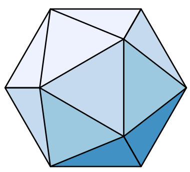
Proofs without words I

Exercises in METAPOST

Toby Thurston

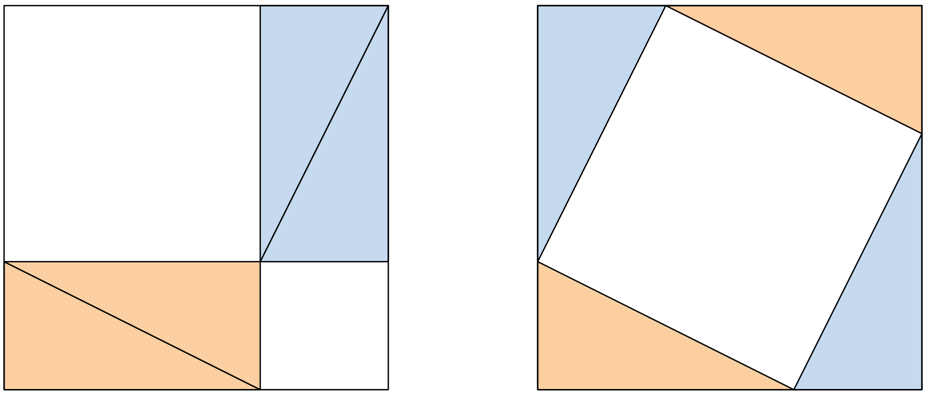
March 2021 —



Contents

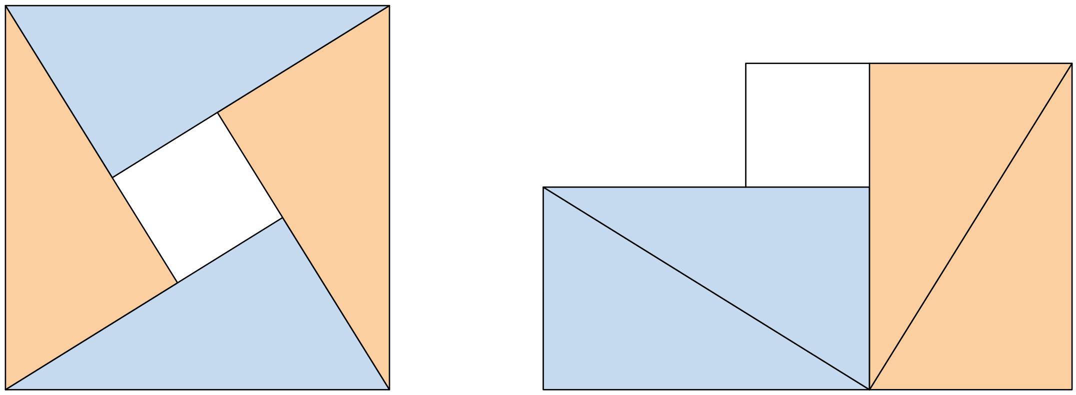
Geometry and Algebra

The Pythagorean theorem I



— adapted from the *Chou pei san ching*

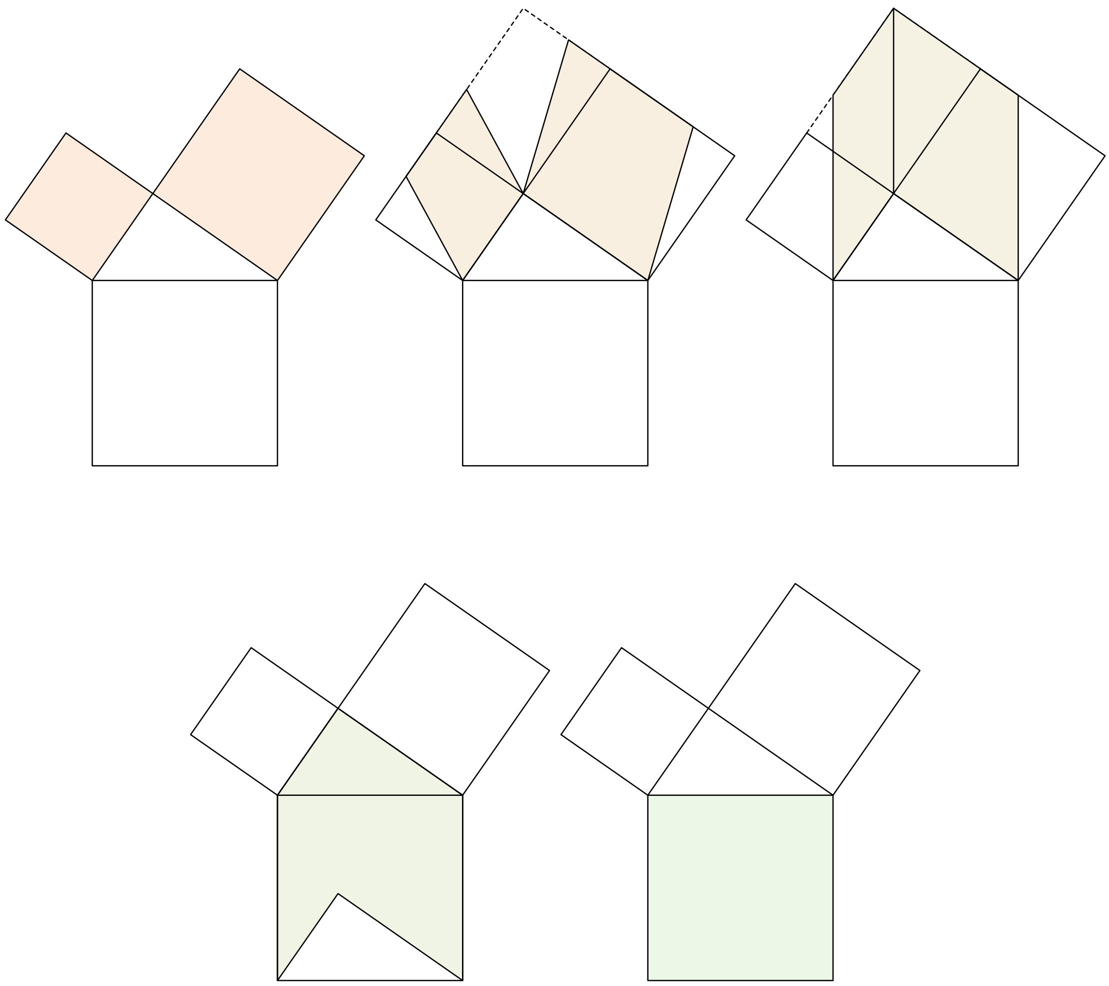
The Pythagorean theorem II



Behold!

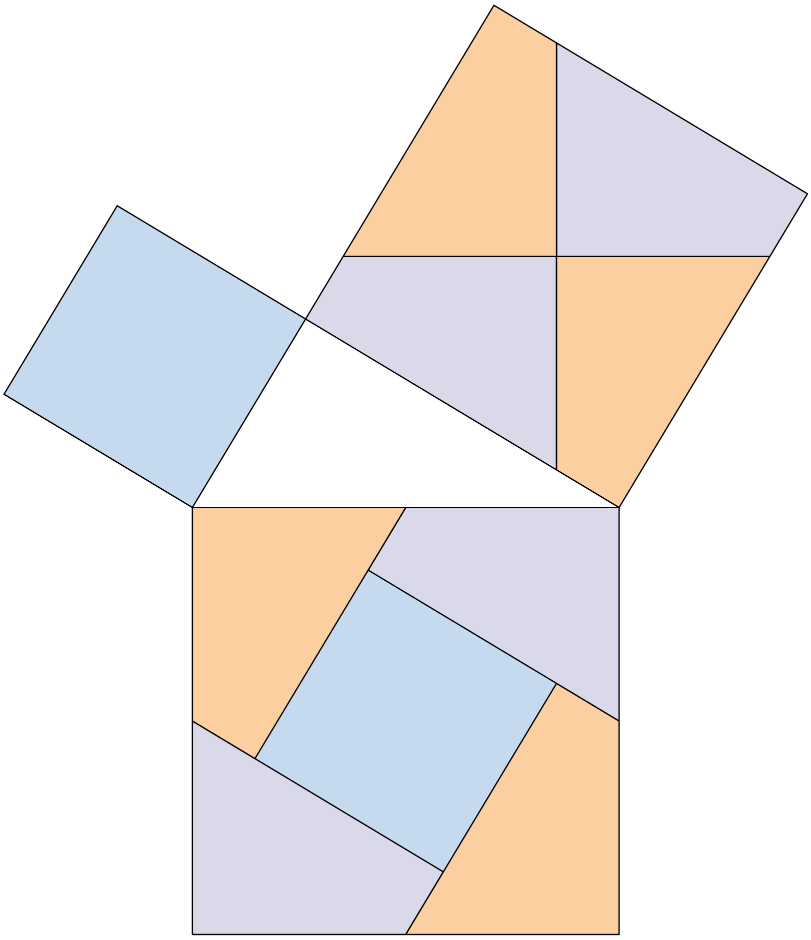
— Bhāskara (12th century)

The Pythagorean theorem III



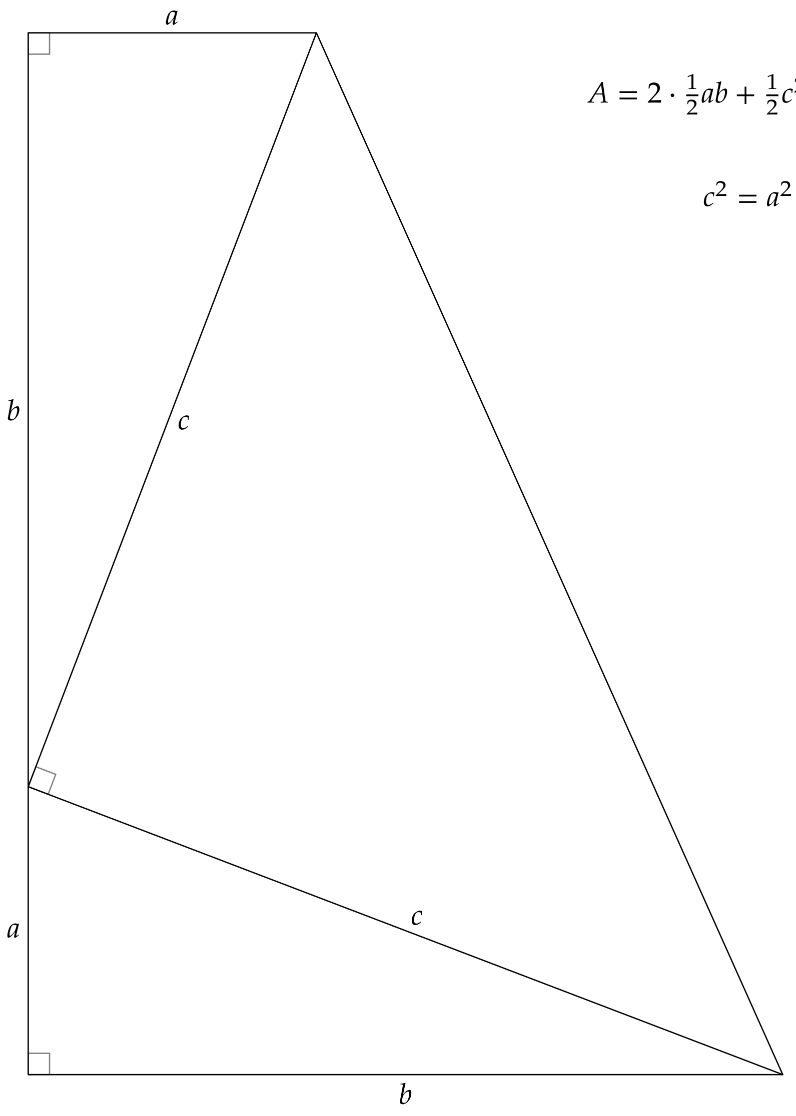
— based on Euclid's proof

The Pythagorean theorem IV



— H. E. Dudeney (1917)

The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}\left(a+b\right)^2$$

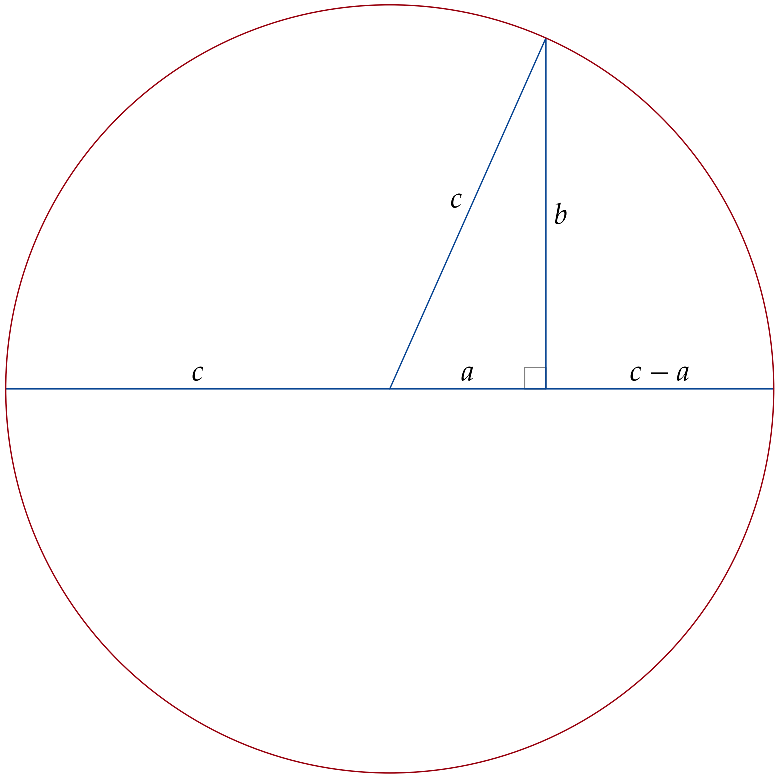
$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)

The Pythagorean theorem VI

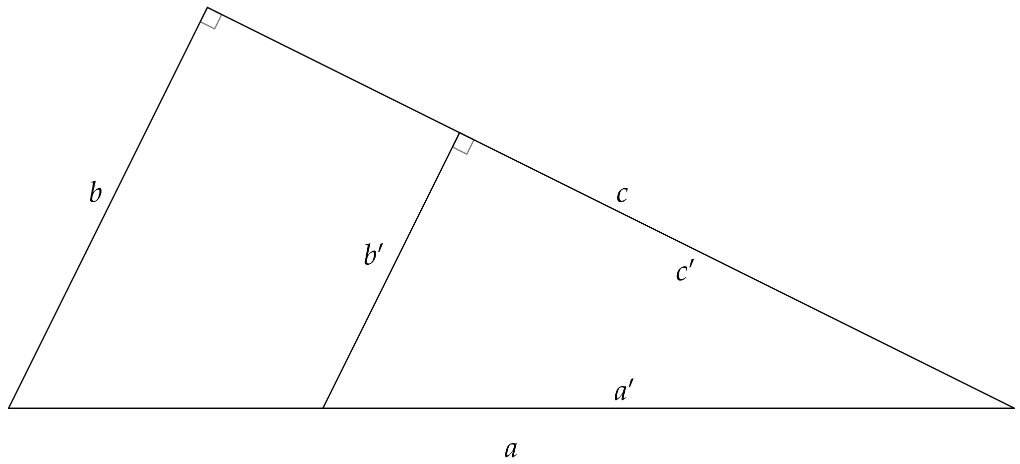
$$\frac{c+a}{b} = \frac{b}{c-a}$$

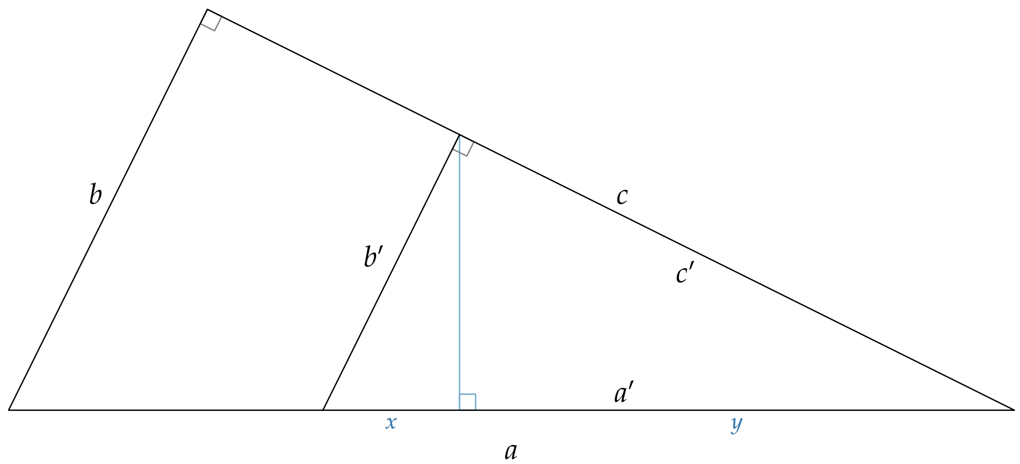
$$a^2 + b^2 = c^2$$



— Michael Hardy

A Pythagorean theorem: $aa' = bb' + cc'$

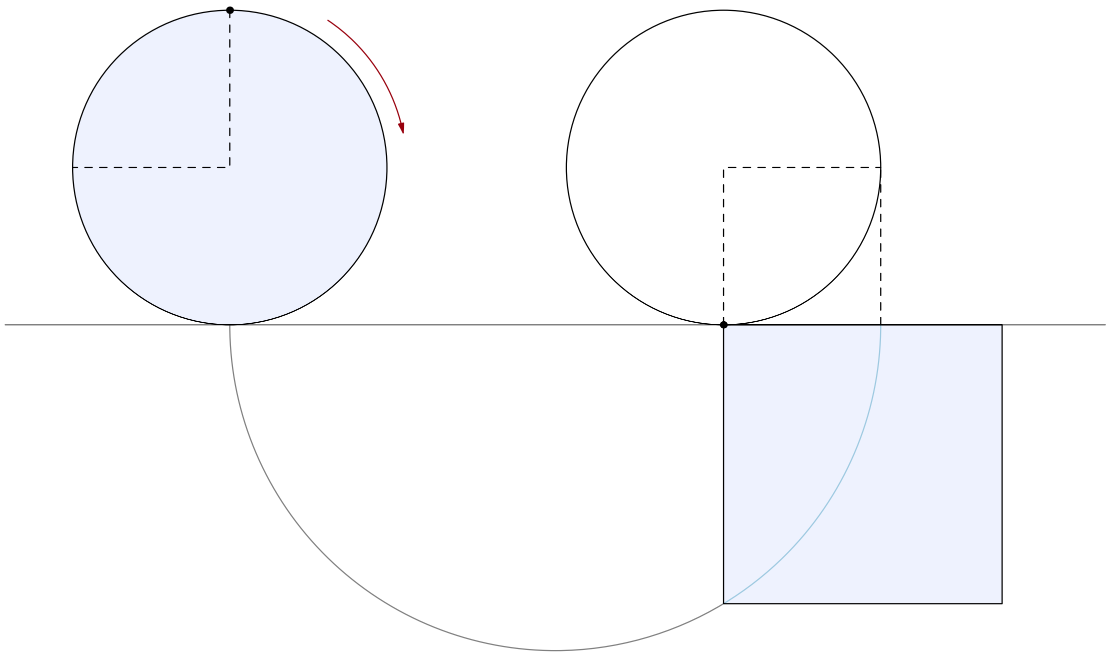




$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$
$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$
$$\therefore \quad aa' = a\left(x + y\right) = bb' + cc'.$$

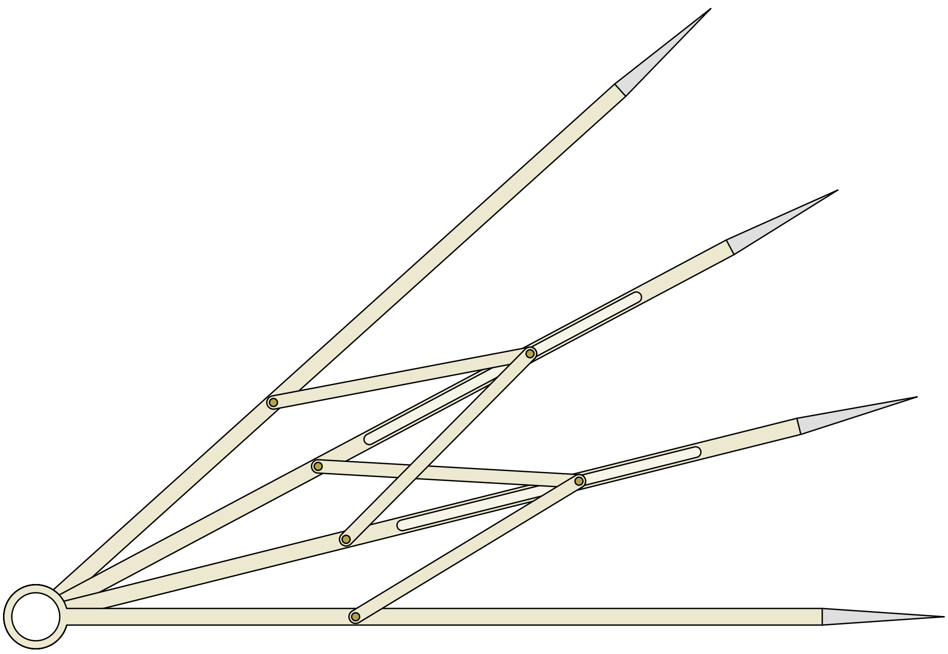
— Enzo R. Gentile

The rolling circle squares itself



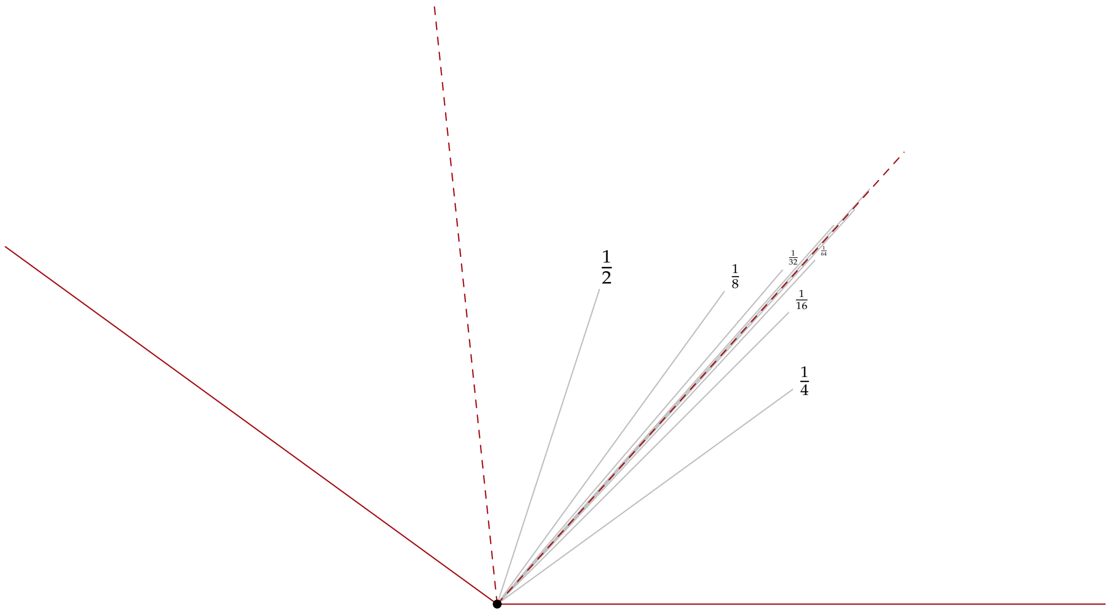
— Thomas Elsner

On trisecting an angle



— Rufus Isaacs

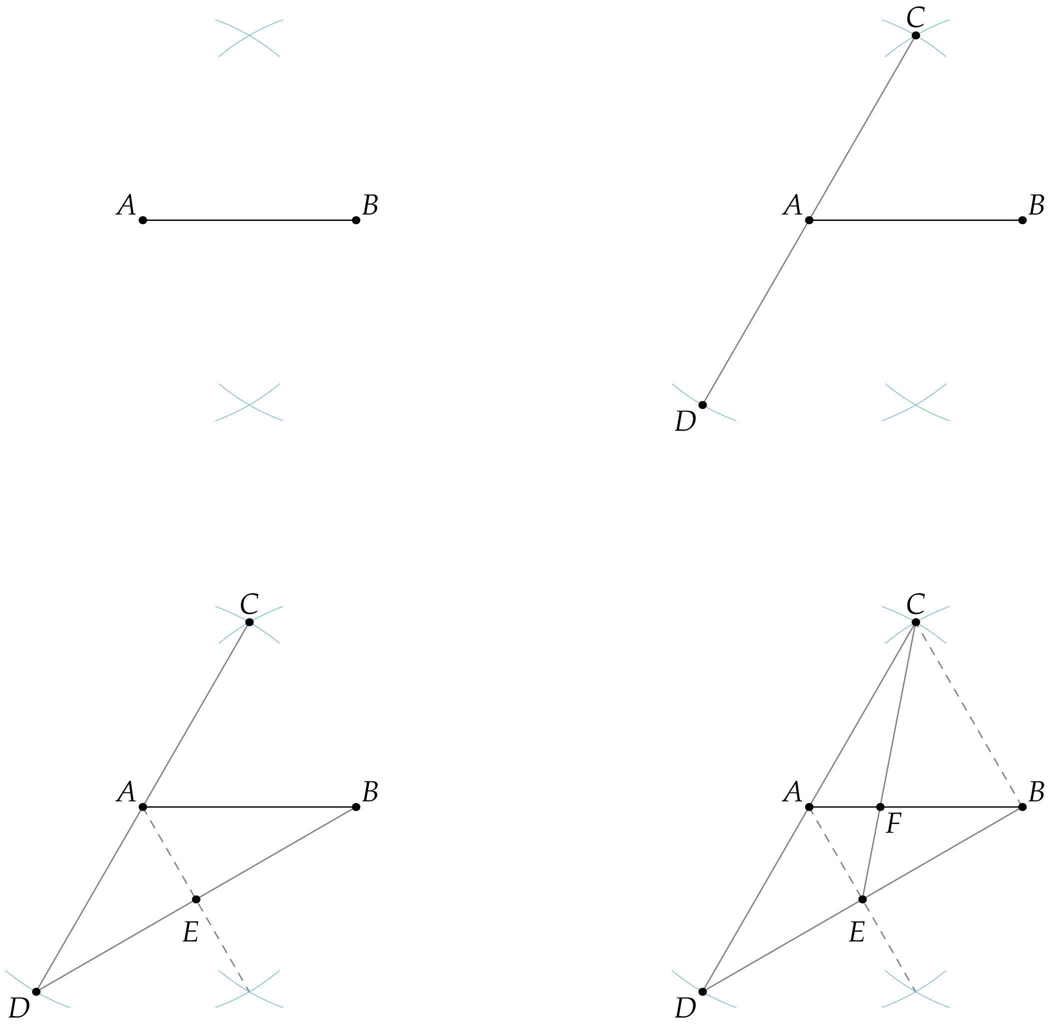
Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

— Eric Kincanon

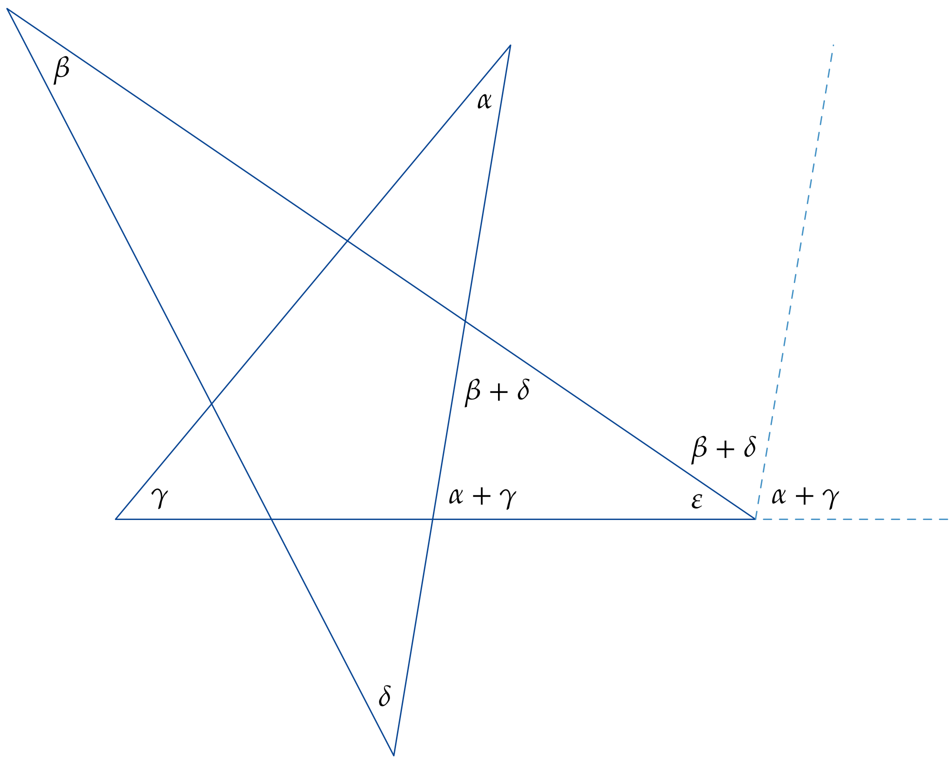
Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

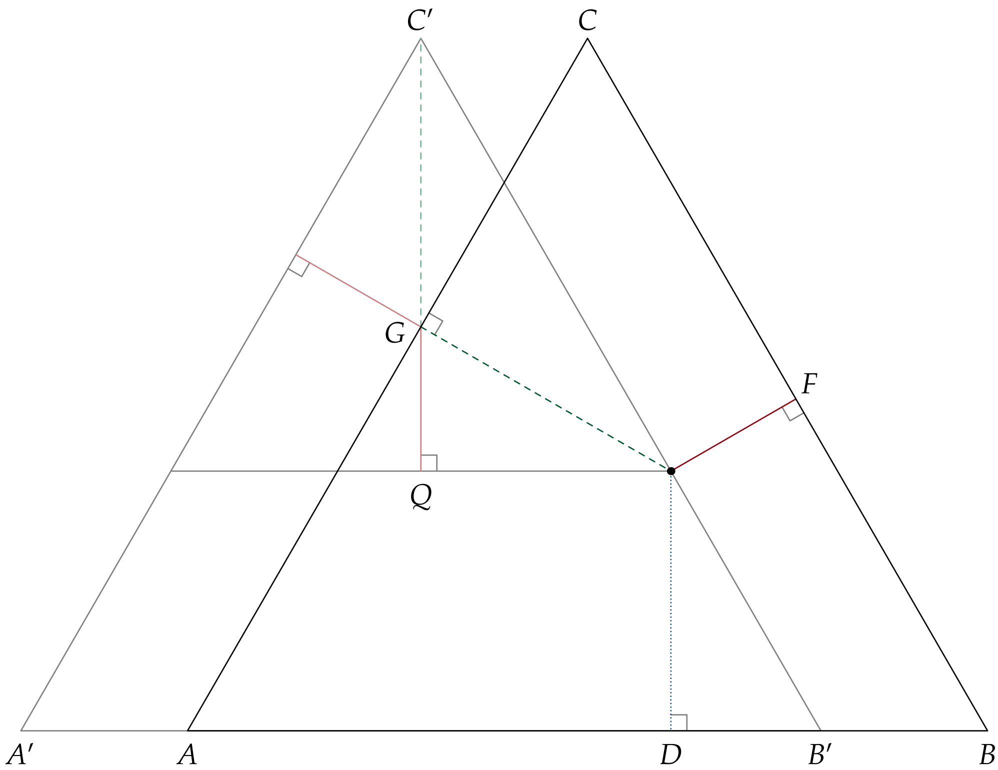
The vertex angles of a star sum to 180°



— Fouad Nakhli

Viviani's theorem I

The perpendiculars to the sides from a point on
the boundary or within an equilateral triangle
add up to the height of the triangle.

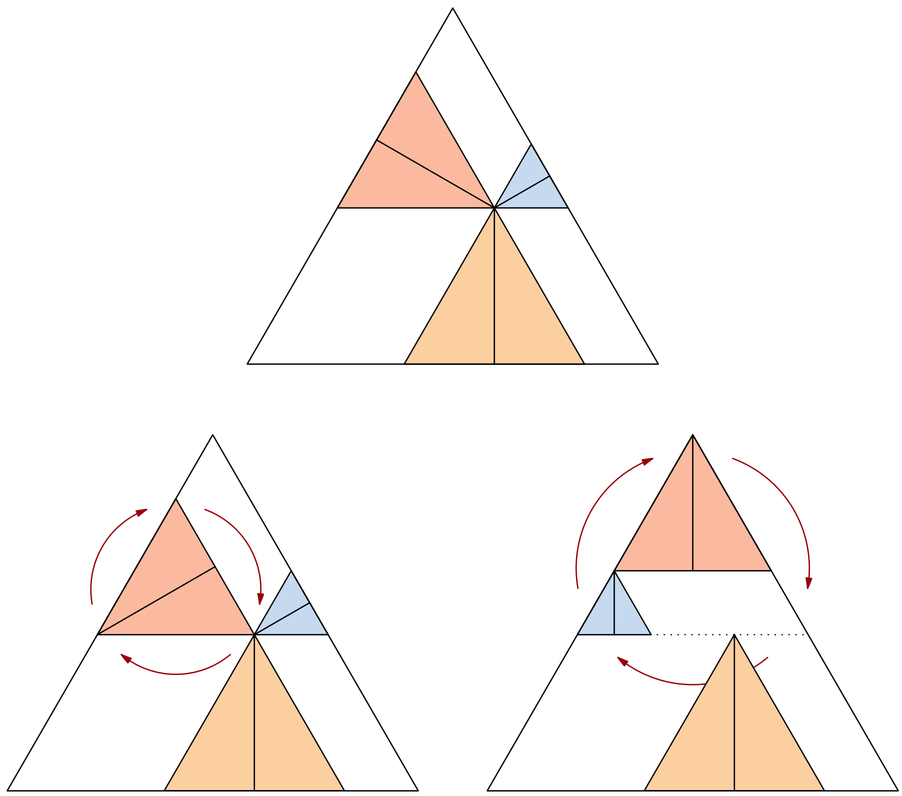


This shows a particular example, with $C'GQ$ collinear, rather than the general case

— Samuel Wolf

Viviani's theorem II

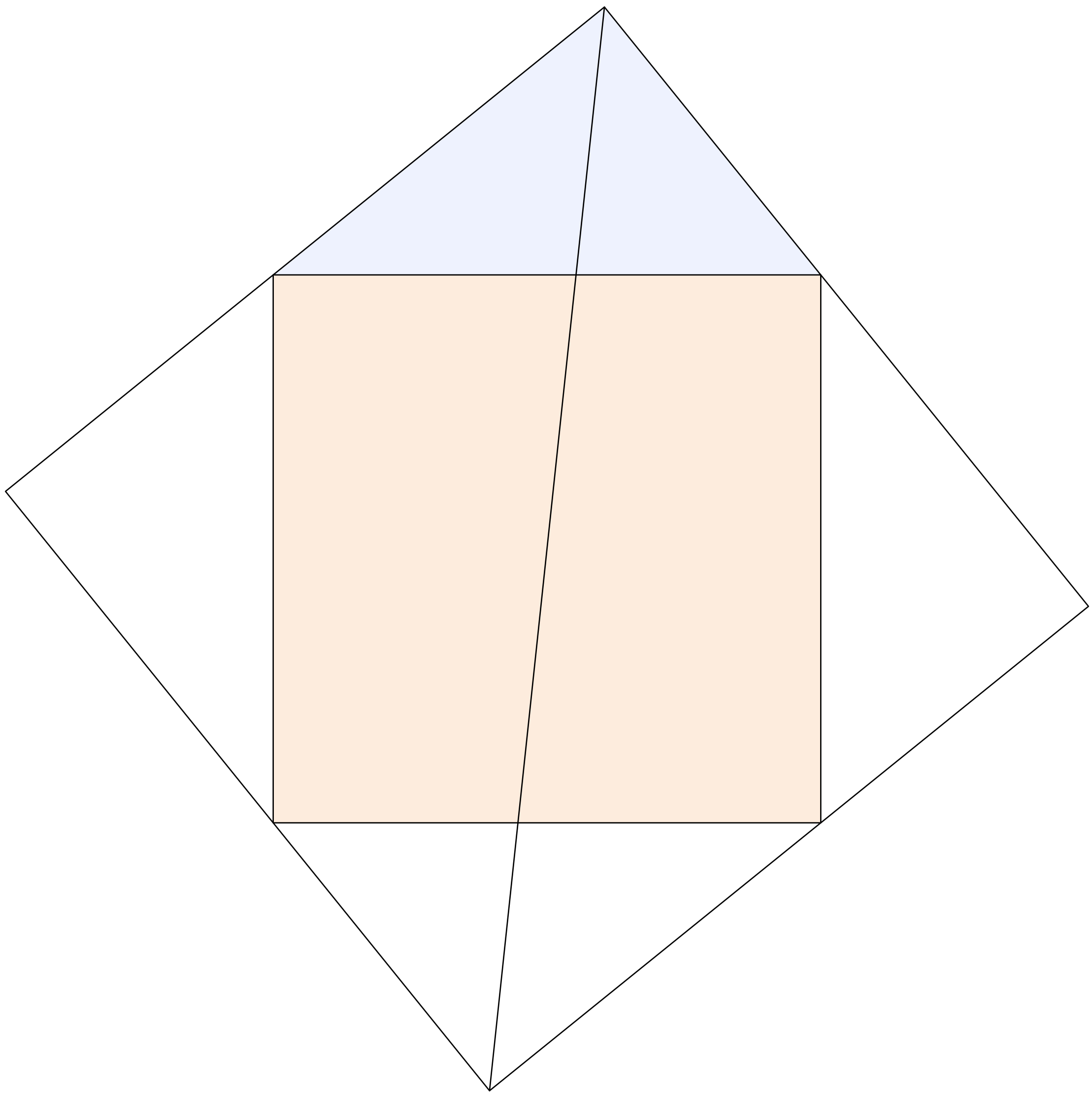
The perpendiculars to the sides from a point on
the boundary or within an equilateral triangle
add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

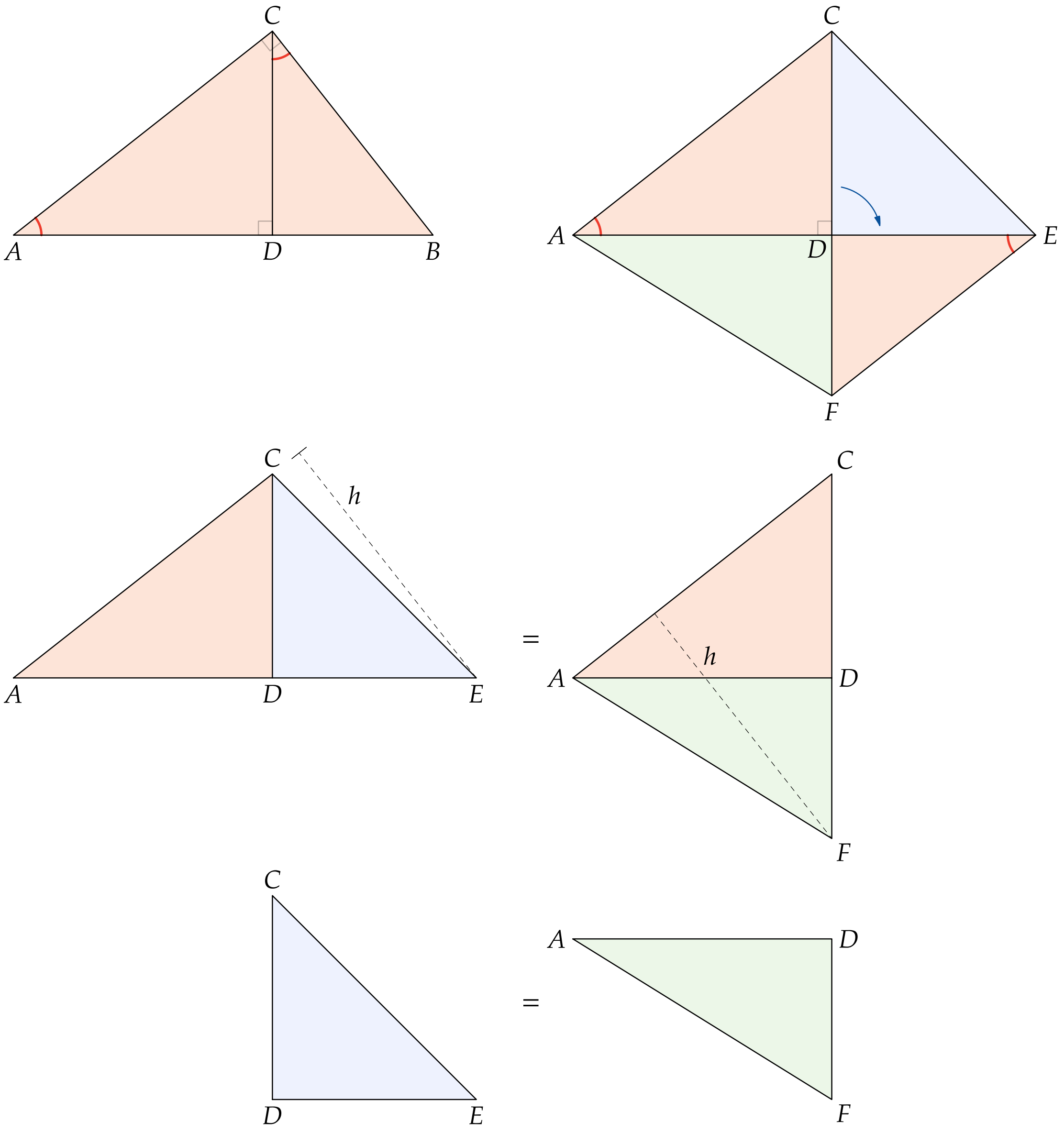
A theorem about right angles

The internal bisector of the right angle of a right
triangle bisects the square on the hypotenuse



— Roland H. Eddy

Area and the projection theorem of a right triangle

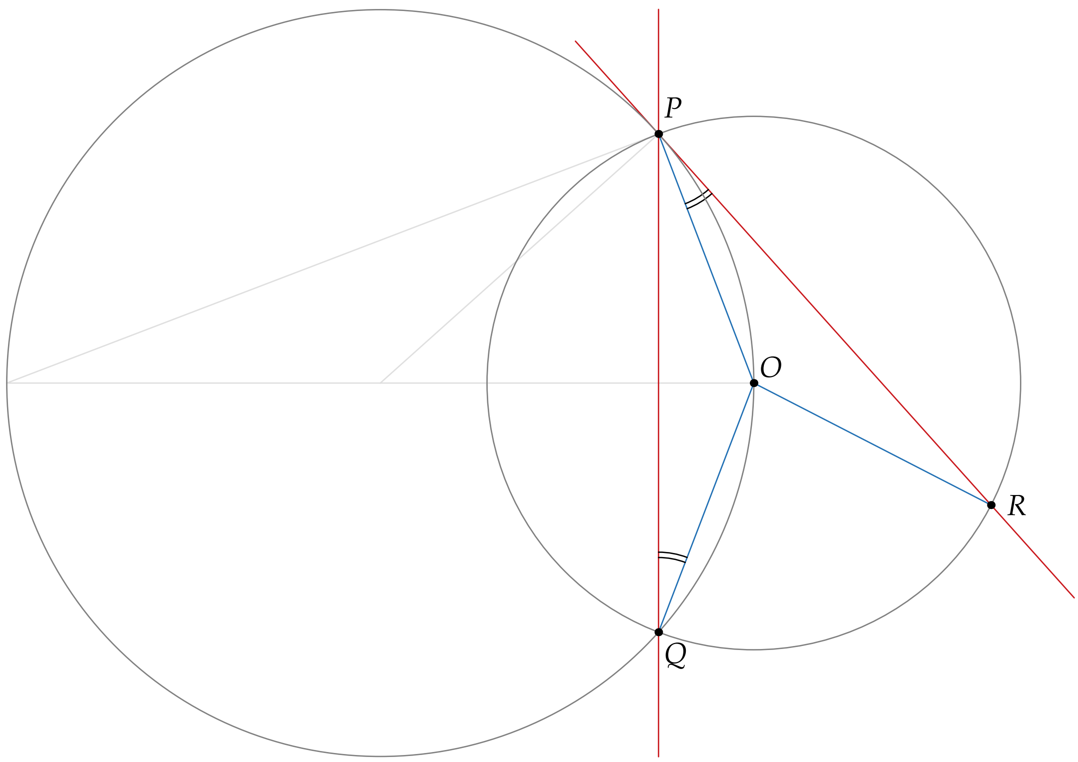


$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

Chords and tangents of equal length

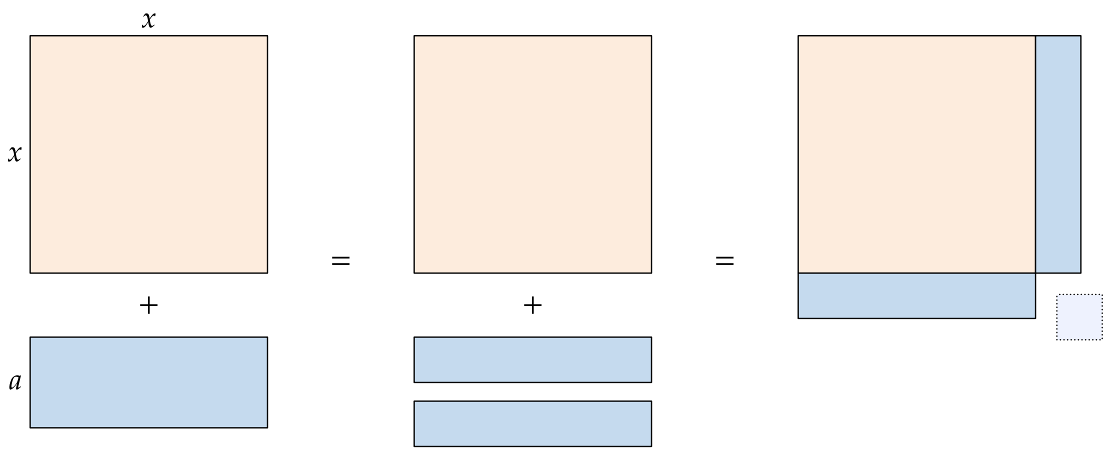
If circle C_1 passes through the center O of circle C_2, the length of the common chord $\overline{PQ}$ is equal to the tangent segment $\overline{PR}$.



— Roland H. Eddy

Completing the square

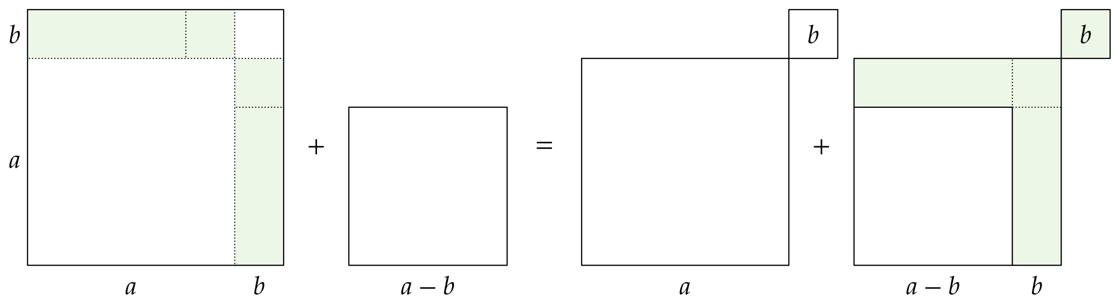
$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

Algebraic areas I

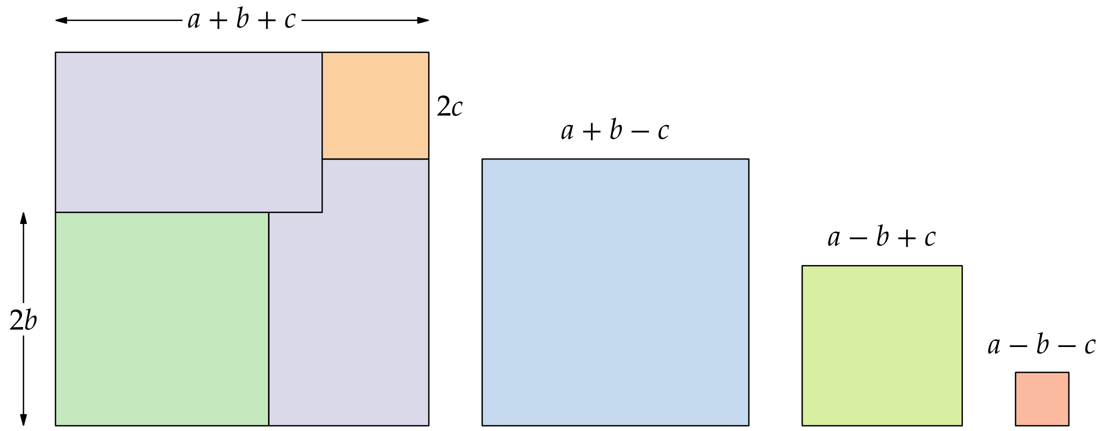
$$(a+b)^2 + (a-b)^2 = 2\left(a^2+b^2\right)$$

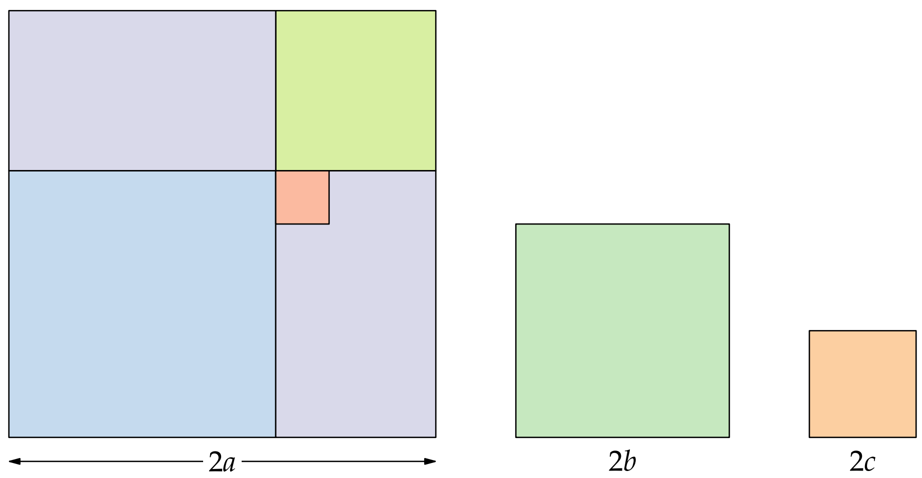


— Shirley Wakin

Algebraic areas II

$$(a+b+c)^2 + (a+b-c)^2 + (a-b+c)^2 + (a-b-c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$

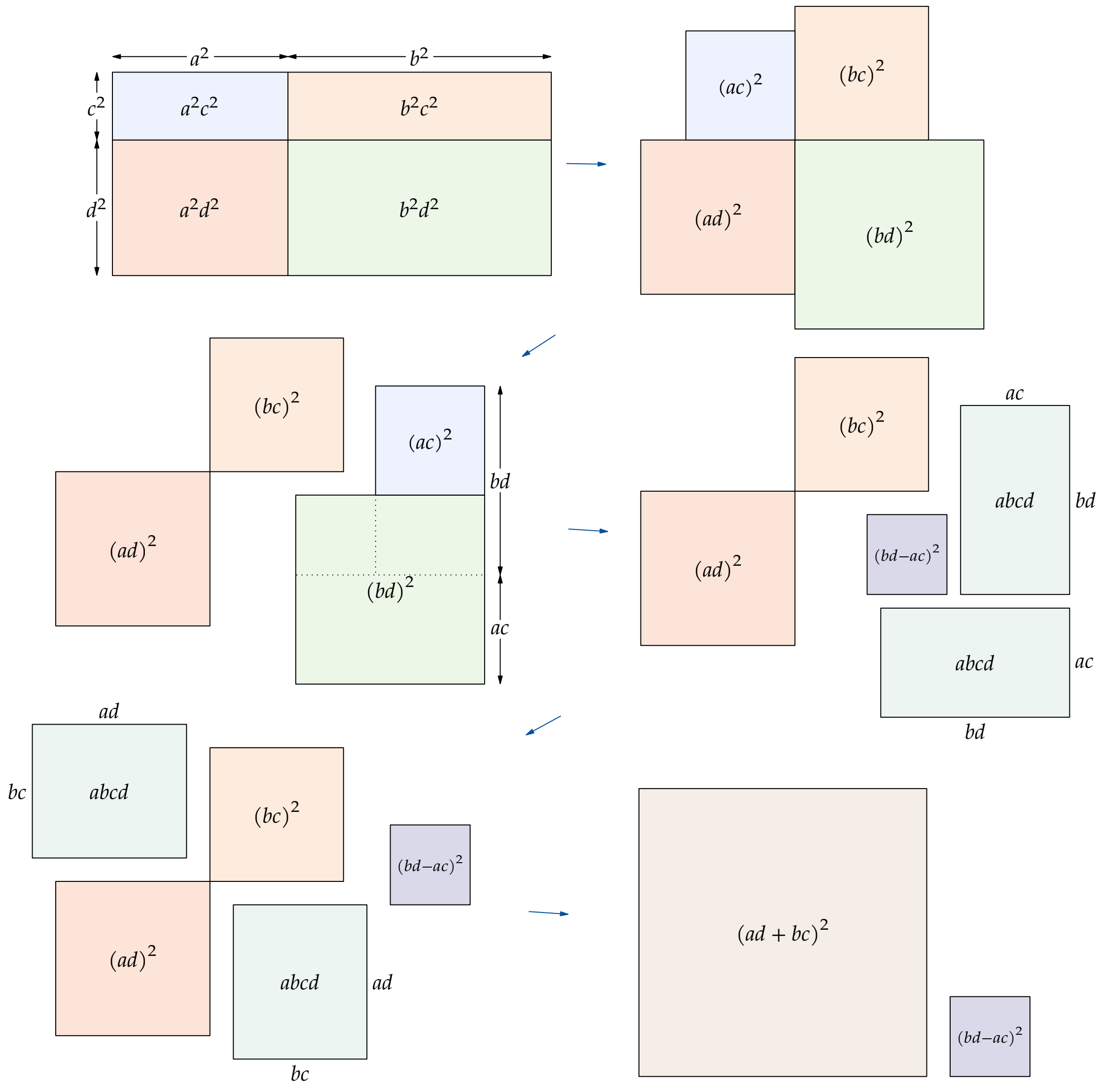




— Sam Pooley and K. Ann Drude

Sum of squares identity

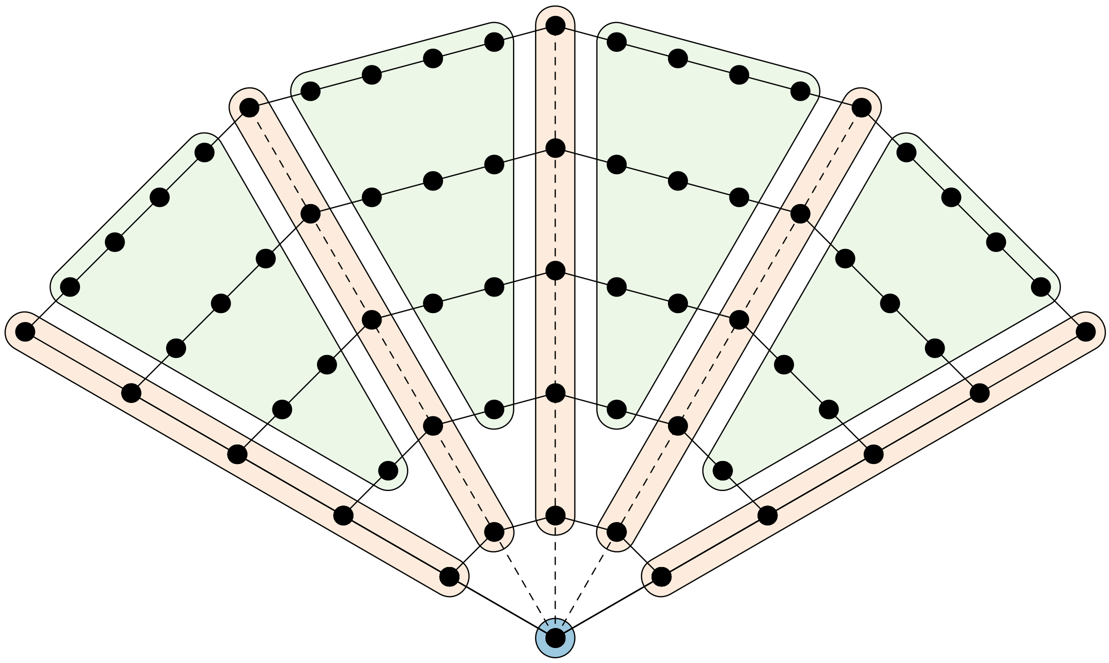
$$\left(a^2+b^2\right)\left(c^2+d^2\right)=(ab+bc)^2+(bd-ac)^2$$



— Diophantus of Alexandria

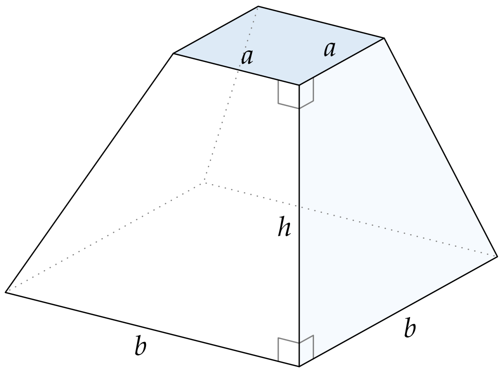
Polygonal numbers

The k^{th} n-gonal number is $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$

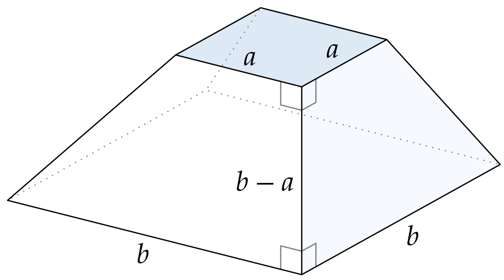


— Dave Logothetti

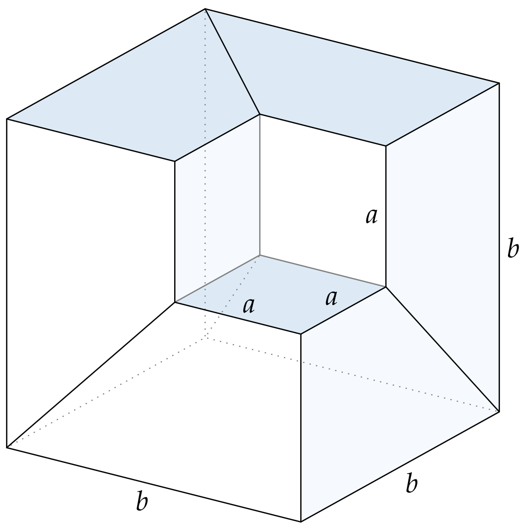
The volume of a frustrum of a square pyramid



P_1



P_2

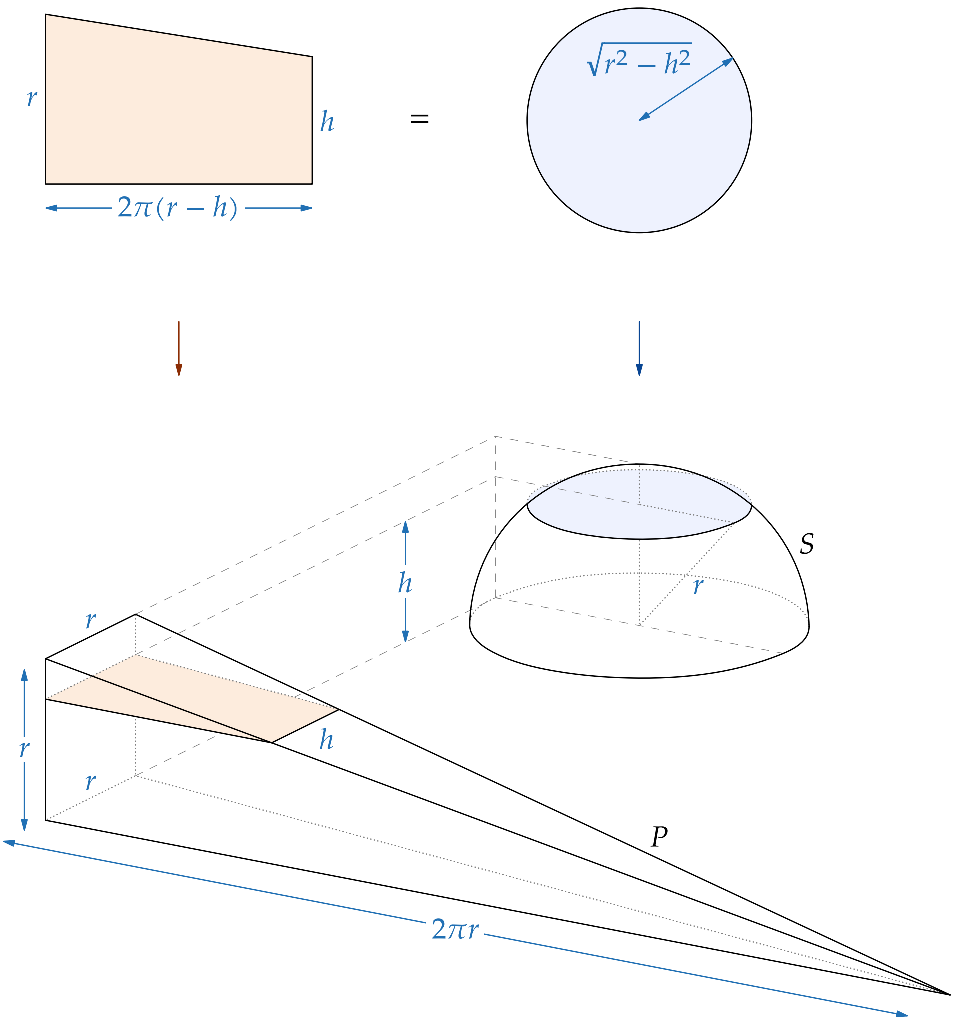


$3P_2$

$$V_{P_1} = \frac{h}{b-a} \cdot V_{P_2} = \frac{h}{b-a} \cdot \frac{1}{3}\left(b^3 - a^3\right) = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

— *The Moscow Papyrus*, c. 1850 BCE

The volume of a hemisphere via Cavalieri's Principle



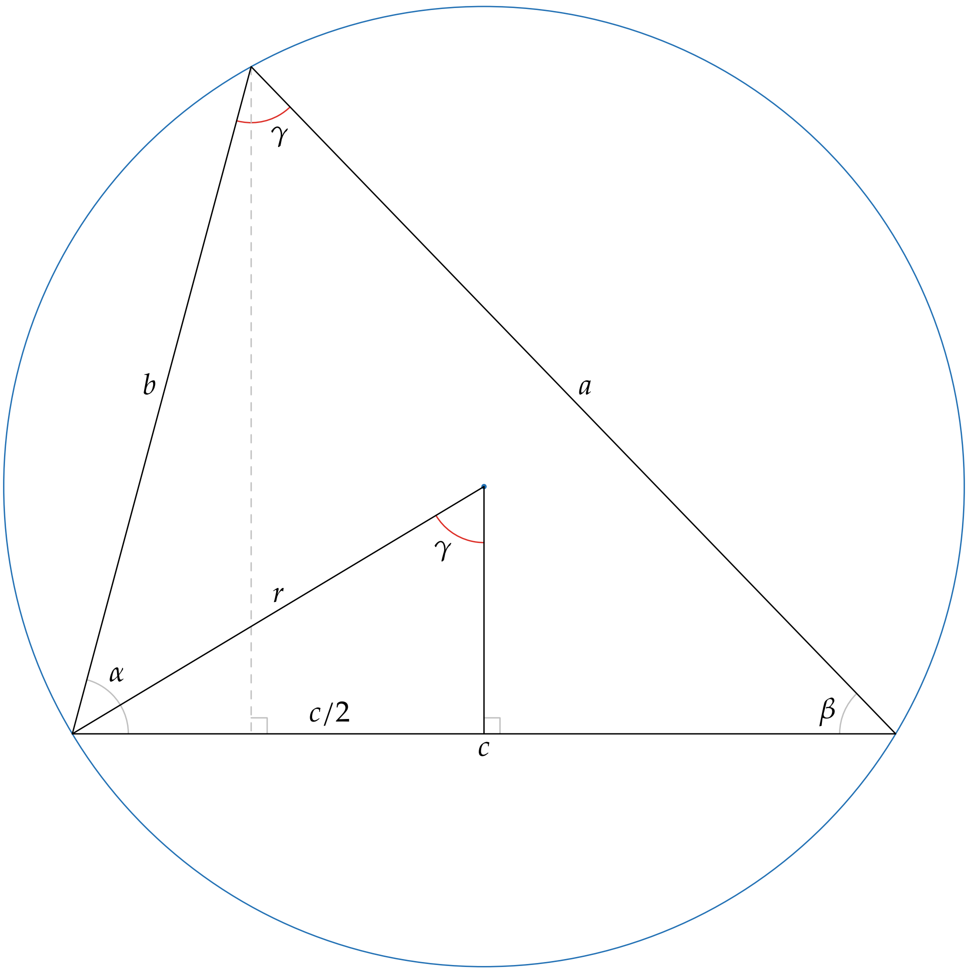
$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

— Sidney H. Kung

Trigonometry, Calculus, & Analytic Geometry

Sine of the sum

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \text{ for } \alpha+\beta<\pi$$



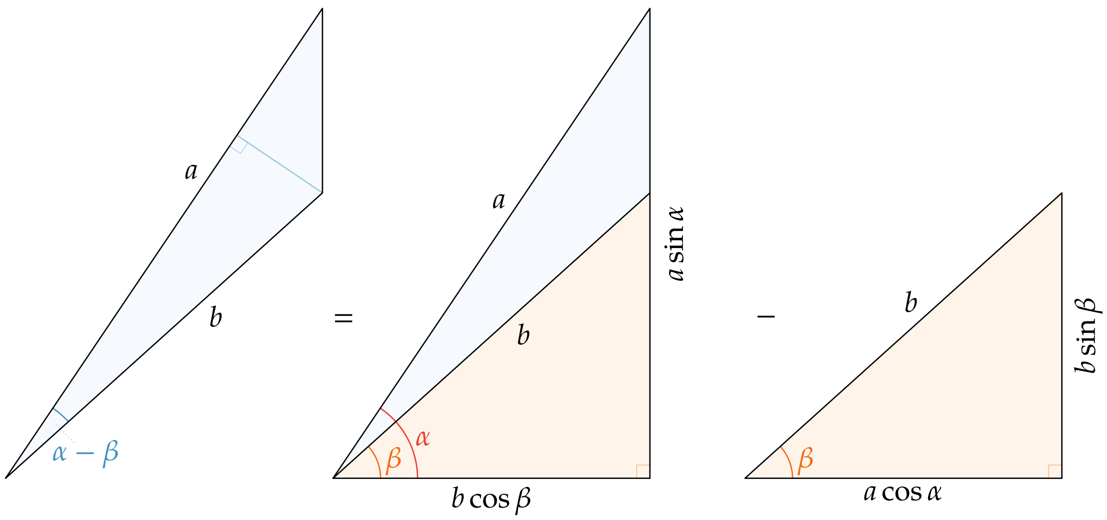
$$c = a\cos\beta + b\cos\alpha$$

$$r = 1/2 \quad\Longrightarrow\quad \sin\gamma = \frac{c/2}{1/2} = c,\ \sin\alpha = a,\ \sin\beta = b$$

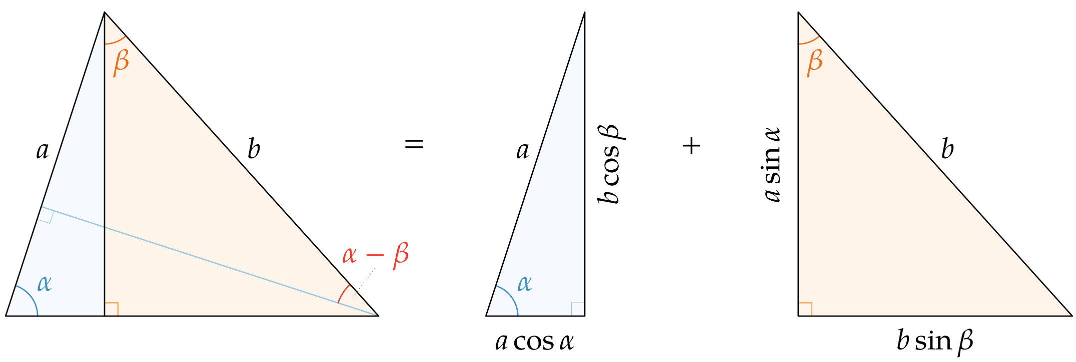
$$\sin(\alpha+\beta) = \sin(\pi-(\alpha+\beta)) = \sin\gamma = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

— Sidney H. Kung

Area and difference formulas



$$\frac{1}{2} \cdot a \cdot b \sin(\alpha - \beta) = \frac{1}{2} \cdot a \sin \alpha \cdot b \cos \beta - \frac{1}{2} \cdot a \cos \alpha \cdot b \sin \beta$$
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\frac{1}{2} \cdot a \cdot b \cos(\alpha - \beta) = \frac{1}{2} \cdot a \cos \alpha \cdot b \cos \beta + \frac{1}{2} \cdot a \sin \alpha \cdot b \sin \beta$$
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

— Sidney H. Kung