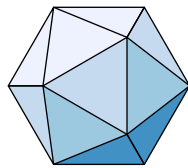


# Proofs without words I

Exercises in METAPOST

Toby Thurston

March 2021 —



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# Geometry and Algebra

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## The Pythagorean theorem I



— adapted from the *Chou pei san ching*

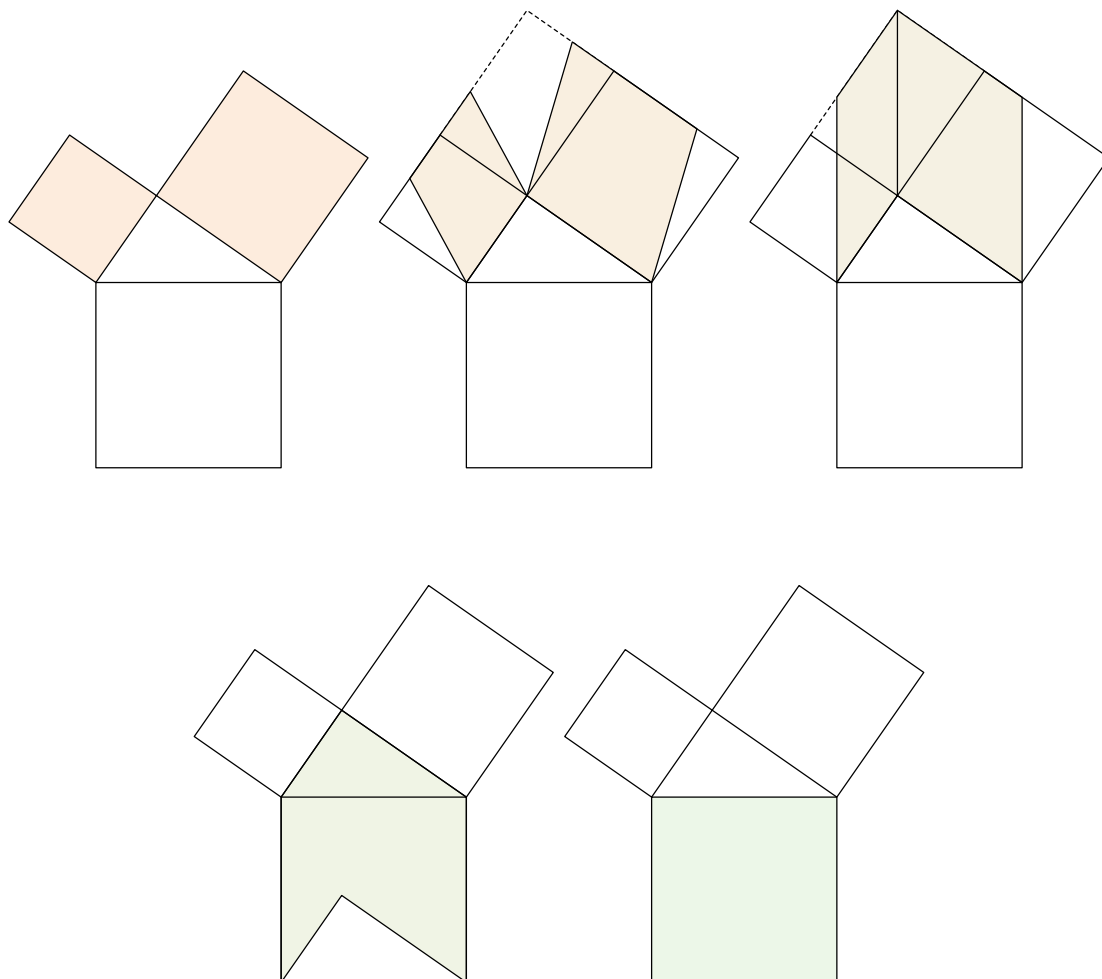
## The Pythagorean theorem II



*Behold!*

— Bhāskara (12th century)

## The Pythagorean theorem III



— based on Euclid's proof

## The Pythagorean theorem IV



— H. E. Dudeney (1917)

## The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a+b)^2$$

$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)



## The Pythagorean theorem VI

$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2 + b^2 = c^2$$



— Michael Hardy

**A Pythagorean theorem:  $aa' = bb' + cc'$**



$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$

$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$

$$\therefore aa' = a(x + y) = bb' + cc'.$$

— Enzo R. Gentile

## The rolling circle squares itself



— Thomas Elsner

## On trisecting an angle



— Rufus Isaacs

## Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

— Eric Kincanon

# Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

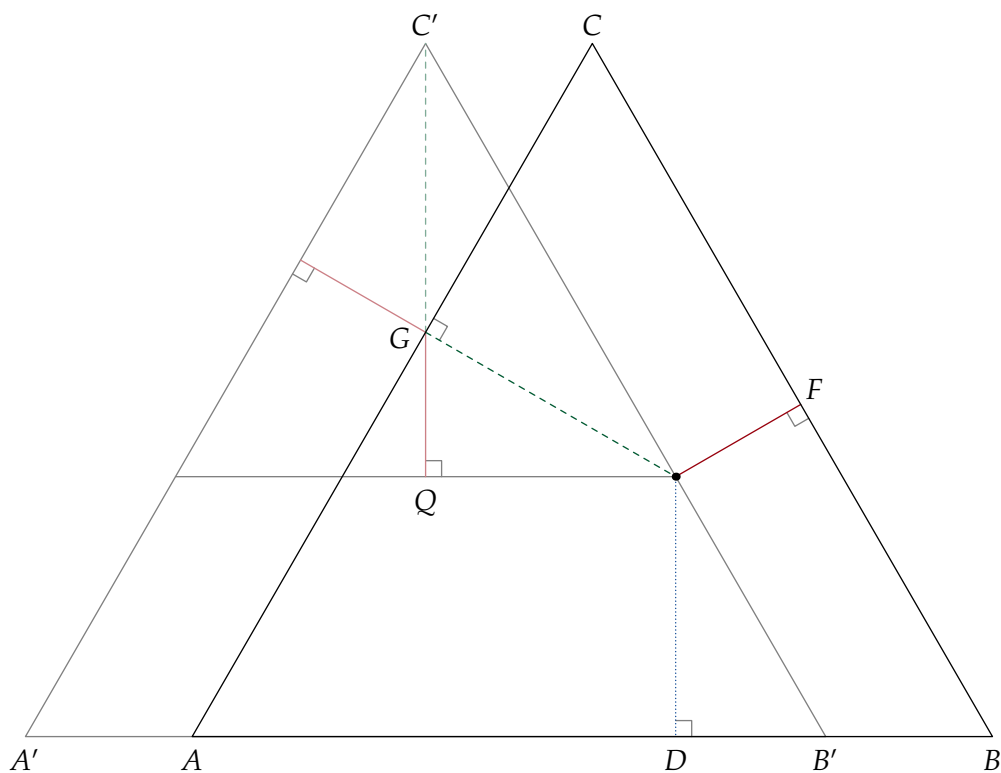
**The vertex angles of a star sum to  $180^\circ$**



— Fouad Nakhli

## Viviani's theorem I

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



*This shows a particular example, with  $C'GQ$  collinear, rather than the general case*

— Samuel Wolf



## Viviani's theorem II

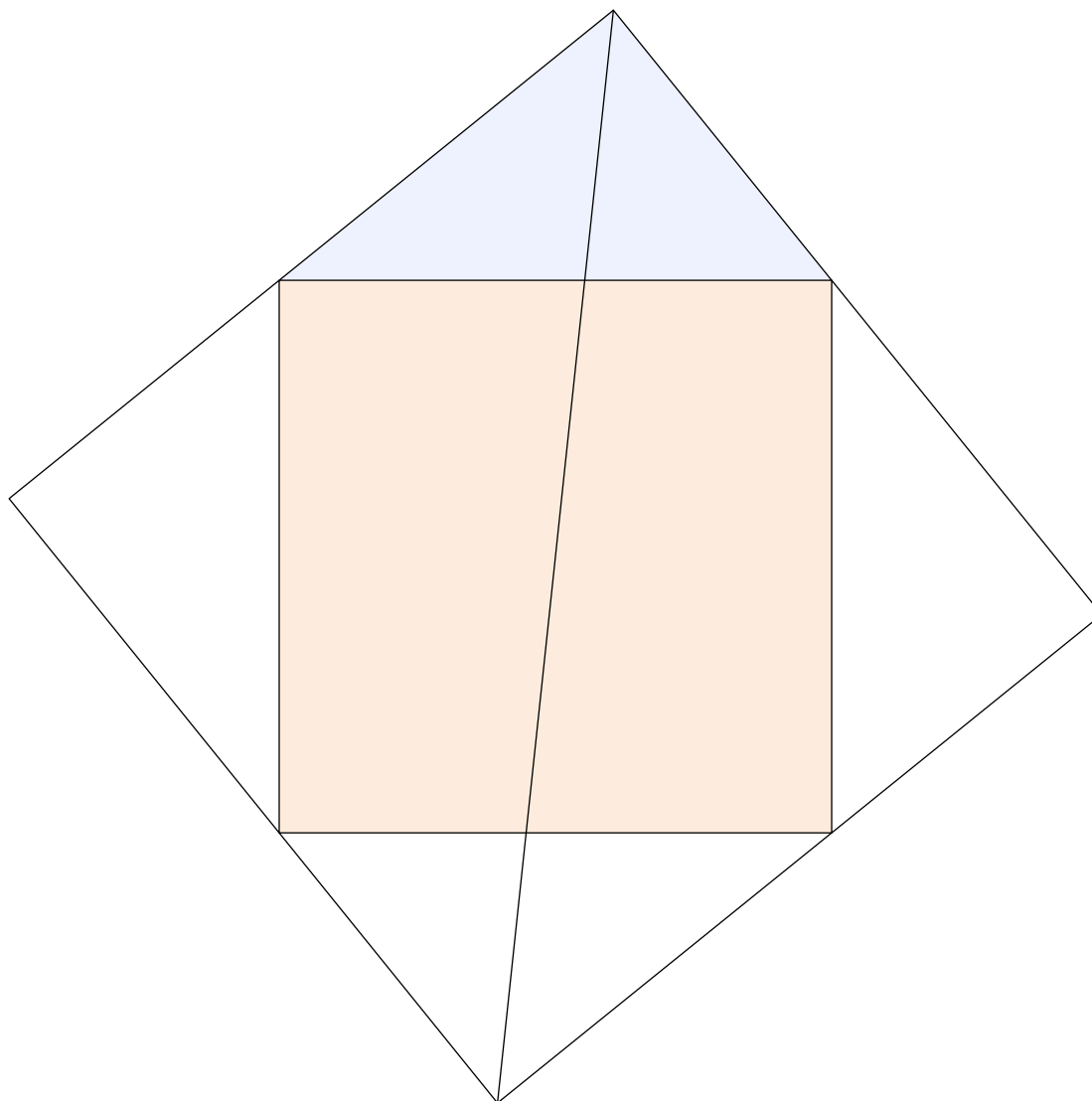
The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

## **A theorem about right angles**

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse



— Roland H. Eddy

# Area and the projection theorem of a right triangle



$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

## Chords and tangents of equal length

If circle  $C_1$  passes through the center  $O$  of circle  $C_2$ , the length of the common chord  $\overline{PQ}$  is equal to the tangent segment  $\overline{PR}$ .



— Roland H. Eddy

## Completing the square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

## Algebraic areas I

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$



— Shirley Wakin

## Algebraic areas II

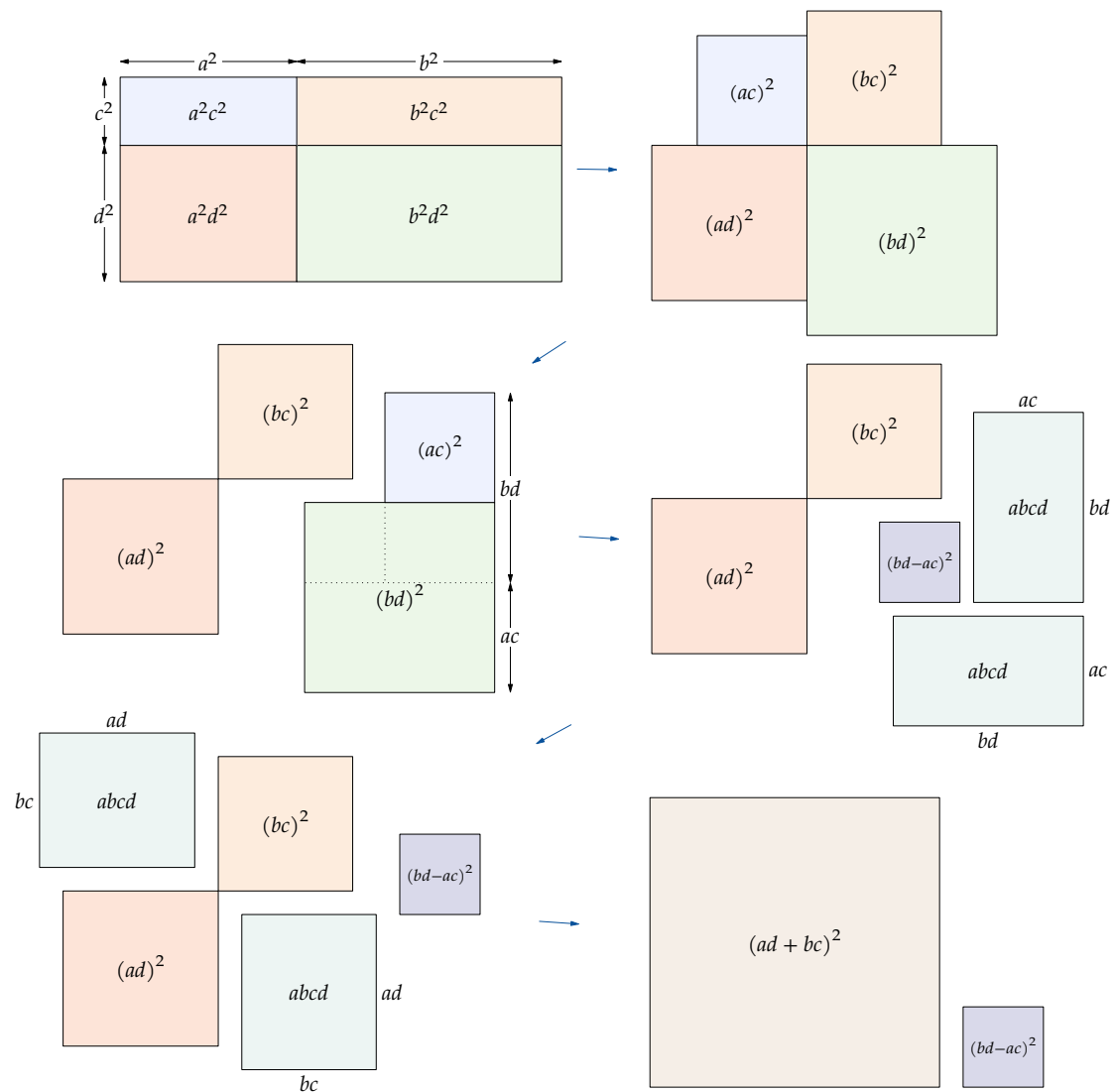
$$(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$



— Sam Pooley and K. Ann Drude

## Sum of squares identity

$$(a^2 + b^2)(c^2 + d^2) = (ab + bc)^2 + (bd - ac)^2$$

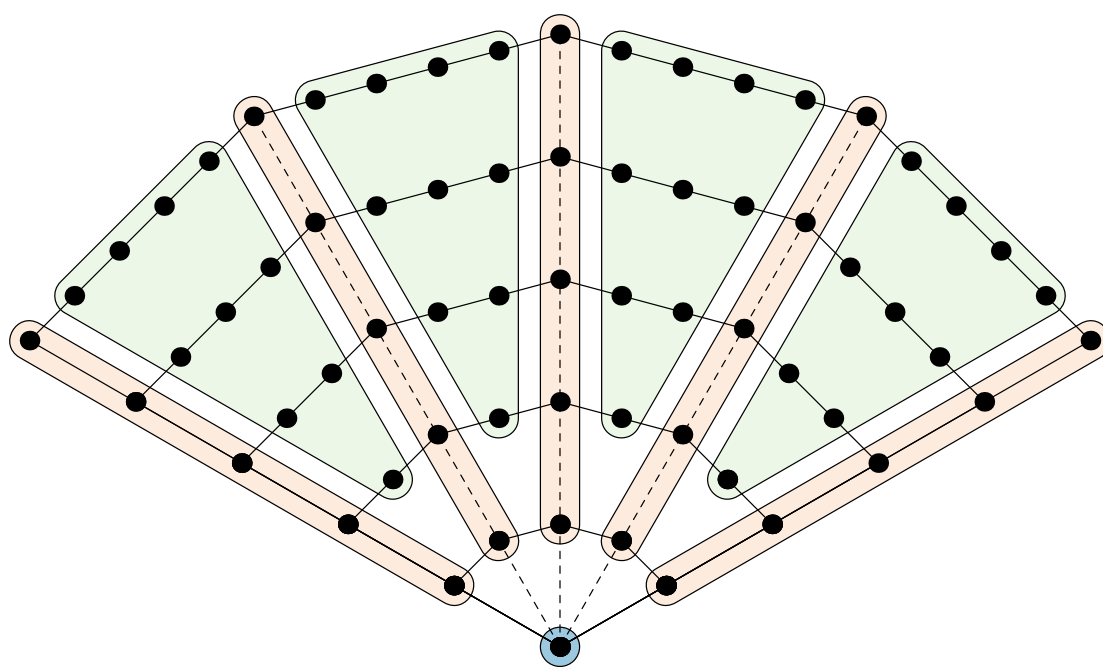


— Diophantus of Alexandria



## Polygonal numbers

The  $k^{\text{th}}$   $n$ -gonal number is  $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$



— Dave Logothetti

## The volume of a frustum of a square pyramid



$$V_{P_1} = \frac{h}{b-a} \cdot V_{P_2} = \frac{h}{b-a} \cdot \frac{1}{3} (b^3 - a^3) = \frac{h}{3} (a^2 + ab + b^2)$$

— *The Moscow Papyrus*, c. 1850 BCE

# The volume of a hemisphere via Cavalieri's Principle



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

— Sidney H. Kung

# Trigonometry, Calculus, & Analytic Geometry

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## Sine of the sum

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \text{ for } \alpha + \beta < \pi$$



$$c = a \cos \beta + b \cos \alpha$$

$$r = 1/2 \implies \sin \gamma = \frac{c/2}{1/2} = c, \sin \alpha = a, \sin \beta = b$$

$$\sin(\alpha + \beta) = \sin(\pi - (\alpha + \beta)) = \sin \gamma = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

— Sidney H. Kung

## Area and difference formulas



$$\frac{1}{2} \cdot a \cdot b \sin(\alpha - \beta) = \frac{1}{2} \cdot a \sin \alpha \cdot b \cos \beta - \frac{1}{2} \cdot a \cos \alpha \cdot b \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\frac{1}{2} \cdot a \cdot b \cos(\alpha - \beta) = \frac{1}{2} \cdot a \cos \alpha \cdot b \cos \beta + \frac{1}{2} \cdot a \sin \alpha \cdot b \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

— Sidney H. Kung

## The law of cosines I



$$\begin{aligned}
 c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\
 &= b^2 \sin^2 \theta + a^2 - 2ab \cos \theta + b^2 \cos^2 \theta \\
 &= a^2 + b^2 (\sin^2 \theta + \cos^2 \theta) - 2ab \cos \theta \\
 &= a^2 + b^2 - 2ab \cos \theta
 \end{aligned}$$

— Timothy A. Sipka

## The law of cosines II



$$(2a \cos \theta - b) \cdot b = (a - c) \cdot (a + c)$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

— Sidney H. Kung



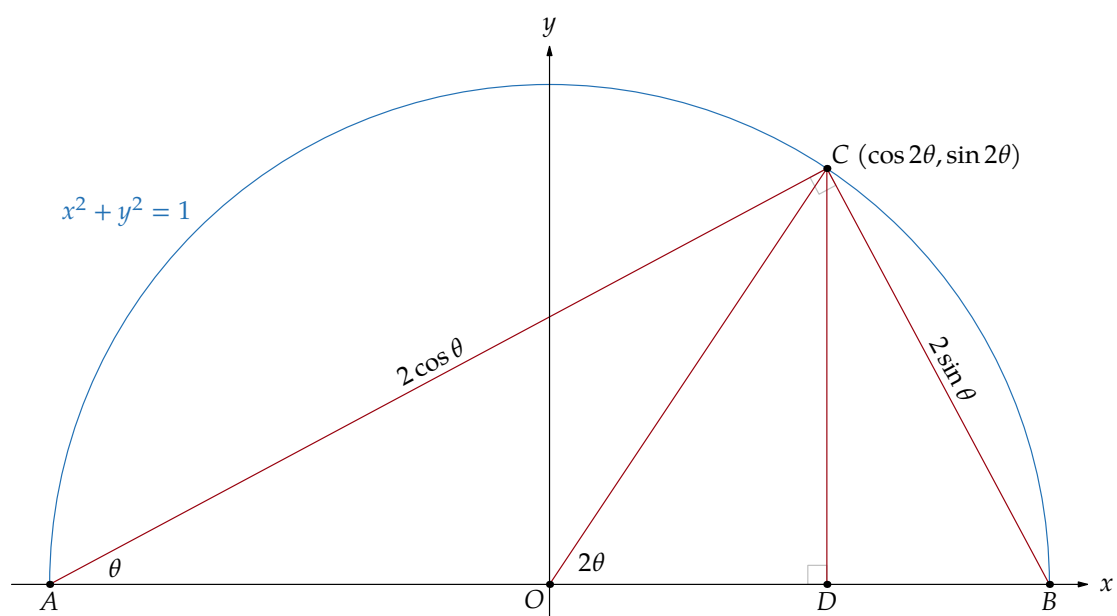
### The law of cosines III (via Ptolemy's theorem)



$$\begin{aligned}c \cdot c &= b \cdot b + (a + 2b \cos(\pi - \theta)) \cdot a \\c^2 &= a^2 + b^2 - 2ab \cos \theta\end{aligned}$$

— Sidney H. Kung

## The double-angle formulae



$$\triangle ACD \sim \triangle ABC$$

$$\begin{aligned} CD/AC &= BC/AB \\ \sin 2\theta / 2 \cos \theta &= 2 \sin \theta / 2 \\ \sin 2\theta &= 2 \sin \theta \cos \theta \end{aligned}$$

$$\begin{aligned} AD/AC &= AC/AB \\ (1 + \cos 2\theta) / 2 \cos \theta &= 2 \cos \theta / 2 \\ \cos 2\theta &= 2 \cos^2 \theta - 1 \end{aligned}$$

— Roger B. Nelsen

## The half-angle tangent formulae



$$\tan \theta/2 = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

— R. J. Walker

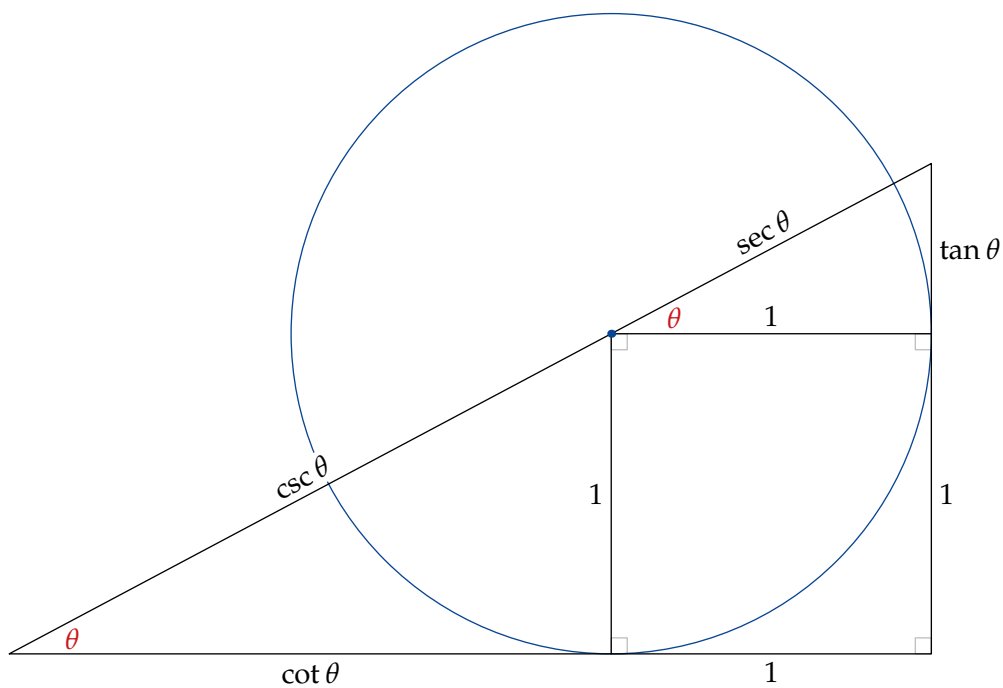
## Mollweide's equation

$$(a - b) \cos \frac{\gamma}{2} = c \sin \left( \frac{\alpha - \beta}{2} \right)$$



— H. Arthur DeKleine

## Tangent, cotangent, secant, and cosecant



$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$(\tan \theta + 1)^2 + (\cot \theta + 1)^2 = (\sec \theta + \csc \theta)^2$$

$$\text{also } \tan \theta = \frac{\tan \theta + 1}{\cot \theta + 1}$$

— William Romaine

## Substitution to make a rational function of sine and cosine



$$z = \tan(\theta/2) \implies \sin \theta = \frac{2z}{1+z^2} \quad \text{and} \quad \cos \theta = \frac{1-z^2}{1+z^2}$$

— Roger B. Nelsen

## Sums of arctangents



$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

— Edward M. Harris

## The distance between a point and a line



$$\frac{d}{1} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

— R. L. Eisenman



# **The midpoint rule is better than the trapezoidal rule for concave functions**



— Frank Burk

## Integration by parts



$$\text{Area } \text{Area 1} + \text{Area } \text{Area 2} = qs - pr$$

$$\int_r^s u \, dv + \int_p^q v \, du = uv \Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) \, dx$$

— Richard Courant

**The graphs of  $f$  and  $f^{-1}$  are reflections about the line  $y = x$**



— Ayoub B. Ayoub



## Area under an arch of the cycloid



$$\frac{1}{2}\pi r \cdot 2r \quad + \quad \pi r^2 \quad + \quad \frac{1}{2}\pi r \cdot 2r$$

therefore  $A = 3\pi r^2$

— Richard M. Beekman

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## The arithmetic mean – geometric mean inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

— Charles D. Gallant

## The arithmetic mean – geometric mean inequality II



$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

— Doris Schattschneider



### The arithmetic mean – geometric mean inequality III

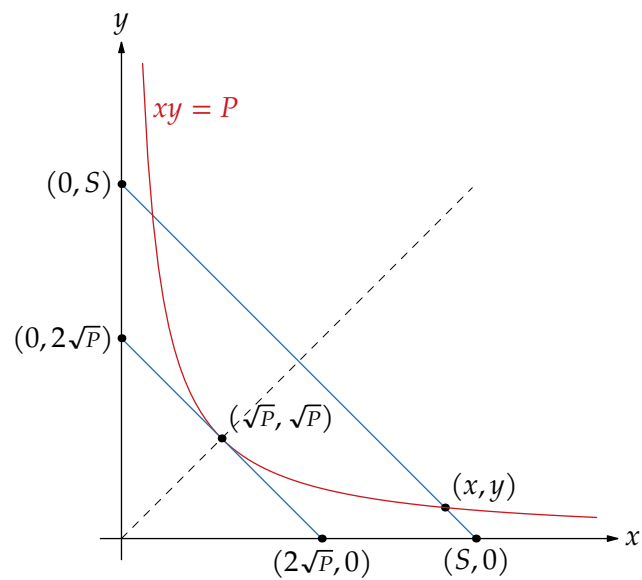
$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \text{with equality iff } a = b$$



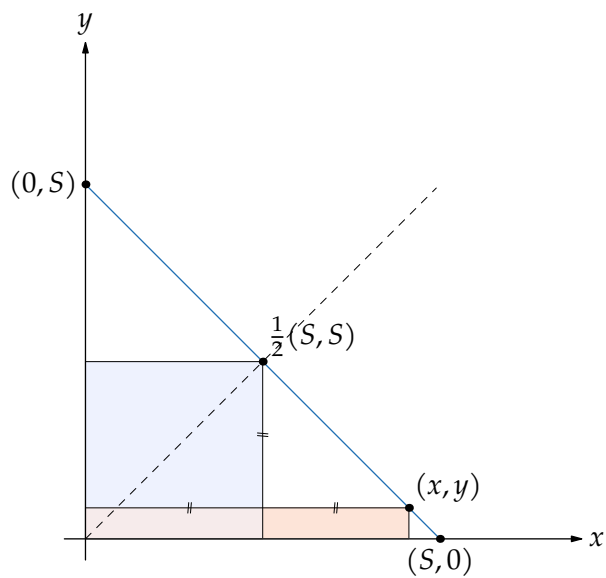
— Roland H. Eddy

## Two extremum problems

For a given product, the sum of two positive numbers is minimal when the numbers are equal.



For a given sum, the product of two positive numbers is maximal when the numbers are equal.



— Paulo Montuchi and Warren Page

# The HM–GM–AM–QM inequalities I



$$PM = a, \quad RM = b, \quad a > b > 0$$

$$HM < GM < AM < QM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{1}{2}(a^2 + b^2)}$$

— Roger B. Nelsen



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# The HM–GM–AM–QM inequalities III



$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\frac{1}{2}(a^2 + b^2)} \geq \frac{a+b}{2}$$



$$(\sqrt{a} + \sqrt{b})^2 \geq 4 \cdot \frac{1}{2} \sqrt{a} \sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$



$$1 \geq 4 \cdot \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

— Roger B. Nelsen

# Five means — and their means



— Roger B. Nelsen

$$e^\pi > \pi^e$$



— Fouad Nakhli

$A^B > B^A$  **for**  $e \leq A < B$



$$\begin{aligned}
 e \leq A < B &\implies m_A > m_B \\
 &\implies \frac{\ln A}{A} > \frac{\ln B}{B} \\
 &\implies A^B > B^A
 \end{aligned}$$

— Charles D. Gallant



## The mediant property

$$\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$



— Richard A. Gibbs

## Regle des nombres moyens – two proofs

$$a, b, c, d > 0; \quad \frac{a}{b} < \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

I.



$$m_1 < m_3 \quad \Rightarrow \quad m_1 < m_2 < m_3$$

— Li Changming

II.



$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

— Roger B. Nelsen

# The sum of a positive number and its reciprocal is at least two

I.



II.



III.



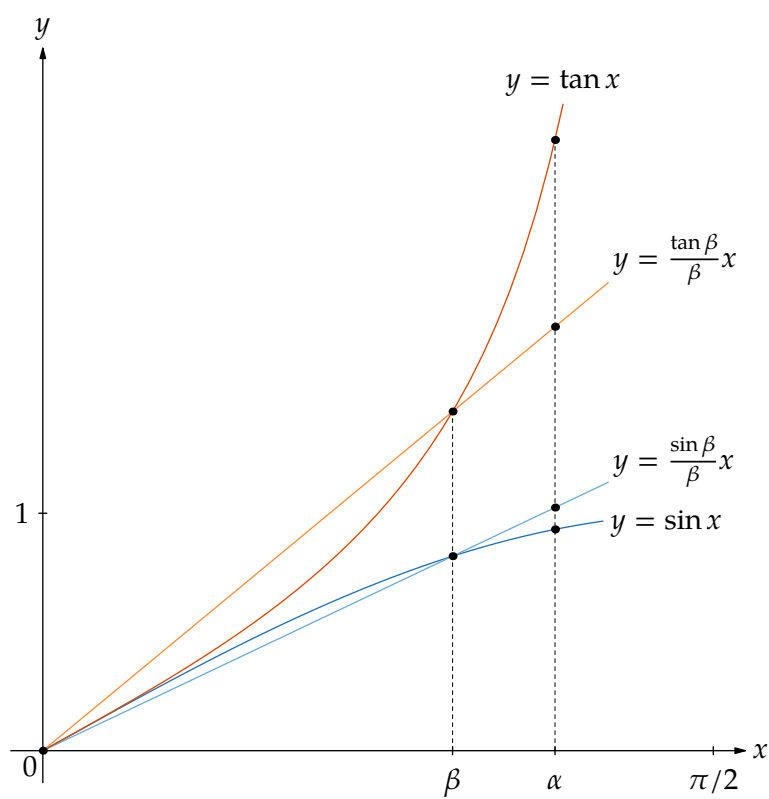
IV.



— Roger B. Nelsen

# Aristarchus' inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \implies \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$



$$\sin \alpha < \frac{\sin \beta}{\beta} \alpha; \quad \tan \alpha > \frac{\tan \beta}{\beta} \alpha$$

$$\therefore \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

— Roger B. Nelsen

## The Cauchy-Schwartz inequality

$$|\langle a, b \rangle \cdot \langle x, y \rangle| \leq \|\langle a, b \rangle\| \|\langle x, y \rangle\|$$



$$(|a| + |y|) (|b| + |x|) \leq 2 \left( \frac{1}{2} |a| |b| + \frac{1}{2} |x| |y| \right) + \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

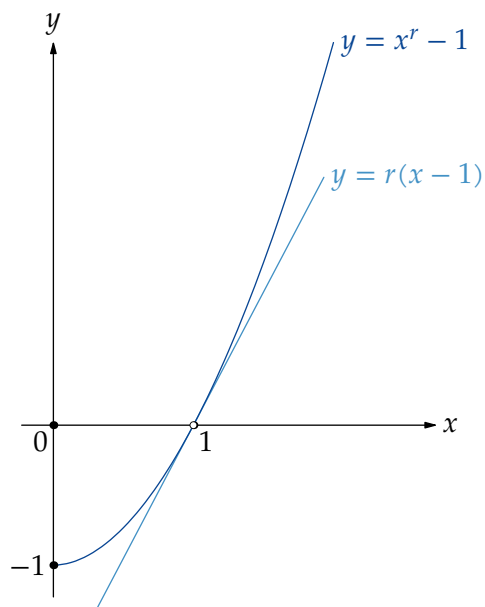
$$\therefore |ax + by| \leq |a||x| + |b||y| \leq \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

— Roger B. Nelsen

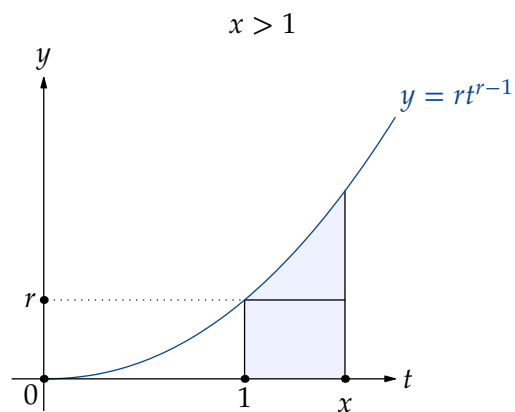
## Bernoulli's inequality

$$x > 0, x \neq 1, r > 1: x^r - 1 > r(x - 1)$$

### I. First semester calculus



### II. Second semester calculus



$$x^r - 1 = \int_1^x rt^{r-1} dt > r(x - 1)$$



$$1 - x^r = \int_x^1 rt^{r-1} dt < r(1 - x)$$

— Roger B. Nelsen

## Napier's inequality

$$b > a > 0 \quad \text{implies} \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

I. First semester calculus



$$m(L_1) < m(L_2) < m(L_3)$$

II. Second semester calculus



$$\frac{1}{b}(b - a) < \int_a^b \frac{1}{x} dx < \frac{1}{a}(b - a)$$

— Roger B. Nelsen

# Integer sums

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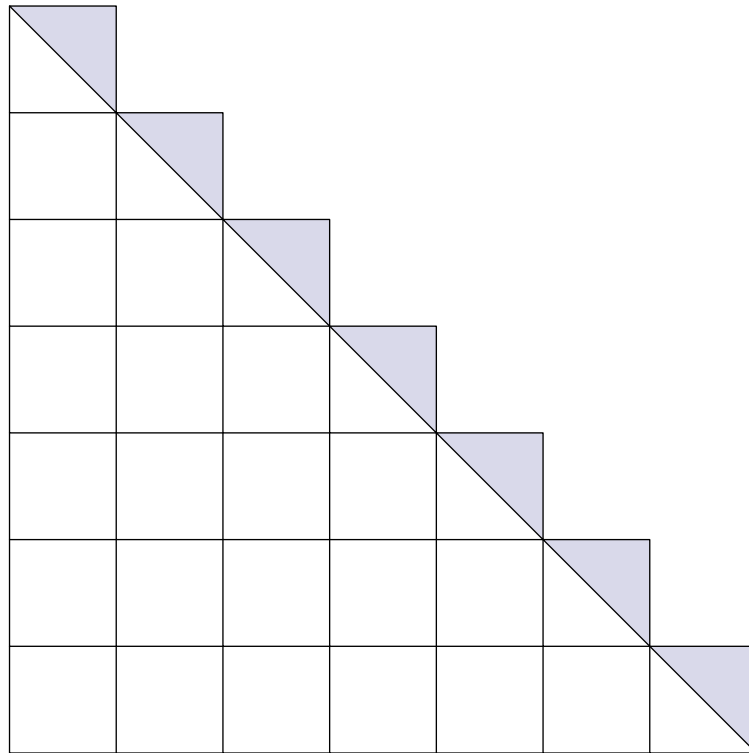
## Sums of integers I



$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$$

— Ancient Greek

## Sums of integers II



$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

— Ian Richards

# Sums of odd integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

— Nichomachus of Gerasa

## Sums of odd integers II



$$1 + 3 + \cdots + (2n - 1) = \frac{1}{4} (2n)^2 = n^2$$

— Roger B. Nelsen

# Sums of odd integers III

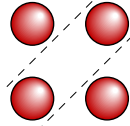


$$\Delta + 3 \cdot \Delta + \dots + (2n - 1) \cdot \Delta = A = n^2 \cdot \Delta$$

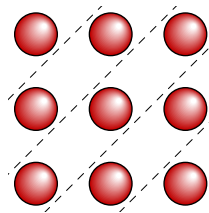
$$\sum_{i=1}^n (2i - 1) = n^2$$

— Jenő Lehel

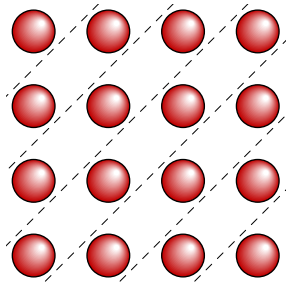
## Squares and sums of integers I



$$1 + 2 + 1 = 2^2$$



$$1 + 2 + 3 + 2 + 1 = 3^2$$

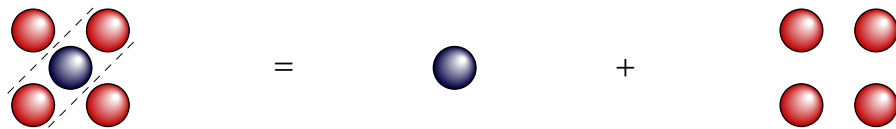


$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

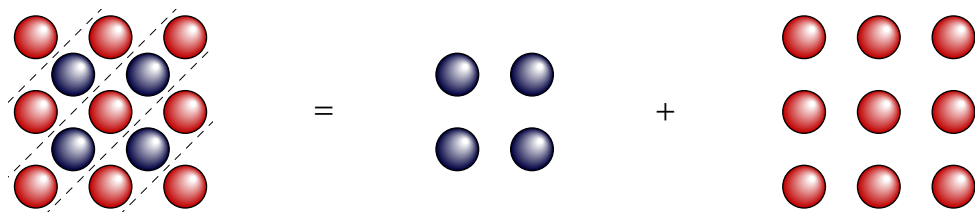
$$1 + 2 + \cdots + (n - 1) + n + (n - 1) + \cdots + 2 + 1 = n^2$$

— Ancient Greek

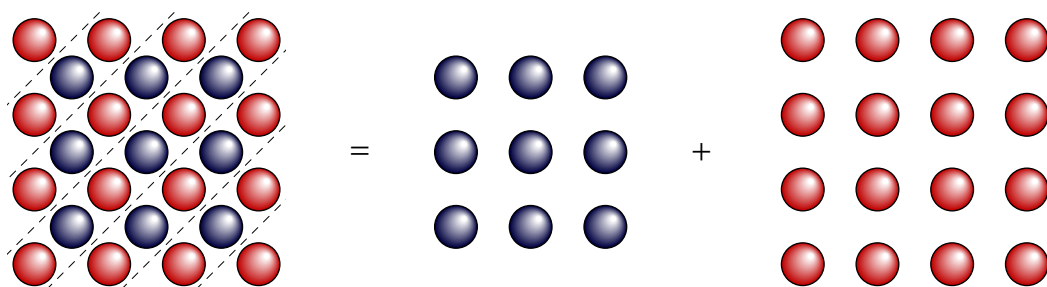
## Squares and sums of integers II



$$1 + 3 + 1 = 1^2 + 2^2$$



$$1 + 3 + 5 + 3 + 1 = 2^2 + 3^2$$



$$1 + 3 + 5 + 7 + 5 + 3 + 1 = 3^2 + 4^2$$

$\vdots$

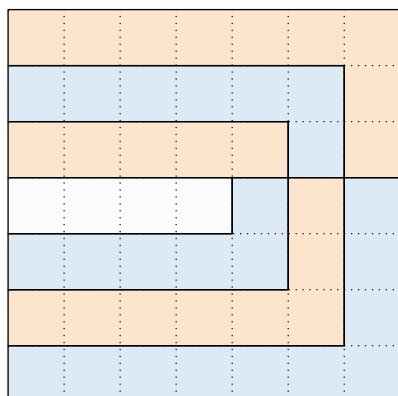
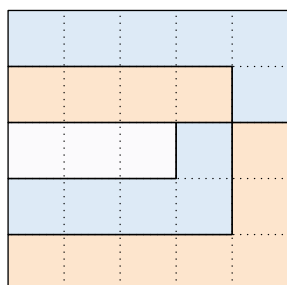
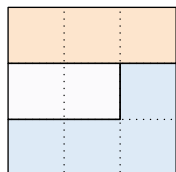
$$1 + 3 + \cdots + (2n - 1) + (2n + 1) + (2n - 1) + \cdots + 3 + 1 = n^2 + (n + 1)^2$$

— Hee Sik Kim

# Arithmetic progressions with sum equal to square of number of terms



$$\sum_{k=n}^{3n-2} k = (2n-1)^2; n = 1, 2, 3, \dots$$



$$n = 4$$

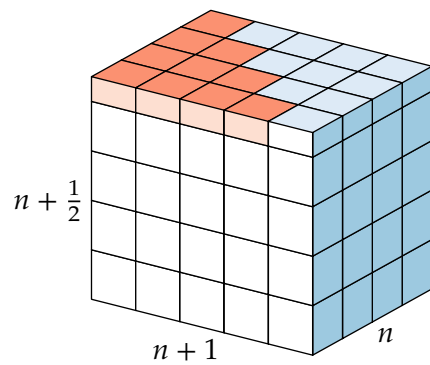
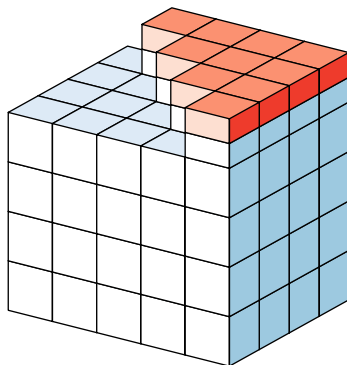
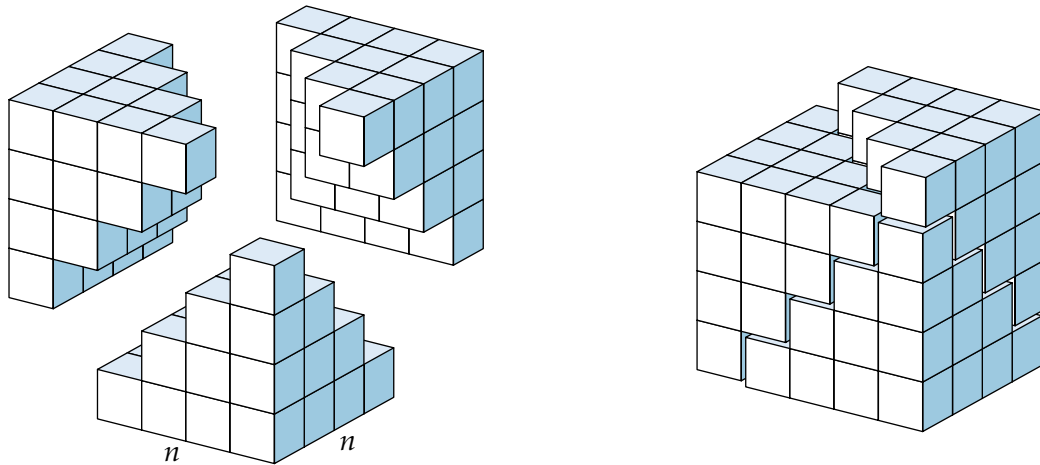
$$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$$

— James O. Chilaka



# Sums of squares I

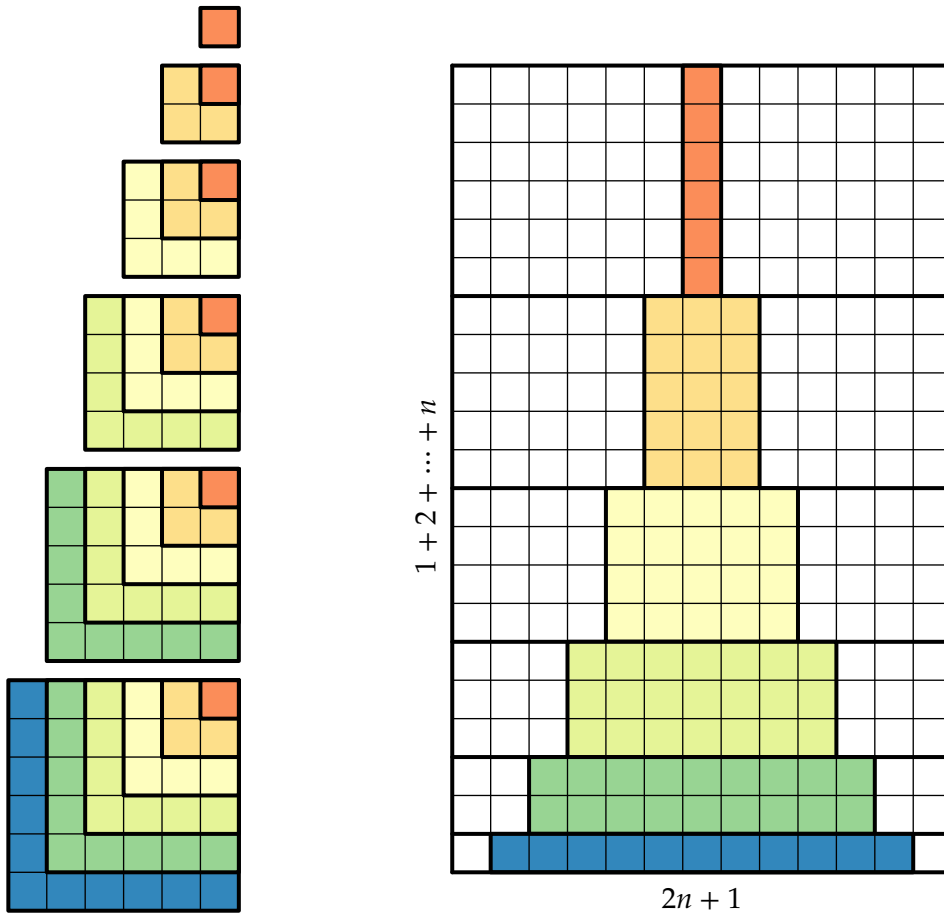
$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)\left(n + \frac{1}{2}\right)$$



— Man-Keung Siu

## Sums of squares II

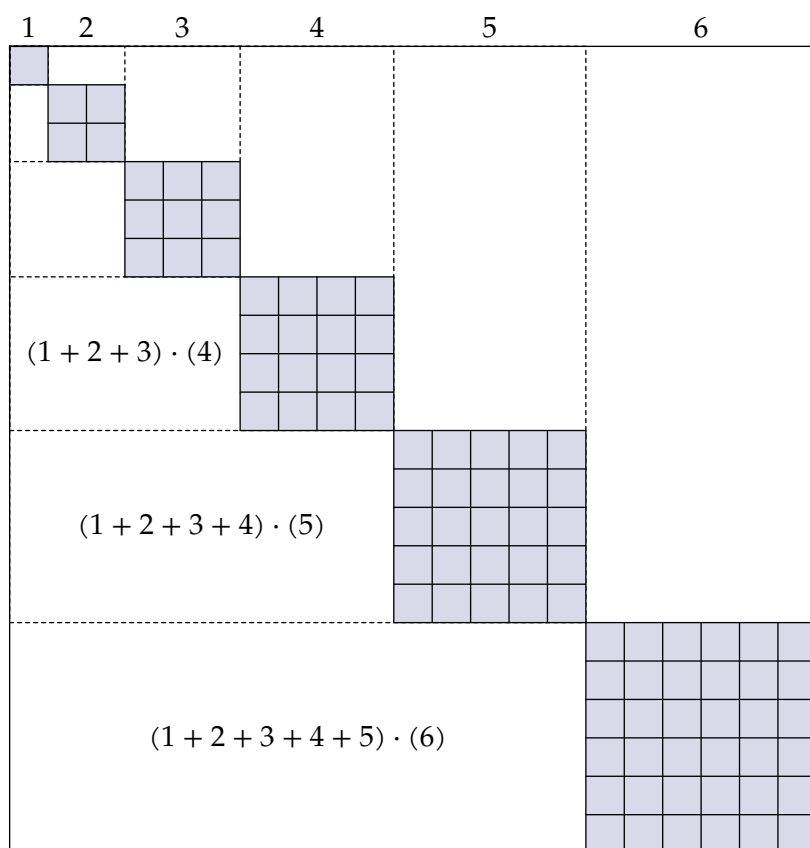
$$3(1^2 + 2^2 + \cdots + n^2) = (2n + 1)(1 + 2 + \cdots + n)$$



— Dan Kalman

## Sums of squares IV

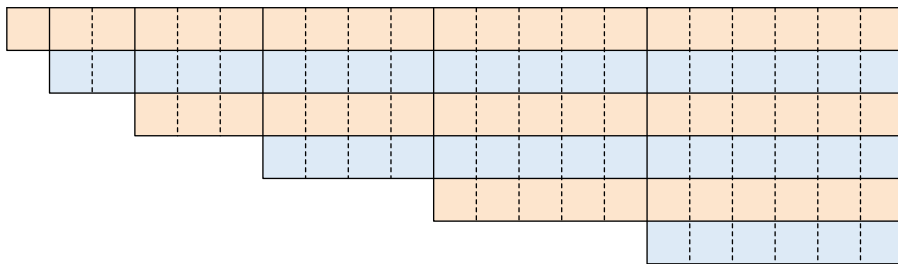
$$\sum_{k=1}^n k^2 = \left( \sum_{k=1}^n k \right)^2 - 2 \sum_{k=1}^{n-1} \left( (k+1) \sum_{i=1}^k i \right)$$



— James O.Chilaka

## Sums of squares V

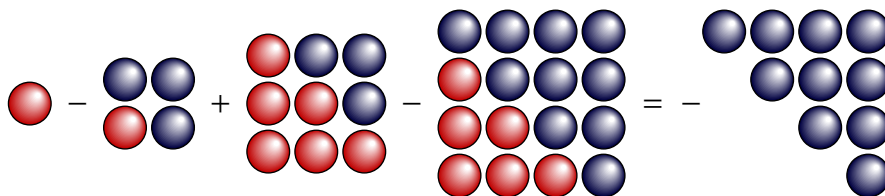
$$\sum_{i=1}^n \sum_{j=i}^n j = \sum_{i=1}^n i^2$$



— Pi-Chun Chuang

## Alternating sums of squares

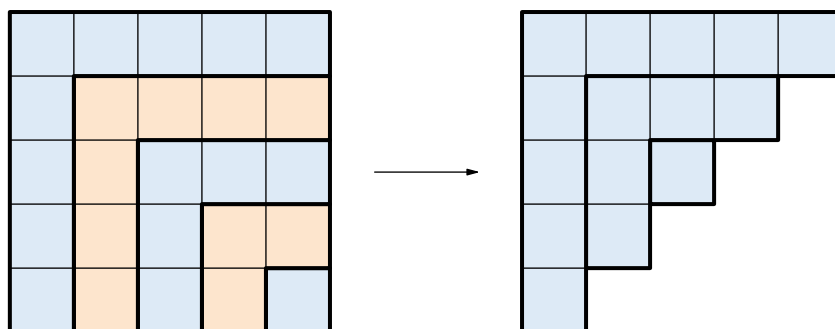
I.



$$\sum_{k=1}^n (-1)^{k+1} k^2 = (-1)^{n+1} T_n = (-1)^{n+1} \frac{n(n+1)}{2}$$

— Dave Logothetti

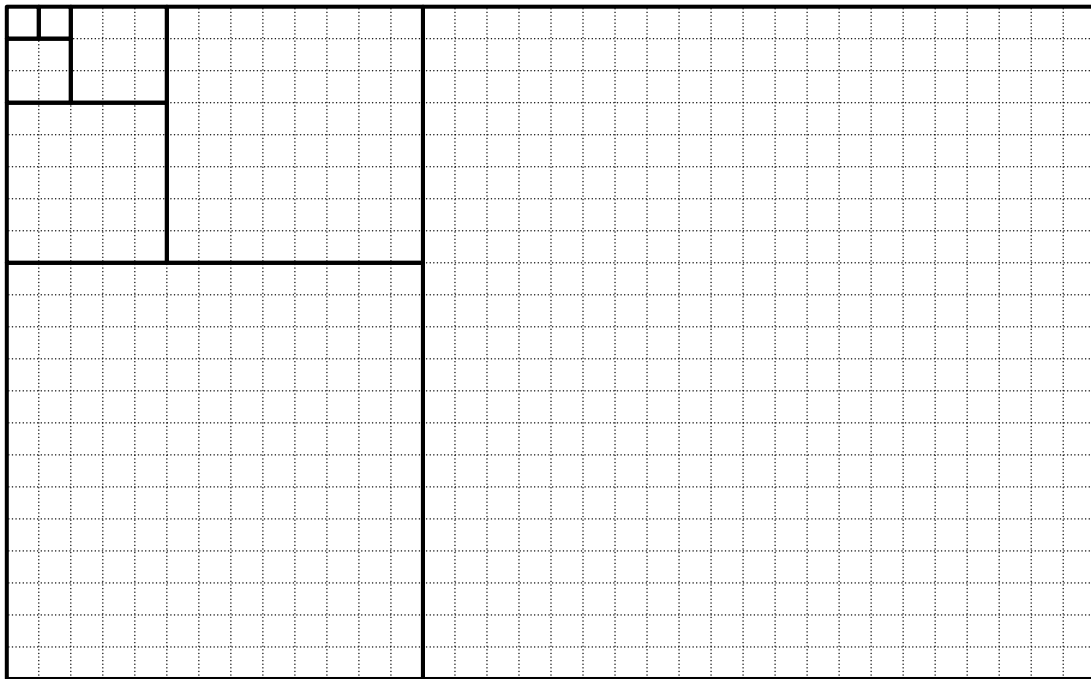
II.



$$n^2 - (n-1)^2 + \cdots + (-1)^{n-1} (1)^2 = \sum_{k=0}^n (-1)^k (n-k)^2 = \frac{n(n+1)}{2}$$

— Steven L. Snover

## Sums of squares of Fibonacci numbers

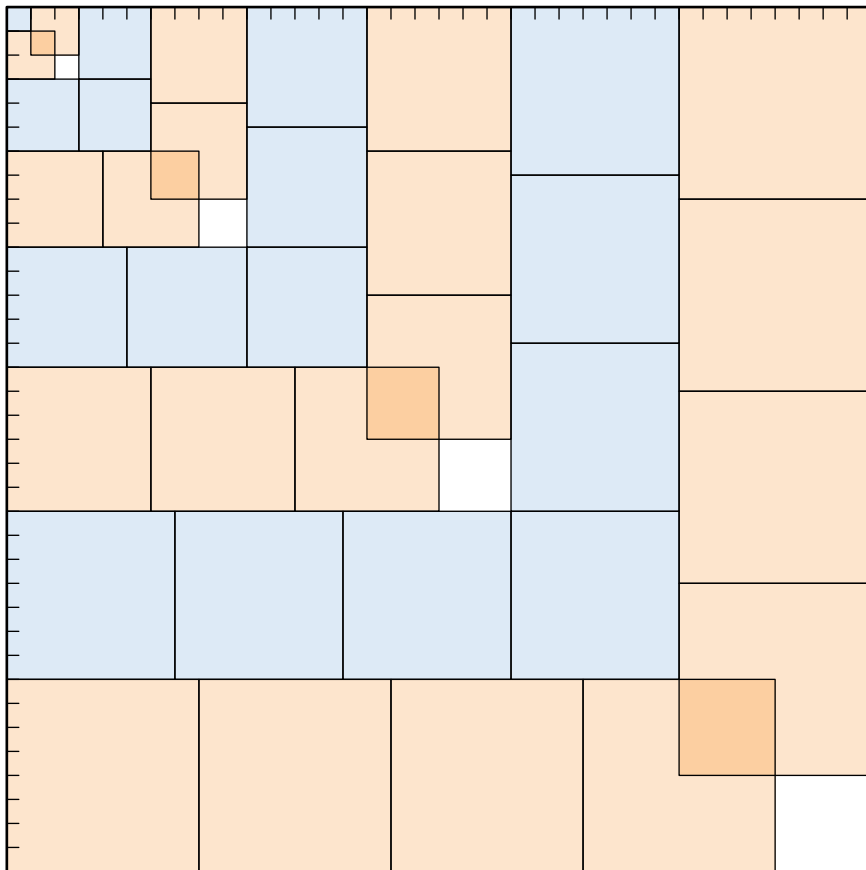


$$F_1 = F_2 = 1; F_{n+2} = F_{n+1} + F_n \quad \text{hence} \quad F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

— Alfred Brousseau

## Sums of cubes I

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



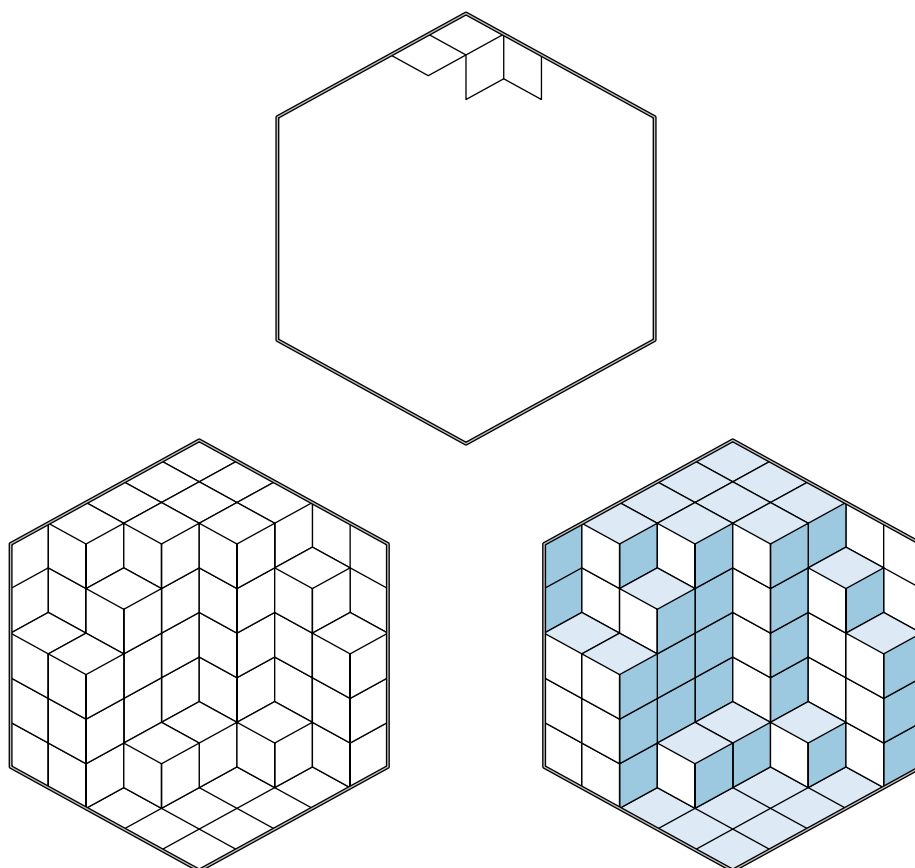
— Solomon W. Golomb

## Miscellaneous

The problem of the calissons . . . . .	81
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## The problem of the calissons



— Guy David and Carlos Tomei