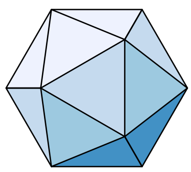
Proofs without words II

More exercises in METAPOST

Toby Thurston

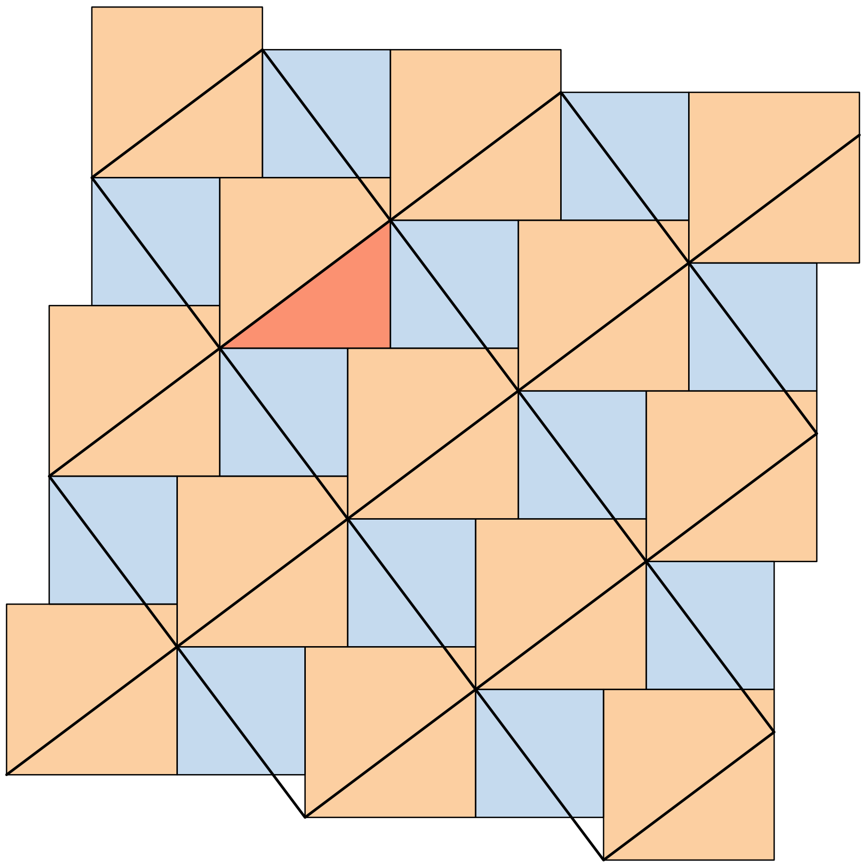
March 2020 — January 2021



Contents

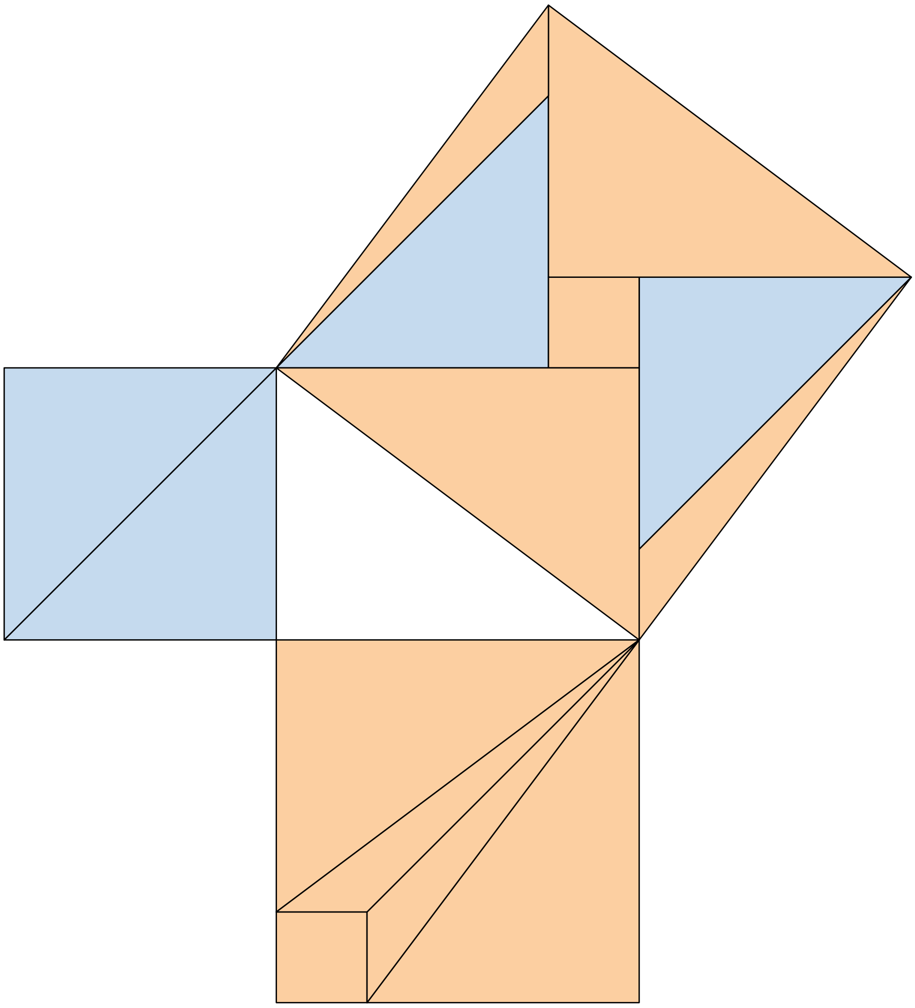
Geometry and Algebra

The Pythagorean theorem VII



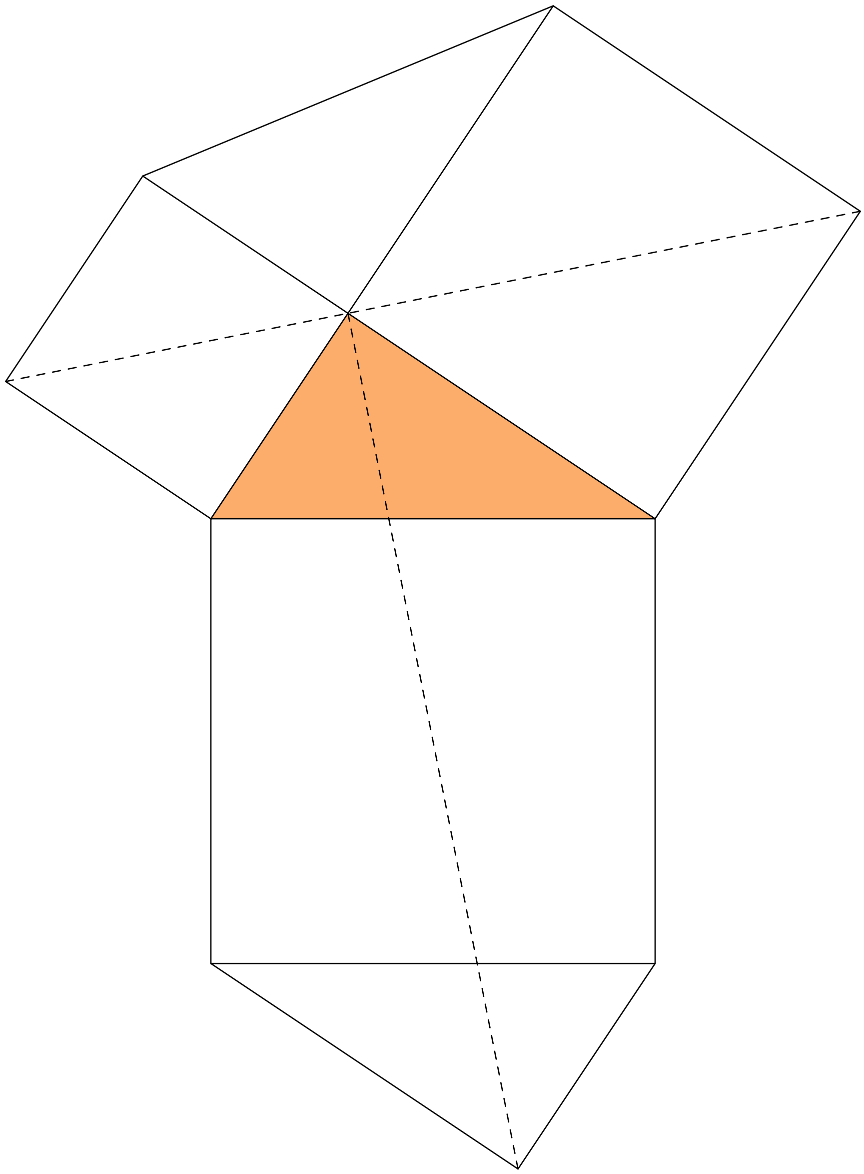
— Annairizi of Arabia (circa 900)

The Pythagorean theorem VIII



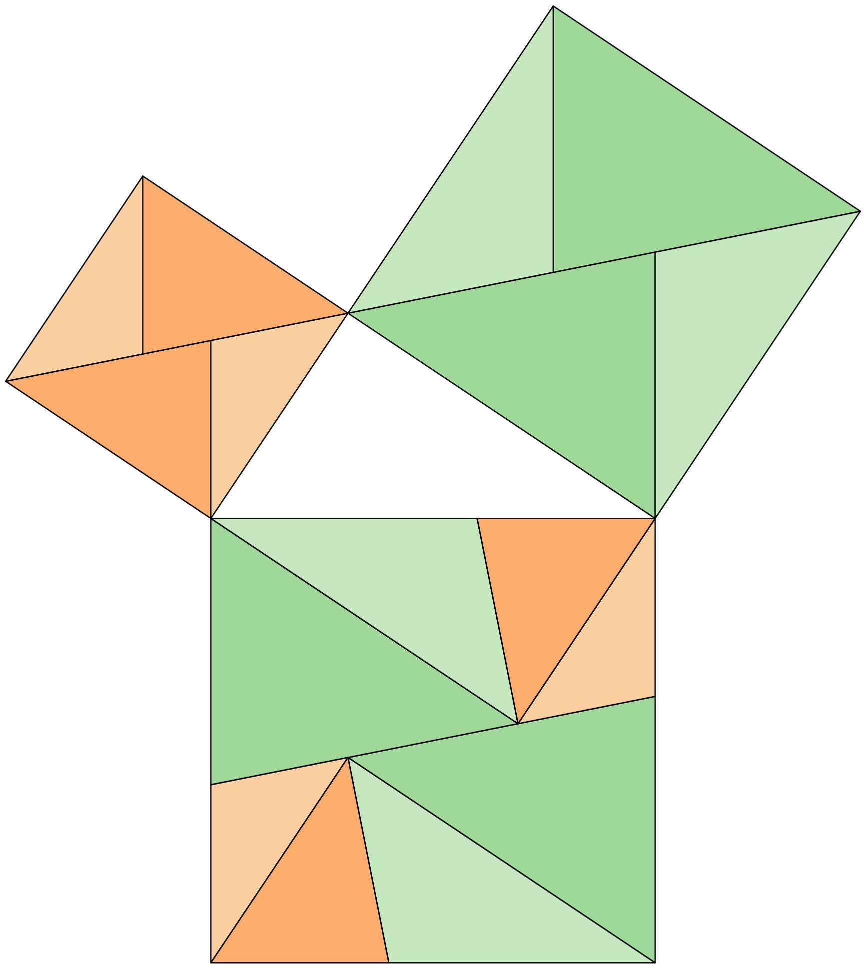
— Liu Hui (3rd century A.D.)

The Pythagorean theorem IX



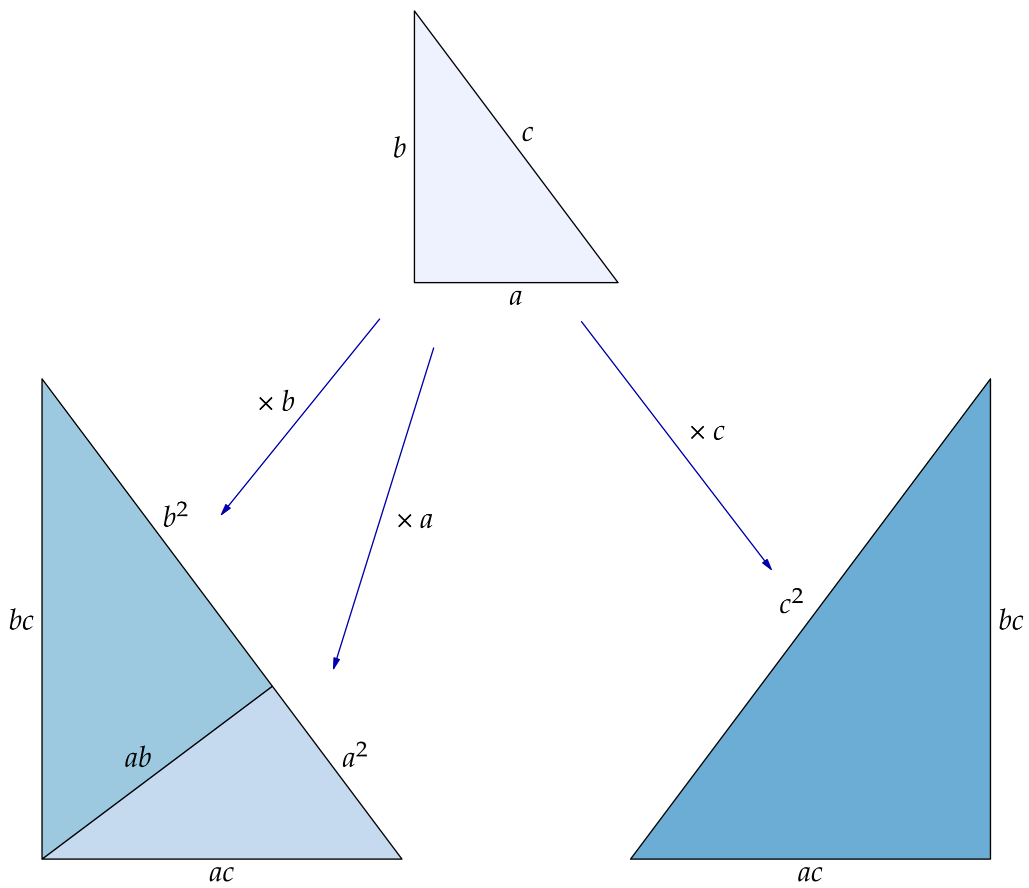
— Leonardo da Vinci (1452–1519)

The Pythagorean theorem X



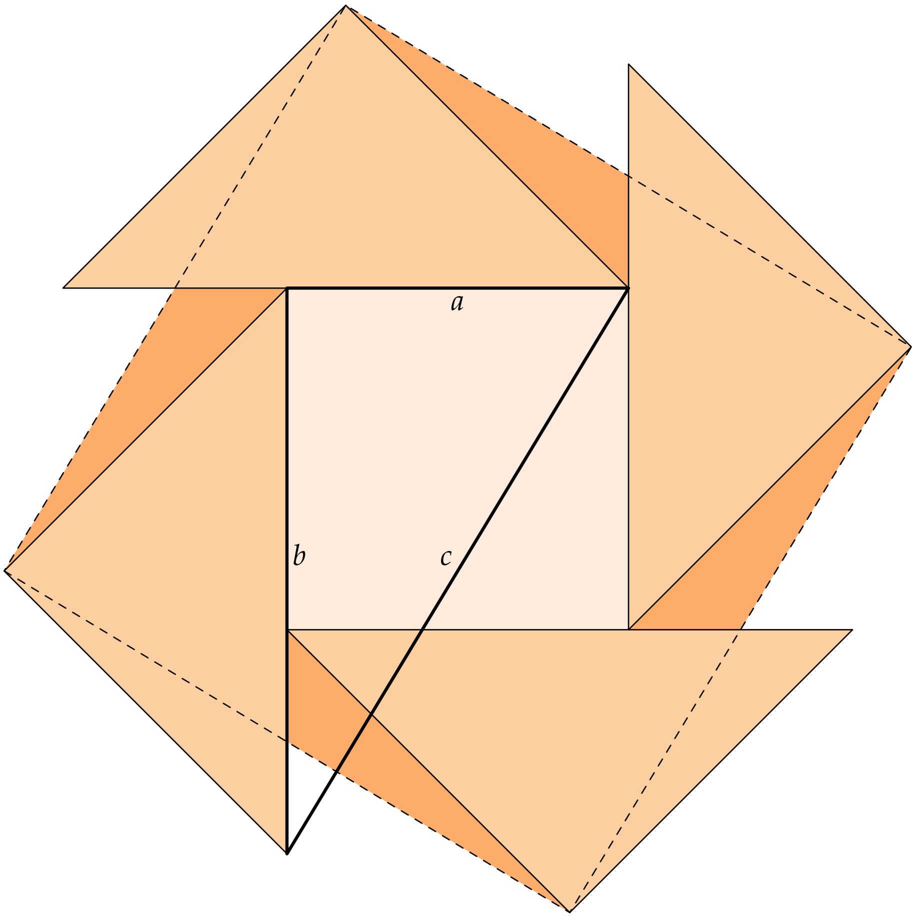
— J. E. Böttcher

The Pythagorean theorem XI



— Frank Burk

The Pythagorean theorem XII

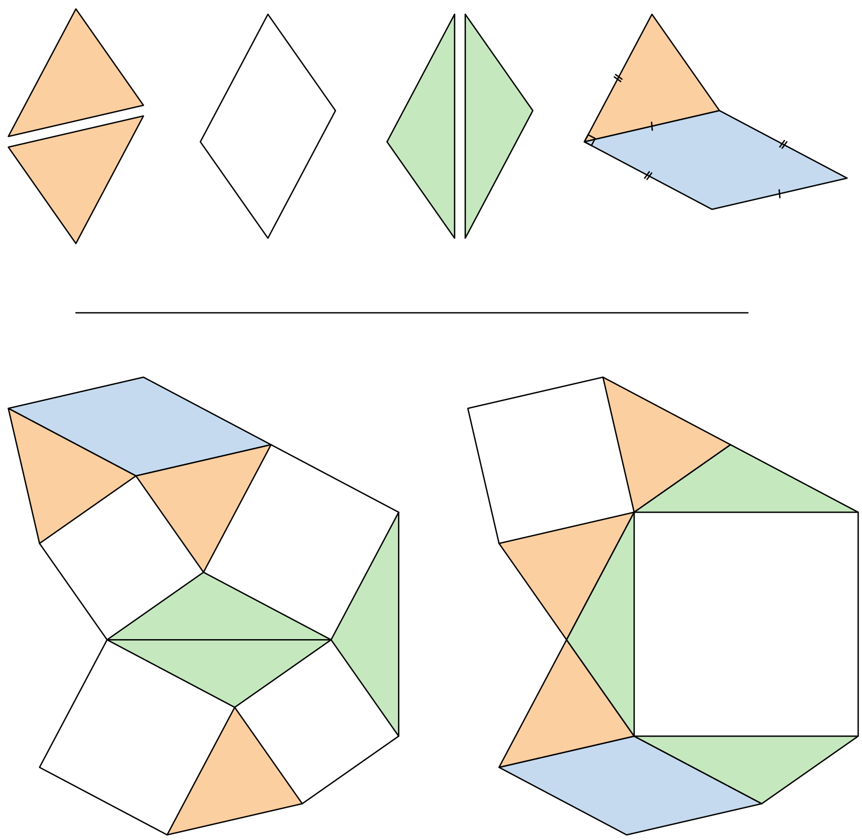


$$a^2 + b^2 = c^2$$

— Poo-Sung Park

A generalization from Pythagoras

The sum of the area of two squares, whose sides are the lengths of two diagonals of a parallelogram, is equal to the sum of the area of four squares, whose sides are its four sides.

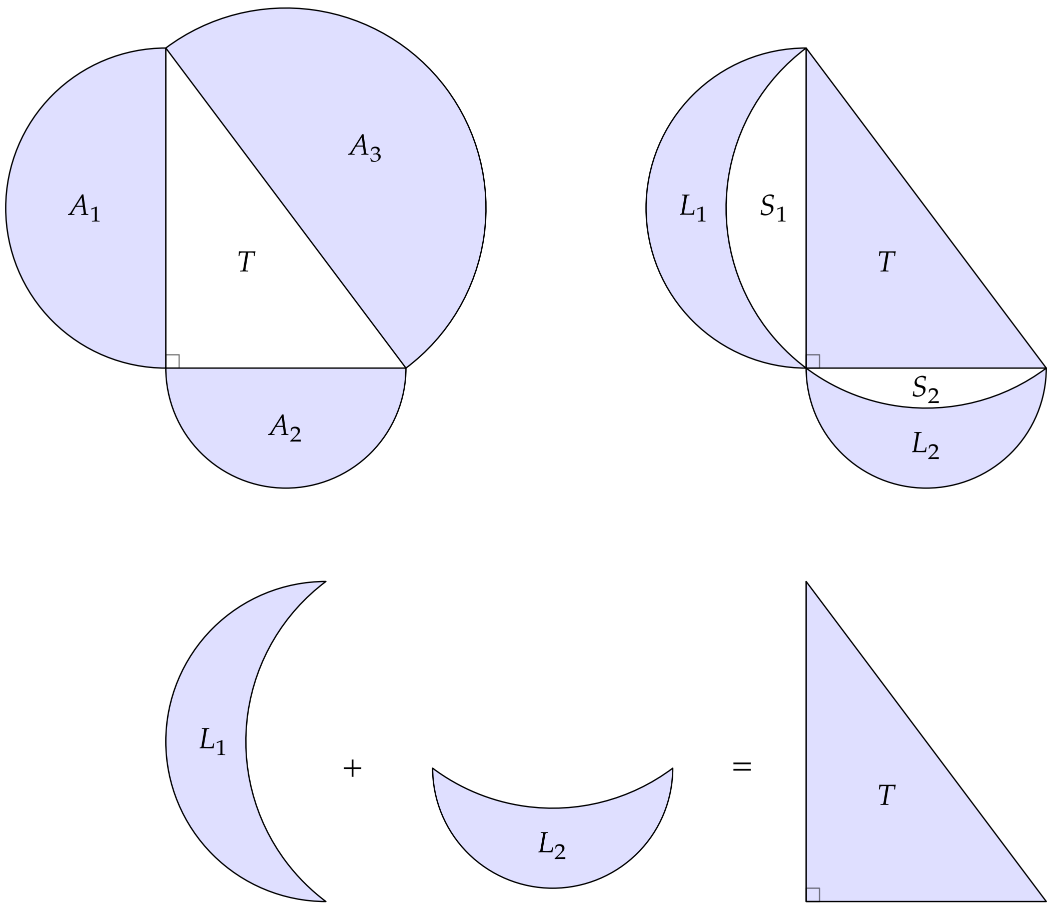


Corollary: The Pythagorean theorem (when the parallelogram is a rectangle).

— David S. Wise

A theorem of Hippocrates of Chios (circa 440 BC)

The combined area of the lunes constructed on the legs of a given right angle triangle is equal to the area of the triangle.

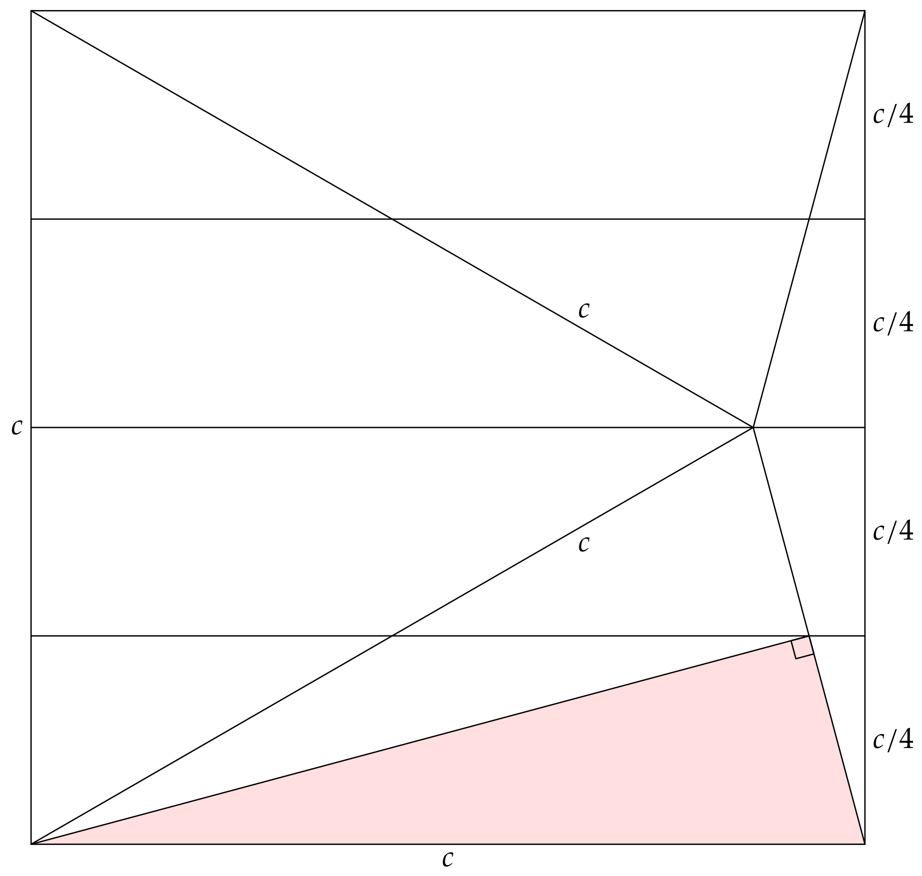


$$A_1 + A_2 = A_3$$
$$(L_1 + S_1) + (L_2 + S_2) = T + S_1 + S_2$$
$$L_1 + L_2 = T$$

— Eugene A. Margerum and Michael M. McDonnell

The area of a right triangle with acute angle $\pi/12$

The area of a right triangle is $\frac{1}{8}(\text{hypotenuse})^2$ if and only if one acute angle is $\pi/12$.



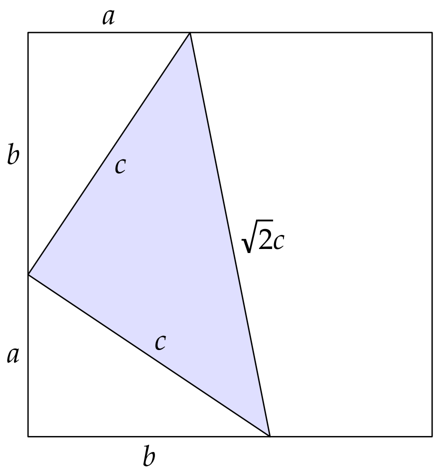
— Klara Pinter

A right angle inequality

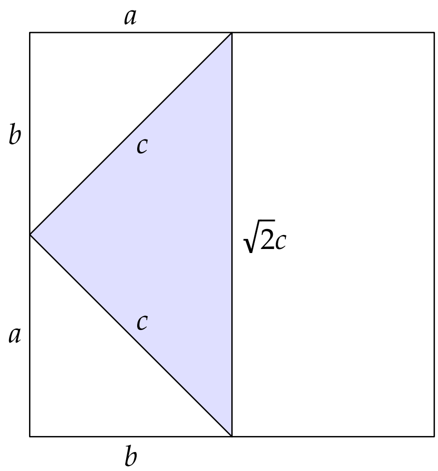
Let c be the hypotenuse of a right triangle whose other two sides are a and b. Prove that

$$a + b \leq \sqrt{2}c.$$

When does equality hold?



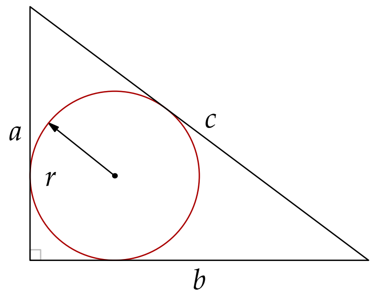
$a + b \leq \sqrt{2}c$



$a + b = \sqrt{2}c \iff a = b$

— Canadian Mathematical Olympiad 1969

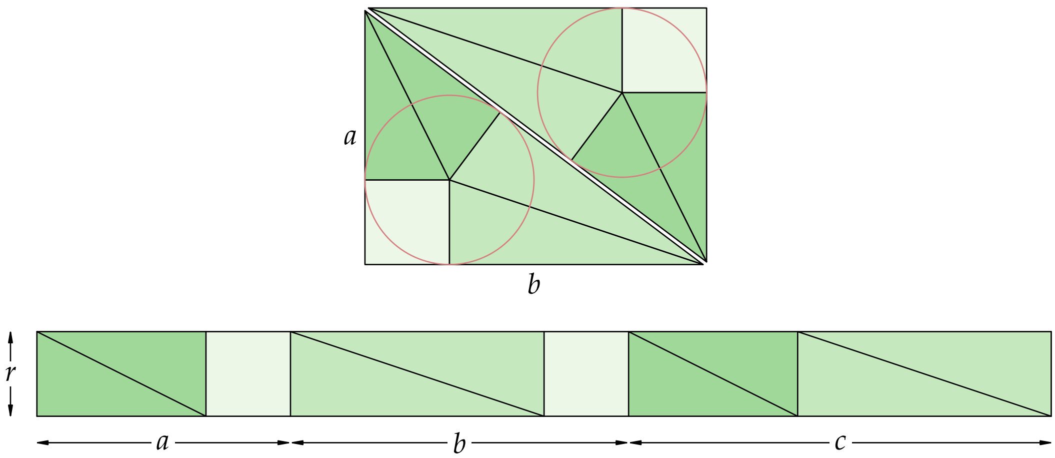
The inradius of a right triangle



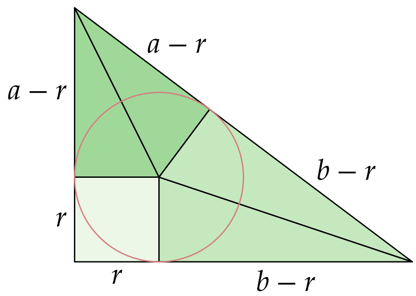
I. $r = \dfrac{ab}{a+b+c}$

II. $r = \dfrac{a+b-c}{2}$

I. $ab = r(a+b+c)$

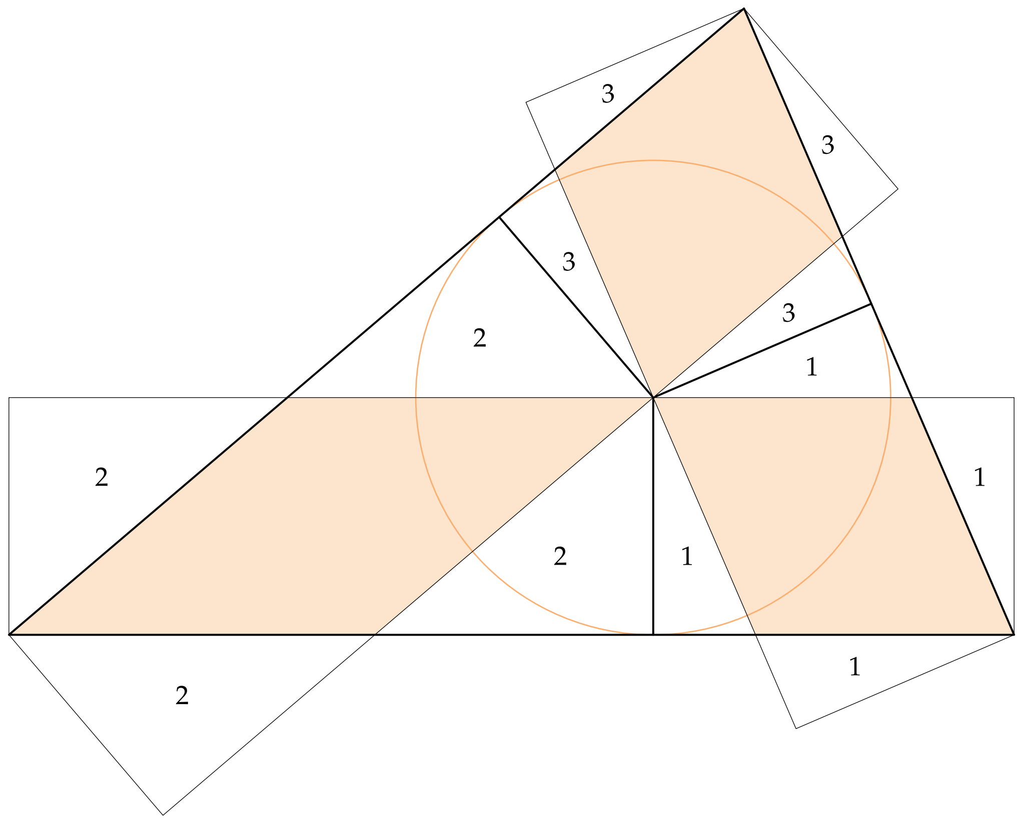


II. $c = a+b-2r$



— Liu Hui (3rd century A.D.)

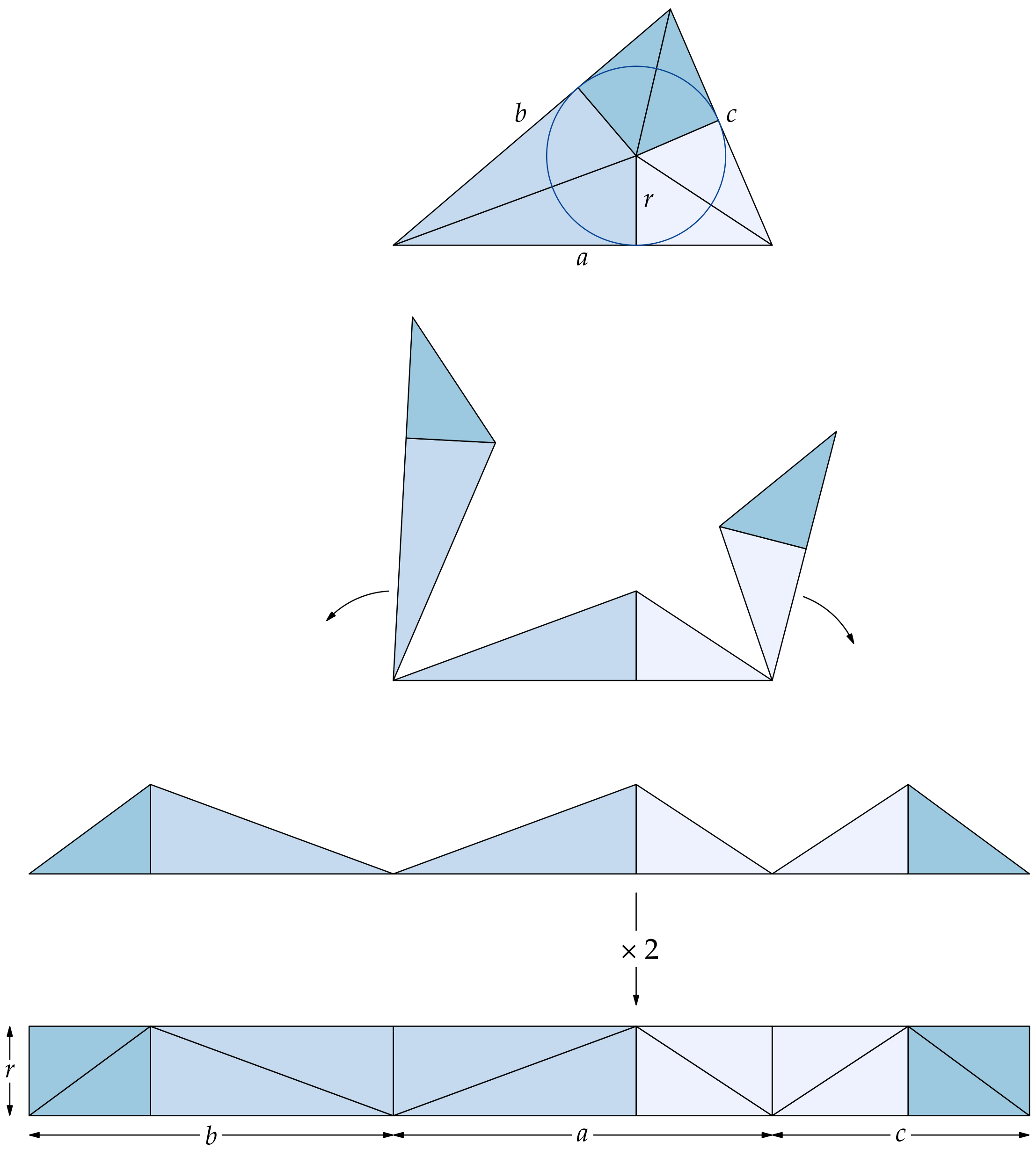
The product of the perimeter of a triangle and its inradius is twice the area of the triangle I



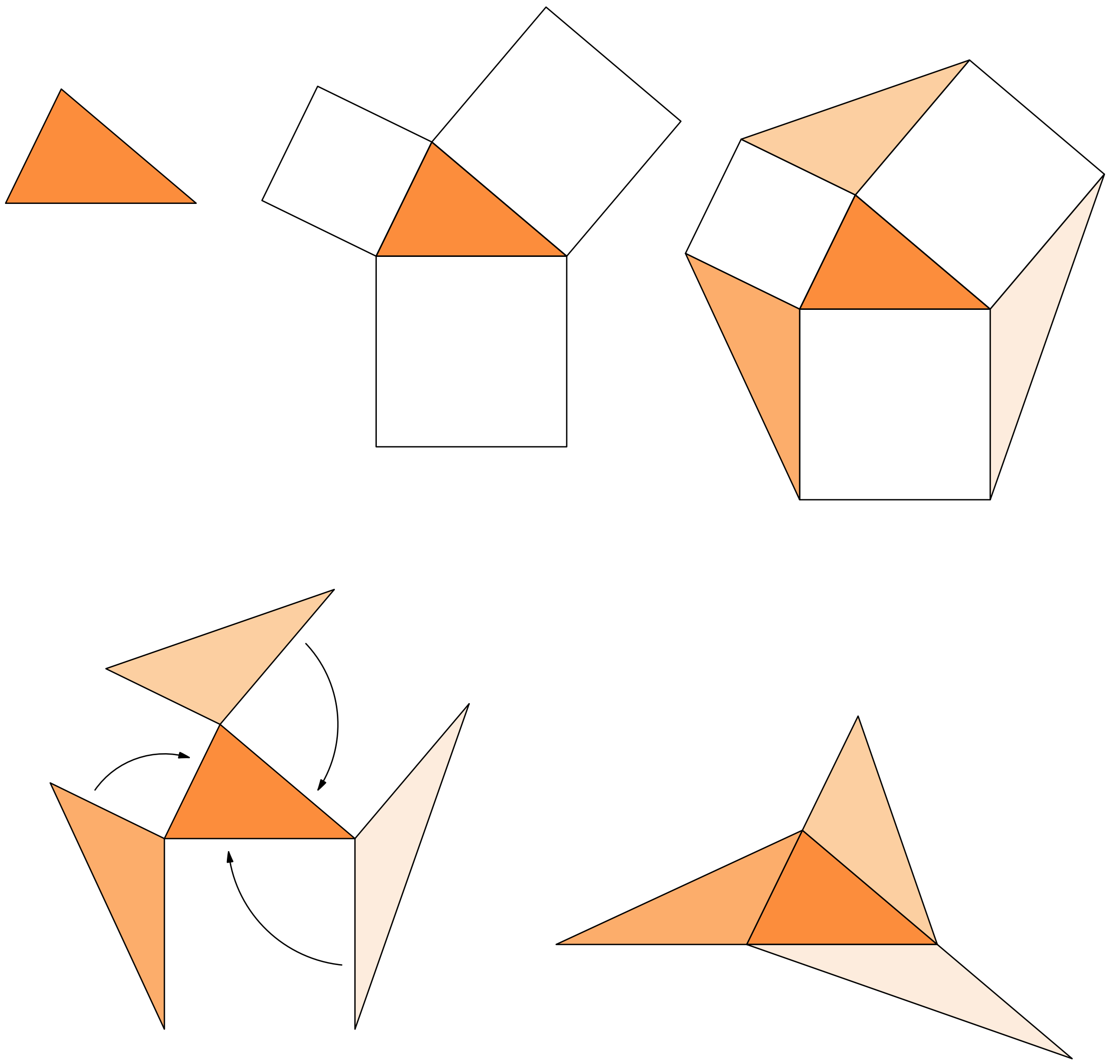
NOTE: *Triangles marked with the same number are equal in area.*

— Grace Lin

The product of the perimeter of a triangle and its inradius is twice the area of the triangle II

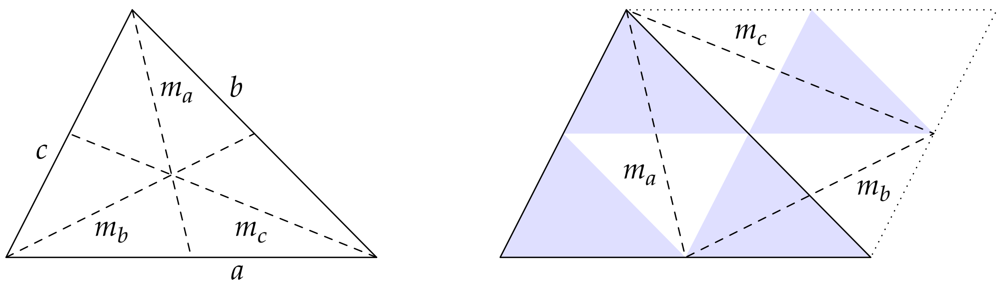


Four triangles with equal area



— Steven L. Snover

The triangle of medians has 3/4 the area of the original triangle

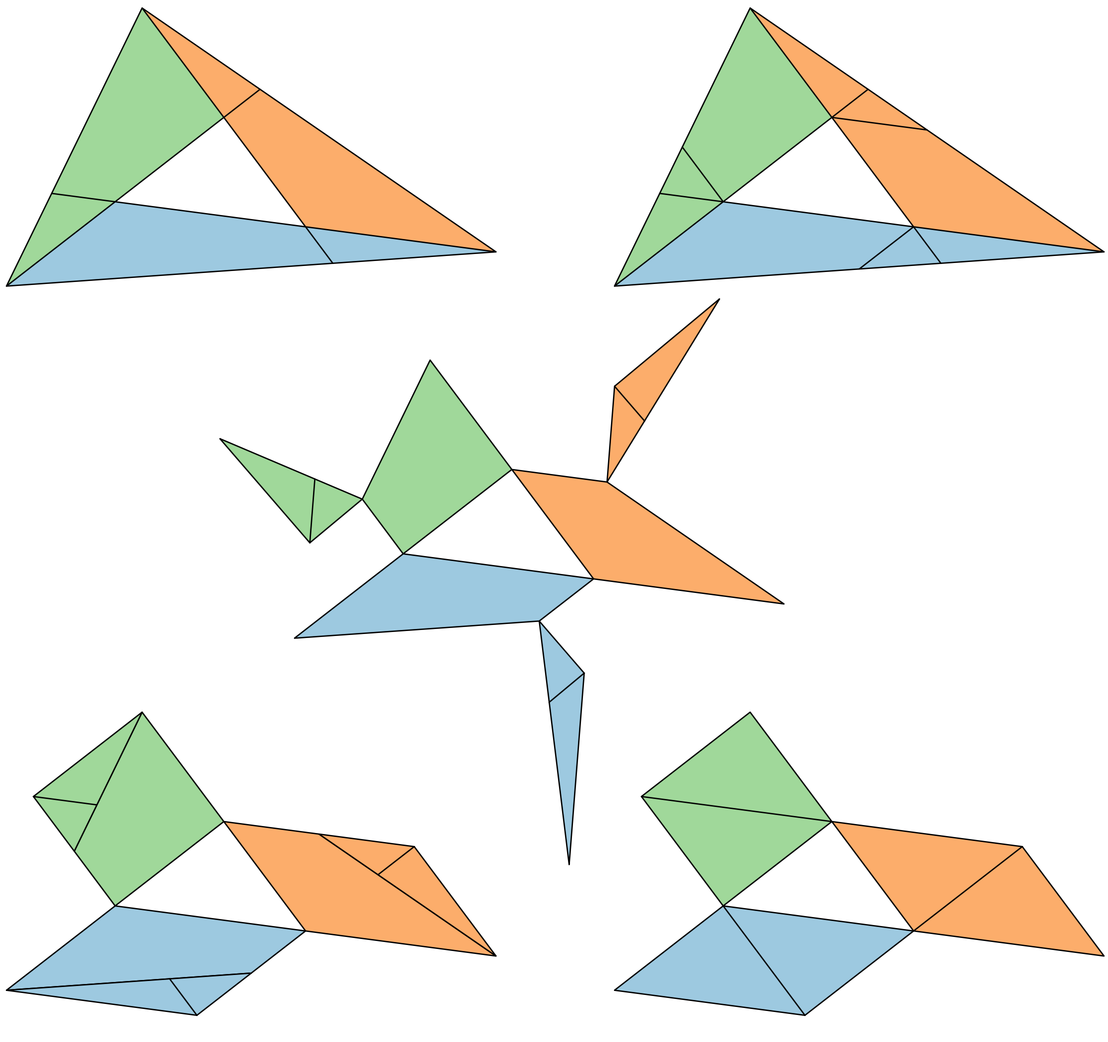


$$\frac{3}{4}\,\text{area}(\triangle abc) = \text{area}(\triangle m_a m_b m_c)$$

— Norbert Hungerbühler

Heptasection of a triangle

If the one-third points on each side of a triangle are joined to opposite vertices, the resulting central triangle is equal in area to one-seventh that of the initial triangle.



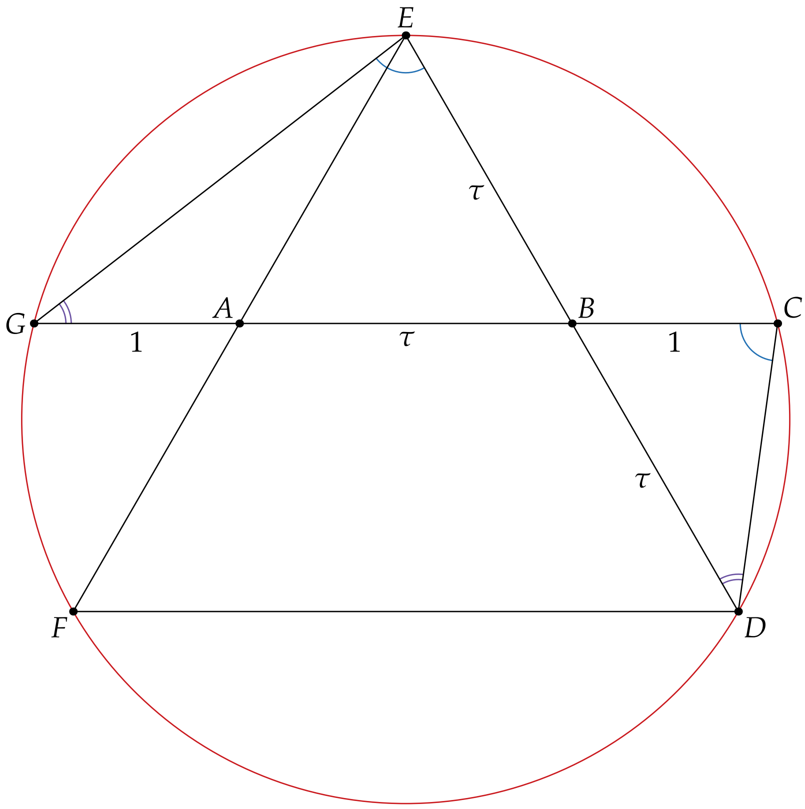
— William Johnston and Joe Kennedy

A Golden Section problem from the *Monthly*

(Problem E3007, *American Mathematical Monthly*, 1983, p.482)

Let A and B be the midpoints of the sides EF and ED of an equilateral triangle DEF. Extend AB to meet the circumcircle (of DEF) at C. Show that B divides AC according to the golden section.

Solution:

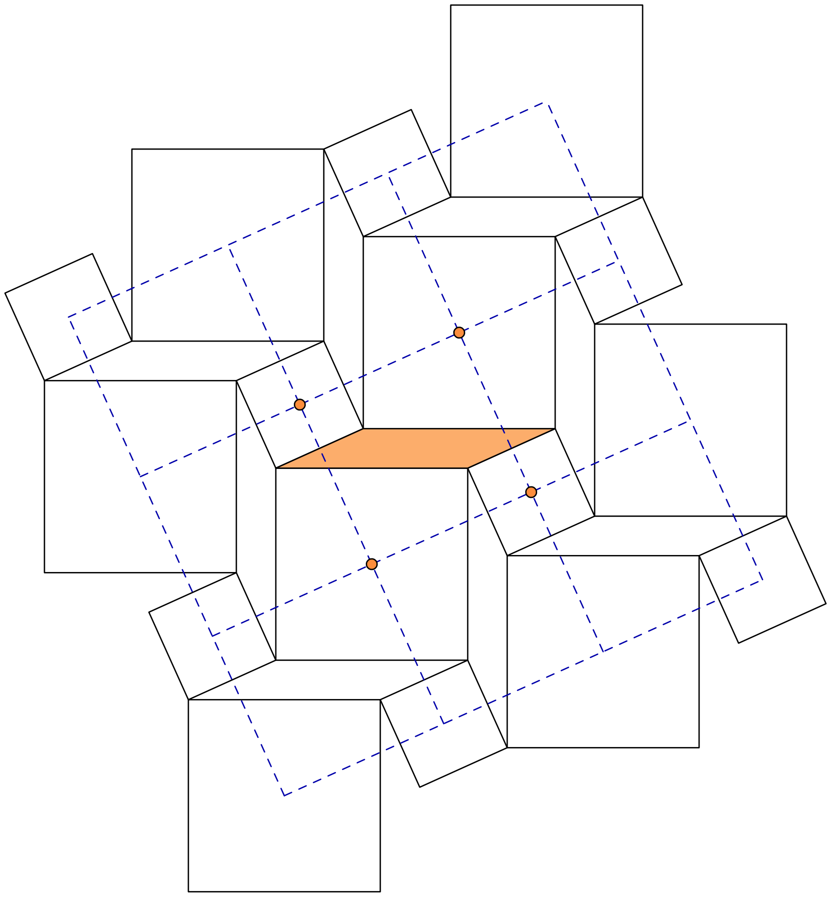


$$\tau^2 = \tau + 1$$

— Jan van de Craats

Tiling with squares and parallelograms

If squares are constructed eternally on the sides of the parallelogram, their centres form a square.

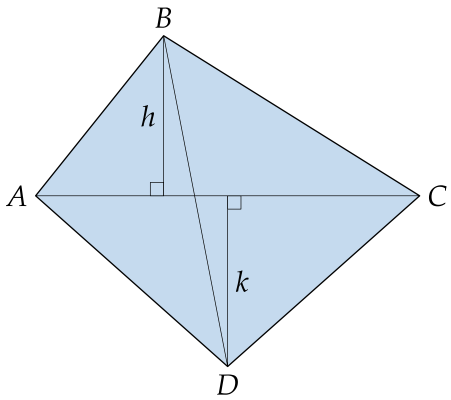


— Alfinio Flores

The area of a quadrilateral I

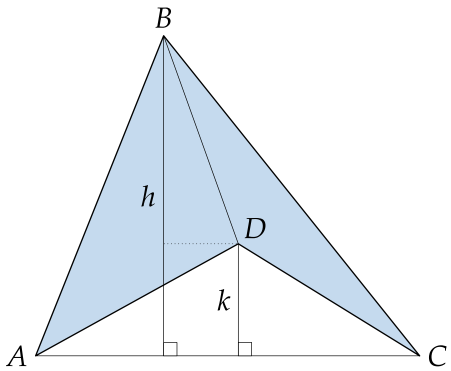
The area of a quadrilateral is less than or equal to half the product of the lengths of its diagonals, with equality if and only if the diagonals are perpendicular.

I. Convex quadrilaterals



$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\overline{AC}\cdot(h+k)\\ &\le \tfrac{1}{2}\overline{AC}\cdot\overline{BD}\end{aligned}$$

II. Concave quadrilaterals



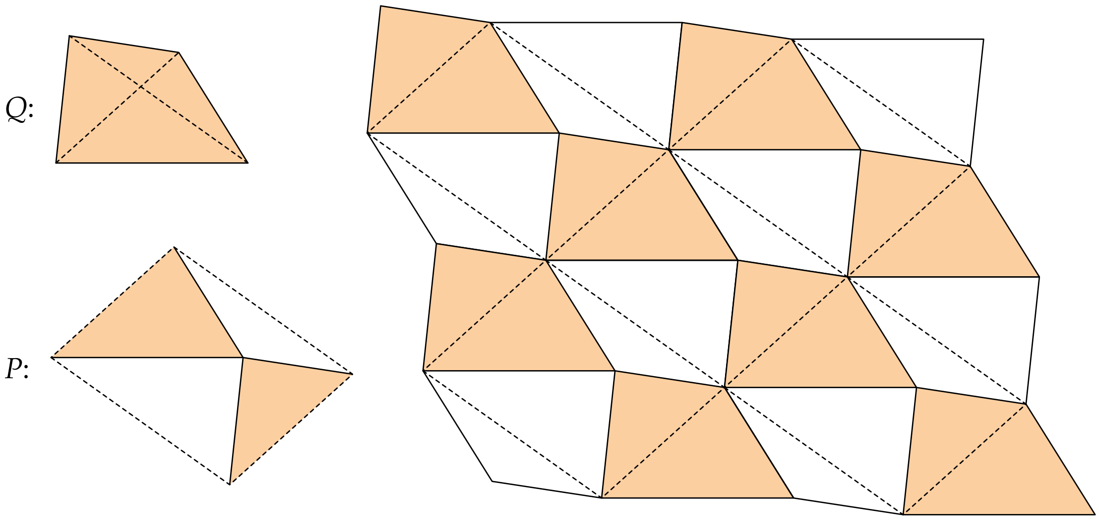
$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\overline{AC}\cdot(h-k)\\ &\le \tfrac{1}{2}\overline{AC}\cdot\overline{BD}\end{aligned}$$

— David B. Sher, Ronald Skurnick, and Dean C. Nataro

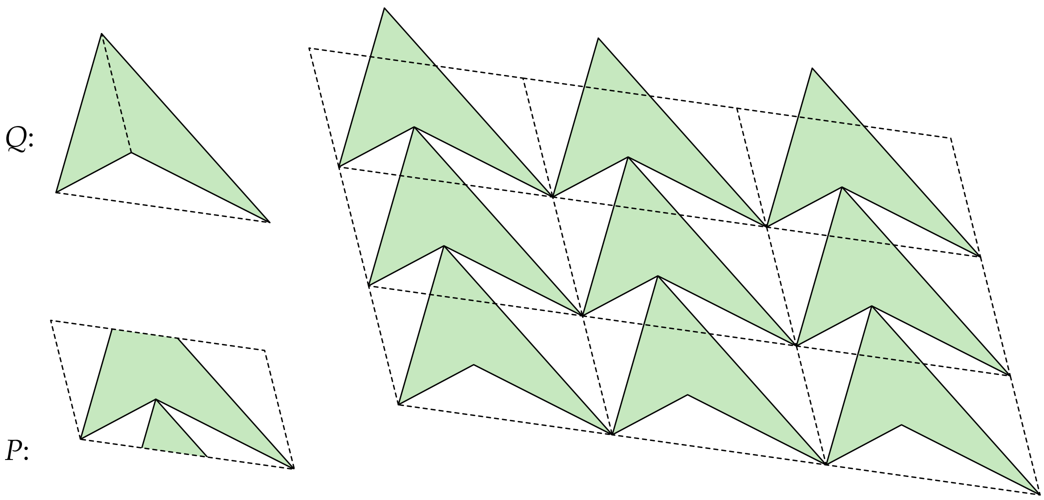
The area of a quadrilateral II

The area of a quadrilateral Q is equal to one-half the area of a parallelogram P whose sides are parallel to and equal in length to the diagonals of Q.

I. Q convex



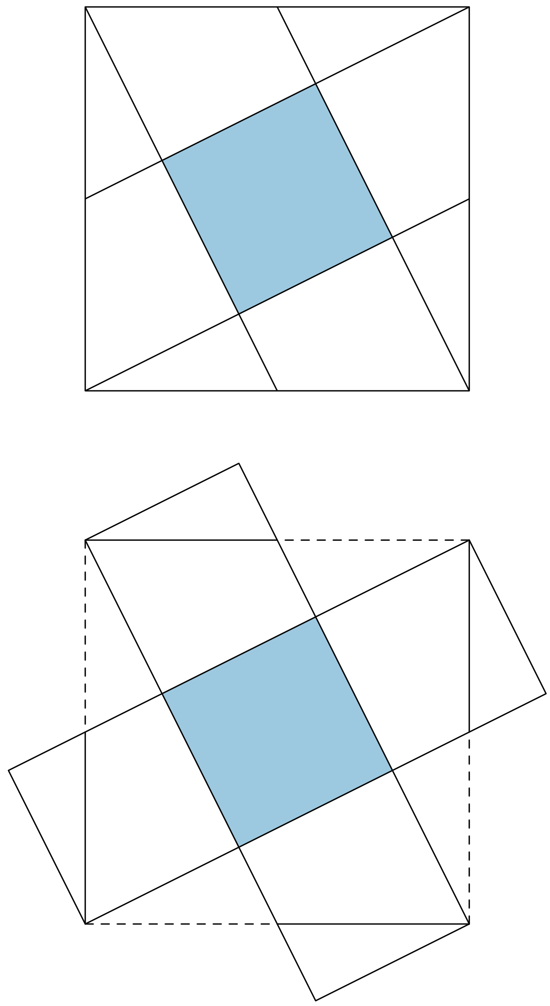
II. Q concave



$$\text{area}(Q) = \frac{1}{2}\,\text{area}(P)$$

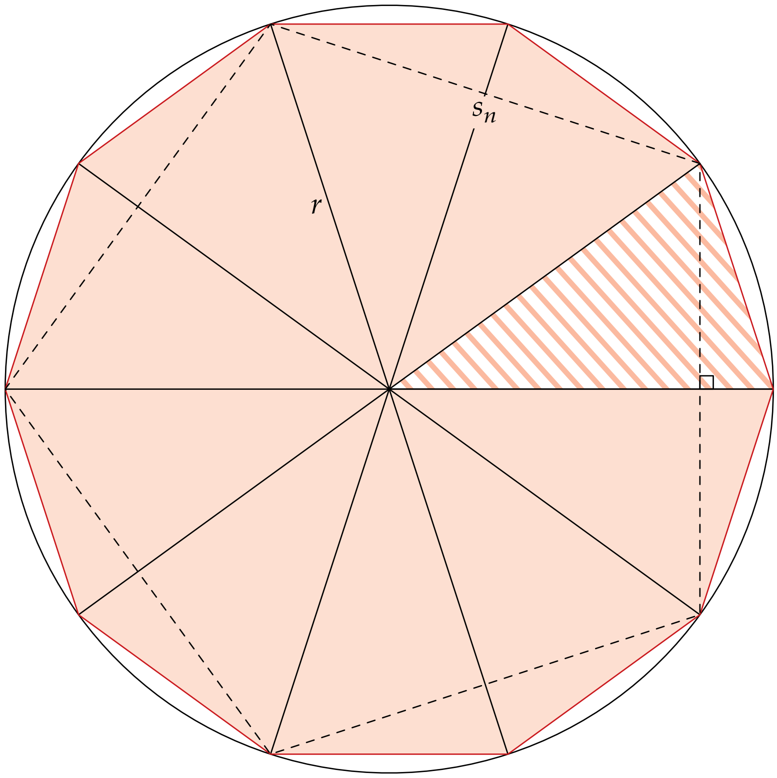
A square within a square

If lines from the vertices of a square are drawn to the mid-points of adjacent sides (as shown in the figure), then the area of the smaller square so produced is one-fifth that of the given square.



Areas and perimeters of regular polygons

The area of a regular $2n$-gon inscribed in a circle is equal to one-half the radius of the circle times the perimeter of a regular n-gon similarly inscribed ($n \geq 3$).



$$
\begin{aligned}
\frac{1}{2n}\,\text{area}\,(P_{2n}) &= \frac{1}{2} \cdot r \cdot \frac{1}{2} s_n \\
\text{area}\,(P_{2n}) &= \frac{1}{2} r \cdot n s_n \\
&= \frac{1}{2} r \cdot \text{perimeter}\,(P_n)
\end{aligned}
$$

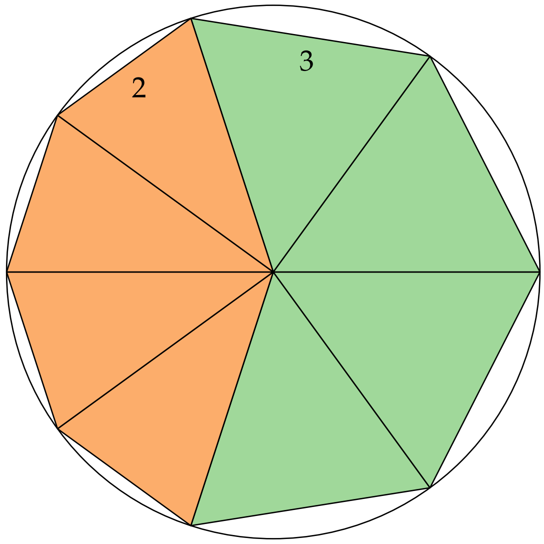
Corollary [Bhāskara, *Litāvati* (India, 12th century AD)]: The area of a circle is equal to one-half the product of its radius and circumference.

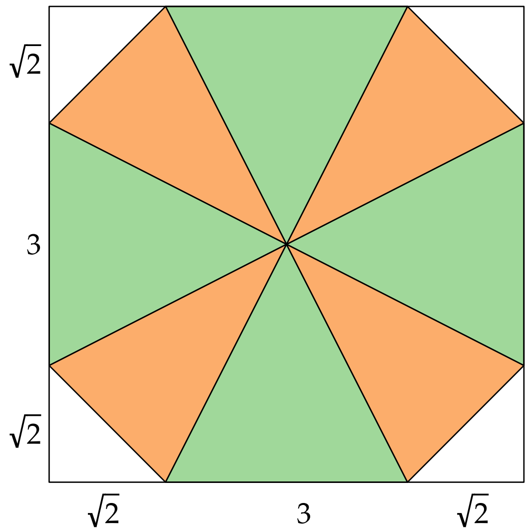
The area of a Putnam octagon

(Problem B1, 39th Annual William Lowell Putnam Mathematical Competition, 1978).

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$, with r, s, and t positive integers.

SOLUTION:





$$A = \left(3 + 2\sqrt{2}\right)^2 - 4 \cdot \frac{1}{2}\left(\sqrt{2}\right)^2 = 9 + 6\sqrt{2} + 6\sqrt{2} + 8 - 4 = 13 + 12\sqrt{2}$$

A Putnam dodecagon

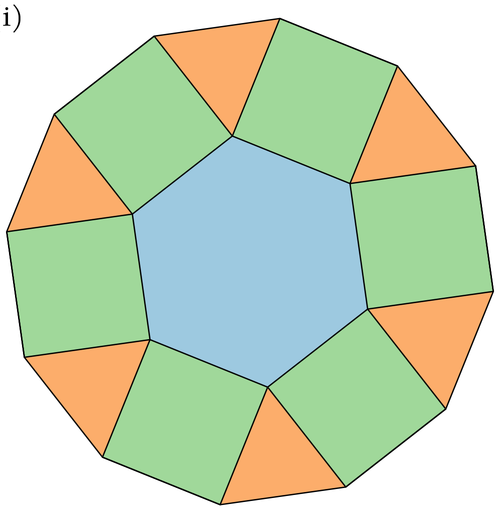
(Problem I-1, 24th Annual William Lowell Putnam Mathematical Competition, 1963)

(i) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.

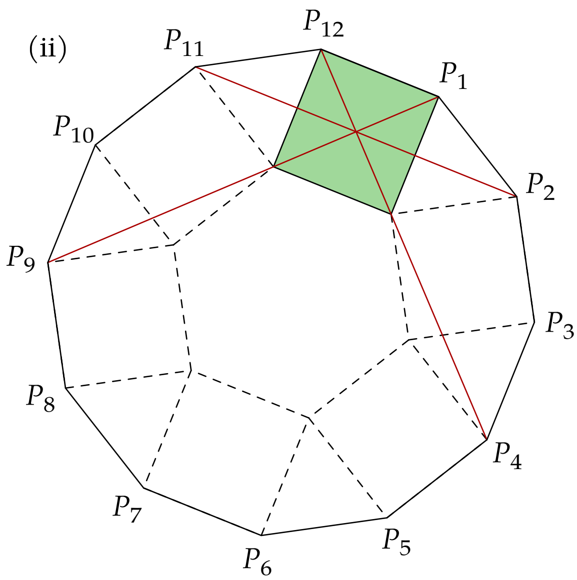
(ii) Let $P_1, P_2, \ldots, P_{12}$ be the successive vertices of a regular dodecagon. Discuss the intersection(s) of the three diagonals P_1P_9, P_2P_{11}, and P_4P_{12}.

SOLUTION:

(i)

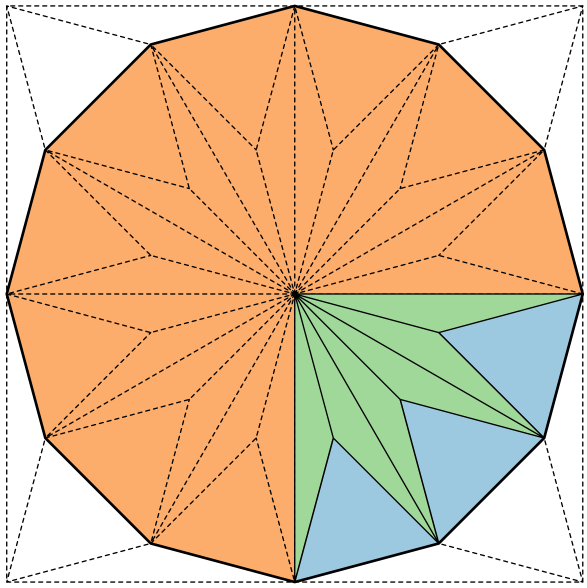


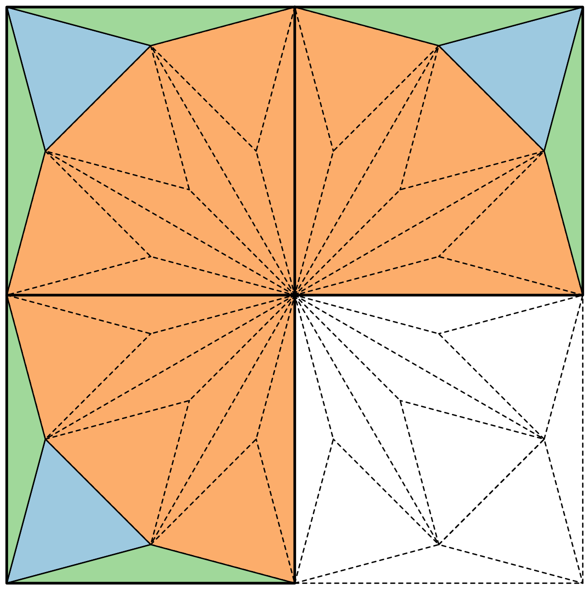
(ii)



The area of a regular dodecagon

A regular dodecagon with circumradius one has area three.



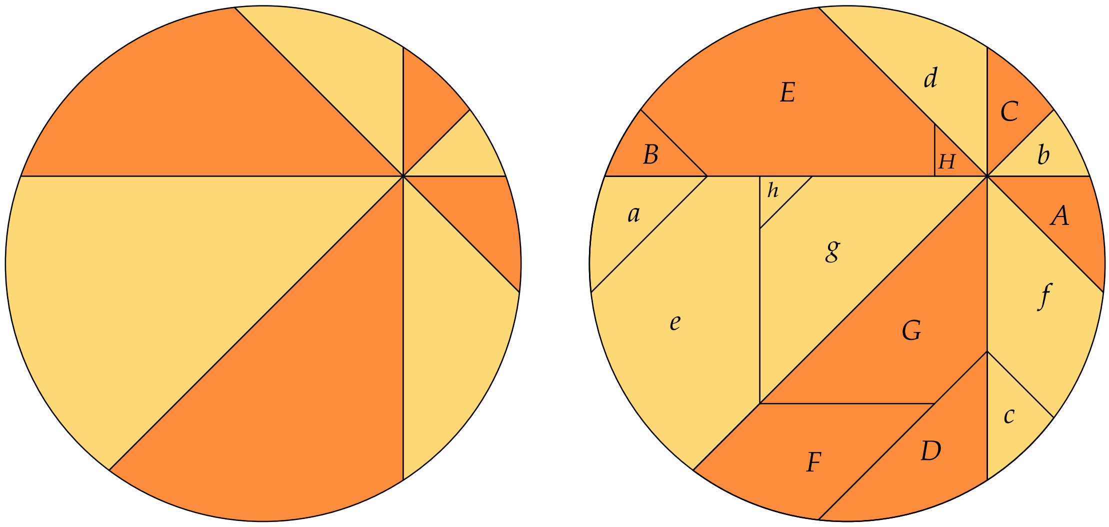


— J. Kürshák

Fair allocation of a pizza

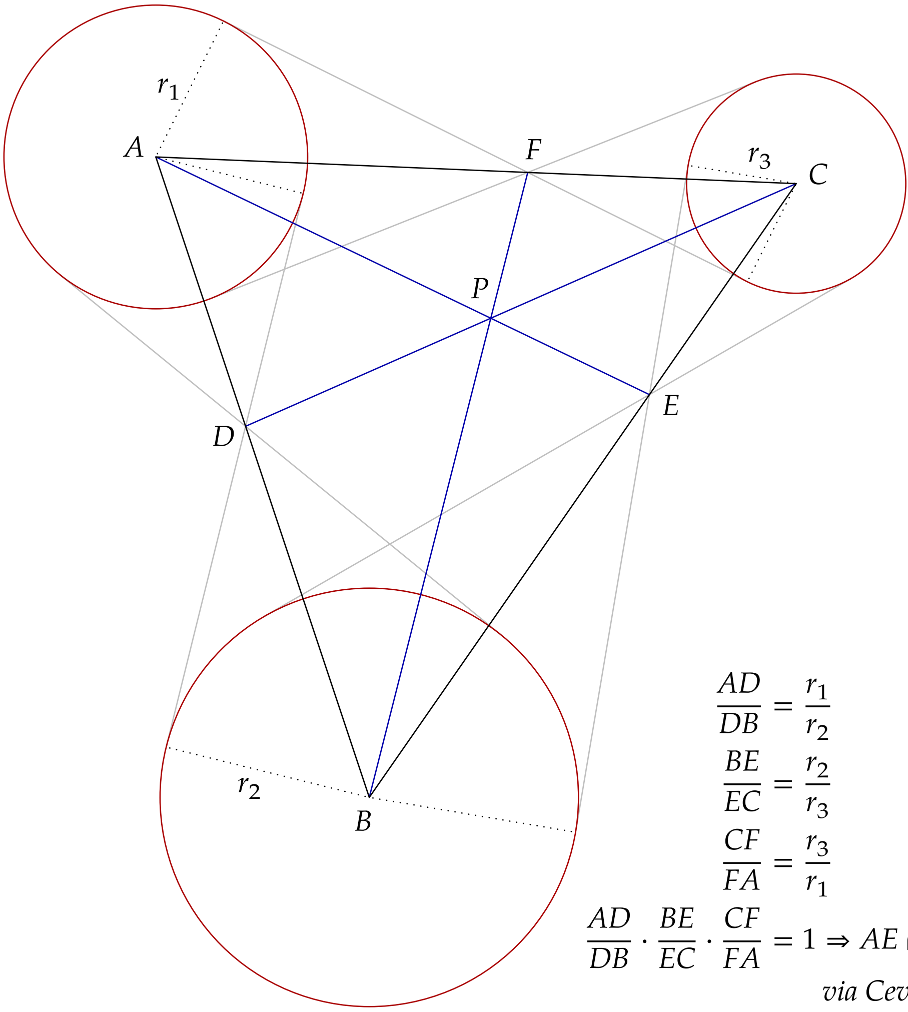
The Pizza Theorem: If a pizza is divided into eight pieces by making cuts at 45° angles through an arbitrary point in the pizza, then the sums of the areas of alternate slices are equal.

Proof:



A three-circle theorem

Given three non-intersecting, mutually external circles, connect the intersection of the internal common tangents of each pair of circles with the centre of the other circle. Then the resulting three line segments are concurrent.



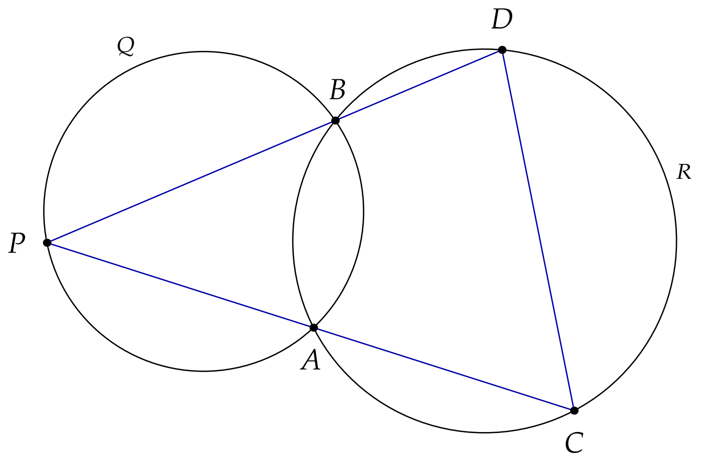
$$\frac{AD}{DB} = \frac{r_1}{r_2}$$
$$\frac{BE}{EC} = \frac{r_2}{r_3}$$
$$\frac{CF}{FA} = \frac{r_3}{r_1}$$
$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1 \Rightarrow AE \cap BF \cap CD = P$$

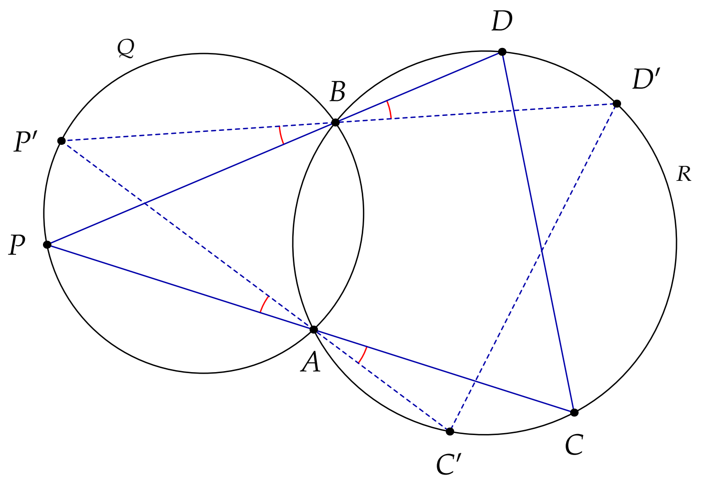
via Ceva's theorem

— R. S. Hu

A constant chord

Suppose two circles Q and R intersect in A and B. A point P on the arc of Q which lies outside R is projected through A and B to determine chord CD of R. Prove that no matter where P is chosen on its arc, the length of chord CD is always the same.





$$\angle C'AC = \angle P'AP = \angle P'BP = \angle D'BD$$

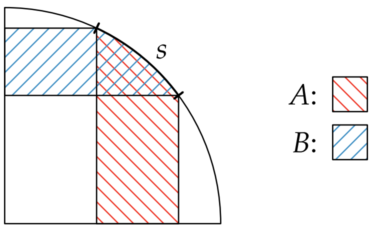
$$\overset{\frown}{C'C} = \overset{\frown}{D'D}, \quad \overset{\frown}{C'D'} = \overset{\frown}{CD}$$

$$C'D' = CD$$

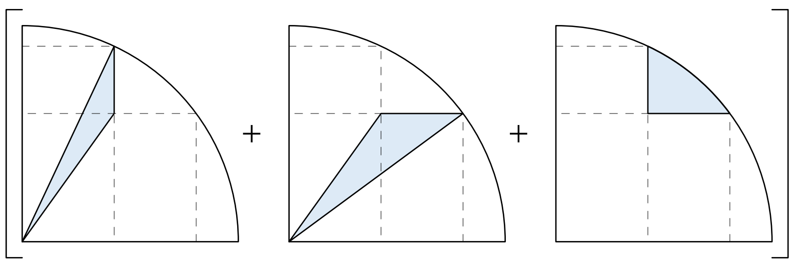
A Putnam area problem

Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis, and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

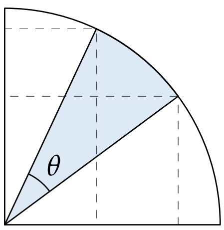
Solution:



$= 2 \times$



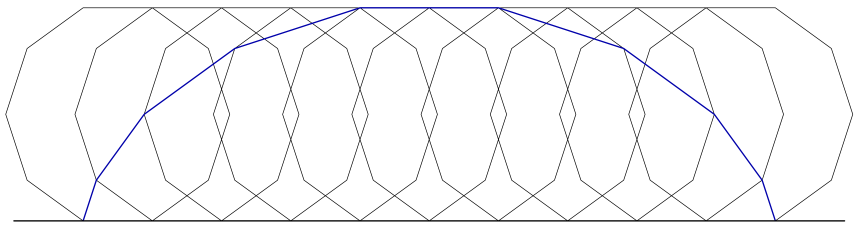
$= 2 \times$

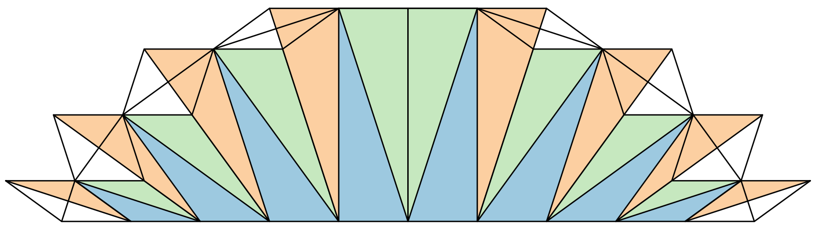


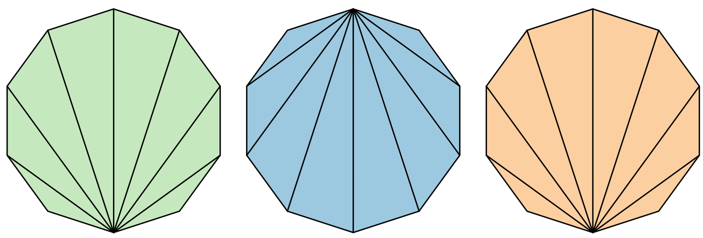
$$A + B = 2 \times \frac{\theta}{2} = \theta = \ell(s)$$

The area under a polygonal arch

The area under a polygonal arch generated by one vertex of a regular n-gon rolling along a straight line is three times the area of the polygon.





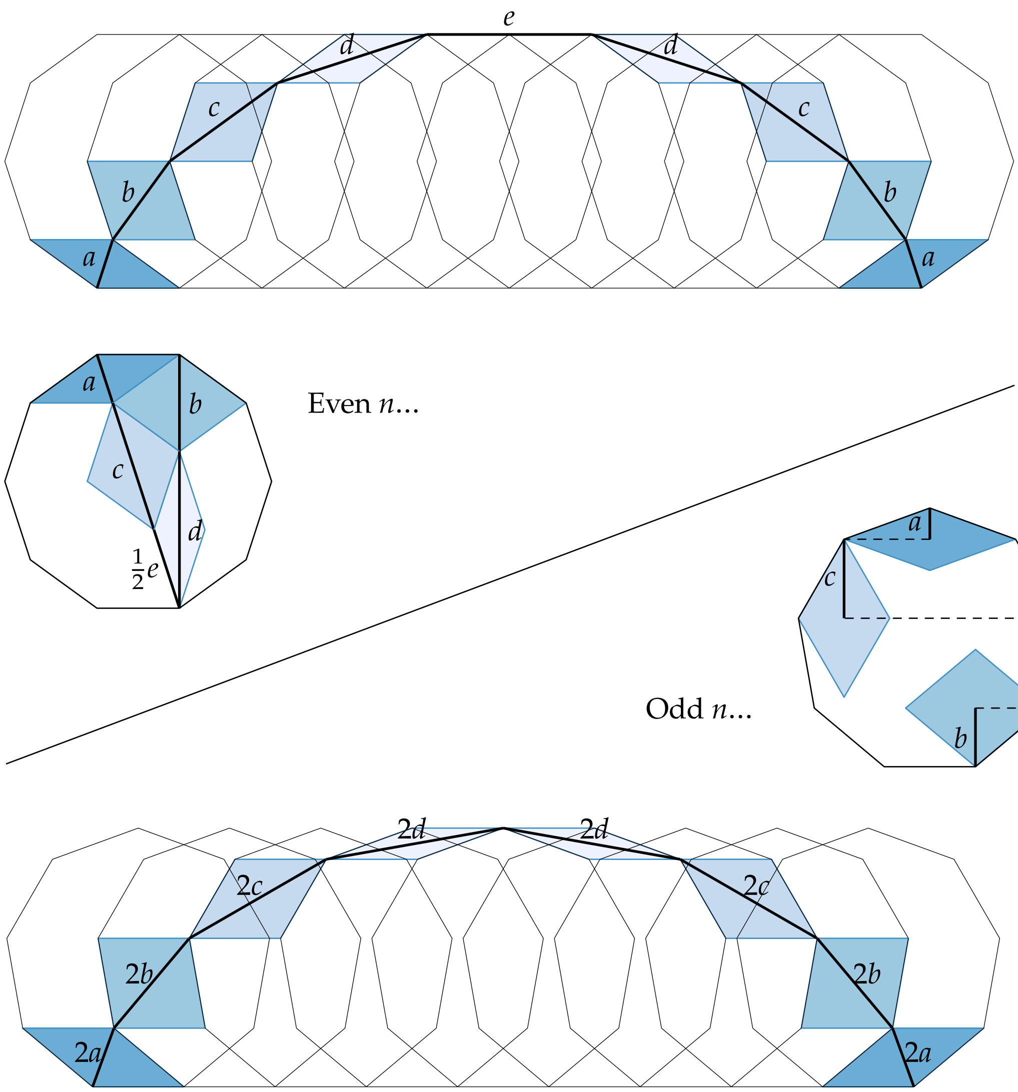


Corollary: The area under one arch of a cycloid is three times the area of the generating circle.

— Philip R. Mallinson

The length of a polygonal arch

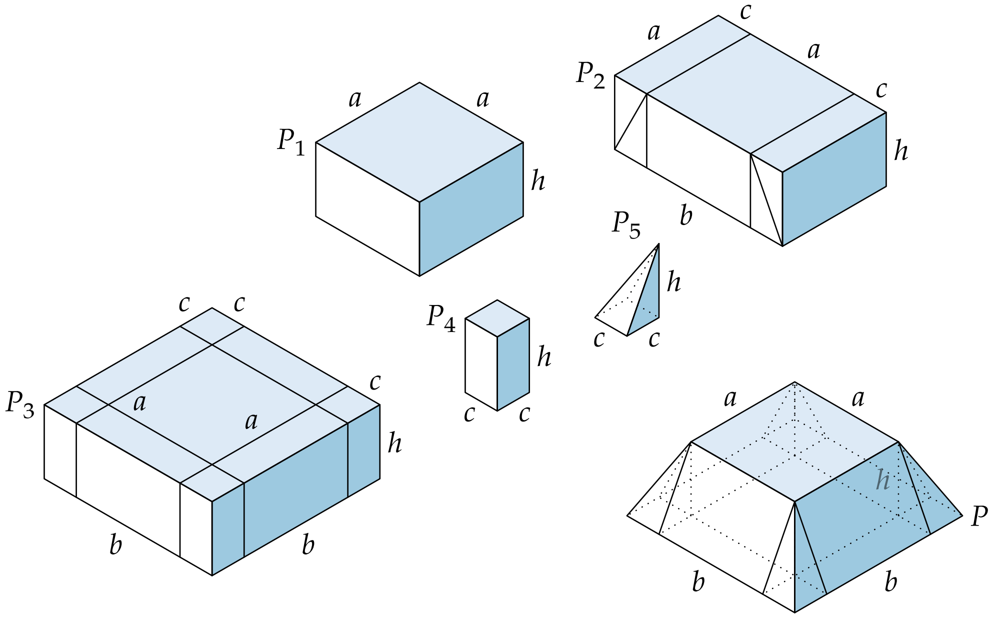
The length of the polygonal arch generated by one vertex of a regular n-gon rolling along a straight line is four times the length of the in-radius plus four times the length of the circum-radius of the n-gon.



Corollary: The arc length of one arch of a cycloid is eight times the radius of the generating circle.

— Philip R. Mallinson

The volume of a frustrum of a square pyramid



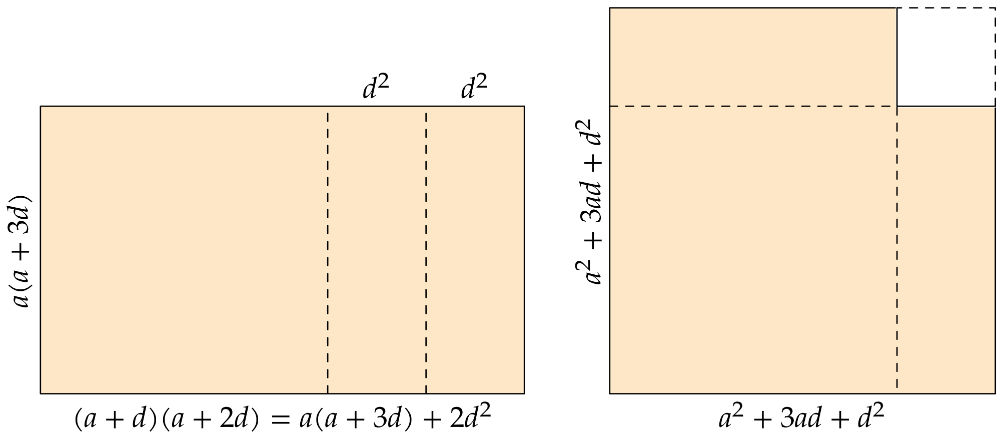
$$P_4 = 3P_5$$

$$P_1 + P_3 = 2P_2 + 4P_4 \quad \Rightarrow \quad P_1 + P_2 + P_3 = 3P_2 + 12P_5 = 3(P_2 + 4P_5) = 3P$$

$$\therefore \quad V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

— Sidney J. Kung

The product of four (positive) numbers in arithmetic progression is always the difference of two squares

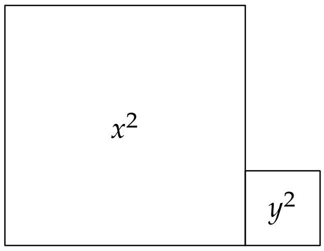


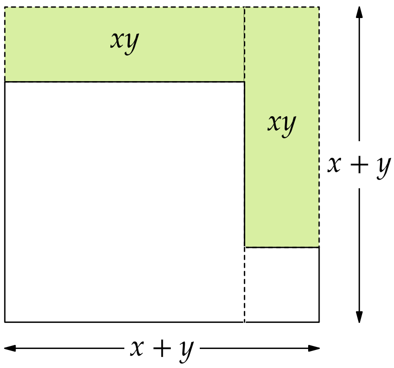
$$a(a+d)(a+2d)(a+3d) = \left(a^2 + 3ad + d^2\right)^2 - \left(d^2\right)^2$$

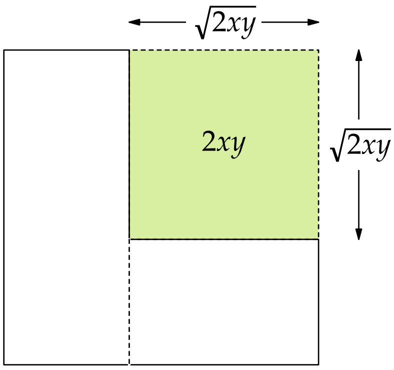
— RBN

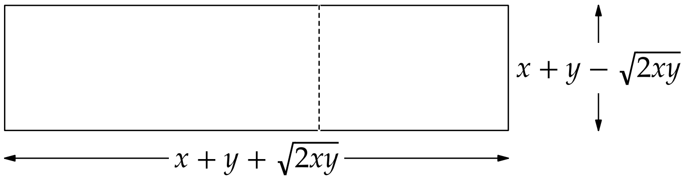
Algebraic areas III: Factoring the sum of two squares

$$x^2 + y^2 = \left(x + \sqrt{2xy} + y\right)\left(x - \sqrt{2xy} + y\right)$$



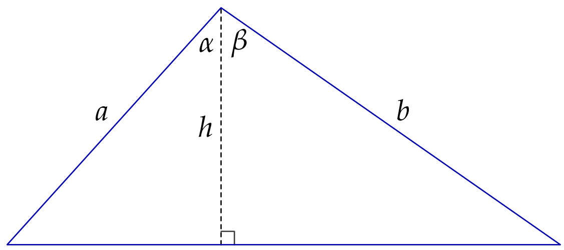




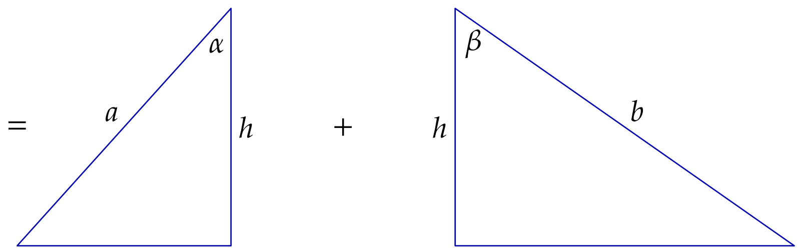


Triognometry, Calculus, & Analytic Geometry

Sine of the sum - II



$$\alpha, \beta \in (0, \pi/2) \quad \Longrightarrow \quad h = a\cos\alpha = b\cos\beta$$



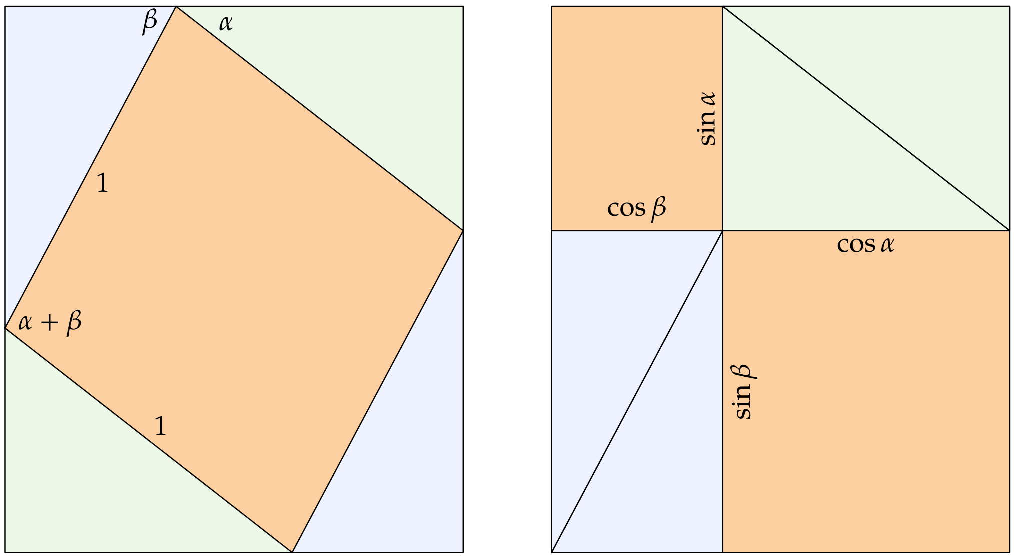
$$\begin{aligned}
\frac{1}{2}ab\sin(\alpha+\beta) &= \frac{1}{2}ah\sin\alpha + \frac{1}{2}bh\sin\beta \\
&= \frac{1}{2}ab\cos\beta\sin\alpha + \frac{1}{2}ba\cos\alpha\sin\beta \\
\therefore \quad \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}$$

— Christopher Brüningsen

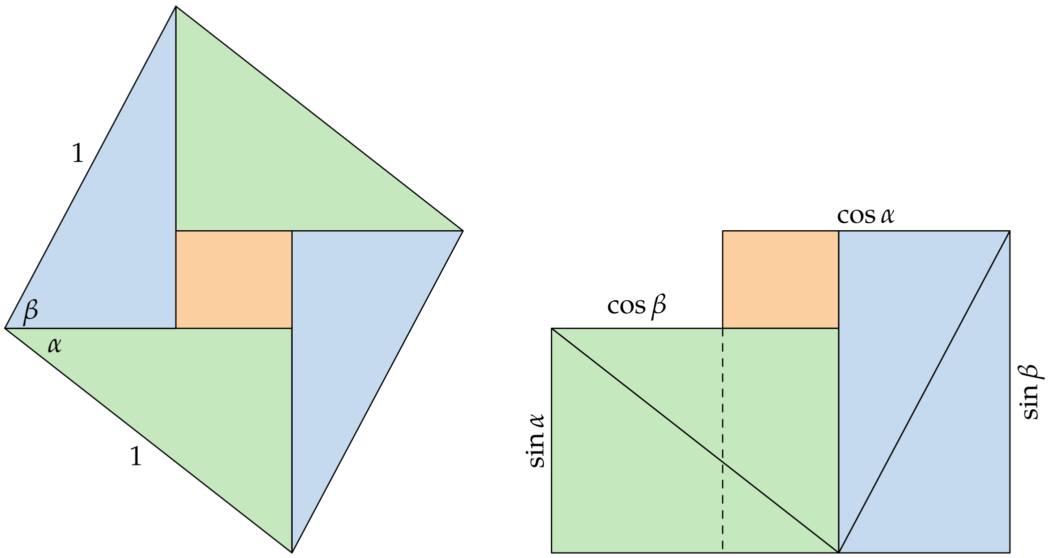
Sine of the sum – III

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

I.

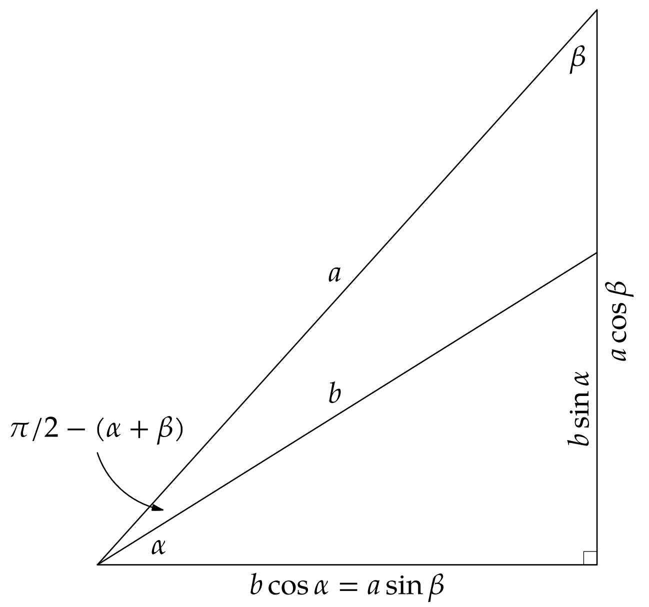


II.



— Volker Priebe and Edgar A. Ramos

Cosine of the sum



$$\frac{1}{2}ab\sin\left(\frac{\pi}{2}-(\alpha+\beta)\right) = \frac{1}{2}b\cos\alpha\cdot a\cos\beta - \frac{1}{2}b\sin\alpha\cdot a\sin\beta$$

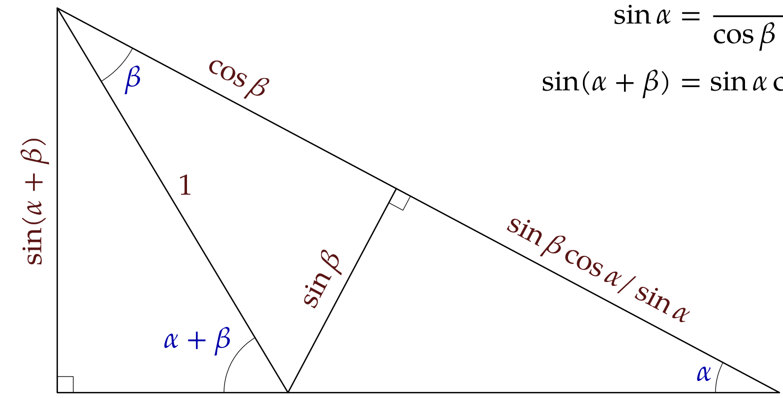
$$\therefore\quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

— Sidney H. Kung

Geometry of addition formulas

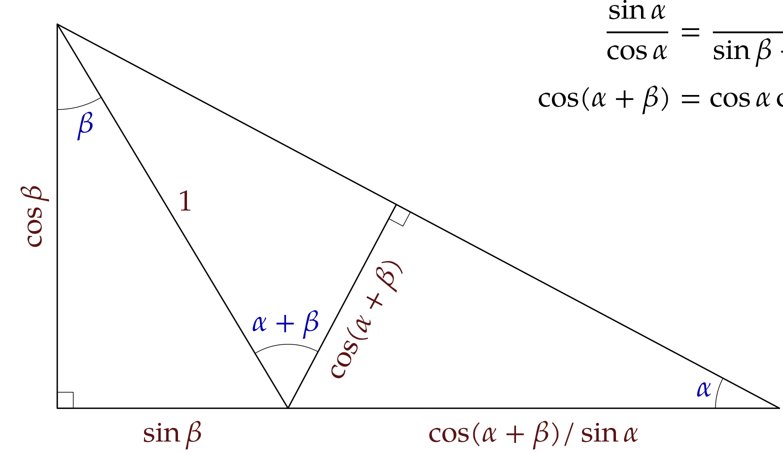
$$\sin\alpha = \frac{\sin(\alpha+\beta)}{\cos\beta + \sin\beta\cos\alpha/\sin\alpha}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$



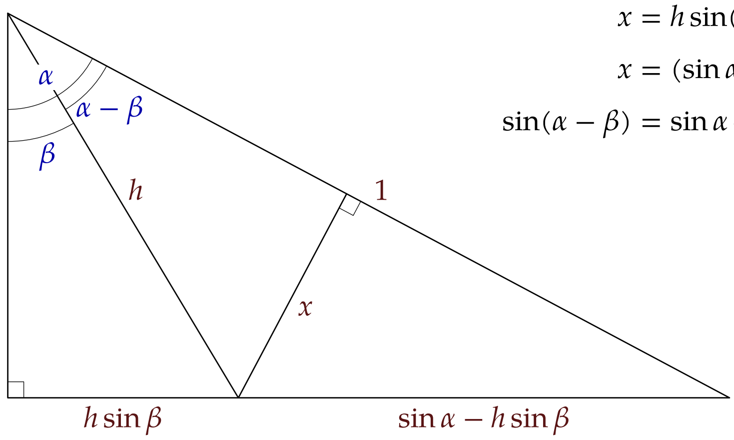
$$\frac{\sin\alpha}{\cos\alpha} = \frac{\cos\beta}{\sin\beta + \cos(\alpha+\beta)/\sin\alpha}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

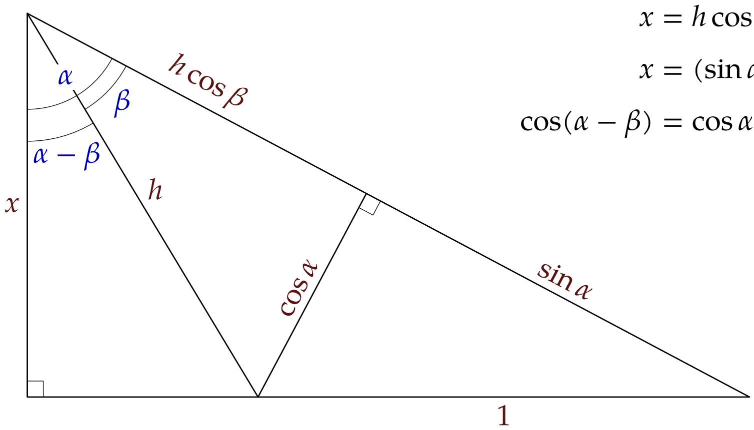


— Leonard M. Smiley

Geometry of subtraction formulas



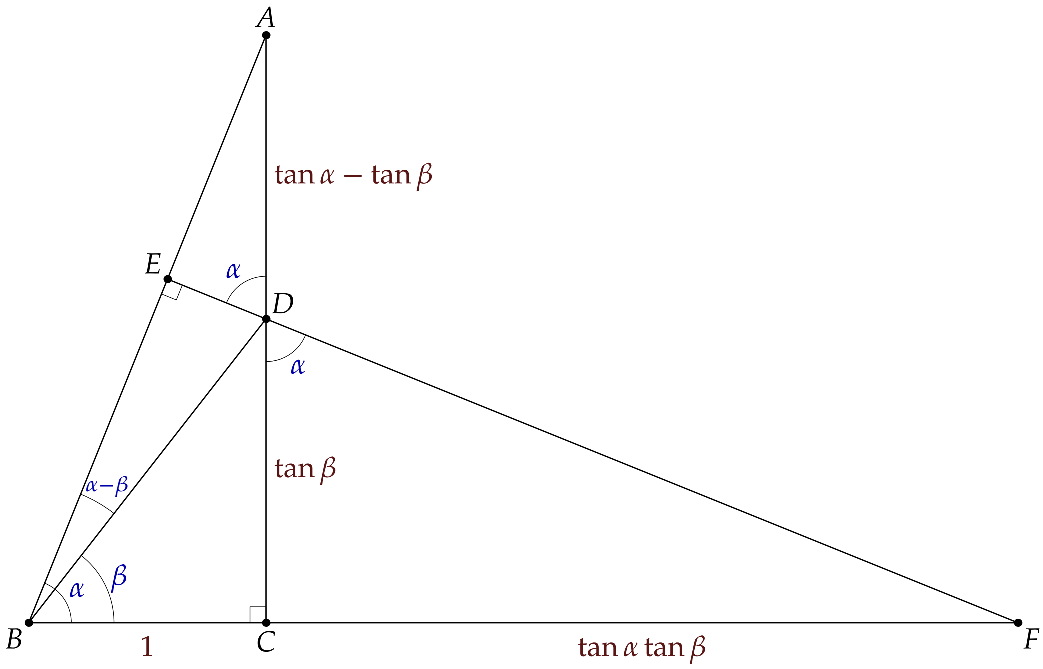
$$h = \cos\alpha / \cos\beta$$
$$x = h \sin(\alpha - \beta)$$
$$x = (\sin\alpha - h\sin\beta)\cos\alpha$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$



$$h = \cos\alpha / \sin\beta$$
$$x = h \cos(\alpha - \beta)$$
$$x = (\sin\alpha + h\cos\beta)\cos\alpha$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

— Leonard M. Smiley

The difference identity for tangents I

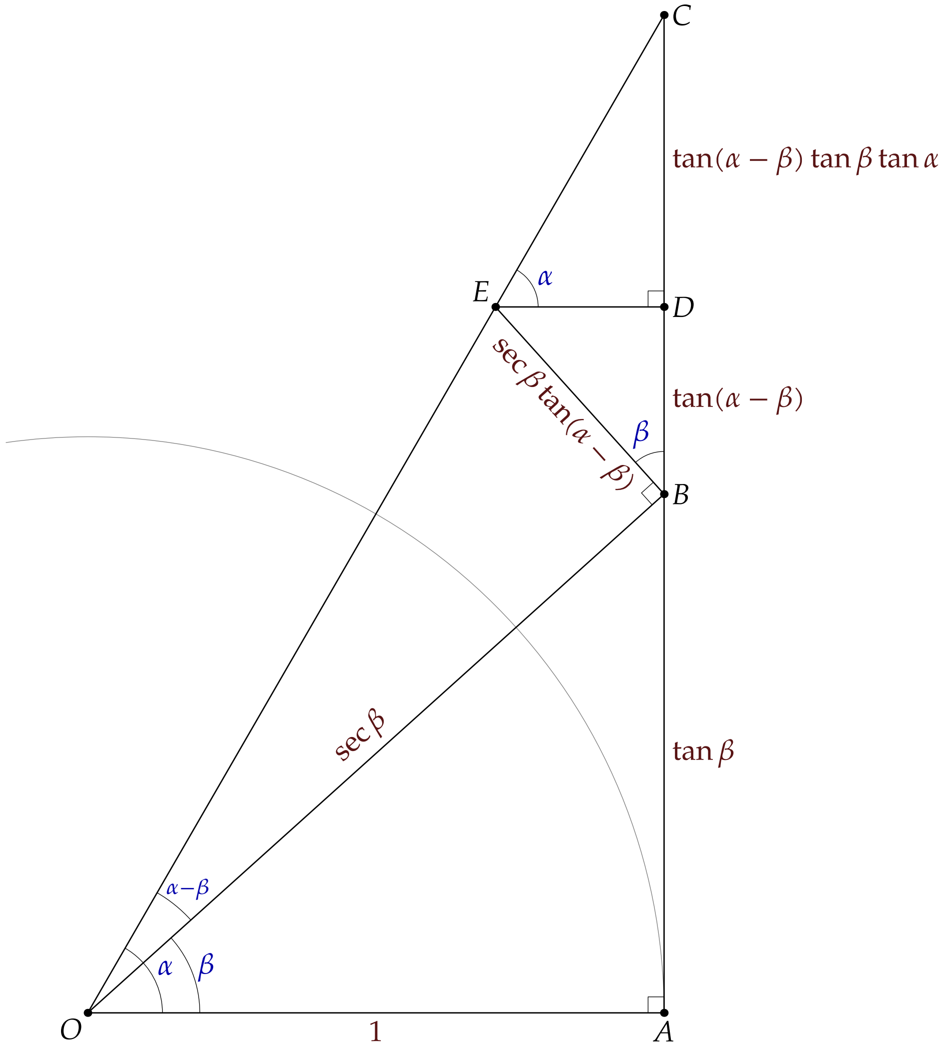


$$\frac{BF}{BE} = \frac{AD}{DE}$$

$$\therefore \quad \tan(\alpha-\beta) = \frac{DE}{BE} = \frac{AD}{BF} = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

— Guanshen Ren

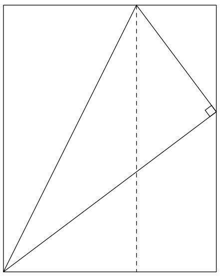
The difference identity for tangents II



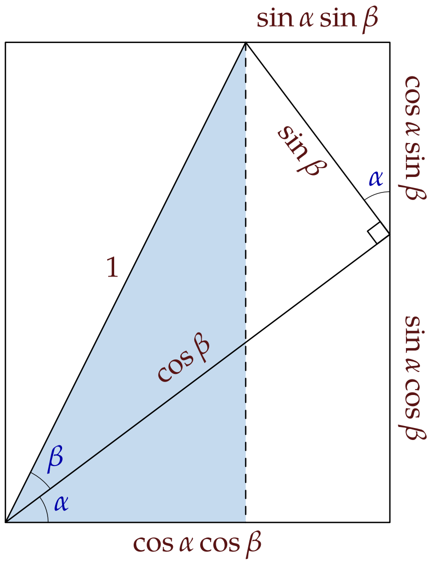
$$
\begin{aligned}
AC - AB &= BD + DC \\
\therefore \tan\alpha - \tan\beta &= \tan(\alpha-\beta) + \tan\alpha\tan\beta\tan(\alpha-\beta) \\
\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}
$$

— Fukuzo Suzuki

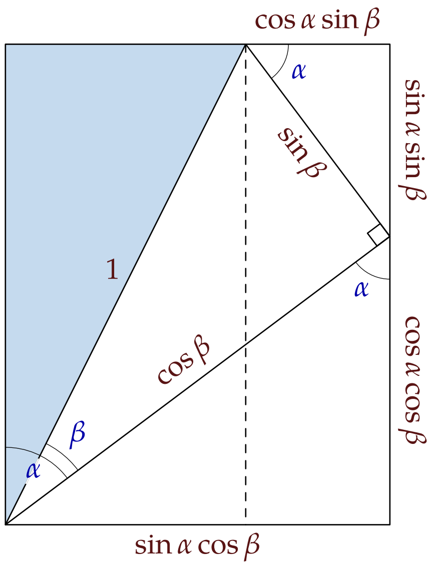
One figure, six identities



The figure

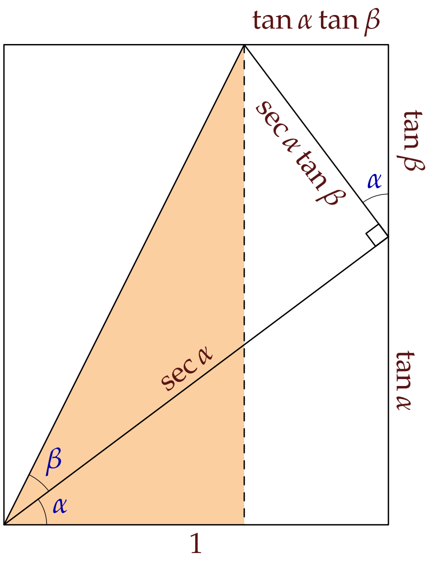


$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

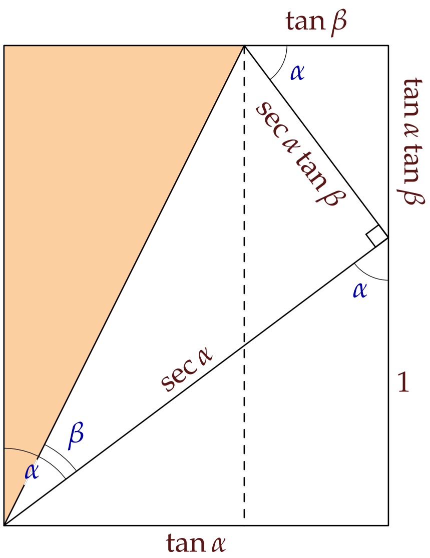


$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 + \tan\alpha\tan\beta}$$

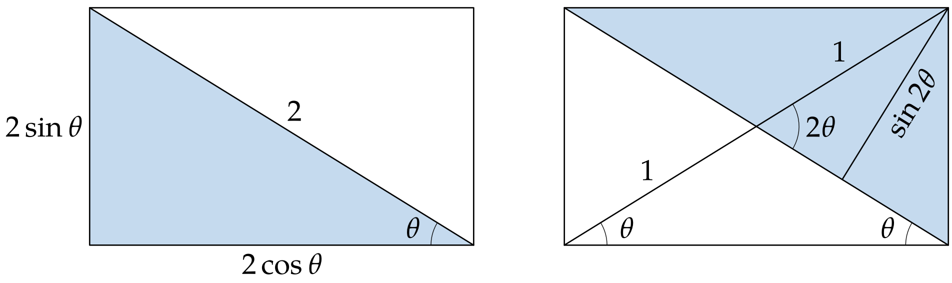


$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

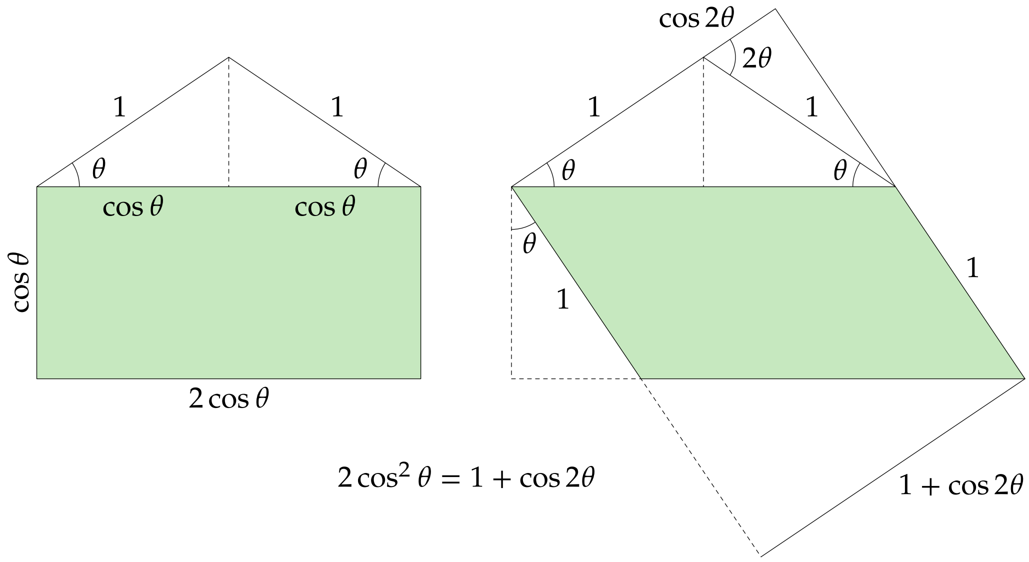


— RBN

The double-angle formulas II



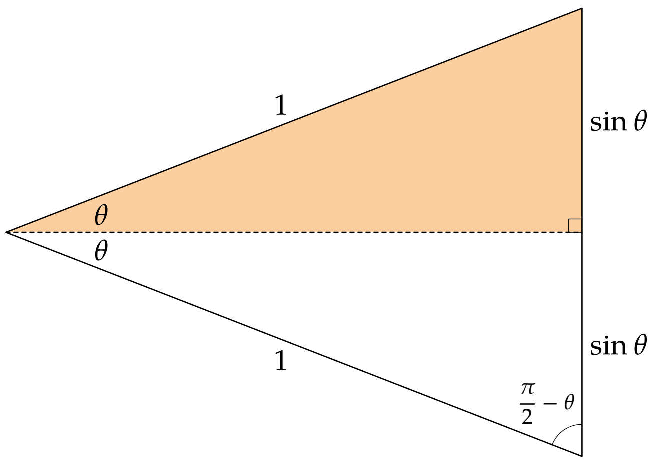
$$2\sin\theta\cos\theta = \sin 2\theta$$



$$2\cos^2\theta = 1 + \cos 2\theta$$

— Yihnan David Gau

The double-angle formulas III (via the laws of sines and cosines)

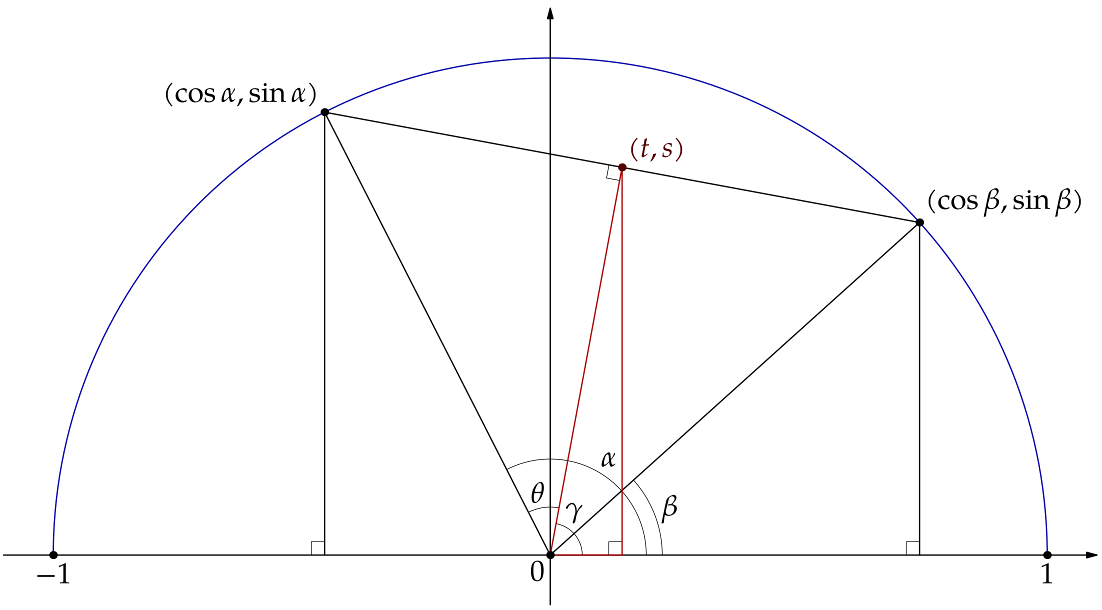


$$\frac{\sin 2\theta}{2\sin\theta} = \frac{\sin(\pi/2-\theta)}{1} = \cos\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$(2\sin\theta)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos 2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

— Sidney H. Kung

The sum-to-product identities I



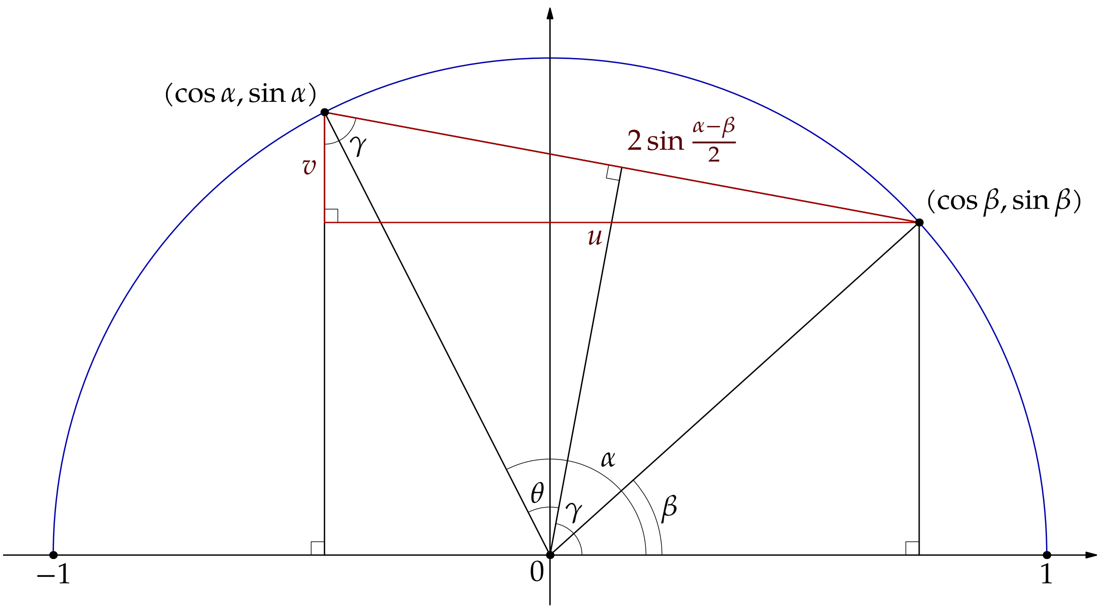
$$\theta = \frac{\alpha-\beta}{2}, \quad \gamma = \frac{\alpha+\beta}{2}$$

$$\frac{\sin\alpha+\sin\beta}{2} = s = \cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

$$\frac{\cos\alpha+\cos\beta}{2} = t = \cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

— Sidney H. Kung

The difference-to-product identities I



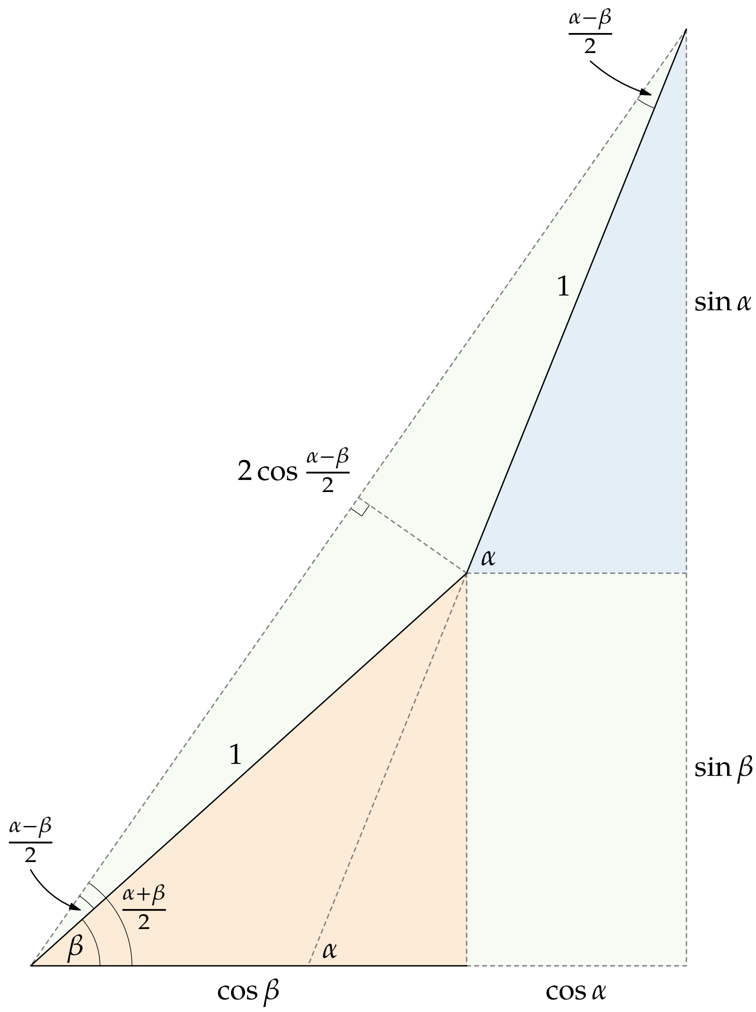
$$\theta = \frac{\alpha-\beta}{2}, \quad \gamma = \frac{\alpha+\beta}{2}$$

$$\sin\alpha - \sin\beta = v = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\beta - \cos\alpha = u = 2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

— Sidney H. Kung

The sum-to-product identities II

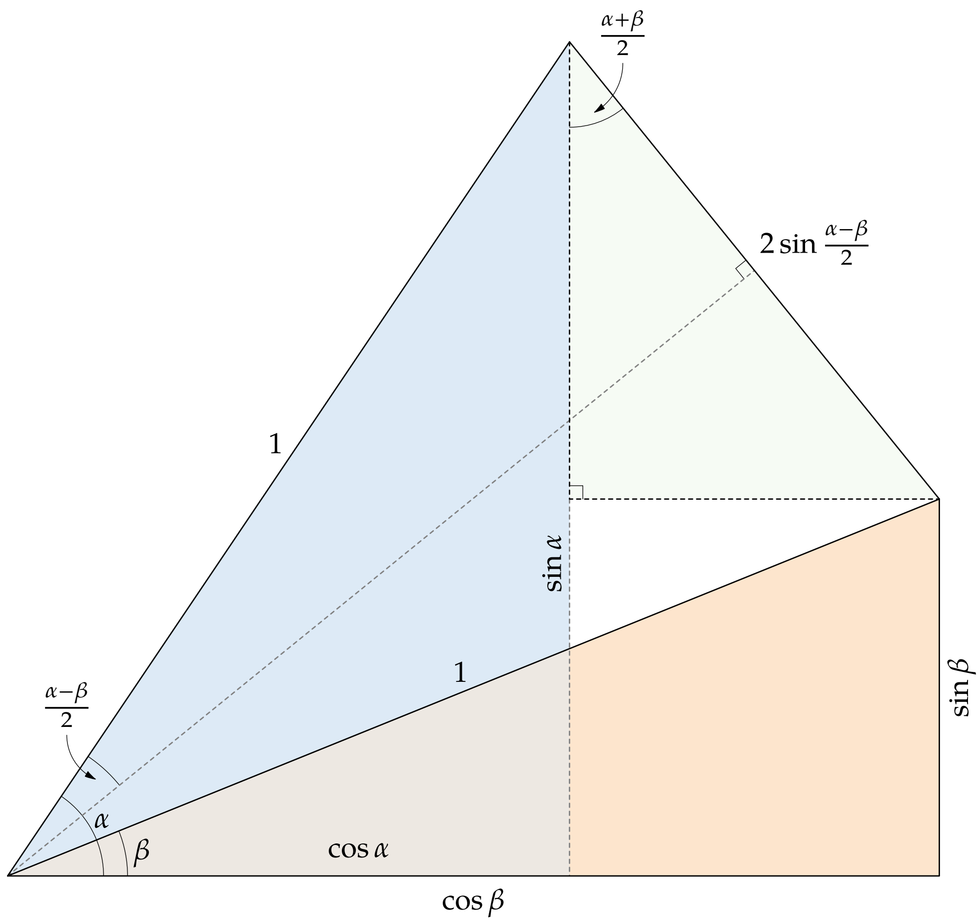


$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha-\beta}{2}\cos\tfrac{\alpha+\beta}{2}$$

$$\sin\alpha + \sin\beta = 2\cos\tfrac{\alpha-\beta}{2}\sin\tfrac{\alpha+\beta}{2}$$

— Yukio Kobayashi

The difference-to-product identities II

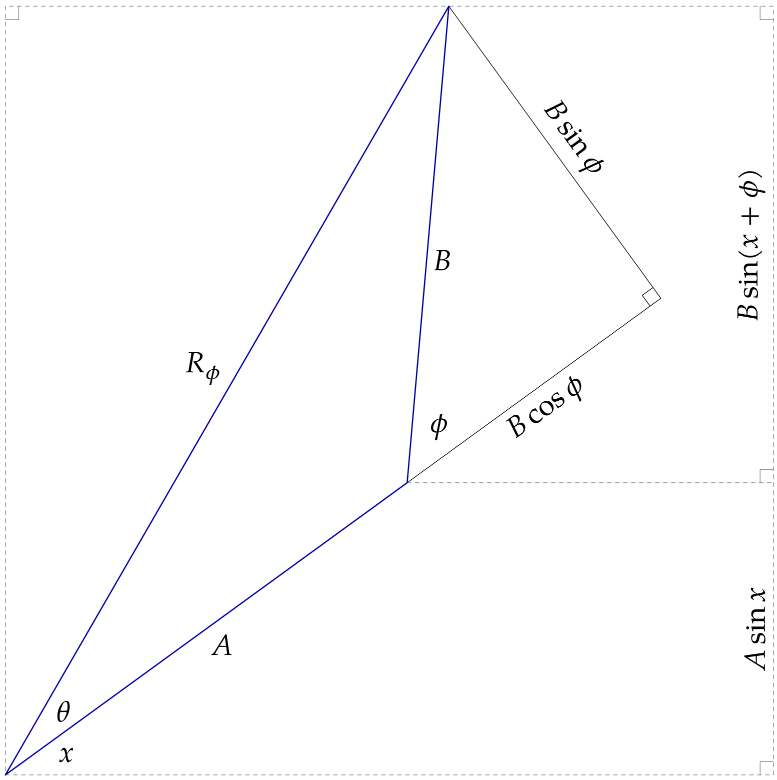


$$\cos\beta - \cos\alpha = 2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

— Yukio Kobayashi

Adding like sines



$$R_\phi = \sqrt{A^2 + B^2 + 2AB\cos\phi}, \quad \tan\theta = \tfrac{B\sin\phi}{A + B\cos\phi}$$

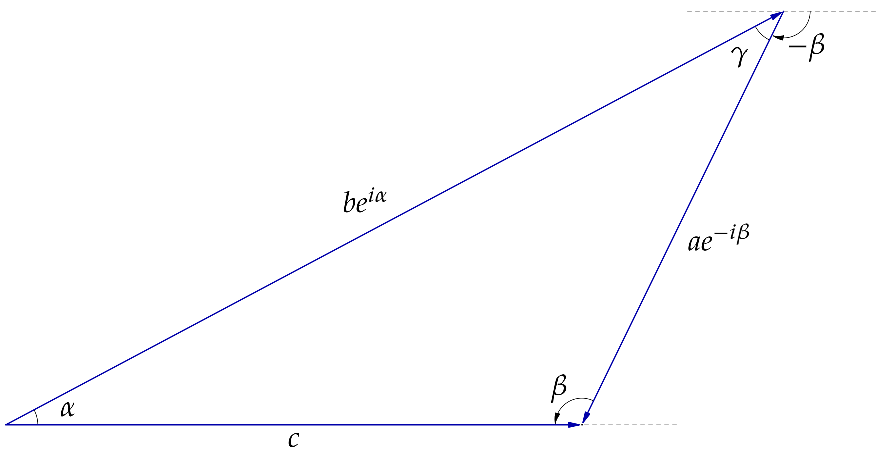
$$A\sin x + B\sin(x+\phi) = R_\phi \sin(x+\theta)$$

$$\phi = \pi/2 \Rightarrow \tan\theta = B/A$$

$$\therefore \quad A\sin x + B\cos x = \sqrt{A^2+B^2}\,\sin(x+\theta)$$

— Rick Mabry and Paul Deiermann

A complex approach to the laws of sines and cosines



$$c = be^{i\alpha} + ae^{-i\beta} = (b\cos\alpha + a\cos\beta) + i(b\sin\alpha - a\sin\beta)$$

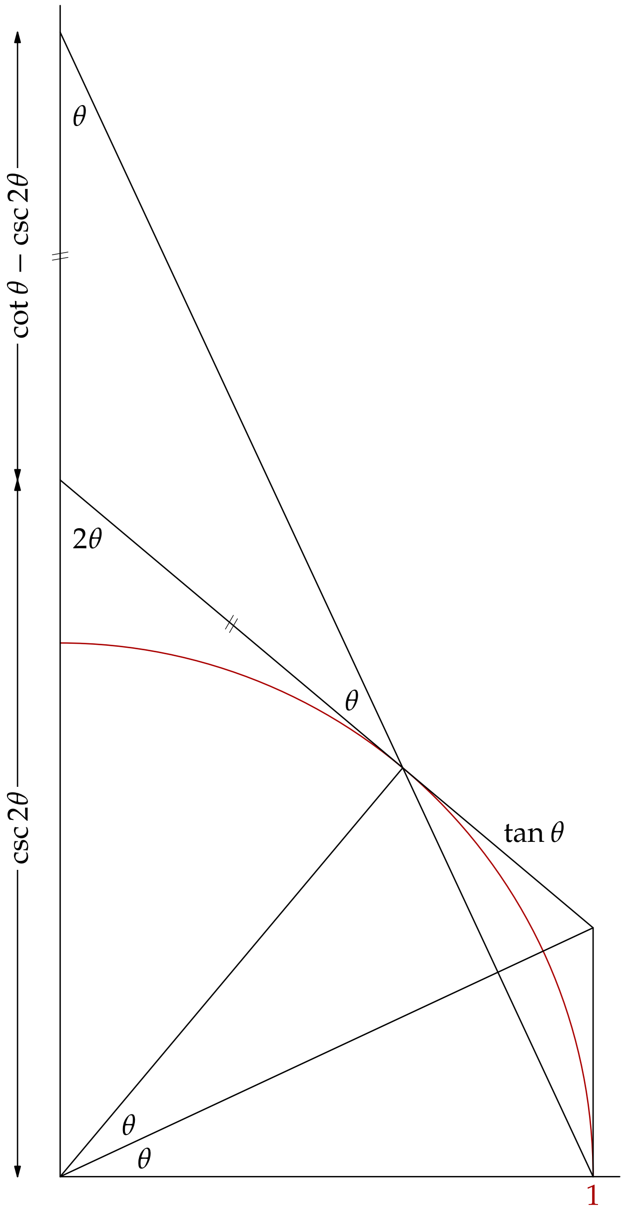
$$\text{if } c \text{ is real, then } b\sin\alpha - a\sin\beta = 0\text{, hence } \frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\begin{aligned} c^2 = |c^2| &= (b\cos\alpha + a\cos\beta)^2 + (b\sin\alpha - a\sin\beta)^2 \\ &= a^2 + b^2 + 2ab\cos(\alpha + \beta) \\ &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}$$

— William V. Grounds

Eisenstein's duplication forumula

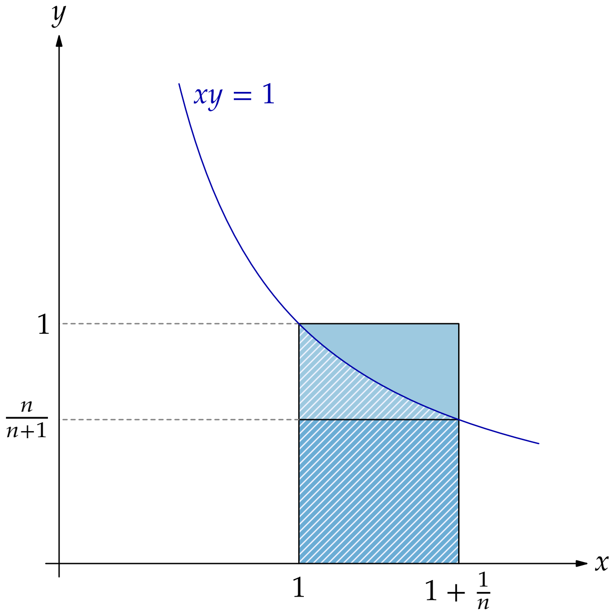
$$2\csc 2\theta = \tan\theta + \cot\theta$$



G. Eisenstein, *Mathematische Werke*, Chelsea, NY. 1975, p.411

A familiar limit for e

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$



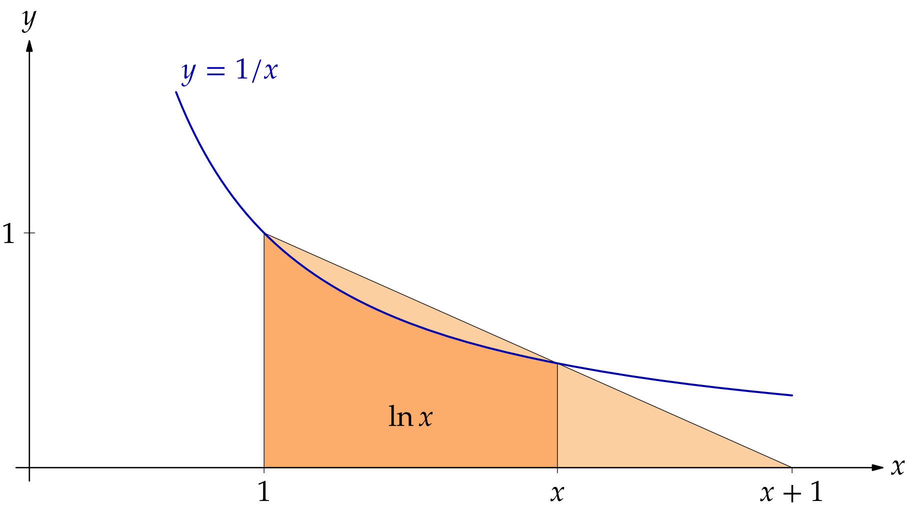
$$\frac{1}{n}\cdot\frac{n}{n+1} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}\cdot 1$$

$$\frac{n}{n+1} \le n\cdot\ln\left(1+\frac{1}{n}\right) \le 1$$

$$\therefore \quad \lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^n\right) = 1$$

A common limit

$$\lim_{x\to\infty} \frac{x}{e^x} = 0$$



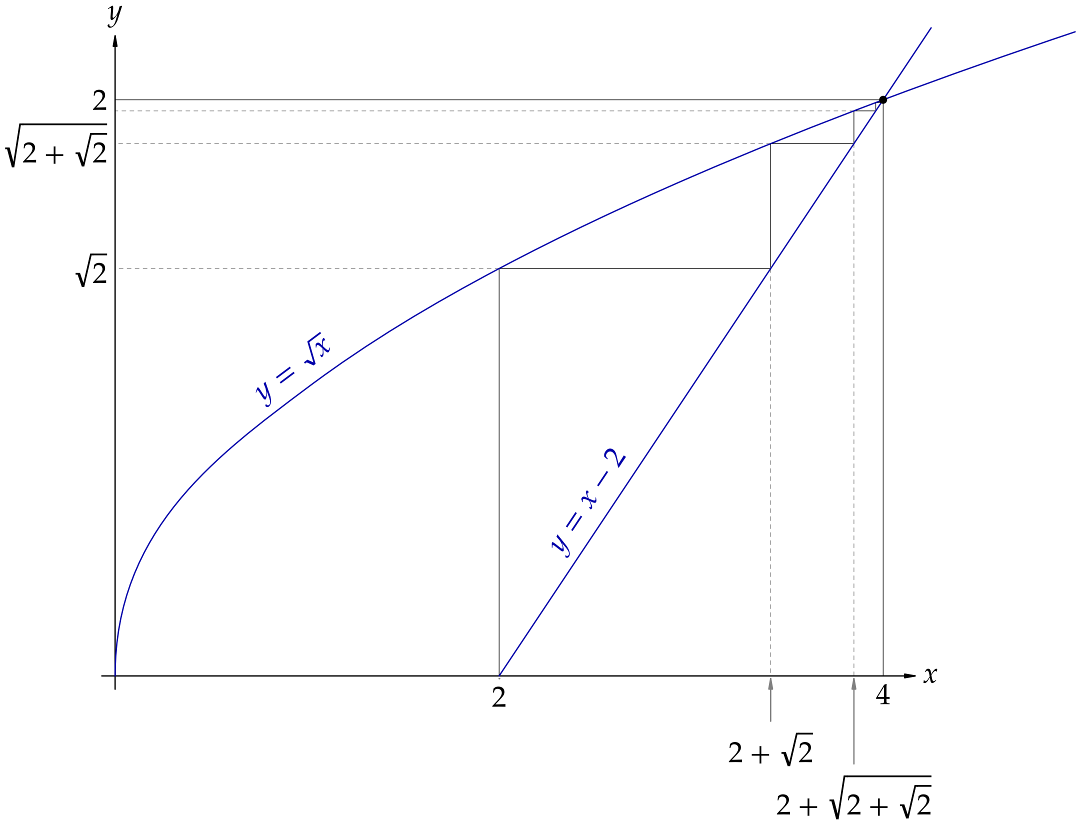
$$\ln x < \frac{1}{2}x$$

$$\therefore \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^{x-\ln x}} = 0$$

— Alan H. Stein and Dennis McGavran

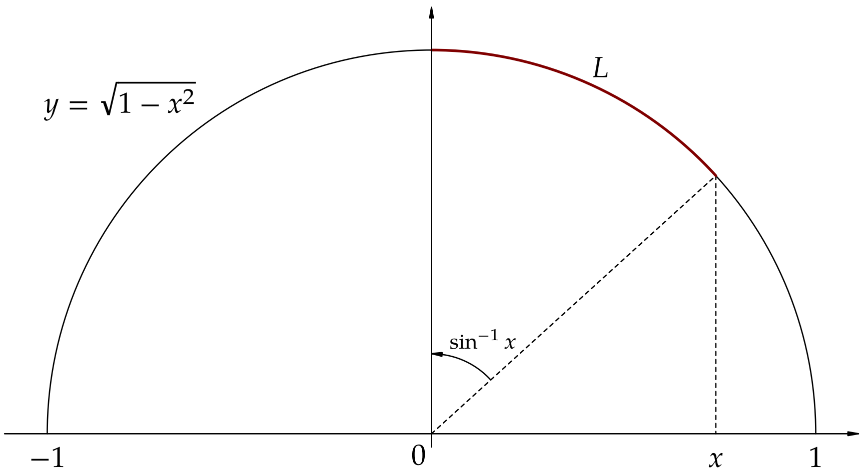
Geometric evaluation of a limit

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}} = 2$$



— Guanshen Ren

The derivative of the inverse sine



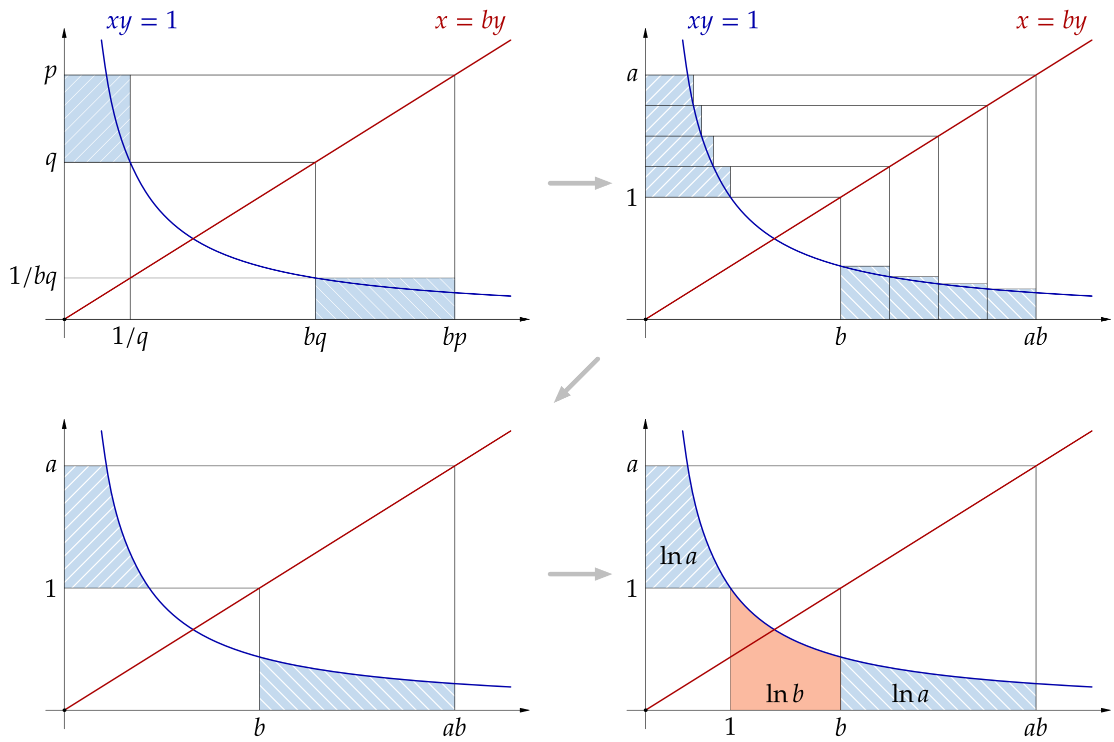
$$L = \sin^{-1} x = \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt$$

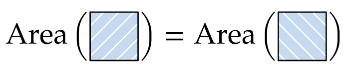
$$\therefore \quad \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

— Craig Johnson

The logarithm of a product

$$\ln ab = \ln a + \ln b$$

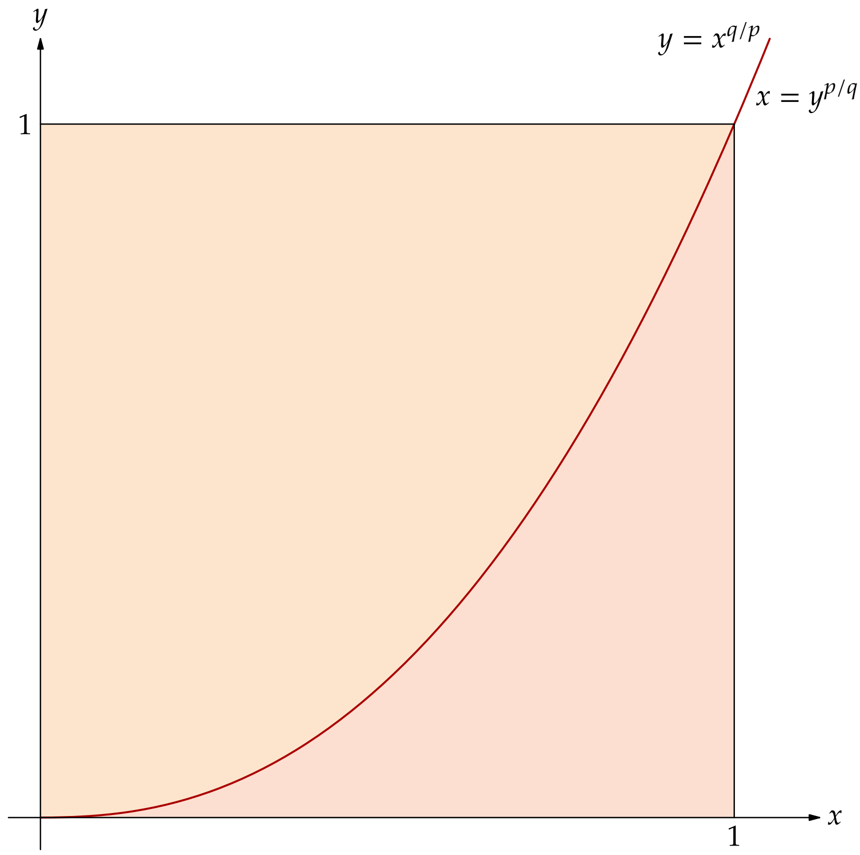




— Jeffery Ely

An integral of a sum of reciprocal powers

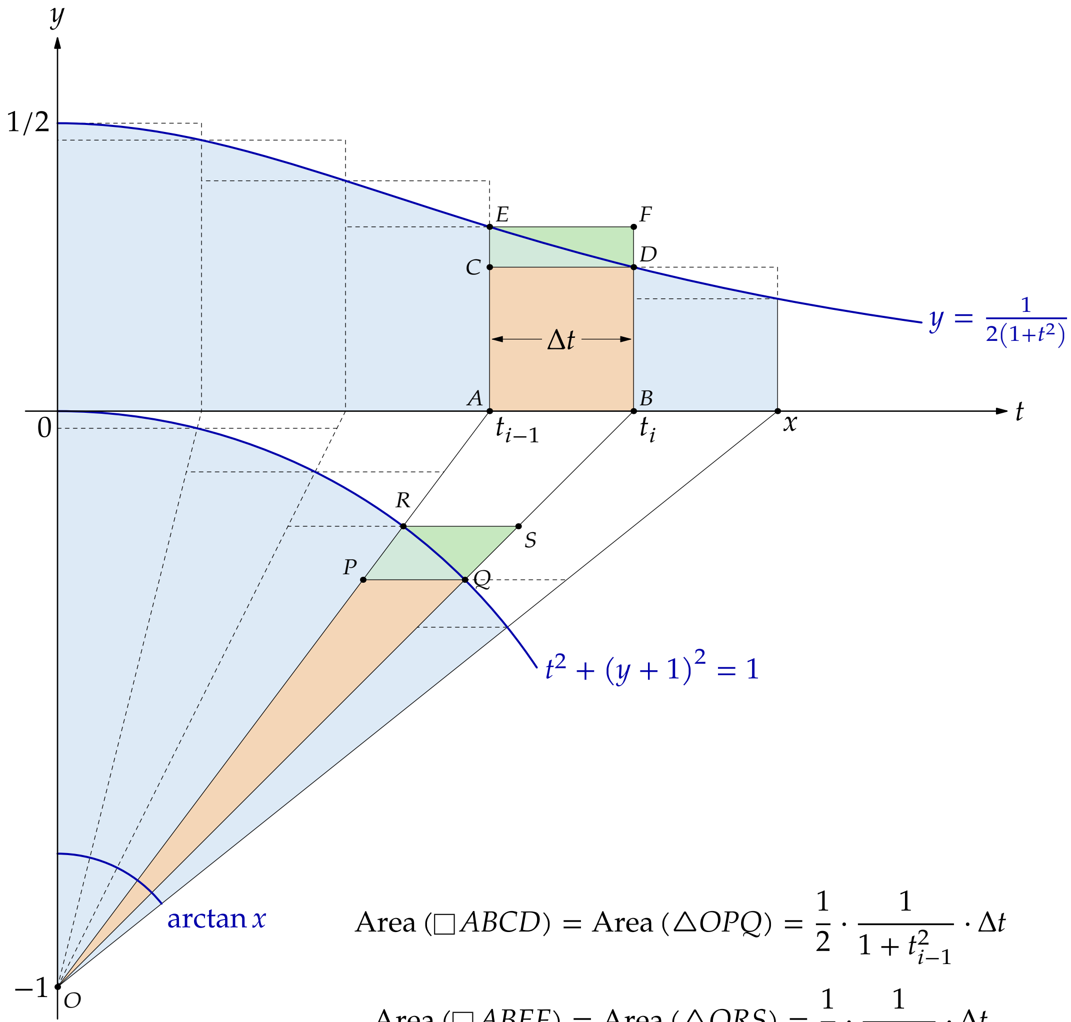
$$\int_0^1 \left(t^{p/q} + t^{q/p}\right)\, dt = 1$$



— Peter R. Newbury

The arctangent integral

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$



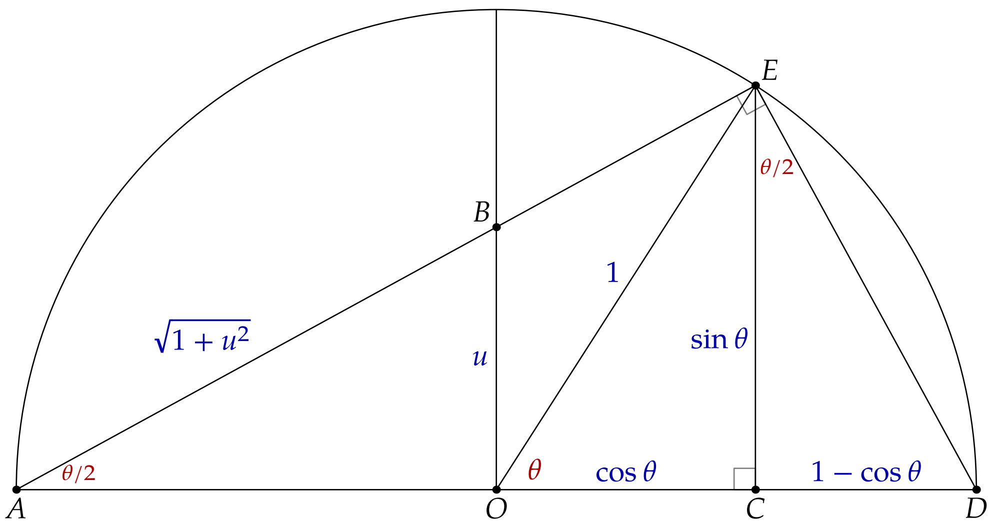
$$\text{Area}\,(\square ABCD) = \text{Area}\,(\triangle OPQ) = \frac{1}{2}\cdot\frac{1}{1+t_{i-1}^2}\cdot\Delta t$$

$$\text{Area}\,(\square ABEF) = \text{Area}\,(\triangle ORS) = \frac{1}{2}\cdot\frac{1}{1+t_{i}^2}\cdot\Delta t$$

$$\therefore\quad \frac{1}{2}\arctan x = \int_0^x \frac{1}{2\left(1+t^2\right)}\,dt$$

— Aage Bondesen

The method of last resort — Weierstrass substitution



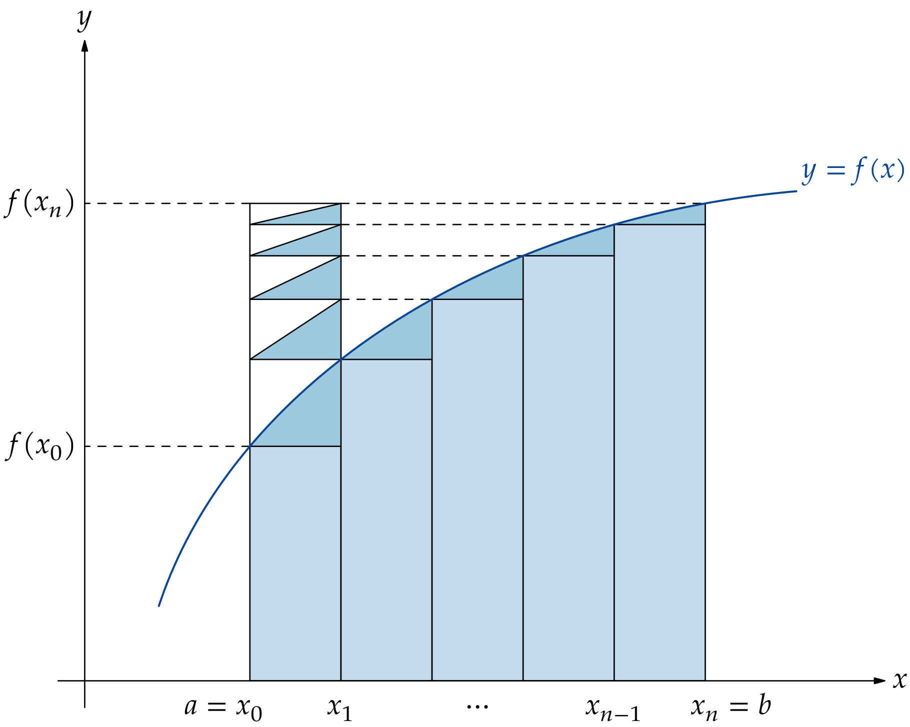
$$u = \tan\frac{\theta}{2}, \quad DE = 2\sin\frac{\theta}{2} = \frac{2u}{\sqrt{1+u^2}}$$

$$\frac{CE}{DE} = \frac{OA}{BA} \quad \Longrightarrow \quad \sin\theta = \frac{2u}{1+u^2}$$

$$\frac{CD}{DE} = \frac{OB}{BA} \quad \Longrightarrow \quad \cos\theta = \frac{1-u^2}{1+u^2}$$

— Paul Deiermann

The trapezoidal rule — for increasing functions

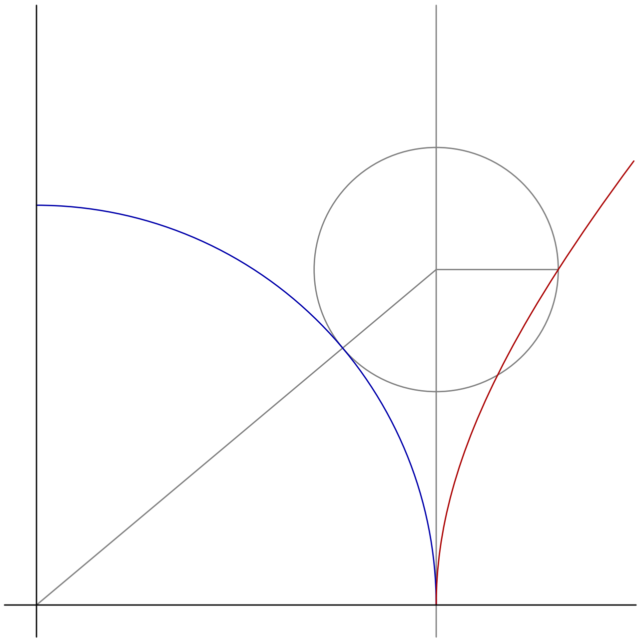


$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)\,\frac{b-a}{n} + \frac{1}{2}\Big(f(x_n) - f(x_0)\Big)\frac{b-a}{n}$$

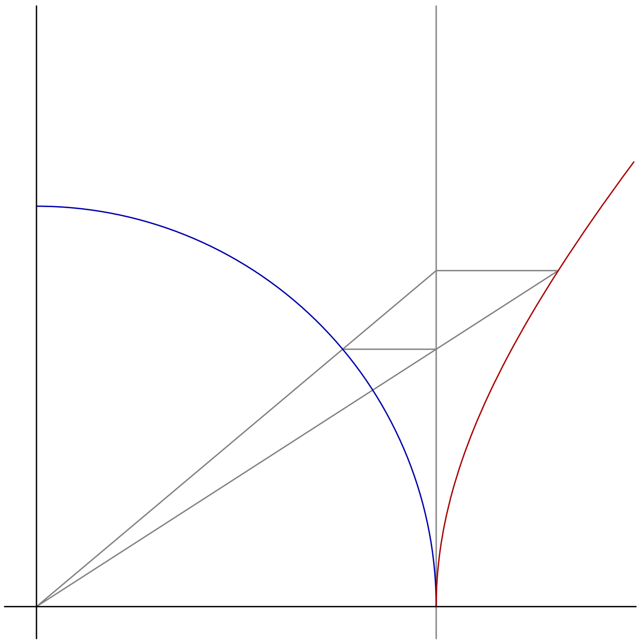
— Jesús Urías

Construction of a hyperbola

I.

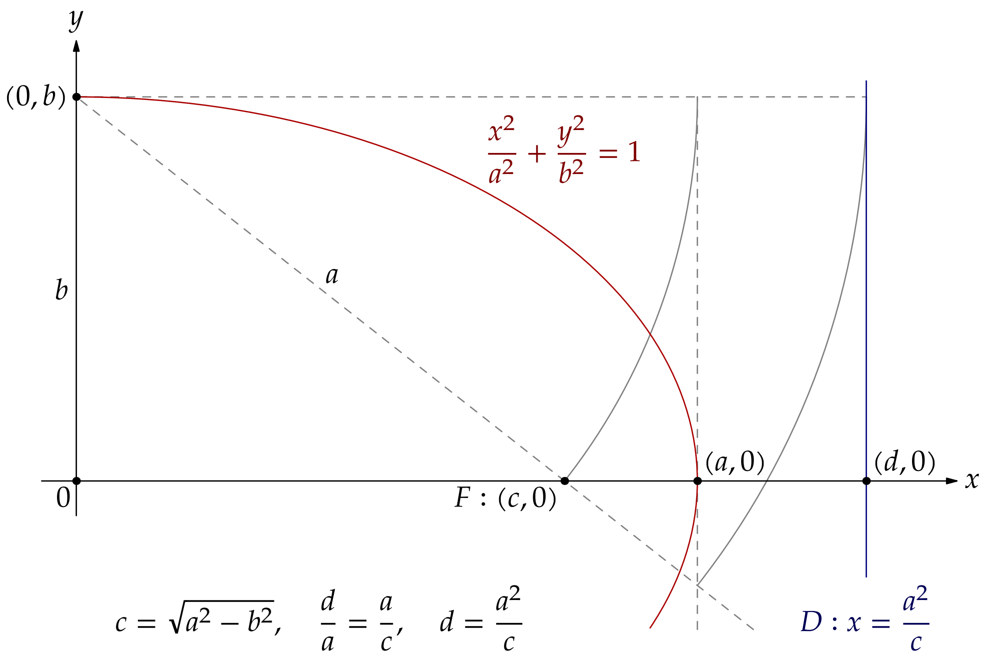


II.



— Ernest J. Eckert

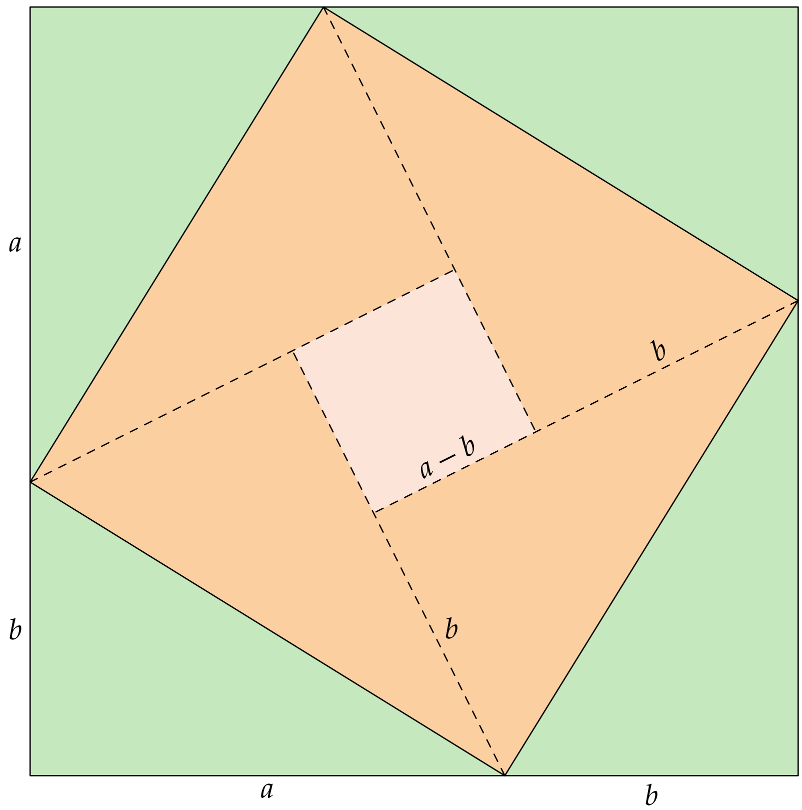
The focus and directrix of an ellipse



— Michel Bataille

Inequalities

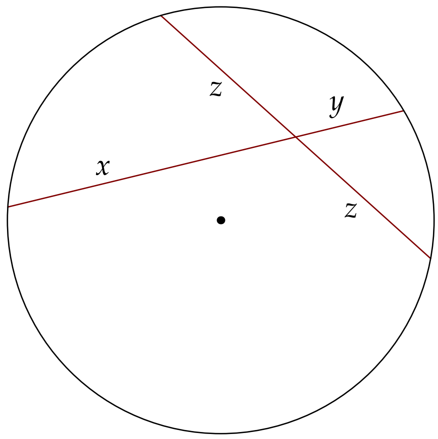
The arithmetic mean – geometric mean inequality IV



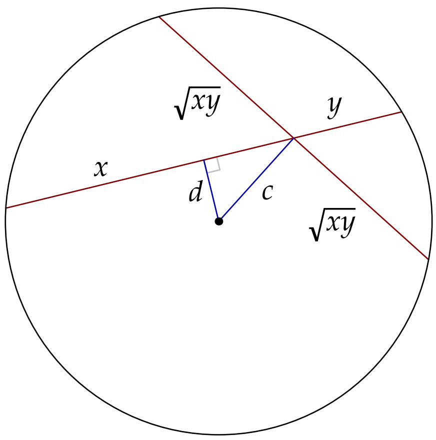
$$(a+b)^2 \geq 4ab \quad \Longrightarrow \quad \frac{a+b}{2} \geq \sqrt{ab}$$

— Ayoub B. Ayoub

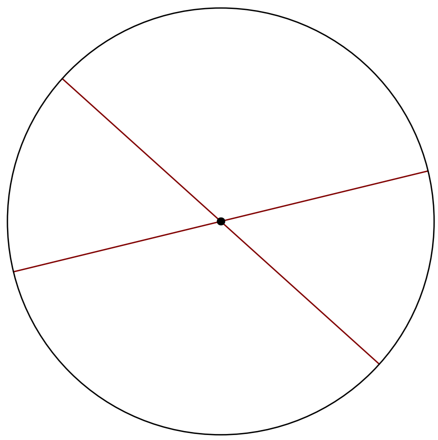
The arithmetic mean – geometric mean inequality V



$$z^2 = xy$$



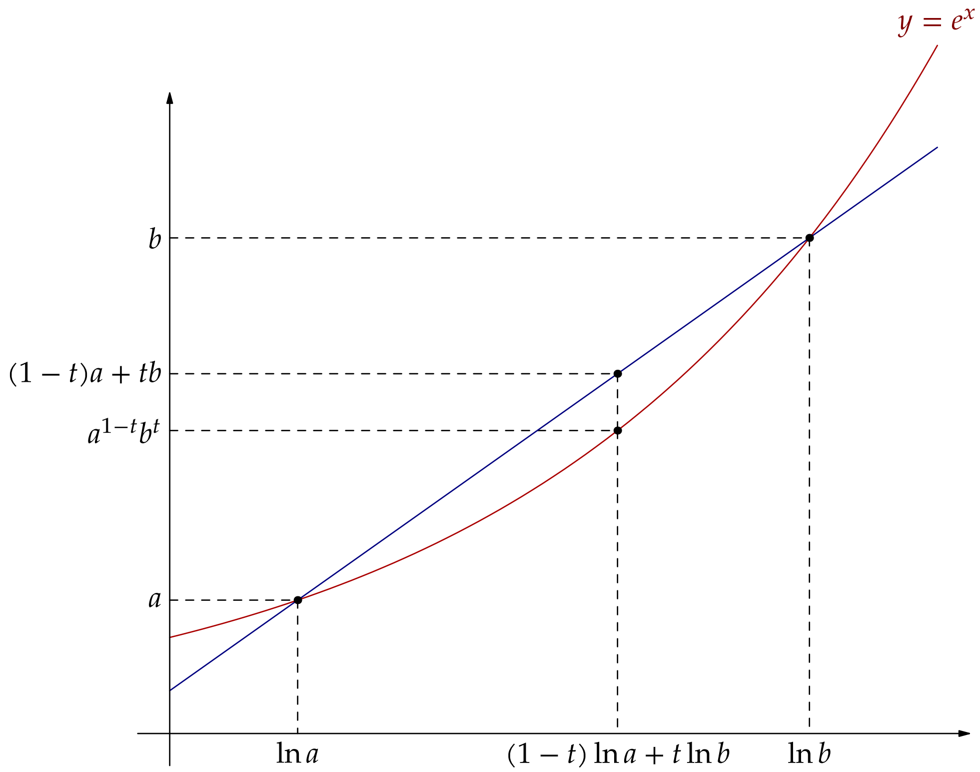
$$d < c \quad \Longrightarrow \quad x + y > 2\sqrt{xy}$$



$$d = c = 0 \quad \Longrightarrow \quad x + y = 2\sqrt{xy}$$

— Sidney H. Kung

The arithmetic mean – geometric mean inequality VI



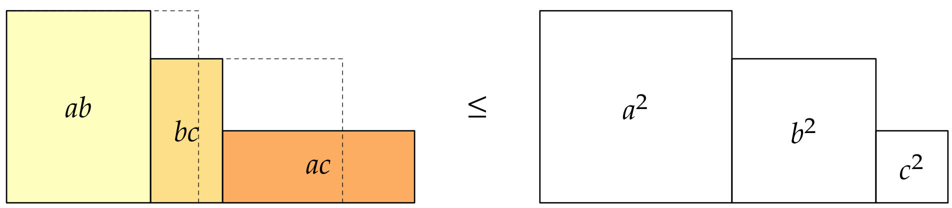
$$0 < a < b, 0 < t < 1 \quad \Rightarrow \quad (1-t)a + tb > a^{1-t}b^t$$

$$t = \frac{1}{2} \quad \Rightarrow \quad \frac{a+b}{2} > \sqrt{ab}$$

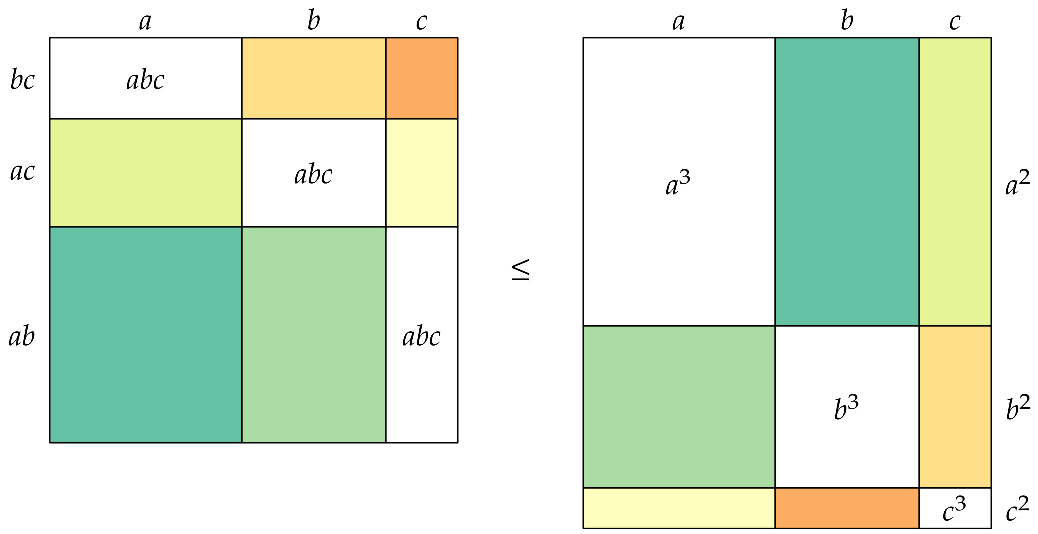
— Michael K. Brozinsky

The arithmetic mean – geometric mean inequality for three positive numbers

LEMMA: $ab + bc + ac \leq a^2 + b^2 + c^2$



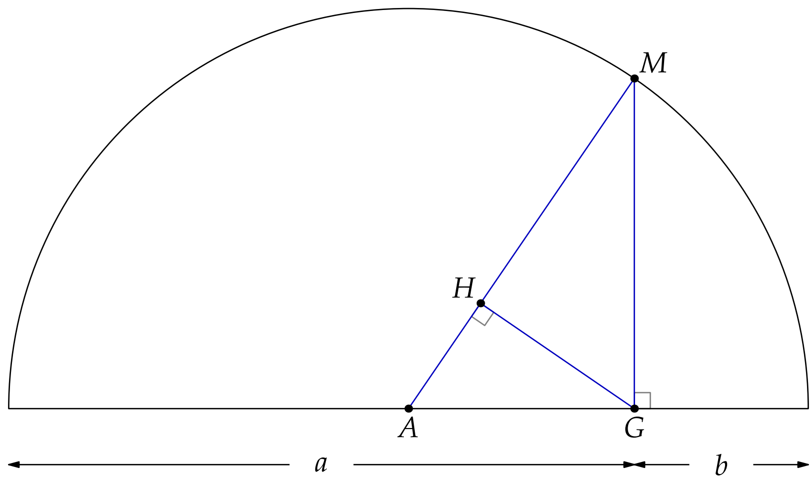
THEOREM: $3abc \leq a^3 + b^3 + c^3$



— Claudi Alsina

The arithmetic-geometric-harmonic mean inequality

$$a, b > 0 \quad \Longrightarrow \quad \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$



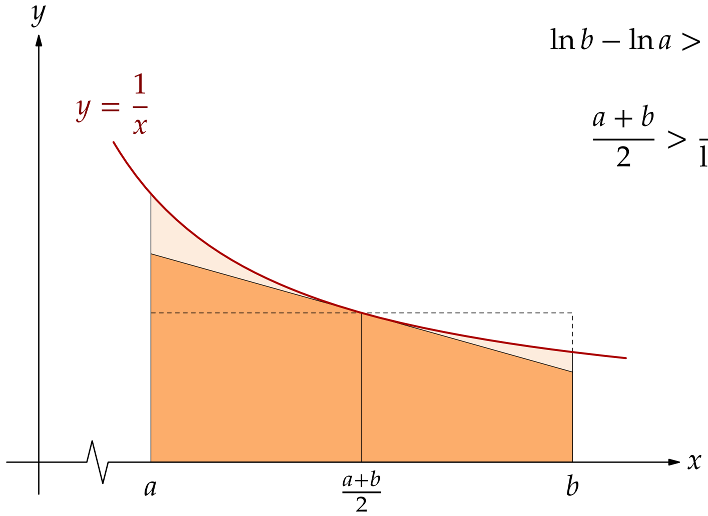
$$\overline{AM} = \frac{a+b}{2}, \quad \overline{GM} = \sqrt{ab}, \quad \overline{HM} = \frac{2ab}{a+b},$$

$$\overline{AM} \geq \overline{GM} \geq \overline{HM}.$$

— Pappus of Alexandria (circa A.D. 320)

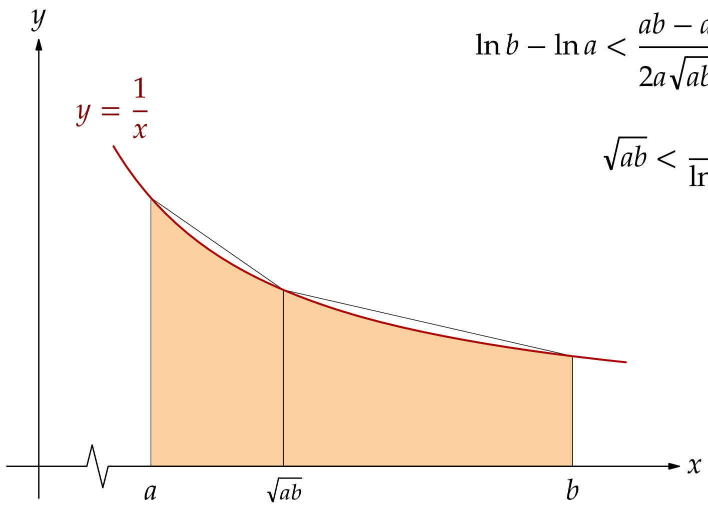
The arithmetic-logarithmic-geometric mean inequality

$$b > a > 0 \implies \frac{a+b}{2} > \frac{b-a}{\ln b - \ln a} > \sqrt{ab}$$



$$\ln b - \ln a > \frac{2}{a+b}(b-a)$$

$$\frac{a+b}{2} > \frac{b-a}{\ln b - \ln a}$$



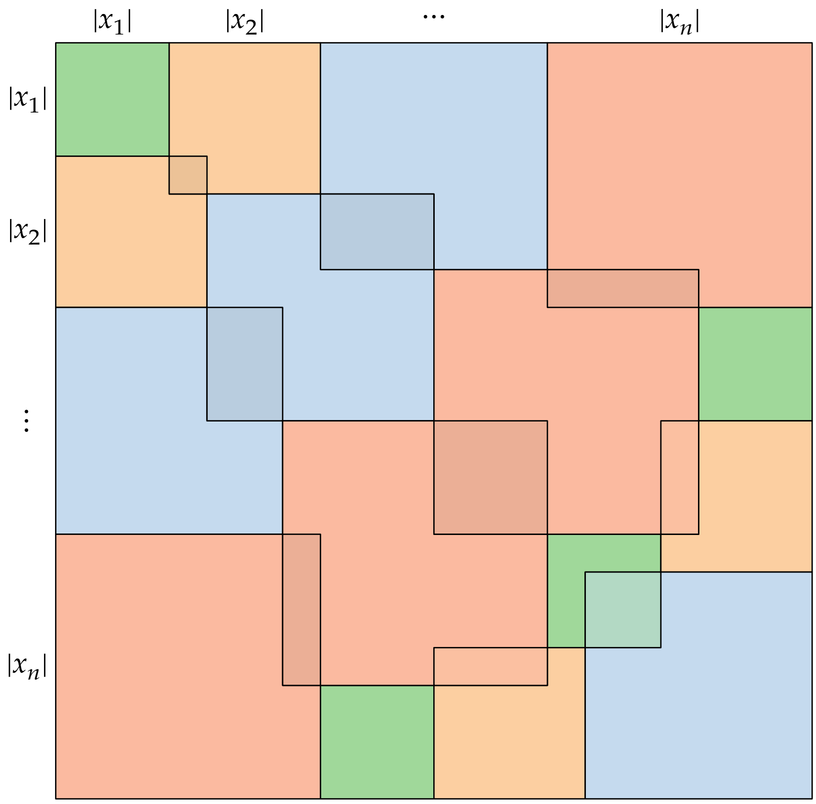
$$\ln b - \ln a < \frac{ab-a}{2a\sqrt{ab}} + \frac{a-ab}{2b\sqrt{ab}} = \frac{b-a}{\sqrt{ab}}$$

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a}$$

— RBN

The mean of the squares exceeds the square of the mean

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$



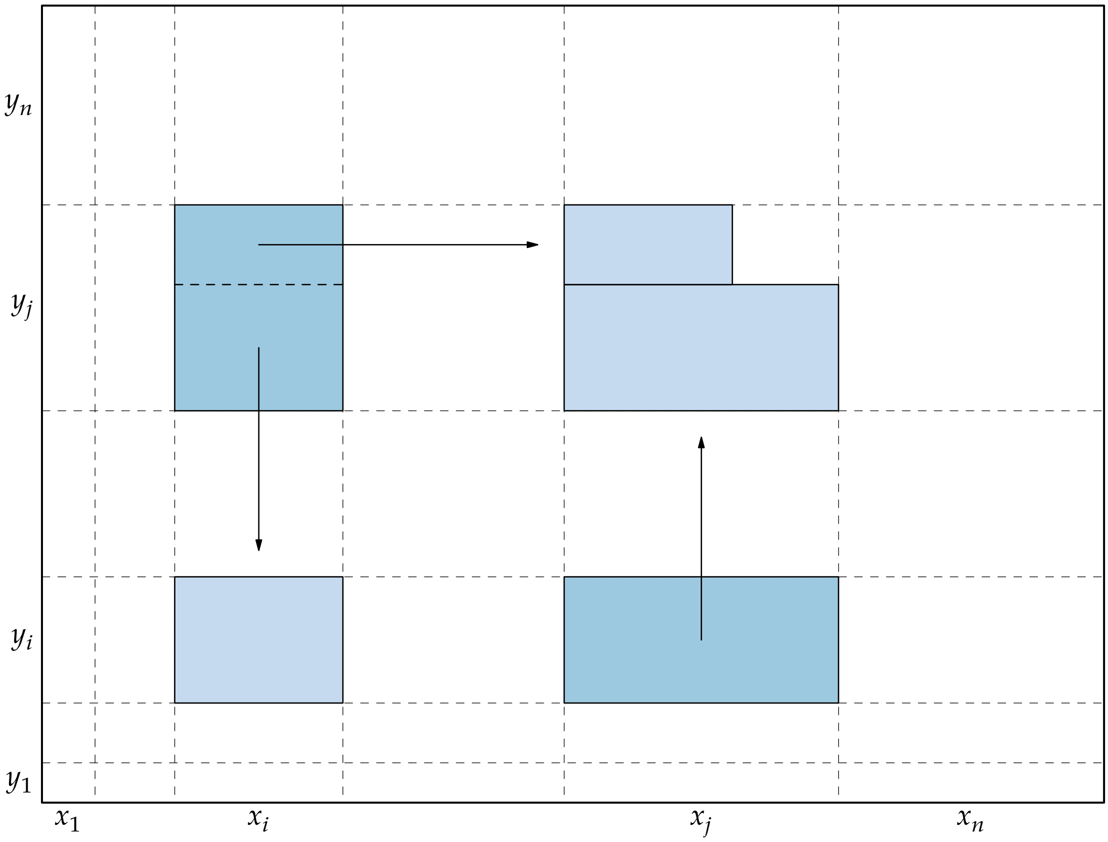
$$n\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \geq \left(|x_1| + |x_2| + \cdots + |x_n|\right)^2 \geq \left(x_1 + x_2 + \cdots + x_n\right)^2$$

$$\therefore \quad \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \geq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2$$

— RBN

The Chebyshev inequality for positive monotone sequences

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \leq \sum_{i=1}^{n} x_i y_i$$



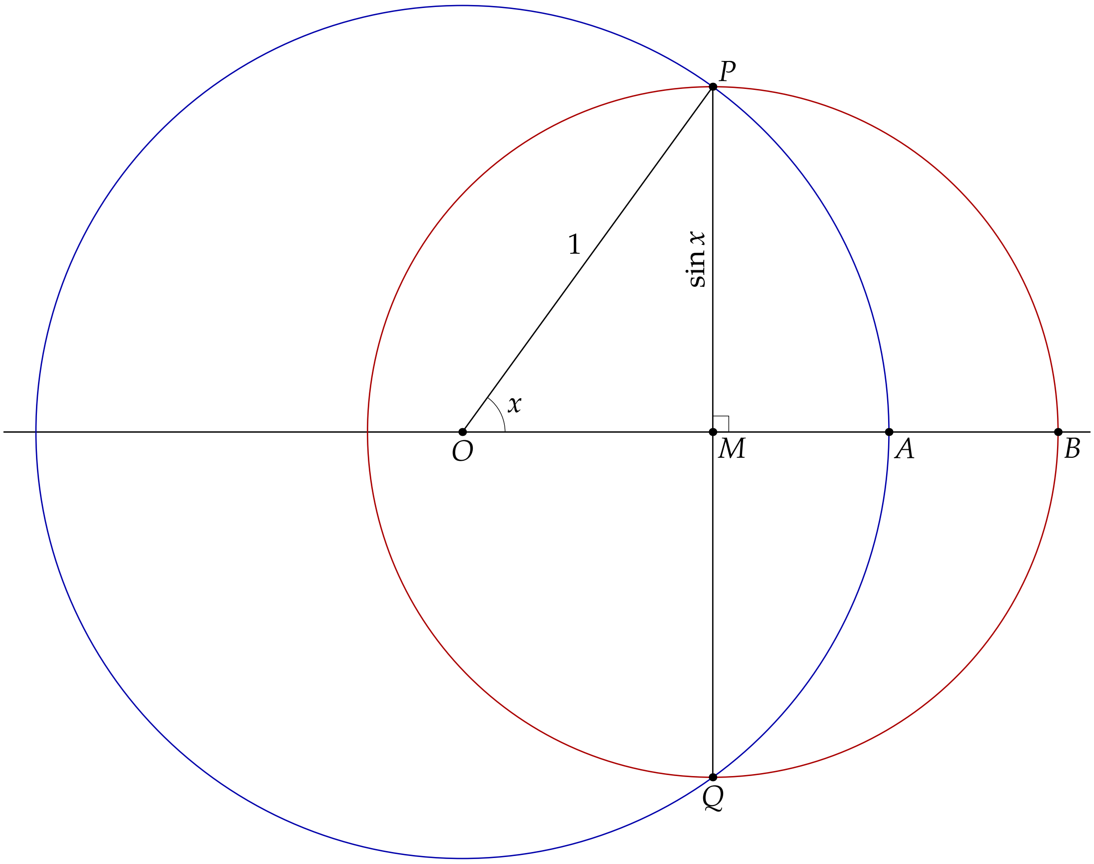
$$x_i < x_j \ \& \ y_i < y_j \quad \Rightarrow \quad x_i y_j + x_j y_i \leq x_i y_i + x_j y_j$$

$$\therefore \quad (x_1 + x_2 + \cdots + x_n)\,(y_1 + y_2 + \cdots + y_n) \leq n\,(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

— RBN

Jordan's inequality

$$0 \le x \le \frac{\pi}{2} \quad \Rightarrow \quad \frac{2x}{\pi} \le \sin x \le x$$



$$\begin{aligned} OB = OM + MP \ge OA \;&\Rightarrow\; \widehat{PBQ} \ge \widehat{PAQ} \ge \overline{PQ} \\ &\Rightarrow\; \pi \sin x \ge 2x \ge 2 \sin x \\ &\Rightarrow\; \frac{2x}{\pi} \le \sin x \le x \end{aligned}$$

— Feng Yuefeng

Young's inequality

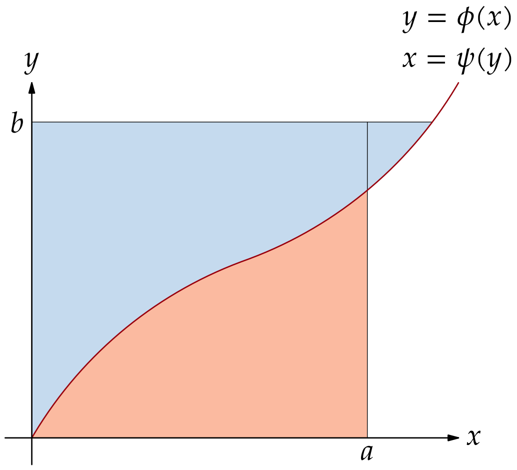
W. H. Young, "On classes of summable functions and their Fourier series", *Proc. Royal Soc.* (A), 87 (1912) 225–229.

Theorem: Let ϕ and ψ be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each others. Then for $a, b \geq 0$ we have

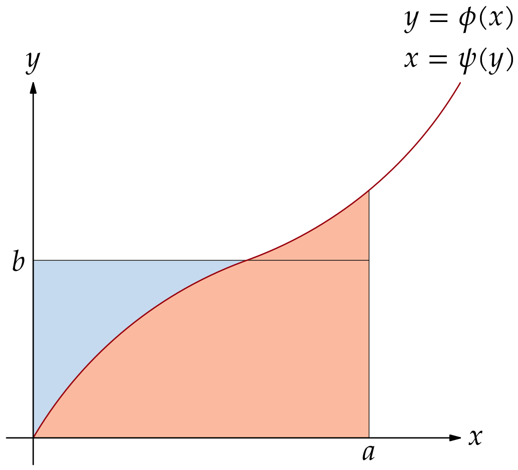
$$ab \leq \int_0^a \phi(x)dx + \int_0^b \psi(y)dy$$

with equality if and only if $b = \phi(a)$.

Proof:



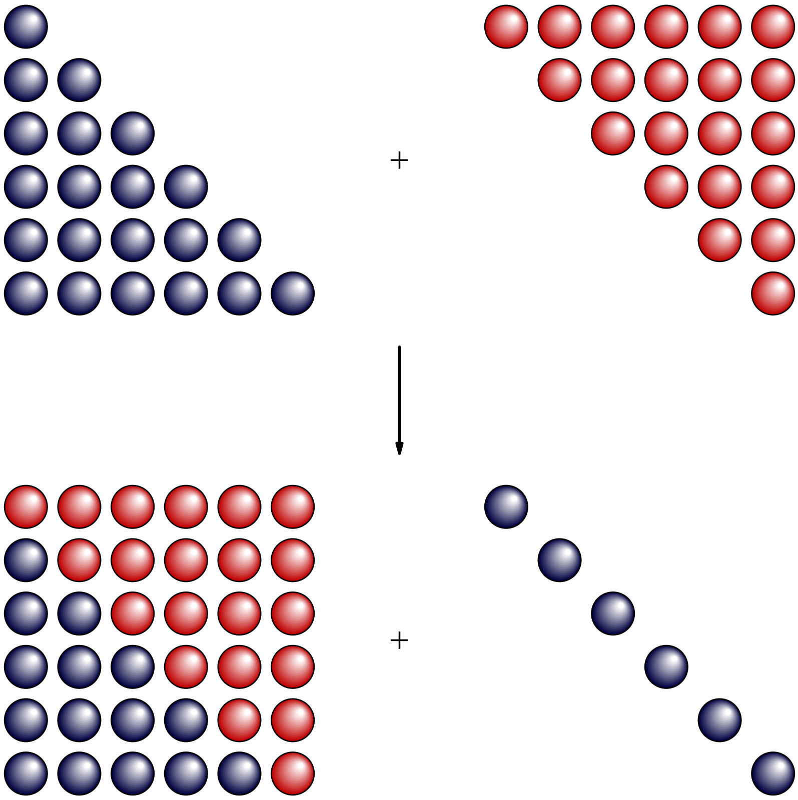
$b > \phi(a)$



$b < \phi(a)$

Integer sums

Sums of integers III



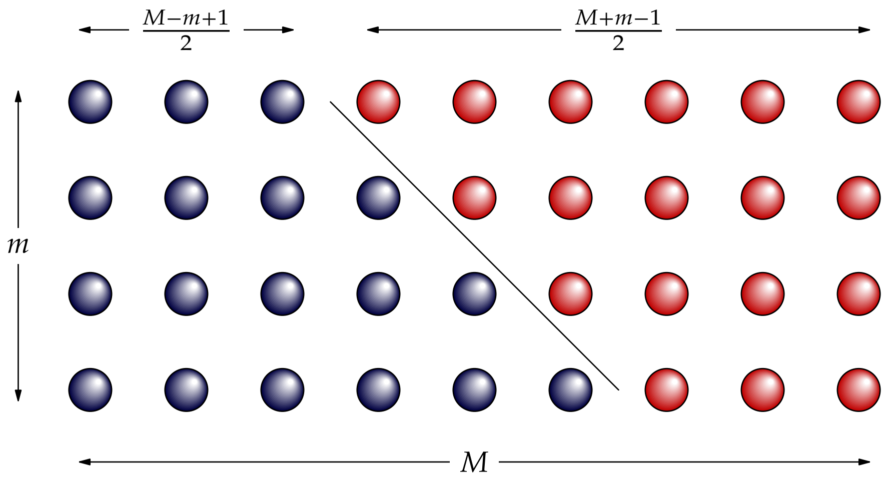
$$1 + 2 + \cdots + n = \frac{1}{2}\left(n^2 + n\right)$$

— S. J. Barlow

Sums of consecutive positive integers

Every integer $N > 1$, not a power of two, can be expressed as the sum of two or more positive integers.

$$N = 2^n(2k+1) \quad (n \geq 0, k \geq 1)$$
$$m = \min\{2^{n+1}, 2k+1\}$$
$$M = \max\{2^{n+1}, 2k+1\}$$
$$2N = mM$$

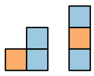


$$N = \left(\frac{M-m+1}{2}\right) + \left(\frac{M-m+1}{2} + 1\right) + \cdots + \left(\frac{M+m-1}{2}\right)$$

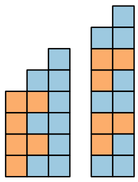
— C. L. Frenzen

Consecutive sums of consecutive integers II

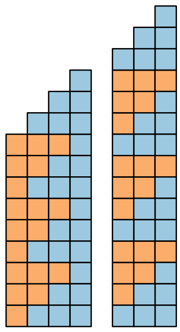
$$T_k = 1 + 2 + \cdots + k \quad \Longrightarrow$$



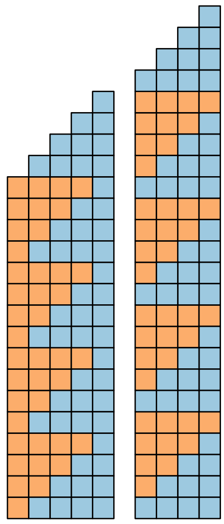
$$\begin{aligned} 1 + 2 &= 3 \\ &= 3T_1 \end{aligned}$$



$$\begin{aligned} 4 + 5 + 6 &= 7 + 8 \\ &= 5T_2 \end{aligned}$$



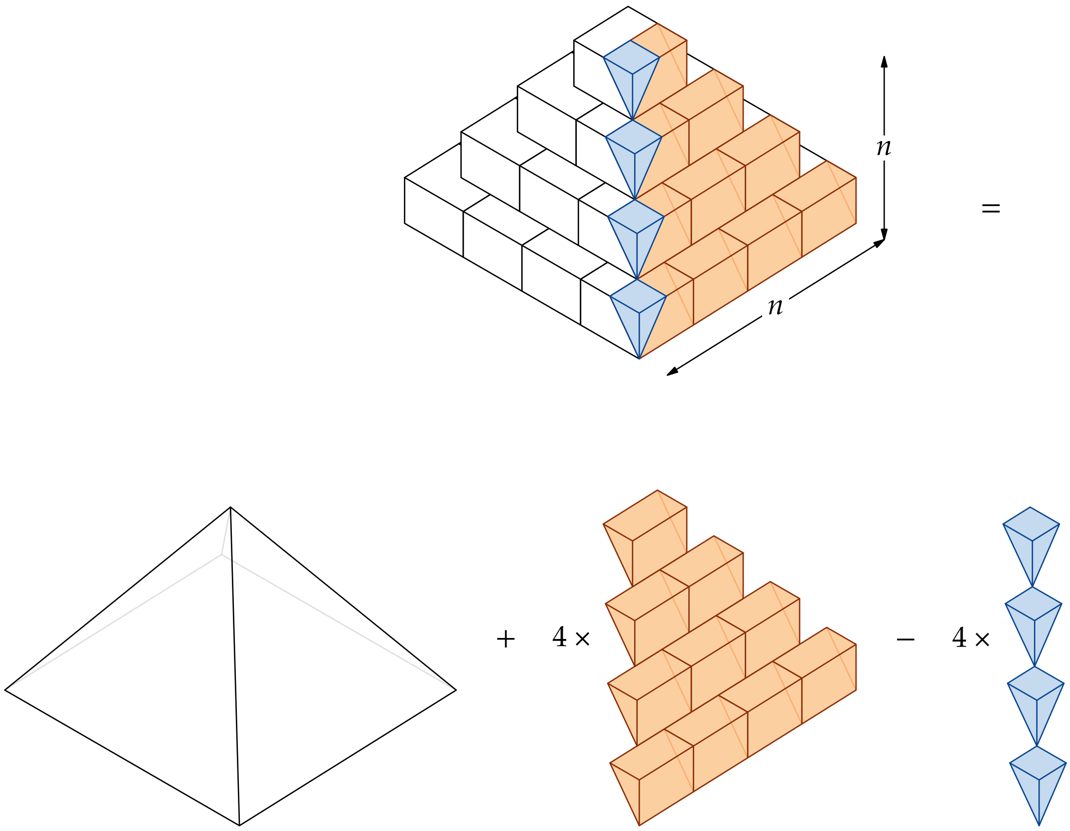
$$\begin{aligned} 9 + 10 + 11 + 12 &= 13 + 14 + 15 \\ &= 7T_3 \end{aligned}$$



$$\begin{aligned} 16 + 17 + 18 + 19 + 20 &= 21 + 22 + 23 + 24 \\ &= 9T_4 \end{aligned}$$

$$\begin{aligned} n^2 + (n^2 + 1) + \cdots + (n^2 + n) &= (n^2 + n + 1) + \cdots + (n^2 + 2n) \\ &= (2n + 1)T_n \end{aligned}$$

Sums of squares VI



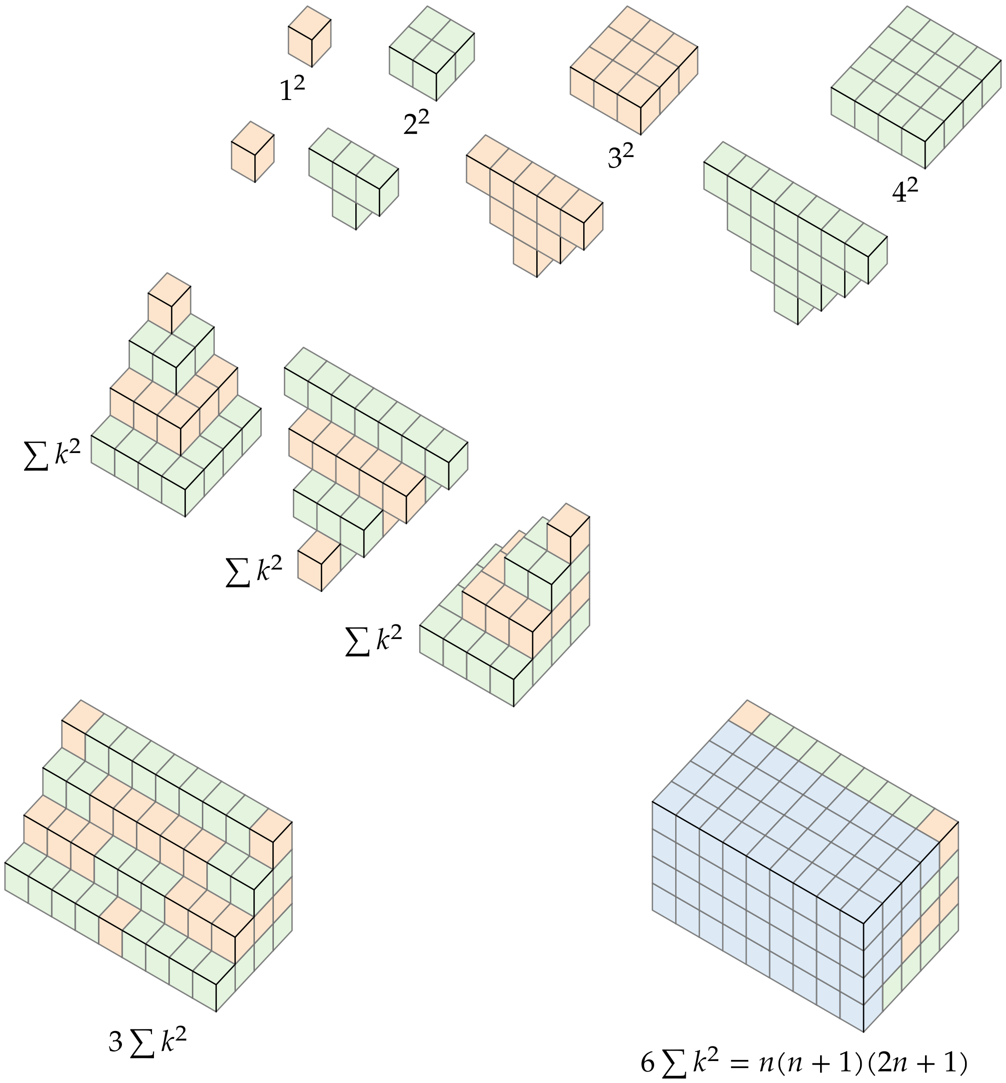
$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n^2 \times n \quad + \quad 4 \times \frac{n(n+1)}{2} \times \frac{1}{4} \quad - \quad 4 \times n \times \frac{1}{12}$$

$$= \frac{1}{6}n(n+1)(2n+1)$$

— I. A. Sakmar

Sums of squares VII

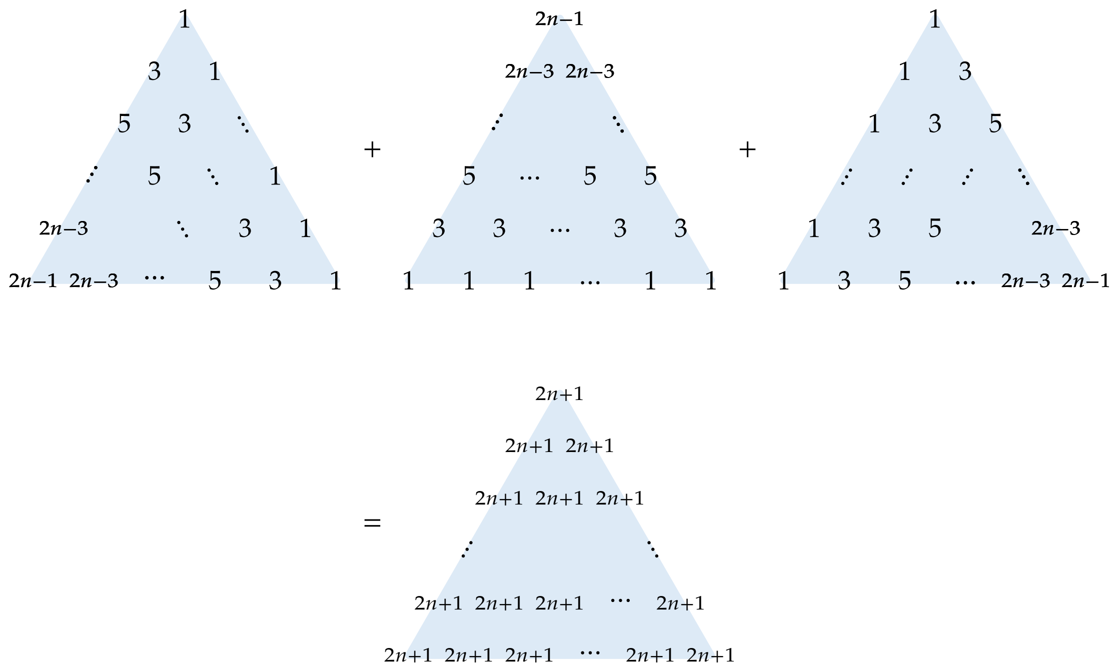
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$



— Nanny Wermuth and Hans-Jürgen Schuh

Sums of squares VIII

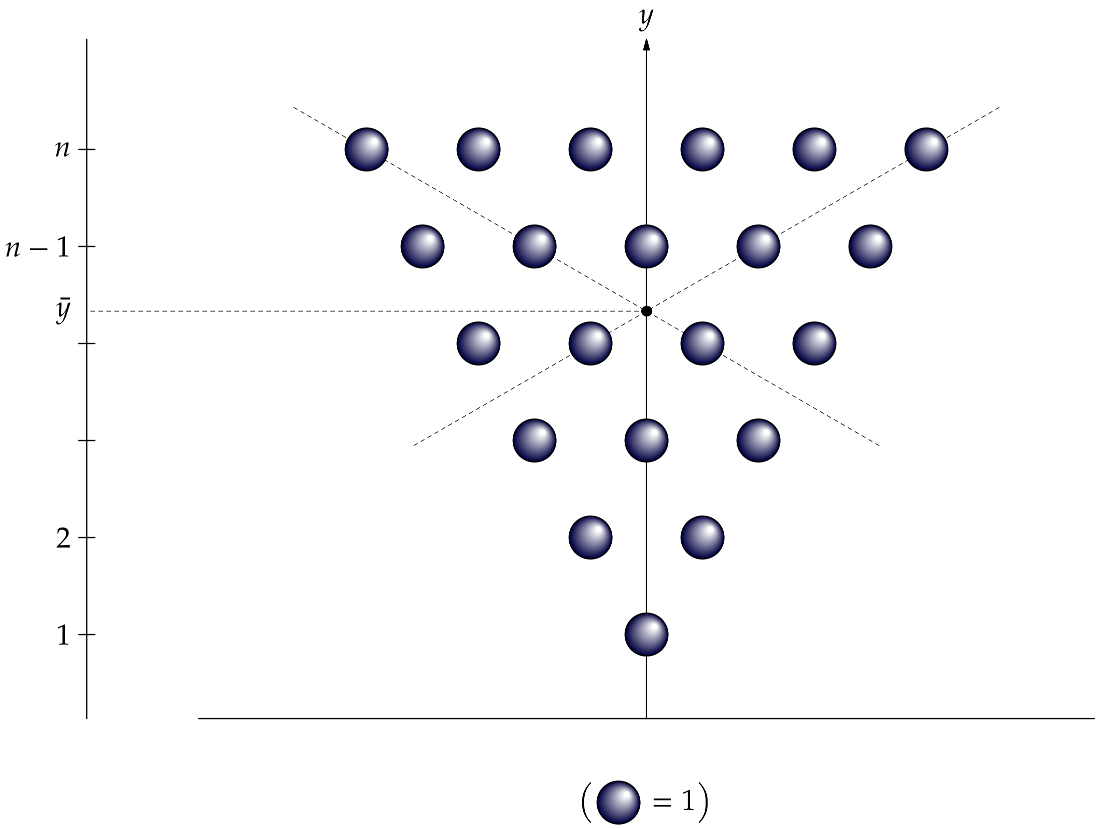
$$k^2 = 1 + 3 + \cdots + (2k-1) \quad \Rightarrow \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$



$$3\left(1^2 + 2^2 + \cdots + n^2\right) = (2n+1)(1 + 2 + \cdots + n)$$

$$\therefore \quad 1^2 + 2^2 + \cdots + n^2 = \frac{2n+1}{3} \cdot \frac{n(n+1)}{2}$$

Sums of squares IX (via centroids)



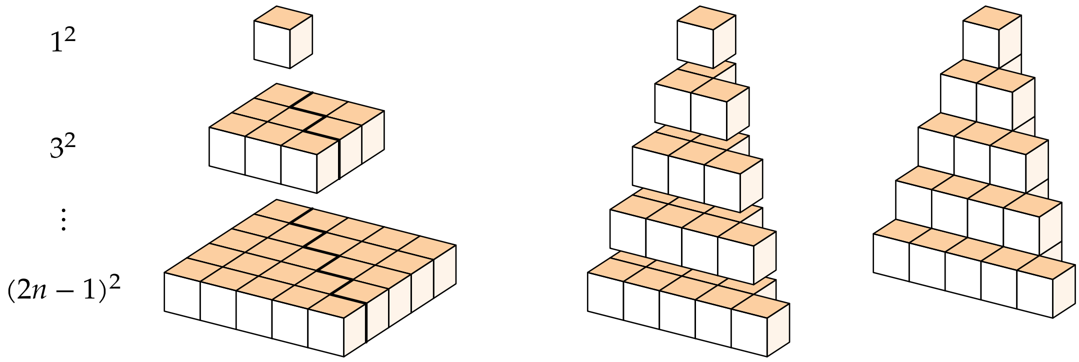
$$\bar{y} = 1 + \frac{2}{3}(n-1) = \frac{1\cdot 1 + 2\cdot 2 + \cdots + n\cdot n}{1+2+\cdots+n}$$

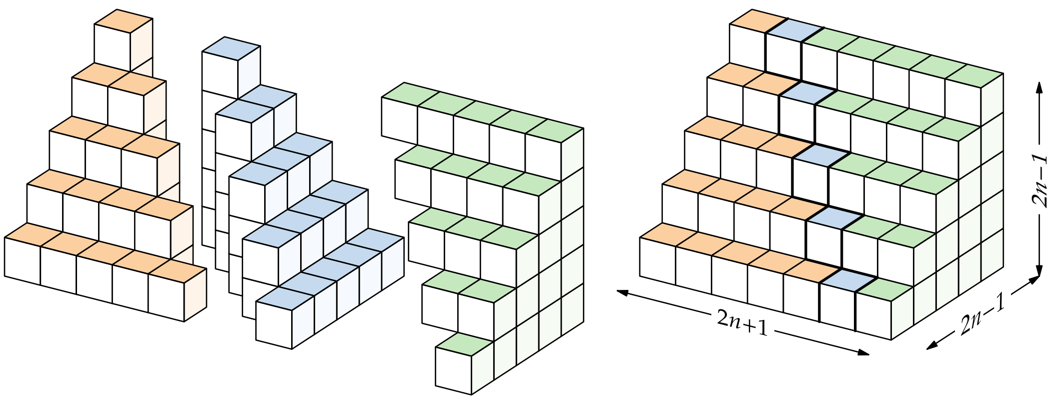
$$\therefore \quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{2}\cdot\frac{1}{3}(2n+1) = \frac{1}{6}n(n+1)(2n+1)$$

— Sidney H. Kung

Sums of odd squares

$$1^2 + 2^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$





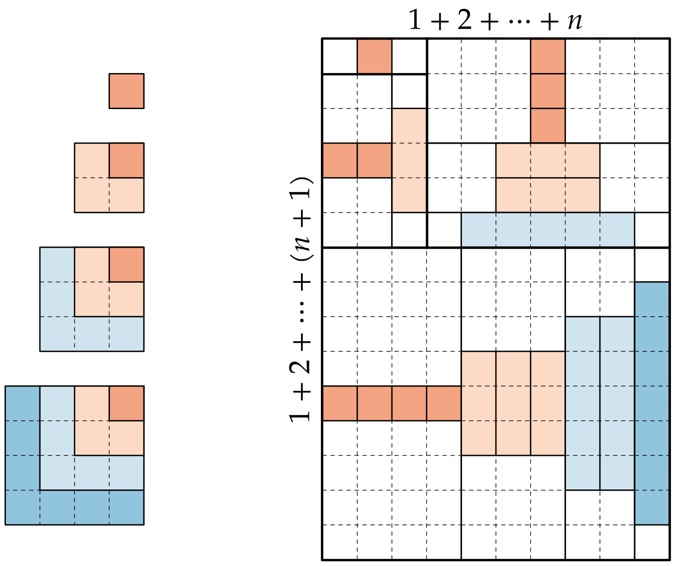
$$3 \times \left(1^2 + 3^2 + \cdots + (2n-1)^2\right) = (1 + 2 + \cdots + (2n-1)) \times (2n+1)$$

$$= \frac{(2n-1)(2n)(2n+1)}{2} = n(2n-1)(2n+1)$$

— RBN

Sums of sums of squares

$$\sum_{k=1}^{n}\sum_{i=1}^{k} i^2 = \frac{1}{3}\binom{n+1}{2}\binom{n+2}{2}$$



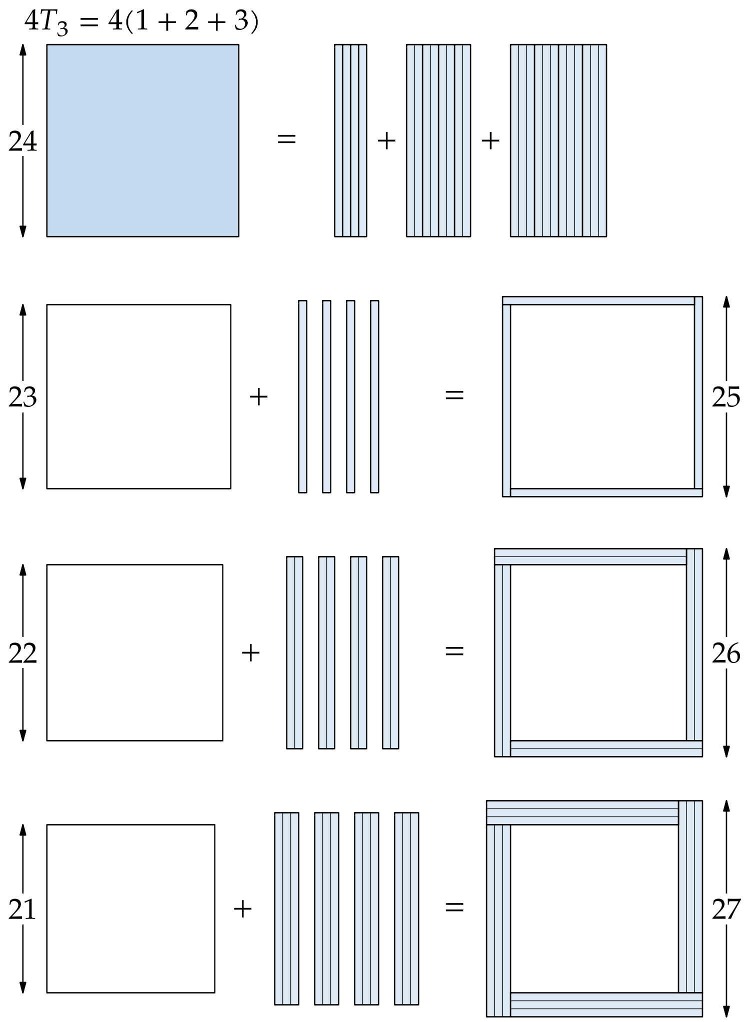
$$3\left(1^2\right)+3\left(1^2+2^2\right)+3\left(1^2+2^2+3^2\right)+\cdots+3\left(1^2+2^2+\cdots+n^2\right)=\binom{n+1}{2}\binom{n+2}{2}$$

— C. G. Wastun

Pythagorean runs

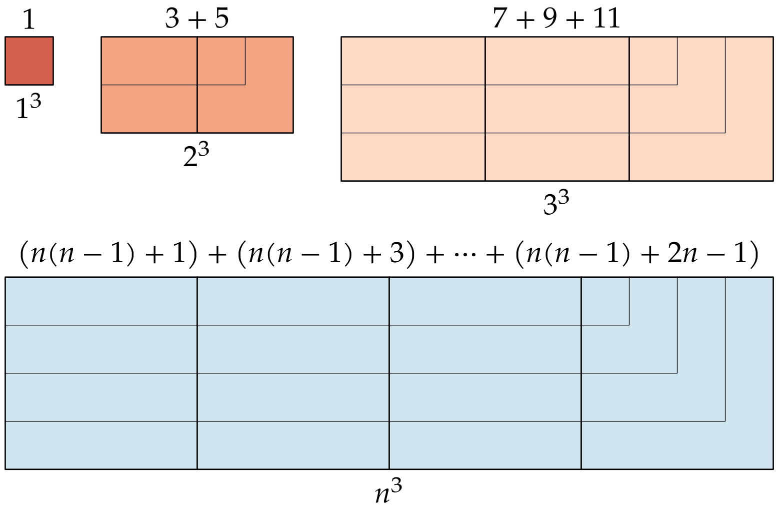
$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$\vdots$$
$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow \quad (4T_n - n)^2 + \cdots + (4T_n)^2 = (4T_n + 1)^2 + \cdots + (4T_n + n)^2$$

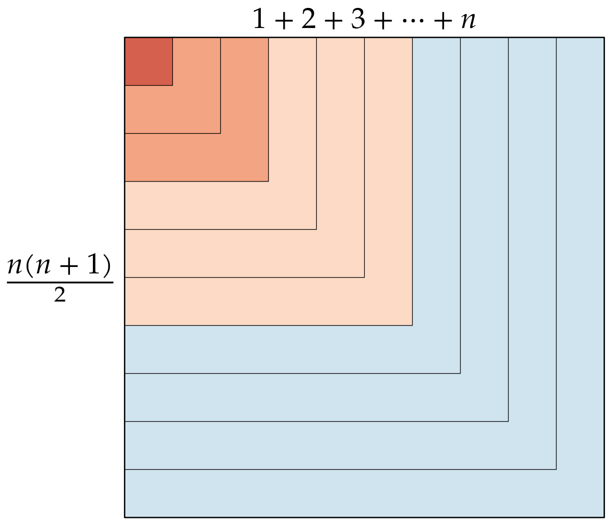
e.g., $n = 3$:



— Michael Boardman

Sums of cubes VII



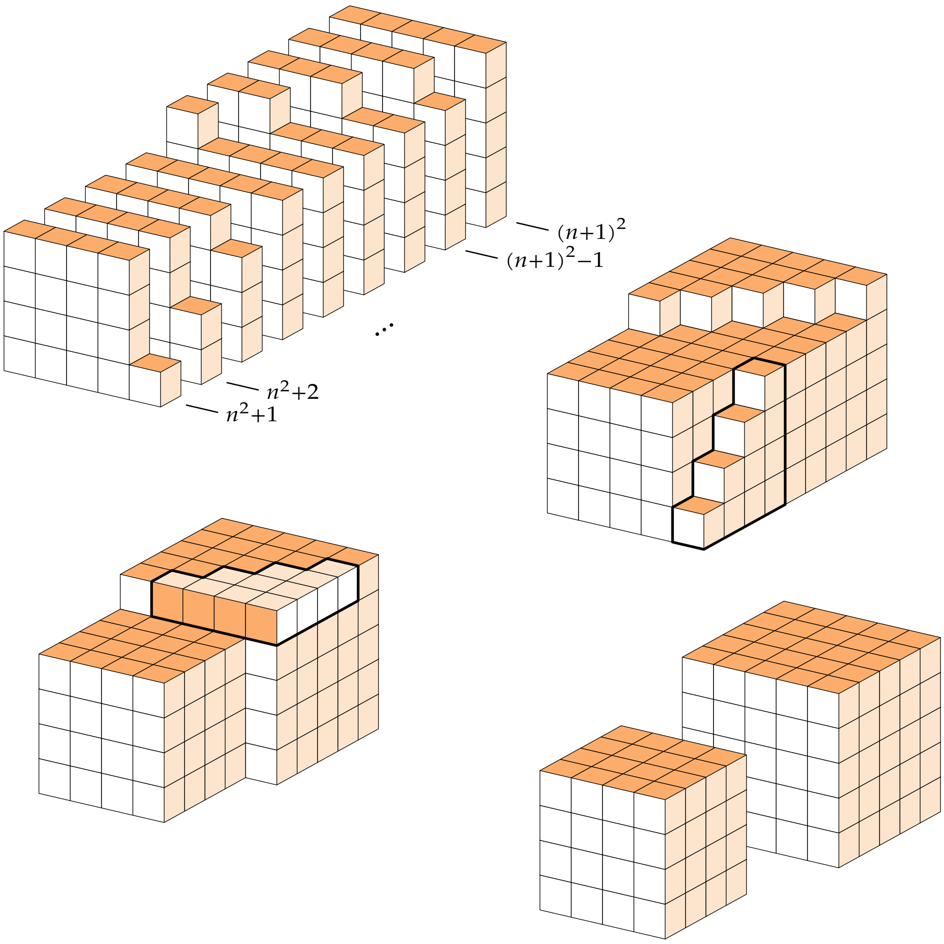


$$1^3+2^3+\cdots+n^3 = 1+3+5+\cdots+2\frac{n(n-1)}{2}-1 = \left(\frac{n(n-1)}{2}\right)^2$$

— Alfinio Flores

Sums of integers as sums of cubes

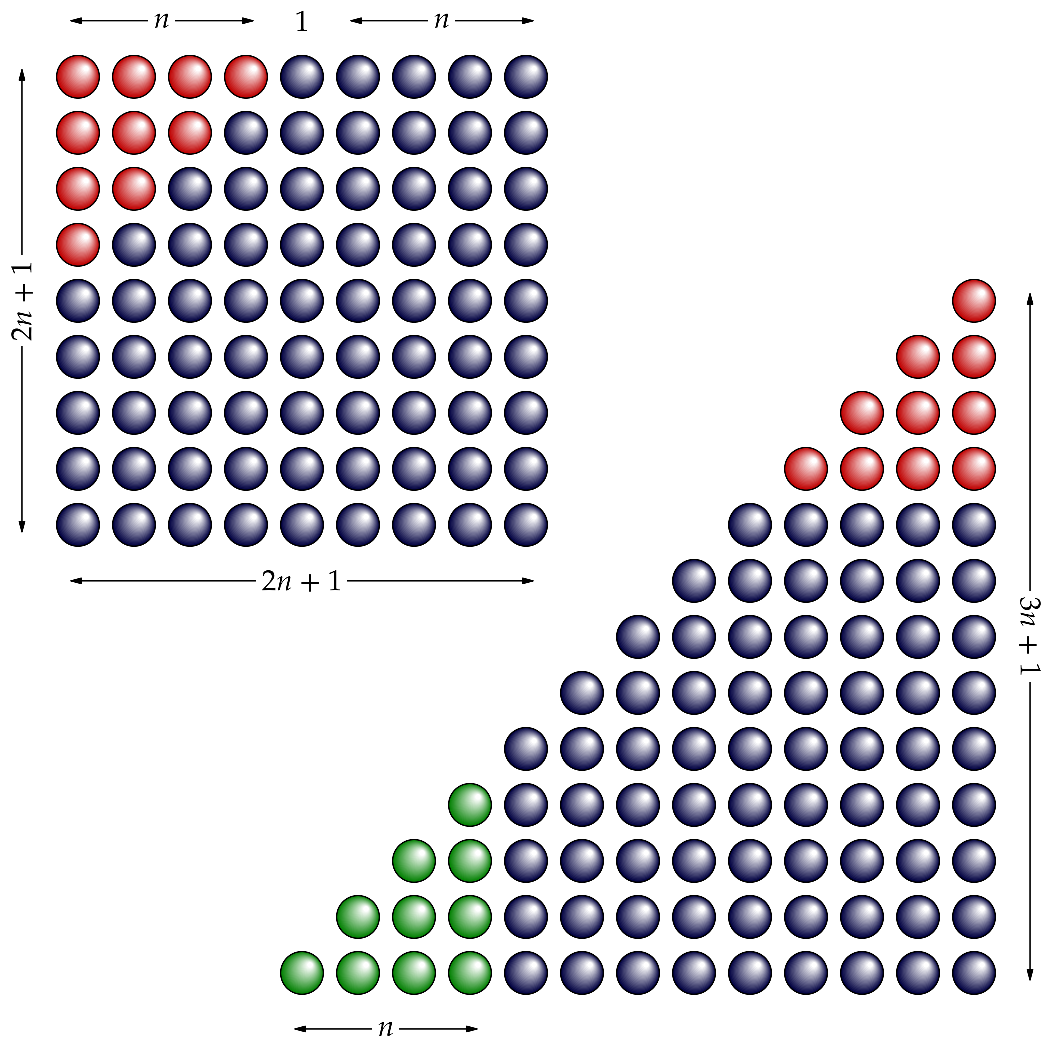
$$
\begin{aligned}
2+3+4 &= 1+8 \\
5+6+7+8+9 &= 8+27 \\
10+11+12+13+14+15+16 &= 27+64 \\
&\vdots \\
\left(n^2+1\right)+\left(n^2+2\right)+\cdots+(n+1)^2 &= n^3+(n+1)^3
\end{aligned}
$$



— RBN

The square of any odd number is the difference between two triangular numbers

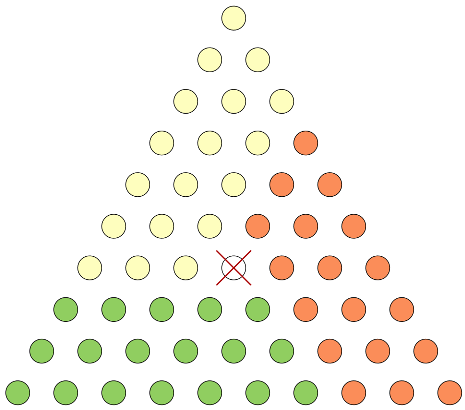
$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad (2n+1)^2 = T_{3n+1} - T_n$$



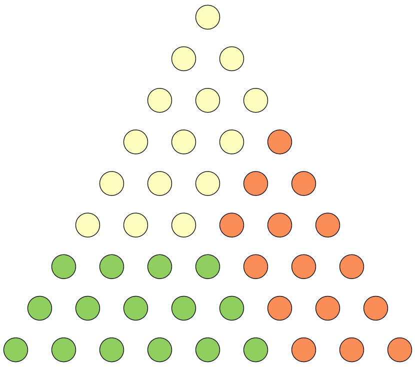
— RBN

Triangular numbers mod 3

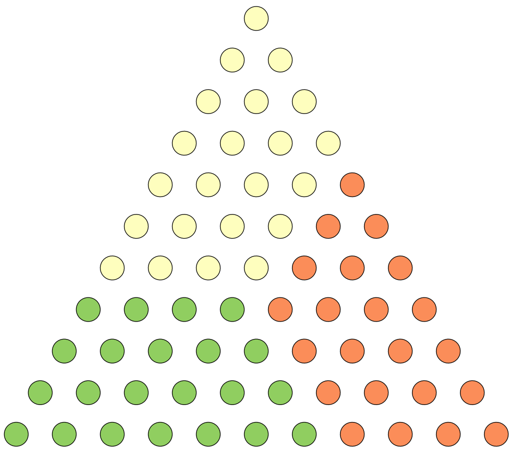
$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad \begin{cases} T_n \equiv 1 \bmod 3, & n \equiv 1 \bmod 3 \\ T_n \equiv 0 \bmod 3, & n \not\equiv 1 \bmod 3 \end{cases}$$



$T_{3k+1} = 1 + 3\,(T_{2k+1} - T_{k+1})$



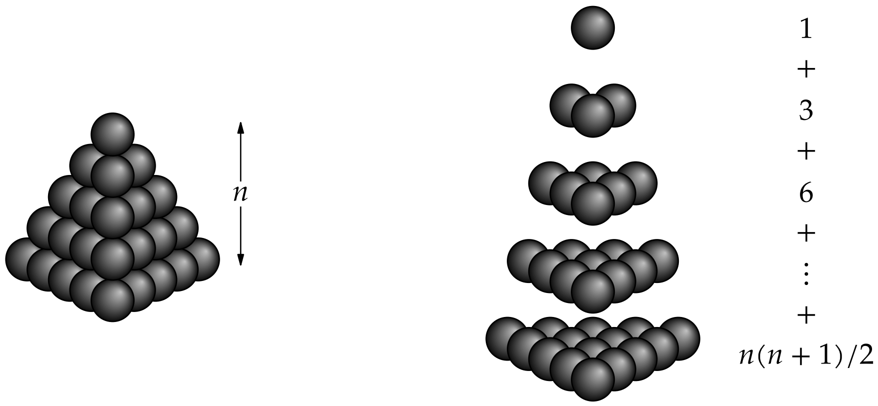
$T_{3k} = 3\,(T_{2k} - T_k)$

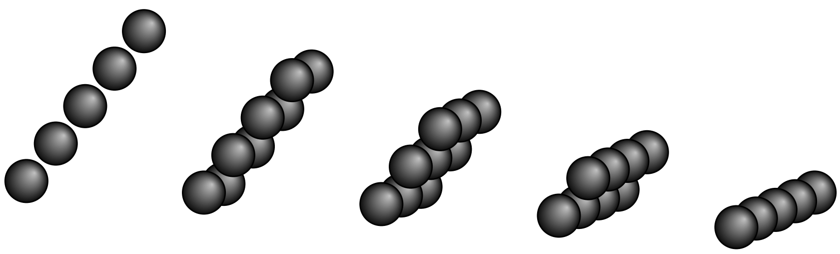


$T_{3k+2} = 3\,(T_{2k+1} - T_k)$

Counting triangular numbers IV: Counting cannonballs

$$1 + 2 + \cdots + k = T_k \quad \Rightarrow \quad \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(n - k + 1)$$



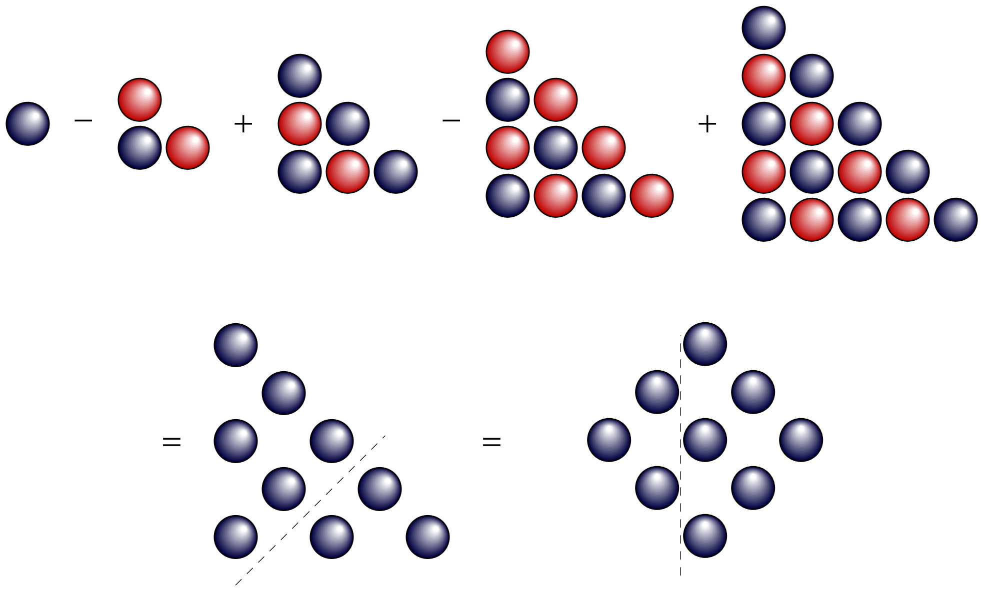


$$1(n) \quad + \quad 2(n-1) \quad + \quad 3(n-2) \quad + \quad \cdots \quad + \quad n(1)$$

— Deanna B. Haunsperger and Stephen F. Kennedy

Alternating sums of triangular numbers

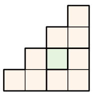
$$1 + 2 + \cdots + k = T_k \;\Rightarrow\; \sum_{k=1}^{2n-1} (-1)^{k+1}\, T_k = n^2$$



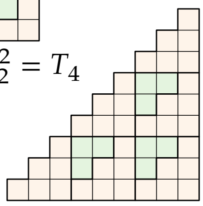
— RBN

The sum of the squares of consecutive triangular numbers is triangular

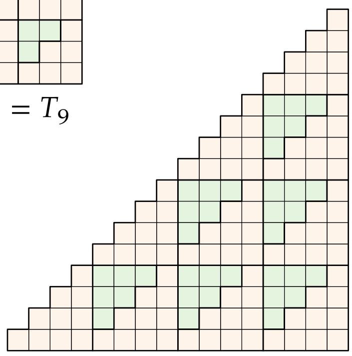
$$1 + 2 + \cdots + n = T_n \;\Rightarrow\; T_{n-1}^2 + T_n^2 = T_{n^2}$$



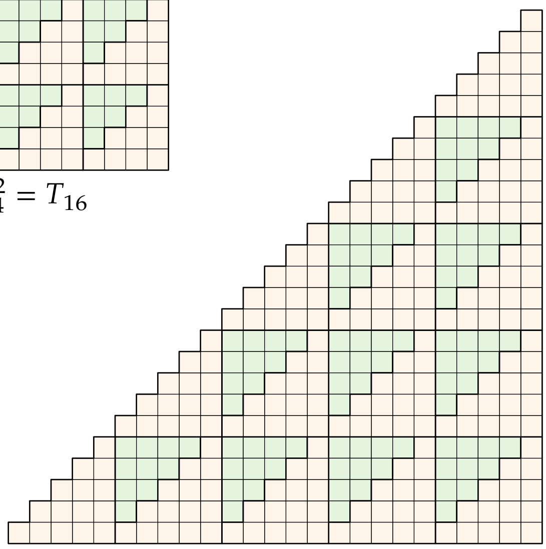
$T_1^2 + T_2^2 = T_4$



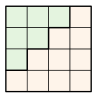
$T_2^2 + T_3^2 = T_9$



$T_3^2 + T_4^2 = T_{16}$



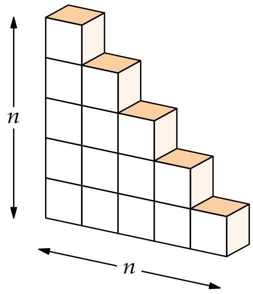
$T_4^2 + T_5^2 = T_{25}$

Note: This is a companion result to the more familiar $T_{n-1} + T_n = n^2 \rightarrow$ 

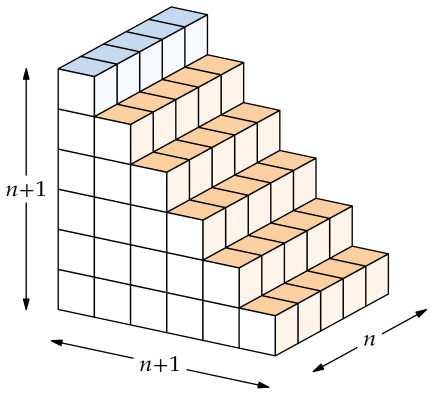
— RBN

Recursion for triangular numbers

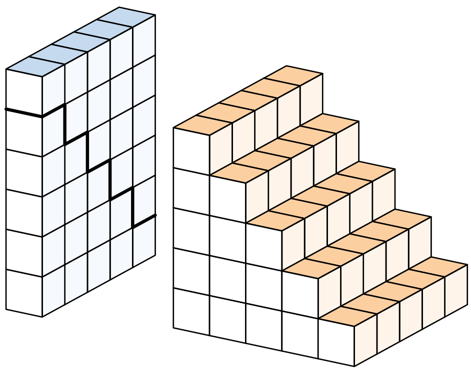
$$1 + 2 + \cdots + n = T_n \quad \Rightarrow \quad T_{n+1} = \frac{n+2}{n} T_n$$



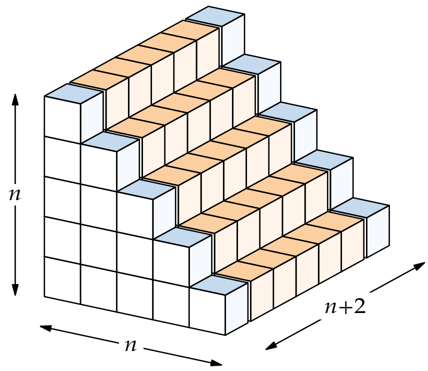
T_n



$n \cdot T_{n+1}$



$2 \cdot T_n + n \cdot T_n$

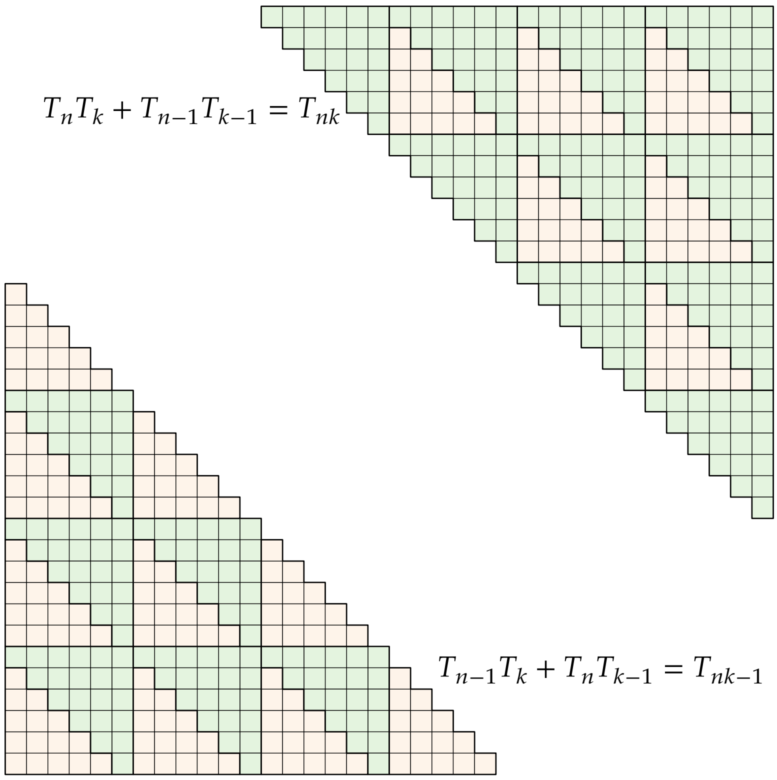


$(n+2) \cdot T_n$

$$n \cdot T_{n+1} = (n+2) \cdot T_n \quad \Rightarrow \quad T_{n+1} = \frac{n+2}{n} T_n$$

Indenties for triangular numbers

$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$



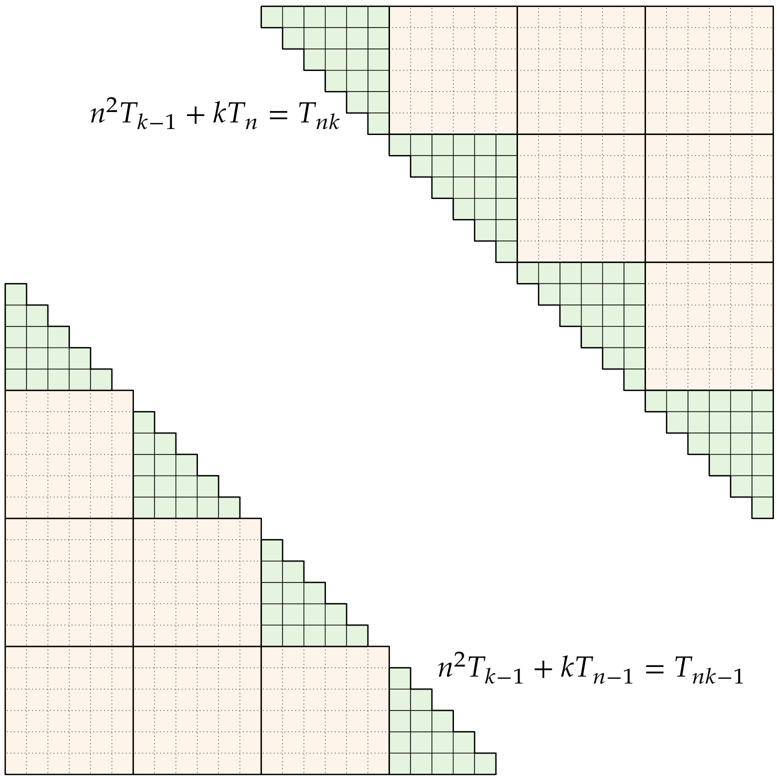
$$T_n T_k + T_{n-1} T_{k-1} = T_{nk}$$

$$T_{n-1} T_k + T_n T_{k-1} = T_{nk-1}$$

— RBN

More identities for triangular numbers

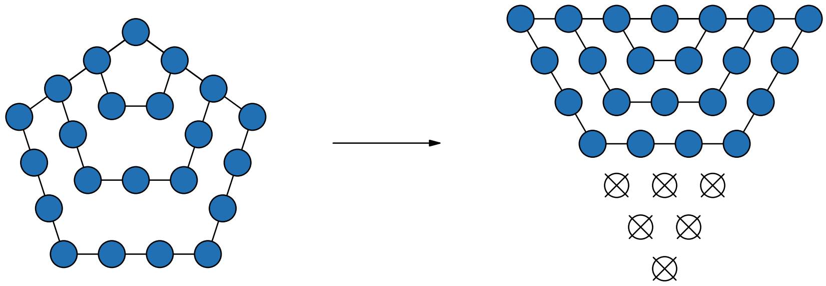
$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$



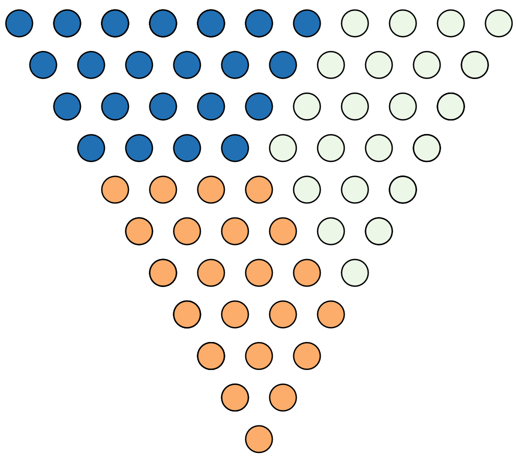
— James O. Chilaka

Identities for pentagonal numbers

$$\left.\begin{aligned} P_n &= 1 + 4 + 7 + \cdots + (3n-2) \\ T_n &= 1 + 2 + 3 + \cdots + n \end{aligned}\right\} \Rightarrow$$

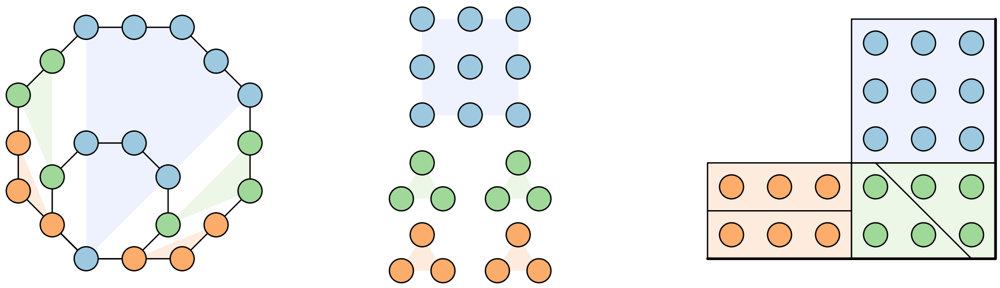


$$P_n = T_{2n-1} - T_{n-1}$$

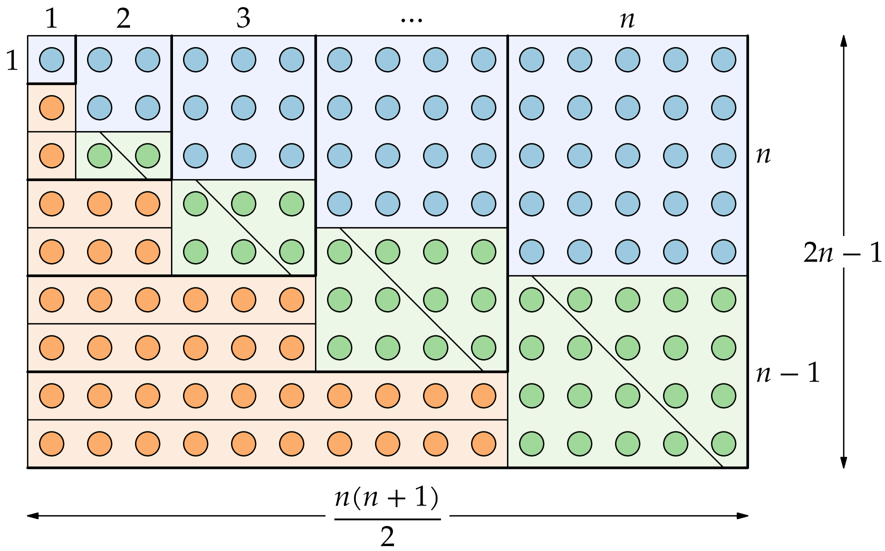


$$3P_n = T_{3n-1}$$

Sums of octagonal numbers



$$T_k = 1 + 2 + \cdots + k \Rightarrow O_k = k^2 + 4T_{k-1}$$

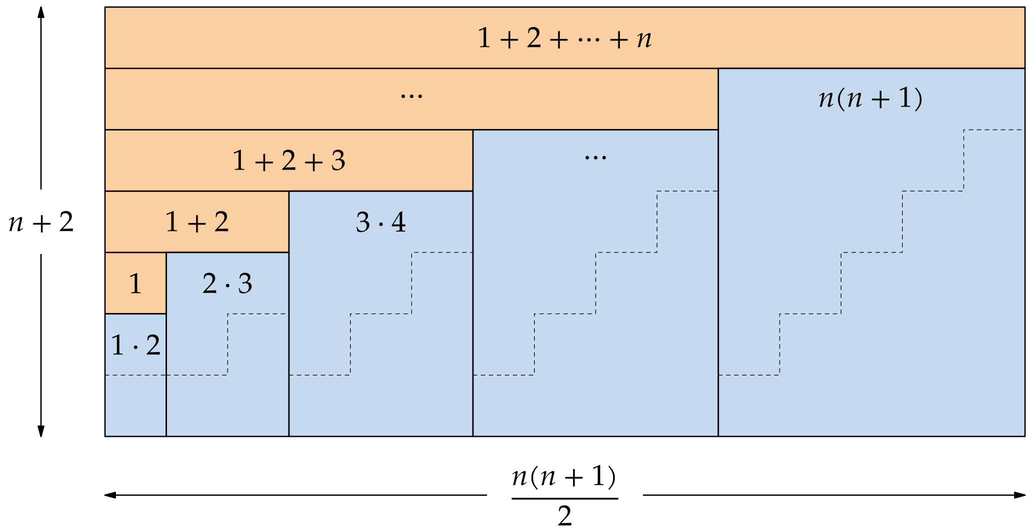


$$\sum_{k=1}^{n} O_k = 1 + 8 + 21 + 40 + \cdots + \left(n^2 + 4T_{n-1}\right) = \frac{n(n+1)(2n-1)}{2}$$

— James O. Chilaka

Sums of products of consecutive integers I

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$



$$T_k = 1 + 2 + \cdots + k \quad \Rightarrow$$

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (T_1 + T_2 + \cdots + T_n) = \frac{n(n+1)(n+2)}{2},$$

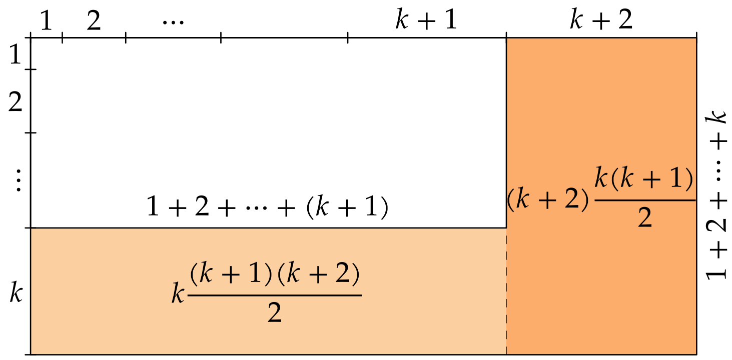
$$(T_1 + T_2 + \cdots + T_n) = \frac{1}{2}\big(1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1)\big),$$

$$\therefore \quad \frac{3}{2}\big(1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1)\big) = \frac{n(n+1)(n+2)}{2}.$$

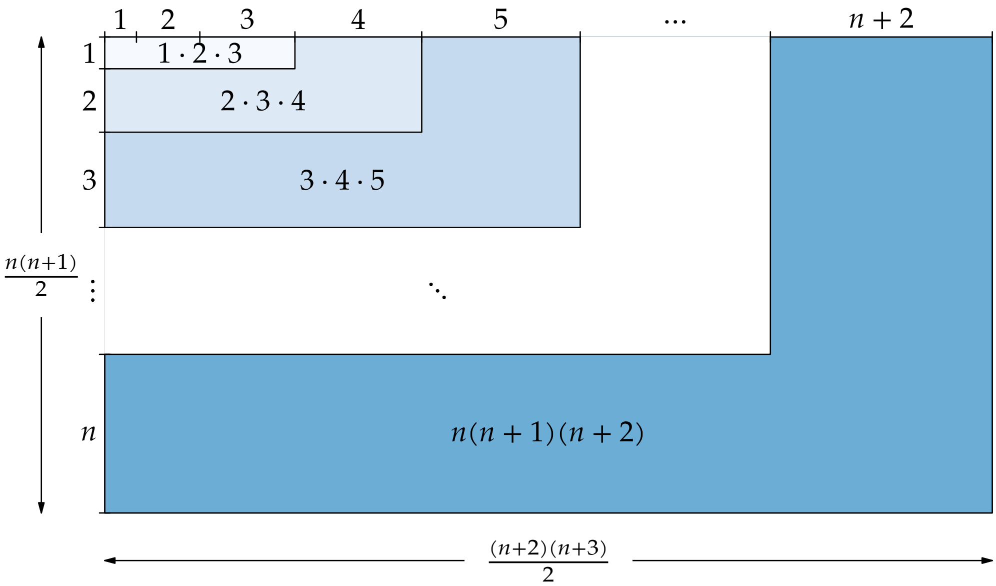
— James O. Chilaka

Sums of products of consecutive integers II

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$



$$k\frac{(k+1)(k+2)}{2} + (k+2)\frac{k(k+1)}{2} = k(k+1)(k+2)$$

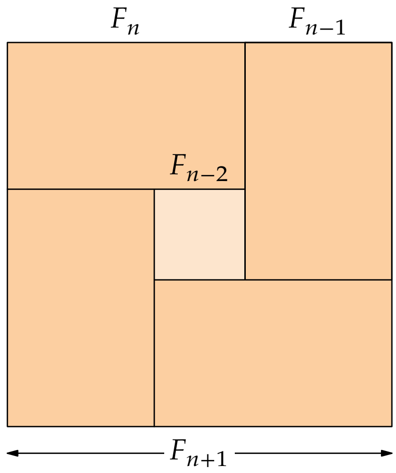


$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2)$$
$$= \frac{n(n+1)}{2} \times \frac{(n+2)(n+3)}{2} = \frac{n(n+1)(n+2)(n+3)}{4}$$

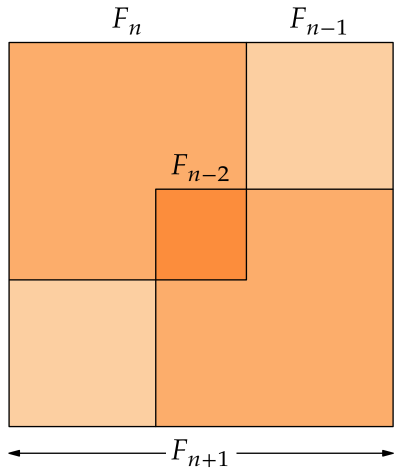
— James O. Chilaka

Fibonacci identities

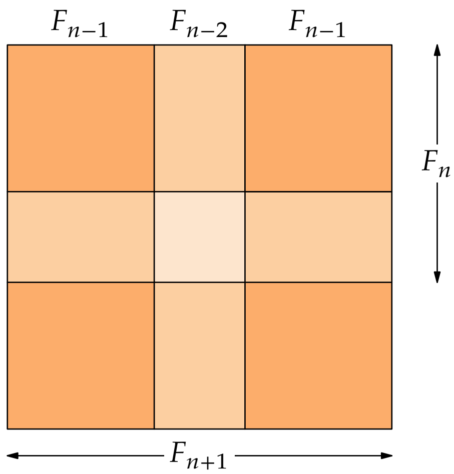
$F_1 = F_2 = 1, \;\; F_n = F_{n-1} + F_{n-2} \quad \Rightarrow$



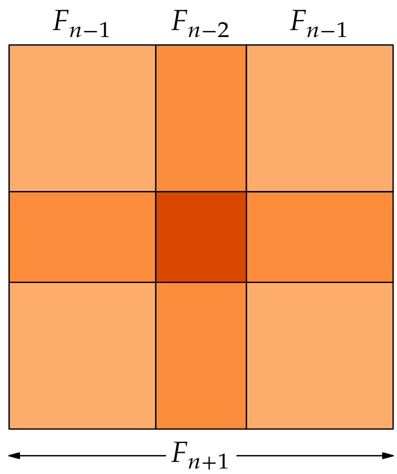
$$F_{n+1}^2 = 4F_nF_{n-1} + F_{n-2}^2$$



$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2$$



$$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-1}F_{n-2} + F_{n-2}^2$$

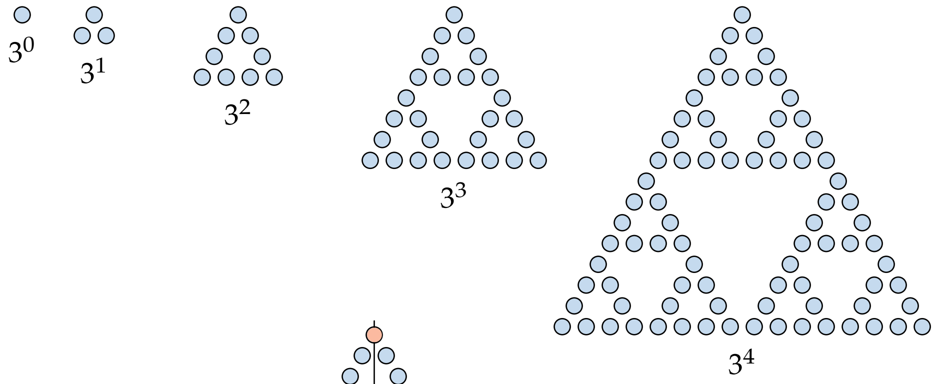


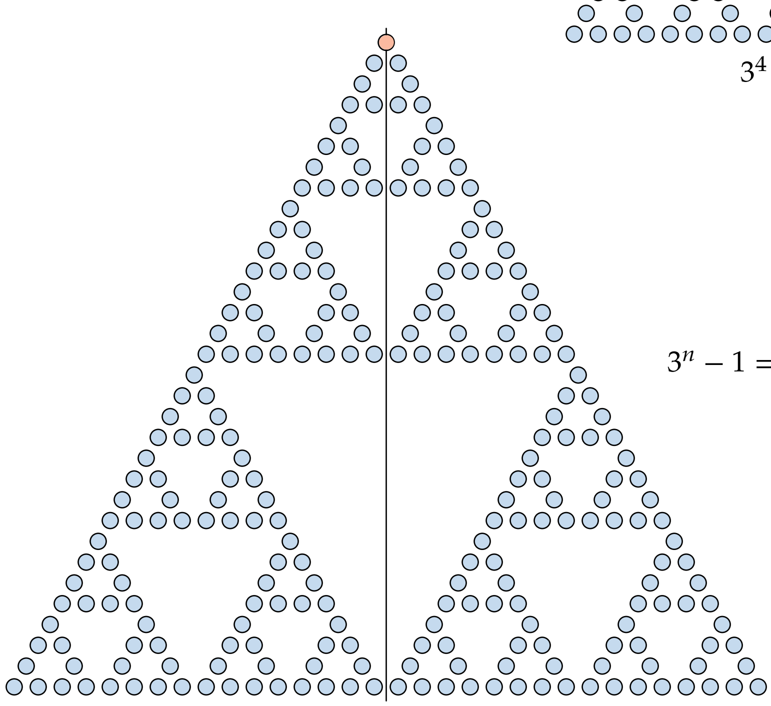
$$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$$

— Alfred Brousseau

Sums of powers of three

$$\sum_{k=0}^{n-1} 3^k = \frac{3^n - 1}{2}$$





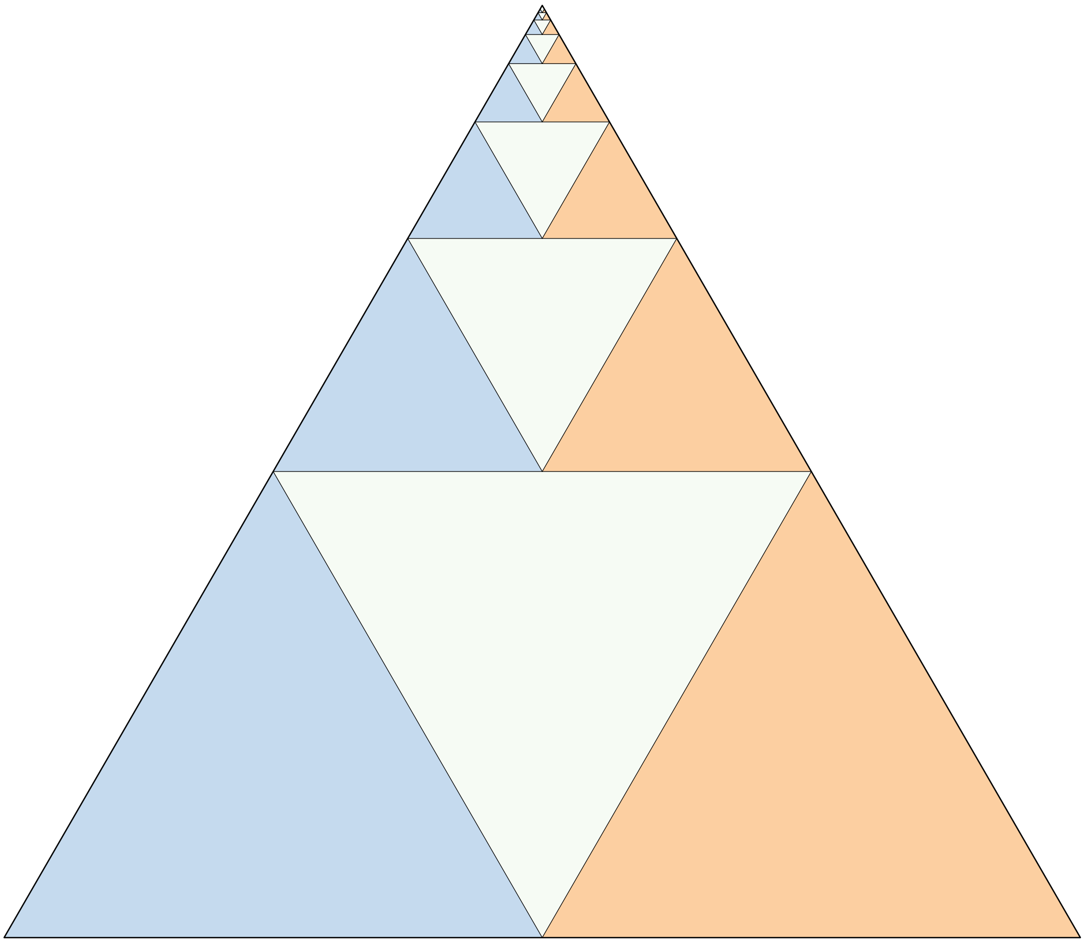
$$3^n - 1 = 2\sum_{k=0}^{n-1} 3^k$$

— David B. Sher

Infinite series, linear algebra, & other topics

A geometric series

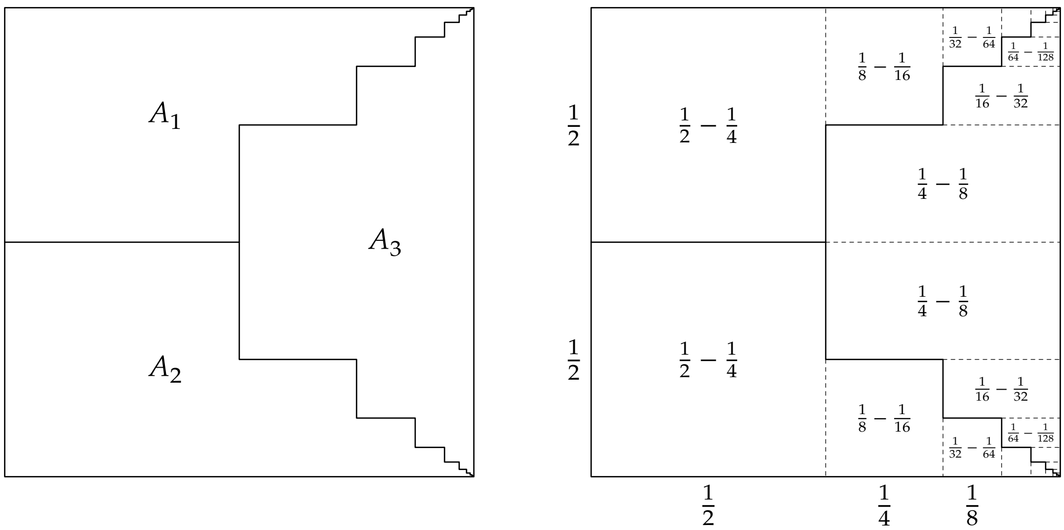
$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$$



— Rick Mabry

An alternating series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots = \frac{1}{3}$$



$$A_1 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots,$$

$$A_1 = A_2 = A_3, \quad A_1 + A_2 + A_3 = 1,$$

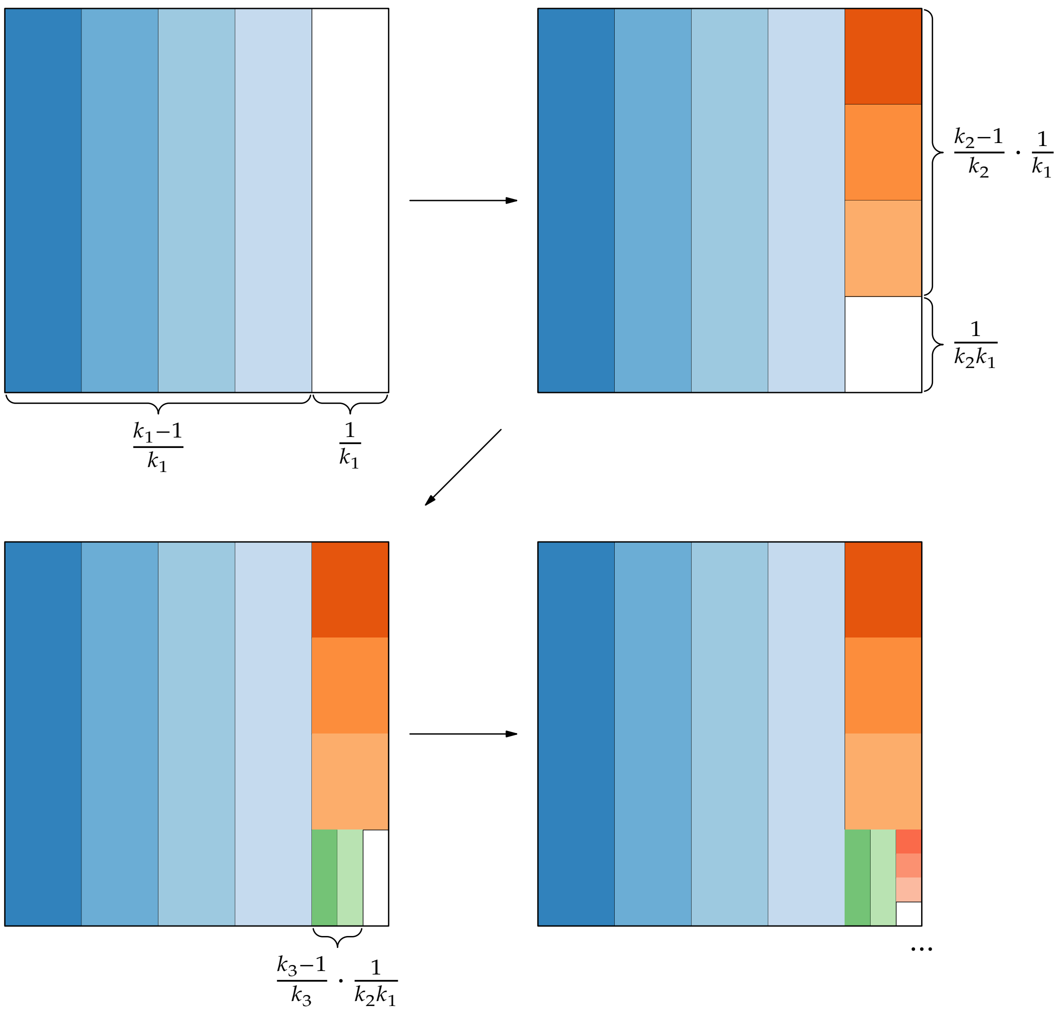
$$\therefore \quad A_1 = \frac{1}{3}.$$

— James O. Chilaka

A generalized geometric series

Let $\{k_1, k_2, k_3\}$ be a sequence of integers, each of which is at least 2. Then

$$\frac{k_1 - 1}{k_1} + \frac{k_2 - 1}{k_2 k_1} + \frac{k_3 - 1}{k_3 k_2 k_1} + \cdots = 1$$

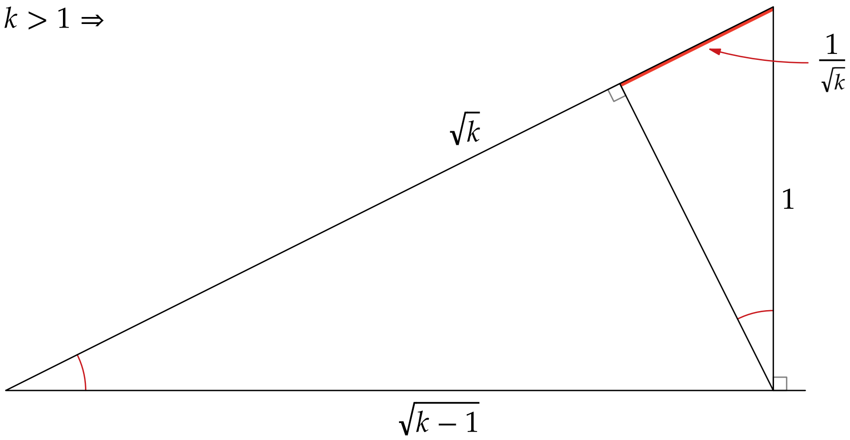


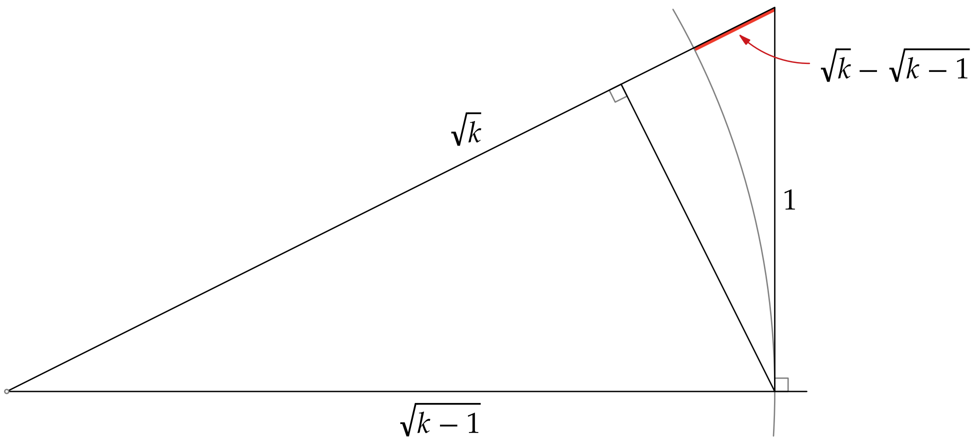
— John Mason

Divergence of a series

$$n > 1 \quad \Rightarrow \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

$k > 1 \Rightarrow$





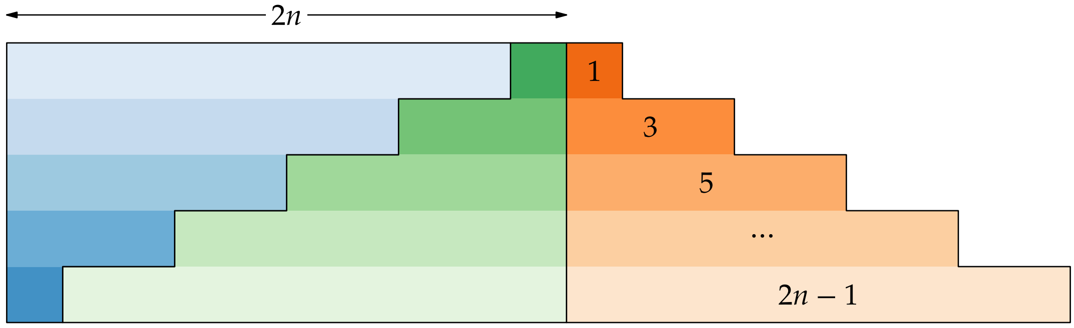
$$\frac{1}{\sqrt{k}} > \sqrt{k} - \sqrt{k-1}$$

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \left(\sqrt{2} - 1\right) + \left(\sqrt{3} - \sqrt{2}\right) + \cdots + \left(\sqrt{n} - \sqrt{n-1}\right)$$

$$\therefore \quad 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

— Sidney H. Kung

Galileo's ratios

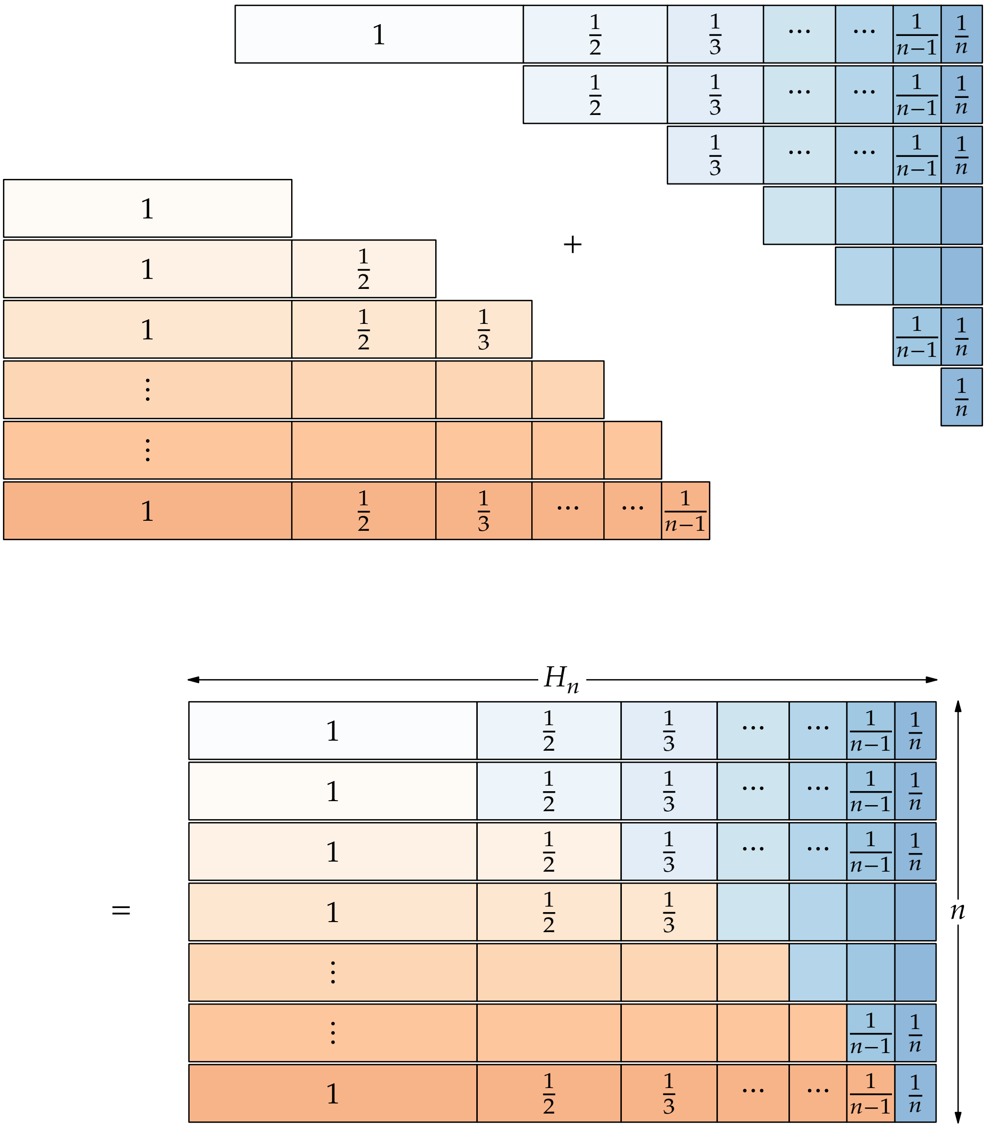


$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+5+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(2n+2n-1)}$$

— Antonio Flores

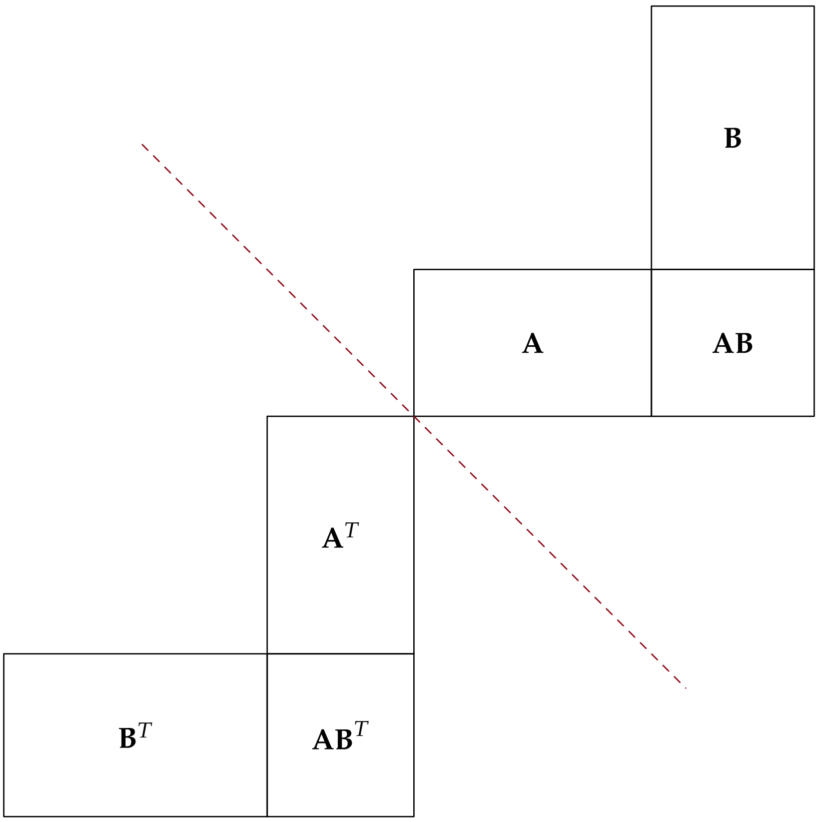
Sums of harmonic numbers

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \;\Rightarrow\; \sum_{k=1}^{n-1} H_k = nH_n - n$$



$$\sum_{k=1}^{n-1} H_k + n = nH_n$$

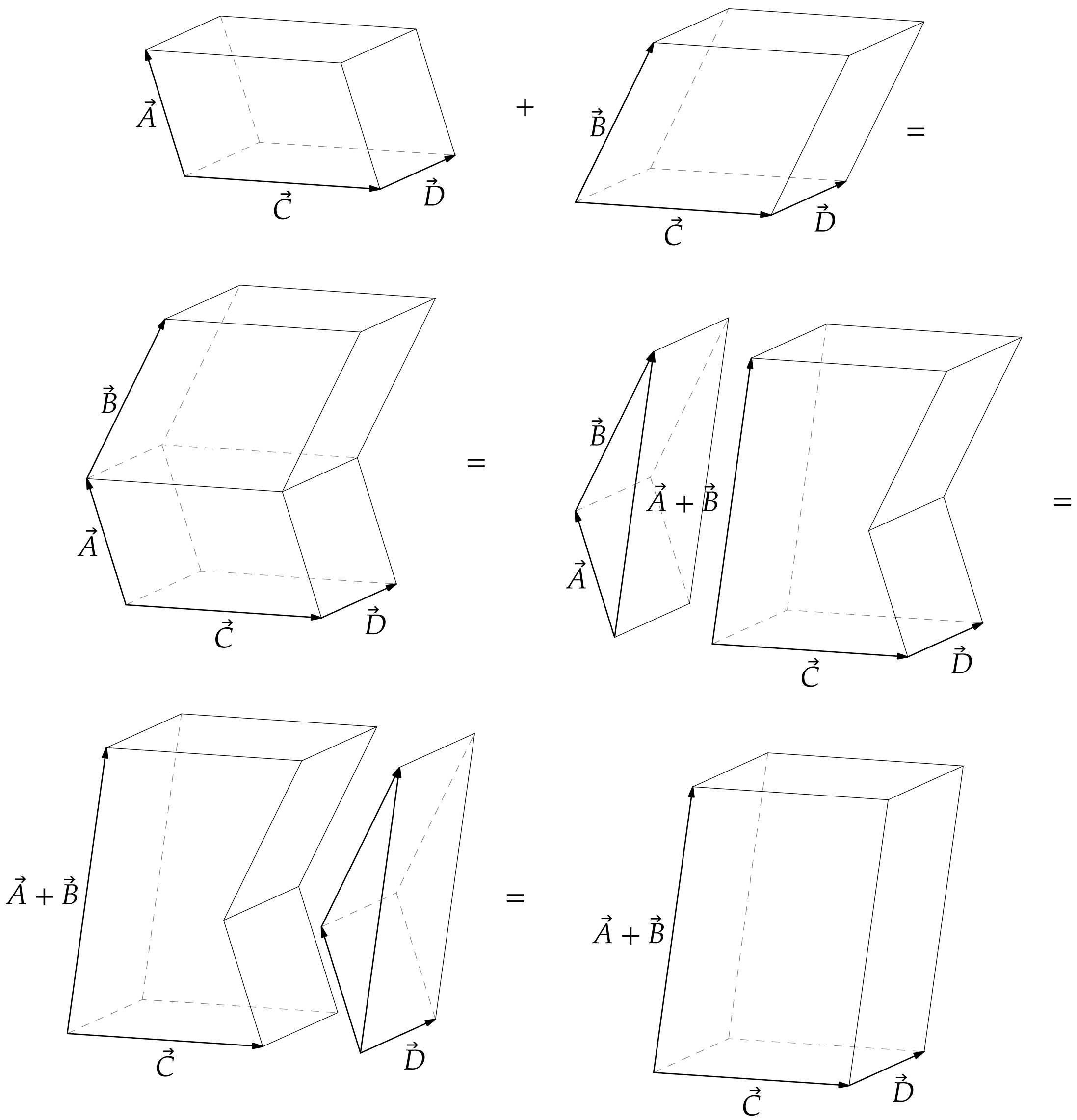
$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, where $\mathbf{A}$ and $\mathbf{B}$ are matrices



— James G. Simmonds

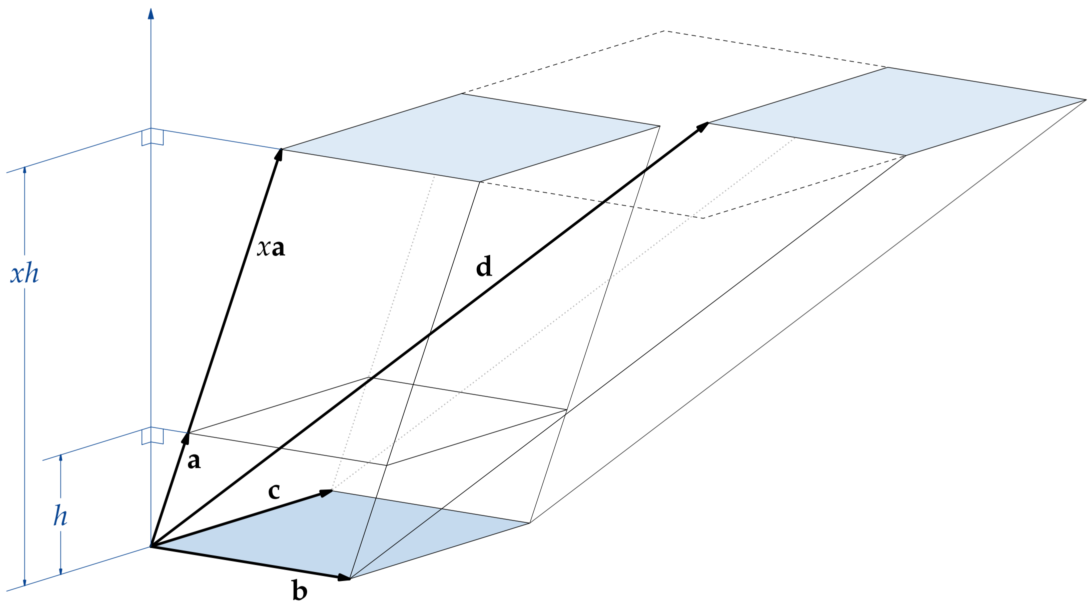
The distributive property of the triple scalar product

$$\vec{A} \cdot \left(\vec{C} \times \vec{D}\right) + \vec{B} \cdot \left(\vec{C} \times \vec{D}\right) = \left(\vec{A} + \vec{B}\right) \cdot \left(\vec{C} \times \vec{D}\right)$$



— Constance C. Edwards and Prashant S. Sansgiry

Cramer's rule



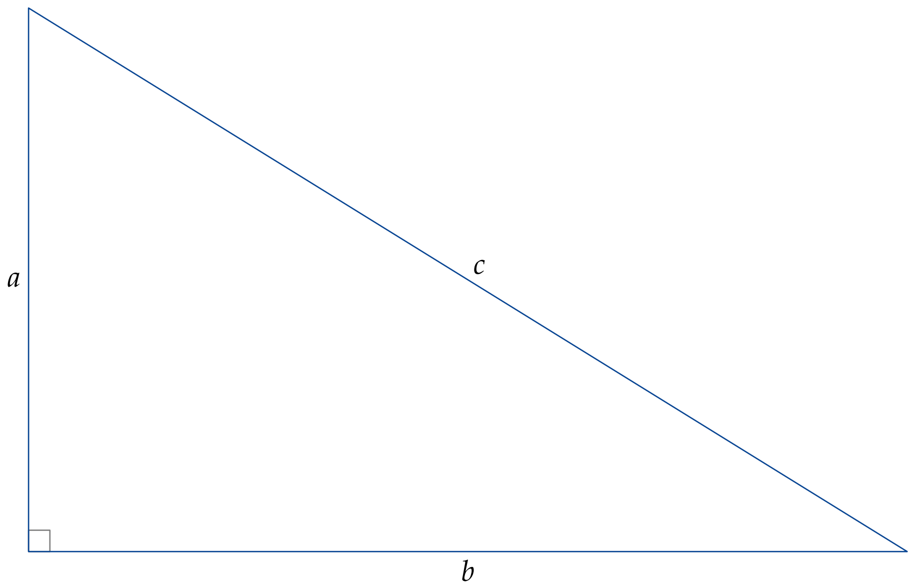
$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d} \quad \Rightarrow \quad \det(\mathbf{d}, \mathbf{b}, \mathbf{c}) = \det(x\mathbf{a}, \mathbf{b}, \mathbf{c}) = x \det(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

$$\therefore \quad x = \frac{\det(\mathbf{d}, \mathbf{b}, \mathbf{c})}{\det(\mathbf{a}, \mathbf{b}, \mathbf{c})}$$

— *The Mathematics Initiative*, Education Development Center

Parametric representation of primitive Pythagorean triples

$$\tfrac{a}{2}, b, c \in \mathbb{Z}^+, \quad (a,b) = 1$$



$$\frac{c+b}{a} = \frac{n}{m}, \ (n,m) = 1 \ \Rightarrow \frac{c-b}{a} = \frac{m}{n},$$

$$\Rightarrow \frac{c}{a} = \frac{n^2+m^2}{2mn}, \ \frac{b}{a} = \frac{n^2-m^2}{2mn},$$

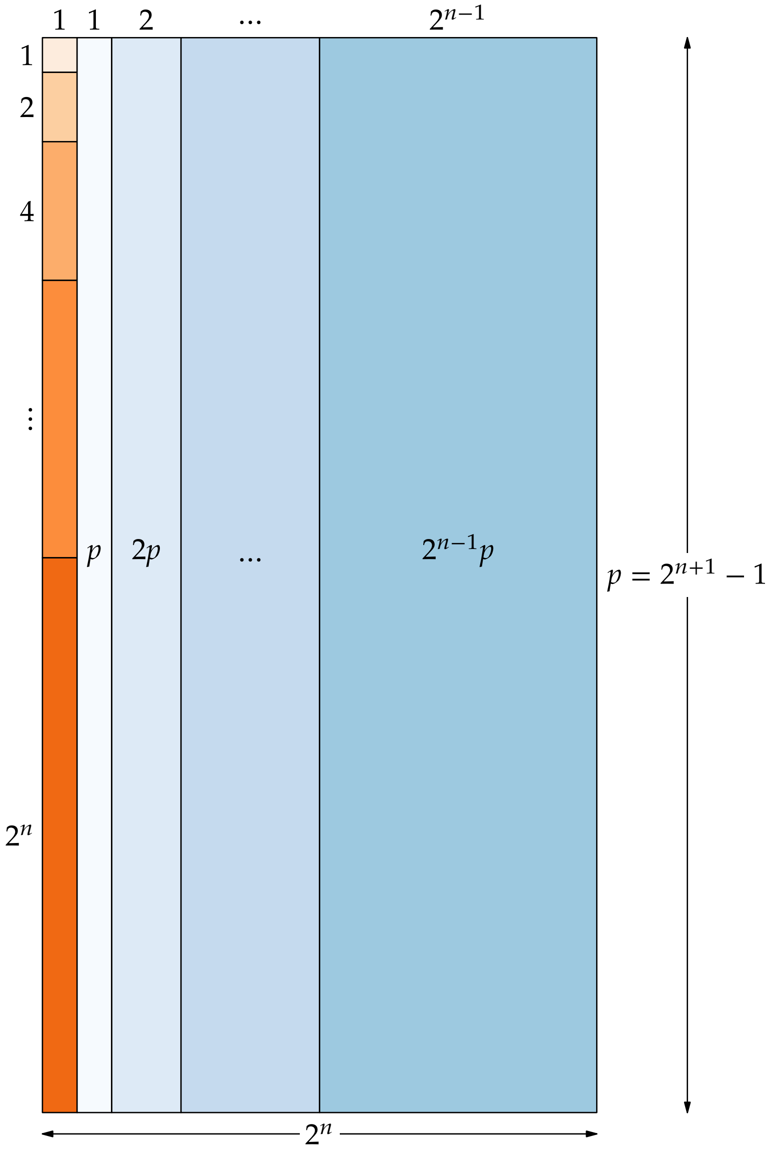
$$\Rightarrow n \not\equiv m \pmod 2.$$

$$\therefore \ (a,b,c) = (2mn, n^2-m^2, n^2+m^2).$$

— Raymond A. Beauregard and E. R. Suryanarayan

On perfect numbers

$$p = 2^{n+1} - 1 \text{ prime} \;\Rightarrow\; N = 2^n p \text{ perfect}$$



$$1 + 2 + \cdots + 2^n + p + 2p + \cdots + 2^{n-1}p = 2^n p = N$$

— Don Goldberg

Self-complementary graphs

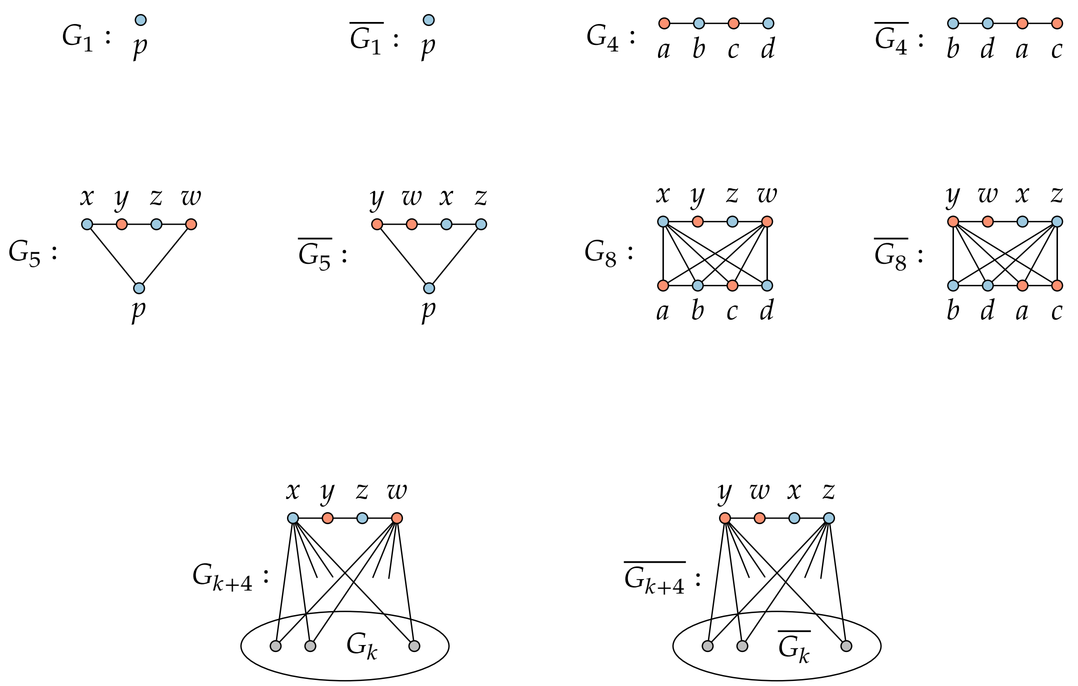
A graph is *simple* if it contains no loops or multiple edges. A simple graph $G = (V, E)$ is *self-complementary* of G is isomorphic to its *complement* $\bar{G} = (V, \bar{E})$, where

$$\bar{E} = \{\{v, w\} : v, w \in V, v \neq w, \text{ and } \{v, w\} \notin E\}.$$

It is a standard exercise to show that if G is a self-complementary simple graph with n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. A converse also holds, as we now show.

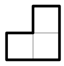
THEOREM: *If n is a positive integer and either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, then there exists a self-complementary simple graph G_n with n vertices.*

PROOF:



— Stephan C. Carlson

Tiling with trominoes

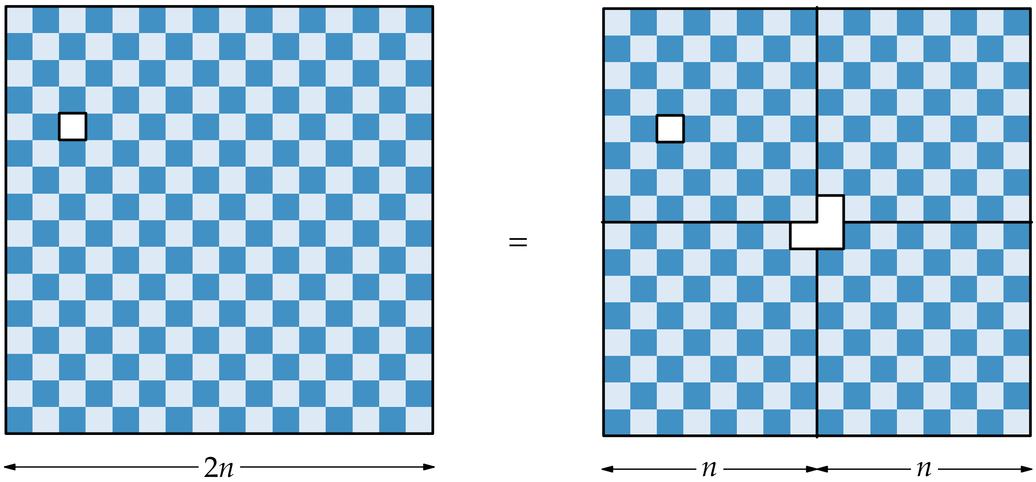
A *tronimo* is a plane figure composed of three squares: 

Theorem: *If n is a power of two, then an $n \times n$ chess board with any one square removed can be tiled with trominoes.*

Proof (by induction):

I.

II.



— Solomon W. Golomb

Note: Except when $n = 5$, an $n \times n$ chessboard with any one square removed can be tiled with tronimoes if and only if $n \not\equiv 0 \pmod 3$. See I-Ping Chu and Richard Johnsonbaugh, “Tiling deficient boards with tronimoes”, *Mathematics Magazine*, 59 (1986) 34–40.