoptions(withcolor Reds 6 5);
label.bot("$1$", 1/2 z.A);
label.rt("$\tan\beta$", 1/2[z.A, z.B]);
label.rt("$\tan(\alpha-\beta)$", 1/2[z.B, z.D]);
label.rt("$\tan(\alpha-\beta)\tan\beta\tan\alpha$", 1/2[z.C, z.D]);
draw thelabel.top("$\sec\beta$", origin)
    rotated beta shifted 1/2 z.B;
draw thelabel.bot("$\sec\beta\tan(\alpha-\beta)$", origin)
    rotated (beta-90) shifted 1/2[z.B, z.E];
drawoptions();

draw origin -- z.A -- z.C -- cycle;
draw origin -- z.B -- z.E -- z.D;
dotlabel.bot ("$A$", z.A);
dotlabel.rt  ("$B$", z.B);
dotlabel.rt  ("$C$", z.C);
dotlabel.rt  ("$D$", z.D);
dotlabel.ulft("$E$", z.E);
dotlabel.llft("$O$", origin);

label.bot(btex \vbox{\openup8pt\halign{\hfil$\displaystyle#$&$\displaystyle{}=#$\hfil\cr
AC-AB&BD+DC\cr
\therefore\tan\alpha-\tan\beta&\tan(\alpha-\beta)+\tan\alpha\tan\beta\tan(\alpha-\beta)\cr
\tan(\alpha-\beta)&{\tan\alpha-\tan\beta\over1+\tan\alpha\tan\beta}\cr
}} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

The difference identity for tangents II



$$AC - AB = BD + DC$$

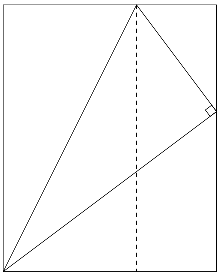
$$\therefore \tan\alpha - \tan\beta = \tan(\alpha-\beta) + \tan\alpha\tan\beta\tan(\alpha-\beta)$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

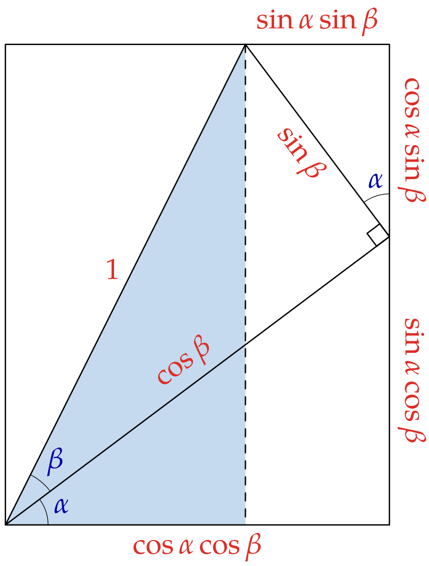
— Fukuzo Suzuki

```
picture F[]; path t, t', t'', s; numeric alpha; alpha = angle(1, 2) - angle (2, 1);
t = (origin -- (180, 0) -- (180, 90) -- cycle) rotated alpha;
t' = t reflectedabout(point 0 of t, point 2 of t);
t'' = t' rotatedabout(point 5/2 of t, 180);
interim bboxmargin := 0; s = bbox t;
vardef angled_label@#(expr t, p, r) = draw thelabel@#(t, origin) rotated r shifted p; enddef;
F0 = image(draw point 2 of t -- (xpart point 2 of t, 0) dashed evenly;
         draw unitsquare scaled 6 rotated (90  + angle point 1 of t) shifted point 1 of t;
         draw t; draw s);
F1 = image(
  fill t'' withcolor Blues 8 3;
  path a[];
  a1 = quartercircle scaled 32 cutafter subpath (0,1) of t;
  a2 = quartercircle scaled 42 cutbefore subpath (0,1) of t cutafter subpath (2,3) of t;
  a3 = quartercircle scaled 32 rotated 90 shifted point 1 of t cutafter subpath (1,2) of t;
  forsuffixes $=1,2,3: draw a$ withpen pencircle scaled 1/4; endfor
  drawoptions(withcolor 2/3 blue);
    label.rt("$\alpha$", point arctime 1/2 arclength a1 of a1 of a1 shifted 2 up);
    label.urt("$\beta$", point arctime 1/2 arclength a2 of a2 of a2);
    label.top("$\alpha$", point arctime 1/2 arclength a3 of a3 of a3);
  drawoptions();
  draw F0;
  drawoptions(withcolor Reds 6 5);
    label.ulft("$1$", point 5/2 of t);
    label.bot("$\cos\alpha\cos\beta$", point 1/2 of s);
    label.top("$\sin\alpha\sin\beta$", 1/2[point 2 of t, point 2 of s]);
    angled_label.top("$\cos\beta$", point 1/2 of t, alpha);
    angled_label.bot("$\sin\beta$", point 3/2 of t, alpha-90);
    angled_label.top("$\cos\alpha\sin\beta$", 1/2[point 1 of t, point 2 of s], -90);
    angled_label.top("$\sin\alpha\cos\beta$", 1/2[point 1 of t, point 1 of s], -90);
  drawoptions();
  label.lft("$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$", (-10, 32));
  label.lft("$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$", (-10, 16));
);
% more of the same for F2 ...
draw F1;
draw F2 shifted 280 down;
```

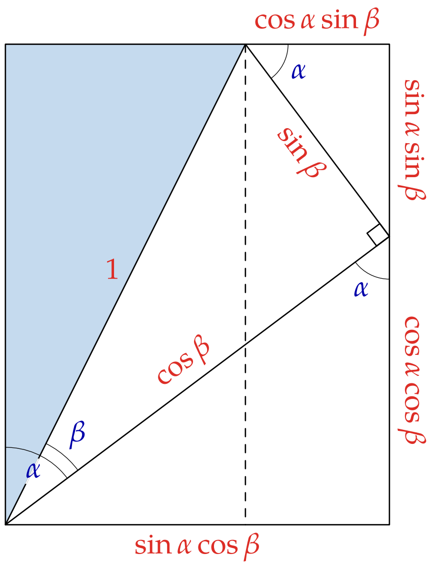
One figure, six identities



The figure



$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

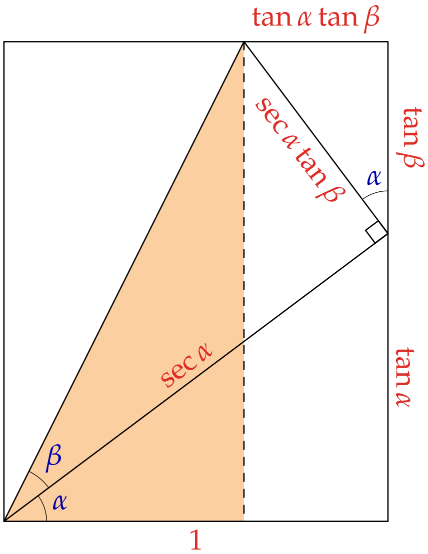


$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

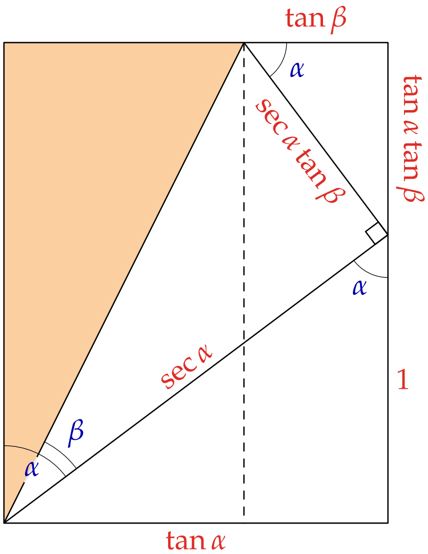
```
% very similar to previous page ...
F2 = image(
  fill t' withcolor Oranges 8 3;
  path a[];
  a1 = quartercircle scaled 58 cutbefore subpath (0,1) of t;
  a2 = quartercircle scaled 68 cutbefore subpath (0,1) of t cutafter subpath (2,3) of t;
  a3 = quartercircle scaled 32 rotated 180 shifted point 1 of t cutbefore subpath (0, 1) of t;
  a4 = quartercircle scaled 32 rotated 270 shifted point 2 of t cutbefore subpath (1, 2) of t;
  forsuffixes $=1,2,3, 4: draw a$ withpen pencircle scaled 1/4; endfor
  draw F0;
  undraw subpath (-35/256, -11/128) of t;
  drawoptions(withcolor 2/3 blue);
  label.bot("$\alpha$", point arctime 1/2 arclength a1 of a1 of a1 shifted 2.4 left);
  label.urt("$\beta$", point arctime 1/2 arclength a2 of a2 of a2) ;
  label.llft("$\alpha$", point arctime 5/8 arclength a3 of a3 of a3);
  label.lrt("$\alpha$", point arctime 5/8 arclength a4 of a4 of a4);
  drawoptions(withcolor Reds 6 5);
  label.bot("$\tan\alpha$", point 1/2 of s);
  label.top("$\tan\beta$", 1/2[point 2 of t, point 2 of s]);
  angled_label.top("$\sec\alpha$", point 1/2 of t, alpha);
  angled_label.bot("$\sec\alpha\tan\beta$", point 3/2 of t, (alpha-90));
  angled_label.top("$\tan\alpha\tan\beta$", 1/2[point 1 of t, point 2 of s], -90);
  label.rt("$1$", 1/2[point 1 of t, point 1 of s]);
  drawoptions();
  label.lft(btex $\displaystyle\tan(\alpha-\beta) =
     {\tan\alpha-\tan\beta\over1 + \tan\alpha\tan\beta}$ etex, (-10, 24));
);

draw F1;
draw F2 shifted 280 down;
```

One figure, six identities

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1+\tan\alpha\tan\beta}$$



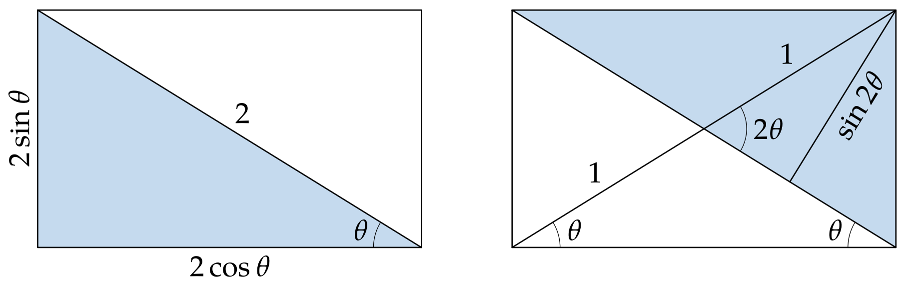
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$



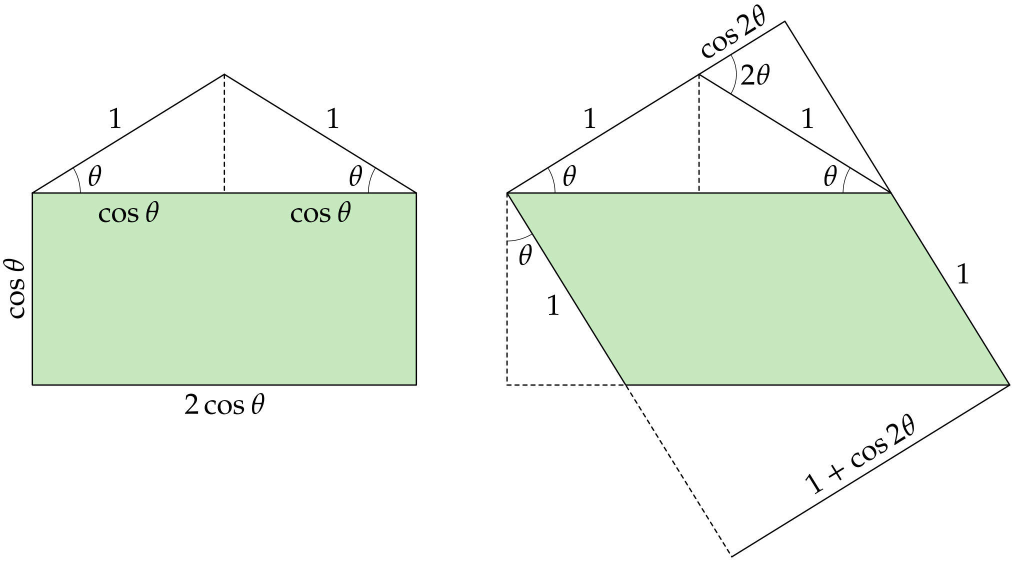
— RBN

```
vardef mark_angle(expr a, o, b, r, rr, t) =
  draw fullcircle scaled 2r rotated angle (a-o) shifted o cutafter (o--b)
      withpen pencircle scaled 1/4;
  label(t, o shifted (rr * unitvector(unitvector(a-o)+unitvector(b-o))));
enddef;
vardef rotated_label@#(expr t, p, r) =
  draw thelabel@#(t, origin) rotated r shifted p
enddef;

path R; R = unitsquare xscaled 144 yscaled 89;
z0 = center R;
picture P[];
P1 = image(
  fill subpath (-1, 1) of R -- cycle withcolor Blues 8 3;
  mark_angle(point 3 of R, point 1 of R, point 0 of R, 18, 24, "$\theta$");
  draw subpath (-1, 3) of R -- point 1 of R;
  label.urt("$2$", z0);
  rotated_label.top("$2\sin\theta$", point -1/2 of R, 90);
  label.bot("$2\cos\theta$", point  1/2 of R);
);
P2 = image(
  pair p; p = whatever[point 1 of R, point 3 of R];
  p - point 2 of R = whatever * (point 1 of R - point 3 of R) rotated 90;
  fill subpath (3, 1) of R -- cycle withcolor Blues 8 3;
  mark_angle(point 3 of R, point 1 of R, point 0 of R, 18, 24, "$\theta$");
  mark_angle(point 1 of R, point 0 of R, point 2 of R, 18, 24, "$\theta$");
  mark_angle(point 1 of R, z0, point 2 of R, 16, 24, "$2\theta$");
  draw subpath (-1, 3) of R -- point 1 of R;
  draw p -- point 2 of R -- point 0 of R;
  label.ulft("$1$", 1/4[point 0 of R, point 2 of R]);
  label.ulft("$1$", 3/4[point 0 of R, point 2 of R]);
  rotated_label.bot("$\sin2\theta$", 1/2[p, point 2 of R], angle (point 2 of R - p));
);
P3 = image(draw P1 shifted 89 left; draw P2 shifted 89 right;
  label.bot("$2\sin\theta\cos\theta = \sin 2\theta$",
  point 1/2 of bbox currentpicture shifted 21 down)
);
% P4, P5, and P6 are similar to P1, P2, and P3
);
P6 = image(draw P4 shifted 89 left; draw P5 shifted 89 right;
          label("$2\cos^2\theta = 1 + \cos 2\theta$",
          point 1/2 of bbox currentpicture shifted 21 down));
label.top(P3, 10 up); label.bot(P6, 10 down);
```

The double-angle formulas II



$$2\sin\theta\cos\theta = \sin 2\theta$$

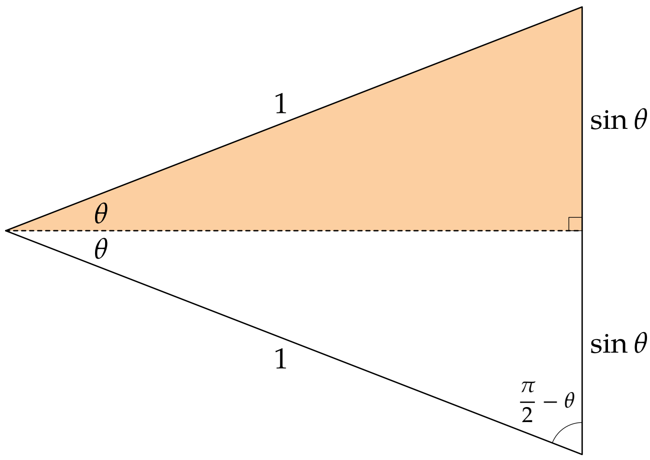


$$2\cos^2\theta = 1 + \cos 2\theta$$

— Yihnan David Gau

```
path t; t = 216 left -- 84 down -- 84 up -- cycle;
fill subpath (3/2, 3) of t -- cycle withcolor Oranges 8 3;

draw unitsquare scaled 5 rotated 90 shifted point 3/2 of t
                withpen pencircle scaled 1/4;

draw fullcircle scaled 24 rotated 90 shifted point 1 of t
                cutafter subpath(0,1) of t
                withpen pencircle scaled 1/4;

draw point 0 of t -- point 3/2 of t dashed evenly scaled 1/2;
draw t;

label.ulft("$1$", point -1/2 of t);
label.llft("$1$", point 1/2 of t);
label.rt("$\sin\theta$", point 5/4 of t);
label.rt("$\sin\theta$", point 7/4 of t);

numeric theta; theta = angle (point 2 of t - point 0 of t);
label("$\theta$", point 0 of t shifted 36 dir 1/2 theta);
label("$\theta$", point 0 of t shifted 36 dir -1/2 theta);
label(TEX("$\displaystyle {\pi\over2}-\theta$") scaled 3/4,
    point 1 of t shifted 24 dir (90 + 1/2 (90 - theta)));

label.bot(btex \vbox{\openup6pt\halign{\hfil$\displaystyle # $\hfil\cr
{\sin2\theta\over2\sin\theta} = {\sin(\pi/2 - \theta) \over 1} = \cos\theta\cr
\sin2\theta = 2\sin\theta\cos\theta\cr
\noalign{\vskip36pt}
\left(2\sin\theta\right)^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos2\theta\cr
\cos2\theta = 1 - 2\sin^2\theta \cr
}} etex, point 1/2 of bbox currentpicture shifted 21 down);
```

The double-angle formulas III (via the laws of sines and cosines)



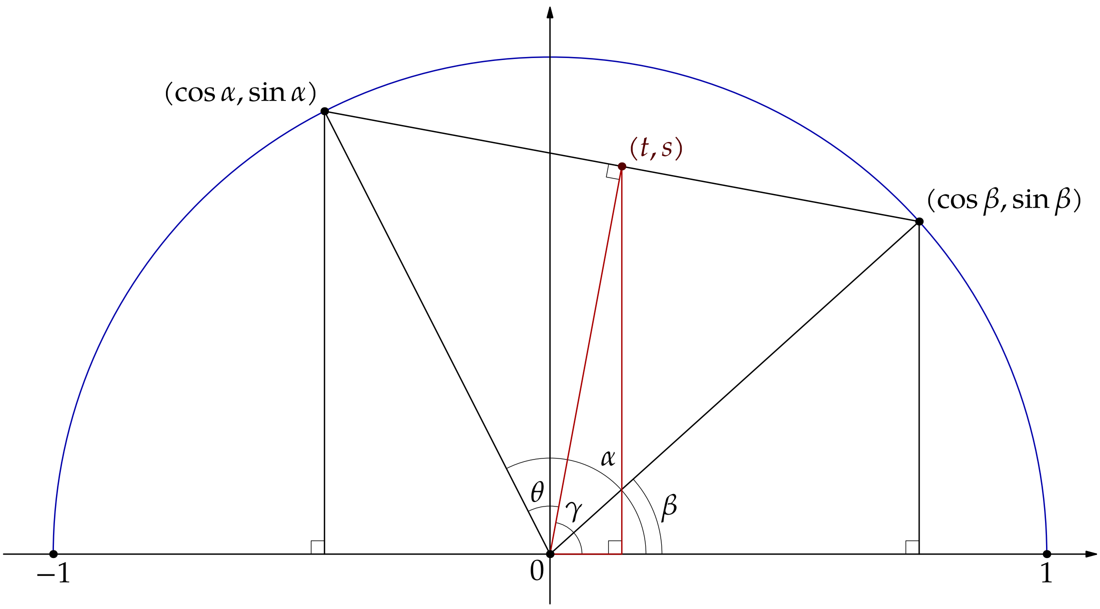
$$\frac{\sin 2\theta}{2\sin\theta} = \frac{\sin(\pi/2-\theta)}{1} = \cos\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$(2\sin\theta)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos 2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

— Sidney H. Kung

```
numeric alpha, beta, gamma, theta, u;
theta = 1/2 (alpha - beta); gamma = 1/2 (alpha + beta); alpha = 117; beta = 42;
u = \mpdim{\hsize} / 2.236;

path xx, yy, hh;
xx = (1.1 left -- 1.1 right) scaled u;
yy = (0.1 down -- 1.1 up) scaled u;
hh = halfcircle scaled 2u;

z0 = point 0 of hh rotated alpha;
z1 = point 0 of hh rotated beta;
z2 = whatever[z0, z1] = whatever * point 0 of hh rotated gamma;

drawoptions(withpen pencircle scaled 1/4);
  draw fullcircle scaled 72 cutafter (origin--z0);
  draw fullcircle scaled 84 cutafter (origin--z1);
  draw fullcircle scaled 24 cutafter (origin--z2);
  draw fullcircle scaled 36 cutafter (origin--z0) cutbefore (origin--z2);
  draw unitsquare scaled 5 rotated 90 shifted (x0, 0);
  draw unitsquare scaled 5 rotated 90 shifted (x1, 0);
  draw unitsquare scaled 5 rotated 90 shifted (x2, 0);
  draw unitsquare scaled 5 rotated angle (z0-z1) shifted z2;
drawoptions();

draw (x0,0) -- z0 -- z1 -- (x1,0);
draw origin -- z0;
draw origin -- z1;

draw hh withcolor 2/3 blue;
drawarrow xx; drawarrow yy;
draw origin -- (x2, 0) -- z2 -- cycle withcolor 2/3 red;
dotlabel.urt("$(t, s)$", z2) withcolor 1/3 red;
dotlabel.urt("$(\cos\beta, \sin\beta)$", z1);
dotlabel.ulft("$(\cos\alpha, \sin\alpha)$", z0);
dotlabel.llft("$0$", origin);
dotlabel.bot("$1$", point 0 of hh);
dotlabel.bot("$-1$", point 4 of hh);
label("$\alpha$", 42 dir 1/2 alpha);
label("$\beta$", 48 dir 1/2 beta);
label("$\gamma$", 18 dir (beta+1/2(gamma-beta)));
label("$\theta$", 24 dir (gamma + 4 + 1/2 (alpha-gamma)));
label.bot(btex \vbox{\openup 12pt\halign{\hfil #\hfil\cr
    $\displaystyle \theta = {\alpha-\beta\over2}$, \quad
    $\displaystyle \gamma = {\alpha + \beta \over 2}$\cr
    $\displaystyle {\sin\alpha+\sin\beta\over2}=s
      =\cos{\alpha-\beta\over2}\sin{\alpha+\beta\over2}$\cr
    $\displaystyle {\cos\alpha+\cos\beta\over2}=t
      =\cos{\alpha-\beta\over2}\cos{\alpha+\beta\over2}$\cr}} etex,
    point 1/2 of bbox currentpicture shifted 34 down);
```

The sum-to-product identities I



$$\theta = \frac{\alpha-\beta}{2}, \quad \gamma = \frac{\alpha+\beta}{2}$$

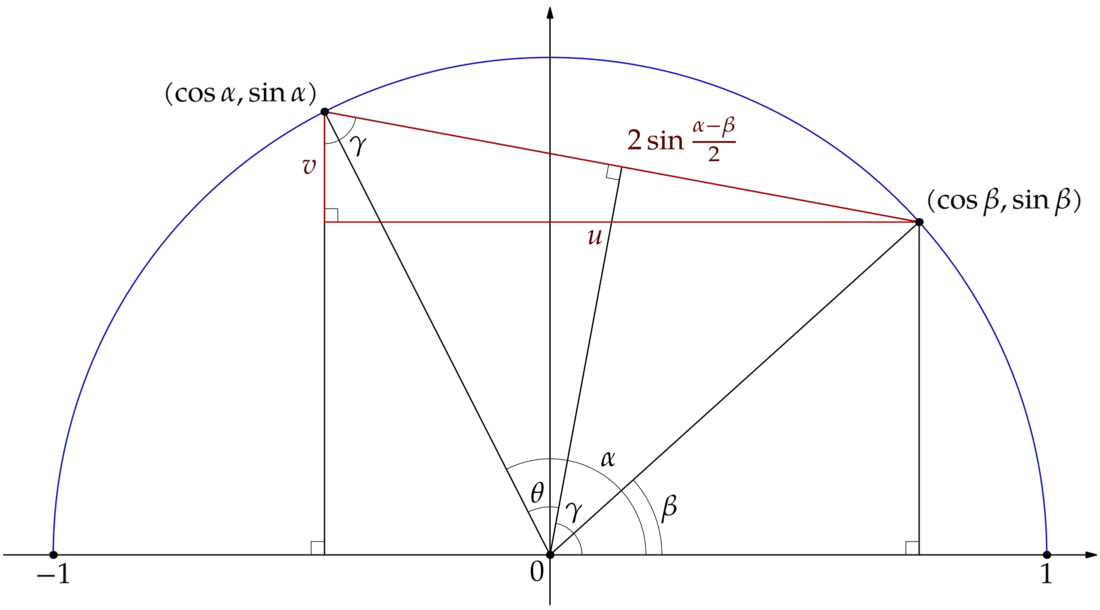
$$\frac{\sin\alpha + \sin\beta}{2} = s = \cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

$$\frac{\cos\alpha + \cos\beta}{2} = t = \cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

— Sidney H. Kung

```
numeric alpha, beta, gamma, theta, u;
theta = 1/2 (alpha - beta); gamma = 1/2 (alpha + beta); alpha = 117; beta = 42;
u = \mpdim{\hsize} / 2.236;
path xx; xx = (1.1 left -- 1.1 right) scaled u;
path yy; yy = (0.1 down -- 1.1 up) scaled u;
path hh; hh = halfcircle scaled 2u;
z0 = point 0 of hh rotated alpha;
z1 = point 0 of hh rotated beta;
z2 = whatever[z0, z1] = whatever * point 0 of hh rotated gamma;
drawoptions(withpen pencircle scaled 1/4);
  draw fullcircle scaled 72 cutafter (origin--z0);
  draw fullcircle scaled 84 cutafter (origin--z1);
  draw fullcircle scaled 24 cutafter (origin--z2);
  draw fullcircle scaled 36 cutafter (origin--z0) cutbefore (origin -- z2);
  draw quartercircle scaled 24 rotated -90 shifted z0 cutafter (z0--z1);
  draw unitsquare scaled 5 shifted (x0, y1);
  draw unitsquare scaled 5 rotated 90 shifted (x0, 0);
  draw unitsquare scaled 5 rotated 90 shifted (x1, 0);
  draw unitsquare scaled 5 rotated angle (z0-z1) shifted z2;
drawoptions();
draw (x0,0) -- z0 -- z1 -- (x1,0);
draw origin -- z0; draw origin -- z1; draw origin -- z2;
draw hh withcolor 2/3 blue;
drawarrow xx; drawarrow yy;
draw (x0, y1) -- z1 -- z0 -- cycle withcolor 2/3 red;
dotlabel.urt("$(\cos\beta, \sin\beta)$", z1);
dotlabel.ulft("$(\cos\alpha, \sin\alpha)$", z0);
label.lft("$v$", (x0, 1/2 y0 + 1/2 y1)) withcolor 1/3 red;
label.bot("$u$", (1/2 x0 + 1/2 x1 - 10 , y1)) withcolor 1/3 red;
label.urt("$2\sin{\alpha-\beta\over2}$", 1/2[z0, z1]) withcolor 1/3 red;
dotlabel.llft("$0$", origin);
dotlabel.bot("$1$", point 0 of hh);
dotlabel.bot("$-1$", point 4 of hh);
label("$\alpha$", 42 dir 1/2 alpha);
label("$\beta$", 48 dir 1/2 beta);
label("$\gamma$", 18 dir (beta+1/2(gamma-beta)));
label("$\theta$", 24 dir (gamma + 4 + 1/2 (alpha-gamma)));
label("$\gamma$", z0 + 18 dir (1/2 gamma-86));
```

The difference-to-product identities I



$$\theta = \frac{\alpha-\beta}{2}, \quad \gamma = \frac{\alpha+\beta}{2}$$

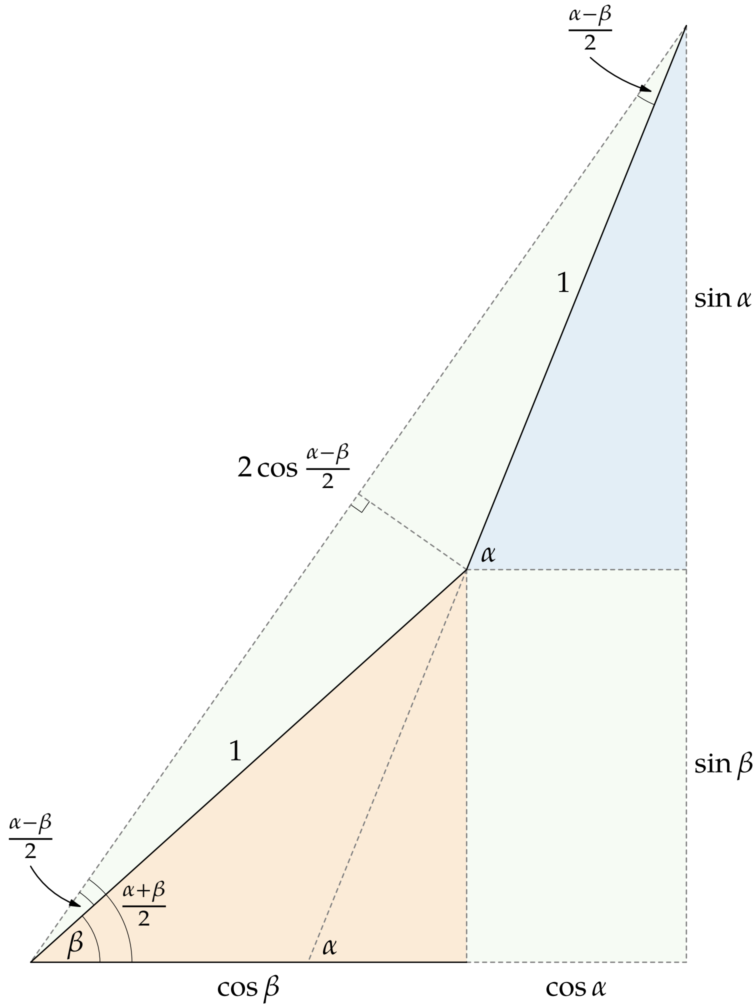
$$\sin\alpha - \sin\beta = v = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\beta - \cos\alpha = u = 2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

— Sidney H. Kung

```
numeric u, alpha, beta; u = 220; alpha = 68; beta = 42;
z1 = right rotated alpha scaled u;
z2 = right rotated beta scaled -u;
z3 = whatever * z1; y3 = y2;
z4 = whatever[z1, z2]; z4 rotated 90 = whatever * (z1-z2);

fill z1 -- z2 -- (x1, y2) -- cycle withcolor Greens 9 1;
fill origin -- z1 -- (x1, 0) -- cycle withcolor 3/4[Greens 9 1, Blues 9 2];
fill origin -- z2 -- (0, y2) -- cycle withcolor 3/4[Greens 9 1, Oranges 9 2];

drawoptions(withpen pencircle scaled 1/4);
draw subpath (1,3) of unitsquare scaled 5 rotated (180 + 1/2 alpha + 1/2 beta) shifted z4;
draw quartercircle scaled 52 shifted z2 cutafter (origin -- z2);
draw quartercircle scaled 76 shifted z2 cutafter (z1 -- z2);

drawoptions(dashed evenly scaled 1/2 withcolor 1/2 white);
draw z1--z2;
draw (0,y2) -- (x1,y2) -- z1;
draw origin -- z3;
draw origin -- z4;
draw origin -- (0, y2);
draw origin -- (x1, 0);
drawoptions();
draw (0,y2) -- z2 -- origin -- z1;

label("$\alpha$", 10 dir 1/2 alpha);
label("$\alpha$", 10 dir 1/2 alpha shifted z3);
label("$\beta$", 18 dir 1/2 beta shifted z2);
label("$\alpha+\beta\over2$", 48 dir 1/4 (alpha + beta) shifted z2);
label.ulft("$2\cos{\alpha-\beta\over2}$", z4);
label.bot("\strut$\cos\alpha$", (1/2 x1, y2));
label.bot("\strut$\cos\beta$", (1/2 x2, y2));
label.rt("$\sin\alpha$", (x1, 1/2 y1));
label.rt("$\sin\beta$", (x1, 1/2 y2));
label.ulft("$1$", 1/2 z1);
label.ulft("$1$", 1/2 z2);

draw quartercircle scaled 64 rotated (180 + 1/2 alpha + 1/2 beta) shifted z1
     cutafter (origin -- z1) withpen pencircle scaled 1/4;
draw quartercircle scaled 64 rotated beta shifted z2
     cutafter (z1 -- z2) withpen pencircle scaled 1/4;

z8 = z1 shifted 36 left shifted 12 down;
z9 = z2 shifted 36 up;

label.top("$\alpha-\beta\over2$", z8);
label.top("$\alpha-\beta\over2$", z9);

drawarrow z8 {dir -42} .. 28 dir (180 + 1/4 beta + 3/4 alpha) shifted z1;
drawarrow z9 {dir -60} .. 28 dir (3/4 beta + 1/4 alpha) shifted z2;
```

The sum-to-product identities II



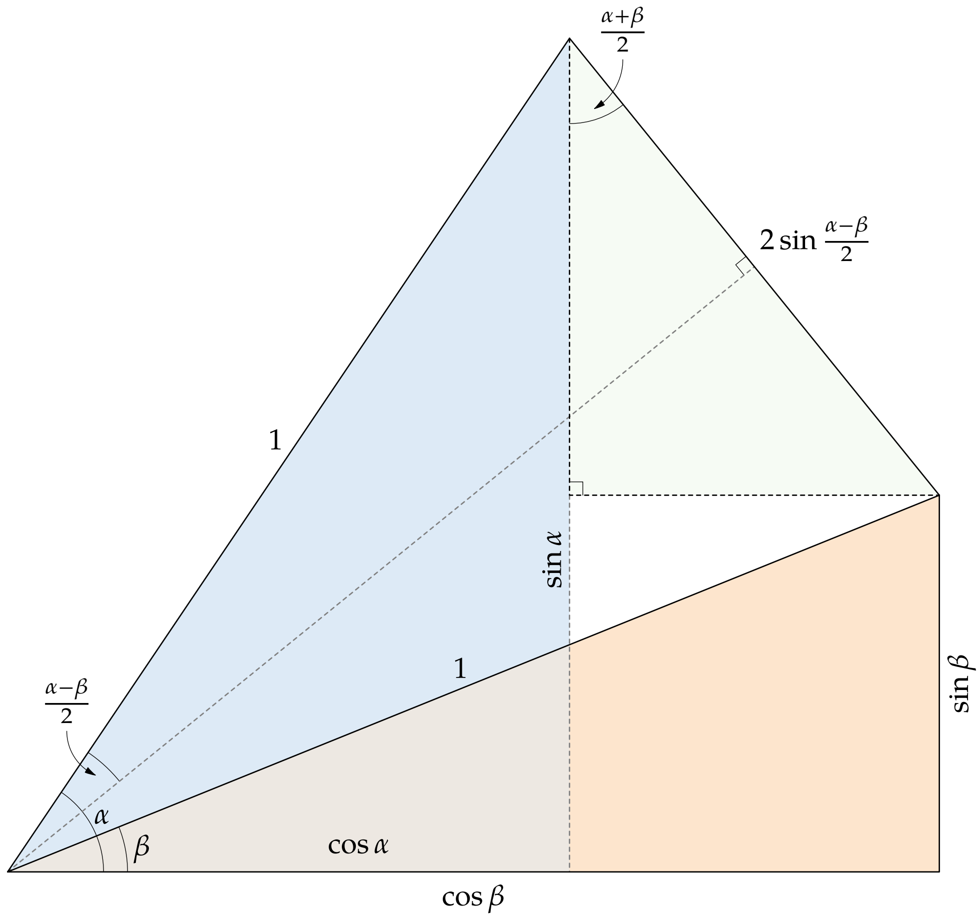
$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha-\beta}{2}\cos\tfrac{\alpha+\beta}{2}$$

$$\sin\alpha + \sin\beta = 2\cos\tfrac{\alpha-\beta}{2}\sin\tfrac{\alpha+\beta}{2}$$

— Yukio Kobayashi

```
numeric u, alpha, beta; u = 377; alpha = 56; beta = 22;
z1 = right rotated alpha scaled u;
z2 = right rotated beta scaled u;
z3 = (x1, y2);
z4 = (x1, 0);
z5 = (x2, 0);
z6 = 1/2[z1, z2];
z7 = whatever[origin, z2] = whatever[z1, z4];

fill origin -- z1 -- z4 -- cycle withcolor Blues 9 2;
fill origin -- z2 -- z5 -- cycle withcolor Oranges 9 2;
fill origin -- z7 -- z4 -- cycle withcolor 1/2[Blues 9 2, Oranges 9 2];
fill z1 -- z2 -- z3 -- cycle withcolor Greens 9 1;

drawoptions(withpen pencircle scaled 1/4);
draw subpath (1,3) of unitsquare scaled 5 shifted z3;
draw subpath (1,3) of unitsquare scaled 5 rotated angle (z1-z2) shifted z6;
path a[];
a1 = quartercircle scaled 72 cutafter (origin -- z1); draw a1;
a2 = quartercircle scaled 90 cutafter (origin -- z2); draw a2;
a3 = quartercircle scaled 108 rotated angle z6 cutafter (origin -- z1); draw a3;
a4 = quartercircle scaled 64 rotated -90 shifted z1 cutafter (z1 -- z2); draw a4;
label.rt("$\alpha$", point arctime 5/8 arclength a1 of a1 of a1);
label.rt("$\beta$", point arctime 1/2 arclength a2 of a2 of a2);

z8 = point infinity of a3 shifted (-8, 8);  label.top("$\alpha-\beta\over2$", z8);
z9 = z1                   shifted (20, -8); label.top("$\alpha+\beta\over2$", z9);
drawarrow z8 {down} .. 49 dir (alpha - 3/8 beta);
drawarrow z9 {down} .. 28 dir (1/4 alpha + 1/4 beta - 90) shifted z1;

drawoptions(dashed evenly scaled 1/2 withcolor 1/2 white);
draw z3 -- z4; draw origin -- z6;
drawoptions(dashed evenly scaled 1/2);
draw z1 -- z3 -- z2;
drawoptions();
draw origin -- z1 -- z2 -- z5 -- origin -- z2;

label.urt("$2\sin{\alpha-\beta\over2}$", z6);
label.top("\strut$\cos\alpha$", 5/8 z4);
label.bot("\strut$\cos\beta$", 1/2 z5);
draw thelabel.top("$\sin\alpha$", origin) rotated 90 shifted 3/8[z4, z1];
draw thelabel.bot("$\sin\beta$", origin) rotated 90 shifted 1/2[z5, z2];
label.ulft("$1$", 1/2 z1);
label.ulft("$1$", 1/2 z2);
```

The difference-to-product identities II



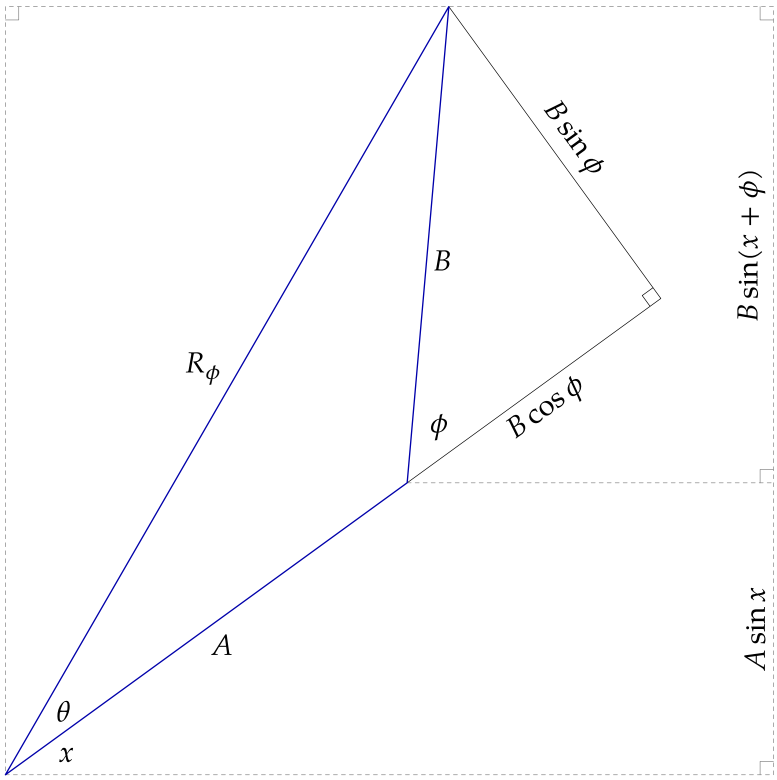
$$\cos\beta - \cos\alpha = 2\sin\tfrac{\alpha-\beta}{2}\sin\tfrac{\alpha+\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\tfrac{\alpha-\beta}{2}\cos\tfrac{\alpha+\beta}{2}$$

— Yukio Kobayashi

```
numeric x, theta, phi, s;
x = 36; theta = 24; phi = 49;
s = 4 in;

path box; box = unitsquare scaled s;
z0 = point 0 of box;
z1 - z0 = whatever * dir (x+theta); y1 = ypart point 2 of box;
z2 - z0 = whatever * dir x;
z1 - z2 = whatever * dir (x + phi);
z3 = whatever [z0, z2]; z1-z3 = whatever * (z2-z0) rotated 90;
x4 = xpart point 1 of box; y4 = y2;

path ra_mark;
ra_mark = subpath (1, 3) of unitsquare scaled 5;

drawoptions(withpen pencircle scaled 1/4 withcolor 1/2 white);
draw ra_mark rotated 90 shifted point 1 of box;
draw ra_mark rotated 90 shifted z4;
draw ra_mark rotated 180 shifted point 2 of box;
draw ra_mark rotated 270 shifted point 3 of box;
draw box dashed evenly scaled 1/2;
draw z2--z4 dashed evenly scaled 1/2;

drawoptions(withpen pencircle scaled 1/4);
draw ra_mark rotated (90+x) shifted z3;
draw z1 -- z3 -- z2;

drawoptions(withcolor 2/3 blue);
draw z0 -- z1 -- z2 -- cycle;

drawoptions();

label("$x$", 24 dir 1/2 x shifted z0);
label("$\theta$", 32 dir (x + 1/2 theta) shifted z0);
label("$\phi$", 24 dir (x + 1/2 phi) shifted z2);

label.ulft("$R_\phi$", 1/2[z0, z1]);
label.lrt("$A$", 1/2[z0, z2]);
label.lrt("$B$", 1/2[z1, z2]);

draw thelabel.top("$B\sin\phi$", origin) rotated angle (z3 - z1) shifted 1/2[z1, z3];
draw thelabel.bot("$B\cos\phi$", origin) rotated angle (z3 - z2) shifted 1/2[z2, z3];
draw thelabel.top("$A\sin x$", origin) rotated 90 shifted 1/2[point 1 of box, z4];
draw thelabel.top("$B\sin (x+\phi)$", origin) rotated 90 shifted 1/2[point 2 of box, z4];
```

Adding like sines



$$R_\phi = \sqrt{A^2 + B^2 + 2AB\cos\phi}, \quad \tan\theta = \frac{B\sin\phi}{A+B\cos\phi}$$

$$A\sin x + B\sin(x+\phi) = R_\phi \sin(x+\theta)$$

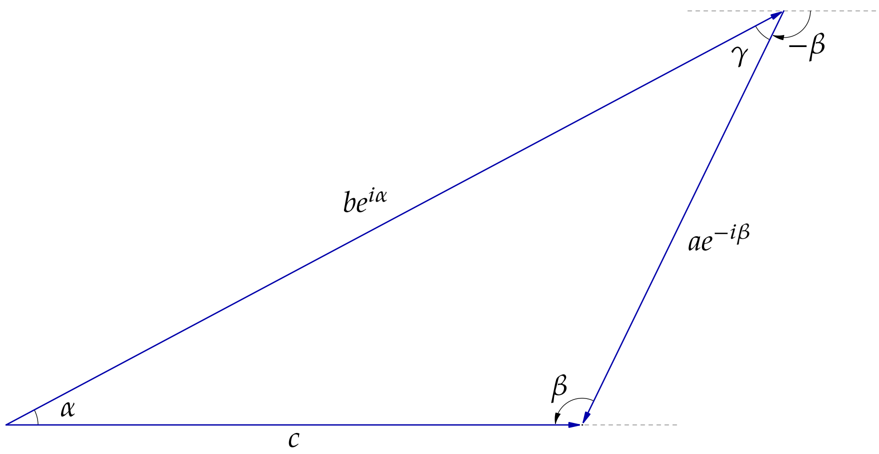
$$\phi = \pi/2 \Rightarrow \tan\theta = B/A$$

$$\therefore \quad A\sin x + B\cos x = \sqrt{A^2+B^2}\sin(x+\theta)$$

— Rick Mabry and Paul Deiermann

```
numeric a, b, c, alpha, beta, gamma;
alpha = 28;
gamma = 36;
beta = 180 - alpha - gamma;
z0 = origin;
z1 = 216 right;
z2 = whatever * dir alpha = whatever * dir (alpha + gamma) shifted z1;

drawoptions(dashed evenly scaled 1/2 withpen pencircle scaled 1/4 withcolor 1/2 white);
draw (right -- 36 right) shifted z1;
draw (left -- right) scaled 36 shifted z2;

drawoptions(withpen pencircle scaled 1/4);
draw quartercircle scaled 24 shifted z0 cutafter (z0--z2);
draw quartercircle scaled 24 rotated (180+alpha) shifted z2 cutafter (z1--z2);
interim ahangle := 25;
drawarrow subpath (8, 8 - beta/45 + 1/16) of fullcircle scaled 20 shifted z2;
drawarrow subpath (4 - beta/45, 4 - 1/16) of fullcircle scaled 20 shifted z1;

drawoptions(withcolor 2/3 blue);
drawarrow z0 -- z1 cutafter fullcircle scaled 2 shifted z1;
drawarrow z0 -- z2 cutafter fullcircle scaled 2 shifted z2;
drawarrow z2 -- z1 cutafter fullcircle scaled 2 shifted z1;
drawoptions();
drawdot z1;

label("$\alpha$", 24 dir 1/2 alpha shifted z0);
label("$-\beta$", 16 dir -1/2 beta shifted z2);
label("$\beta$",  16 dir (alpha + gamma + 1/2 beta) shifted z1);
label("$\gamma$", 24 dir (180 + alpha + 1/2 gamma) shifted z2);

label.bot("$c$", 1/2[z0, z1]);
label.ulft("$be^{i\alpha}$", 1/2[z0, z2]);
label.lrt("$ae^{-i\beta}$", 1/2[z1, z2]);

label(btex $c=be^{i\alpha} + ae^{-i\beta}
             =(b\cos\alpha + a\cos\beta) + i(b\sin\alpha - a\sin\beta)$ etex,
    point 1/2 of bbox currentpicture shifted 36 down);

label(btex if $c$ is real, then $b\sin\alpha-a\sin\beta=0$,
      hence $\displaystyle {a\over\sin\alpha}={b\over\sin\beta}$ etex,
    point 1/2 of bbox currentpicture shifted 36 down);

label(btex \vbox{\openup 6pt\halign{\hfil $##$&${}=#$\hfil\cr
c^2 & \left|c^2\right| = \left(b\cos\alpha + a\cos\beta\right)^2 +
                         \left(b\sin\alpha - a\sin\beta\right)^2\cr
    & a^2 + b^2 + 2ab\cos(\alpha + \beta)\cr
    & a^2 + b^2 - 2ab\cos\gamma\cr}} etex,
    point 1/2 of bbox currentpicture shifted 42 down);
```

A complex approach to the laws of sines and cosines



$$c = be^{i\alpha} + ae^{-i\beta} = (b\cos\alpha + a\cos\beta) + i(b\sin\alpha - a\sin\beta)$$

if c is real, then $b\sin\alpha - a\sin\beta = 0$, hence $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

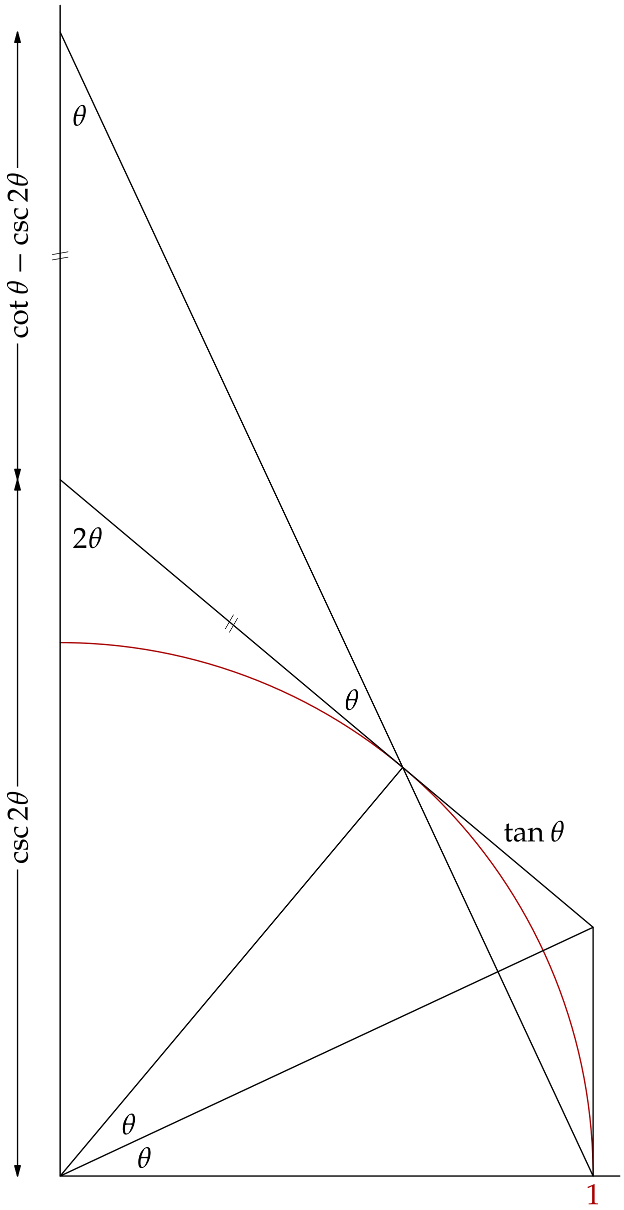
$$\begin{aligned} c^2 = \left|c^2\right| &= (b\cos\alpha + a\cos\beta)^2 + (b\sin\alpha - a\sin\beta)^2 \\ &= a^2 + b^2 + 2ab\cos(\alpha+\beta) \\ &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}$$

— William V. Grounds

```
numeric theta;
theta = 25;
z0 = origin;
z1 = 200 right;
z2 = z1 rotated 2 theta;
z3 - z2 = whatever * (z2-z0) rotated 90; x3 = x1;
z4 = whatever[z2, z3];
z5 = whatever[z1, z2];
x5 = x4 = x0;

draw quartercircle scaled 2 abs(z1) shifted z0 withcolor 2/3 red;
draw z1 -- z5;
draw z1 -- z3 -- z4;
draw z0 -- z2;
draw z0 -- z3;
draw z1 shifted 10 right -- z0 -- z5 shifted 10 up;

label.bot("$1$", z1) withcolor 2/3 red;

label("$\theta$", 32 dir 1/2 theta);
label("$\theta$", 32 dir 3/2 theta);
label("$\theta$", 32 dir (1/2 theta - 90) shifted z5);
label("$\theta$", 32 dir (3/2 theta + 90) shifted z2);
label("$2\theta$", 24 dir (theta - 90) shifted z4);

label.urt("$\tan\theta$", 1/2[z2, z3]);
picture mark; mark = image(for i=-1,1: draw (up--down) scaled 3 rotated 10 shifted (i,0)
    withpen pencircle scaled 1/4; endfor);
draw mark rotated angle (z2-z4) shifted 1/2[z2, z4];
draw mark rotated angle (z4-z5) shifted 1/2[z4, z5];

path a[];
a1 = (z0 -- z4) shifted 16 left;
a2 = (z4 -- z5) shifted 16 left;

picture t[];
t1 = thelabel(TEX("$\csc 2\theta$") rotated 90, point 1/2 of a1);
t2 = thelabel(TEX("$\cot\theta - \csc 2\theta$") rotated 90, point 1/2 of a2);

forsuffixes $=1,2:
    drawdblarrow a$; unfill bbox t$; draw t$;
endfor

label.top("$2\csc 2\theta = \tan\theta + \cot\theta$", point 5/2 of bbox currentpicture);
```

Eisenstein's duplication forumula

$$2\csc 2\theta = \tan\theta + \cot\theta$$

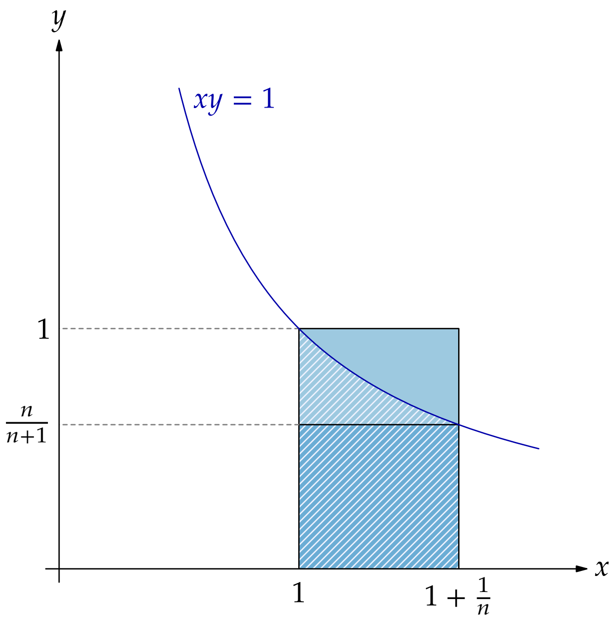


G. Eisenstein, *Mathematische Werke*, Chelsea, NY. 1975, p.411

```
numeric u; u = 90;

path xx, yy, ff;
xx = 5 left -- 2.2 u * right;
yy = xx rotated 90;

% f(x) = 1/x, f' = -1/x^2
ff = ((1/2, 2){1, -4} .. (1, 1){1, -1} .. (3/2, 2/3){1, -4/9} .. (2, 1/2){1, -1/4}) scaled u;

numeric n; n = 3/2;
z1 = (1, n/(n+1)) scaled u;
z2 = ((n+1)/n, 1) scaled u;

fill z1 -- (x2, y1) -- z2 -- (x1, y2) -- cycle withcolor Blues 7 3;
fill z1 -- (x2, y1) -- (x2, 0) -- (x1, 0) -- cycle withcolor Blues 7 4;
input thatch
thatch_space := 2;
rule buildcycle(xx, yy shifted (x1, 0), ff, yy shifted (x2, 0)) withcolor Blues 7 1;

draw (x1, 0) -- (x1, y2) -- z2 -- (x2, 0);
draw z1 -- (x2, y1);
draw z1 -- (0, y1) dashed evenly scaled 1/2 withcolor 1/2 white;
draw (x1,y2) -- (0, y2) dashed evenly scaled 1/2 withcolor 1/2 white;

draw ff withcolor 2/3 blue; label.urt("$xy=1$", point 1/8 of ff) withcolor 2/3 blue;

drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);


label.bot("$\strut 1$", (x1, 0));
label.bot("$\strut 1 + {1\over n}$", (x2, 0));

label.lft("$1$", (0, y2));
label.lft("$n\over n+1$", (0, y1));

label.top("$\displaystyle \lim_{n\to\infty} \left(1+{1\over n}\right)^n = e$",
    point 5/2 of bbox currentpicture shifted 12 up);

label.bot(btex \vbox{\openup 12pt\halign{\hfil $\displaystyle #$ \hfil\cr
    {1\over n}\cdot{n\over n+1} \le \ln\left(1+{1\over n}\right) \le {1\over n}\cdot 1\cr
    {n\over n+1} \le n \cdot \ln\left(1+{1\over n}\right) \le 1\cr
    \therefore \quad \lim_{n\to\infty} \ln\left(\left(1+{1\over n}\right)^n\right) = 1\cr
    }} etex, point 1/2 of bbox currentpicture shifted 36 down);
```

A familiar limit for e

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$



$$\frac{1}{n}\cdot\frac{n}{n+1} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}\cdot 1$$

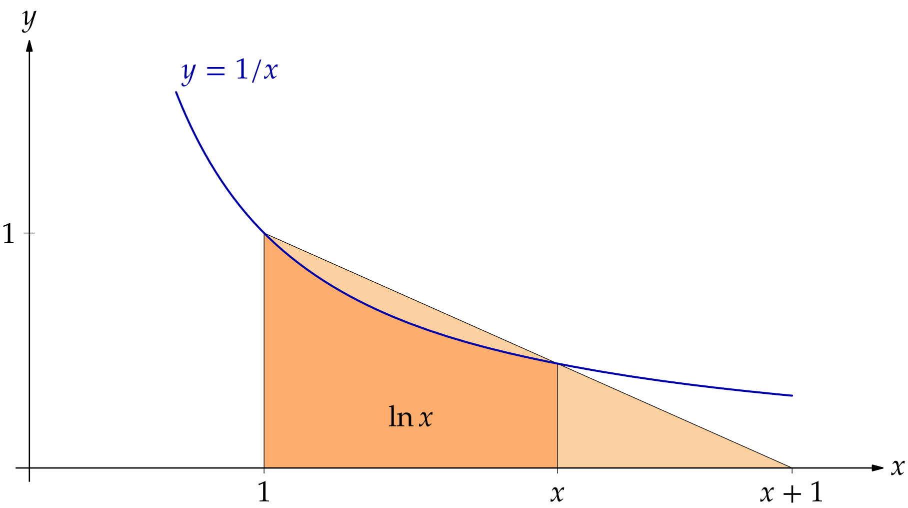
$$\frac{n}{n+1} \le n\cdot\ln\left(1+\frac{1}{n}\right) \le 1$$

$$\therefore \quad \lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^n\right) = 1$$

```
path xx, yy, ff;
xx = 5 left -- 320 right;
yy = 5 down -- 160 up;

% f=1/x f'= -1/x^2
numeric u; u = 88;
ff = ((5/8, 8/5){25, -64} .. (1, 1){1, -1}
   .. (9/4, 4/9){81, -16} .. (13/4, 4/13){169, -16}) scaled u;

z1 = point 1 of ff;
z2 = point 2 of ff;
z3 = whatever [z1, z2]; y3 = 0;

path ln, trig;
ln = (x1, 0) -- (x2, 0) -- subpath(2, 1) of ff -- cycle;
trig = (x1, 0) -- z3 -- z1 -- cycle;

fill trig withcolor Oranges 7 2;
fill ln withcolor Oranges 7 3;


drawoptions(withpen pencircle scaled 1/4);
draw (x1, 0) -- z1 -- z3;
draw (x2, 0) -- z2;
draw (left--right) scaled 2 shifted (0, u); label.lft("$1$", (-2, u));
draw (2 down -- origin) shifted (x1, 0); label.bot("\strut$1$", (x1, 0));
draw (2 down -- origin) shifted (x2, 0); label.bot("\strut$x$", (x2, 0));
draw (2 down -- origin) shifted (x3, 0); label.bot("\strut$x+1$", (x3, 0));

drawoptions(withpen pencircle scaled 3/4 withcolor 2/3 blue);
draw ff; label.urt("$y=1/x$", point 0 of ff);

drawoptions();
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);

label("$\ln x$", 1/2[(x1,0), z2]);
label("$\displaystyle \lim_{x\to\infty} {x\over e^x} = 0$",
    point 5/2 of bbox currentpicture shifted 72 up);
label(btex \vbox{\openup 8pt\halign{\hfil $\displaystyle # $ \hfil\cr
    \ln x < {1\over2}x \cr
    \therefore\: \lim_{x\to\infty} {x \over e^x} =
                 \lim_{x\to\infty} {1 \over e^{x-\ln x}} = 0 \cr
    }} etex, point 1/2 of bbox currentpicture shifted 36 down);
```

A common limit

$$\lim_{x\to\infty} \frac{x}{e^x} = 0$$



$$\ln x < \frac{1}{2}x$$

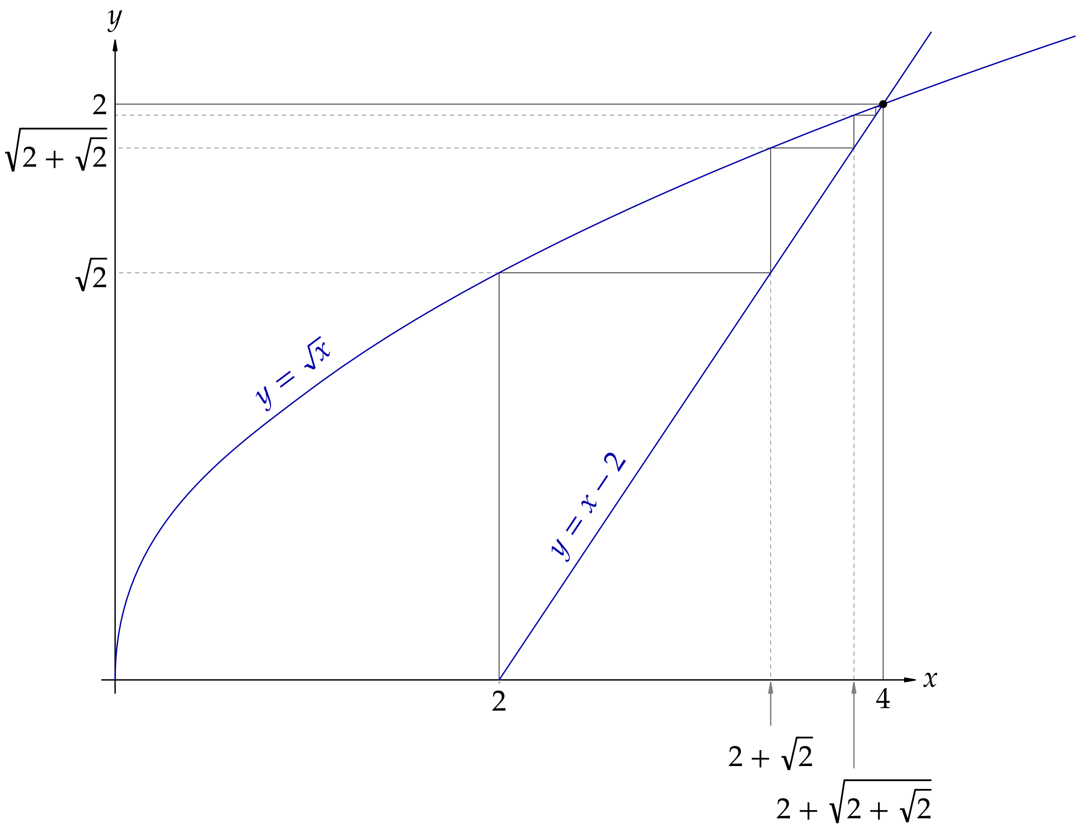
$$\therefore \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^{x-\ln x}} = 0$$

— Alan H. Stein and Dennis McGavran

```
numeric u, v; u = 72; v = 3/2 u;
path xx; xx = 5 left -- 300 right;
path yy; yy = 5 down -- 240 up;
path ff; ff = (origin {up} for x=1 upto 5: .. (x,sqrt(x)){1, 1/(2sqrt(x))} endfor)
         xscaled u yscaled v;
path ll; ll = ((2,0) -- (4.25,2.25)) xscaled u yscaled v;

interim ahangle := 20;

numeric x, y; x = 2; y = 0;
string xlabel, ylabel; xlabel = "2"; ylabel = "\sqrt{2}";
draw ((x, y)
    for i=1 upto 10:
      hide(if odd i:
        if i < 6:
          drawarrow (x*u, 7-8i) -- (x*u, -1) withpen pencircle scaled 1/2 withcolor 1/2;
          label.bot("$" & xlabel & "$", (x*u, 7-8i));
          xlabel := "2 + \sqrt{" & xlabel & "}";
        fi y := sqrt(x);
      else:
        if i < 6:
          label.lft("$" & ylabel & "$", (0, y*v));
          ylabel := "\sqrt{2+" & ylabel & "}";
        fi x := 2 + y;
      fi) -- (x,y)
      hide(if i < 6:
        draw ((x,y) -- if odd i: (0,y) else: (x,0) fi)
        xscaled u yscaled v
        dashed evenly scaled 1/2
        withpen pencircle scaled 1/4
        withcolor 1/2 white;
      fi)
    endfor) xscaled u yscaled v withpen pencircle scaled 1/4;
draw((0,2v)--(4u,2v)--(4u,0)) withpen pencircle scaled 1/4;
label.lft("$2$", (0, 2v));
label.bot("$4$", (4u, 0));

drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);
drawoptions(withcolor 2/3 blue);
draw ff; picture t; t = thelabel.top("$y=\sqrt x$", origin);
    draw t rotated angle direction 1 of ff shifted point 1 of ff;
draw ll; picture t; t = thelabel.top("$y=x-2$", origin);
    draw t rotated angle direction 1/4 of ll shifted point 1/4 of ll;
drawoptions();
fill fullcircle scaled dotlabeldiam shifted (4u, 2v);

label("$\displaystyle \sqrt{2+\sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}} = 2$",
    point 5/2 of bbox currentpicture shifted 42 up);
```

Geometric evaluation of a limit

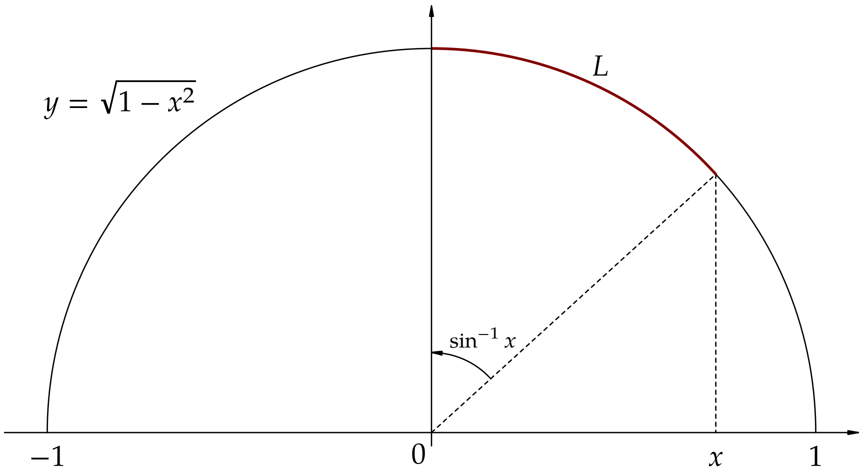
$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}}=2$$



— Guanshen Ren

```
interim ahangle := 20;
path xx, yy, hh, arc;
hh = halfcircle scaled 288;
xx = (left--right) scaled 160;
yy = 5 down -- 160 up;
numeric t; t = 0.94;
arc = quartercircle scaled 60 cutbefore (origin -- point t of hh);

drawarrow arc;
draw origin -- point t of hh -- (xpart point t of hh, 0) dashed evenly scaled 1/2;
draw hh;  draw subpath (t, 2) of hh withpen pencircle scaled 1 withcolor 1/2 red;
drawarrow xx;
drawarrow yy;


label.bot("\strut $1$", point 0 of hh);
label.bot("\strut $x$", (xpart point t of hh, 0));
label.bot("\strut $-1$", point 4 of hh);
label.llft("\strut $0$", origin);

label.urt("$L$", point 1+1/2t of hh);
label.ulft("$y=\sqrt{1-x^2}$", point 2.818 of hh);

label.urt("$\scriptstyle \sin^{-1}x$", point 1.8 of arc);


label.bot(btex \vbox{\openup12pt\halign{\hfil $\displaystyle #$\hfil\cr
L = \sin^{-1} x = \int_0^x {1\over \sqrt{1-t^2}}\,dt\cr
\therefore\quad {d\over dx}\sin^{-1}x = {1\over\sqrt{1-x^2}}\cr}} etex,
point 1/2 of bbox currentpicture shifted 36 down);
```

The derivative of the inverse sine



$$L = \sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt$$

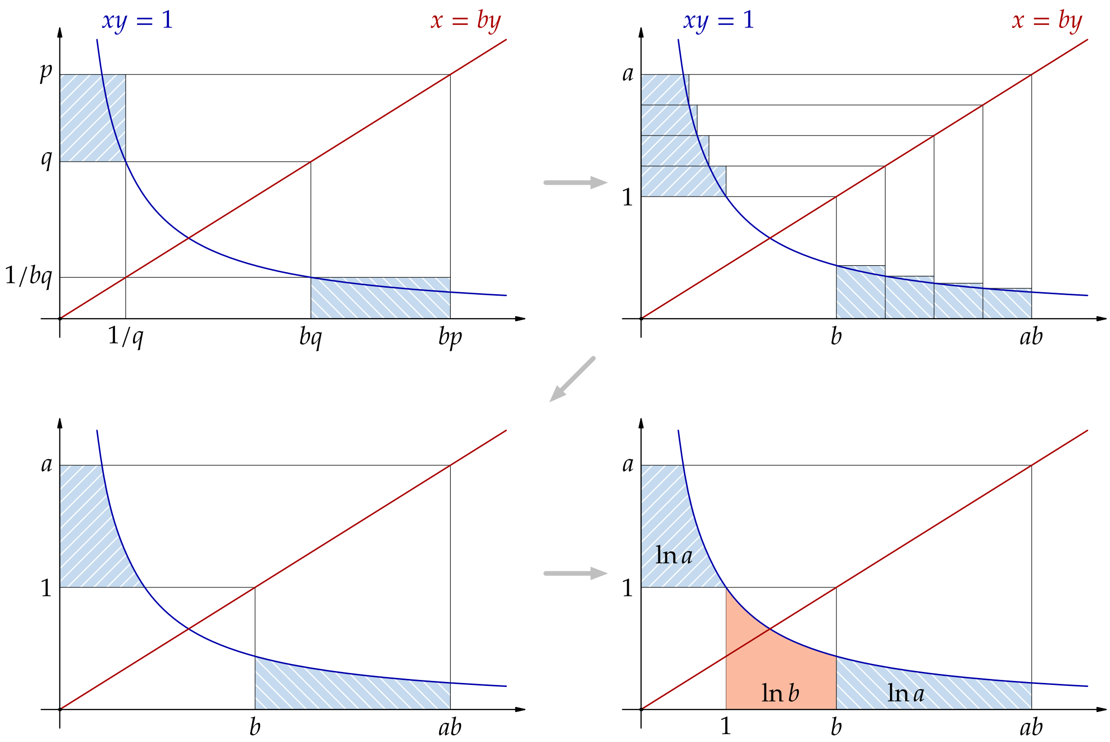
$$\therefore \quad \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

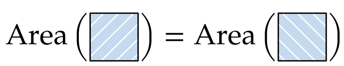
— Craig Johnson

```
interim ahangle := 25;
path xx, yy; xx = 8 left -- 200 right; yy = 8 down -- 125 up;
% f=1/x f'=-1/x^2
path ff; ff = ((1/4, 4){1, -16} .. (1/2, 2){1, -4} .. (1, 1){1, -1} .. (3/2, 2/3){9, -4}
         .. (2, 1/2){4, -1} .. (5/2, 2/5){25, -4} .. (3, 1/3){9, -1}) xscaled 64 yscaled 30;
path ll; ll = origin--(xpart point infinity of ff, ypart point 0 of ff); z1=point 7/8 of ll;
picture p[]; p0 = image(
    draw (x1,0) -- z1 -- (0, y1) withpen pencircle scaled 1/4;
    draw ff withpen pencircle scaled 5/8 withcolor 2/3 blue;
    draw ll withpen pencircle scaled 5/8 withcolor 2/3 red;
    drawarrow xx; drawarrow yy;
    draw origin withpen pencircle scaled 1/2 dotlabeldiam;
);
input thatch
p1 = image(
    z11 = point 9/16 of ll;
    z12 = ff intersectionpoint (xx shifted (0, y11));
    z13 = ff intersectionpoint (yy shifted (x11, 0));
    path a; a = (0, y12) -- z12 -- (x12, y1) -- (0, y1) -- cycle;
    path b; b = (x13, 0) -- (x1, 0) -- (x1, y13) -- z13 -- cycle;
    fill a withcolor Blues 8 3; thatch_angle :=  45; rule a withcolor white;
    fill b withcolor Blues 8 3; thatch_angle := -45; rule b withcolor white;
    draw (x11, 0) -- z11 -- (0, y11) withpen pencircle scaled 1/4;
    draw (x12, 0) -- (x12, y1) withpen pencircle scaled 1/4;
    draw (0, y13) -- (x1, y13) withpen pencircle scaled 1/4;
    draw p0;
    label.ulft("$x=by$", point infinity of ll) withcolor 2/3 red;
    label.urt("$xy=1$", point 0 of ff) withcolor 2/3 blue;
    label.lft("$p$", (0, y1)); label.bot("$bp$", (x1, 0));
    label.lft("$q$", (0, y11)); label.bot("$bq$", (x11, 0));
    label.lft("$1/bq$", (0, y13)); label.bot("$1/q$", (x12, 0));
);
% ... similar for p2, p3, p4
draw p1 shifted (-125, +84) shifted - center bbox p0;
draw p2 shifted (+125, +84) shifted - center bbox p0;
draw p3 shifted (-125, -84) shifted - center bbox p0;
draw p4 shifted (+125, -84) shifted - center bbox p0;

drawoptions(withpen pencircle scaled 2 withcolor 3/4 white);
interim ahangle := 42;
interim linejoin := mitered;
drawarrow (left--right) scaled 12 shifted 84 up;
drawarrow (left--right) scaled 12 rotated 225;
drawarrow (left--right) scaled 12 shifted 84 down;
drawoptions();

label("$\ln ab = \ln a + \ln b$", point 5/2 of bbox currentpicture shifted 42 up);
% See source for how the legend is done
```

The logarithm of a product

$$\ln ab = \ln a + \ln b$$



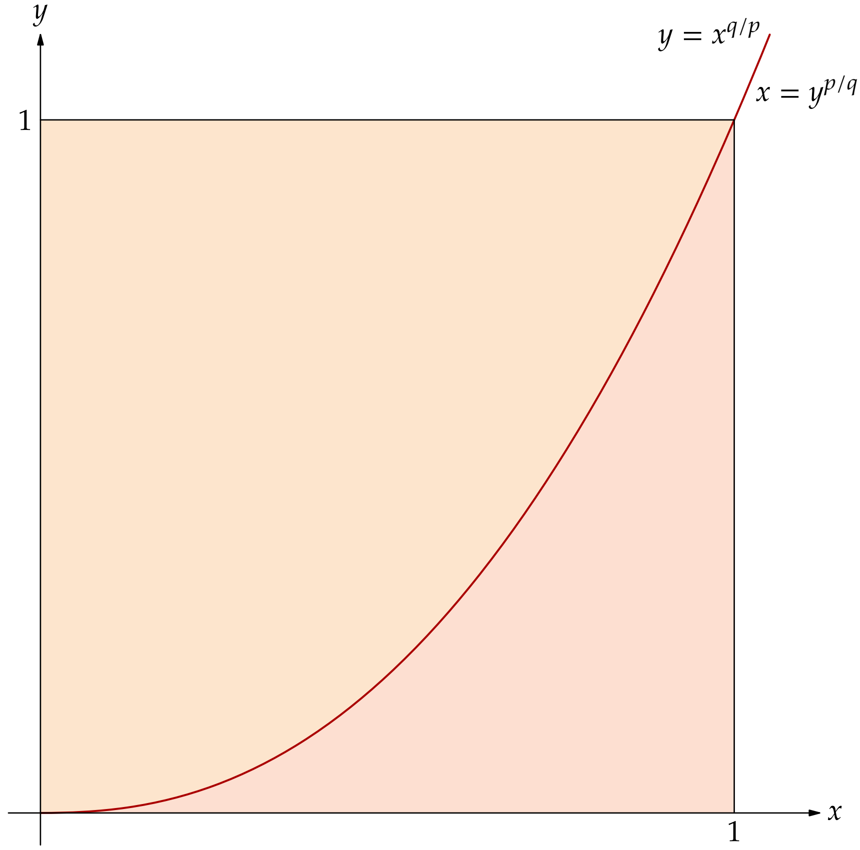


— Jeffery Ely

```
numeric u; u = 260;

vardef f(expr x) = x**(7/3) enddef;

path xx, yy, ff;
xx = 12 left -- (u + 32) * right;
yy = xx rotated 90;
ff = (origin{right} for x=1/8 step 1/8 until 9/8:
    .. (x, f(x)) endfor) scaled u
    cutafter (xx shifted point 1 of yy);

fill subpath(0, 8) of ff -- (0, u) -- cycle withcolor Oranges 8 2;
fill subpath(0, 8) of ff -- (u, 0) -- cycle withcolor Reds 8 2;
draw ff withcolor 2/3 red withpen pencircle scaled 3/4;
draw (0, u) -- (u, u) -- (u, 0);
drawarrow xx; label.rt ("$x$", point 1 of xx); label.bot("$1$", (u,0));
drawarrow yy; label.top("$y$", point 1 of yy); label.lft("$1$", (0,u));

label.urt("$x = y^{p/q}$", point 8 of ff shifted (6, 2));
label.lft("$y = x^{q/p}$", point 9 of ff);

label.top("$\displaystyle \int_0^1 \left(t^{p/q} + t^{q/p} \right)\, dt = 1$",
    point 5/2 of bbox currentpicture);
```

An integral of a sum of reciprocal powers

$$\int_0^1 \left(t^{p/q} + t^{q/p}\right)\, dt = 1$$

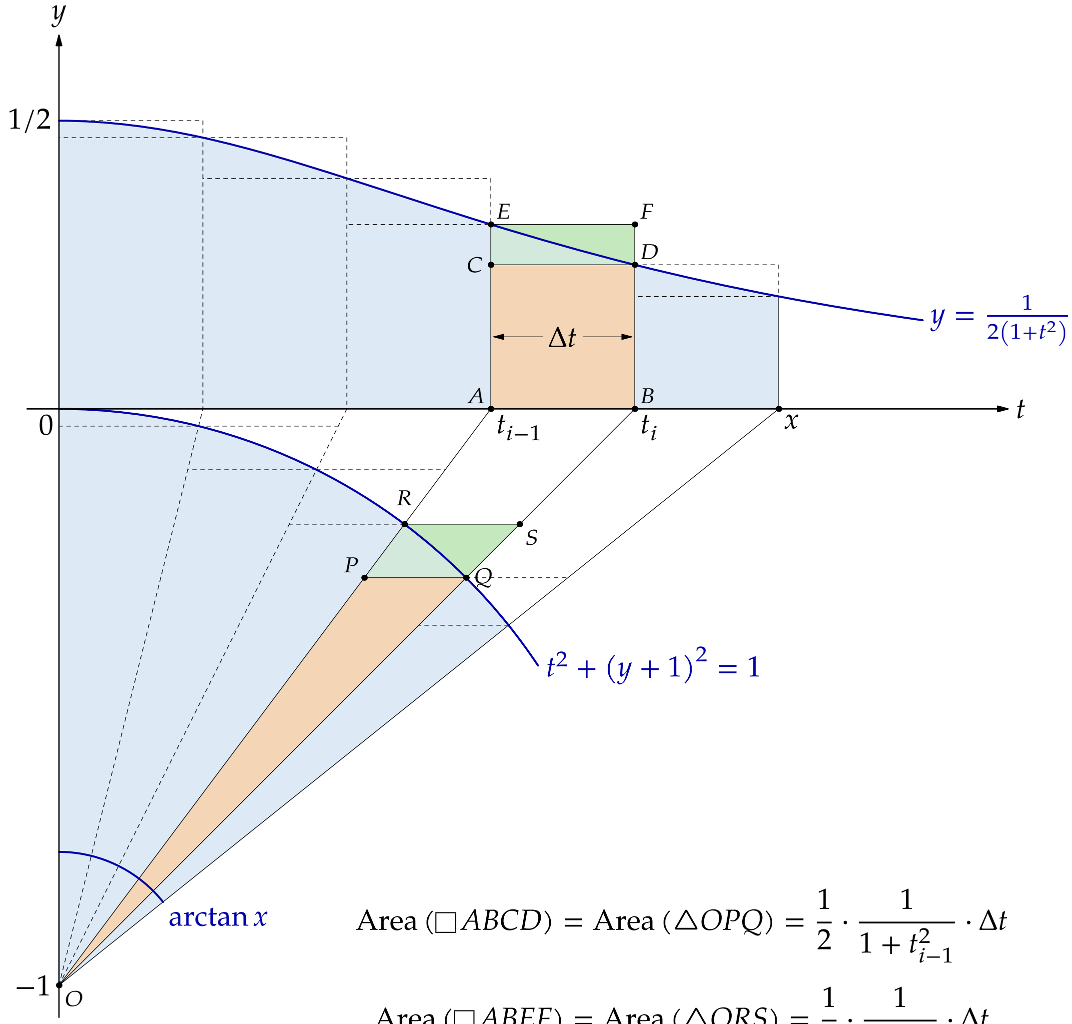


— Peter R. Newbury

```
numeric u; u = 3in; z.0 = u * down;
path tt; tt = 12 left -- (3/2u + 32) * right;
path yy; yy = (u + 12) * down -- (1/2u + 32) * up;
path ff; ff = ((0, 1/2){1,0} for t=1/4 step 1/4 until 3/2:
    .. (t, 1/(2 * (1+t**2))){1, -t/(t**2 +1)/(t**2 +1)} endfor) scaled u;
path gg; gg = origin for i=1 upto 6:
    .. u * dir angle (xpart point i of ff, u) shifted z.0 endfor;
path aa; aa = quartercircle scaled 100 shifted z.0 cutbefore (z.0--point 5 of gg);

fill origin -- (xpart point 5 of ff, 0) -- subpath (5, 0) of ff -- cycle withcolor Blues 8 2;
fill z.0 -- subpath (5, 0) of gg -- cycle withcolor Blues 8 2;
z.P = whatever[z.0, point 3 of gg]; y.P = ypart point 4 of gg;
z.S = whatever[z.0, point 4 of gg]; y.S = ypart point 3 of gg;
fill z.0 -- point 4 of gg -- z.P -- cycle withcolor 3/4[Blues 8 2, Oranges 7 2];
fill subpath (3, 4) of gg -- z.P -- cycle withcolor 1/2[Blues 8 2, Greens 7 2];
fill subpath (3, 4) of gg -- z.S -- cycle withcolor Greens 7 2;
z.C = (xpart point 3 of ff, ypart point 4 of ff);
z.F = (xpart point 4 of ff, ypart point 3 of ff);
fill (xpart point 3 of ff, 0) -- (xpart point 4 of ff, 0) --
    point 4 of ff -- z.C -- cycle withcolor 3/4[Blues 8 2, Oranges 7 2];
fill subpath (3, 4) of ff -- z.C -- cycle withcolor 1/2[Blues 8 2, Greens 7 2];
fill subpath (3, 4) of ff -- z.F -- cycle withcolor Greens 7 2;

drawoptions(withpen pencircle scaled 1/4);
  for i=1 upto 5:
    draw subpath (-2, 1) of unitsquare
        xscaled (xpart point i-1 of ff - xpart point i of ff)
        yscaled (ypart point i-1 of ff - ypart point i of ff)
        shifted point i of ff if i <> 4: dashed evenly scaled 1/2 fi;
    draw z.0 -- (xpart point i of ff, 0) -- point i of ff
        if i < 3: dashed evenly scaled 1/2 fi;
    draw (origin -- u * left) shifted point i of gg
        cutafter (z.0 -- point i-1 of gg)
        if i <> 4: dashed evenly scaled 1/2 fi;
    draw (origin -- u * right) shifted point i-1 of gg
        cutafter (z.0 -- (xpart point i of ff, 0))
        if i <> 4: dashed evenly scaled 1/2 fi;
  endfor
drawoptions(withpen pencircle scaled 3/4 withcolor 2/3 blue);
  draw ff; draw gg; draw aa;
  label.rt("$y={1\over 2\left(1+t^2\right)}$", point 6 of ff);
  label.rt("$t^2 + \left(y+1\right)^2=1$", point 6 of gg);
  label.lrt("$\arctan x$", point 0 of aa);
drawoptions();

drawarrow tt; label.rt("$t$", point 1 of tt);
drawarrow yy; label.top("$y$", point 1 of yy);
% ... the rest is labels ...
```

The arctangent integral

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$



$$\text{Area}\,(\square ABCD) = \text{Area}\,(\triangle OPQ) = \frac{1}{2} \cdot \frac{1}{1+t_{i-1}^2} \cdot \Delta t$$

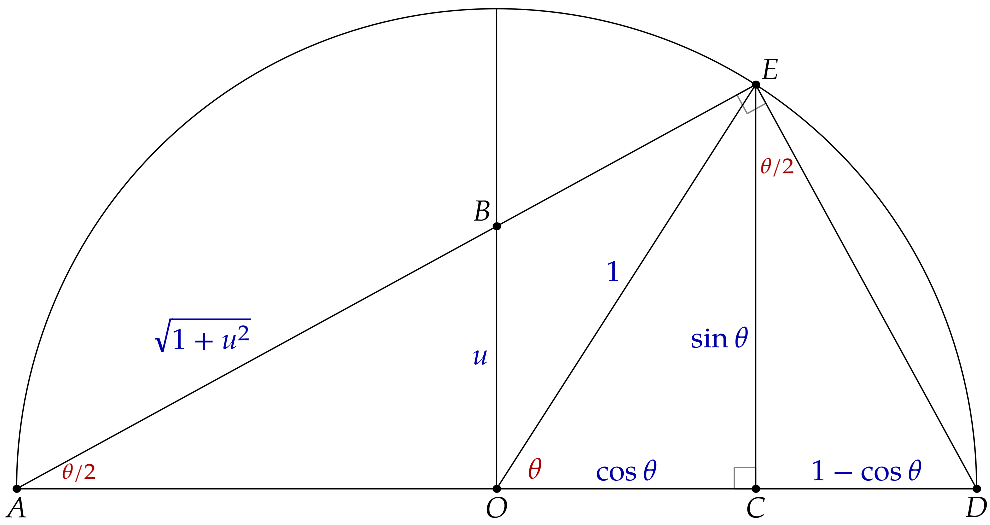
$$\text{Area}\,(\square ABEF) = \text{Area}\,(\triangle ORS) = \frac{1}{2} \cdot \frac{1}{1+t_i^2} \cdot \Delta t$$

$$\therefore \quad \frac{1}{2}\arctan x = \int_0^x \frac{1}{2\left(1+t^2\right)}\,dt$$

— Aage Bondesen

```
path hh;
hh = halfcircle scaled 5in;
z0 = origin;
z1 = point 4 of hh;
z4 = point 0 of hh;
z5 = point 1.273 of hh;
z3 = (x5, 0);
z2 = whatever * up = whatever [z1, z5];

draw unitsquare scaled 8 rotated 90 shifted z3 withcolor 1/2 white;
draw unitsquare scaled 8 rotated angle (z1-z5) shifted z5 withcolor 1/2 white;
forsuffixes $=0, 1, 3, 4: draw z5 -- z$; endfor
draw hh -- cycle;
draw z0 -- point 2 of hh;

dotlabel.bot("$O$", z0);
dotlabel.bot("$A$", z1);
dotlabel.bot("$C$", z3);
dotlabel.bot("$D$", z4);

dotlabel.ulft("$B$", z2);
dotlabel.urt("$E$", z5);

drawoptions(withcolor 2/3 red);
label("$\theta$", 16 dir 1/2 angle z5);
label("$\scriptstyle\theta/2$", 24 dir 1/4 angle z5 shifted z1);
label("$\scriptstyle\theta/2$", 32 dir (270 + 1/4 angle z5) shifted z5);

drawoptions(withcolor 2/3 blue);
label.lft("$u$", 1/2[z0, z2]);
label.ulft("$1$", 1/2[z0, z5]);
label.top("$\cos\theta$", 1/2[z0, z3]);
label.lft("$\sin\theta$", 5/8[z5, z3]);
label.top("$1-\cos\theta$", 1/2[z3, z4]);
label.ulft("$\sqrt{1+u^2}$", 1/2[z1, z2]);

drawoptions();

label.bot(btex \vbox{\openup 12pt\halign{\hfil $\displaystyle #$ \hfil\cr
  u=\tan\frac{\theta}{2}, \quad DE=2\sin\frac{\theta}{2}=\frac{2u}{\sqrt{1+u^2}}\cr
  \frac{CE}{DE}=\frac{OA}{BA} \quad\Longrightarrow\quad \sin\theta=\frac{2u}{1+u^2}\cr
  \frac{CD}{DE}=\frac{OB}{BA} \quad\Longrightarrow\quad \cos\theta=\frac{1-u^2}{1+u^2}\cr
}} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

The method of last resort — Weierstrass substitution



$$u = \tan\frac{\theta}{2}, \quad DE = 2\sin\frac{\theta}{2} = \frac{2u}{\sqrt{1+u^2}}$$

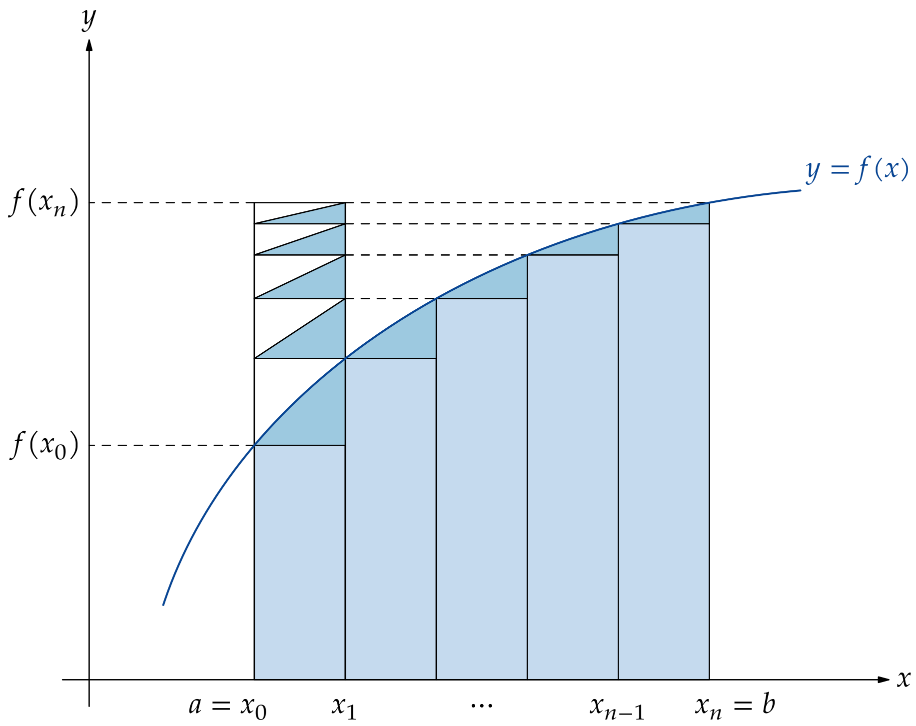
$$\frac{CE}{DE} = \frac{OA}{BA} \quad\Longrightarrow\quad \sin\theta = \frac{2u}{1+u^2}$$

$$\frac{CD}{DE} = \frac{OB}{BA} \quad\Longrightarrow\quad \cos\theta = \frac{1-u^2}{1+u^2}$$

— Paul Deiermann

```
path xx, yy; xx = 10 left -- 300 right; yy = 10 down -- 240 up;
path ff; ff = subpath (3.7, 2.15) of fullcircle xscaled 560 yscaled 410 shifted (300, -20);

numeric t[];
for i=0 upto 6:
  (t[i], whatever) = ff intersectiontimes
          yy shifted (i/7)[(xpart point 0 of ff, 0), (xpart point infinity of ff, 0)];
  z[10 + i] = point t[i] of ff;
endfor
for i=0 upto 5:
    z[i] = (xpart point t[i+1] of ff, ypart point t[i] of ff);
endfor
for i=1 upto 5:
    fill subpath (t[i], t[i+1]) of ff -- z[i] -- cycle withcolor Blues 8 4;
    fill z[i] -- (x[i-1], y[i]) -- (x[i-1], 0) -- (x[i], 0) -- cycle withcolor Blues 8 3;
    draw z[i] -- (x[i-1], y[i]) -- (x[i-1], 0) -- (x[i], 0) -- cycle;
endfor
for i=2 upto 5:
    path trig;
    trig = (z[10+i] -- z[11+i] -- z[i] -- cycle) shifted (x1-x[i], 0);
    fill trig withcolor Blues 8 4;
    draw trig;
    draw point 1 of trig -- z[11+i] dashed evenly;
endfor
draw z5 -- z16; draw (x1, y16) -- (x0, y16) -- z11;
draw z11 -- (0, y11) dashed evenly;
draw (x0, y16) -- (0, y16) dashed evenly;
draw ff withpen pencircle scaled 3/4 withcolor Blues 8 8;
drawarrow xx; label.rt("$x$", point 1 of xx);
drawarrow yy; label.top("$y$", point 1 of yy);

label.lft("$f\left(x_0\right)$", (0, y11));
label.lft("$f\left(x_n\right)$", (0, y16));
label.urt("$y=f\left(x\right)$", point infinity of ff) withcolor Blues 8 8;

label.bot("\strut\llap{$a={}$}$x_0$", (x0, 0));
label.bot("\strut$x_1$", (x1, 0));
label.bot("\strut$\cdots$", (1/2(x2+x3), 0));
label.bot("\strut$x_{n-1}$", (x4, 0));
label.bot("\strut$x_n$\rlap{${}=b$}", (x5, 0));

label.bot(btex $\displaystyle
\int_a^b f\left(x\right)dx = \sum_{i=0}^{n-1} f\left(x_i\right) {b-a\over n}
+ {1\over2} \biggl(\,f\left(x_n\right) - f\left(x_0\right) \biggr)   {b-a\over n}
$ etex, point 1/2 of bbox currentpicture shifted 42 down);
```

The trapezoidal rule — for increasing functions



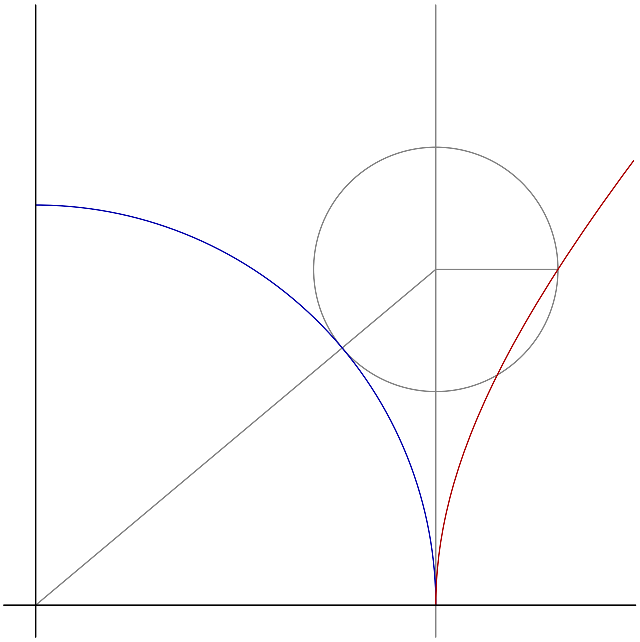
$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)\,\frac{b-a}{n} + \frac{1}{2}\Big(f(x_n) - f(x_0)\Big)\frac{b-a}{n}$$

— Jesús Urías

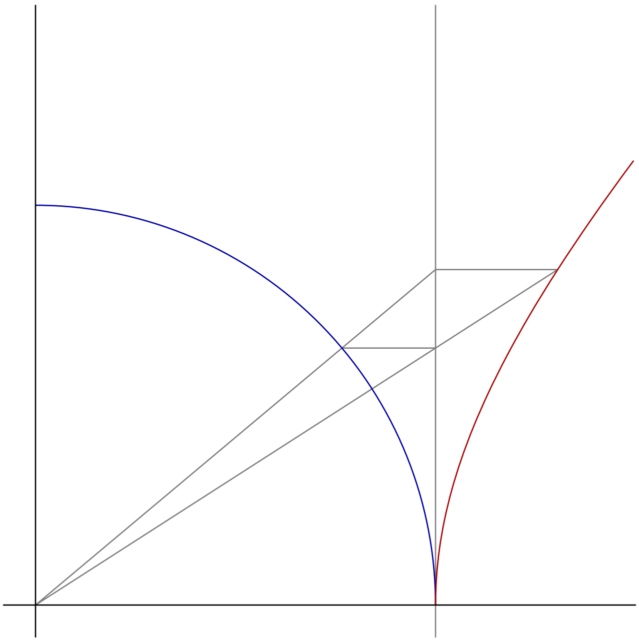
```
numeric r; r = 150;
path xx; xx = 12 left -- 3/2 r * right;
path yy; yy = xx rotated 90;
path base; base = quartercircle scaled 2r;
path hh; hh = point 0 of base for t=4 step 4 until 48:
    hide(numeric a; a = ypart dir t / xpart dir t;)
    .. (a ++ 1, a) scaled r
endfor;
z1 = point 10 of hh; z2 = (r, y1);

picture P[];
P0 = image(
    draw base withcolor 2/3 blue;
    draw hh withcolor 2/3 red;
    draw xx; draw yy;
);
P1 = image(
    draw origin -- z2 -- z1 withcolor 1/2 white;
    draw fullcircle scaled 2 abs(z1-z2) shifted z2 withcolor 1/2 white;
    draw yy shifted point 0 of base withcolor 1/2 white;
    draw P0;
    label.lft("I.", point 3 of bbox P0 shifted 30 left);
);
P2 = image(
    draw origin -- z2 -- z1 -- cycle withcolor 1/2 white;
    z3 = whatever * z1;
    z4 = whatever * z2;
    x3 = r; y3 = y4;
    draw z3 -- z4 withcolor 1/2 white;
    draw yy shifted point 0 of base withcolor 1/2 white;
    draw P0;
    label.lft("II.", point 3 of bbox P0 shifted 30 left);
);

draw P1;
draw P2 shifted (1.8r * down);
```

Construction of a hyperbola

I.



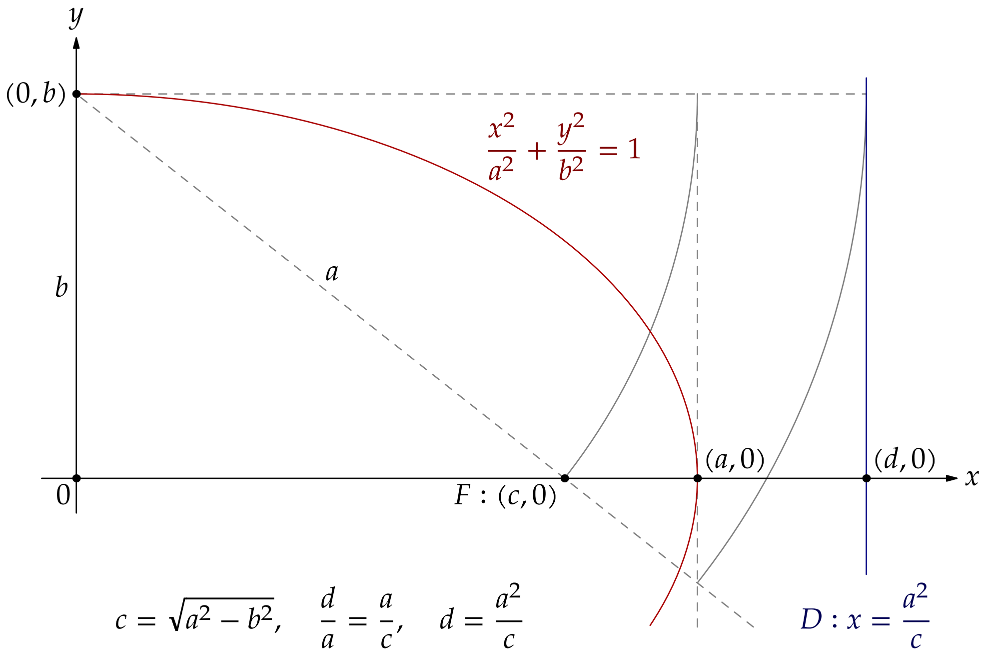
II.



— Ernest J. Eckert

```
numeric a, b, c, d;
a = 233; b = 144; c = a +-+ b; d/a = a/c;

path xx, yy, ee, directrix, cf, cd;
xx = 13 left -- (d+34) * right;
yy = 13 down -- (b+21) * up;
ee = fullcircle xscaled 2a yscaled 2b;
directrix = (36 down -- (b+6) * up) shifted (d, 0);
cf = fullcircle scaled 2a shifted (0, b);
cd = fullcircle scaled 2d shifted (0, b);

z1 = whatever[(0,b), (c,0)]; x1 = a + 21; x2 = a; y2 = y1;
drawoptions(withcolor 1/2 white);
    draw z1 -- (0, b) -- point 0 of cd dashed evenly;
    draw z2 --           point 0 of cf dashed evenly;
    draw subpath (8, 6) of cf cutafter ((0,b) -- z1);
    draw subpath (8, 6) of cd cutafter ((0,b) -- z1);
drawoptions();

draw directrix withcolor 1/2 blue;
draw subpath (-1/2, 2) of ee withcolor 2/3 red;
drawarrow xx;
drawarrow yy;

label.urt("$a$", 1/2[(0,b), (c,0)]);
label.lft("$b$", 1/2(0,b));

dotlabel.llft("$0$", (0,0));
dotlabel.llft("$F:(c,0)$", (c,0));
dotlabel.lft("$(0, b)$", (0, b));
dotlabel.urt("$(a, 0)$", (a, 0));
dotlabel.urt("$(d, 0)$", (d, 0));

label.rt("$x$", point 1 of xx);
label.top("$y$", point 1 of yy);

label.urt("$\displaystyle {x^2\over a^2} + {y^2\over b^2} = 1$", point 1.1 of ee)
    withcolor 1/2 red;
label.bot("$\displaystyle D : x = {a^2\over c}$", point 0 of directrix)
    withcolor 1/3 blue;
label.bot(btex $\displaystyle c=\sqrt{a^2-b^2}$, \quad
               $\displaystyle{d\over a}={a\over c}$, \quad
               $\displaystyle d={a^2\over c}$ etex,
    (1/2c, ypart point 0 of directrix));
```

The focus and directrix of an ellipse

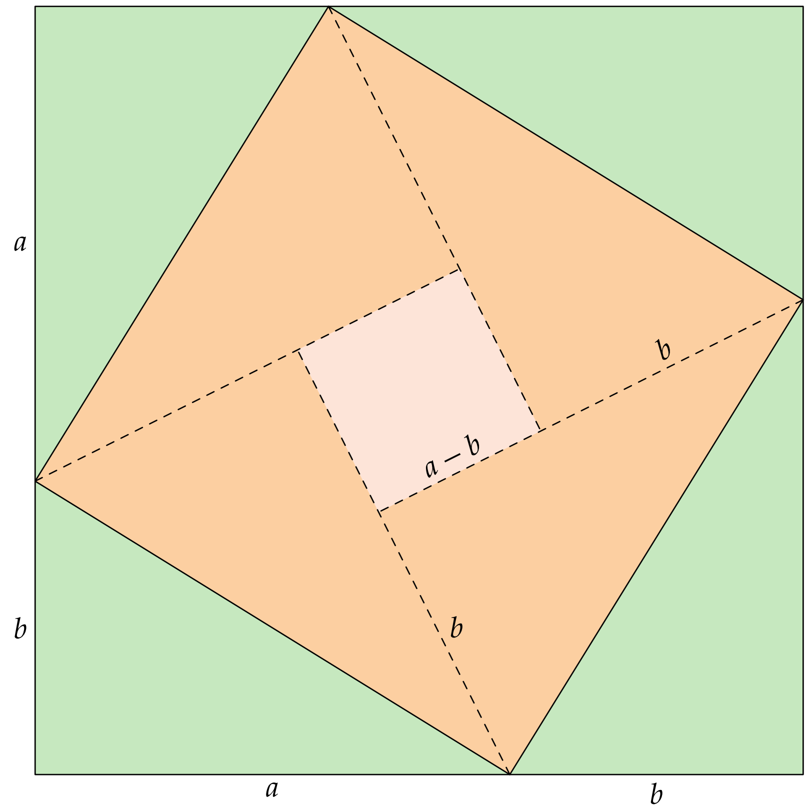


— Michel Bataille

Inequalities

```
numeric a, b; a = 233; b = 144;
path S[];
S0 = unitsquare scaled 288;
S1 = for i=0 upto 3:
    point i + a/(a+b) of S0 --
  endfor cycle;
S2 = for i=0 upto 3:
    point i of S0 reflectedabout(point i-1 of S1, point i of S1) --
  endfor cycle;

fill S0 withcolor Greens 7 2;
fill S1 withcolor Oranges 7 2;
fill S2 withcolor Reds 7 1;

for i=0 upto 3:
  draw point i of S1 -- point i of S2 dashed evenly;
endfor
draw S1; draw S0;

label.lft("$a$", 1/2[point 3 of S1, point 3 of S0]);
label.lft("$b$", 1/2[point 3 of S1, point 0 of S0]);
label.bot("$a$", 1/2[point 0 of S1, point 0 of S0]);
label.bot("$b$", 1/2[point 0 of S1, point 1 of S0]);
label.urt("$b$", 1/2[point 0 of S1, point 1 of S2]);

draw thelabel.top(TEX("$b$"), origin)
    rotated (90 - 2 angle (a,b))
    shifted 1/2[point 1 of S1, point 2 of S2];

draw thelabel.top(TEX("$a-b$"), origin)
    rotated (90 - 2 angle (a,b))
    shifted point 3/2 of S2;

label.bot(btex $\displaystyle \left(a+b\right)^2 \ge 4ab$
               \quad\Longrightarrow\quad
               $\displaystyle {a+b\over2} \ge \sqrt{ab}$
          etex, point 1/2 of bbox currentpicture shifted 42 down);
```

The arithmetic mean – geometric mean inequality IV

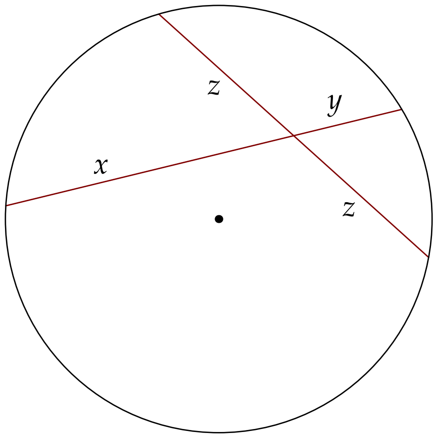


$$(a+b)^2 \geq 4ab \quad \Longrightarrow \quad \frac{a+b}{2} \geq \sqrt{ab}$$

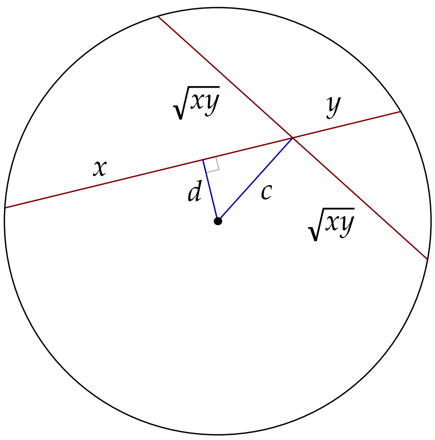
— Ayoub B. Ayoub

```
path base; base = fullcircle scaled 160;
path zz; zz = (point 0 of base -- point 4 of base)
    scaled 4 shifted 42 up rotated -42
    cutbefore subpath (4, 8) of base
    cutafter subpath (0, 4) of base;
numeric s; s = 11/16;
numeric t; (t, whatever) = base intersectiontimes
    (point 1/2 of zz -- 16[point s of base, point 1/2 of zz]);
path xy; xy = point s of base -- point t of base;
% z0 = intersection of xy and zz, z1 = closest point on xy to center, z2 = for label
z0 = whatever[point s of base, point t of base] = whatever[point 0 of zz, point 1 of zz];
z1 = whatever[point s of base, point t of base];
z1 - center base = whatever * (point s of base - point t of base) rotated 90;
z2 = point 6 of base shifted 21 down;
picture P[];
P1 = image(
  draw xy withcolor 1/2 red; draw zz withcolor 1/2 red;
  draw base; drawdot center base withpen pencircle scaled dotlabeldiam;
  label.ulft("$x$", 3/8[point t of base, z0]);
  label.ulft("$y$", 1/2[point s of base, z0]);
  label.llft("$z$", point 1/4 of zz);
  label.llft("$z$", point 3/4 of zz);
  label.bot("$z^2 = xy$", z2);
);
P2 = image(
  draw unitsquare scaled 5 rotated angle (center base - z1) shifted z1 withcolor 3/4 white;
  draw xy withcolor 1/2 red; draw zz withcolor 1/2 red;
  draw z0 -- center base -- z1 withcolor 2/3 blue; draw base;
  draw center base withpen pencircle scaled dotlabeldiam;
  label.ulft("$x$", 3/8[point t of base, z0]);
  label.ulft("$y$", 1/2[point s of base, z0]);
  label.llft("$\sqrt{xy}$", point 1/4 of zz);
  label.llft("$\sqrt{xy}$", point 3/4 of zz);
  label.lft("$d$", 1/2[z1, center base]);
  label.lrt("$c$", 1/2[z0, center base]);
  label.bot("$d<c$\quad\Longrightarrow\quad$x+y>2\sqrt{xy}$", z2);
);
P3 = image(
  for p = xy, zz:
    draw (point 0 of base -- point 4 of base)
      rotated angle (point 1 of p - point 0 of p) withcolor 1/2 red;
  endfor
  draw base; draw center base withpen pencircle scaled dotlabeldiam;
  label.bot("$d=c=0$\quad\Longrightarrow\quad$x+y=2\sqrt{xy}$", z2);
);
draw P1 shifted 240 up; draw P2 shifted 120 left; draw P3 shifted 120 right;
```

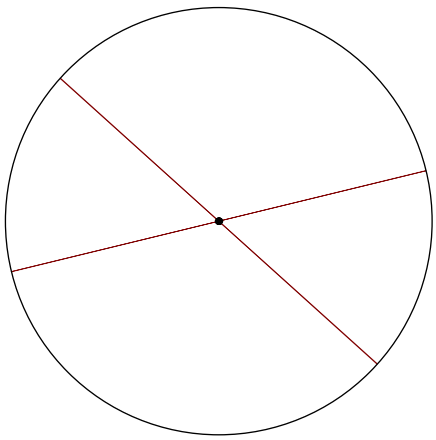
The arithmetic mean – geometric mean inequality V



$$z^2 = xy$$



$$d < c \quad \Longrightarrow \quad x + y > 2\sqrt{xy}$$

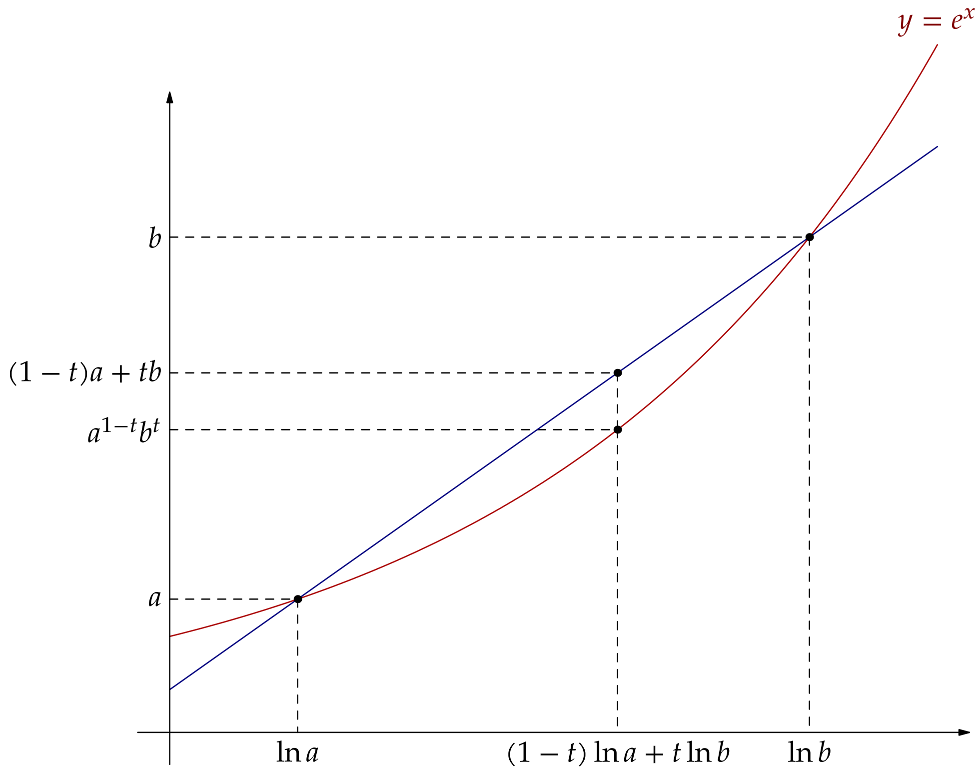


$$d = c = 0 \quad \Longrightarrow \quad x + y = 2\sqrt{xy}$$

— Sidney H. Kung

```
path xx, yy, ee, ll;
xx = 12 left -- 300 right;
yy = 12 down -- 240 up;

ee = ((0, 1) for x=1 upto 36: .. (8x, mexp(14x)) endfor) yscaled 36;
z1 = point 6 of ee;
z2 = point 30 of ee;
z0 = whatever[z1, z2]; x0 = 0;
z3 = whatever[z1, z2]; x3 = xpart point 1 of xx - 12;
z4 = point 21 of ee;
z5 = whatever[z1, z2]; x5 = x4;

ll = z0 -- z3;

draw (0, y1) -- z1 -- (x1, 0) dashed evenly;
draw (0, y2) -- z2 -- (x2, 0) dashed evenly;
draw (0, y4) -- z4 -- (x4, 0) dashed evenly;
draw (0, y5) -- z5 -- z4      dashed evenly;


draw ee withcolor 2/3 red;
draw ll withcolor 1/2 blue;

forsuffixes $=1, 2, 4, 5: drawdot z$ withpen pencircle scaled dotlabeldiam; endfor

drawarrow xx;
drawarrow yy;

label.top("$y=e^x$", point infinity of ee) withcolor 1/2 red;
label.lft("$a$", (0, y1)); label.bot("$\ln a$", (x1, 0));
label.lft("$b$", (0, y2)); label.bot("$\ln b$", (x2, 0));
label.lft("$a^{1-t} b^t$", (0, y4));
label.lft("$(1-t)a + tb$", (0, y5));
label.bot("$(1-t)\ln a + t\ln b$", (x4, 0));

label.bot(btex
  $0 < a < b$,\quad $0 < t < 1$ \quad\Longrightarrow\quad $(1-t)a + tb > a^{1-t}b^t$
  etex, point 1/2 of bbox currentpicture shifted 36 down);
label.bot(btex
  $t=\frac12$ \quad\Longrightarrow\quad $\displaystyle \frac{a+b}{2} > \sqrt{ab}$
  etex, point 1/2 of bbox currentpicture shifted 36 down);
```

The arithmetic mean – geometric mean inequality VI



$$0 < a < b, \quad 0 < t < 1 \quad \Longrightarrow \quad (1-t)a + tb > a^{1-t}b^t$$

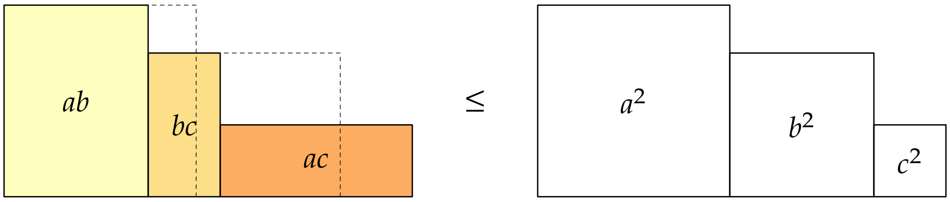
$$t = \tfrac{1}{2} \quad \Longrightarrow \quad \frac{a+b}{2} > \sqrt{ab}$$

— Michael K. Brozinsky

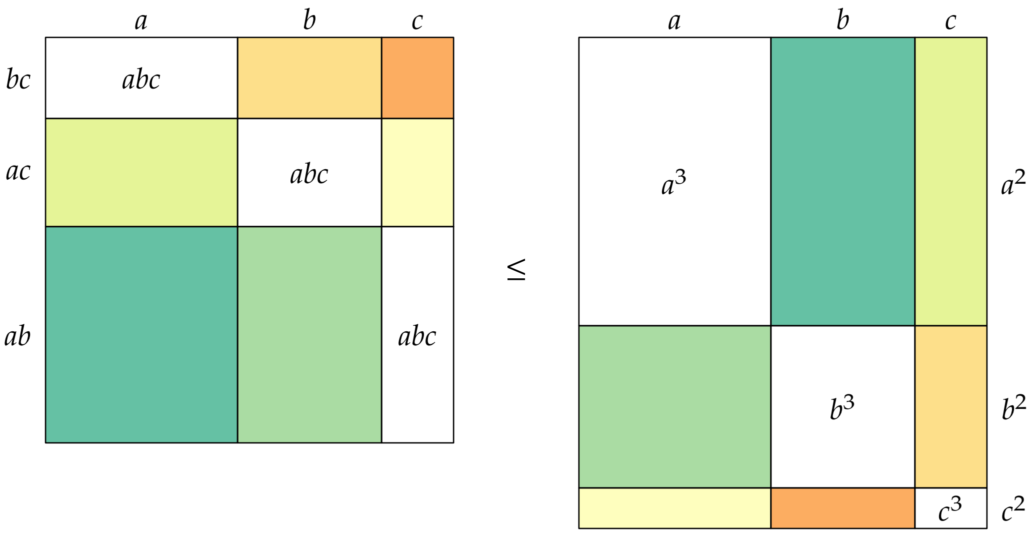
```
numeric a, b, c, k;
k = 48; k * a = 72; b = 3/4 a; c = 3/8 a;

path ab, bc, ac, asq, bsq, csq;
ab = unitsquare xscaled b yscaled a scaled k;
bc = unitsquare xscaled c yscaled b scaled k shifted point 1 of ab;
ac = unitsquare xscaled a yscaled c scaled k shifted point 1 of bc;

asq = unitsquare scaled a scaled k;
bsq = unitsquare scaled b scaled k shifted point 1 of asq;
csq = unitsquare scaled c scaled k shifted point 1 of bsq;

picture P[];
P1 = image(
    fill ab withcolor Spectral 9 5;
    fill bc withcolor Spectral 9 4;
    fill ac withcolor Spectral 9 3;
    forsuffixes $=asq, bsq:
        draw subpath (1, 3) of $ dashed evenly scaled 1/2 withpen pencircle scaled 1/4;
    endfor
    forsuffixes $=ab, bc, ac:
        draw $;
        label("$" & str $ & "$", center $);
    endfor
);
P2 = image(
    forsuffixes $=asq, bsq, csq:
        draw $;
        label("$" & substring (0,1) of str $ & "^2$", center $);
    endfor
);

draw P1 shifted (100 left  - center bbox P1);
draw P2 shifted (100 right - center bbox P2);
label("${}\le{}$", origin);
label("\textsc{Lemma}: $ab + bc + ac \le a^2 + b^2 + c^2$", 64 up);

% and much the same for the Theorem pictures
```

The arithmetic mean – geometric mean inequality for three positive numbers

LEMMA: $ab + bc + ac \leq a^2 + b^2 + c^2$



THEOREM: $3abc \leq a^3 + b^3 + c^3$

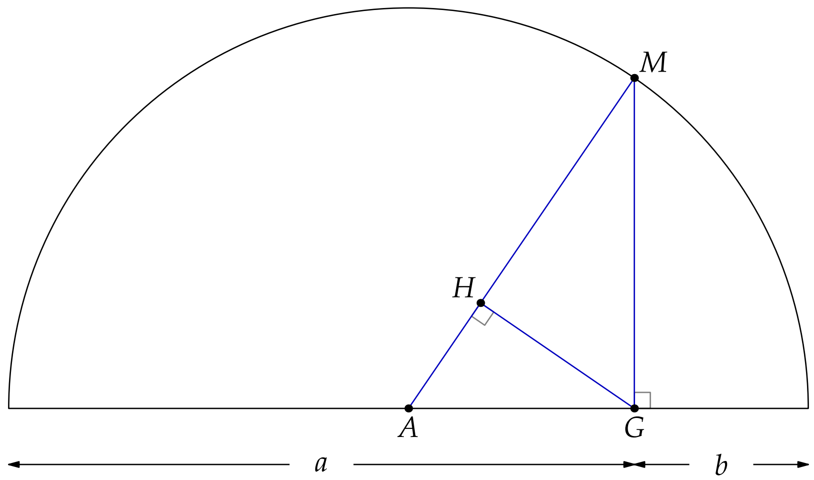


— Claudi Alsina

```
path sc; sc = (halfcircle -- cycle) scaled 300;
z.A = origin;
z.M = point 1.2345 of sc;
x.G = x.M; y.G = y.A;
z.H = whatever [z.A, z.M];
z.H - z.G = whatever * (z.M - z.A) rotated 90;

draw unitsquare scaled 6                            shifted z.G withcolor 1/2;
draw unitsquare scaled 6 rotated angle (z.A - z.H) shifted z.H withcolor 1/2;
draw z.A -- z.M -- z.G -- z.H withcolor 3/4 blue;
draw sc;

path a, b;
a = (point 4 of sc -- z.G) shifted 21 down;
b = (z.G -- point 0 of sc) shifted 21 down;
forsuffixes n = a, b:  % suffixes so that str works...
  drawdblarrow n;
  unfill fullcircle scaled 24 shifted point 1/2 of n;
  label("$" & str n & "$", point 1/2 of n);
endfor

dotlabel.bot ("$A$", z.A);
dotlabel.bot ("$G$", z.G);
dotlabel.ulft("$H$", z.H);
dotlabel.urt ("$M$", z.M);

interim bboxmargin := 32;
label.top(btex
  $a,b > 0$ \quad\Longrightarrow\quad
  $\displaystyle {a+b\over2}\ge\sqrt{ab}\ge{2ab\over a+b}$
  etex, point 5/2 of bbox currentpicture);
label.bot(btex \vbox{\openup 12pt\halign{\hfil #\hfil\cr
  $\displaystyle \overline{AM} = {a+b\over2}$, \quad
  $\displaystyle \overline{GM} = \sqrt{ab}$, \quad
  $\displaystyle \overline{HM} = {2ab\over a+b}$,\cr
  $\overline{AM} \ge \overline{GM} \ge \overline{HM}$.\cr
  }} etex, point 1/2 of bbox currentpicture);
```

The arithmetic-geometric-harmonic mean inequality

$$a, b > 0 \quad \Longrightarrow \quad \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$



$$\overline{AM} = \frac{a+b}{2}, \quad \overline{GM} = \sqrt{ab}, \quad \overline{HM} = \frac{2ab}{a+b},$$

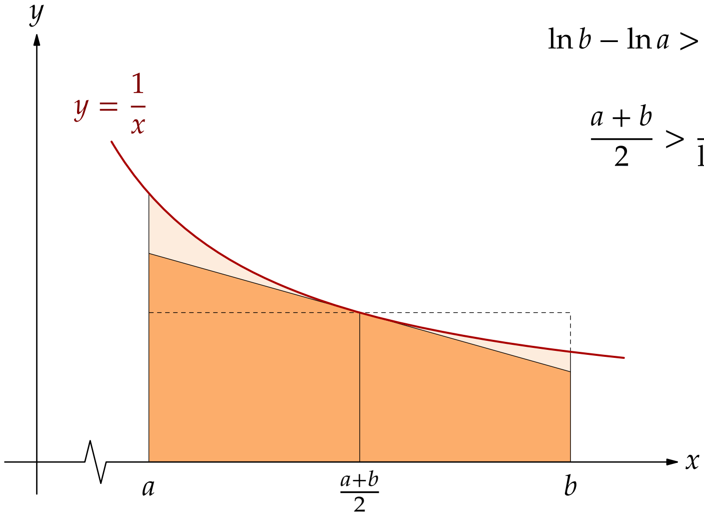
$$\overline{AM} \geq \overline{GM} \geq \overline{HM}.$$

— Pappus of Alexandria (circa A.D. 320)

```
path xx, yy, ff; numeric t; t=2;
xx = 12 left -- (18,0) -- (18+t, 4t) -- (18+3t, -4t) -- (18+4t, 0) -- (240, 0);
yy = 12 down -- 160 up;
ff = ((1,1){1, -1} .. (2, 1/2){4, -1} .. (3, 1/3){9, -1} .. (4, 1/4){16, -1})
  xscaled 64 yscaled 108 shifted (-36, 12);
numeric a, b, m, s, t, u, v, w;
a = 42;          z1 = (a,0);  (t,whatever) = ff intersectiontimes (yy shifted z1);
b = 200;         z2 = (b,0);  (u,whatever) = ff intersectiontimes (yy shifted z2);
m = 1/2a + 1/2b; z3 = (m,0);  (v,whatever) = ff intersectiontimes (yy shifted z3);
s = sqrt(a*b);   z4 = (s,0);  (w,whatever) = ff intersectiontimes (yy shifted z4);
picture P[];
P0 = image(
  draw ff withcolor 2/3 red withpen pencircle scaled 3/4;
  label.top("$\displaystyle y={1\over x}$", point 0 of ff) withcolor 1/2 red;
  drawarrow xx; label.rt("$x$", point infinity of xx);
  drawarrow yy; label.top("$y$", point infinity of yy);
  label.bot("\strut $a$", z1); label.bot("\strut $b$", z2);
);
P1 = image(
  x5 = x1; z5 = whatever * direction v of ff shifted point v of ff;
  x6 = x2; z6 = whatever * direction v of ff shifted point v of ff;
  fill z2 -- z1 -- subpath (t, u) of ff -- cycle withcolor Oranges 7 1;
  fill z2 -- z1 -- z5 -- z6 -- cycle withcolor Oranges 7 3;
  draw z1 -- point t of ff withpen pencircle scaled 1/4;
  draw z2 -- point u of ff withpen pencircle scaled 1/4;
  draw z3 -- point v of ff withpen pencircle scaled 1/4;
  draw z5 -- z6 withpen pencircle scaled 1/4;
  draw point u of ff -- (xpart point u of ff, ypart point v of ff)
      -- (xpart point t of ff, ypart point v of ff)
      dashed evenly scaled 1/2 withpen pencircle scaled 1/4;
  draw P0; label.bot("\strut ${a+b\over2}$", z3);
  label.bot(btex \vbox{\openup 12pt\halign{\hfil$\displaystyle # $\hfil\cr
    \ln b - \ln a > {2\over a+b} (b - a)\cr
    {a+b\over2} > {b-a\over \ln b - \ln a}\cr}} etex, point 2 of bbox P0);
);
P2 = image(
  fill z2 -- z1 -- subpath (t, u) of ff -- cycle withcolor Oranges 7 2;
  draw z1 -- point t of ff -- point w of ff -- point u of ff -- z2 withpen pencircle scaled 1/4;
  draw z4 --                  point w of ff                       withpen pencircle scaled 1/4;
  draw P0; label.bot("\strut $\scriptstyle\sqrt{ab}$", z4);
  label.bot(btex \vbox{\openup 12pt\halign{\hfil$\displaystyle # $\hfil\cr
  \ln b - \ln a < {ab-a\over 2a\sqrt{ab}} + {a-ab\over2b\sqrt{ab}} = {b - a\over\sqrt{ab}}\cr
  \sqrt{ab} < {b-a\over \ln b - \ln a}\cr}} etex, point 2 of bbox P0);
);
draw P1; draw P2 shifted 240 down;
label.top(btex $\displaystyle
  b > a > 0 \;\Longrightarrow\; {a+b\over2} > {b-a\over \ln b - \ln a} > \sqrt{ab}
  $ etex , point 5/2 of bbox currentpicture shifted 24 up);
```

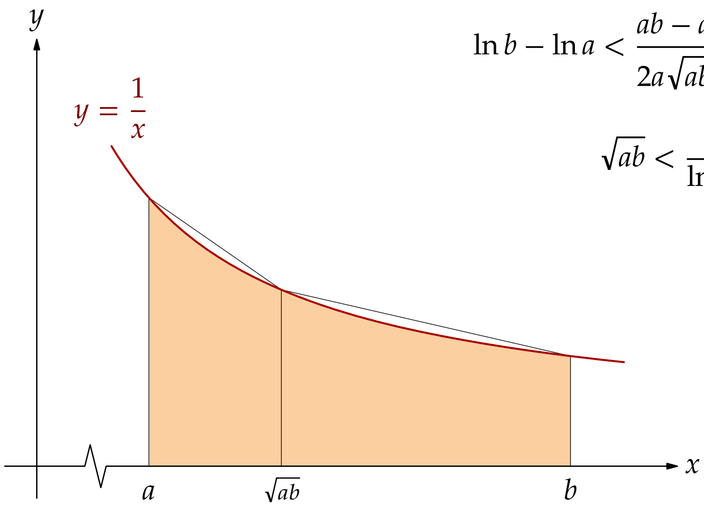
The arithmetic-logarithmic-geometric mean inequality

$$b > a > 0 \implies \frac{a+b}{2} > \frac{b-a}{\ln b - \ln a} > \sqrt{ab}$$



$$\ln b - \ln a > \frac{2}{a+b}(b-a)$$

$$\frac{a+b}{2} > \frac{b-a}{\ln b - \ln a}$$



$$\ln b - \ln a < \frac{ab-a}{2a\sqrt{ab}} + \frac{a-ab}{2b\sqrt{ab}} = \frac{b-a}{\sqrt{ab}}$$

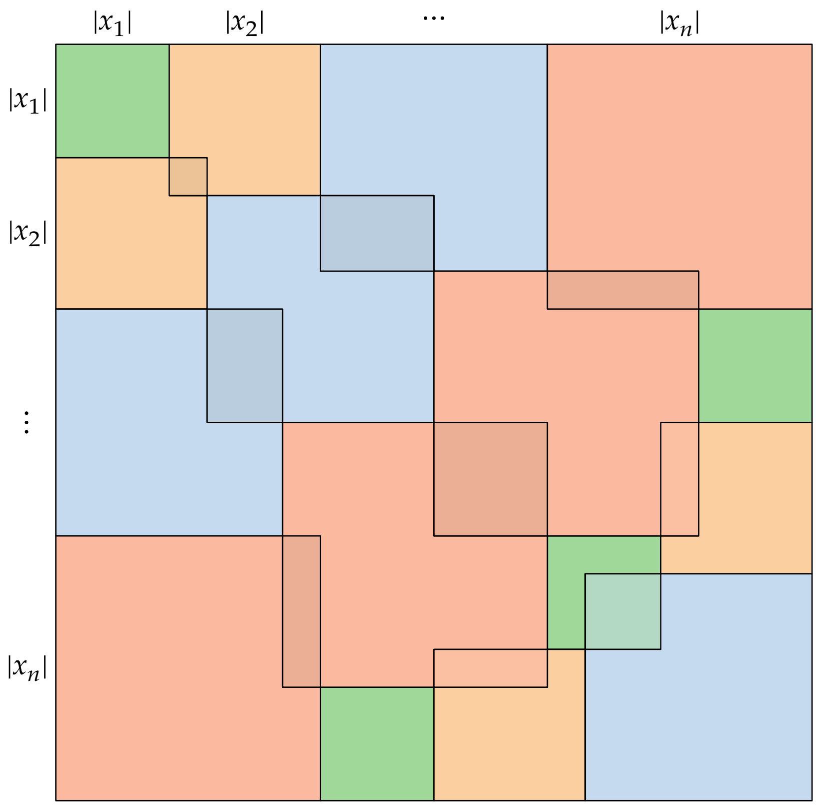
$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a}$$

— RBN

```
path a[], b[], c[], d[];   color sh.a, sh.b, sh.c, sh.d;
a0 = unitsquare scaled 42.5; sh.a = Greens 7 3;
b0 = unitsquare scaled 56.7; sh.b = Oranges 7 2;
c0 = unitsquare scaled 85.0; sh.c = Blues 7 2;
d0 = unitsquare scaled 99.2; sh.d = Reds 7 2;

a1 = a0 rotated -90;
b1 = b0 rotated -90 shifted point 1 of a1;
b2 = b0 rotated -90 shifted point 3 of a1;
c1 = c0 rotated -90 shifted point 1 of b1;
c2 = c0 rotated -90 shifted point 3 of b2;
d1 = d0 rotated -90 shifted point 1 of c1;
d2 = d0 rotated -90 shifted point 3 of c2;
a2 = a0 shifted point 2 of d1;
b3 = b0 shifted point 1 of a2;
c3 = c0 shifted point 1 of b3;
a3 = a0 rotated 180 shifted point 2 of d2;
b4 = b0 rotated 180 shifted point 3 of a3;
c4 = c0 rotated -90 shifted (subpath (2,3) of b1 intersectionpoint subpath (1,2) of b2);
d3 = d0 rotated -90 shifted (subpath (2,3) of c1 intersectionpoint subpath (1,2) of c4);
d4 = d0 rotated -90 shifted (subpath (2,3) of c4 intersectionpoint subpath (1,2) of c2);
a4 = a0 rotated -90 shifted (subpath (2,3) of d3 intersectionpoint subpath (1,2) of d4);

forsuffixes $=a,b,c,d: forsuffixes @=1,2,3,4:
  fill $.@ withcolor sh.$; draw $.@;
endfor endfor

path o[];
o1 = buildcycle(b1, b2); fill o1 withcolor 15/16 sh.b;
o2 = buildcycle(c3, a4); fill o2 withcolor 1/2[sh.c, sh.a];
o3 = buildcycle(c1, c4); fill o3 withcolor 15/16 sh.c;
o4 = buildcycle(c2, c4); fill o4 withcolor 15/16 sh.c;
o5 = buildcycle(d1, d3); fill o5 withcolor 15/16 sh.d;
o6 = buildcycle(d4, d3); fill o6 withcolor 15/16 sh.d;
o7 = buildcycle(d2, d4); fill o7 withcolor 15/16 sh.d;
o8 = buildcycle(b3, d3); fill o8 withcolor 3/4[sh.b, sh.d];
o9 = buildcycle(b4, d4); fill o9 withcolor 3/4[sh.b, sh.d];

for i=1 upto 9: draw o[i]; endfor

label.top("$|x_1|$", point -1/2 of a1);
label.lft("$|x_1|$", point +1/2 of a1);
label.top("$|x_2|$", point -1/2 of b2);
label.lft("$|x_2|$", point +1/2 of b1);
label.top("\strut$\cdots$", point -1/2 of c2);
label.lft("\hbox to 16pt{\hss$\vdots$\hss}", point +1/2 of c1);
label.top("$|x_n|$", point -1/2 of d2);
label.lft("$|x_n|$", point +1/2 of d1);

% plus the display labels...
```

The mean of the squares exceeds the square of the mean

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$



$$n\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \geq \left(|x_1| + |x_2| + \cdots + |x_n|\right)^2 \geq \left(x_1 + x_2 + \cdots + x_n\right)^2$$

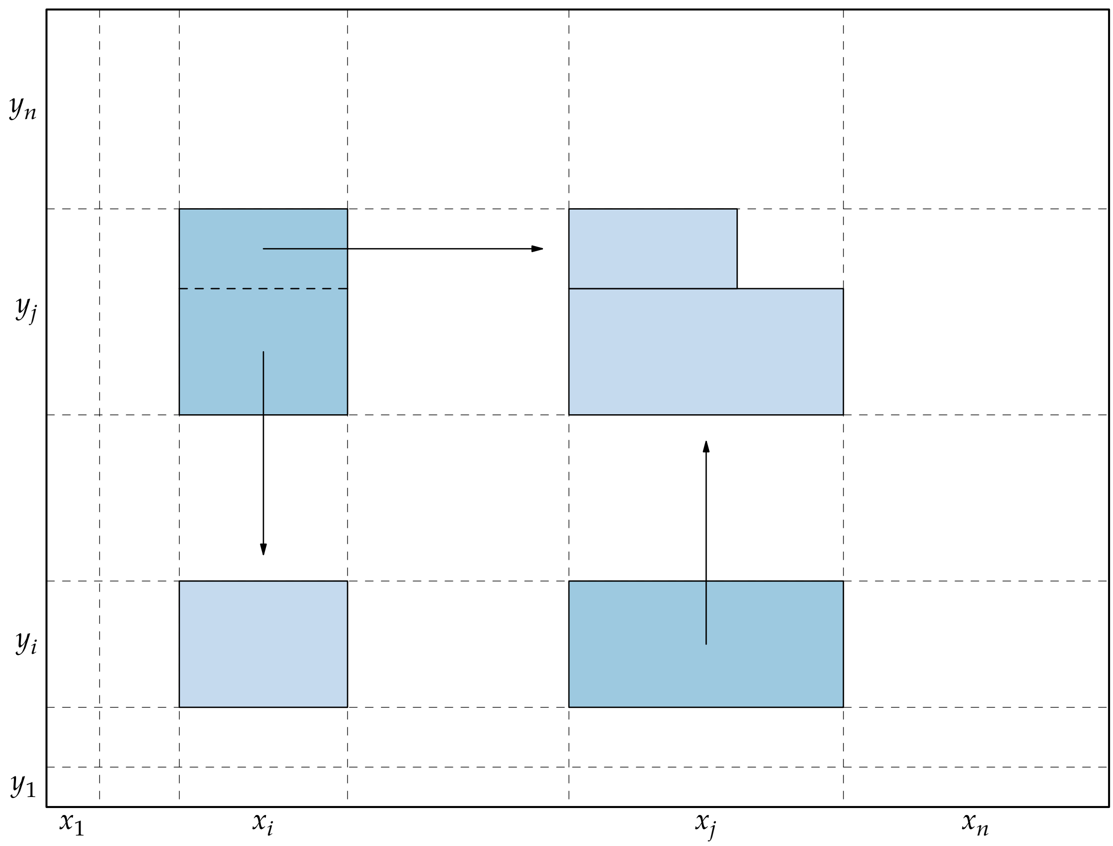
$$\therefore \quad \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \geq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2$$

— RBN

```
numeric wd, ht; wd = 400; ht = 300;
path box; box = unitsquare xscaled wd yscaled ht;
for i=6,15,34,59,90:
  z[i] = (i/120)*(wd, ht);
  draw subpath (0,1) of box shifted (0, y[i]) dashed evenly withpen pencircle scaled 1/4;
  draw subpath (3,4) of box shifted (x[i], 0) dashed evenly withpen pencircle scaled 1/4;
endfor
path s[];
s1 = unitsquare xscaled (x34-x15) yscaled (y34-y15);
s2 = unitsquare xscaled (x90-x59) yscaled (y34-y15);
s3 = unitsquare xscaled (x34-x15) yscaled (y90-y59-y34+y15);
s11 = s1 shifted z15;
s21 = s2 shifted z59;
s31 = s3 shifted (x59, y59+y34-y15);
s12 = s1 shifted (x15,y59);
s22 = s2 shifted (x59,y15);
s32 = s3 shifted (x15,y59+y34-y15);
fill s11 withcolor Blues 6 2; draw s11;
fill s21 withcolor Blues 6 2; draw s21;
fill s31 withcolor Blues 6 2; draw s31;
fill s12 withcolor Blues 6 3; draw subpath (-1,2) of s12;
fill s22 withcolor Blues 6 3; draw s22;
fill s32 withcolor Blues 6 3; draw subpath (1,4) of s32; draw subpath (0,1) of s32 dashed evenly;

drawarrow center s12 -- point 5/2 of s11 shifted 10 up;
drawarrow center s22 -- point 1/2 of s21 shifted 10 down;
drawarrow center s32 -- point 7/2 of s31 shifted 10 left;

draw box withpen pencircle scaled 3/4;

label.bot("$x_1$", 1/2(0+x6, 0));
label.bot("$x_i$", 1/2(x15+x34, 0));
label.bot("$x_j$", 1/2(x59+x90, 0));
label.bot("$x_n$", 1/2(x90+wd, 0));
label.lft("$y_1$", 1/2(0,   0+ y6));
label.lft("$y_i$", 1/2(0, y15+y34));
label.lft("$y_j$", 1/2(0, y59+y90));
label.lft("$y_n$", 1/2(0, y90+ ht));

label.top("$\displaystyle \sum_{i=1}^n x_i \sum_{i=1}^n y_i \le \sum_{i=1}^n x_i y_i$",
  point 5/2 of bbox currentpicture shifted 24 up);
label.bot(btex \vbox{\openup12pt\halign{\hfil$#$\hfil\cr
  x_i < x_j \;\&\; y_i < y_j \quad\Rightarrow\quad x_i y_j + x_j y_i \le x_i y_i + x_j y_j \cr
  \therefore\quad
  \left(x_1 + x_2 + \cdots + x_n\right)\left(y_1 + y_2 + \cdots + y_n\right)
  \le n\left(x_1y_1 + x_2y_2 + \cdots + x_ny_n\right) \cr
  }} etex, point 1/2 of bbox currentpicture shifted 24 down);
```

The Chebyshev inequality for positive monotone sequences

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \leq \sum_{i=1}^{n} x_i y_i$$



$$x_i < x_j \ \&\ y_i < y_j \quad \Rightarrow \quad x_i y_j + x_j y_i \leq x_i y_i + x_j y_j$$

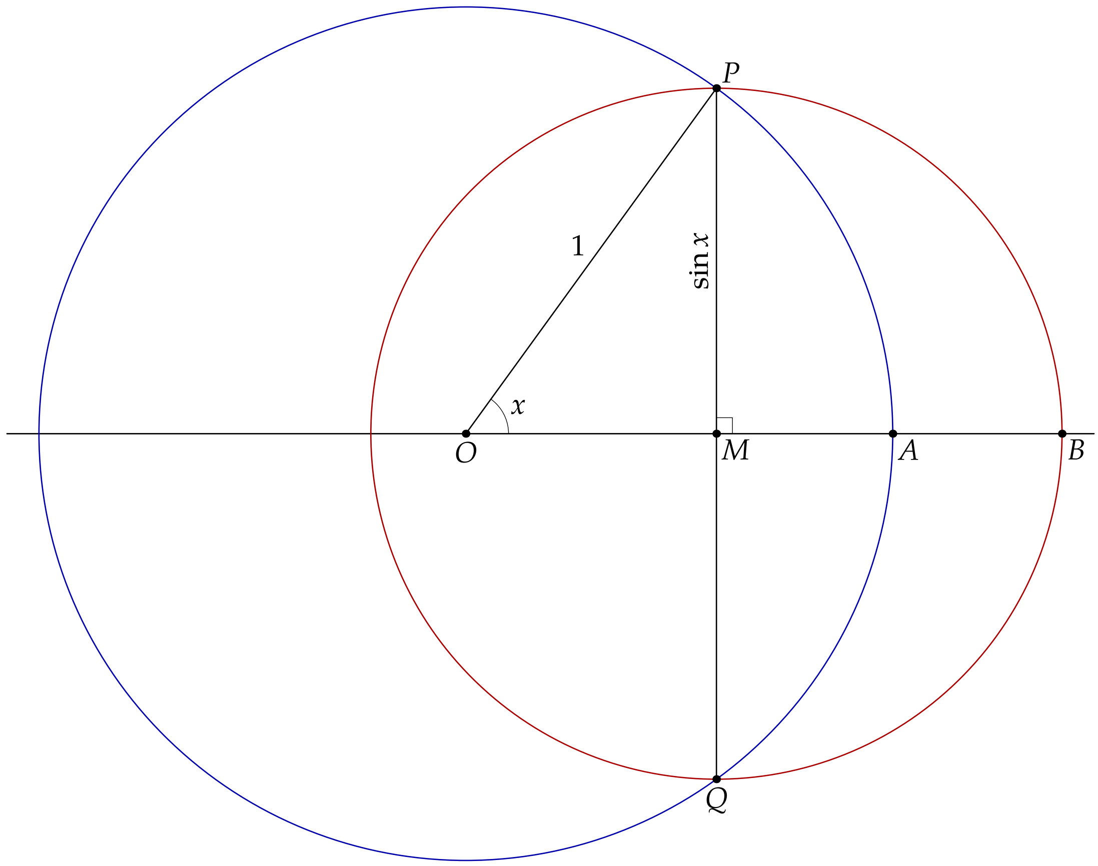
$$\therefore \quad (x_1 + x_2 + \cdots + x_n)\,(y_1 + y_2 + \cdots + y_n) \leq n\,(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

— RBN

```
path C, C';
pair A, B, M, O, P, Q; numeric t; t = 1.2;
C = fullcircle scaled 320; O = origin;
A = point 0 of C; P = point t of C; Q = point 8-t of C;
ypart M = ypart O; xpart M = xpart P;
path C'; C' = fullcircle scaled abs (P-Q) shifted M;
B = point 0 of C';

% angle marks
drawoptions(withpen pencircle scaled 1/4);
  draw unitsquare scaled 6 shifted M;
  draw fullcircle scaled 32 shifted O cutafter (O--P);
  label("$x$", 22 dir 1/2 angle (P-O));
drawoptions();

draw point 4 of C + 12 left -- B + 12 right; draw O -- P -- Q;
draw C withcolor 2/3 blue; draw C' withcolor 2/3 red;

forsuffixes $=P: dotlabel.urt("$" & str $ & "$", $); endfor
forsuffixes $=O, Q: dotlabel.bot("$" & str $ & "$", $); endfor
forsuffixes $=A,B,M: dotlabel.lrt("$" & str $ & "$", $); endfor

label.ulft("$1$", 1/2[O,P]);
draw thelabel.top("$\sin x$", origin) rotated 90 shifted 1/2[M,P];

label.top(btex $\displaystyle
  0 \le x \le {\pi\over2} \quad\Rightarrow\quad {2x\over\pi}\le\sin x \le x
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);

label.bot(btex \vbox{\openup 12pt\halign{\hfil$#$&\quad\Rightarrow\quad $#$\hfil\cr
  OB=OM+MP \ge OA & \overarc5 PBQ \ge \overarc{6.5} PAQ \ge \overline{PQ}\cr
  &\pi\sin x \ge 2x \ge 2\sin x\cr
  &2x/\pi \le \sin x \le x\cr
  }} etex, point 1/2 of bbox currentpicture shifted 24 down);
```

Jordan's inequality

$$0 \leq x \leq \frac{\pi}{2} \quad \Rightarrow \quad \frac{2x}{\pi} \leq \sin x \leq x$$



$$\begin{aligned} OB = OM + MP \geq OA \quad &\Rightarrow \quad \widehat{PBQ} \geq \widehat{PAQ} \geq \overline{PQ} \\ &\Rightarrow \quad \pi \sin x \geq 2x \geq 2 \sin x \\ &\Rightarrow \quad 2x/\pi \leq \sin x \leq x \end{aligned}$$

— Feng Yuefeng

```
z0 = (160, 133);
path xx, yy, ff;
xx = 10 left -- (x0, 0);
yy = 10 down -- (0, y0);
ff = origin .. 1/2 z0 {dir 1/2 angle z0} .. z0;

numeric a, b, c;
a = arctime  3/4  arclength ff of ff; z1 = point a of ff;
b = arctime 11/12 arclength ff of ff; z2 = point b of ff;
c = arctime  1/2  arclength ff of ff; z3 = point c of ff;

picture P[];
P0 = image(
  draw z1 -- (x1, 0) withpen pencircle scaled 1/4; label.bot("$a$", (x1, 0));
  draw ff withcolor Reds 8 8;
  drawarrow xx;
  drawarrow yy;
  label.rt("$x$", point 1 of xx);
  label.top("$y$", point 1 of yy);
  label.top(btex\vbox{\openup2pt\halign{\hfil$#$\hfil\cr
    y = \phi(x)\cr
    x = \psi(y)\cr
  }} etex, point infinity of ff);
);
P1 = image(
  fill subpath(0, b) of ff -- (0, y2) -- cycle withcolor Blues 7 2;
  fill subpath(0, a) of ff -- (x1, 0) -- cycle withcolor Reds 7 2;
  draw z2 -- (0, y2) withpen pencircle scaled 1/4; label.lft("$b$", (0, y2));
  draw z1 -- (x1,y2) withpen pencircle scaled 1/4;
  draw P0;
  label.bot("$b > \phi(a)$", point 1/2 of bbox P0 shifted 8 down);
);
P2 = image(
  fill subpath(0, c) of ff -- (0, y3) -- cycle withcolor Blues 7 2;
  fill subpath(0, a) of ff -- (x1, 0) -- cycle withcolor Reds 7 2;
  draw (x1, y3) -- (0, y3) withpen pencircle scaled 1/4; label.lft("$b$", (0, y3));
  draw P0;
  label.bot("$b < \phi(a)$", point 1/2 of bbox P0 shifted 8 down);
);
label.lft(P1, 13 left); label.rt(P2, 13 right);
```

Young's inequality

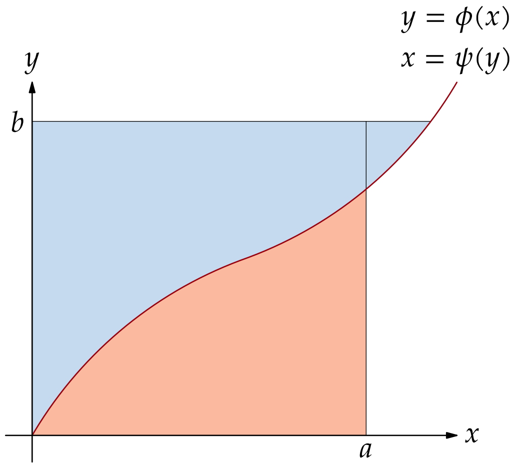
W. H. Young, "On classes of summable functions and their Fourier series", *Proc. Royal Soc.* (A), 87 (1912) 225–229.

THEOREM: Let ϕ and ψ be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each others. Then for $a, b \geq 0$ we have

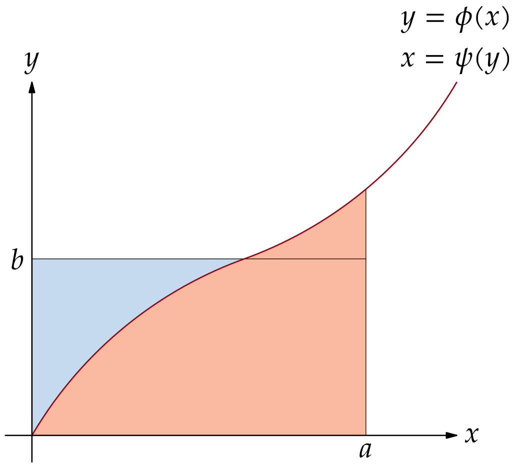
$$ab \leq \int_0^a \phi(x)dx + \int_0^b \psi(y)dy$$

with equality if and only if $b = \phi(a)$.

PROOF:



$b > \phi(a)$

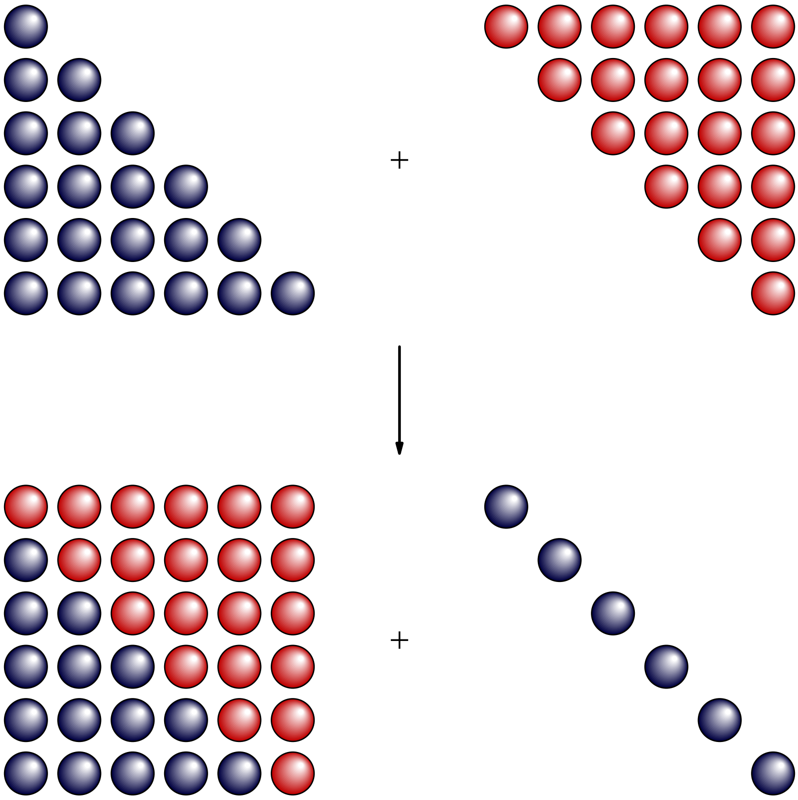


$b < \phi(a)$

Integer sums

```
input paintball

for j=1 upto 6:
  for i=0 upto 6:
    draw if j > i:
      bball shifted (20i - 140, -20j)
    else:
      rball shifted (20i +  20, -20j)
    fi;
  endfor
endfor
for j=0 upto 5:
  for i=0 upto 5:
    draw if j > i: bball else: rball fi shifted (20i - 140, -20j - 200);
  endfor
  draw bball shifted(40 + 20j, -20j - 200);
endfor
label("${}+{}$", (0, -70));
label("${}+{}$", (0, -250));
drawarrow 140 down -- 180 down withpen pencircle scaled 1;

label.bot("$\displaystyle 1+2+\cdots+n={1\over2}\left(n^2 + n\right)$",
  point 1/2 of bbox currentpicture shifted 36 down);
```

Sums of integers III



$$1 + 2 + \cdots + n = \frac{1}{2}\left(n^2 + n\right)$$

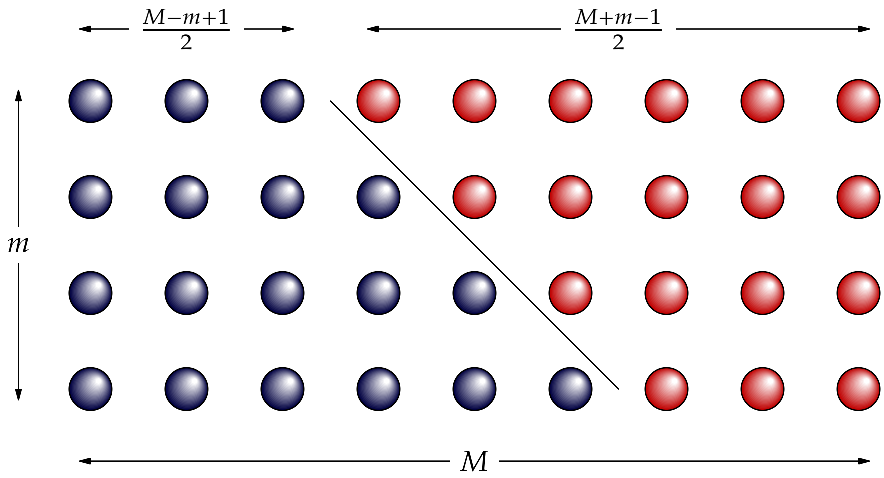
— S. J. Barlow

```
input paintball

numeric u; u = 36;
for i=-4 upto 4:
  for j=0 upto 3:
    draw if i+j > 1: rball else: bball fi shifted ((i, j) scaled u);
  endfor
endfor

draw (-3/2u, 3u) -- (3/2u, 0);

vardef mark_dimen(expr S, a, b) =
  save t; t = 1 + 4/abs(a-b); drawdblarrow t[b,a] -- t[a,b];
  save P; picture P; P = thelabel(S, origin);
  unfill bbox P shifted 1/2[a,b];
  draw P shifted 1/2[a,b];
enddef;

interim bboxmargin := 4;
mark_dimen("$M$", (-4u, -3/4u), (4u, -3/4u));
mark_dimen("$m$", (-4.75u, 0), (-4.75u, 3u));
mark_dimen("${M-m+1\over2}$", (-4u, 3.75u), (-2u, 3.75u));
mark_dimen("${M+m-1\over2}$", (-1u, 3.75u), (+4u, 3.75u));

label.top(btex \vbox{\openup8pt\halign{\hfil $##$&${} = #$\hfil\cr
  N & 2^n(2k+1)\quad \hbox{($n\ge0$, $k\ge1$)}\cr
  m & \min\left\{ 2^{n+1}, 2k+1\right\}\cr
  M & \max\left\{ 2^{n+1}, 2k+1\right\}\cr
  2N & mM\cr
  }} etex, point 5/2 of bbox currentpicture shifted 24 up);

label.bot(btex $\displaystyle
  N = \left({M-m+1\over2}\right) + \left({M-m+1\over2}+1\right) + \cdots +
  \left({M+m-1\over2}\right)
  $ etex, point 1/2 of bbox currentpicture shifted 24 down);
```

Sums of consecutive positive integers

Every integer $N > 1$, not a power of two, can be expressed as the sum of two or more positive integers.

$$\begin{aligned} N &= 2^n(2k+1) \quad (n \geq 0, k \geq 1) \\ m &= \min\left\{2^{n+1}, 2k+1\right\} \\ M &= \max\left\{2^{n+1}, 2k+1\right\} \\ 2N &= mM \end{aligned}$$



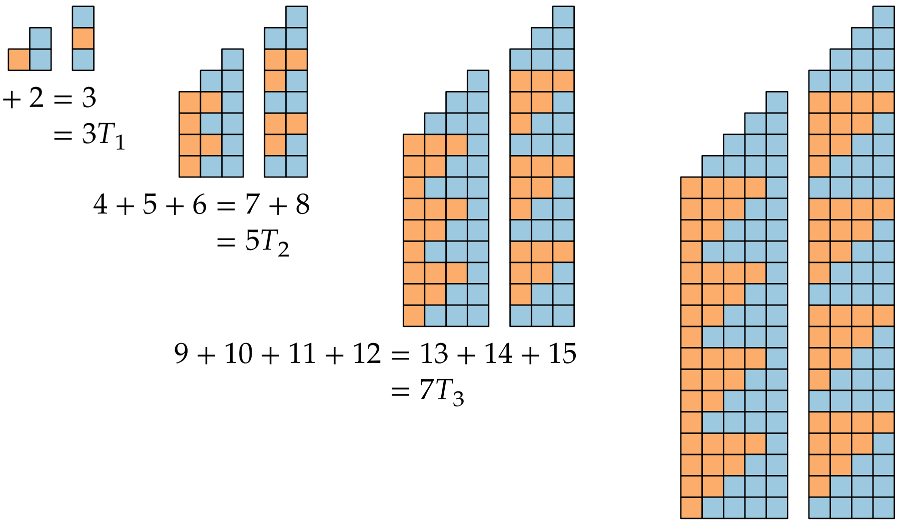
$$N = \left(\frac{M-m+1}{2}\right) + \left(\frac{M-m+1}{2}+1\right) + \cdots + \left(\frac{M+m-1}{2}\right)$$

— C. L. Frenzen

```
vardef f(expr n, c) = save s; image(
  for y = 0 upto n-1:
    for x = 0 upto y:
      path s; s = unitsquare shifted (x, y) scaled 8;
      fill s withcolor c; draw s;
    endfor
  endfor)
enddef;
numeric x, dy; x = dy = 0;
for i=1 upto 4:
  picture F, J;
  F = f(i, Oranges 7 3);
  J = f(i, Blues 7 3); J := J rotatedabout(center J, 180) shifted 8 right;
  x := x + 20i + 24;
  dy := - 8(i**2+2i);
  for y=0 upto i-1:
    draw F shifted (x,           dy + y*i*8);
    draw F shifted (x + (i+2)*8, dy + y*i*8 + (y+1)*8);
  endfor
  for y=0 upto i:
    draw J shifted (x,           dy + y*i*8);
    draw J shifted (x + (i+1)*8, dy + y*i*8 + (y)*8);
  endfor
  string s, t;
  s = decimal (i**2) for k = i**2 + 1 upto (i+1)**2 - 1:
    & if k=(i+1)**2-i: "&" else: "+" fi & decimal k
  endfor & "\cr";
  t = "&" & decimal (2i+1) & "T_{" & decimal i & "}\hidewidth\cr";
  label.llft("$\vbox{\halign{\hfil$#$&${}=#$\hfil\cr" & s & t & "}}$",
    (x + (2i + 2.4) * 8, dy - 4));
endfor
label.urt("$T_k = 1 + 2 + \cdots + k$ \quad \Longrightarrow",
  point -1 of bbox currentpicture shifted 32 up);
label.bot(btex \vbox{\openup 8pt\halign{\hfil$#$&${}=#$\hfil\cr
  n^2 + (n^2+1) + \cdots + (n^2 + n)&(n^2+n+1) + \cdots + (n^2+2n)\cr
  &(2n+1)T_n\cr
}} etex, point 1/2 of bbox currentpicture shifted 32 down);
```

Consecutive sums of consecutive integers II

$T_k = 1 + 2 + \cdots + k \quad \Longrightarrow$



$1 + 2 = 3$
$= 3T_1$

$4 + 5 + 6 = 7 + 8$
$= 5T_2$

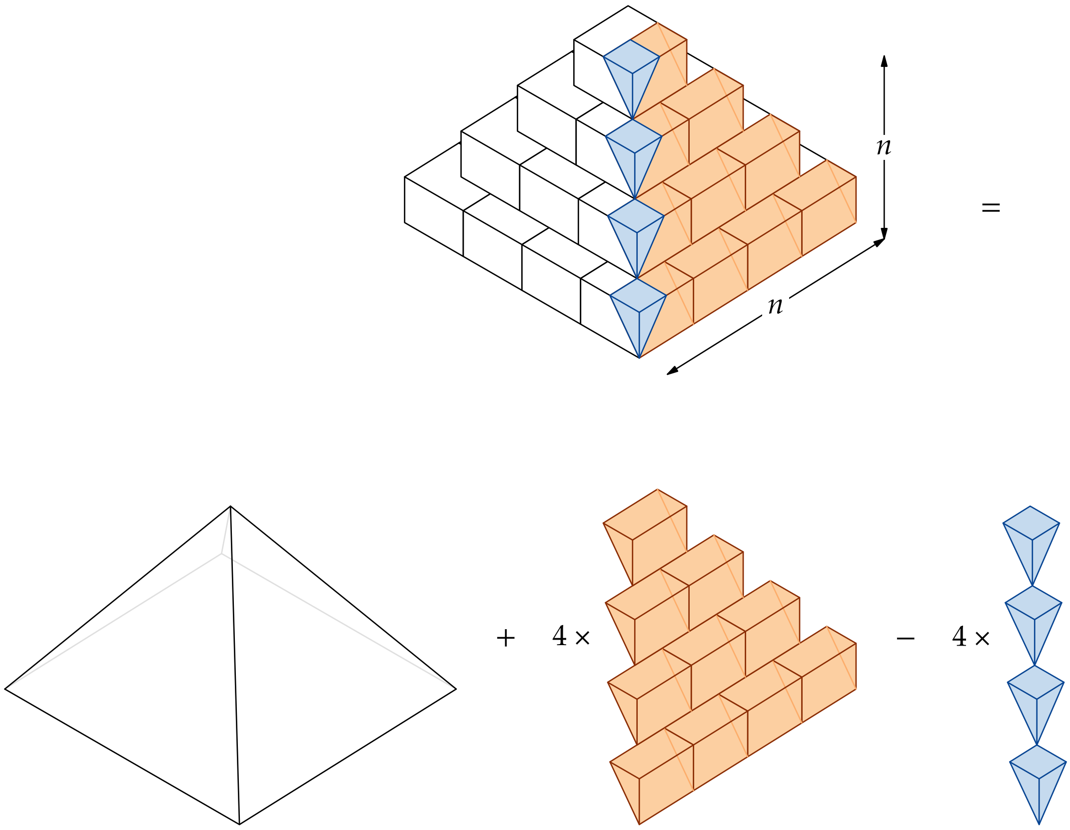
$9 + 10 + 11 + 12 = 13 + 14 + 15$
$= 7T_3$

$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$
$= 9T_4$

$$\begin{aligned} n^2 + (n^2+1) + \cdots + (n^2+n) &= (n^2+n+1) + \cdots + (n^2+2n) \\ &= (2n+1)T_n \end{aligned}$$

```
% quick & dirty Isometric projection
vardef p(expr x, y, z) = 0.50824829 * (1.73205x + 1.6z, -x + 1.8y + z) enddef;
numeric q, s, h; s = 100; h = 3/4 s; q = 1/4 s;
path base; base = p(0,0,0)--p(s,0,0)--p(s,0,s)--p(0,0,s)--cycle;
pair apex; apex = p(1/2s,h,1/2s);
path ft; ft = p(0,0,0)--p(q,0,0)--p(q,1/4h,0)--p(q,1/4h,q)--p(0,1/4h,q)--p(0,1/4h,0)--cycle;
path vv; vv = p(0,0,0)--p(0,0,q)--p(0,1/4h,q)--p(-1/2q,1/4h,q)--p(-1/2q,1/4h,0)--cycle;
path ww; ww = p(0,0,0)--p(0,1/4h,1/2q)--p(-1/2q,1/4h,1/2q)--p(-1/2q,1/4h,0)--cycle;
picture cube, wedge, corner;
cube = image(
  fill ft withcolor white; draw ft; draw point 2 of ft -- point -1 of ft;
);
wedge = image(
  fill vv withcolor Oranges 8 3; draw vv withcolor Oranges 8 8;
  draw point 1 of vv -- point 3 of vv withcolor Oranges 8 4;
  forsuffixes $=0, 2, 4: draw p(0,1/4h,0) -- point $ of vv withcolor Oranges 8 8; endfor
);
corner = image(
  fill ww withcolor Blues 8 3; draw ww withcolor Blues 8 8;
  forsuffixes $=0, 1, 3: draw p(0,1/4h,0) -- point $ of ww withcolor Blues 8 8; endfor
);
picture P[];
P1 = image(
  draw subpath(2, 4) of base   withcolor 7/8 white;
  draw point 3 of base -- apex withcolor 7/8 white;
  draw subpath(0, 2) of base;
  for i=0 upto 2: draw point i of base -- apex; endfor
);
P2 = image(
  for k=0 upto 3:
    for i=0 upto 3-k:
      for j=3-k downto 0:
        draw cube shifted p(i*q + 1/2q*k, k/4*h, j*q + 1/2q*k);
      endfor
    endfor
  endfor
  picture nn; nn = thelabel("$n$", origin); numeric o; o = 12; path a[];
  a1 = p(s+o,0,0) -- p(s+o,0,s);
  a2 = p(s+o,h,s) -- p(s+o,0,s);
  forsuffixes $=1,2:
    drawdblarrow a$;
    unfill bbox nn shifted point 1/2 of a$;
    draw nn shifted point 1/2 of a$;
  endfor
);
% ... plus simpler loops for P3 and P4
draw P2 shifted (0, 7/4s); draw P3 shifted (0,7/4s); draw P4 shifted (0,7/4s);
draw P1 shifted (-6/4s,0); draw P3 shifted (0,0);    draw P4 shifted (6/4s,0);
label("$=$", (2s+20, 2s-20));
label("${}+\quad4\times{}$", (52, 20));
label("${}-\quad4\times{}$", (202,20));
```

Sums of squares VI



$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n^2 \times n \quad + \quad 4 \times \frac{n(n+1)}{2} \times \frac{1}{4} \quad - \quad 4 \times n \times \frac{1}{12}$$

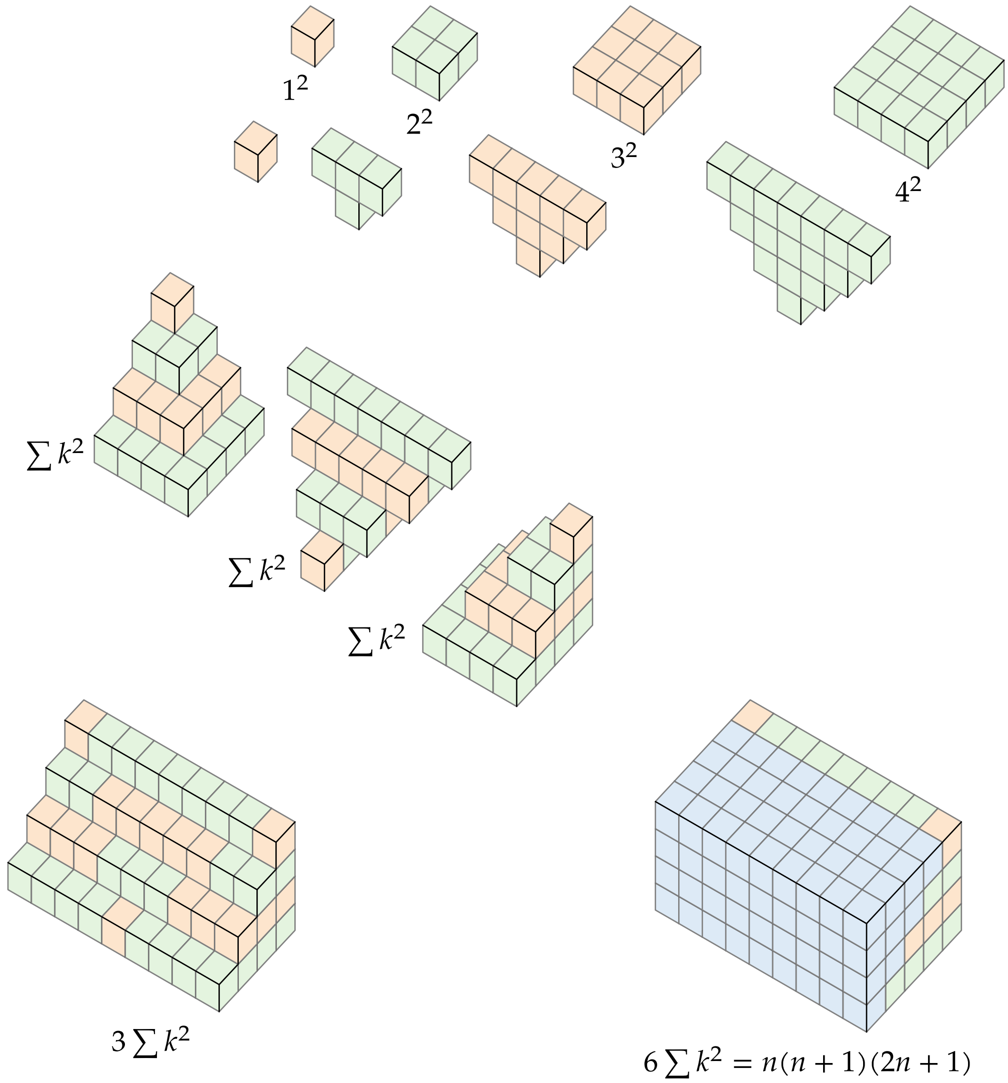
$$= \frac{1}{6}n(n+1)(2n+1)$$

— I. A. Sakmar

```
vardef p(expr x, y, z) = 5.0824829 * (1.73205x + 1.4z, -x + 2y + 1.5z) enddef;
path pcube; pcube = origin--p(1,0,0)--p(1,0,1)--p(1,1,1)--p(0,1,1)--p(0,1,0)--cycle;
vardef make_cube(expr shade) = image(
  fill pcube withcolor shade; draw pcube withcolor 1/2 white;
  for t=1,3,5: draw point t of pcube -- p(1,1,0); endfor
) enddef;
picture cube[], slab[], tee[];
cube0 = make_cube(Greens 8 2); cube1 = make_cube(Oranges 8 2);
for k=1 upto 4:
  slab[k] = image(for j=1 upto k: for i=1 upto k:
    draw cube[k mod 2] shifted p(i, 4, -j);
  endfor endfor);
  tee[k] = image(for j=1 upto k: for i=1 upto 2j-1:
    draw cube[k mod 2] shifted p(i-j+1, j-k+2, -k-3);
  endfor endfor);
  pair t; t = (1 + 3/4(k**2), 5) scaled 20;
  draw slab[k] shifted t; draw tee[k] shifted t;
  label("$" & decimal k & "^2$", p(k+1, 4, -k-1)) shifted t;
endfor
for k=4 downto 1:
  draw slab[k] shifted p(-2, -k, 0);
  draw tee[k]  shifted p( 6, k-4, 2k-1);
  draw slab[k] shifted p(16-k, -k, 0);
endfor
picture kk; kk = thelabel("$\sum k^2$", origin);
for x=-5.1, 3.5, 8.6: draw kk shifted p(x, -4, -1); endfor
picture bank; bank = image(
  for k=4 downto 1:
    draw slab[k] shifted p(0, -k, 0);
    for j=k+1 upto 9-k:
      draw cube[(5-k) mod 2] shifted p(j, 4-k, -k);
    endfor
    draw slab[k] shifted p(9-k, -k, 0);
  endfor);
cube2 = make_cube(Blues 8 2);
picture block; block = image(
  draw bank;
  for k=0 upto 3:
    for j=k-5 downto -5:
      for i=1 upto 9:
        draw cube2 shifted p(i, k, j);
      endfor
    endfor
  endfor
  label.bot("$6\sum k^2=n(n+1)(2n+1)$", point 1/2 of bbox currentpicture));
bank := image(draw bank; label.bot("$3\sum k^2$", point 1/2 of bbox currentpicture));
draw bank shifted (-50, -150); draw block shifted (200, -150);
label.top("$\displaystyle \sum_{k=1}^n k^2 = {n(n+1)(2n+1)\over 6}$",
  point 5/2 of bbox currentpicture shifted 32 up);
```

Sums of squares VII

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$



— Nanny Wermuth and Hans-Jürgen Schuh

```
path t[]; t0 = for i=0 upto 2: 72 dir (90+120i) -- endfor cycle;
numeric s; s = 30 + arclength subpath (0, 1) of t0;
for i=1 upto 3:
  t[i] = t0 rotated 120(2-i) shifted ((i-2)*s, s);
endfor
for i=0 upto 3:
  filldraw t[i] withpen pencircle scaled 3 withcolor Blues 8 2;
endfor
picture elips; elips = thelabel("$\cdots$", origin);
for i=1, 3, 5:
  numeric n; n = 5 - 1/2(i-1);
  for k=1 upto 3:
    for j=0 upto n:
      label(if j=2: elips rotated 60(1+k) else: "$" & decimal i & "$" fi,
      ((i-1)/10)[(j/n)[point 2 of t[k], point 1 of t[k]], point 0 of t[k]]);
    endfor
    label("$\scriptstyle 2n-1$", point 0 of t[k]);
    label("$\scriptstyle 2n-3$", 4/5[point 1 of t[k], point 0 of t[k]]);
    label("$\scriptstyle 2n-3$", 4/5[point 2 of t[k], point 0 of t[k]]);
    label(elips rotated (60k-60), 3/5[point 1 of t[k], point 0 of t[k]]);
    label(elips rotated (60k), 3/5[point 2 of t[k], point 0 of t[k]]);
  endfor
endfor
label("$\scriptstyle 2n+1$", point 0 of t0);
label(elips rotated 60, 3/5[point 0 of t0, point 1 of t0]);
label(elips rotated -60, 3/5[point 0 of t0, point 2 of t0]);
for i=1, 2, 4, 5:
  for j=0 upto i:
    label(if j=3: elips else: "$\scriptstyle 2n+1$" fi,
      (i/5)[point 0 of t0, (j/i)[point 1 of t0, point 2 of t0]]);
  endfor
endfor
z1 = 1/2[center t1, center t2];
z2 = 1/2[center t2, center t3];
z3 = center t0;
label("${}+{}$", z1);
label("${}+{}$", z2);
label("${}={}$", (x1, y3));

label.top(btex $\displaystyle
k^2 = 1+3+\cdots+(2k-1) \quad\implies\quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}
$ etex, point 5/2 of bbox currentpicture shifted 42 up);

label.bot(btex \vbox{\openup 12pt\halign{\hss $\displaystyle #$&$\displaystyle {}=#$\hss\cr
  3\left(1^2 + 2^2 + \cdots + n^2 \right) & (2n+1)(1+2+\cdots+n)\cr
  \therefore\quad 1^2 + 2^2 + \cdots + n^2& \frac{2n+1}{3}\cdot\frac{n(n+1)}{2}\cr
}} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

Sums of squares VIII

$$k^2 = 1 + 3 + \cdots + (2k-1) \quad \Rightarrow \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$
\begin{array}{c}
1 \\
3 \quad 1 \\
5 \quad 3 \quad \ddots \\
\iddots \quad 5 \quad \ddots \quad 1 \\
2n{-}3 \quad \ddots \quad 3 \quad 1 \\
2n{-}1 \quad 2n{-}3 \quad \cdots \quad 5 \quad 3 \quad 1
\end{array}
\; + \;
\begin{array}{c}
2n{-}1 \\
2n{-}3 \quad 2n{-}3 \\
\iddots \qquad \ddots \\
5 \quad \cdots \quad 5 \quad 5 \\
3 \quad 3 \quad \cdots \quad 3 \quad 3 \\
1 \quad 1 \quad 1 \quad \cdots \quad 1 \quad 1
\end{array}
\; + \;
\begin{array}{c}
1 \\
1 \quad 3 \\
1 \quad 3 \quad 5 \\
\iddots \quad \iddots \quad \iddots \quad \ddots \\
1 \quad 3 \quad 5 \quad 2n{-}3 \\
1 \quad 3 \quad 5 \quad \cdots \quad 2n{-}3 \quad 2n{-}1
\end{array}
$$

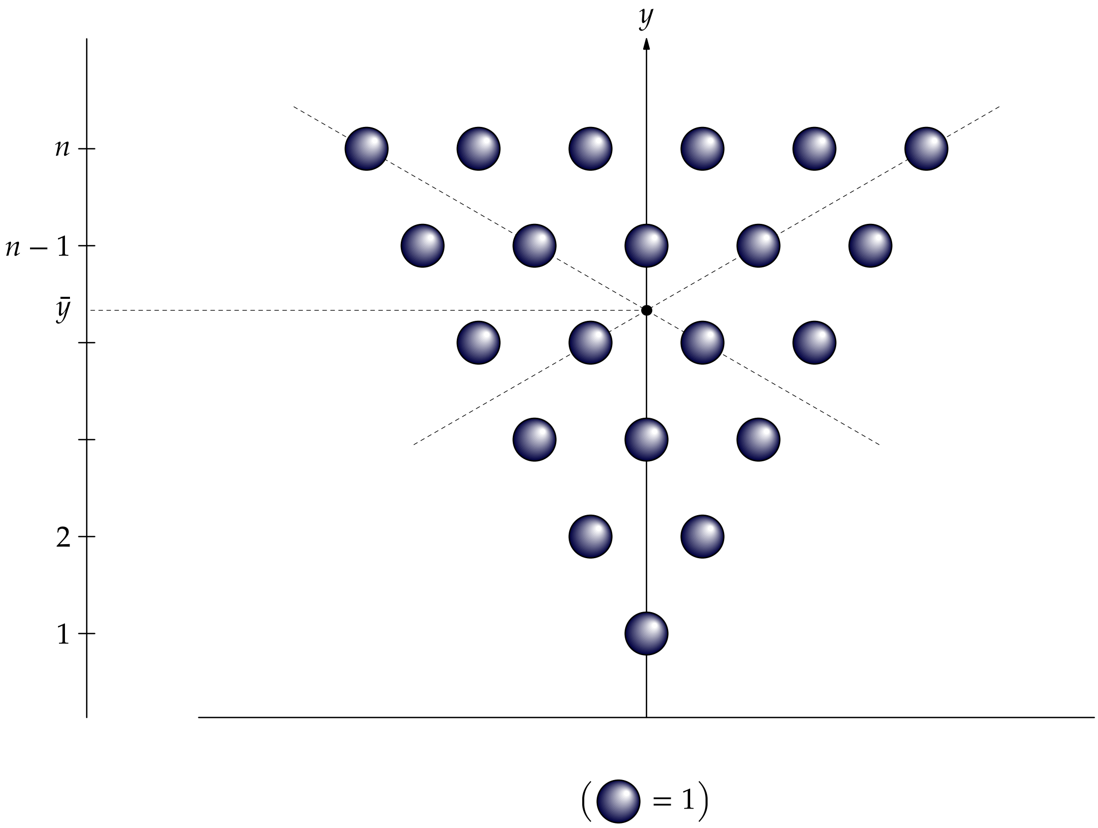
$$
= \;
\begin{array}{c}
2n{+}1 \\
2n{+}1 \quad 2n{+}1 \\
2n{+}1 \quad 2n{+}1 \quad 2n{+}1 \\
\iddots \qquad\qquad \ddots \\
2n{+}1 \quad 2n{+}1 \quad 2n{+}1 \quad \cdots \quad 2n{+}1 \\
2n{+}1 \quad 2n{+}1 \quad 2n{+}1 \quad \cdots \quad 2n{+}1 \quad 2n{+}1
\end{array}
$$

$$3\left(1^2 + 2^2 + \cdots + n^2\right) = (2n+1)(1 + 2 + \cdots + n)$$

$$\therefore \quad 1^2 + 2^2 + \cdots + n^2 = \frac{2n+1}{3} \cdot \frac{n(n+1)}{2}$$

```
numeric u, n; n = 6; u = 42;
pair uu, vv; uu = right scaled u; vv = uu rotated 120;
path xx, yy, ss;
xx = (left -- right) scaled 4u shifted (3/4 u * down);
yy = point 1/2 of xx -- point 1/2 of xx shifted (7 vv + 3.5 uu);
ss = yy shifted (5u * left);
x0 = 0; y0 = y1 = 2/3(n-1) * ypart vv;
x1 = xpart point 0 of ss;

drawoptions(dashed evenly scaled 1/2 withpen pencircle scaled 1/4);
  for t=1,2: draw yy rotatedabout(z0, 120t); endfor
  draw z0 -- z1;
drawoptions();
draw xx; drawarrow yy; draw ss;
draw z0 withpen pencircle scaled 4;
label.top("$y$", point 1 of yy);

y = -1;
for i=1 upto n:
  draw (left--right) scaled 3 shifted (x1, incr y * ypart vv);
  if i <= 2:
    label.lft("$" & decimal i & "$", (x1 -3, y * ypart vv));
  elseif i = n-1:
    label.lft("$n-1$", (x1 -3, y * ypart vv));
  elseif i = n:
    label.lft("$n$", (x1 -3, y * ypart vv));
  fi
endfor
label.lft("$\bar{y}$", (x1 - 3, y1));

input paintball
for i=0 upto n-1:
  for j=0 upto i:
    draw bball shifted (i * vv + j * uu);
  endfor
endfor

picture b; b = bball shifted (-1/4u, -3/2u);
label.lft("$\Bigl($", point -1/2 of bbox b shifted 4 right);
draw b;
label.rt("${}=1\Bigr)$", point 3/2 of bbox b shifted 4 left);

label.bot(btex \vbox{\openup 12pt\halign{\hfil$\displaystyle # $\hfil\cr
  \bar{y} = 1 + \frac23 (n-1) = \frac{1\cdot1+2\cdot2+\cdots+n\cdot n}{1+2+\cdots+n}\cr
  \therefore\quad
  1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{2}\cdot \frac13 (2n+1) = \frac16 n(n+1)(2n+1)\cr
  }} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

Sums of squares IX (via centroids)



$$\bar{y} = 1 + \frac{2}{3}(n-1) = \frac{1 \cdot 1 + 2 \cdot 2 + \cdots + n \cdot n}{1 + 2 + \cdots + n}$$

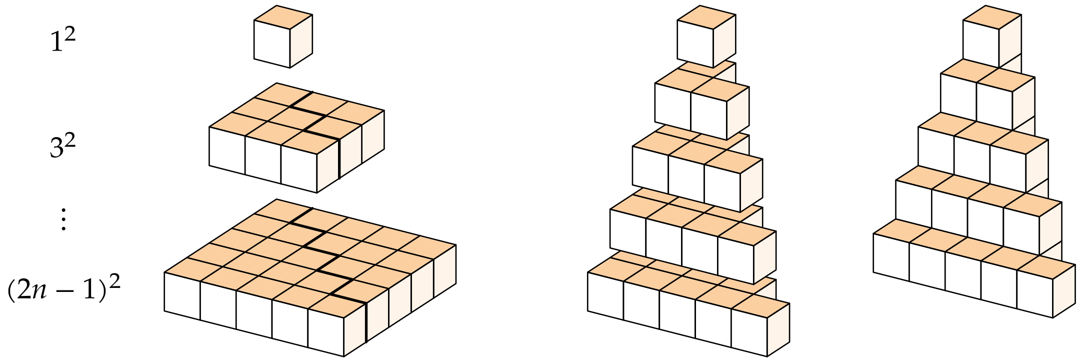
$$\therefore \quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{2} \cdot \frac{1}{3}(2n+1) = \frac{1}{6}n(n+1)(2n+1)$$

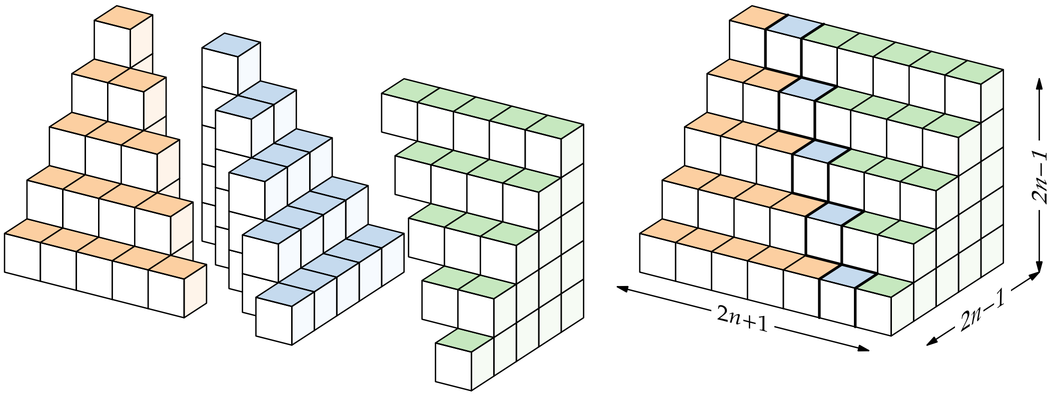
— Sidney H. Kung

```
input isometric_projection
set_projection(24, -32); ipscale := 16;
path upper; upper = p(0,1,0) -- p(1,1,0) -- p(1,1,1) -- p(0,1,1) -- cycle;
path front; front = p(0,0,0) -- p(1,0,0) -- p(1,1,0) -- p(0,1,0) -- cycle;
path side; side  = p(1,0,0) -- p(1,0,1) -- p(1,1,1) -- p(1,1,0) -- cycle;
vardef make_cube(expr u, f, s) =  image(
  fill upper withcolor u; fill front withcolor f; fill side withcolor s;
  draw upper; draw front; draw side) enddef;
picture Cube[];
Cube1 = make_cube(Oranges 8 3, background, Oranges 8 1);
Cube2 = make_cube(Blues   8 3, background, Blues   8 1);
Cube3 = make_cube(Greens  8 3, background, Greens  8 1);
picture ix, xxv;
ix = image(
  for i=0 downto -2: for j=0 upto 2: draw Cube1 shifted p(j,0,i); endfor endfor
  draw p(3, 0, -1) -- p(3, 1, -1) -- p(2,1,-1) -- p(2,1,0) -- p(1,1,0) -- p(1,1,1)
    withpen pencircle scaled 1);
xxv = image(
  for i=0 downto -4: for j=0 upto 4: draw Cube1 shifted p(j,0,i); endfor endfor
  draw p(5, 0, -3) for t=4 downto 1: -- p(t+1, 1, 1-t) -- p(t,1,1-t) -- p(t,1,2-t) endfor
    withpen pencircle scaled 1);
picture P[];
% P0 is the three "slabs" plus the labels on the left
P0 = image(draw Cube1; draw ix shifted p(0,-2,0); draw xxv shifted p(0, -5, 0);
  label.top("$1^2$",      (-72, ypart p(0,  0,  0)));
  label.top("$3^2$",      (-72, ypart p(0, -2, -2)));
  label.top("$\vdots$",   (-72, ypart p(0, -3.5, -3)));
  label.top("$(2n-1)^2$", (-72, ypart p(0, -5, -4))));

% P1 is the two stacks of cubes on the right
P1 = image(
for j=-4 upto 0:
  for k=0 downto j:
    for i=0 upto -k:
      draw Cube1 shifted p(i, 5/4 j, k);
      draw Cube1 shifted p(i, j, k) shifted 108 right;
    endfor
  endfor
endfor);

% P2 and P3 are similar, but longer...

draw P0; draw P1 shifted (160,0); draw P2 shifted (-40, -180); draw P3 shifted (200, -180);

label.top("$\displaystyle 1^2 + 2^2 + \cdots + \left(2n-1\right)^2 = {n(2n-1)(2n+1)\over3}$",
  point 5/2 of bbox currentpicture shifted 42 up);
label.bot(btex \vbox{\openup 12pt\halign{\hfil$\displaystyle #$&$\displaystyle{}=#$\hfil\cr
  3 \times \left(1^2 + 3^2 + \cdots + \left(2n-1\right)^2\right)
  &\left(1+2+\cdots+(2n-1)\right) \times (2n+1)\cr
  &{(2n-1)(2n)(2n+1)\over2} = n(2n-1)(2n+1)\cr
  }} etex, point 1/2 of bbox currentpicture shifted 24 down);
```

Sums of odd squares

$$1^2 + 2^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$





$$3 \times \left(1^2 + 3^2 + \cdots + (2n-1)^2\right) = (1 + 2 + \cdots + (2n-1)) \times (2n+1)$$

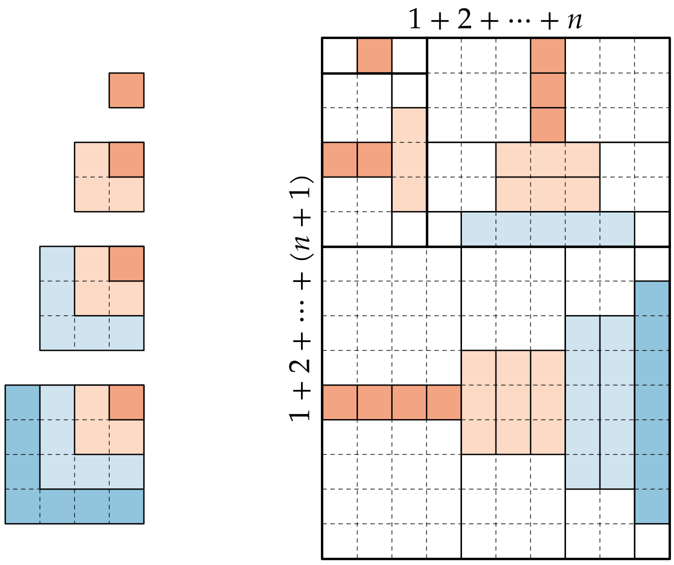
$$= \frac{(2n-1)(2n)(2n+1)}{2} = n(2n-1)(2n+1)$$

— RBN

```
numeric u; u = 13; def dotty = dashed evenly scaled 1/2 withpen pencircle scaled 1/4 enddef;
picture P[];
P0 = image(for i=1 upto 4: for j=i downto 1:
  path f; f = unitsquare scaled (-j*u) shifted (0, -u * i * (i+1)/2);
  fill f withcolor RdBu[8][2+j];
  for k=1 upto j-1:
    draw subpath(1,2) of f shifted (k*u, 0) dotty;
    draw subpath(2,3) of f shifted (0, k*u) dotty;
  endfor
  draw f;
endfor endfor);
for n=1 upto 4:
  P[n] = image(pair start; start = origin;
    for i=1 upto n:
      numeric k; k = n + 1 - i;
      path s; s = unitsquare scaled u xscaled k yscaled (2i-1) shifted start;
      path t; t = subpath (0,1) of s shifted (0, -k*u) --
                  subpath (2,3) of s shifted (0, +k*u) -- cycle;
      fill s withcolor RdBu[8][2+i];
      for j=1 upto k-1:
        draw subpath (0, -1) of t shifted (j*u, 0) dotty;
        draw subpath (0, -1) of s shifted (j*u, 0);
      endfor
      for j=1 upto 2n:
        draw subpath (0, 1) of t shifted (0, j*u) dotty;
      endfor
      draw s; draw t;
      start := point 1 of s shifted (0, -u);
    endfor) if odd n: rotated -90 fi;
endfor
P2 := P2 shifted (-u, -4u); P3 := P3 shifted (5u, 0); P4 := P4 shifted (-u, -11u);
interim bboxmargin := 0;
draw P0 shifted 80 left;
draw P1; draw bbox P1 withpen pencircle scaled 3/4;
draw P2; draw subpath (-1, 2) of bbox P2 withpen pencircle scaled 3/4;
draw P3; draw subpath (0, 3)  of bbox P3 withpen pencircle scaled 3/4;
draw P4; draw subpath (-1, 2) of bbox P4 withpen pencircle scaled 3/4;
label.top("$1+2+\cdots+n$", point 19/7 of bbox P3);
draw thelabel.top("$1+2+\cdots+(n+1)$", origin) rotated 90 shifted point -5/6 of bbox P4;

label.top(btex $\displaystyle
  \sum_{k=1}^n \sum_{i=1}^k i^2 =\frac13 {n+1\choose 2}{n+2 \choose 2}
  $ etex, point 5/2 of bbox currentpicture shifted 42 up);
label.bot(btex $\displaystyle
  3\left(1^2\right) + 3\left(1^2 + 2^2\right) + 3\left(1^2 + 2^2 + 3^2\right) + \cdots +
  3\left(1^2 + 2^2 + \cdots + n^2\right) = {n+1\choose2}{n+2\choose2}
  $ etex, point 1/2 of bbox currentpicture shifted 24 down);
```

Sums of sums of squares

$$\sum_{k=1}^{n}\sum_{i=1}^{k} i^2 = \frac{1}{3}\binom{n+1}{2}\binom{n+2}{2}$$



$$3\left(1^2\right)+3\left(1^2+2^2\right)+3\left(1^2+2^2+3^2\right)+\cdots+3\left(1^2+2^2+\cdots+n^2\right)=\binom{n+1}{2}\binom{n+2}{2}$$

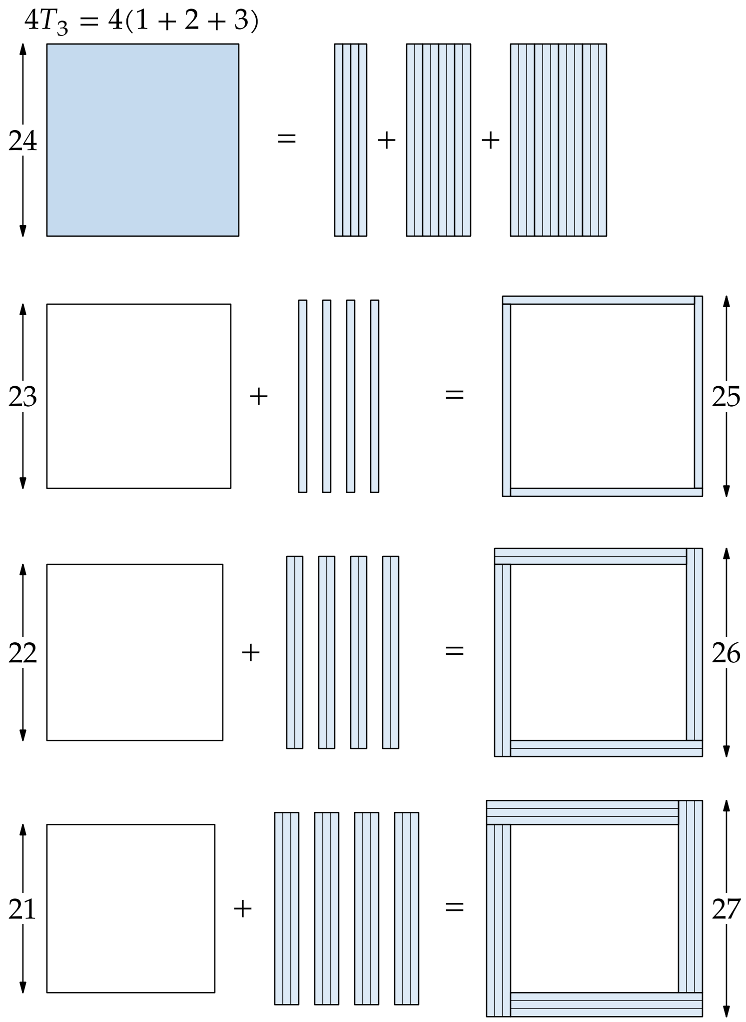
— C. G. Wastun

```
numeric u; u = 3;
path base; base = unitsquare shifted 1/2 down scaled 24u;
picture piece[];
forsuffixes @=1,2,3:
  piece@ = image(
    path p; p = subpath (0, @/24) of base -- subpath (3-@/24, 3) of base -- cycle;
    fill p withcolor Blues 8 2;
    for i=1 upto @-1:
      draw subpath (0, -1) of p shifted (i*u,0) withpen pencircle scaled 1/4;
    endfor
    draw p;
  );
endfor

vardef mark_dimen(expr p, t) =
  save pp, tt;
  path pp; pp = p shifted (unitvector(direction 1/2 of p) rotated 90 scaled 3 labeloffset);
  picture tt; tt = thelabel(t, point 1/2 of pp);
  drawdblarrow pp; unfill bbox tt; draw tt;
enddef;

label.top("$4T_3 = 4(1+2+3)$", point 5/2 of base);
mark_dimen(subpath(0, -1) of base, "$24$");
fill base withcolor Blues 8 3; draw base;
for i=0 upto 3:
  draw piece1 shifted (36u +  i*u, 0);
  draw piece2 shifted (45u + 2i*u, 0);
  draw piece3 shifted (58u + 3i*u, 0);
endfor
label("$=$", (30u, 0)); label("$+$", (42.5u, 0)); label("$+$", (55.5u, 0));

for i=1 upto 3:
  draw image(
    path s; s = unitsquare shifted 1/2 down scaled (24u - i*u);
    path t; t = s shifted (58u, 0);
    draw s;
    mark_dimen(subpath (0, -1) of s, decimal (24-i));
    mark_dimen(point 2 of t + (i,i)*u -- point 1 of t + (i,-i)*u, decimal (24+i));
    for j=0 upto 3:
      draw piece[i] shifted (36u + (2j-3)*(1/2i*u+u), 0);
      draw piece[i] shifted (0, 1/2i*u) rotated 90j shifted point j+3/2 of t;
    endfor
    label("$=$", (51u, 0)); label("$+$", (27.5u - i*u, 0));
  ) shifted (0, -32u * i);
endfor

label.ulft("e.g., $n=3$:", point -1 of bbox currentpicture shifted 6 up);

% see source for the TeX label at the top
```

Pythagorean runs

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$\vdots$$
$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow \quad (4T_n - n)^2 + \cdots + (4T_n)^2 = (4T_n + 1)^2 + \cdots + (4T_n + n)^2$$

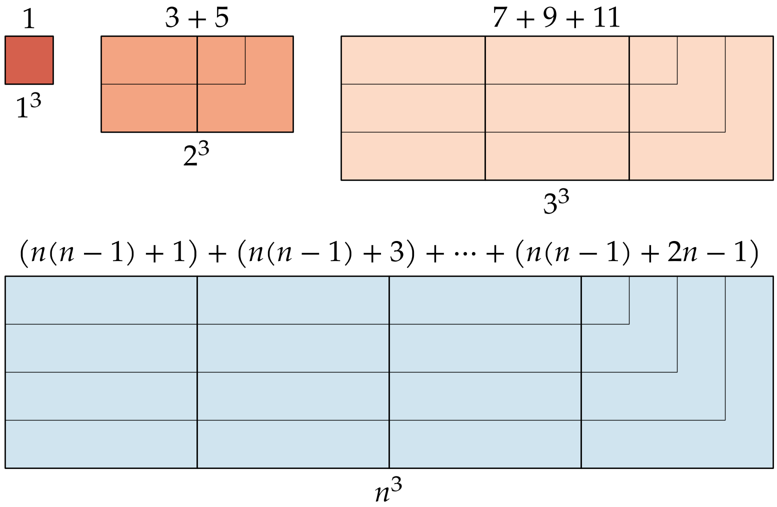
e.g., $n = 3$:

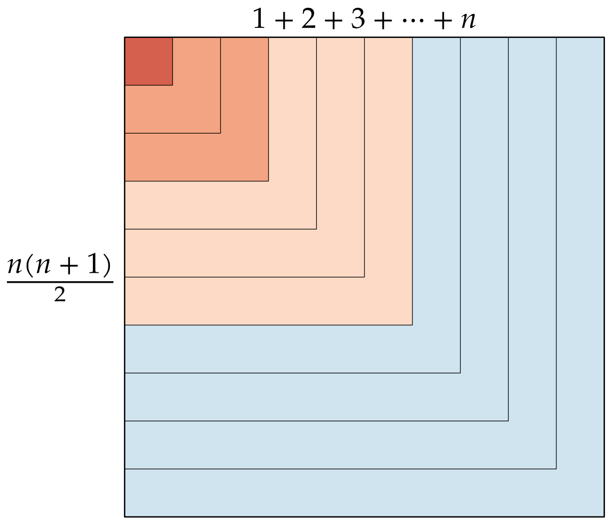


— Michael Boardman

```
numeric c; u = 18;
numeric x, y; x = y = 0;
for i=1 upto 4:
  path s; s = unitsquare shifted down scaled (i*u);
  if i<4:
    label.top("$" & decimal (i*(i-1)+1) for j=i*(i-1)+3 step 2 until i*(i-1)+2i-1:
      & "+" & decimal j
    endfor & "$", (x + i*i/2 * u, 0));
    label.bot("$" & decimal i & "^3$", (x + i*i/2 * u, y - i * u));
  else:
    label.top("$\bigl(n(n-1)+1\bigr) + \bigl(n(n-1)+3\bigr) + \cdots" &
              "+ \bigl(n(n-1)+2n-1\bigr)$", (x + i*i/2 * u, -5u));
    label.bot("$n^3$", (x + i*i/2 * u, y - i * u));
  fi
  for j=1 upto i:
    fill s shifted (x, y) withcolor RdBu[8][1+i];
    for k=1 upto i-1:
      draw
        if j < i:
          subpath (0,1) of s shifted (x, y+k*u)
        else:
          subpath (0, 1-k/i) of s shifted (x, y+k*u) -- point 2+k/i of s shifted (x, y)
        fi
        withpen pencircle scaled 1/4;
    endfor
    draw s shifted (x, y);
    x := x + (i*u);
  endfor
  x := x + u; if i=3: x := 0; y := -5u; fi
endfor
def trrt(expr x) = ceiling 1/2(sqrt(8x + 1) - 1) enddef;
for i=10 downto 1:
  path s; s = unitsquare shifted down scaled (i*u) shifted (3u, -12u);
  fill s withcolor RdBu[8][1 + trrt(i)];
  draw subpath (0, 2) of s withpen pencircle scaled 1/4;
endfor
path s; s = unitsquare shifted down scaled (10u) shifted (3u, -12u);
draw s;
label.top("$1+2+3+\cdots+n$", point 5/2 of s);
label.lft("$\displaystyle n(n+1)\over 2$", point 7/2 of s);
label.bot("$\displaystyle 1^3 + 2^3 + \cdots + n^3=1+3+5+\cdots+2{n(n-1)\over2}-1 =" &
          "\left(n(n-1)\over2\right)^2$", point 1/2 of bbox currentpicture shifted 16 down);
```

Sums of cubes VII





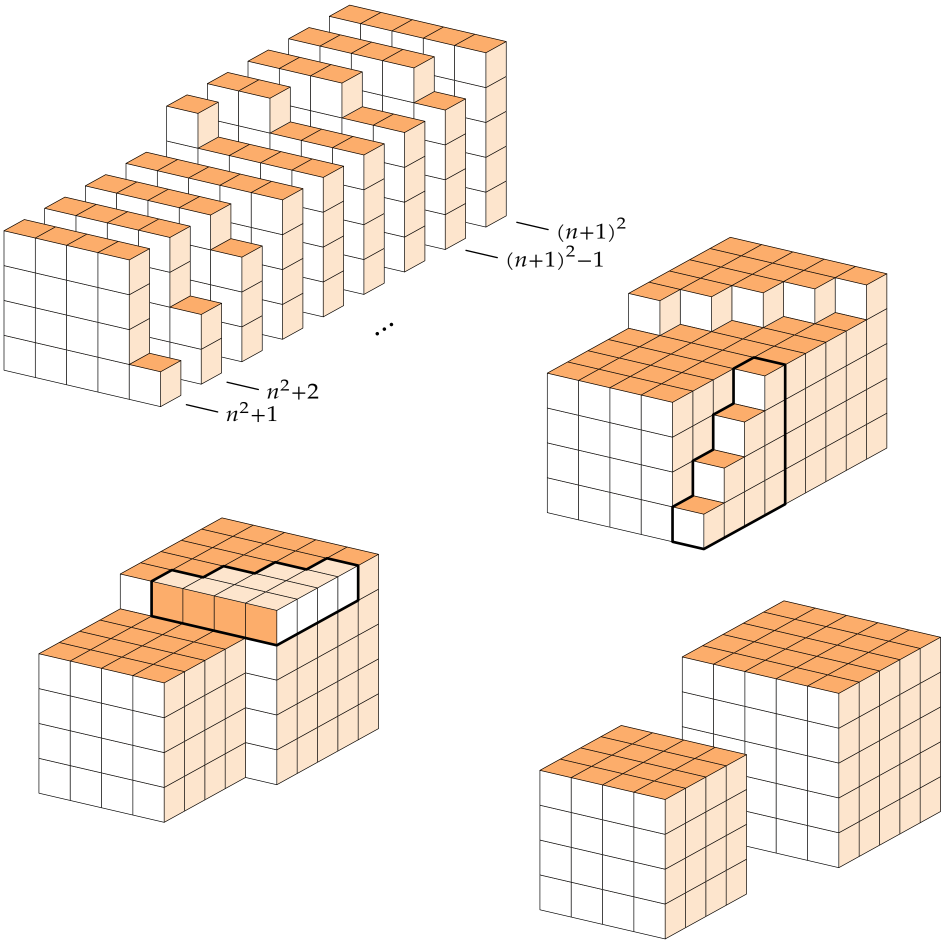
$$1^3+2^3+\cdots+n^3 = 1+3+5+\cdots+2\frac{n(n-1)}{2}-1 = \left(\frac{n(n-1)}{2}\right)^2$$

— Alfinio Flores

```
input isometric_projection
set_projection(21, -33); ipscale := 14;
picture Cube, sideways_cube;
path face; face = p(0,0,0) -- p(1, 0, 0) -- p(1,1,0) -- p(0, 1, 0) -- cycle;
path side; side = p(1,0,0) -- p(1, 0, 1) -- p(1,1,1) -- p(1, 1, 0) -- cycle;
path lid;  lid  = p(0,1,0) -- p(1, 1, 0) -- p(1,1,1) -- p(0, 1, 1) -- cycle;
vardef make_cube(expr f, s, l) = image(
  fill face withcolor f; fill side withcolor s; fill lid withcolor l;
  draw face; draw subpath (0, 3) of side; draw subpath (2, 4) of lid;
) enddef;
drawoptions(withpen pencircle scaled 1/4);
  Cube = make_cube(white, Oranges 8 2, Oranges 8 4);
  sideways_cube = make_cube(Oranges 8 4, white, Oranges 8 2);
drawoptions();
% spaced out...
for z=9 downto 5:
  for y=0 upto 4:
    for x=0 upto 4:
      if 5y + x < 16 + z:
        draw Cube shifted p(x, y, 2z);
      fi
    endfor
  endfor
endfor
for z=4 downto 1:
  for x=0 upto 4:
    for y=0 upto if x=4: z-1 else: 3 fi:
      draw Cube shifted p(x, y, 2z);
    endfor
  endfor
endfor
for z=1,2:
  draw p(5.5, 0, 2z + 1/2) -- p(6.5, 0, 2z + 1/2);
  label.rt("$\scriptstyle n^2 +" & decimal z & "$", p(6.5, 0, 2z + 1/2));
endfor
draw p(5.5, 0, 16.25) -- p(6.5, 0, 16.25);
draw p(5.5, 0, 18.75) -- p(6.5, 0, 18.75);
label.rt("$\scriptstyle \left(n+1\right)^2-1$", p(6.5, 0, 16.25));
label.rt("$\scriptstyle \left(n+1\right)^2$", p(6.5, 0, 18.75));
draw thelabel("$\cdots$", origin) rotated 33 shifted p(7, 0, 10);

% ... and so on
```

Sums of integers as sums of cubes

$$
\begin{aligned}
2+3+4 &= 1+8 \\
5+6+7+8+9 &= 8+27 \\
10+11+12+13+14+15+16 &= 27+64 \\
&\vdots \\
\left(n^2+1\right)+\left(n^2+2\right)+\cdots+(n+1)^2 &= n^3+(n+1)^3
\end{aligned}
$$

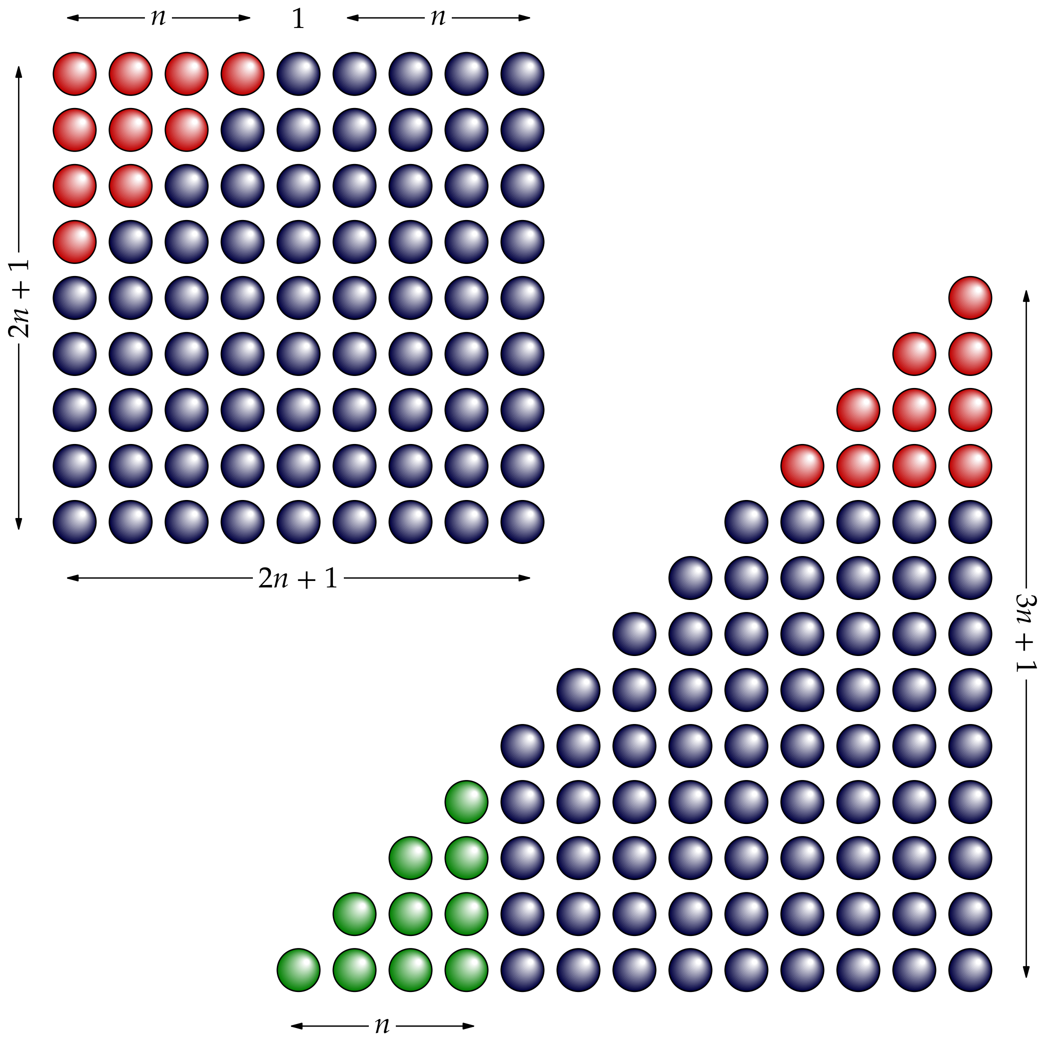


— RBN

```
input paintball

numeric n, s; n = 4; s = 21;
picture P[], txt[], arr[];
txt0 = thelabel("$1$", origin);
txt1 = thelabel("$n$", origin);
txt2 = thelabel("$2n+1$", origin);
txt3 = thelabel("$3n+1$", origin);
arr1 = image(drawdblarrow (left--right) scaled (3/8n * s + 1/8s); unfill bbox txt1; draw txt1);
arr2 = image(drawdblarrow (left--right) scaled (   n * s + 1/8s); unfill bbox txt3; draw txt2);
arr3 = image(drawdblarrow (left--right) scaled (3/2n * s + 1/8s); unfill bbox txt3; draw txt3);

P1 = image(
  for i=-n upto n:
    for j = -n upto n:
      draw if j < i + 5: bball else: rball fi shifted ((i, j) scaled s);
    endfor
  endfor
  draw txt0 shifted (       (0, n+1) scaled s);
  draw arr1 shifted ((+(n+1)/2, n+1) scaled s);
  draw arr1 shifted ((-(n+1)/2, n+1) scaled s);
  draw arr2 shifted (      (0, -n-1) scaled s);
  draw arr2 rotated 90 shifted ((-n-1, 0) scaled s);
);
P2 = image(
  for j=-2n upto n:
    for i=j upto n:
      draw if i < -n: gball elseif j > 0: rball else: bball fi shifted ((i, j) scaled s);
    endfor
  endfor
  draw arr1 shifted ((-13/8n, -2n-1) scaled s);
  draw arr3 rotated -90 shifted ((n+1, -1/2n) scaled s);
);

draw P1; draw P2 shifted (2n*s, -n*s);

label.top("$1+2+\cdots+n=T_n\quad\Rightarrow\quad\left(2n+1\right)^2 = T_{3n+1} - T_n$",
  point 5/2 of bbox currentpicture shifted 16 up);
```

The square of any odd number is the difference between two triangular numbers

$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad (2n+1)^2 = T_{3n+1} - T_n$$

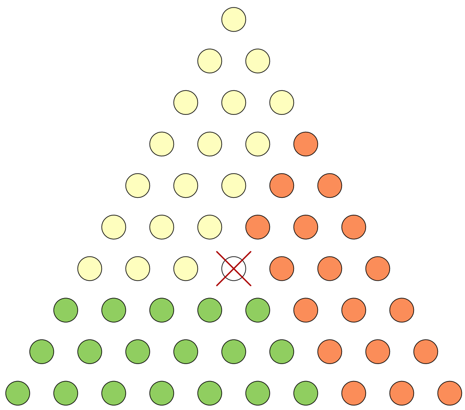


— RBN

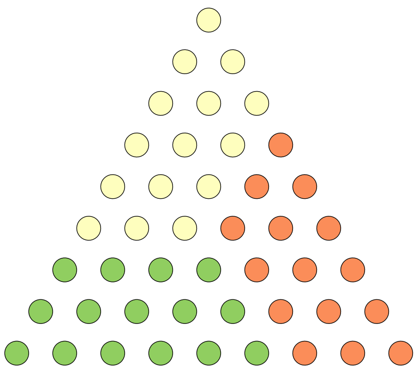
```
numeric u, v; u = 18; v = ypart ((u,0) rotated 60);
picture b[], P[]; path o; o = fullcircle scaled 1/2 u;
forsuffixes $=1,2,3:
  b$ = image(fill o withcolor RdYlGn 3 $;
  draw o withpen pencircle scaled 1/4);
endfor
P1 = image(
  for i=1,2,3:
    for j=6,5,4:
      for k=1 upto j:
        draw b[i] shifted (k*u - 1/2j*u - 2u, (10/3-j)*v) rotated 120i;
      endfor
    endfor
  endfor
  label.bot("$T_{3k} = 3\left(T_{2k}-T_k\right)$", (0, -10/3v));
);
P2 = image(
  for i=1,2,3:
    for j=7,6,5:
      for k=1 upto j:
        draw b[i] shifted (k*u - 1/2j*u - 2u, (12/3-j)*v) rotated 120i;
      endfor
    endfor
  endfor
  draw o withpen pencircle scaled 1/4;
  for t=-45, 45: draw (left -- right) scaled 1/2 u rotated t withcolor 2/3 red; endfor
  label.bot("$T_{3k+1} = 1 + 3\left(T_{2k+1}-T_{k+1}\right)$", (0, -11/3v));
);
P3 = image(
  for i=1,2,3:
    for j=7,6,5,4:
      for k=1 upto j:
        draw b[i] shifted (k*u - 1/2j*u - 5/2u, (11/3-j)*v) rotated 120i;
      endfor
    endfor
  endfor
  label.bot("$T_{3k+2} = 3\left(T_{2k+1}-T_k\right)$", (0, -4v));
);
draw P1; draw P2 shifted (6u, 11v); draw P3 shifted (12u, -3v);
label.top(btex $1+2+\cdots+n=T_n \enspace\Rightarrow\enspace
  \left\{\vcenter{\vbox{\openup8pt\halign{#\hfil\cr
  $T_n \equiv 1 \bmod 3$, \quad $n\equiv 1 \bmod 3$\cr
  $T_n \equiv 0 \bmod 3$, \quad $n\not\equiv 1 \bmod 3$\cr
  }}}\right.$ etex, point 5/2 of bbox currentpicture shifted 21 up);
```

Triangular numbers mod 3

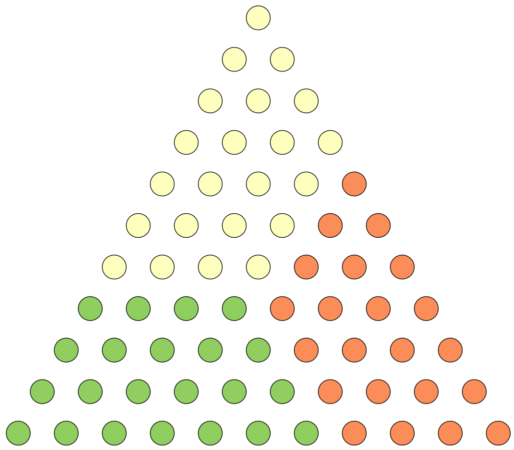
$$1+2+\cdots+n=T_n \quad \Rightarrow \quad \begin{cases} T_n \equiv 1 \bmod 3, & n \equiv 1 \bmod 3 \\ T_n \equiv 0 \bmod 3, & n \not\equiv 1 \bmod 3 \end{cases}$$



$$T_{3k+1}=1+3\left(T_{2k+1}-T_{k+1}\right)$$



$$T_{3k}=3\left(T_{2k}-T_k\right)$$

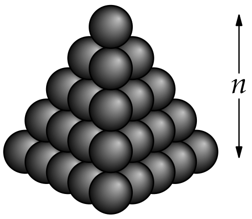


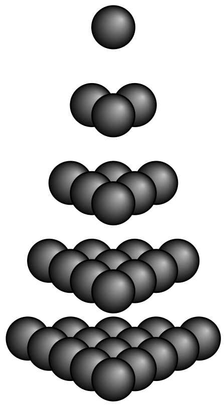
$$T_{3k+2}=3\left(T_{2k+1}-T_k\right)$$

```
input paintball
input isometric_projection
set_projection(-16, 0); ipscale := 16;

picture P[];
P1 = image(
  for k=4 downto 0:
    for j=k downto 0:
      for i=0 upto j:
        draw cball shifted p(i - 0.5 j, -0.866 k, -0.866 j + 0.5k);
      endfor
    endfor
  endfor
  path a; picture n; a = p(3,0,-1) -- p(3, -4*.866, -1);
  n = thelabel("$n$", point 1/2 of a); drawdblarrow a; unfill bbox n; draw n;
);
P2 = image(
  for k=4 downto 0:
    for j=k downto 0:
      for i=0 upto j:
        draw cball shifted p(i - 0.5 j, -2 k, -0.866 j + 0.5k);
      endfor
    endfor
    if k < 4:
      label("$" & if k<3: decimal ((k+1)*(k+2)/2) else: "\vdots" fi & "$", p(5, -2k, 0));
      label("$+$",                                                        p(5, -2k-1, 0));
    else:
      label("$n(n+1)/2$", p(5, -2k, 0));
    fi
  endfor
);
P3 = image(
  set_projection(-16, 32);
  for i=4 downto 0:
    for j=i downto 0:
      for k=0 upto 4-i:
        draw cball shifted p(i-1/2k, -0.866 k, -0.866j + 0.5k) shifted (42i, -16i);
      endfor
    endfor
  endfor
  label.bot(btex $
  1(n)   \quad + \quad
  2(n-1) \quad + \quad
  3(n-2) \quad + \quad
  \cdots \quad + \quad n(1)$ etex, (92, -88));
);
draw P1; draw P2 shifted (180, 40); draw P3 shifted (0, -144);
label.top(btex $\displaystyle
  1+2+\cdots+k=T_k \enspace\Rightarrow\enspace
  \sum_{k=1}^n T_k = \sum_{k=1}^n k(n-k+1)
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);
```

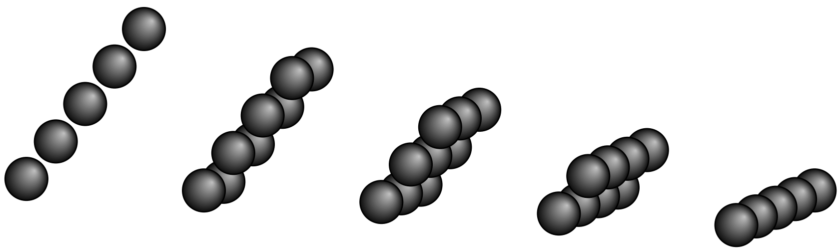
Counting triangular numbers IV: Counting cannonballs

$$1+2+\cdots+k=T_k \quad \Rightarrow \quad \sum_{k=1}^{n} T_k=\sum_{k=1}^{n} k(n-k+1)$$





$$\begin{array}{c} 1 \\ + \\ 3 \\ + \\ 6 \\ + \\ \vdots \\ + \\ n(n+1)/2 \end{array}$$



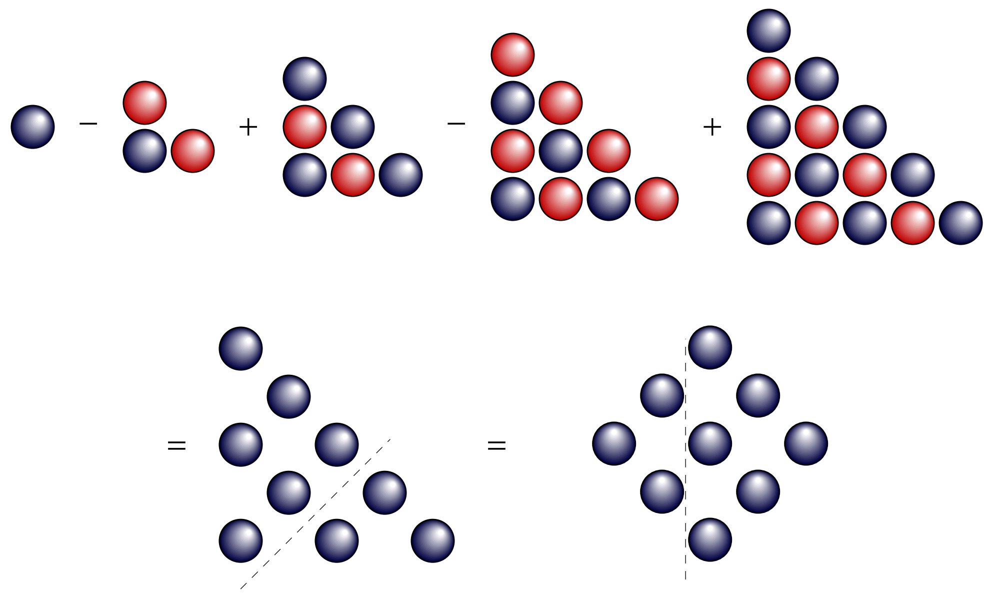
$$1(n) \quad + \quad 2(n-1) \quad + \quad 3(n-2) \quad + \quad \cdots \quad + \quad n(1)$$

— Deanna B. Haunsperger and Stephen F. Kennedy

```
input paintball
picture P[];
P1 = image(
  z1 = 9 up;
  for k=1 upto 5:
    for i=1 upto k:
      for j=1 upto i:
        draw if odd i: bball else: rball fi shifted (18 * (i-j, j)) shifted z1;
      endfor
    endfor
    x1 := x1 + 18k + 24; y1 := y1 - 9;
    if k < 5:
      label.lft("$" & if odd k: "-" else: "+" fi & "$", (-14,  9k+18) + z1);
    fi
  endfor
);
P2 = image(
  for i=1 step 2 until 5:
    for j=1 upto i:
      draw bball shifted (18 * (i-j, j));
    endfor
  endfor
  draw origin -- (56, 56) dashed evenly withpen pencircle scaled 1/4;
);
P3 = image(
  for i=-1 upto 1:
    for j=-1 upto 1:
      draw bball shifted (36/sqrt(2) * (i,j));
    endfor
  endfor
  draw (2 left--right) scaled 30 rotated 45 shifted 13 up
    dashed evenly withpen pencircle scaled 1/4;
);

label(P1, origin);
label("$=$",       (-120, -120));
label(P2,           (-60, -124));
label("$=$",          (0, -120));
label(P3 rotated 45, (80, -122));

label.top(btex $\displaystyle
  1+2+\cdots+k=T_k \enspace\Rightarrow\enspace
  \sum_{k=1}^{2n-1} \left(-1\right)^{k+1} T_k = n^2
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);
```

Alternating sums of triangular numbers

$$1 + 2 + \cdots + k = T_k \;\Rightarrow\; \sum_{k=1}^{2n-1} (-1)^{k+1}\, T_k = n^2$$

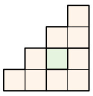


— RBN

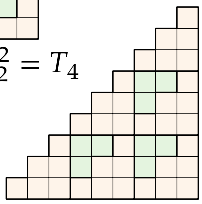
```
numeric u; u = 8;

vardef T(expr n) = save m, r, s;
  numeric m, r; m = abs(n); r = n/m; % don't use n=0
  picture s; s = image(
    fill unitsquare scaled u withcolor if r > 0: Oranges 8 1 else: Greens 8 2 fi;
    draw unitsquare scaled u withpen pencircle scaled 1/4;
  );
  image(
    for i=0 upto m - 1:
      for j=0 upto m - 1 - i:
        draw s shifted (u*(i, j));
      endfor
    endfor
    draw (origin --
      for i=0 upto m-1: (m-i, i) -- (m-i, i+1) -- endfor
      (0, m) -- cycle) scaled u;
  ) rotated 90r
enddef;

picture P[];
for n=1 upto 4:
  P[n] = image(
    for i=0 upto n:
      for j=0 upto n-i:
        if (i < n) and (j  < n - i):
          draw T(-n) shifted (u*(n+1)*(-i-1,j+1));
        fi
        draw T(n+1) shifted (u*(n+1)*(-i,j));
      endfor
    endfor
    label.bot("$T_" & decimal n & "^2 + T_" & decimal (n+1) &
              "^2 = T_{" & decimal (n*n+2n+1) & "}$", (-u/2 * (n+1) * (n+1), 0));
  );
  draw P[n] shifted (5/2u * n * n * (1,-1));
endfor

P5 = image(draw T(4); draw T(-3) shifted 4(-u,u))
  shifted lrcorner bbox currentpicture shifted 80 down;
draw P5;
label.lft("\textsc{Note}: This is a companion result to the more familiar" &
          "$T_{n-1} + T_n = n^2 \,\rightarrow$", point -1/2 of bbox P5);

label.top(btex $\displaystyle
  1+2+\cdots+n=T_n \enspace\Rightarrow\enspace
  T_{n-1}^2 + T_n^2 = T_{n^2}
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);
```

The sum of the squares of consecutive triangular numbers is triangular

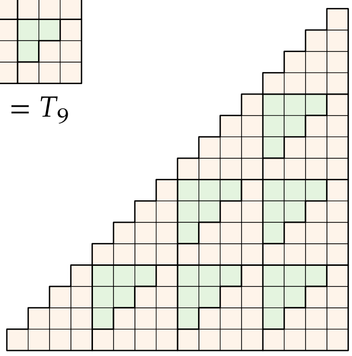
$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad T_{n-1}^2 + T_n^2 = T_{n^2}$$



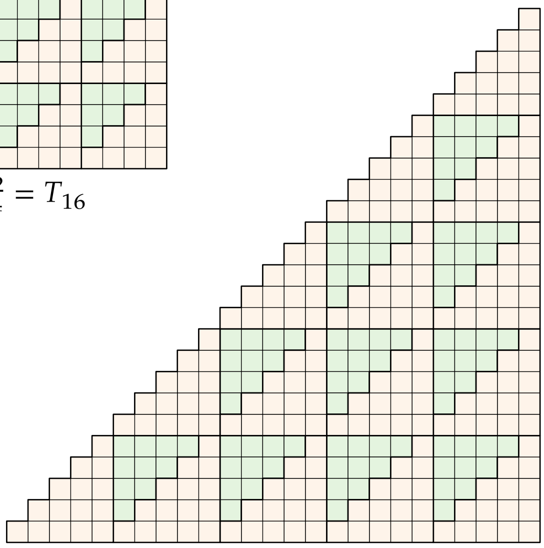
$T_1^2 + T_2^2 = T_4$



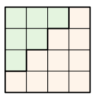
$T_2^2 + T_3^2 = T_9$



$T_3^2 + T_4^2 = T_{16}$



$T_4^2 + T_5^2 = T_{25}$

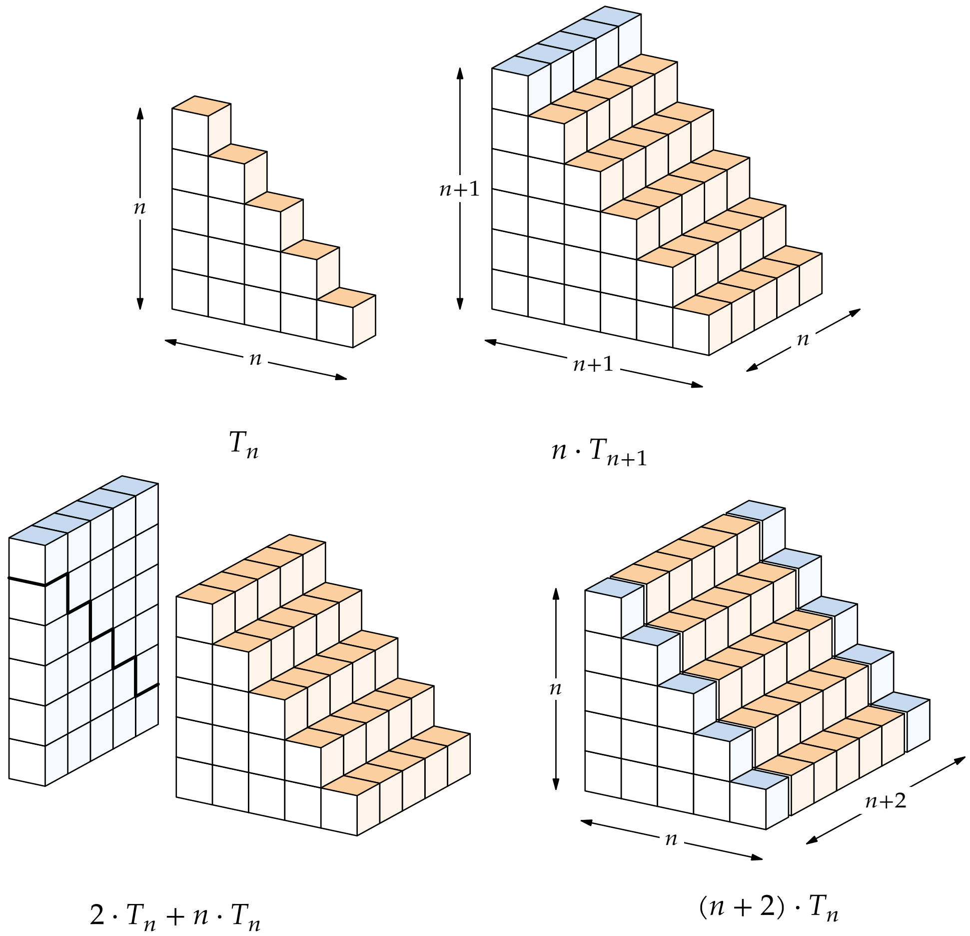
NOTE: This is a companion result to the more familiar $T_{n-1} + T_n = n^2$ $\rightarrow$ 

— RBN

```
input isometric_projection
set_projection(20, -32); ipscale := 16;
path face; face = p(0,0,0) -- p(1,0,0) -- p(1,1,0) -- p(0,1,0) -- cycle;
path side; side = p(1,0,0) -- p(1,0,1) -- p(1,1,1) -- p(1,1,0) -- cycle;
path lid; lid = p(0,1,0) -- p(1,1,0) -- p(1,1,1) -- p(0,1,1) -- cycle;
vardef make_cube(expr f, s, l) = image(
  fill face withcolor f; fill side withcolor s; fill lid withcolor l;
  draw face; draw subpath (0, 3) of side; draw subpath (2, 4) of lid;
) enddef;
picture o_cube; o_cube = make_cube(background, Oranges 8 1, Oranges 8 3);
picture b_cube; b_cube = make_cube(background, Blues   8 1, Blues   8 3);
vardef show_dim(expr p, s) = save pp, tt;
  path pp; pp = p shifted (unitvector(direction 1/2 of p) rotated 90 scaled 4 labeloffset);
  picture tt; tt = thelabel("$\scriptstyle " & s & "$", point 1/2 of pp);
  drawdblarrow pp; unfill bbox tt; draw tt;
enddef;
picture P[];
P1 = image(
  for x=0 upto 4:
    for y=0 upto 4-x:
      draw o_cube shifted p(x,y,0);
    endfor
  endfor
  show_dim(p(0,0,0) -- p(0,5,0), "n");
  show_dim(p(5,0,0) -- p(0,0,0), "n");
  label("$T_n$", p(2,-3,0));
);
P2 = image(
  for x=0 upto 5:
    for y=0 upto 5-x:
      for z=4 downto 0:
        draw if x=0: b_cube else: o_cube fi shifted p(x,y,z);
      endfor
    endfor
  endfor
  show_dim(p(0,0,0) -- p(0,6,0), "n+1");
  show_dim(p(6,0,0) -- p(0,0,0), "n+1");
  show_dim(p(6,0,6) -- p(6,0,1), "n");
  label("$n\cdot T_{n+1}$", p(3,-3,0));
);

% ... and so on ...

draw P1 shifted 60 left;     draw P2 shifted 60 right;
draw P3 shifted (-72, -180); draw P4 shifted (+96, -180);
label.top(btex $\displaystyle
  1+2+\cdots+n=T_n \enspace\Rightarrow\enspace T_{n+1} = {n+2\over n}T_n
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);
label.bot(btex $\displaystyle
  n\cdot T_{n+1} = (n+2) \cdot T_n \enspace\Rightarrow\enspace T_{n+1} = {n+2\over n}T_n
  $ etex, point 1/2 of bbox currentpicture shifted 21 down);
```

Recursion for triangular numbers

$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad T_{n+1} = \frac{n+2}{n} T_n$$

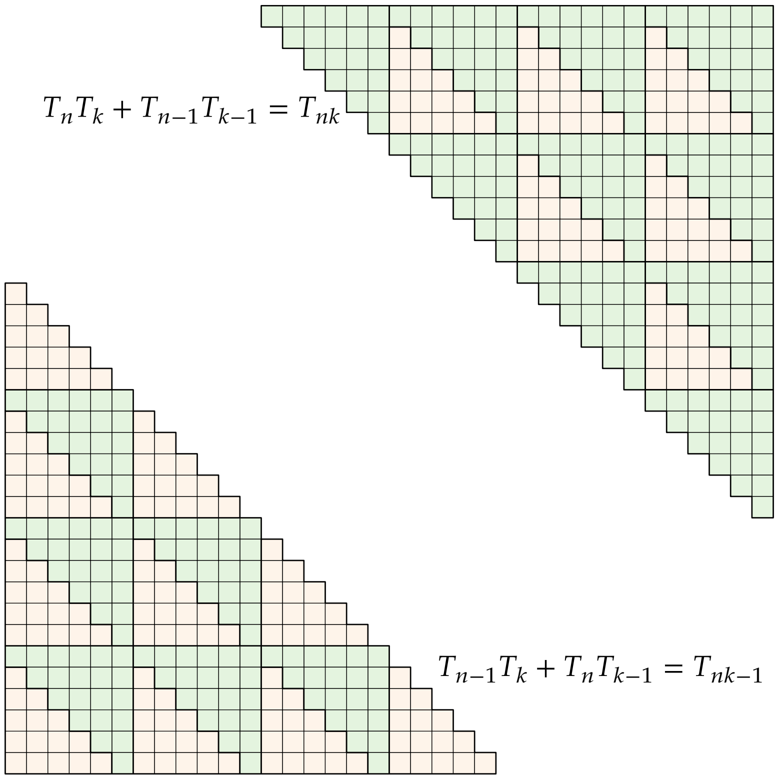


$$n \cdot T_{n+1} = (n+2) \cdot T_n \quad \Rightarrow \quad T_{n+1} = \frac{n+2}{n} T_n$$

```
numeric u; u = 8;
vardef T(expr n) = save m, r, s;
  numeric m; m = abs(n);
  numeric r; r = n/m; % don't use n=0
  picture s; s = image(
    fill unitsquare scaled u withcolor if r < 0: Oranges 8 1 else: Greens 8 2 fi;
    draw unitsquare scaled u withpen pencircle scaled 1/4;
  );
  image(
    for i=0 upto m - 1:
      for j=0 upto m - 1 - i:
        draw s shifted (u*(i, j));
      endfor
    endfor
    draw (origin -- for i=0 upto m-1:
        (m-i, i) -- (m-i, i+1) --
      endfor (0, m) -- cycle) scaled u;
  ) rotated (90+90r)
enddef;
picture t[]; t5 = T(-5); t6 = T(6);
picture P[];
P1 = image(
  for x=0 upto 3:
    for y=0 upto 3-x:
      draw t5 shifted (6u*(x,y));
      if x+y < 3:
        draw t6 shifted (6u*(x+1, y+1));
      fi
    endfor
  endfor
  label.urt("$T_{n-1}T_k + T_n T_{k-1} = T_{nk-1}$", (20u, 4u));
);
P2 = image(
  for x=0 upto 3:
    for y=0 upto 3-x:
      draw t6 shifted (-6u*(x,y));
      if x+y < 3:
        draw t5 shifted (-6u*(x+1, y+1));
      fi
    endfor
  endfor
  label.llft("$T_{n}T_k + T_{n-1} T_{k-1} = T_{nk}$", -(20u, 4u));
);

draw P1; draw P2 shifted 36(u, u);

label.top("$T_n=1+2+\cdots+n\enspace\Rightarrow$",
  point 5/2 of bbox currentpicture shifted 42 up);
```

Identities for triangular numbers

$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$



$$T_n T_k + T_{n-1} T_{k-1} = T_{nk}$$

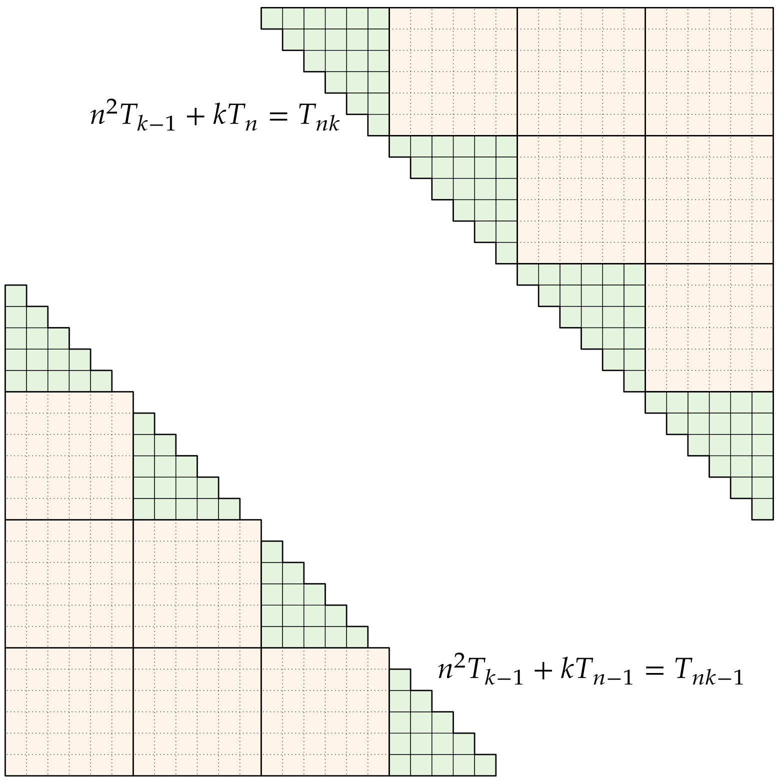
$$T_{n-1} T_k + T_n T_{k-1} = T_{nk-1}$$

— RBN

```
numeric u; u = 8;
vardef T(expr n) = save m, r, s;
  numeric m; m = abs(n);
  numeric r; r = n/m; % don't use n=0
  picture s; s = image(
    fill unitsquare scaled u withcolor if r < 0: Oranges 8 1 else: Greens 8 2 fi;
    draw unitsquare scaled u withpen pencircle scaled 1/4;
  );
  image(
    for i=0 upto m - 1:
      for j=0 upto m - 1 - i:
        draw s shifted (u*(i, j));
      endfor
    endfor
    draw (origin -- for i=0 upto m-1:
        (m-i, i) -- (m-i, i+1) --
      endfor (0, m) -- cycle) scaled u;
  ) rotated (90+90r)
enddef;
picture t[]; t6 = T(6); t5 = T(5) rotated 180;
picture sq; sq = image(
  fill unitsquare scaled 6u withcolor Oranges 8 1;
  for i=1 upto 5:
    draw ((i, 0) -- (i, 6)) scaled u dashed withdots scaled 1/4 withpen pencircle scaled 1/4;
    draw ((0, i) -- (6, i)) scaled u dashed withdots scaled 1/4 withpen pencircle scaled 1/4;
  endfor
  draw unitsquare scaled 6u;
);
picture P[];
P1 = image(
  for x=0 upto 3:
    for y=0 upto 3-x:
      draw if x+y < 3: sq else: t5 fi shifted (6u*(x,y));
    endfor
  endfor
  label.urt("$n^2T_{k-1} + kT_{n-1} = T_{nk-1}$", (20u, 4u));
);
P2 = image(
  for x=0 upto 3:
    for y=0 upto 3-x:
      draw if x+y < 3: sq rotated 180 else: t6 fi shifted (-6u*(x,y));
    endfor
  endfor
  label.llft("$n^2T_{k-1} + kT_{n} = T_{nk}$", -(20u, 4u));
);

draw P1; draw P2 shifted 36(u, u);

label.top("$T_n=1+2+\cdots+n\enspace\Rightarrow$",
  point 5/2 of bbox currentpicture shifted 42 up);
```

More identities for triangular numbers

$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$

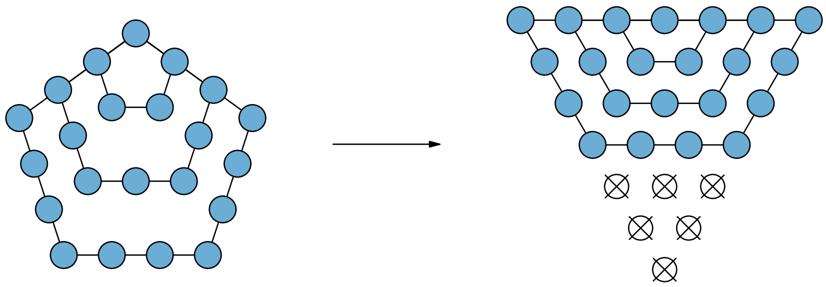


— James O. Chilaka

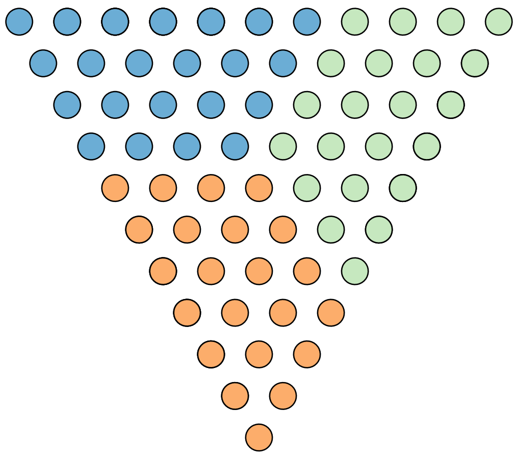
```
numeric s, r, d; s = 18; 1.8d = s; r = s / 2 / cosd 54;
picture xo; xo = image(draw fullcircle scaled 0.9 d;
  draw (left--right) scaled 0.6 d rotated 45;
  draw (left--right) scaled 0.6 d rotated -45;
);
picture o[];
o0 = image(fill fullcircle scaled d withcolor Blues 7 4; draw fullcircle scaled d);
o1 = image(fill fullcircle scaled d withcolor Oranges 7 3; draw fullcircle scaled d);
o2 = image(fill fullcircle scaled d withcolor Greens 7 2; draw fullcircle scaled d);
path p[];
p1 = (for t=0 upto 4: up scaled r rotated  72t -- endfor cycle) shifted (0, -r);
p2 = origin -- for t=0 upto 3: left scaled s rotated 60t -- endfor cycle;
pair u, v; u = point 2 of p2; v = point 3 of p2;
picture P[];
for k=1 upto 2: P[k] = image(
  for i = 1 upto 3:
    draw p[k] scaled i;
  endfor
  for i = 1 upto 3:
    for t = i upto i * length(p[k]) - i:
      draw o0 shifted (point t/i of p[k] scaled i);
    endfor
  endfor
  draw o0;
  if k=2:
    draw xo shifted 3u shifted v;  draw xo shifted 2u shifted 2v;
    draw xo shifted 3u shifted 2v; draw xo shifted 2u shifted 3v;
    draw xo shifted 3u shifted 3v; draw xo shifted  u shifted 3v;
  fi
); endfor
P3 = image(
  for i=0 upto 2:
    for j=1 upto 3:
      for k=1 upto j * length p2:
        draw o[i] shifted (point k/j of p2 scaled j)
          shifted -11/3 v shifted 1/3 u rotated 120i;
      endfor
    endfor
  endfor
  label.bot("$3P_n = T_{3n-1}$", 4(u+v));
);

label.lft(P1, 42 left); drawarrow (left--right) scaled 20; label.rt(P2, 42 right);
label.bot("$P_n = T_{2n-1} - T_{n-1}$", 64 down);
draw P3 shifted 180 down;

label.top(btex $\left. \vcenter{\openup 4pt\halign{$#$\hfill\cr
  P_n = 1 + 4 + 7 + \cdots + (3n-2)\cr
  T_n = 1 + 2 + 3 + \cdots + n\cr}} \right\} \Longrightarrow {}
  $ etex, point 5/2 of bbox currentpicture shifted 42 up);
```

Identities for pentagonal numbers

$$\left.\begin{aligned} P_n &= 1 + 4 + 7 + \cdots + (3n-2) \\ T_n &= 1 + 2 + 3 + \cdots + n \end{aligned}\right\} \Longrightarrow$$



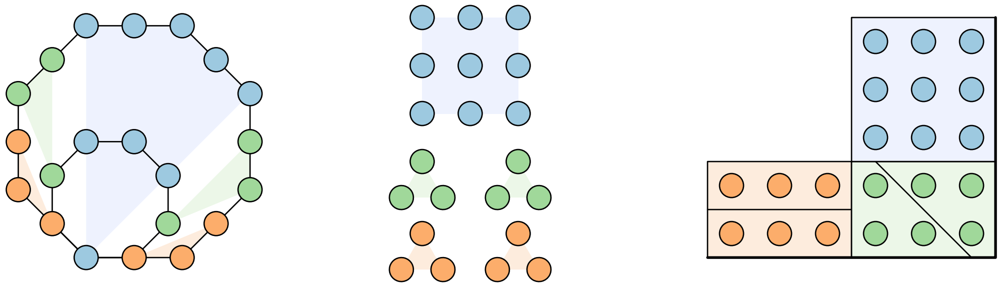
$$P_n = T_{2n-1} - T_{n-1}$$



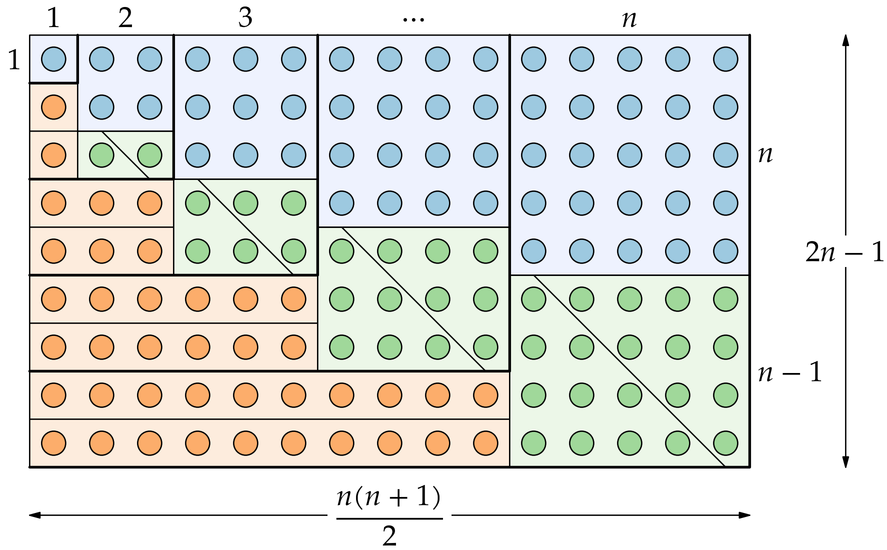
$$3P_n = T_{3n-1}$$

```
numeric s, r, d; s = 18; 2d = s; 2r * cosd 67.5 = s;
color cf[]; cf0 = Blues 7 1; cf1 = Oranges 7 1;  cf2 = Greens 7 1;
color cb[]; cb0 = Blues 7 3; cb1 = Oranges 7 3;  cb2 = Greens 7 3;
picture P[], o[];
for i=0 upto 2:
  o[i] = image(fill fullcircle scaled d withcolor cb[i]; draw fullcircle scaled d);
endfor
P1 = image(path G[];
  G0 = (for t=0 upto 7: down scaled r rotated (45t-45/2) -- endfor cycle) shifted (0, -r);
  G1 = G0 shifted - point 0 of G0; G2 = G1 scaled 2;
  fill point 0 of G2 -- subpath (3,5) of G2 -- cycle withcolor cf0;
  for p = -1, 1:
    fill subpath (1/2p, 3/2p) of G2 -- cycle withcolor cf1;
    fill subpath (2p, 5/2p) of G2 -- point 2p of G1 -- cycle withcolor cf2;
  endfor
  draw G1; draw G2;
  for t = 10, 13, 14, 15, 23, 23.5, 24, 24.5, 25:
    draw o0 shifted point t mod 10 of G[floor (t/10)];
  endfor
  for t = 20.5, 21, 21.5, 26.5, 27, 27.5:
    draw o1 shifted point t mod 10 of G[floor (t/10)];
  endfor
  for t = 12, 22, 22.5, 25.5, 26, 16:
    draw o2 shifted point t mod 10 of G[floor (t/10)];
  endfor
);
P2 = image(
  picture t[], q;
  for i=1,2: t[i] = image(
    fill for j=0 upto 2: (0, d) rotated 120j -- endfor cycle withcolor cf[i];
    for j=0 upto 2: draw o[i] shifted (0,d) rotated 120j; endfor
  ); endfor
  q = image(
    fill unitsquare shifted -(1/2, 1/2) scaled 2s withcolor cf0;
    for j=-1 upto 1: for k=-1 upto 1: draw o0 shifted (j*s, k*s); endfor endfor
  );
  draw t1 shifted (-s, 0);    draw t1 shifted (s, 0);
  draw t2 shifted (-s, 3/2s); draw t2 shifted (s, 3/2s);
  draw q shifted (0, 4s);
);
% see source for octaframe routine
P3 = image(draw octaframe(3, nullpicture));
P4 = image(
  draw octaframe(5, "$n$");
  draw octaframe(4, "$\cdots$") shifted (0, 2s);
  draw octaframe(3, "$3$")      shifted (0, 4s);
  draw octaframe(2, "$2$")      shifted (0, 6s);
  draw octaframe(1, "$1$")      shifted (0, 8s);
  % ... and the labels ...
);
% ... finally place P[] and labels
```

Sums of octagonal numbers



$$T_k = 1 + 2 + \cdots + k \Rightarrow O_k = k^2 + 4T_{k-1}$$



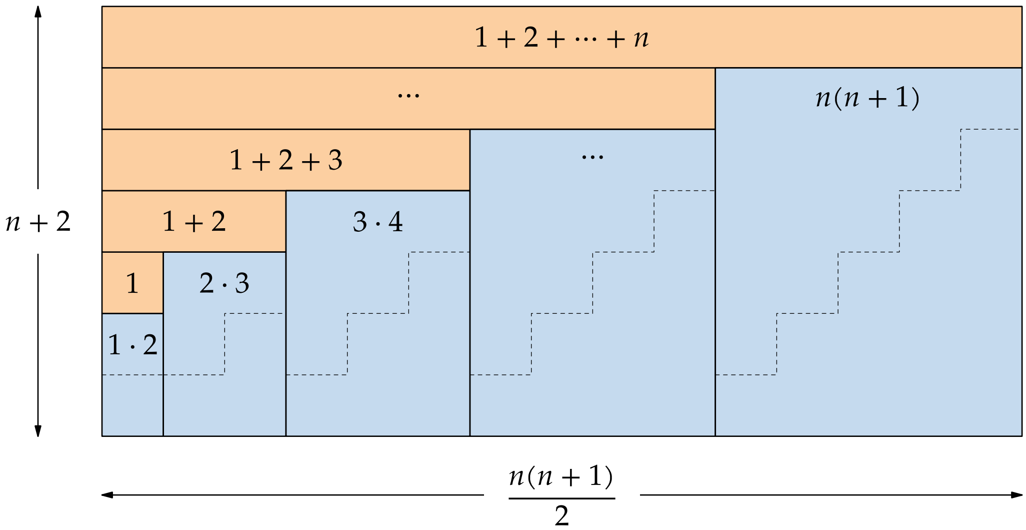
$$\sum_{k=1}^{n} O_k = 1 + 8 + 21 + 40 + \cdots + \left(n^2 + 4T_{n-1}\right) = \frac{n(n+1)(2n-1)}{2}$$

— James O. Chilaka

```
numeric s; s = 23;
vardef box(expr n) =
  path a, b;
  a = unitsquare xscaled n yscaled (n + 1) scaled s;
  b = unitsquare xscaled 1 yscaled (n*(n+1)/2) scaled s rotated 90 shifted point 2 of a;
  image(
    fill a withcolor Blues 7 2; fill b withcolor Oranges 7 2;
    label(if n > 4: "$n(n+1)$"
      elseif n = 4: "$\cdots$"
              else: "$" & decimal n & "\cdot" & decimal (n+1) & "$"
                fi, (xpart center a, ypart center b - s));
    label(if n > 4: "$1+2+\cdots+n$"
      elseif n = 4: "$\cdots$"
              else: "$1" for i=2 upto n: & "+" & decimal i endfor & "$"
                fi, center b);
    path t; t = origin -- right scaled s;
    draw t shifted (0,s) for i=2 upto n: -- t shifted (s*(i-1,i)) endfor
      dashed evenly scaled 1/2 withpen pencircle scaled 1/4;
    draw a; draw b;
  )
enddef;

for n=1 upto 5:
    draw box(n) shifted (s*n*(n-1)/2, 0);
endfor

vardef show_dim(expr s, p) = save t; picture t; t = thelabel(s, point 1/2 of p);
  drawdblarrow p; interim bboxmargin := 8; unfill bbox t; draw t;
enddef;
show_dim("$\displaystyle {n(n+1)\over2}$", (origin -- 15 right * s) shifted 22 down);
show_dim("$n+2$", (origin --  7 up * s) shifted 24 left);

label.top("$\displaystyle \sum_{k=1}^n k (k+1) = {n(n+1)(n+2)\over3}$",
  point 5/2 of bbox currentpicture shifted 42 up);
label.bot(btex \vbox{\openup 16pt\halign{#\hfill\cr
  $T_k = 1 + 2 + \cdots + k \enspace\Rightarrow$\cr
  \hfill $\displaystyle 1\cdot2 + 2\cdot3 + \cdots + n(n+1) +
    \left(T_1+T_2+\cdots+T_n\right)={n(n+1)(n+2)\over2}$,\cr
  \hfill $\displaystyle \left(T_1+T_2+\cdots+T_n\right) =
    {1\over2}\left( 1\cdot2 + 2\cdot3 + \cdots + n(n+1)\strut\right)$,\cr
  \hfill \therefore\quad
    $\displaystyle {3\over2}\left( 1\cdot2 + 2\cdot3 + \cdots + n(n+1)\strut\right) =
    {n(n+1)(n+2)\over2}$.\cr
 }} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

Sums of products of consecutive integers I

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$



$$T_k = 1+2+\cdots+k \quad \Rightarrow$$

$$1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) + (T_1+T_2+\cdots+T_n) = \frac{n(n+1)(n+2)}{2},$$

$$(T_1+T_2+\cdots+T_n) = \frac{1}{2}\big(1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)\big),$$

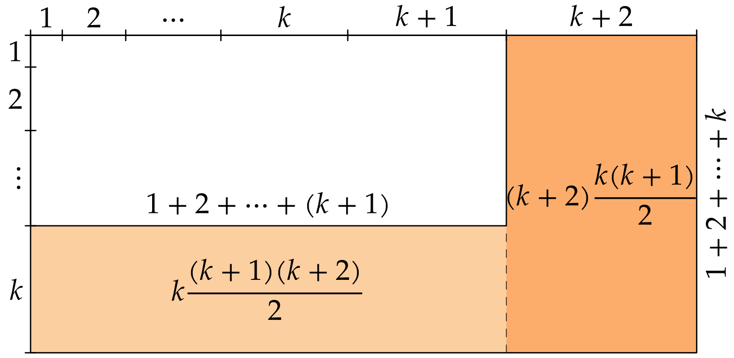
$$\therefore \quad \frac{3}{2}\big(1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)\big) = \frac{n(n+1)(n+2)}{2}.$$

— James O. Chilaka

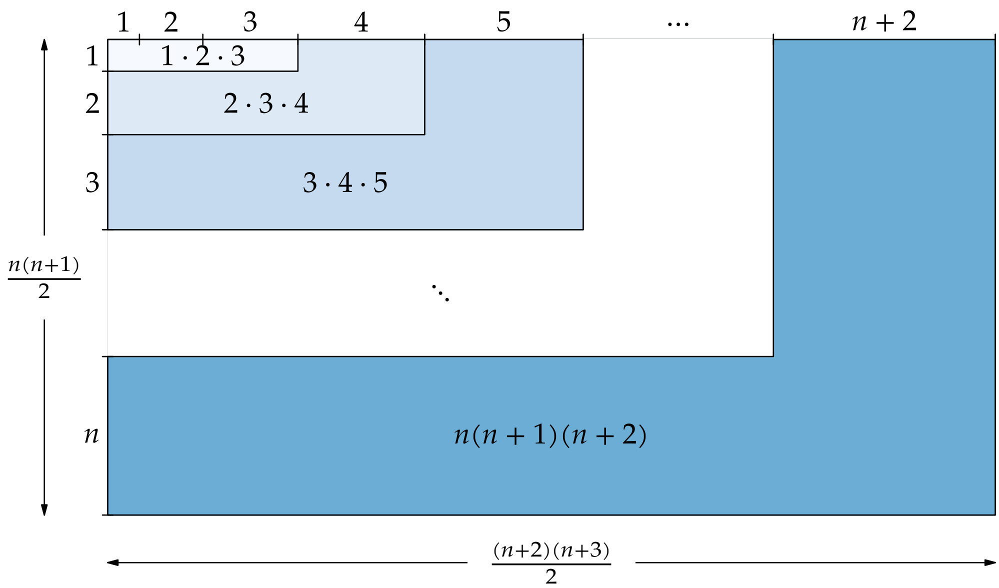
```
numeric u, n; u = 12; n = 5;
vardef t(expr n) = n*(n+1)/2 enddef;
path b[];
for i=1 upto n:
  b[i] = unitsquare xscaled t(i + 2) yscaled -t(i) scaled u;
endfor

picture P[];
P1 = image(
  fill b4 withcolor Oranges 8 4; fill b3 withcolor background;
  z0 = (xpart point 2 of b3, ypart point 2 of b4);
  fill subpath (2,3) of b3 -- point 3 of b4 -- z0 -- cycle
    withcolor Oranges 8 3;
  draw z0 -- point 2 of b3 dashed evenly withpen pencircle scaled 1/4;
  draw b3; draw b4;
  draw 2 left -- origin -- 2 up; % first ticks
  numeric i, a, b; i = a = b = 0;
  for s = "1", "2", "\cdots", "k", "k+1", "k+2":
    a := b; b := t(incr i) * u;
    draw (up--down) scaled 2 shifted (b, 0);
    label.top("$" & s & "$", (1/2(a+b), 0));
  endfor
  numeric i, a, b; i = a = b = 0;
  for s = "1", "2", "\vdots", "k":
    a := b; b := t(incr i) * u;
    draw (left--right) scaled 2 shifted (0, -b);
    label.lft("$" & s & "$", (0, -1/2(a+b)));
  endfor
  label.top("$1+2+\cdots+(k+1)$", point 5/2 of b3);
  label("$\displaystyle k{(k+1)(k+2)\over2}$", 1/2[point 2 of b3, point 3 of b4]);
  label("$\displaystyle (k+2){k(k+1)\over2}$", 1/2[point 1 of b4, z0]);
  label.rt(TEX("$1+2+\cdots+k$") rotated 90, point 3/2 of b4);
);

% P2 is similiar...

label.top(P1, 2u * up); label.bot(P2, 3u * down + u * left);
label(btex $\displaystyle
  k{(k+1)(k+2)\over2}+(k+2){k(k+1)\over2}=k(k+1)(k+2)
  $ etex, origin);
label.top(btex $\displaystyle
  \sum_{k=1}^n k(k+1)(k+2) = {n(n+1)(n+2)(n+3)\over4}
  $ etex, point 5/2 of bbox currentpicture shifted 21 up);
label.bot(btex \vbox{%
  \leftline{$1\cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2)$}
  \bigskip
  \rightline{$\displaystyle {} = \frac{n(n+1)}{2}\times\frac{(n+2)(n+3)}{2}
                              = \frac{n(n+1)(n+2)(n+3)}{4}$}
  } etex, point 1/2 of bbox currentpicture);
```

Sums of products of consecutive integers II

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$



$$k\frac{(k+1)(k+2)}{2} + (k+2)\frac{k(k+1)}{2} = k(k+1)(k+2)$$



$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n(n+1)(n+2)$$

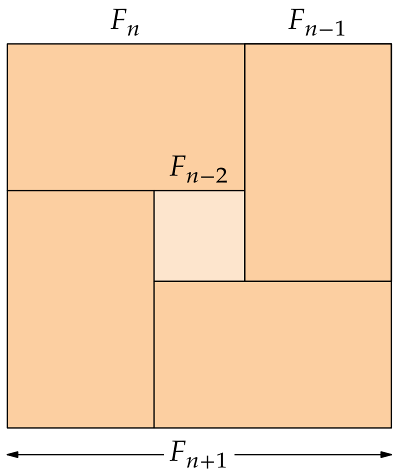
$$= \frac{n(n+1)}{2} \times \frac{(n+2)(n+3)}{2} = \frac{n(n+1)(n+2)(n+3)}{4}$$

— James O. Chilaka

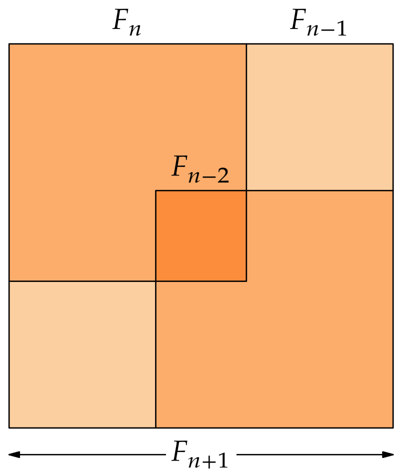
```
path p, q, r, s, t, u;
p = unitsquare shifted -(1/2, 1/2) scaled 34;
q = unitsquare shifted -(1/2, 1/2) scaled 55;
r = unitsquare shifted -(1/2, 1/2) scaled 89;
s = unitsquare shifted -(1/2, 1/2) scaled 144;
t = unitsquare shifted -(1/2, 1/2) xscaled 34 yscaled 55;
u = unitsquare shifted -(1/2, 1/2) xscaled 55 yscaled 89;

picture P[];
P0 = image(
  label.top("$F_{n-1}$", point 5/2 of u shifted (point 1 of p - point 0 of u));
  path a; a = subpath(0, 1) of s shifted 10 down;
  picture t; t = thelabel("$F_{n+1}$", point 1/2 of a);
  drawdblarrow a; unfill bbox t; draw t;
);
P1 = image(
  fill s withcolor Oranges 8 3; fill p withcolor Oranges 8 2;
  for i=0 upto 4:
    draw subpath (1,4) of u shifted (point 1 of p - point 0 of u) rotated 90i;
  endfor
  draw P0;
  label.top("$F_{n-2}$", point 5/2 of p);
  label.top("$F_{n}$",   point 3/2 of u shifted (point 1 of p - point 0 of u) rotated 90);
  label.bot("$F_{n+1}^2 = 4 F_n F_{n-1} + F_{n-2}^2$",
    point 1/2 of bbox currentpicture shifted 12 down);
);
P2 = image(
  path a, b;
  a = r shifted (point 1 of p - point 1 of r);
  b = r shifted (point 3 of p - point 3 of r);
  fill s withcolor Oranges 8 3; fill a withcolor Oranges 8 4;
  fill b withcolor Oranges 8 4; fill p withcolor Oranges 8 5;
  draw subpath (0, 2) of a; draw subpath (2, 4) of b; draw s;
  label.top("$F_{n-2}$", point 5/2 of p);
  label.top("$F_{n}$",   point 3/2 of u shifted (point 1 of p - point 0 of u) rotated 90);
  draw P0;
  label.bot("$F_{n+1}^2 = 2 F_n^2 + 2 F_{n-1}^2 - F_{n-2}^2$",
    point 1/2 of bbox currentpicture shifted 12 down);
);

% .. and so on for P3 and P4

draw P1 shifted (-89, +144); draw P2 shifted (+89, +144);
draw P3 shifted (-89, -89);  draw P4 shifted (+89, -89);
```

Fibonacci identities

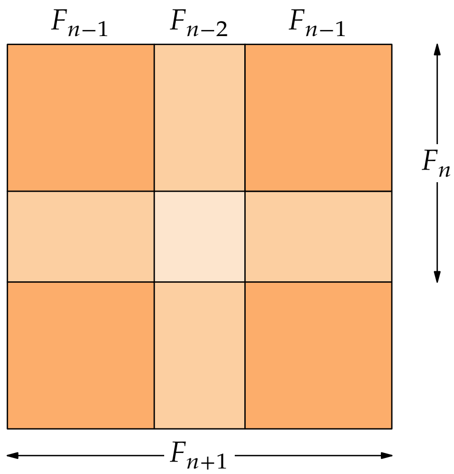
$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \Rightarrow$



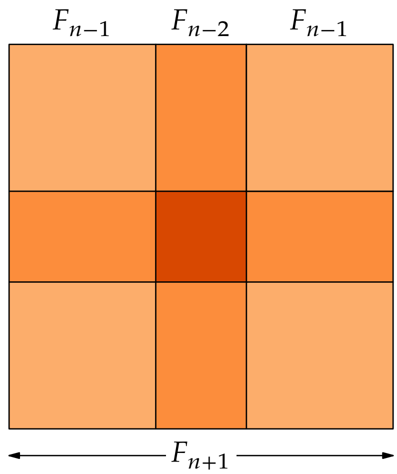
$F_{n+1}^2 = 4F_nF_{n-1} + F_{n-2}^2$



$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2$



$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-1}F_{n-2} + F_{n-2}^2$



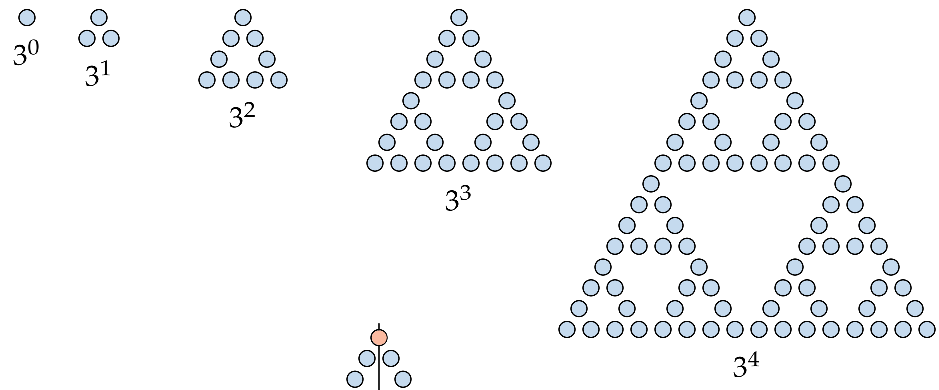
$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$

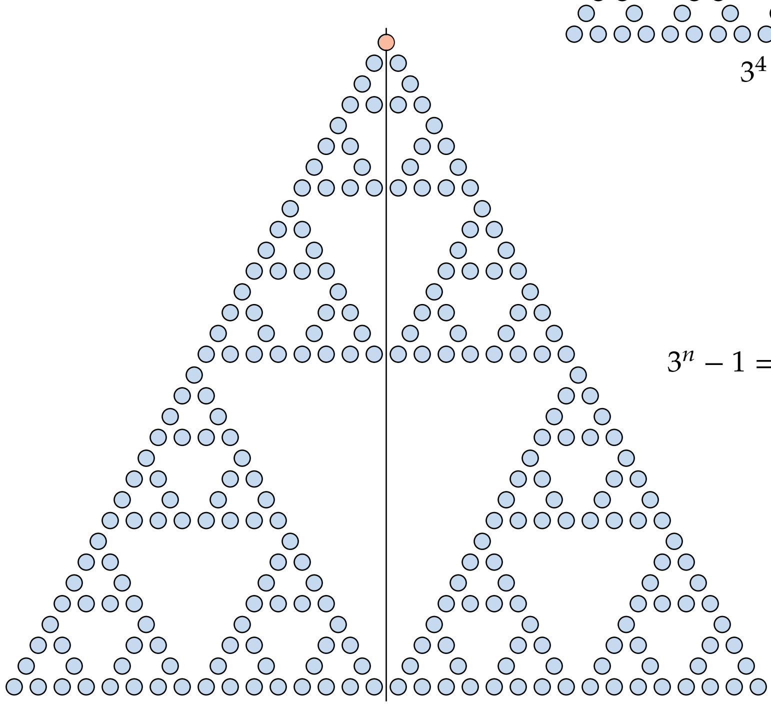
— Alfred Brousseau

```
numeric s; s = 6;
picture t[];
t0 = image(
  fill fullcircle scaled s withcolor Blues 8 3;
  draw fullcircle scaled s;
);

for i=1 upto 5:
    numeric delta; delta = 2**(i-1);
    t[i] = image(
        draw t[i-1];
        draw t[i-1] shifted ((3/2s * delta, 0) rotated -60);
        draw t[i-1] shifted ((-3/2s * delta, 0) rotated 60);
    );
endfor

numeric x; x = 0;
for i=0 upto 4:
    draw t[i] shifted (x, 0);
    label.bot("$3^" & decimal i & "$", point 1/2 of bbox t[i] + (x, 0));
    x := x + 4.5s * (i+1);
endfor

z0 = (22, -20) * s;
t5 := t5 shifted z0;
draw t5;
draw point 1/2 of bbox t5 -- point 5/2 of bbox t5;
drawdot z0 withpen pencircle scaled 0.9s withcolor Reds 8 3;
label("$\displaystyle 3^n-1 = 2\sum_{k=0}^{n-1} 3^k$", point 1.5 of bbox t5);

label.top("$\displaystyle \sum_{k=0}^{n-1} 3^k = {3^n-1 \over 2} $",
  point 5/2 of bbox currentpicture shifted 12 up);
```

Sums of powers of three

$$\sum_{k=0}^{n-1} 3^k = \frac{3^n - 1}{2}$$





$$3^n - 1 = 2\sum_{k=0}^{n-1} 3^k$$

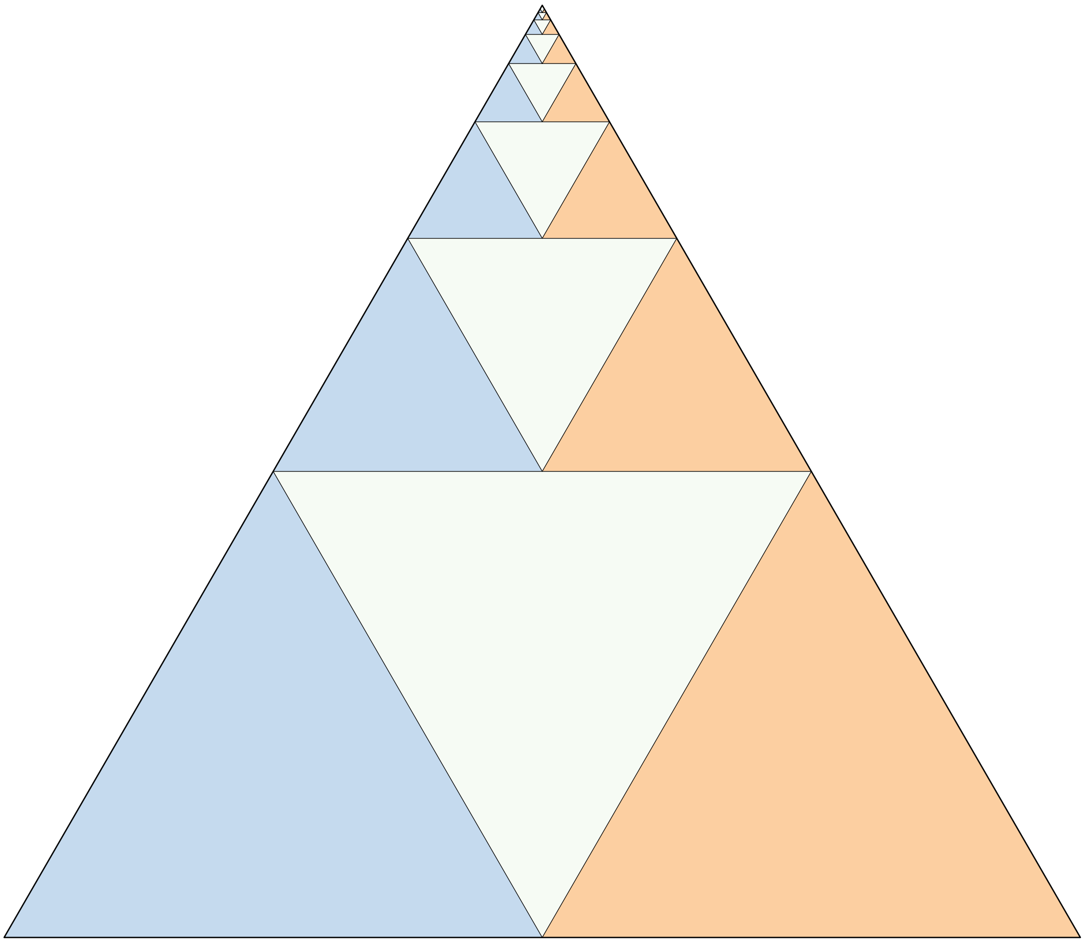
— David B. Sher

Infinite series, linear algebra, & other topics

```
path t; t = for i = 0 upto 2: up scaled 233 rotated 120i -- endfor cycle;
fill t withcolor Greens 8 1;
for i=1 upto 8:
  numeric s; s = 2 ** (i-1);
  path a; a = t shifted - point 1 of t
                scaled (1/2 ** i)
                shifted point 1/s of t;
  path b; b = a reflectedabout(up, down);
  fill a withcolor Blues   7 2; draw a withpen pencircle scaled 1/4;
  fill b withcolor Oranges 7 2; draw b withpen pencircle scaled 1/4;
endfor
draw t;

label.top(btex $\displaystyle
 \frac14 + \left(\frac14\right)^2 + \left(\frac14\right)^3 + \cdots = \frac13
 $ etex, point 5/2 of bbox currentpicture shifted 21 up);
```

A geometric series

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$$

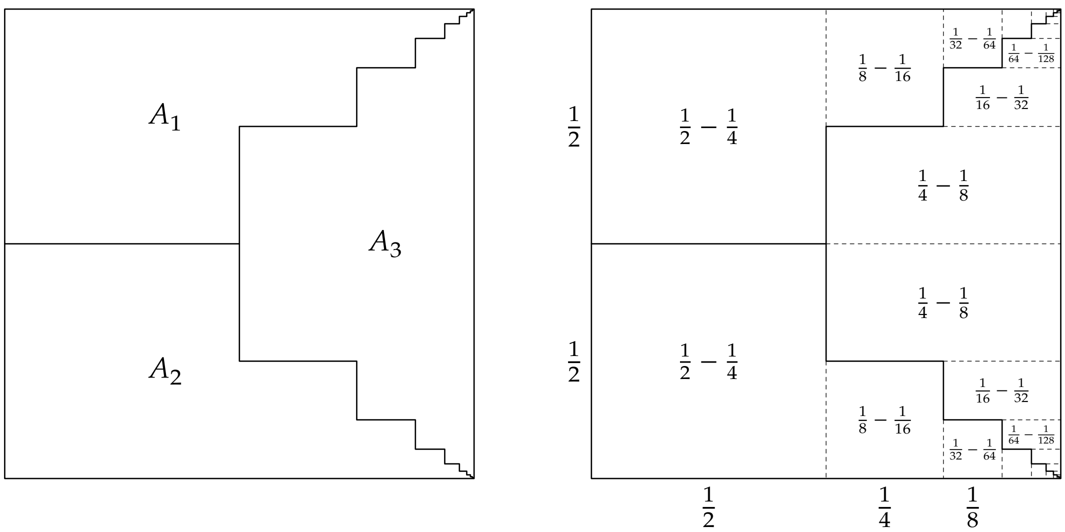


— Rick Mabry

```
numeric u; u = 88; numeric x, y; x = y = 1;
path a; a = (origin for i=1 upto 8:
  hide(y := y * 1/2) -- (1-x, 1-y)
  hide(x := x * 1/2) -- (1-x, 1-y)
endfor) scaled u;
path b; b = a reflectedabout(left, right);
picture P[];
P0 = image(
  draw unitsquare shifted (-1/2, -1/2) scaled 2u;
  draw u * left -- origin; draw a; draw b;
);
P1 = image(
  draw P0;
  for i=1 upto 3:
    label("$A_" & decimal i & "$", 5/8u * dir 120i);
  endfor
);
P2 = image(
  draw P0;
  for t=1 upto 6:
    string p, q; p = decimal (2 ** t); q = decimal (2 ** (t+1));
    picture ff;
    ff = thelabel("$\frac1{" & p & "} - \frac1{" & q & "}$", origin) scaled (7/8 ** t);
    forsuffixes $=a, b:
      draw ff shifted if odd t:
        (xpart point t of $ - (1/2 ** ((t+1)/2)) * u, ypart point t of $)
      else:
        (xpart point t of $, 1/2(ypart point t of $ + ypart point t-2 of $))
      fi;
    endfor
  endfor
  drawoptions(dashed evenly scaled 1/2 withpen pencircle scaled 1/4);
  draw origin -- u * right;
  for t = 1 upto length a:
    forsuffixes $=a, b:
      numeric signum; signum = ypart point t of $ / abs(ypart point t of $);
      draw point t of $ --
        if odd t: (xpart point t of $, u * signum)
            else: (u, ypart point t of $) fi;
    endfor
  endfor
  drawoptions();
  label.lft("$1\over2$", (-u, 1/2u)); label.lft("$1\over2$", (-u, -1/2u));
  label.bot("$1\over2$", (-1/2u, -u));
  label.bot("$1\over4$", (+1/4u, -u)); label.bot("$1\over8$", (+5/8u, -u));
);
draw P1 shifted (-5/4u, 0); draw P2 shifted (+5/4u, 0);
```

An alternating series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots = \frac{1}{3}$$



$$A_1 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots,$$

$$A_1 = A_2 = A_3, \quad A_1 + A_2 + A_3 = 1,$$

$$\therefore \quad A_1 = \frac{1}{3}.$$

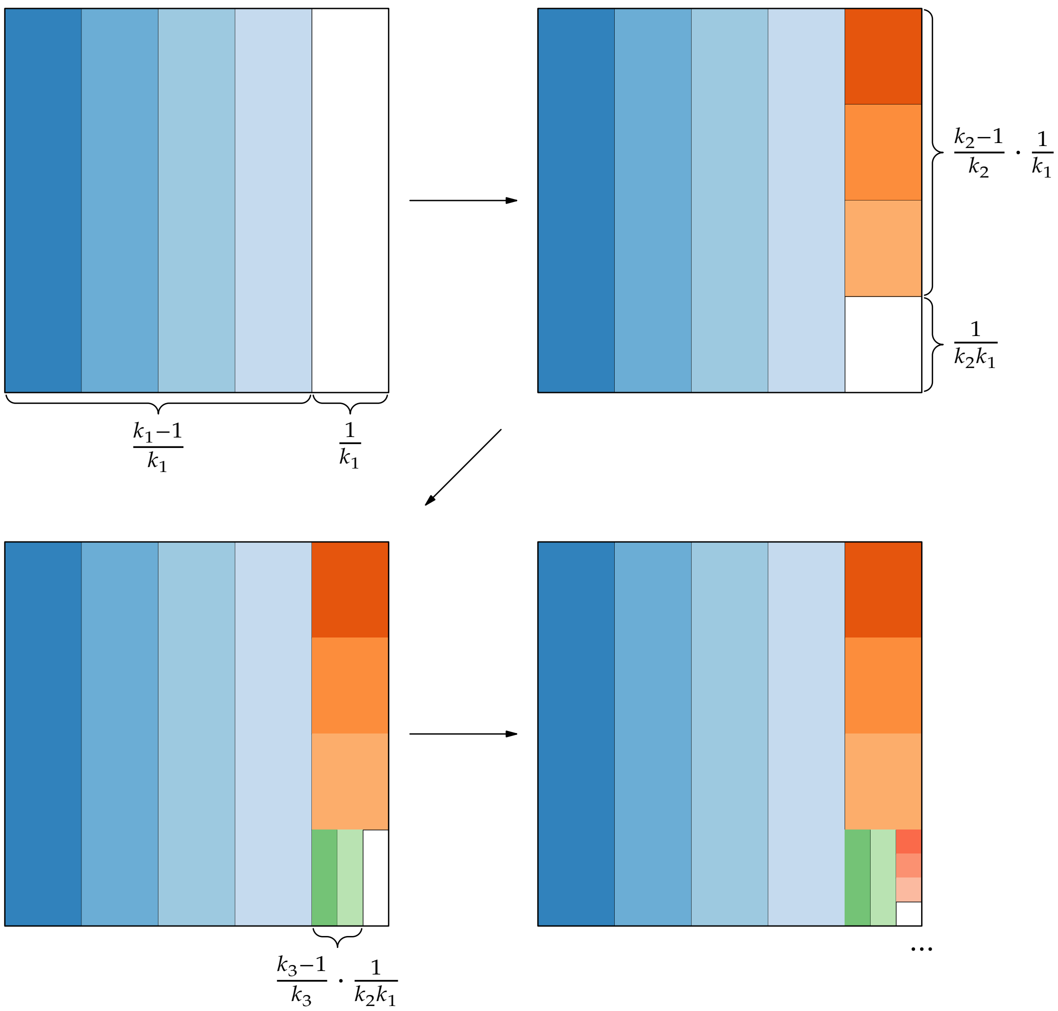
— James O. Chilaka

```
    m shifted (-4u - 1v) {u} .. {v} m {-v} .. {u} m shifted (4u - 1v)
    --
    b shifted (4v-4u) {u} .. {-v rotated -10} b shifted (v-1/2u);
label@#(port, m);
enddef;
numeric u; u = 144;
picture P[]; path s; s = unitsquare shifted -(1/2, 1/2) scaled u;
P1 = image(
for i=0 upto 3:
    path t; t = unitsquare xscaled 1/5u yscaled u
                shifted point 1/5i of s;
    fill t withcolor Blues[6][5-i];
    draw subpath (1,2) of t withpen pencircle scaled 1/4;
endfor
);
P2 = image(
draw P1;
for i=0 upto 2:
    path t; t = unitsquare xscaled -1/5u yscaled -1/4u shifted point 2-1/4i of s;
    fill t withcolor Oranges[6][5-i];
    draw subpath (2,3) of t withpen pencircle scaled 1/4;
endfor
);
P3 = image(
draw P2;
for i=0 upto 1:
    path t; t = unitsquare xscaled (1/3 * 1/5u) yscaled 1/4u shifted point 4/5 + 1/15i of s;
    fill t withcolor Greens[4][3-i];
    draw subpath (1,2) of t withpen pencircle scaled 1/4;
endfor
);
P4 = image(
draw P3;
for i=0 upto 2:
    path t; t = unitsquare xscaled -1/15u yscaled -1/16u shifted point 5/4-1/16i of s;
    fill t withcolor Reds[6][4-i];
    draw subpath (2,3) of t withpen pencircle scaled 1/4;
endfor
);
P11 = image(
draw P1; draw s;
brace_label.bot("$k_1-1\over k_1$", point 0 of s, point 4/5 of s);
brace_label.bot("$1\over k_1$", point 4/5 of s, point 1 of s);
);
P12 = image(
draw P2; draw s;
brace_label.rt("${k_2-1\over k_2}\cdot{1\over k_1}$", point 5/4 of s, point 2 of s);
brace_label.rt("${1\over k_2 k_1}$", point 1 of s, point 5/4 of s);
);
P13 = image(
draw P3; draw s;
brace_label.bot("${k_3-1\over k_3}\cdot{1\over k_2 k_1}$", point 4/5 of s, point 14/15 of s);
);
P14 = image(
draw P4; draw s; label.bot("$\cdots$", point 1 of s shifted 5 down);
);

draw P11 shifted 100 left shifted 100 up;
draw P12 shifted 100 right shifted 100 up;
draw P13 shifted 100 left shifted 100 down;
draw P14 shifted 100 right shifted 100 down;

drawarrow (left--right) scaled 20 shifted 100 up;
drawarrow (left--right) scaled 20 rotated 225;
drawarrow (left--right) scaled 20 shifted 100 down;

label.top(btex $\displaystyle
{k_1-1\over k_1} +
{k_2-1\over k_2 k_1} +
{k_3-1\over k_3 k_2 k_1} + \cdots = 1$ etex, point 5/2 of bbox currentpicture shifted 21 up);
```

A generalized geometric series

Let $\{k_1, k_2, k_3\}$ be a sequence of integers, each of which is at least 2. Then

$$\frac{k_1 - 1}{k_1} + \frac{k_2 - 1}{k_2 k_1} + \frac{k_3 - 1}{k_3 k_2 k_1} + \cdots = 1$$



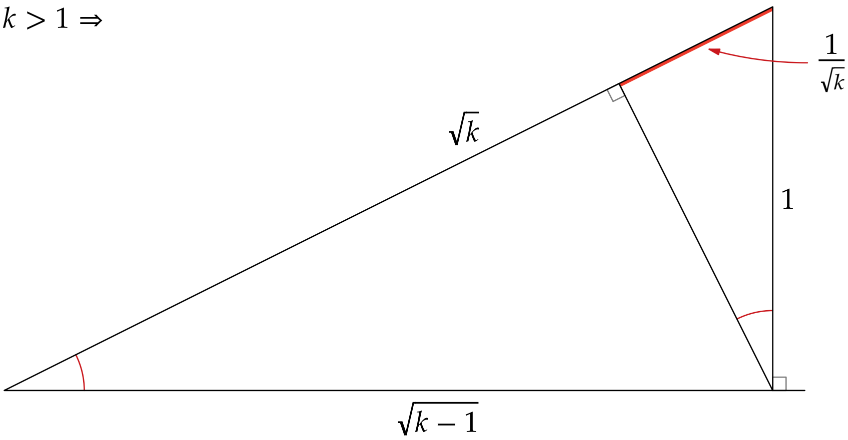
— John Mason

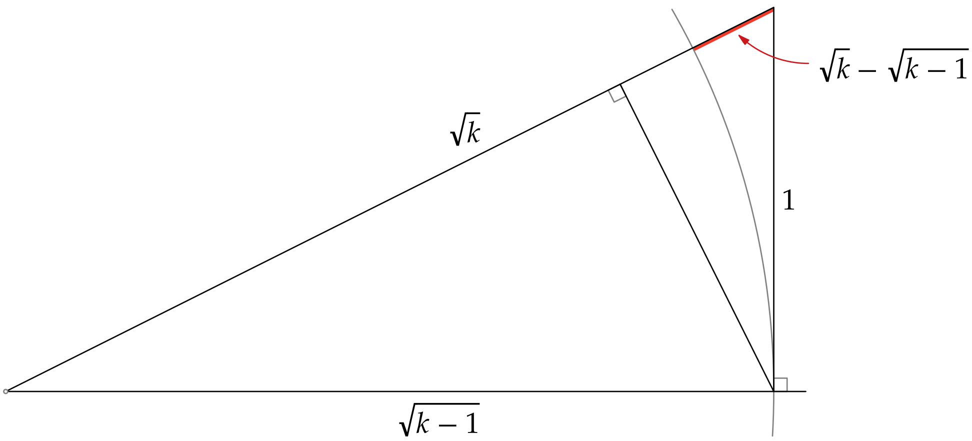
```
numeric u;
u = 144;
z0 = left * u;
z1 = right * u;
z2 = up * u shifted z1;
z3 = whatever[z0, z2]; z3-z1 = whatever * (z2-z0) rotated 90;
z4 = z2 shifted 13 right shifted 21 down;
picture P[];
P0 = image(
draw unitsquare scaled 5 shifted z1 withcolor 1/2 white;
draw unitsquare scaled 5 rotated angle (z0-z3) shifted z3 withcolor 1/2 white;
draw z1 shifted 12 right -- z0 -- z2 -- z1 -- z3;
label.ulft("$\sqrt{k}$", 5/8[z0, z2]);
label.bot ("$\sqrt{k-1}$", 9/16[z0, z1]);
label.rt  ("$1$", 1/2[z1, z2]);
);
P1 = image(
path a; a = quartercircle scaled 60 shifted z0 cutafter (z0--z2);
draw a withcolor Reds 7 6;
draw a shifted -z0 rotated 90 shifted z1 withcolor Reds 7 6;
fill z2 -- z3 -- ((z3--z2) shifted 3/2 down cutbefore (z3--z1)) -- cycle
    withcolor Reds 7 5;
draw P0;
label.rt("\rlap{$\displaystyle 1\over\sqrt{k}$}", z4);
drawarrow z4 {left} .. 1/2[z2, z3] cutafter ((z3--z2) shifted 4 down) withcolor Reds 7 6;
);
P2 = image(
numeric theta; theta = angle (z2-z0);
path a, b;
a = subpath(0, 5/4 theta / 45) of fullcircle scaled 4u rotated -1/8 theta shifted z0;
b = buildcycle(z2 -- z3, a, (z3--z2) shifted 3/2 down, z1--z2);
fill b withcolor Reds 7 5;
draw a withcolor 1/2;
draw P0;
fill fullcircle scaled 2 shifted z0 withcolor 1/2;
fill fullcircle scaled 1 shifted z0 withcolor 1;
label.rt("\rlap{$\sqrt{k} - \sqrt{k-1}$}", z4);
drawarrow z4 {left} .. center b cutafter ((z3--z2) shifted 4 down) withcolor Reds 7 6;
);
label.top(P1, 7 up);
label.bot(P2, 7 down);
label.lrt("$\hbox{$k > 1$} \Rightarrow {}$", ulcorner bbox currentpicture);
label.top("$\displaystyle n > 1 \enspace\Rightarrow\enspace \sum_{k=1}^n {1\over\sqrt{k}} > \sqrt{n}
      point 5/2 of bbox currentpicture shifted 13 up);
label.bot(btex \vbox{\openup 8pt\def\osr#1{{1\over\sqrt{#1}}}\halign{\hss $\displaystyle #$\hss\cr
\osr{k} > \sqrt{k} - \sqrt{k-1}\cr
\osr2 + \osr3 + \cdots + \osr{n} > \left(\sqrt2-1\right) + \left(\sqrt3-\sqrt2\right) + \cdots + \le
\therefore\quad 1 + \osr2 + \osr3 + \cdots + \osr{n} > \sqrt{n}\cr
}} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

Divergence of a series

$$n > 1 \quad \Rightarrow \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

$k > 1 \Rightarrow$





$$\frac{1}{\sqrt{k}} > \sqrt{k} - \sqrt{k-1}$$

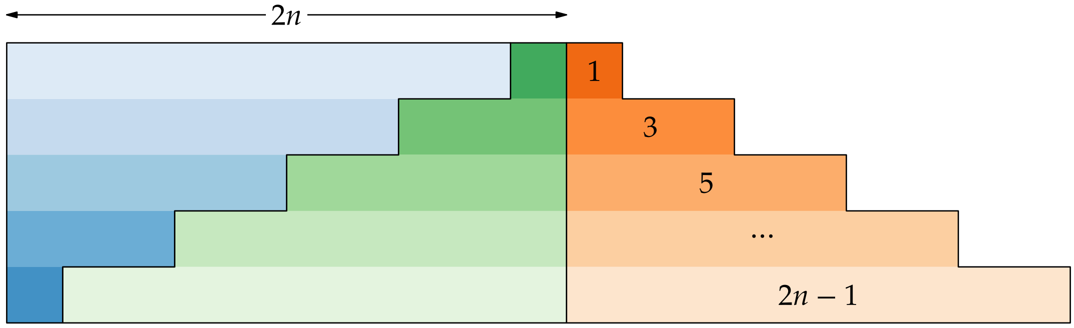
$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \left(\sqrt{2} - 1\right) + \left(\sqrt{3} - \sqrt{2}\right) + \cdots + \left(\sqrt{n} - \sqrt{n-1}\right)$$

$$\therefore \quad 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

— Sidney H. Kung

```
numeric u, n; u = 21; n = 5;
picture a, b, c;
path gg;
gg = (origin for i=1 upto n: -- (2i-1, 1-i) -- (2i-1, -i) endfor -- (0, -n) -- cycle) scaled u;
vardef go(suffix shade) =
save p; picture p; p = image(
    for i=1 upto n:
        fill unitsquare xscaled ((1-2i)*u) yscaled -u shifted point 2i-1 of gg withcolor shade[9][7-
    endfor
    draw gg;
); p enddef;
a = go(Oranges);
b = go(Greens) reflectedabout(up, down);
c = go(Blues) reflectedabout(left, right) shifted (-2n*u, -n*u);
draw a; draw b; draw c;
for i=1 upto n:
label("$" & if i=n: "2n-1" elseif i=n-1: "\cdots" else: decimal (2i-1) fi & "$", u * (i-1/2, 1/2-i))
endfor
path nn; nn = (origin -- 2n * u * left) shifted (1/2u * up);
picture pp; pp = thelabel("$2n$", point 1/2 of nn);
drawdblarrow nn; unfill bbox pp; draw pp;

label.bot(btex $\displaystyle
{1\over3} = {1+3\over5+7} = {1+3+5\over7+9+11} = \cdots =
{1+3+5+\cdots+(2n-1)\over (2n+1)+(2n+3)+\cdots+(2n+2n-1)}$ etex, point 1/2 of bbox currentpicture sh
```

Galileo's ratios



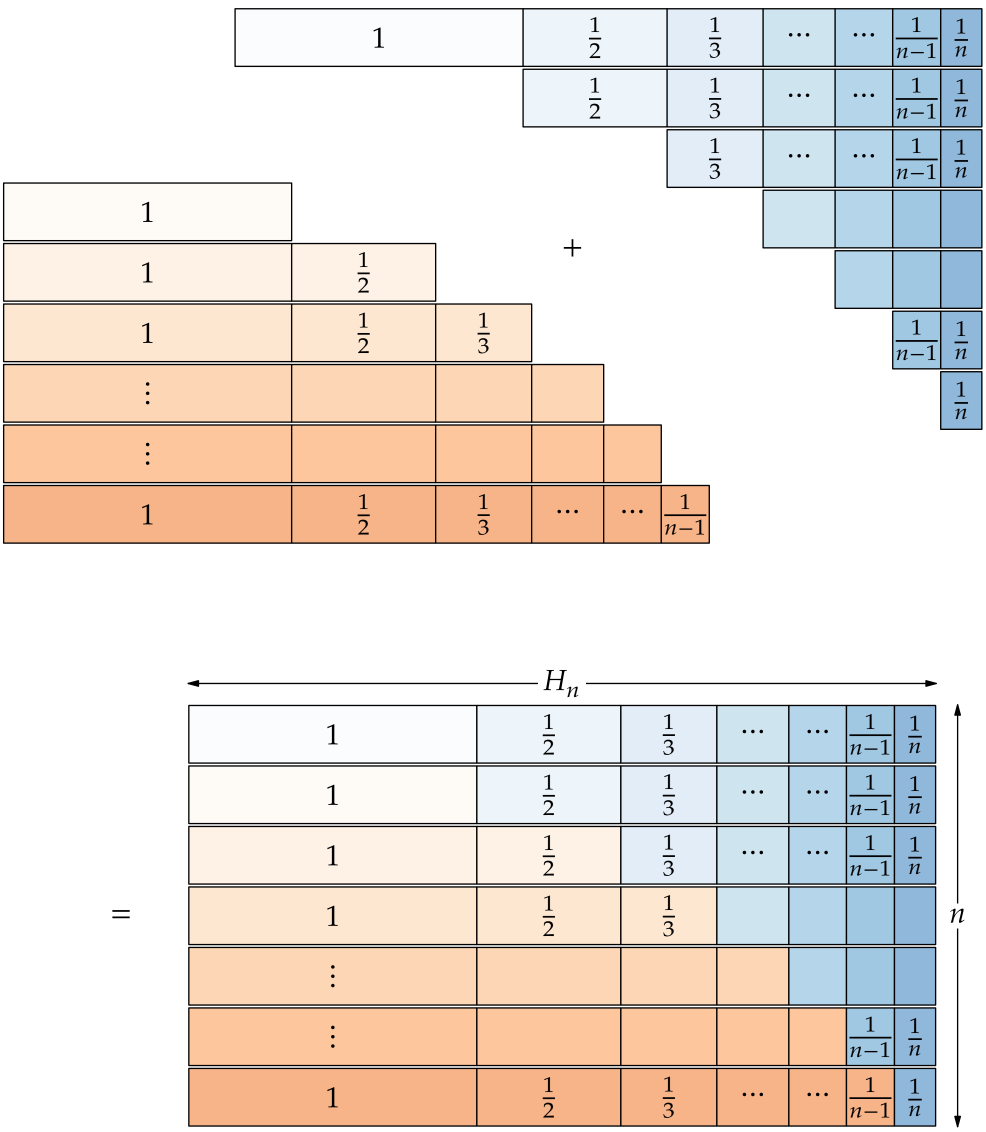
$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+5+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(2n+2n-1)}$$

— Antonio Flores

```
for i=1 upto n:
s[i] = unitsquare xscaled (1/i) yscaled r scaled u;
endfor

picture P[];
P1 = image(
for i=1 upto n-1:
    numeric x, y;
    x = 0; y = -r * i * 1.05 u;
    for j=1 upto i:
        fill s[j] shifted (x, y) withcolor 1/2[Oranges[9][i], white];
        draw s[j] shifted (x, y);
        if (3 < i) and (i < n-1):
            if j=1: label("$\vdots$", center s1 shifted (x, y)); fi
        else:
            label("$" & if j=1: "1" elseif j=n-1: "{1\over n-1}"
            elseif j > 3: "\cdots" else: "{1\over " & decimal j & "}" fi & "$",
            center s[j] shifted (x, y));
        fi
        x := x + u/j;
    endfor
endfor
);
P2 = image(
for i=1 upto n:
    numeric x, y;
    x = 0; y = -r * i * 1.05 u;
    for j=1 upto i-1:
        x := x + u/j;
    endfor
    for j=i upto n:
        fill s[j] shifted (x, y) withcolor 1/2[Blues[9][j], white];
        draw s[j] shifted (x, y);
        if (3 < i) and (i < n-1):
            if j=1: label("$\vdots$", center s1 shifted (x, y)); fi
        else:
            label("$" & if j=1: "1" elseif j=n-1: "{1\over n-1}" elseif j=n: "{1\over n}"
            elseif j > 3: "\cdots" else: "{1\over " & decimal j & "}" fi & "$",
            center s[j] shifted (x, y));
        fi
        x := x + u/j;
    endfor
endfor
);
P3 = image(
draw P1;
draw P2 shifted (0, r * 1.05u);
label.lft("${}={}$", point -1/2 of bbox currentpicture shifted 13 left);
begingroup; interim bboxmargin := 0;
path a, b;
a = subpath (2, 3) of bbox P2 shifted (0, 8 + 1.05 * r * u);
b = subpath (1, 2) of bbox P2 shifted (8, 0 + 1.05 * r * u);
drawdblarrow a;
drawdblarrow b;
endgroup;
picture hh, nn;
hh = thelabel("$H_n$", point 1/2 of a);
nn = thelabel("$n$", point 1/2 of b);
unfill bbox hh; draw hh;
unfill bbox nn; draw nn;
);

draw P1 shifted - center bbox P1 shifted (-3/4u, -2r * u);
draw P2 shifted - center bbox P2 shifted ( 1/8u,  1/2r * u);
label("${}+{}$", origin);

draw P3 shifted - center bbox P3 shifted (-1/8u, -9/4u);
label.bot("$\displaystyle \sum_{k=1}^{n-1} H_k + n = nH_n$", point 1/2 of bbox currentpicture shifte
label.top("$\displaystyle H_k = 1 + {1\over2} + {1\over3} + \cdots + {1\over k} \enspace\Rightarrow\
point 5/2 of bbox currentpicture shifted 21 up);
```

Sums of harmonic numbers

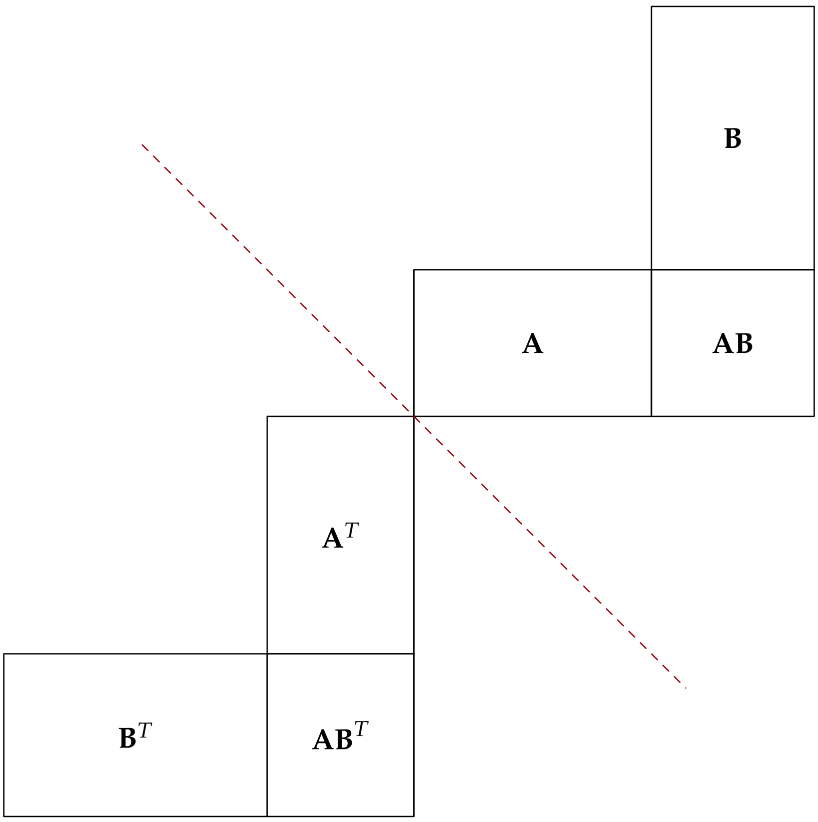
$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \quad \Rightarrow \quad \sum_{k=1}^{n-1} H_k = nH_n - n$$



$$\sum_{k=1}^{n-1} H_k + n = nH_n$$

```
path A, B, AB, A', B', AB';
numeric ax, ay, bx, by;
ax = 89; ay = 55; bx = 61; by = 98.7;

A = unitsquare xscaled ax yscaled ay;
B = unitsquare xscaled bx yscaled by shifted point 2 of A;
AB = unitsquare xscaled bx yscaled ay shifted point 1 of A;
A' = A reflectedabout((-1,1), (1,-1));
AB' = AB reflectedabout((-1,1), (1,-1));
B' = B reflectedabout((-1,1), (1,-1));

forsuffixes $=A,B,AB:
draw $; label("$\mathbf{" & str $ & "}$", center $);
draw $'; label("$\mathbf{" & str $ & "}^T$", center $');
endfor
draw (left--right) scaled 144 rotated -45 dashed evenly withcolor Reds 7 7;
```

$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, where $\mathbf{A}$ and $\mathbf{B}$ are matrices

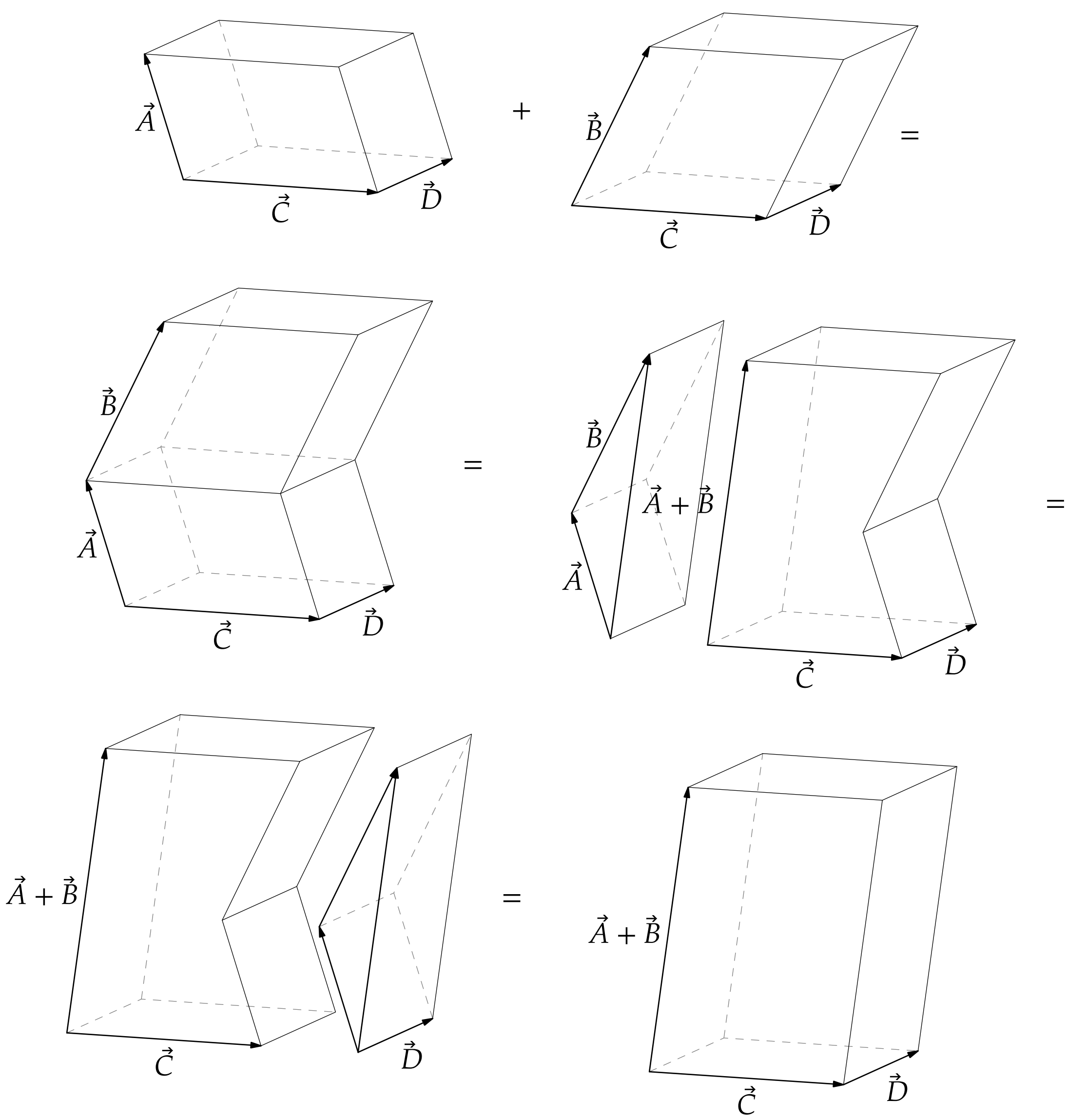


— James G. Simmonds

```
drawoptions(dashed evenly withpen pencircle scaled 1/4 withcolor 1/2);
draw origin -- D -- D+C;
draw D -- A+D -- A+B+D;
draw A -- A+D -- A+C+D;
drawoptions(withpen pencircle scaled 1/4);
draw A -- A+C -- A+C+D;
draw C -- A+C -- A+B+C -- A+B;
draw C+D -- A+C+D -- A+B+C+D -- A+B+D;
draw A+B -- A+B+D;
draw A+B+C -- A+B+C+D;
drawoptions();
drawarrow origin -- A; label.lft("$\vA$", 1/2A);
drawarrow A -- A+B; label.lft("$\vB$", 1/2[A, A+B]);
drawarrow origin -- C; label.bot("$\vC$", 1/2C);
drawarrow C -- C + D; label.lrt("$\vD$", 1/2[C, C+D]);
);
P4 = image(
pair dx; dx = -1/2C;
drawoptions(dashed evenly withpen pencircle scaled 1/4 withcolor 1/2);
draw origin -- D -- D+C;
draw D -- A+B+D;
draw A + dx -- A+D + dx;
draw D + dx -- A+D + dx -- A+B+D + dx;
drawoptions(withpen pencircle scaled 1/4);
draw C -- A+C -- A+B+C -- A+B;
draw C+D -- A+C+D -- A+B+C+D -- A+B+D;
draw A+B -- A+B+D;
draw A+C -- A+C+D;
draw A+B+C -- A+B+C+D;
draw origin + dx -- D+ dx -- A+B+D+ dx -- A+B+ dx;
drawoptions();
drawarrow origin -- A+B;
drawarrow origin -- C;
drawarrow C -- C + D;
drawarrow origin + dx -- A + dx;
drawarrow origin + dx -- A+B + dx;
drawarrow A + dx -- A+B + dx;
label("$\vA+\vB$", 1/2(A+B+ dx));
label.bot("$\vC$", 1/2C);
label.lft("$\vA$", 1/2A+ dx);
label.lft("$\vB$", 1/2[A+ dx, A+B+ dx]);
label.lrt("$\vD$", 1/2[C, C+D]);
);
P5 = image(
pair dx; dx = 3/2C;
drawoptions(dashed evenly withpen pencircle scaled 1/4 withcolor 1/2);
draw origin -- D -- D+C;
draw D -- A+B+D;
draw A + dx -- A+D + dx;
draw D + dx -- A+D + dx -- A+B+D + dx;
drawoptions(withpen pencircle scaled 1/4);
draw C -- A+C -- A+B+C -- A+B;
draw C -- C+D -- A+C+D -- A+B+C+D -- A+B+D;
draw A+B -- A+B+D;
draw A+C -- A+C+D;
draw A+B+C -- A+B+C+D;
draw D + dx -- A+B+D+ dx -- A+B+ dx;
drawoptions();
drawarrow origin + dx -- D + dx;
drawarrow origin -- A+B;
drawarrow origin -- C;
drawarrow + dx -- A+ dx;
drawarrow + dx -- A+B+ dx;
drawarrow A+ dx -- A+B+ dx;
label.lft("$\vA+\vB$", 1/2(A+B));
label.bot("$\vC$", 1/2C);
label.lrt("$\vD$", 1/2[dx, dx+D]);
);
P6 = image(
drawoptions(dashed evenly withpen pencircle scaled 1/4 withcolor 1/2);
```

The distributive property of the triple scalar product

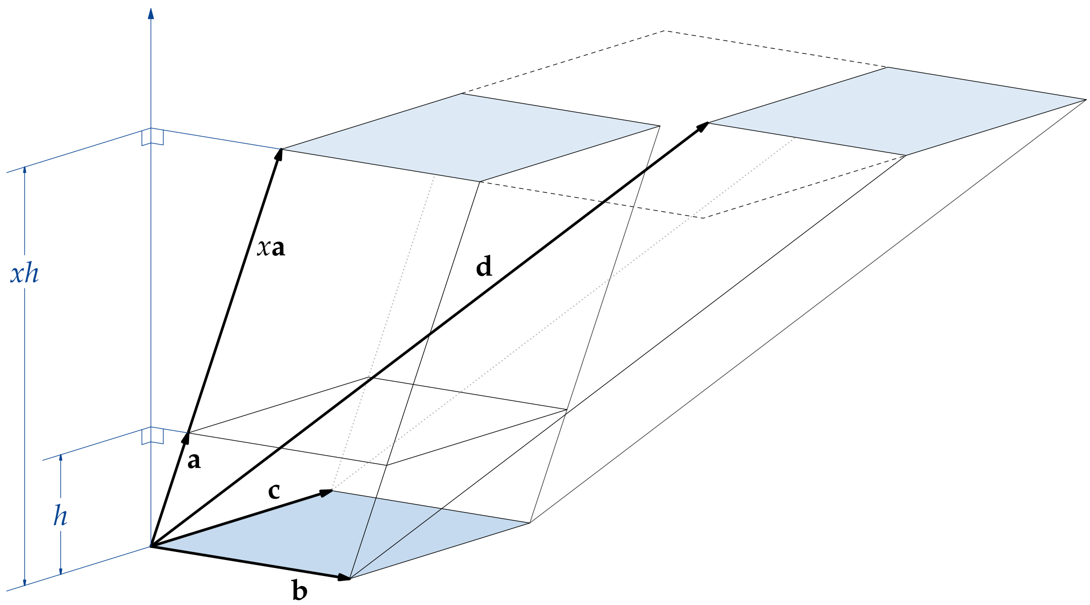
$$\vec{A}\cdot\left(\vec{C}\times\vec{D}\right)+\vec{B}\cdot\left(\vec{C}\times\vec{D}\right)=\left(\vec{A}+\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)$$



— Constance C. Edwards and Prashant S. Sansgiry

```
set_projection(13, -36); ipscale := 23;

numeric x, y, z;
x = 7/2;
y = 9/8;
z = 9/8;

pair a, b, c, d, o;
o = origin;
a = p(3/4, 2, 0);
b = p(4, 0, 0);
c = p(0, 0, 5);
d = x*a + y*b + z*c;

path base[];
base0 = o -- b -- b + c -- c -- cycle;
base1 = base0 shifted a;
base2 = base0 shifted (x*a);
base3 = base0 shifted d;

draw point 3 of base2 -- point 3 of base3 - y*b -- point 3 of base3 dashed evenly scaled 1/2 withpen
draw point 1 of base2 -- point 1 of base3 - z*c -- point 1 of base3 dashed evenly scaled 1/2 withpen

fill base0 withcolor Blues 8 3;
draw base0 withpen pencircle scaled 1/4;
forsuffixes $=2, 3:
draw point 1 of base 0 -- point 1 of base$ withpen pencircle scaled 1/4 withcolor 0;
draw point 2 of base 0 -- point 2 of base$ withpen pencircle scaled 1/4 withcolor 1/4;
draw point 3 of base 0 -- point 3 of base$ dashed withdots scaled 1/4 withcolor 3/4;
endfor
draw base1 withpen pencircle scaled 1/4;
forsuffixes $=2, 3:
fill base$ withcolor Blues[8][2];
draw base$ withpen pencircle scaled 1/4;
endfor
forsuffixes $=a,b,c,d:
drawarrow origin -- $ withpen pencircle scaled 1;
endfor
drawarrow a -- x*a withpen pencircle scaled 1;

drawoptions(withpen pencircle scaled 1/4 withcolor Blues 8 8);
drawarrow p(0, 0,  -4) -- o -- p(0, 9, 0);
draw      p(0, 2, -3) -- p(0, 2, 0) -- a;
draw      p(0, 2x, -4) -- p(0, 2x, 0) -- x*a;
numeric s; s = 1/4;
draw p(0,2,-s)--p(0,2-s,-s)--p(0,2-s,0)--p(s,2-s,0)--p(s,2,0);
draw p(0,2x,-s)--p(0,2x-s,-s)--p(0,2x-s,0)--p(s,2x-s,0)--p(s,2x,0);
path hx, h; picture lhx, lh;
h = p(0, 0, -5/2)--p(0, 2, -5/2);
hx = p(0, 0, -7/2)--p(0, 2x, -7/2);
interim ahangle := 20; ahlength := 2;
drawdblarrow h; lh = thelabel("$h$", point 1/2 of h); unfill bbox lh; draw lh;
drawdblarrow hx; lhx = thelabel("$xh$", point 3/4 of hx); unfill bbox lhx; draw lhx;
drawoptions();

label.rt("$\textbf{a}$", 3/4a);
label.rt("$x\textbf{a}$", 3/4a*x);
label.bot("$\textbf{b}$", 3/4b);
label.ulft("$\textbf{c}$", 3/4c);
label.ulft("$\textbf{d}$", 5/8d);

label.bot(btex \vbox{\openup 12pt \def\va{\textbf{a}}\def\vb{\textbf{b}}\def\vc{\textbf{c}}\def\vd{\
\halign{\hss $\displaystyle # $\hss\cr
x\va+ y\vb+ z\vc = \vd \enspace\Rightarrow\enspace \det(\vd,\vb,\vc) = \det(x\va,\vb,\vc) = x\det(\v
\therefore\quad x={\det(\vd,\vb,\vc)\over\det(\va,\vb,\vc)}\cr}} etex, point 1/2 of bbox currentpict
```

Cramer's rule



$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d} \quad \Rightarrow \quad \det(\mathbf{d}, \mathbf{b}, \mathbf{c}) = \det(x\mathbf{a}, \mathbf{b}, \mathbf{c}) = x \det(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

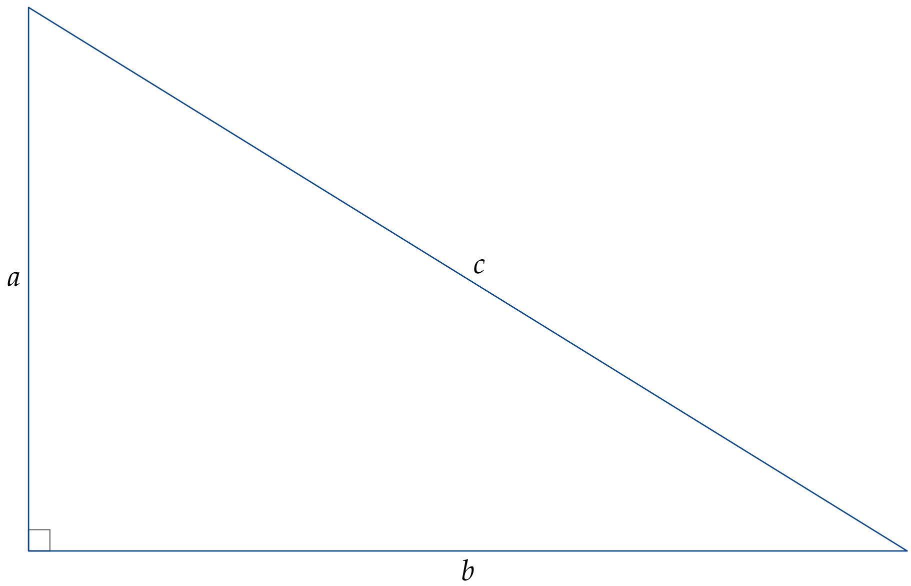
$$\therefore \quad x = \frac{\det(\mathbf{d}, \mathbf{b}, \mathbf{c})}{\det(\mathbf{a}, \mathbf{b}, \mathbf{c})}$$

— *The Mathematics Initiative*, Education Development Center

```
path t; t = (origin -- 144 up -- 233 right -- cycle) scaled 1.414;
draw unitsquare scaled 8 withcolor 1/2; draw t withcolor Blues 8 8;
label.lft("$a$", point 1/2 of t);
label.bot("$b$", point -1/2 of t);
label.urt("$c$", point 3/2 of t);

label.top("${a\over2}, b, c \in \mathbb{Z}^+$,\quad $(a,b)=1$", point 5/2 of bbox currentpicture shi
label.bot(btex \vbox{\openup 12 pt\halign{#\hfil&$\displaystyle {}\Rightarrow #$\hfil\cr
$\displaystyle {c+b\over a}={n\over m}$, \enspace $(n, m) = 1$
& {c-b\over a} = {m\over n},\cr
& {c \over a} = {n^2 + m^2 \over 2mn},\enspace
  {b \over a} = {n^2 - m^2 \over 2mn},\cr
& n \not\equiv m \pmod2.\cr
\therefore\enspace $(a,b,c) = \bigl(2mn, n^2-m^2, n^2+m^2\bigr)$.\hidewidth&\omit\cr
}} etex, point 1/2 of bbox currentpicture shifted 42 down);
```

Parametric representation of primitive Pythagorean triples

$$\frac{a}{2}, b, c \in \mathbb{Z}^+, \quad (a,b) = 1$$



$$\begin{aligned}
\frac{c+b}{a} = \frac{n}{m}, \ \ (n,m) = 1 \ &\Rightarrow \frac{c-b}{a} = \frac{m}{n}, \\
&\Rightarrow \frac{c}{a} = \frac{n^2+m^2}{2mn}, \ \ \frac{b}{a} = \frac{n^2-m^2}{2mn}, \\
&\Rightarrow n \not\equiv m \pmod 2.
\end{aligned}$$

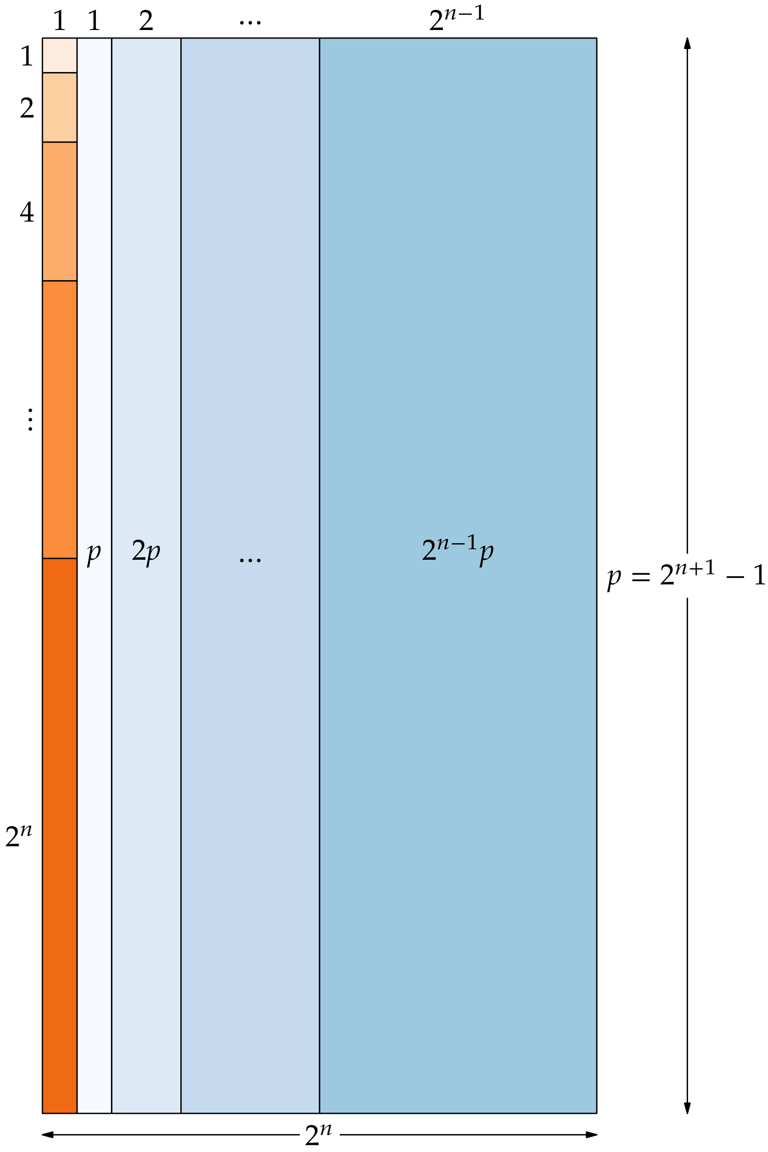
$$\therefore \ (a,b,c) = (2mn, n^2-m^2, n^2+m^2).$$

— Raymond A. Beauregard and E. R. Suryanarayan

```
numeric u, n; u = 13; n = 4;
for i=0 upto n:
path s; s = unitsquare xscaled -1 yscaled -(2**i) scaled u shifted ((0, 2**i-1)*-u);
fill s withcolor Oranges[7][i+1]; draw s;
label.lft("$" & if i=n: "2^n" elseif i=n-1: "\vdots" else: decimal (2**i) fi & "$", point 3/2 of s);
if i=0: label.top("$1$", point 1/2 of s); fi
endfor
for i=0 upto n-1:
path s; s = unitsquare xscaled (2**i) yscaled -(2**(n+1)-1) scaled u shifted ((2**i-1, 0)*u);
fill s withcolor Blues [8][i+1];
draw s;
label.top("$" & if i=n-1: "2^{n-1}" elseif i=n-2: "\cdots" else: decimal (2**i) fi & "$", point 1/2
label.top("$" & if i=n-1: "2^{n-1}p" elseif i=n-2: "\cdots" elseif i=0: "p" else: decimal (2**i) & "
endfor
path ax, ay; picture lax, lay;
ax = ((1, 2**(n+1)-1) -- (-(2**n-1), 2**(n+1)-1)) scaled -u shifted 8 down;
ay = ((2**n, 0) -- (2**n, -(2**(n+1)-1))) scaled u shifted 21 right;
lax = thelabel("$2^n$", point 1/2 of ax);
lay = thelabel("$p=2^{n+1}-1$", point 1/2 of ay);
drawdblarrow ax; unfill bbox lax; draw lax;
drawdblarrow ay; unfill bbox lay; draw lay;

label.top("$p=2^{n+1}-1$ prime ${}\Rightarrow{}$ $N=2^np$ perfect",
point 5/2 of bbox currentpicture shifted 13 up);
label.bot("$1+2+\cdots+2^n+p+2p+\cdots+2^{n-1}p = 2^np = N$",
point 1/2 of bbox currentpicture shifted 13 down);
```

On perfect numbers

$$p = 2^{n+1} - 1 \text{ prime } \Rightarrow N = 2^n p \text{ perfect}$$



$$1 + 2 + \cdots + 2^n + p + 2p + \cdots + 2^{n-1}p = 2^n p = N$$

— Don Goldberg

```
path pp; pp = for i=0 upto 3: (i*u, 0) -- endfor (3/2u, v) -- cycle; draw pp;
numeric t; t = -1;
forsuffixes s=x, y, z, w, p:
    draw if odd ASCII str s: rdot else: bdot fi shifted point incr t of pp;
    label("$\strut " & str s & "$", point t of pp shifted if t>3: 8 down else: 10 up fi);
endfor
label.lft("$G_5 : {}$", point -1/2 of bbox currentpicture)
);
G51 = image(
path pp; pp = for i=0 upto 3: (i*u, 0) -- endfor (3/2u, v) -- cycle; draw pp;
numeric t; t = -1;
forsuffixes s=y, w, x, z, p:
    draw if odd ASCII str s: rdot else: bdot fi shifted point incr t of pp;
    label("$\strut " & str s & "$", point t of pp shifted if t>3: 8 down else: 10 up fi);
endfor
label.lft("$\overline{G_5} : {}$", point -1/2 of bbox currentpicture)
);
G8 = image(
path pp; pp = for i=0 upto 3: (i*u, 0) -- endfor for i=3 downto 0: (i*u, v) -- endfor cycle; draw pp
for t = 4 upto 7: draw point 0 of pp -- point t of pp -- point 3 of pp; endfor
numeric t; t = -1;
forsuffixes s=x, y, z, w, d, c, b, a:
    draw if odd ASCII str s: rdot else: bdot fi shifted point incr t of pp;
    label("$\strut " & str s & "$", point t of pp shifted if t>3: 10 down else: 10 up fi);
endfor
label.lft("$G_8 : {}$", point -1/2 of bbox currentpicture);
);
G81 = image(
path pp; pp = for i=0 upto 3: (i*u, 0) -- endfor for i=3 downto 0: (i*u, v) -- endfor cycle; draw pp
for t = 4 upto 7: draw point 0 of pp -- point t of pp -- point 3 of pp; endfor
numeric t; t = -1;
forsuffixes s = y, w, x, z, c, a, d, b:
    draw if odd ASCII str s: rdot else: bdot fi shifted point incr t of pp;
    label("$\strut " & str s & "$", point t of pp shifted if t>3: 10 down else: 10 up fi);
endfor
label.lft("$\overline{G_8} : {}$", point -1/2 of bbox currentpicture);
);
G0 = image(
path pp, ee;
ee = fullcircle xscaled 6u yscaled 2u;
draw ee;
pp = (origin for i=1 upto 3: -- (i*u, 0) endfor) shifted (-3/2u, -2v); draw pp;
for i=1 upto 5:
    z[i] = (i/6)[point 4 of ee, point 0 of ee];
endfor
for i=1, 2, 5:
    draw point 0 of pp -- z[i] -- point 3 of pp;
    draw dot shifted z[i];
endfor
for i=3, 4:
    draw point 0 of pp -- z[i] cutafter ee scaled 2;
    draw point 3 of pp -- z[i] cutafter ee scaled 2;
endfor
numeric t; t = -1;
forsuffixes s=x, y, z, w:
    draw if odd ASCII str s: rdot else: bdot fi shifted point incr t of pp;
    label("$\strut " & str s & "$", point t of pp shifted 10 up);
endfor
label("$G_k$", 7/12[point 4 of ee, point 0 of ee]);
label("$G_{k+4} : {}$", point -1/2 of bbox currentpicture);
);
G10 = image(
path pp, ee;
ee = fullcircle xscaled 6u yscaled 2u;
draw ee;
pp = (origin for i=1 upto 3: -- (i*u, 0) endfor) shifted (-3/2u, -2v); draw pp;
for i=1 upto 5:
    z[i] = (i/6)[point 4 of ee, point 0 of ee];
endfor
for i=1, 2, 5:
    draw point 0 of pp -- z[i] -- point 3 of pp;
```

Self-complementary graphs

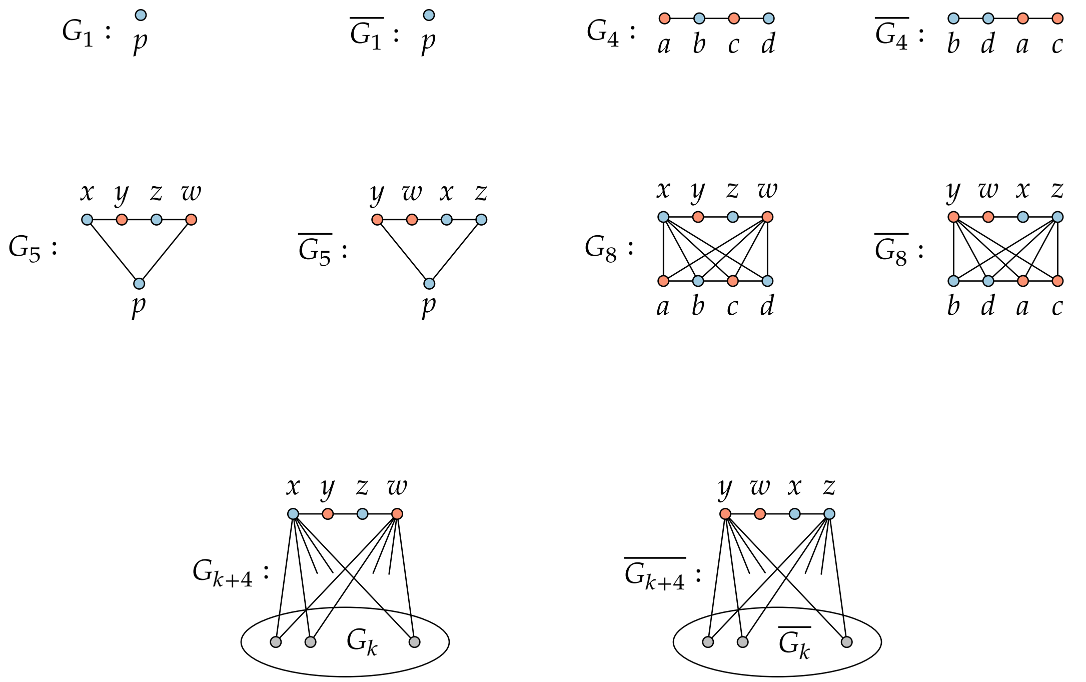
A graph is *simple* if it contains no loops or multiple edges. A simple graph $G = (V, E)$ is *self-complementary* of G is isomorphic to its *complement* $\bar{G} = (V, \bar{E})$, where

$$\bar{E} = \{\{v, w\} : v, w \in V, v \neq w, \text{and } \{v, w\} \notin E\}.$$

It is a standard exercise to show that if G is a self-complementary simple graph with n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. A converse also holds, as we now show.

THEOREM: *If n is a positive integer and either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, then there exists a self-complementary simple graph G_n with n vertices.*

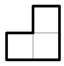
PROOF:



— Stephan C. Carlson

```
draw point 5/2 of t -- origin -- point 7/2 of t withpen pencircle scaled 1/4 withcolor 1/2;
draw t withpen pencircle scaled 1; \end{mplibcode}

\bigskip\noindent
\textsc{Theorem}: \textit{If $n$ is a power of two, then an $n\times n$ chess board with any one squ
can be tiled with trominoes}.

\bigskip\noindent
\textsc{Proof} (by induction):

\bigskip\noindent
I. $$
fill t withcolor Blues 8 6; fill origin -- subpath (5/2, 7/2) of t -- cycle withcolor Blues 8 2;
draw t withpen pencircle scaled 1; \end{mplibcode}
$$

\bigskip\noindent
II.
$$
numeric n, u; n = 16; u = 10;
picture P[];
P0 = image(
for i=0 upto n-1: for j = 0 upto n-1:
    fill unitsquare scaled u shifted (u*(i, j)) withcolor Blues[8][if odd (i+j): 2 else: 6 fi];
endfor endfor
unfill unitsquare scaled u shifted (2u,11u);
draw   unitsquare scaled u shifted (2u,11u) withpen pencircle scaled 1;
draw unitsquare scaled (u*n) withpen pencircle scaled 1;
);
P2 = image(draw P0;
begingroup; interim bboxmargin := 0;
for t=-1/2, 1/2:
    draw point t of bbox P0 -- point 2+t of bbox P0 withpen pencircle scaled 1;
endfor
path hole; hole = (origin -- (0, u) -- (u, u) -- (u, -u) -- (-u, -u) -- (-u, 0) -- cycle) shifted ce
unfill hole; draw hole withpen pencircle scaled 1;
path a, b; picture p, q;
a = subpath (0, 1/2) of bbox P0 shifted 12 down;
b = subpath (1/2, 1) of bbox P0 shifted 12 down;
endgroup;
drawdblarrow a;
drawdblarrow b;
p = thelabel("$n$", point 1/2 of a); unfill bbox p; draw p;
q = thelabel("$n$", point 1/2 of b); unfill bbox q; draw q;
);
P1 = image(draw P0; path a; picture p;
begingroup; interim bboxmargin := 0;
a = subpath (0, 1) of bbox P0 shifted 12 down;
endgroup; drawdblarrow a;
p = thelabel("$2n$", point 1/2 of a); unfill bbox p; draw p;
);

label(P1, 112 left);
label(P2, 112 right);
label("$=$", origin);
```

Tiling with trominoes

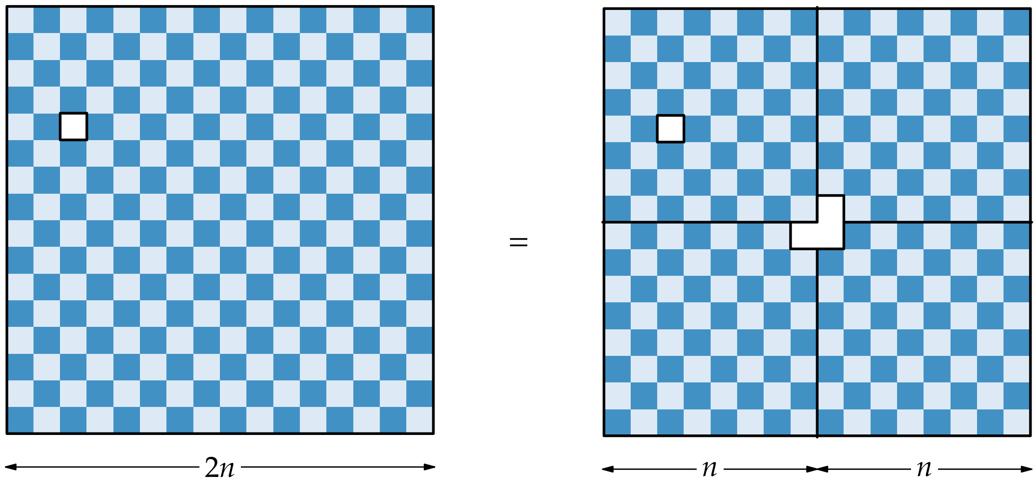
A *tronimo* is a plane figure composed of three squares: 

Theorem: *If n is a power of two, then an $n \times n$ chess board with any one square removed can be tiled with trominoes.*

Proof (by induction):

I.

II.



— Solomon W. Golomb

Note: Except when $n = 5$, an $n \times n$ chessboard with any one square removed can be tiled with tronimoes if and only if $n \not\equiv 0 \pmod 3$. See I-Ping Chu and Richard Johnsonbaugh, "Tiling deficient boards with tronimoes", *Mathematics Magazine*, 59 (1986) 34–40.