

# Proofs without words I

Exercises in METAPOST

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# Geometry and Algebra

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## The Pythagorean theorem I



— adapted from the *Chou pei san ching*

## The Pythagorean theorem II



*Behold!*

— Bhāskara (12th century)

## The Pythagorean theorem III



— based on Euclid's proof

## The Pythagorean theorem IV



— H. E. Dudeney (1917)

## The Pythagorean theorem V



— James A. Garfield (1876)



## The Pythagorean theorem VI

$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2 + b^2 = c^2$$



— Michael Hardy

**A Pythagorean theorem:  $aa' = bb' + cc'$**



$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$

$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$

$$\therefore aa' = a(x + y) = bb' + cc'.$$

— Enzo R. Gentile

## The rolling circle squares itself



— Thomas Elsner

## On trisecting an angle



— Rufus Isaacs

## Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

— Eric Kincanon

# Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

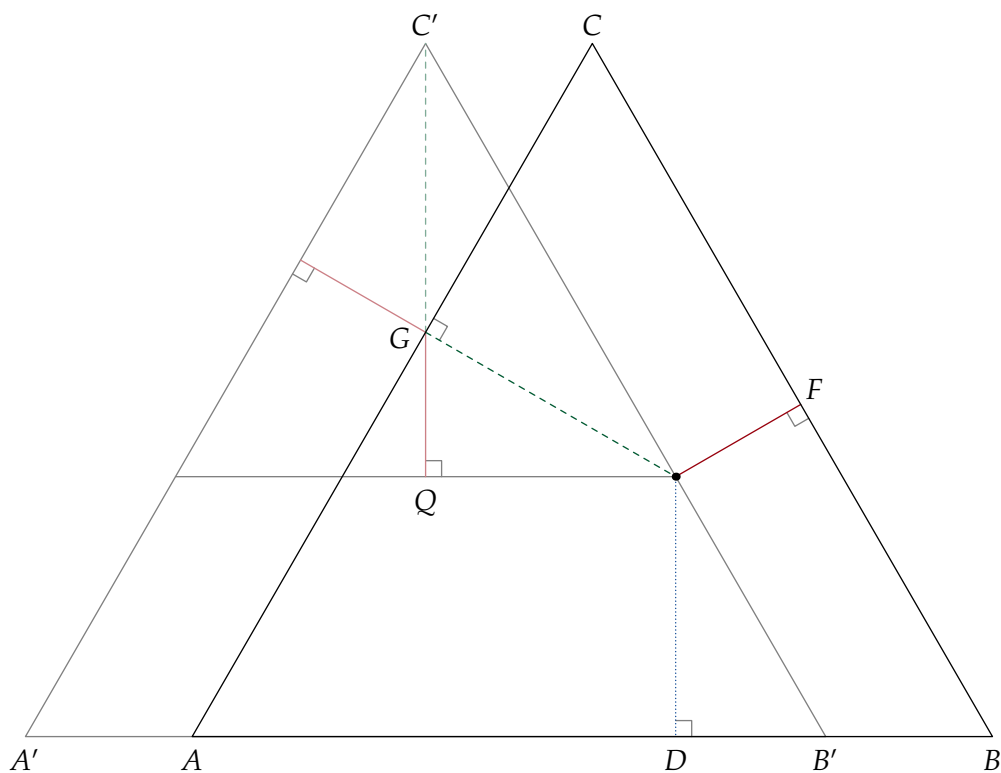
**The vertex angles of a star sum to  $180^\circ$**



— Fouad Nakhli

## Viviani's theorem I

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



*This shows a particular example, with  $C'GQ$  collinear, rather than the general case*

— Samuel Wolf



## Viviani's theorem II

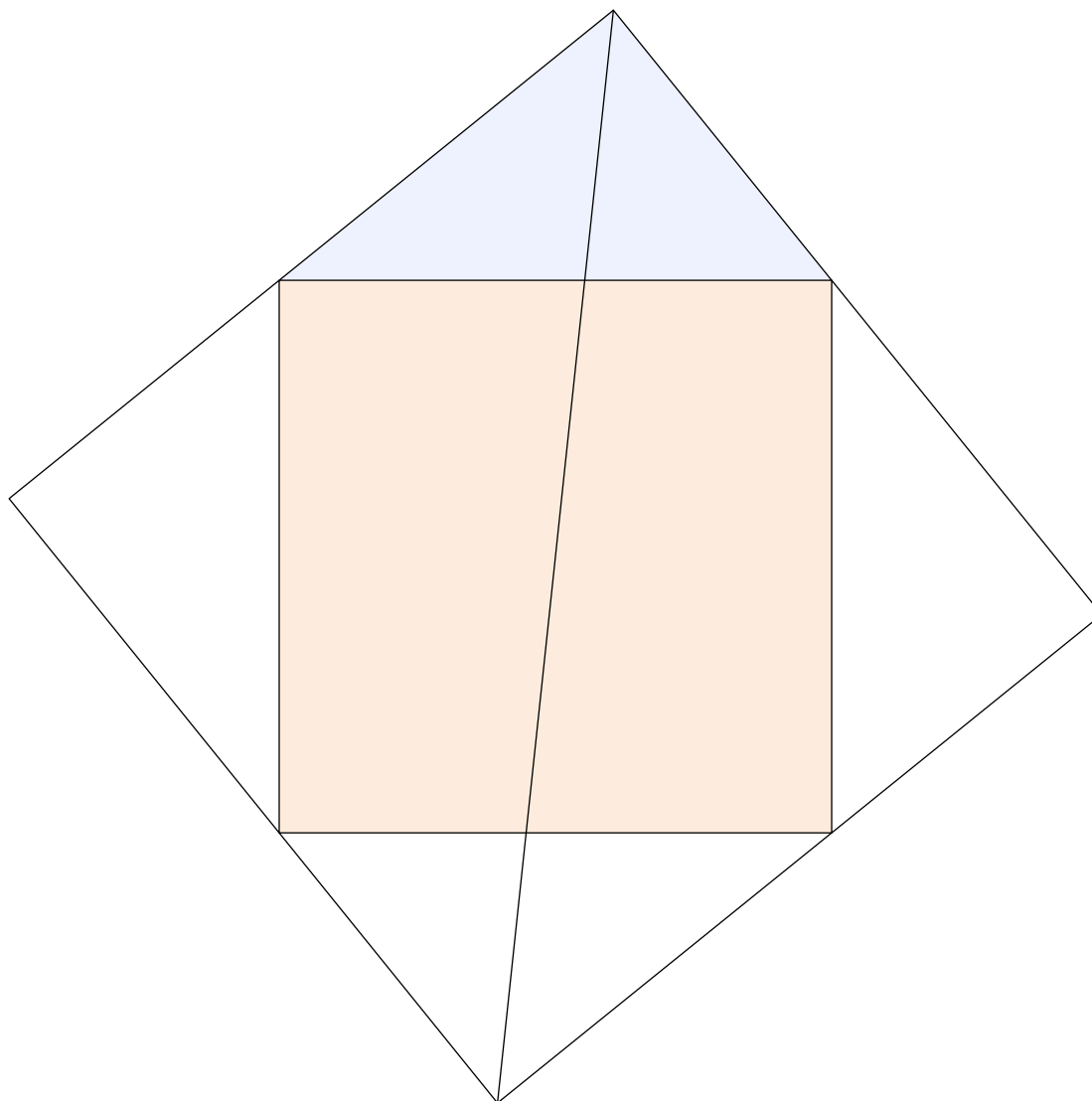
The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

## **A theorem about right angles**

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse



— Roland H. Eddy

# Area and the projection theorem of a right triangle

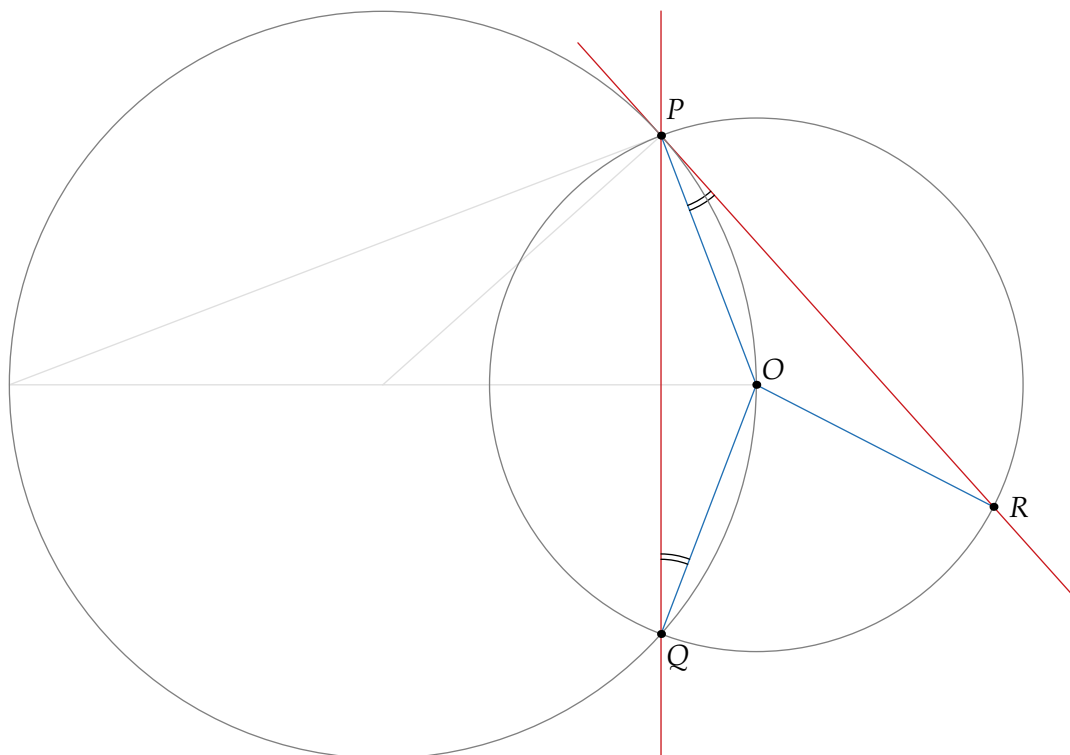


$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

## Chords and tangents of equal length

If circle  $C_1$  passes through the center  $O$  of circle  $C_2$ , the length of the common chord  $\overline{PQ}$  is equal to the tangent segment  $\overline{PR}$ .



— Roland H. Eddy

## Completing the square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

## Algebraic areas I

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$



— Shirley Wakin

## Algebraic areas II

$$(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$



— Sam Pooley and K. Ann Drude

## Sum of squares identity

$$(a^2 + b^2)(c^2 + d^2) = (ab + bc)^2 + (bd - ac)^2$$

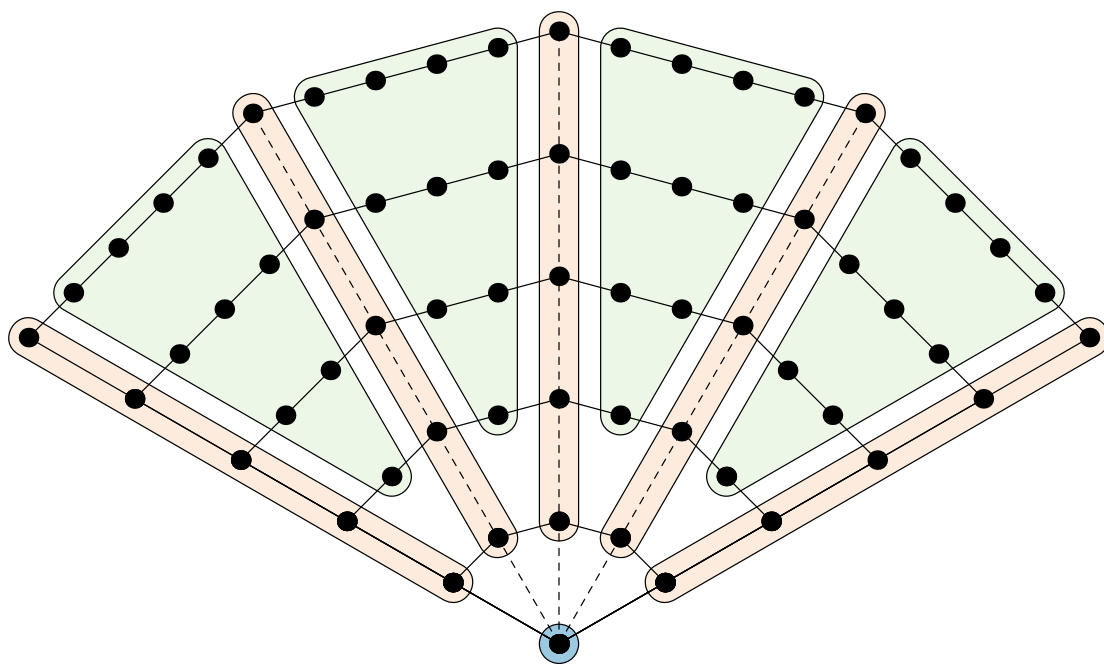


— Diophantus of Alexandria



## Polygonal numbers

The  $k^{\text{th}}$   $n$ -gonal number is  $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$



— Dave Logothetti

## The volume of a frustum of a square pyramid



$$V_{P_1} = \frac{h}{b-a} \cdot V_{P_2} = \frac{h}{b-a} \cdot \frac{1}{3} (b^3 - a^3) = \frac{h}{3} (a^2 + ab + b^2)$$

— The Moscow Papyrus, c. 1850 BCE

# The volume of a hemisphere via Cavalieri's Principle



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

— Sidney H. Kung

# Trigonometry, Calculus, & Analytic Geometry

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## Sine of the sum

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \text{ for } \alpha + \beta < \pi$$



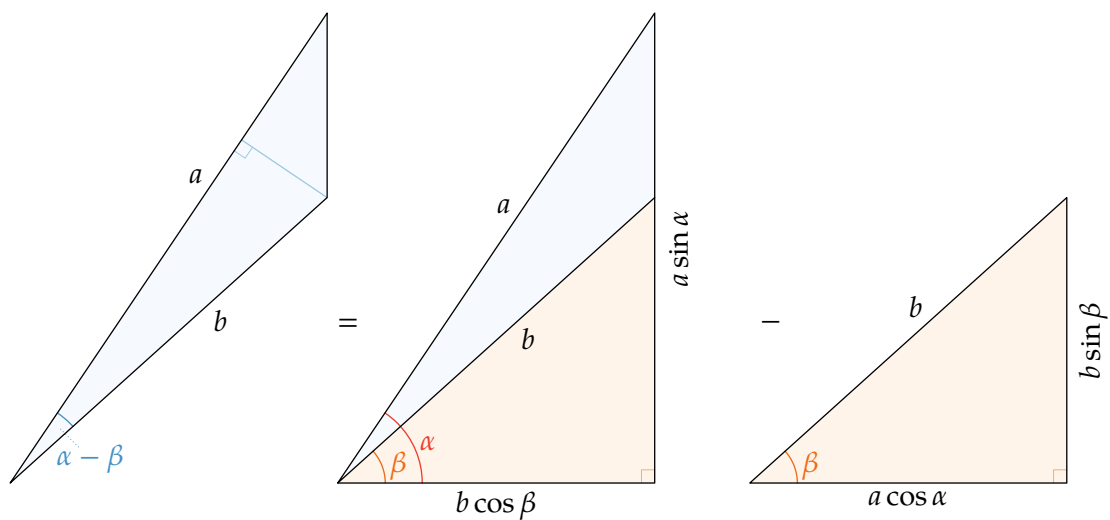
$$c = a \cos \beta + b \cos \alpha$$

$$r = 1/2 \implies \sin \gamma = \frac{c/2}{1/2} = c, \sin \alpha = a, \sin \beta = b$$

$$\sin(\alpha + \beta) = \sin(\pi - (\alpha + \beta)) = \sin \gamma = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

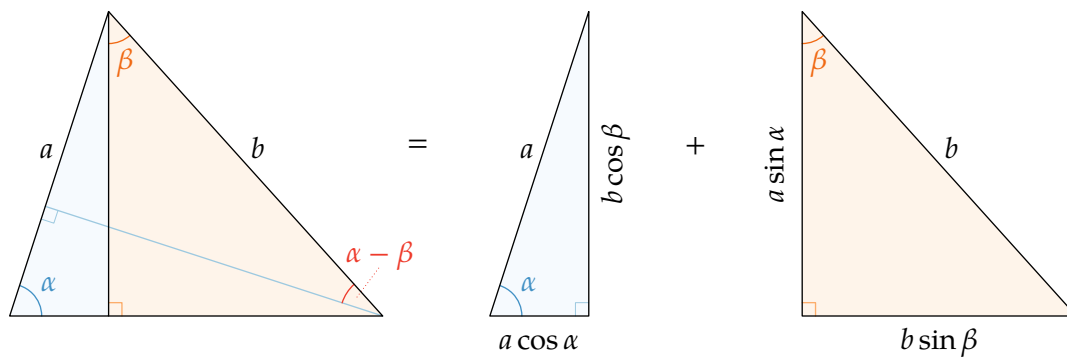
— Sidney H. Kung

## Area and difference formulas



$$\frac{1}{2} \cdot a \cdot b \sin(\alpha - \beta) = \frac{1}{2} \cdot a \sin \alpha \cdot b \cos \beta - \frac{1}{2} \cdot a \cos \alpha \cdot b \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\frac{1}{2} \cdot a \cdot b \cos(\alpha - \beta) = \frac{1}{2} \cdot a \cos \alpha \cdot b \cos \beta + \frac{1}{2} \cdot a \sin \alpha \cdot b \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

— Sidney H. Kung

## The law of cosines I



$$\begin{aligned}
 c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\
 &= b^2 \sin^2 \theta + a^2 - 2ab \cos \theta + b^2 \cos^2 \theta \\
 &= a^2 + b^2 (\sin^2 \theta + \cos^2 \theta) - 2ab \cos \theta \\
 &= a^2 + b^2 - 2ab \cos \theta
 \end{aligned}$$

— Timothy A. Sipka

## The law of cosines II



$$(2a \cos \theta - b) \cdot b = (a - c) \cdot (a + c)$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

— Sidney H. Kung



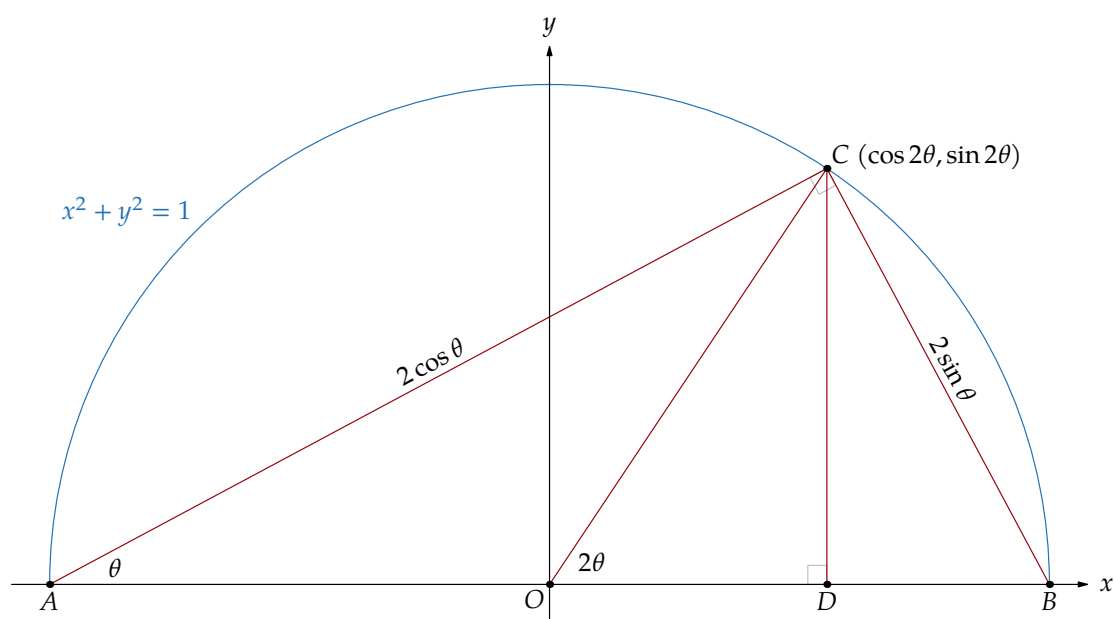
### The law of cosines III (via Ptolemy's theorem)



$$\begin{aligned}c \cdot c &= b \cdot b + (a + 2b \cos(\pi - \theta)) \cdot a \\c^2 &= a^2 + b^2 - 2ab \cos \theta\end{aligned}$$

— Sidney H. Kung

## The double-angle formulae



$$\triangle ACD \sim \triangle ABC$$

$$CD/AC = BC/AB$$

$$\sin 2\theta / 2 \cos \theta = 2 \sin \theta / 2$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$AD/AC = AC/AB$$

$$(1 + \cos 2\theta) / 2 \cos \theta = 2 \cos \theta / 2$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

— Roger B. Nelsen

## The half-angle tangent formulae



$$\tan \theta/2 = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

— R. J. Walker

## Mollweide's equation

$$(a - b) \cos \frac{\gamma}{2} = c \sin \left( \frac{\alpha - \beta}{2} \right)$$



— H. Arthur DeKleine

## Tangent, cotangent, secant, and cosecant



$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$(\tan \theta + 1)^2 + (\cot \theta + 1)^2 = (\sec \theta + \csc \theta)^2$$

$$\text{also } \tan \theta = \frac{\tan \theta + 1}{\cot \theta + 1}$$

— William Romaine

## Substitution to make a rational function of sine and cosine



$$z = \tan(\theta/2) \implies \sin \theta = \frac{2z}{1+z^2} \quad \text{and} \quad \cos \theta = \frac{1-z^2}{1+z^2}$$

— Roger B. Nelsen

## Sums of arctangents



$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

— Edward M. Harris

## The distance between a point and a line



$$\frac{d}{1} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

— R. L. Eisenman



**The midpoint rule is better than the trapezoidal rule for concave functions**



— Frank Burk

## Integration by parts



$$\text{Area } \text{blue} + \text{Area } \text{orange} = qs - pr$$

$$\int_r^s u \, dv + \int_p^q v \, du = uv \Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) \, dx$$

— Richard Courant

**The graphs of  $f$  and  $f^{-1}$  are reflections about the line  $y = x$**



— Ayoub B. Ayoub



## Area under an arch of the cycloid



$$\frac{1}{2}\pi r \cdot 2r \quad + \quad \pi r^2 \quad + \quad \frac{1}{2}\pi r \cdot 2r$$

therefore  $A = 3\pi r^2$

— Richard M. Beekman

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## The arithmetic mean – geometric mean inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

— Charles D. Gallant

## The arithmetic mean – geometric mean inequality II



$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

— Doris Schattschneider



### The arithmetic mean – geometric mean inequality III

$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \text{with equality iff } a = b$$



— Roland H. Eddy

## Two extremum problems

For a given product, the sum of two positive numbers is minimal when the numbers are equal.

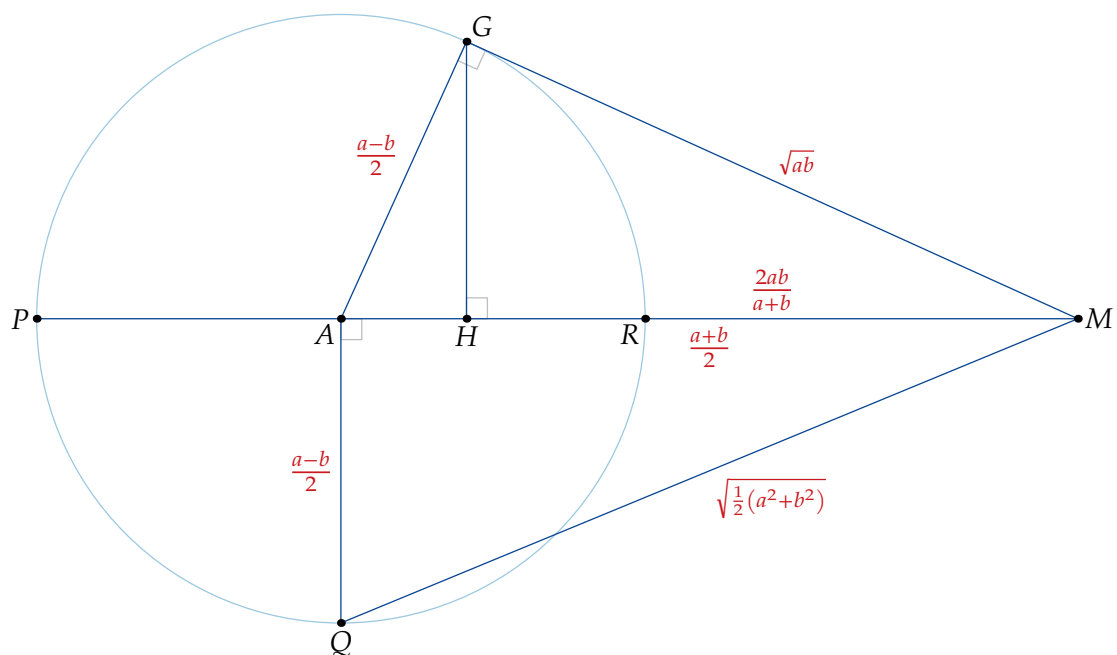


For a given sum, the product of two positive numbers is maximal when the numbers are equal.



— Paulo Montuchi and Warren Page

## The HM–GM–AM–QM inequalities I



$$PM = a, \quad RM = b, \quad a > b > 0$$

$$HM < GM < AM < QM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{1}{2}(a^2 + b^2)}$$

— Roger B. Nelsen

## The HM–GM–AM–QM inequalities II



$$AB = a, \quad BC = b, \quad AD = DC = \frac{a+b}{2}$$

$$BE \perp AC, \quad DE = AD$$

$$FE \perp ED, \quad FB \parallel ED, \quad EG = BD = \frac{b-a}{2}$$

— Sidney H. Kung

# The HM–GM–AM–QM inequalities III



$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\frac{1}{2}(a^2 + b^2)} \geq \frac{a+b}{2}$$



$$(\sqrt{a} + \sqrt{b})^2 \geq 4 \cdot \frac{1}{2} \sqrt{a} \sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$



$$1 \geq 4 \cdot \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

— Roger B. Nelsen

# Five means — and their means



— Roger B. Nelsen

$$e^\pi > \pi^e$$



— Fouad Nakhli

$A^B > B^A$  **for**  $e \leq A < B$



$$\begin{aligned}
 e \leq A < B &\implies m_A > m_B \\
 &\implies \frac{\ln A}{A} > \frac{\ln B}{B} \\
 &\implies A^B > B^A
 \end{aligned}$$

— Charles D. Gallant



## The mediant property

$$\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$



— Richard A. Gibbs

## Regle des nombres moyens – two proofs

$$a, b, c, d > 0; \quad \frac{a}{b} < \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

I.



$$m_1 < m_3 \quad \Rightarrow \quad m_1 < m_2 < m_3$$

— Li Changming

II.



$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

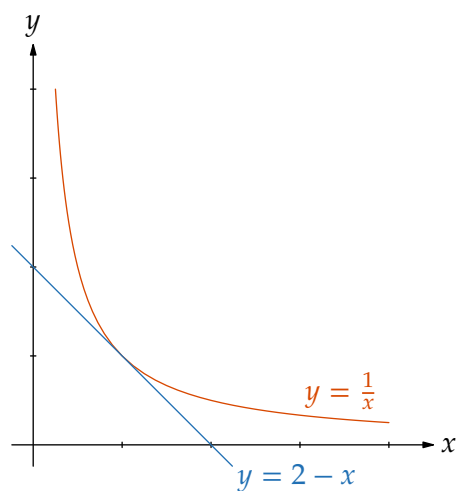
— Roger B. Nelsen

# The sum of a positive number and its reciprocal is at least two

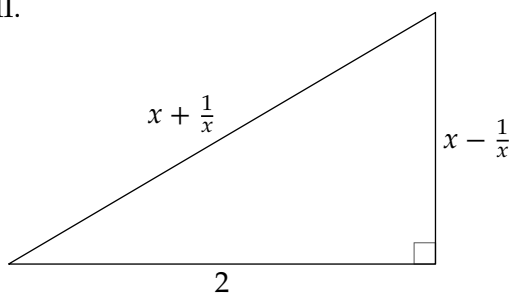
I.



II.



III.



IV.



— Roger B. Nelsen

# Aristarchus' inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \implies \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$



$$\sin \alpha < \frac{\sin \beta}{\beta} \alpha; \quad \tan \alpha > \frac{\tan \beta}{\beta} \alpha$$

$$\therefore \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

— Roger B. Nelsen

## The Cauchy-Schwartz inequality

$$|\langle a, b \rangle \cdot \langle x, y \rangle| \leq \|\langle a, b \rangle\| \|\langle x, y \rangle\|$$



$$(|a| + |y|) (|b| + |x|) \leq 2 \left( \frac{1}{2} |a| |b| + \frac{1}{2} |x| |y| \right) + \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

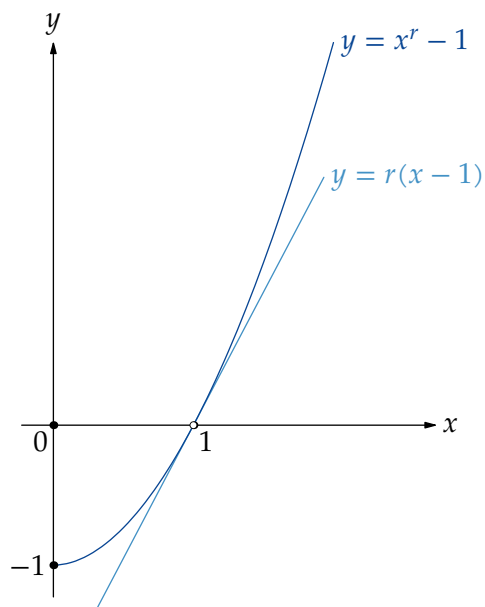
$$\therefore |ax + by| \leq |a||x| + |b||y| \leq \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

— Roger B. Nelsen

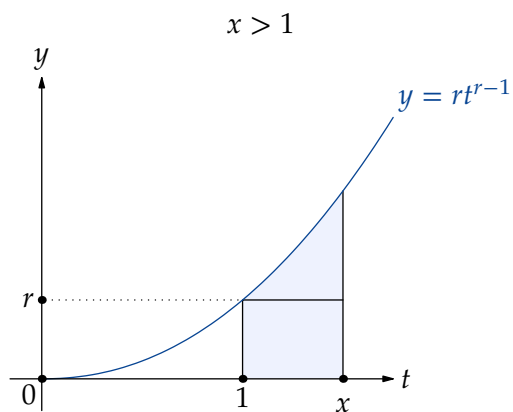
## Bernoulli's inequality

$$x > 0, x \neq 1, r > 1: x^r - 1 > r(x - 1)$$

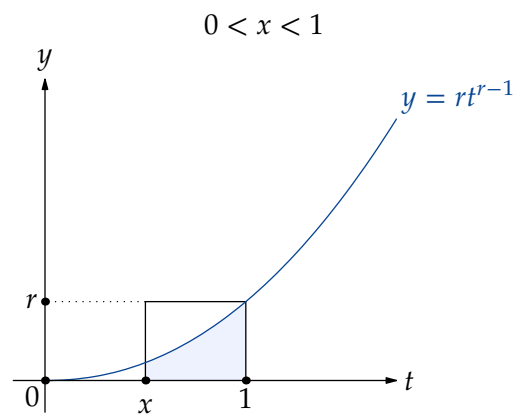
I. First semester calculus



II. Second semester calculus



$$x^r - 1 = \int_1^x rt^{r-1} dt > r(x - 1)$$



$$1 - x^r = \int_x^1 rt^{r-1} dt < r(1 - x)$$

— Roger B. Nelsen

## Napier's inequality

$$b > a > 0 \quad \text{implies} \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

I. First semester calculus



$$m(L_1) < m(L_2) < m(L_3)$$

II. Second semester calculus



$$\frac{1}{b}(b - a) < \int_a^b \frac{1}{x} dx < \frac{1}{a}(b - a)$$

— Roger B. Nelsen

# Integer sums

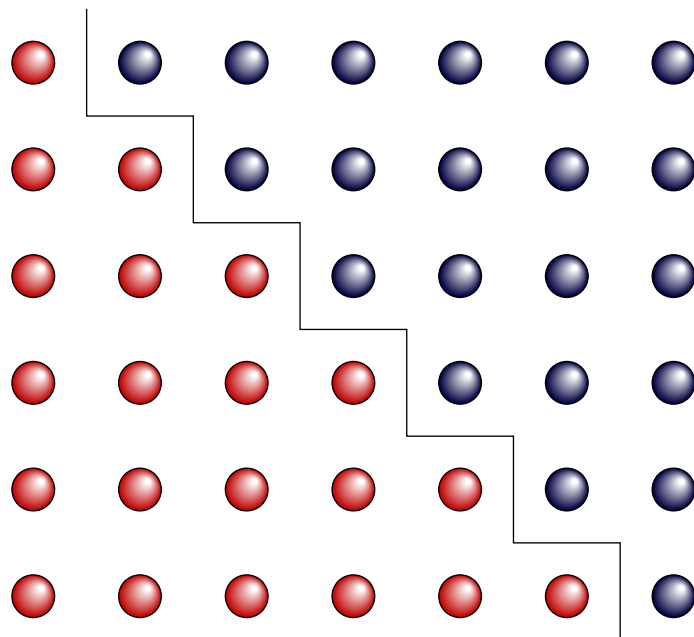
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## *Integer sums*

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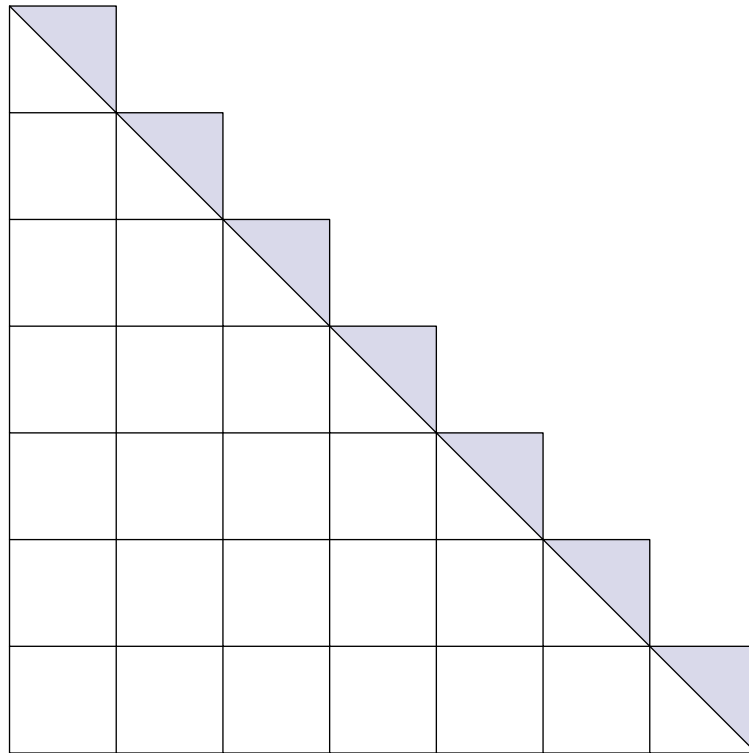
## Sums of integers I



$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$$

— Ancient Greek

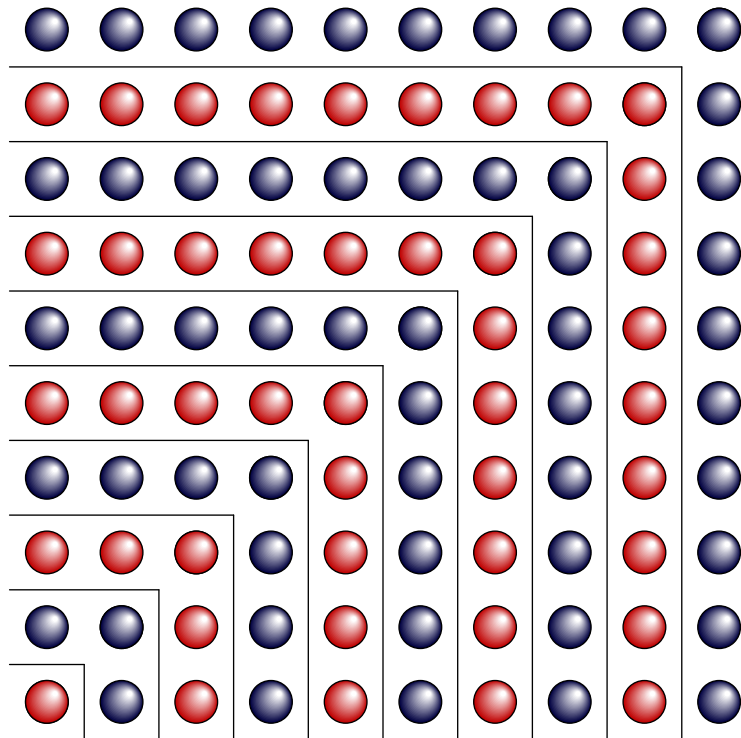
## Sums of integers II



$$1 + 2 + \dots + n = \frac{n^2}{2} + \frac{n}{2}$$

— Ian Richards

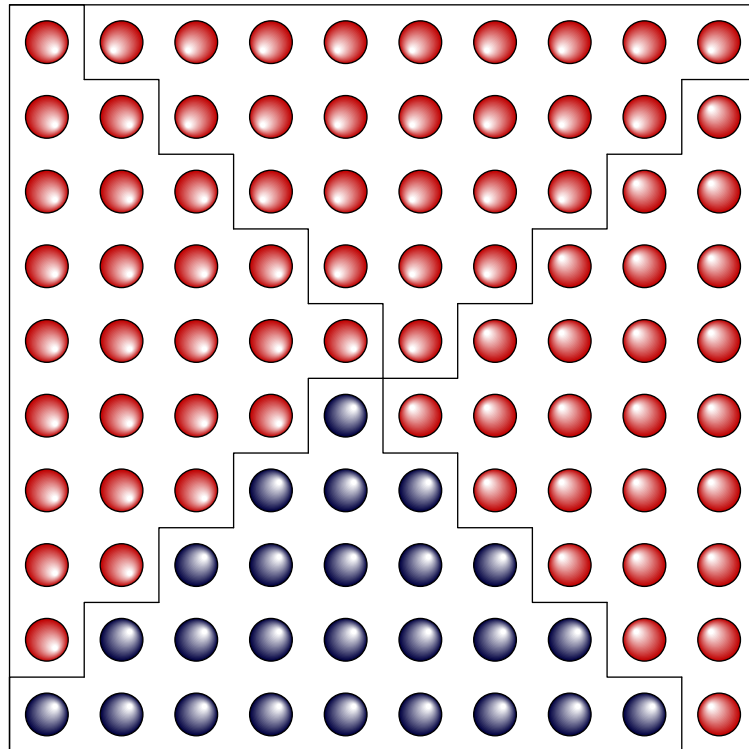
## Sums of odd integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

— Nichomachus of Gerasa

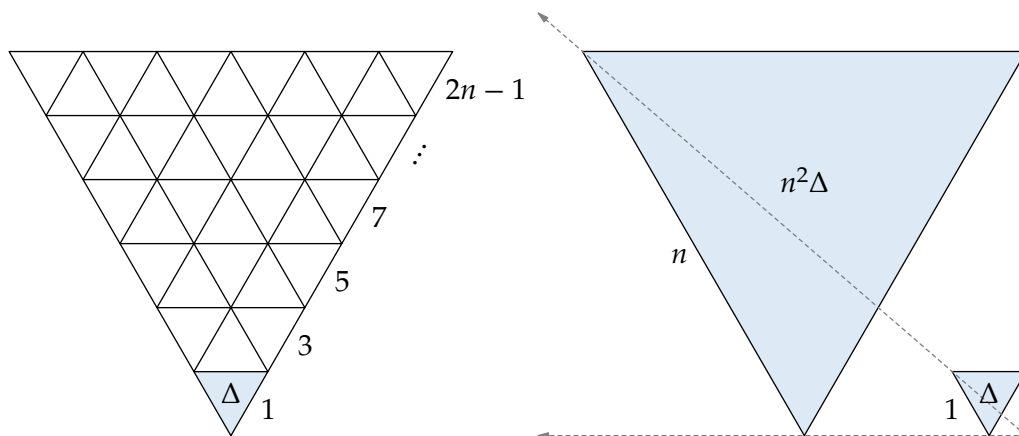
## Sums of odd integers II



$$1 + 3 + \cdots + (2n - 1) = \frac{1}{4} (2n)^2 = n^2$$

— Roger B. Nelsen

### Sums of odd integers III

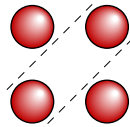


$$\Delta + 3 \cdot \Delta + \dots + (2n - 1) \cdot \Delta = A = n^2 \cdot \Delta$$

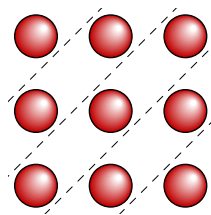
$$\sum_{i=1}^n (2i - 1) = n^2$$

— Jenő Lehel

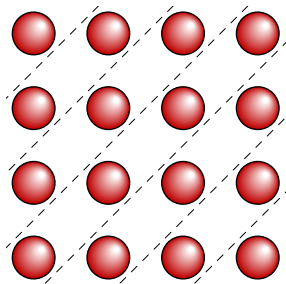
## Squares and sums of integers I



$$1 + 2 + 1 = 2^2$$



$$1 + 2 + 3 + 2 + 1 = 3^2$$

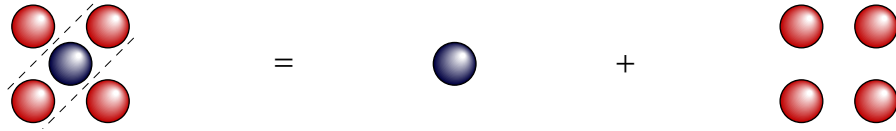


$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

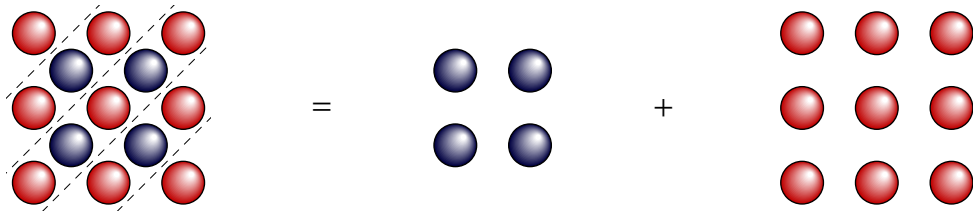
$$1 + 2 + \cdots + (n - 1) + n + (n - 1) + \cdots + 2 + 1 = n^2$$

— Ancient Greek

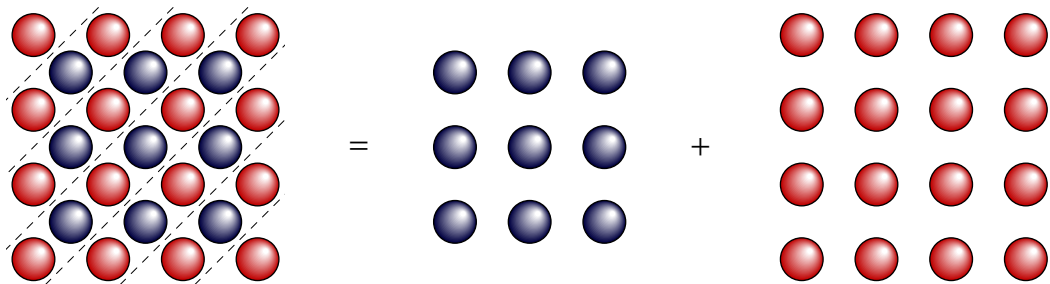
## Squares and sums of integers II



$$1 + 3 + 1 = 1^2 + 2^2$$



$$1 + 3 + 5 + 3 + 1 = 2^2 + 3^2$$



$$1 + 3 + 5 + 7 + 5 + 3 + 1 = 3^2 + 4^2$$

$\vdots$

$$1 + 3 + \cdots + (2n - 1) + (2n + 1) + (2n - 1) + \cdots + 3 + 1 = n^2 + (n + 1)^2$$

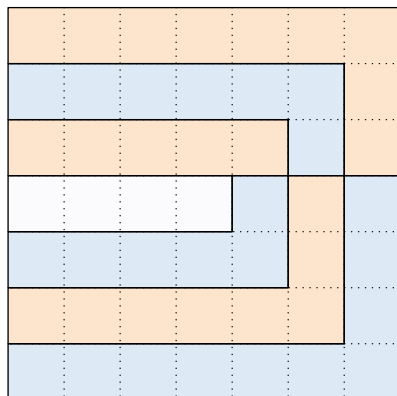
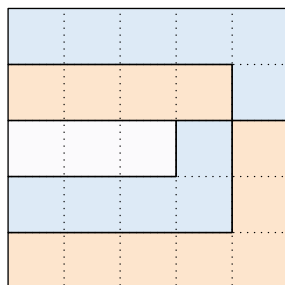
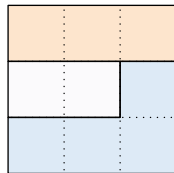
— Hee Sik Kim



# Arithmetic progressions with sum equal to square of number of terms



$$\sum_{k=n}^{3n-2} k = (2n-1)^2; n = 1, 2, 3, \dots$$



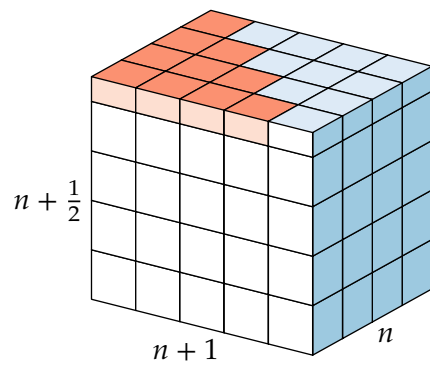
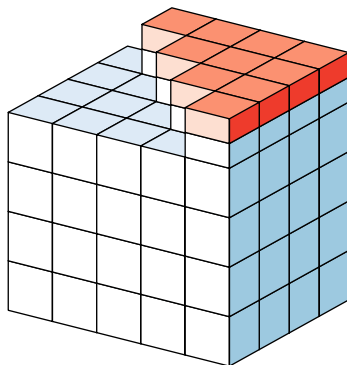
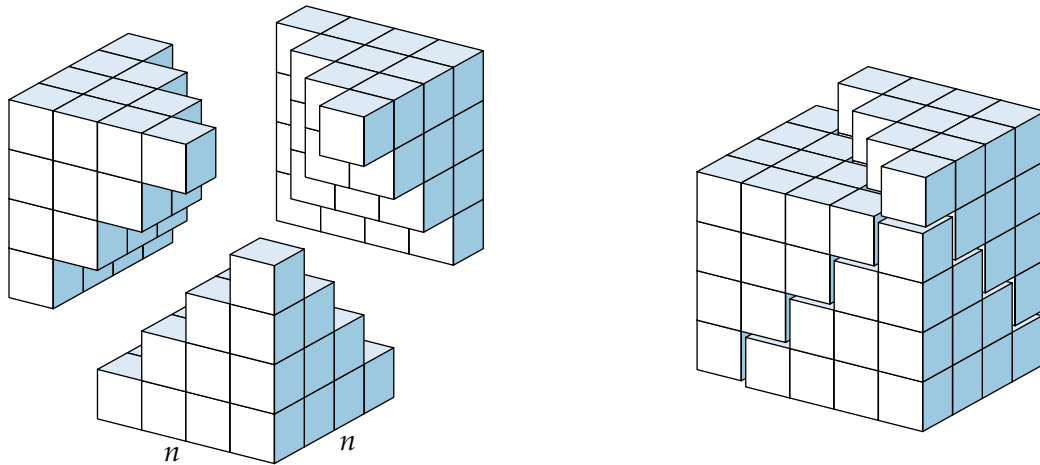
$$n = 4$$

$$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$$

— James O. Chilaka

## Sums of squares I

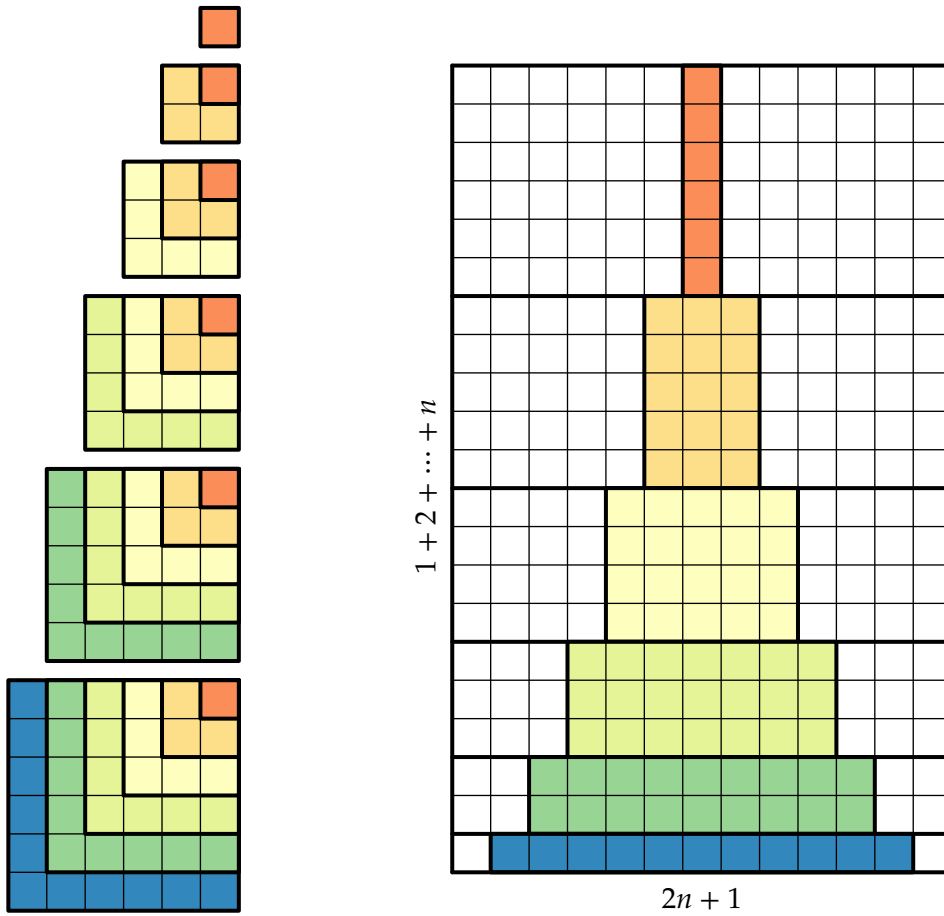
$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)\left(n + \frac{1}{2}\right)$$



— Man-Keung Siu

## Sums of squares II

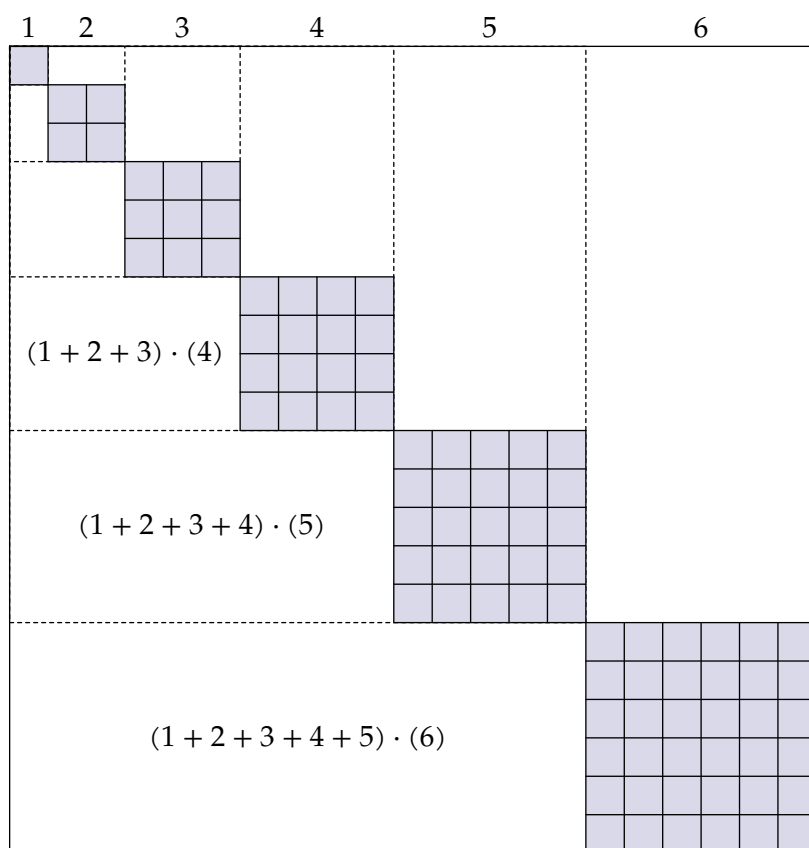
$$3(1^2 + 2^2 + \cdots + n^2) = (2n + 1)(1 + 2 + \cdots + n)$$



— Dan Kalman

## Sums of squares IV

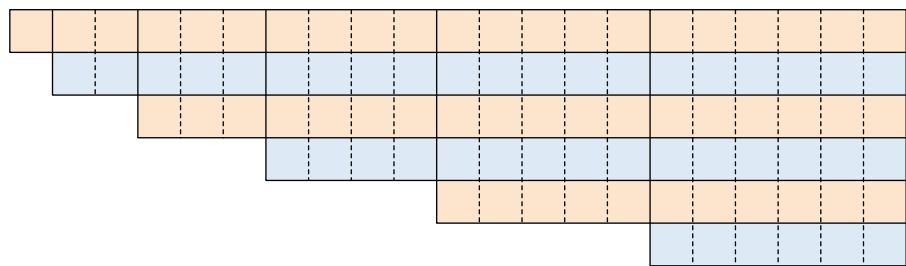
$$\sum_{k=1}^n k^2 = \left( \sum_{k=1}^n k \right)^2 - 2 \sum_{k=1}^{n-1} \left( (k+1) \sum_{i=1}^k i \right)$$



— James O.Chilaka

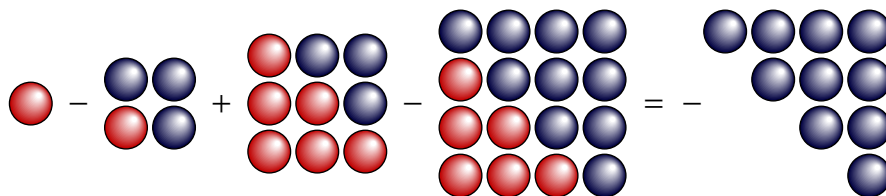
Sums of squares V

$$\sum_{i=1}^n \sum_{j=i}^n j = \sum_{i=1}^n i^2$$



## Alternating sums of squares

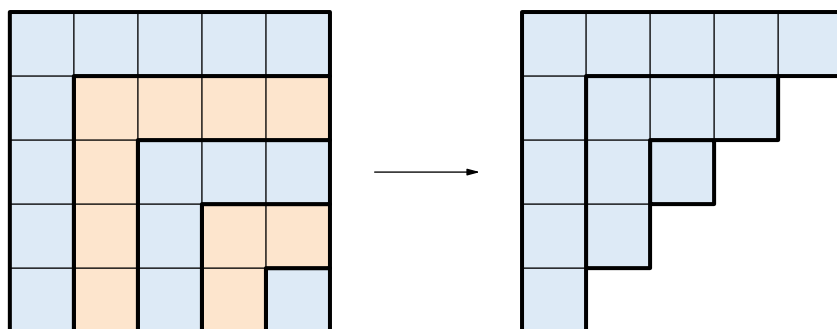
I.



$$\sum_{k=1}^n (-1)^{k+1} k^2 = (-1)^{n+1} T_n = (-1)^{n+1} \frac{n(n+1)}{2}$$

— Dave Logothetti

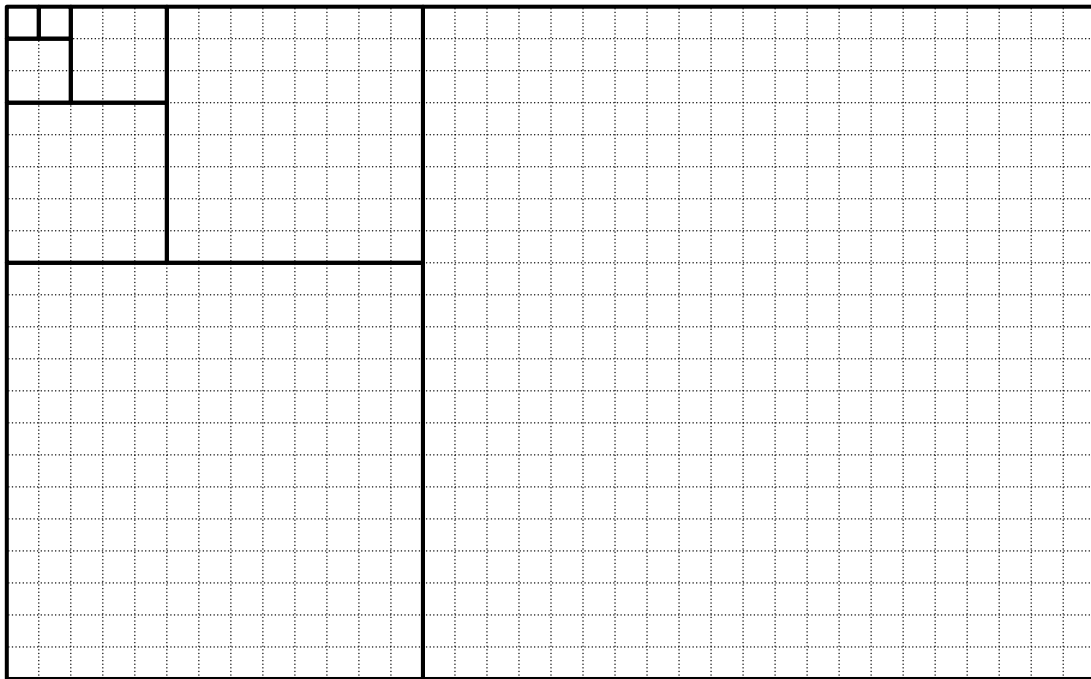
II.



$$n^2 - (n-1)^2 + \cdots + (-1)^{n-1} (1)^2 = \sum_{k=0}^n (-1)^k (n-k)^2 = \frac{n(n+1)}{2}$$

— Steven L. Snover

## Sums of squares of Fibonacci numbers

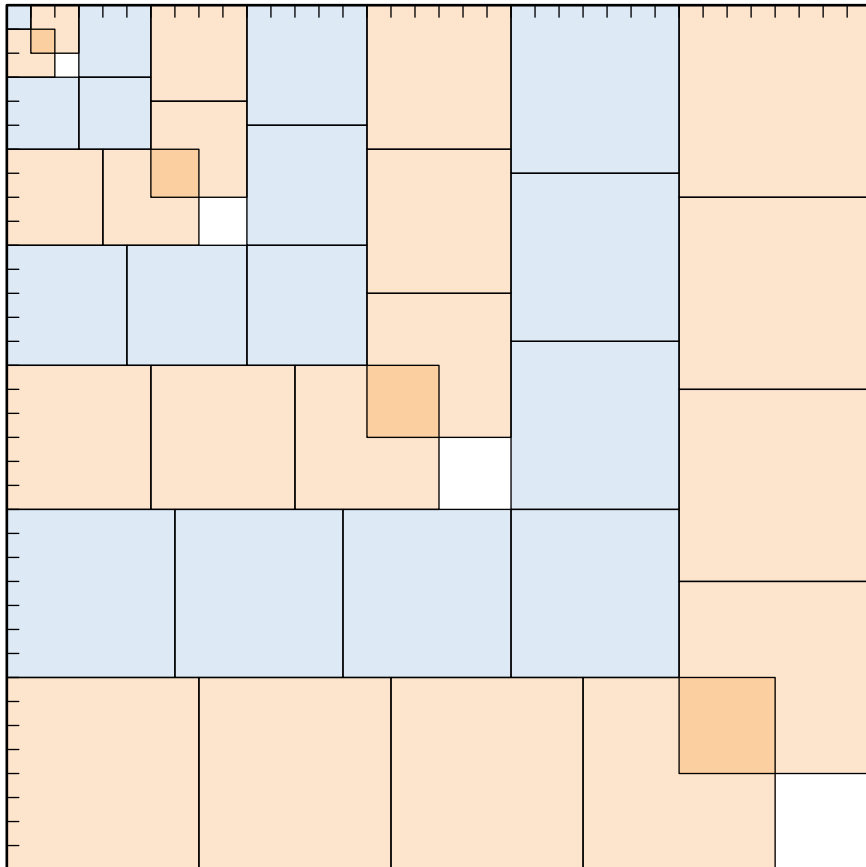


$$F_1 = F_2 = 1; F_{n+2} = F_{n+1} + F_n \quad \text{hence} \quad F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

— Alfred Brousseau

## Sums of cubes I

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

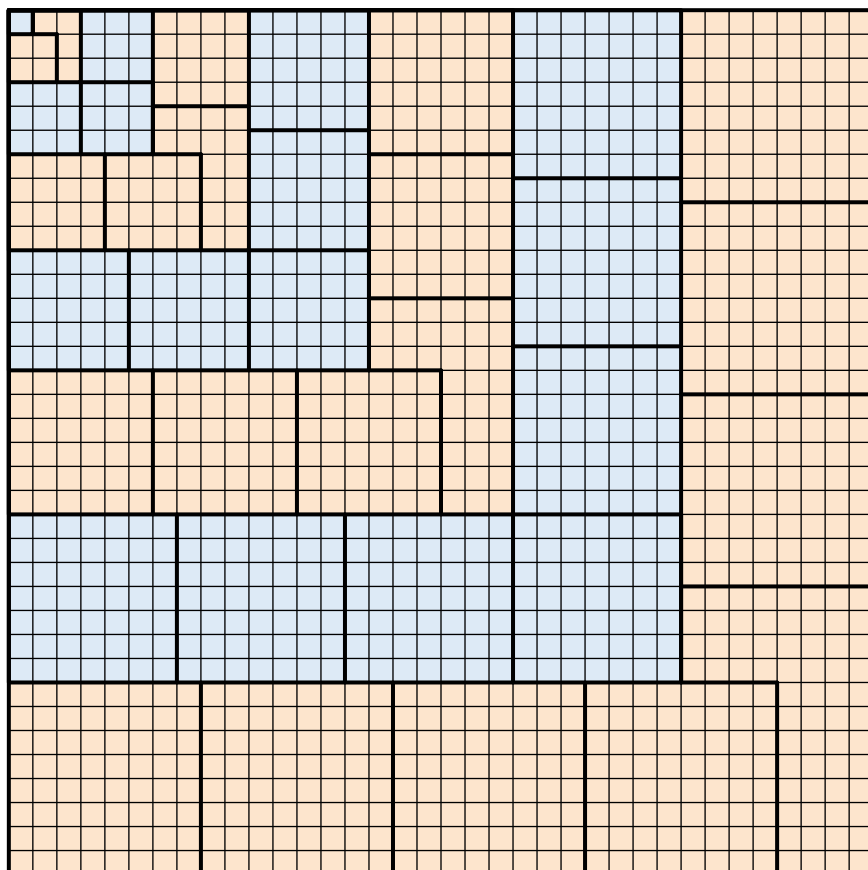


— Solomon W. Golomb



## Sums of cubes II

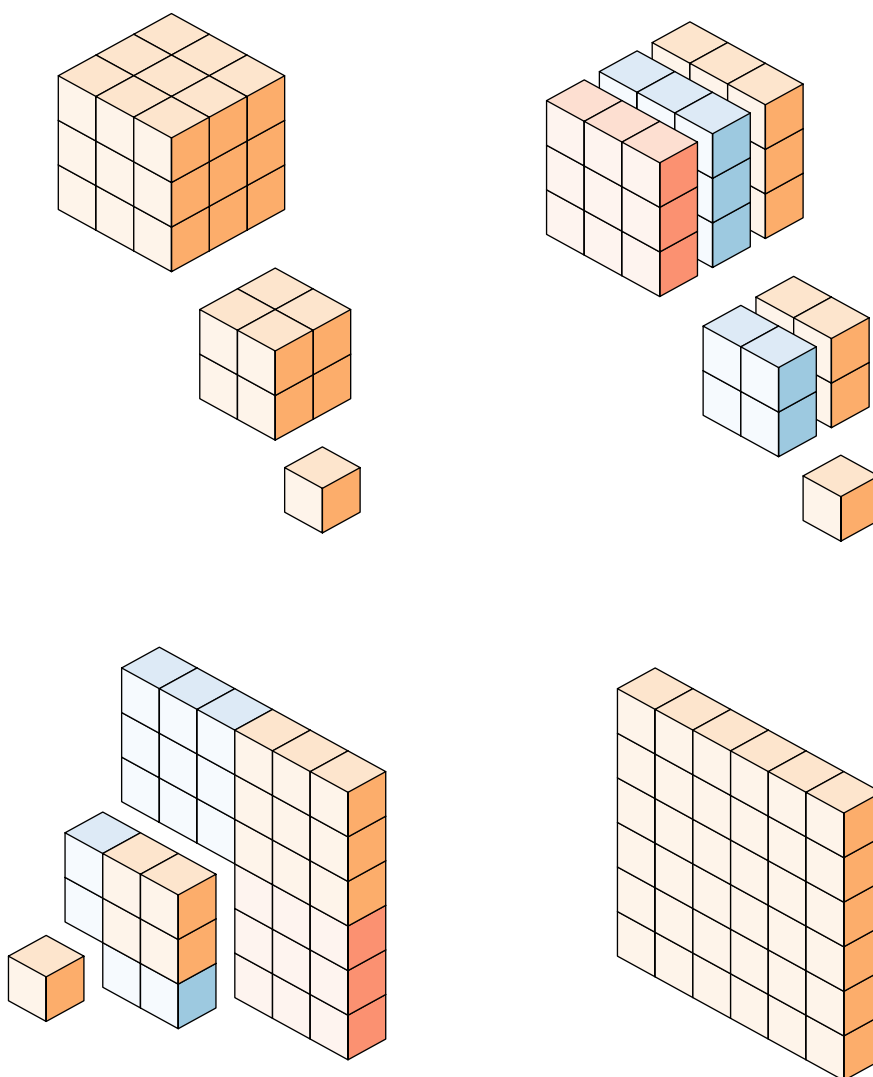
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— J. Barry Love

# Sums of cubes III

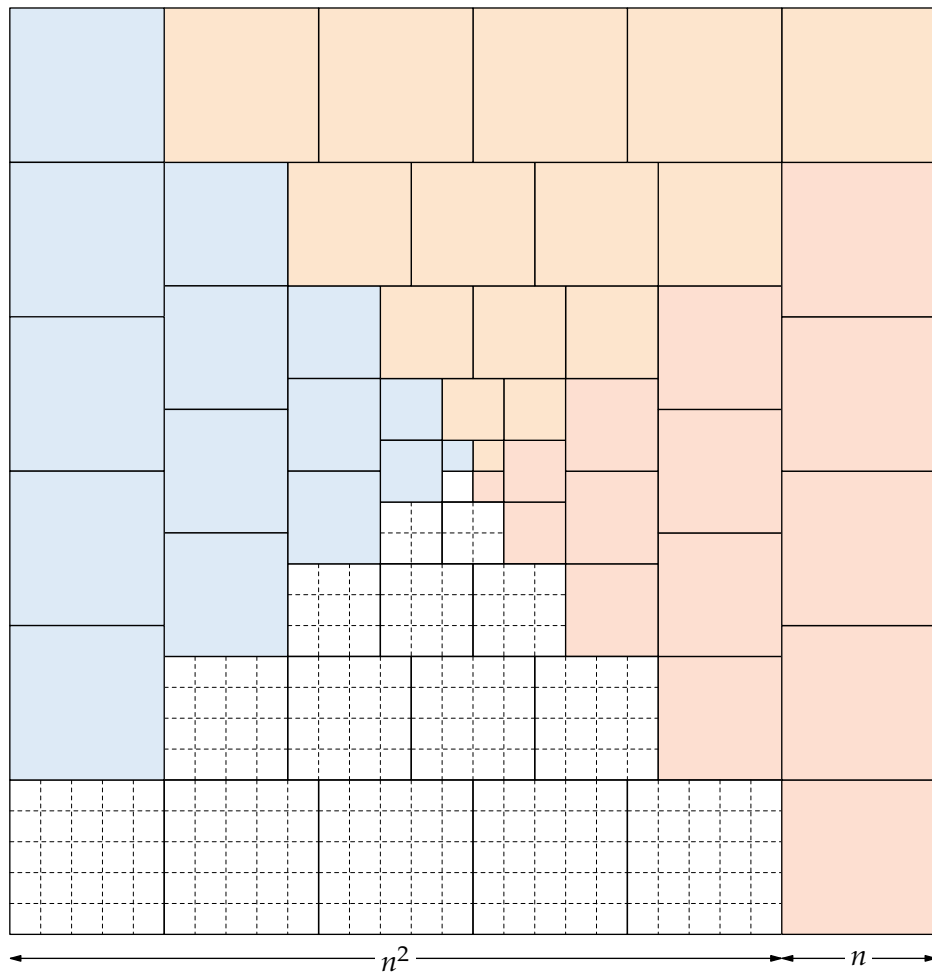
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— Alan L. Fry

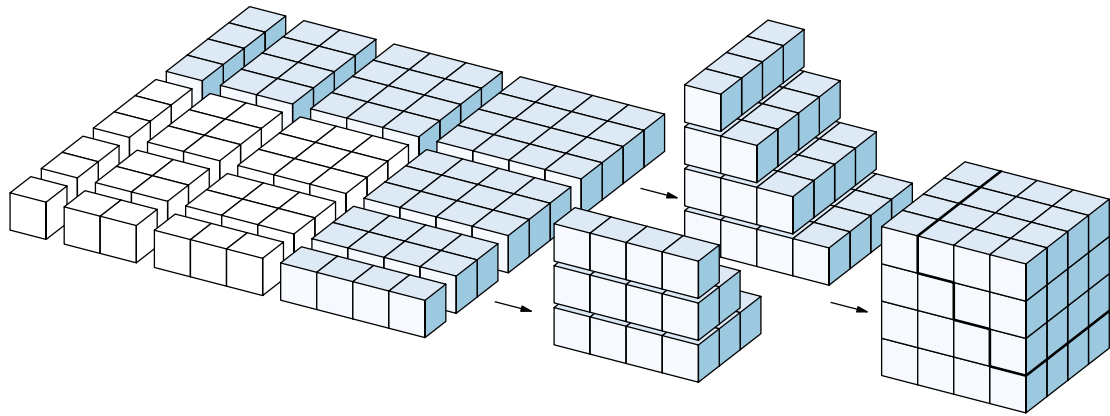
## Sums of cubes IV

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4} (n(n+1))^2$$

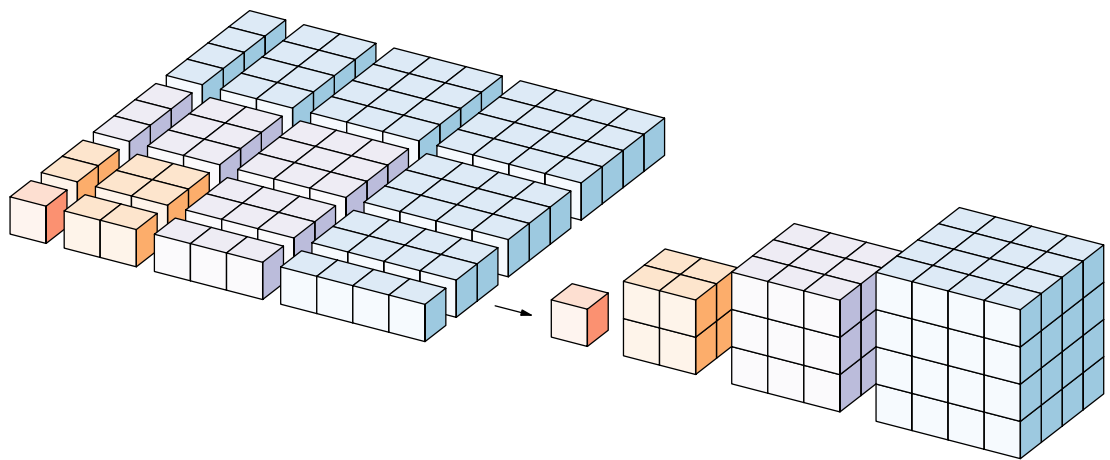


— Antonella Cupillari

## Sums of cubes V



$$t_n = 1 + 2 + \dots + n \Rightarrow t_n^2 - t_{n-1}^2 = n^3$$



$$t_n^2 = (1 + 2 + \dots + n)^2 = 1^3 + 2^3 + \dots + n^3$$

— Roger Nelsen

## Sums of cubes VI

$$\begin{array}{r}
 \boxed{1 \quad 2 \quad 3 \quad \dots \quad n} \\
 + \quad \boxed{2 \quad 4 \quad 6 \quad \dots \quad 2n} \\
 + \quad \boxed{3 \quad 6 \quad 9 \quad \dots \quad 3n} \\
 + \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
 + \quad \boxed{n \quad 2n \quad 3n \quad \dots \quad n^2} \\
 \\
 = \quad \sum_{i=1}^n i + 2 \sum_{i=1}^n i + \dots + n \sum_{i=1}^n i \\
 = \quad \left( \sum_{i=1}^n i \right)^2
 \end{array}$$

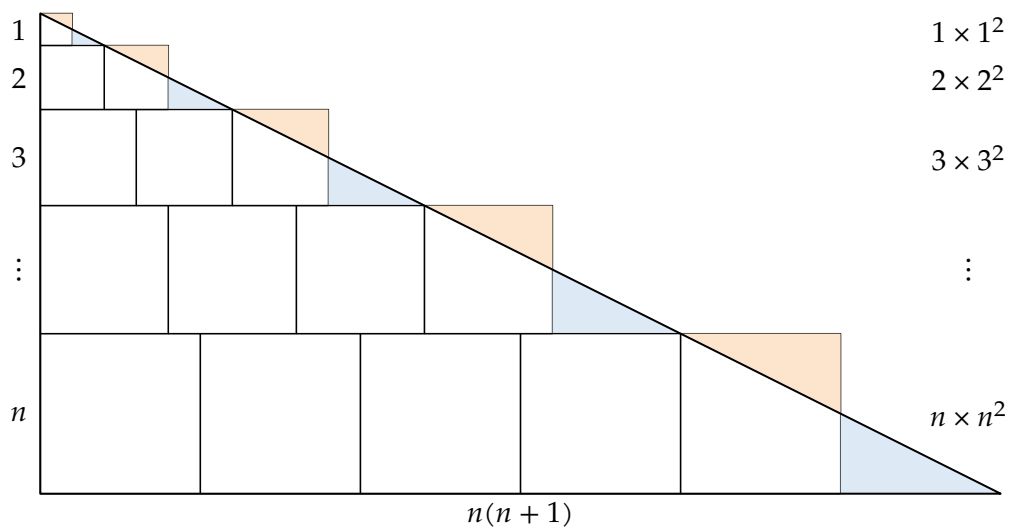
$$\begin{array}{r}
 \boxed{1} \quad \boxed{2} \quad \boxed{3} \quad \dots \quad \boxed{n} \\
 + \quad \boxed{2} \quad \boxed{4} \quad \boxed{6} \quad \dots \quad \boxed{2n} \\
 + \quad \boxed{3} \quad \boxed{6} \quad \boxed{9} \quad \dots \quad \boxed{3n} \\
 + \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
 + \quad \boxed{n} \quad \boxed{2n} \quad \boxed{3n} \quad \dots \quad \boxed{n^2} \\
 \\
 = \quad 1 + 2 \cdot 2^2 + 3 \cdot 3^2 + \dots + n \cdot n^2 \\
 = \quad \sum_{i=1}^n i^3
 \end{array}$$

— Farhood Pouryoussefi

## Sums of integers and sums of cubes

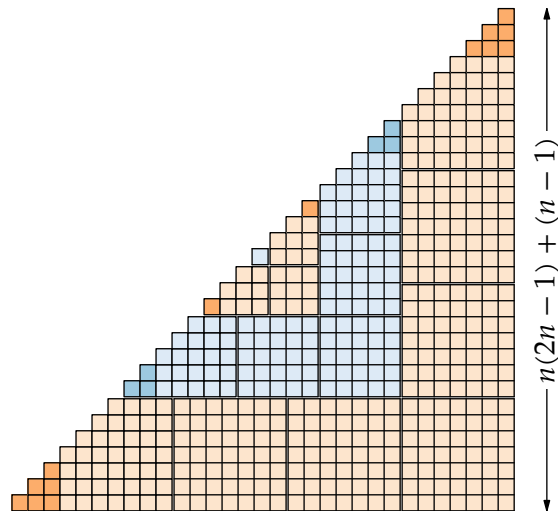
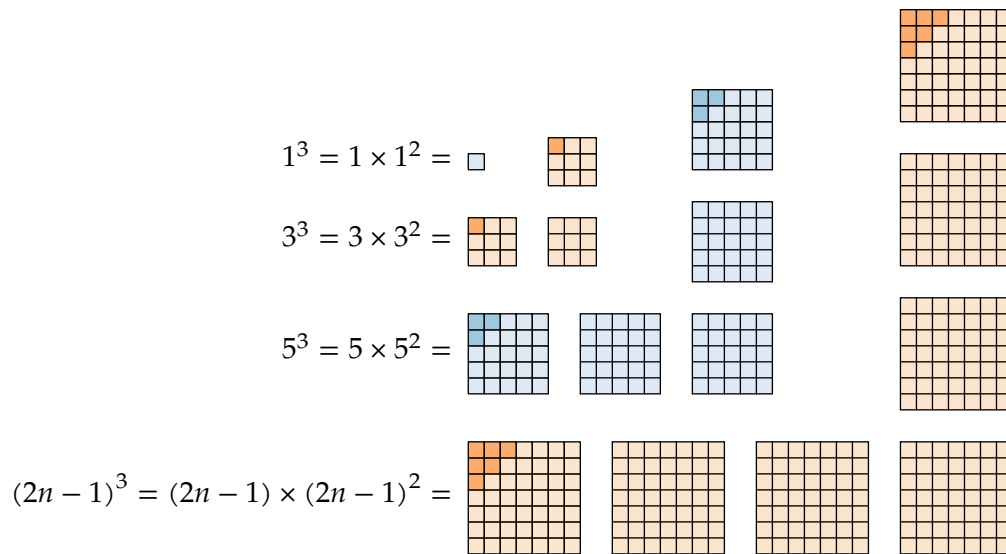
$$1 + 2 + \dots + n = \frac{1}{2}n(n+1)$$

$$1^3 + 2^3 + \dots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$$



— Georg Schrage

## Sums of odd cubes are triangular numbers

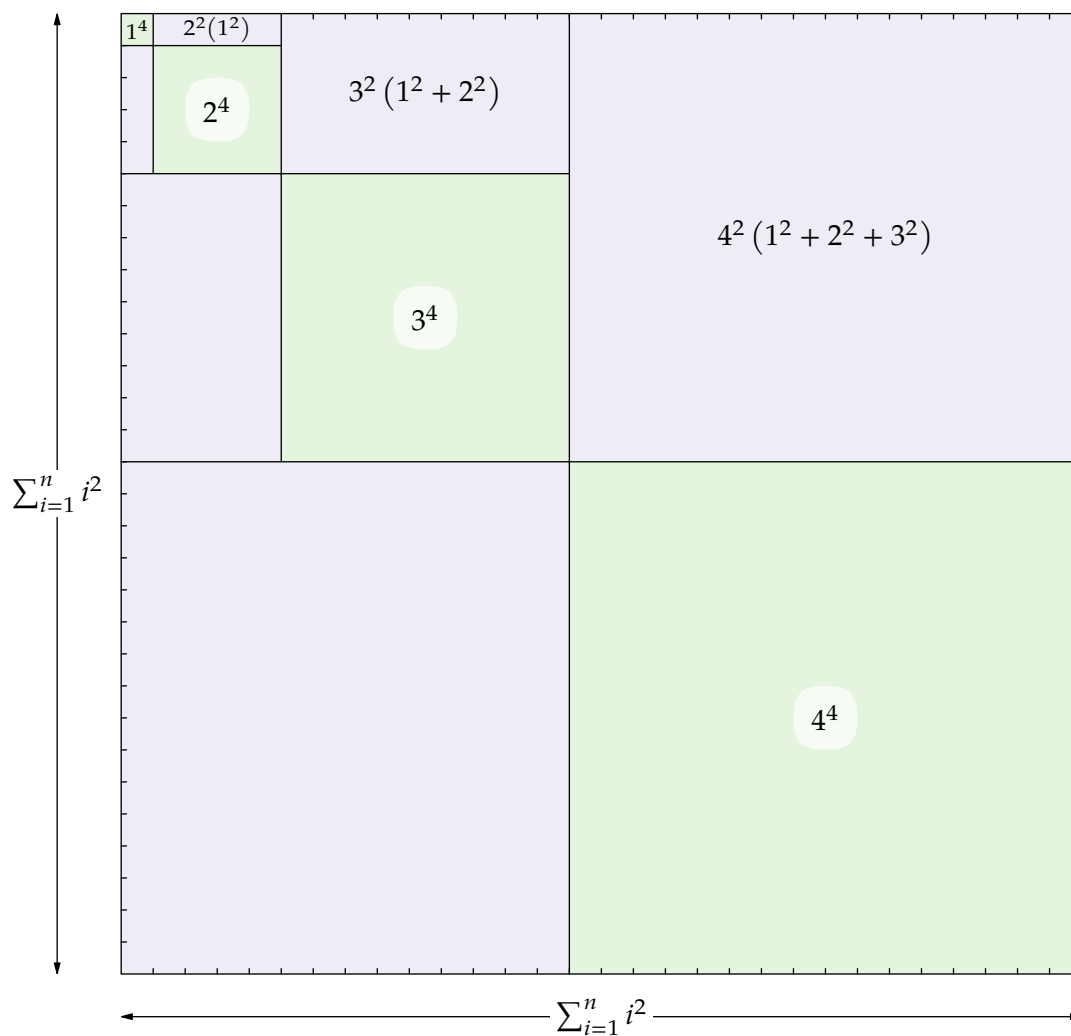


$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = 1 + 2 + 3 + \cdots + (2n^2 - 1) = n^2 (2n^2 - 1)$$

— Monte J. Zenger

## Sums of fourth powers

$$\sum_{i=1}^n i^4 = \left(\sum_{i=1}^n i^2\right)^2 - 2\left(\sum_{k=2}^n \left(k^2 \sum_{i=1}^{k-1} i^2\right)\right)$$

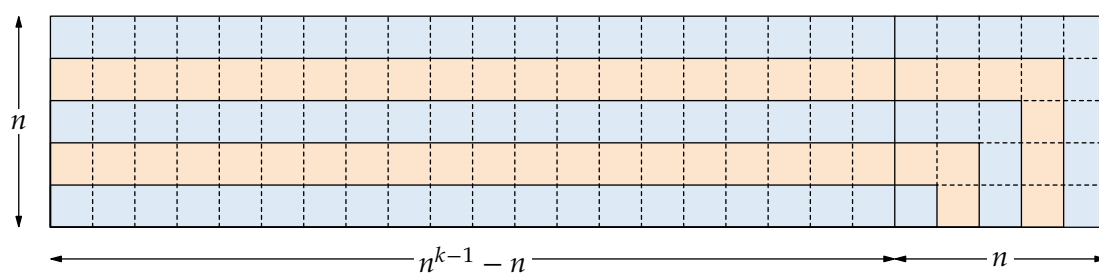


— Elizabeth M. Markham



# **$k$ -th powers as sums of consecutive odd numbers**

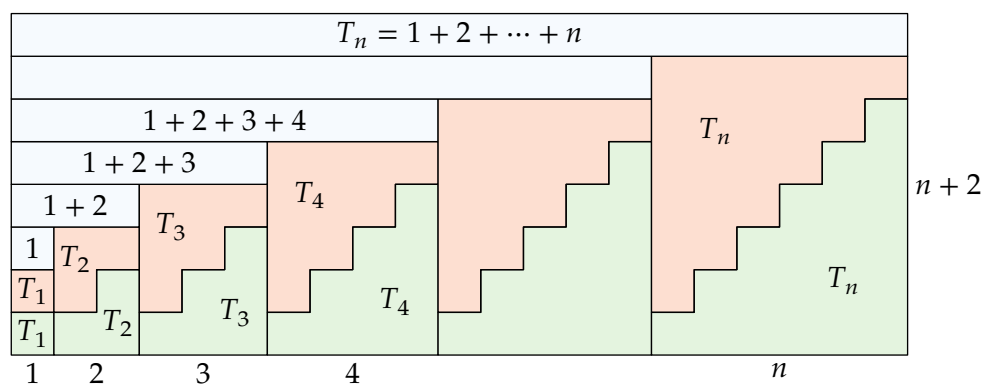
$$n^k = (n^{k-1} - n + 1) + (n^{k-1} - n + 3) + \cdots + (n^{k-1} - n + 2n - 1) \text{ for } k = 2, 3, \dots$$



— N. Gopalakrishnan Nair

## Sums of triangular numbers I

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



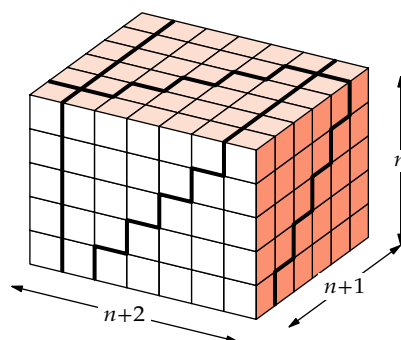
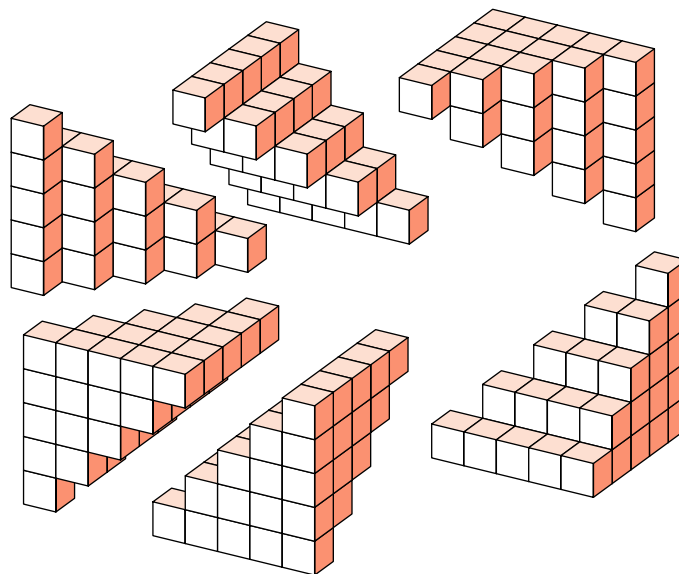
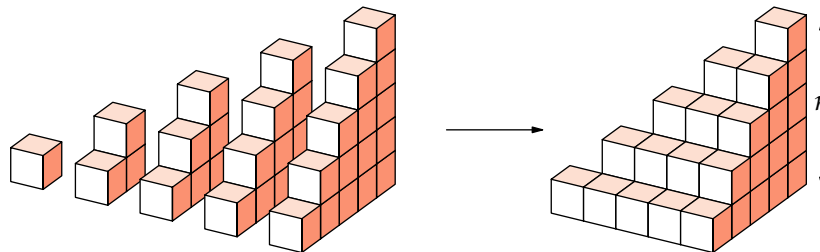
$$3(T_1 + T_2 + \cdots + T_n) = (n+2) \cdot T_n$$

$$T_1 + T_2 + \cdots + T_n = \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

— Monte J. Zenger

## Sums of triangular numbers II

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



— Roger B. Nelsen

## Sums of triangular numbers III

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{1}{6}n(n+1)(n+2)$$

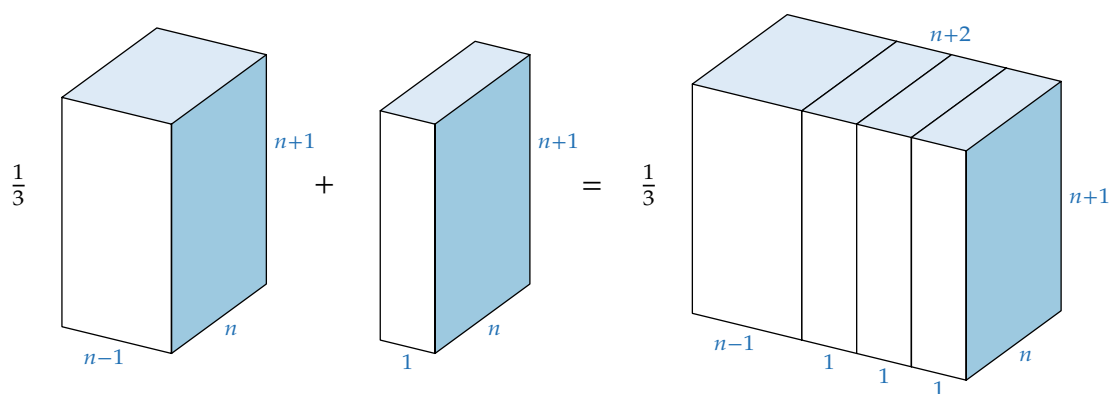
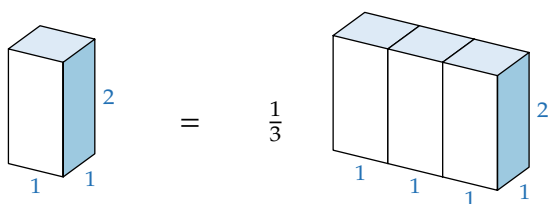
$$\begin{array}{ccccccc}
 & & 1 & & & & n & & & & & & 1 & & & \\
 & & 1 & 2 & & & n-1 & n-1 & & & & & 2 & 1 & & \\
 & & 1 & 2 & 3 & & & \cdot & \cdot & \cdot & & & 3 & 2 & 1 & \\
 & 1 & 2 & 3 & \cdot & & & \cdot & \cdot & \cdot & \cdot & & \cdot & 3 & 2 & 1 \\
 & 1 & 2 & 3 & \cdot & \cdot & & 3 & 3 & 3 & 3 & 3 & & \cdot & \cdot & 3 & 2 & 1 \\
 1 & 2 & 3 & \cdot & \cdot & n-1 & & 2 & 2 & 2 & 2 & 2 & 2 & n-1 & \cdot & \cdot & 3 & 2 & 1 \\
 1 & 2 & 3 & \cdot & \cdot & n-1 & n & 1 & 1 & 1 & 1 & 1 & 1 & 1 & n & n-1 & \cdot & \cdot & 3 & 2 & 1
 \end{array}$$

$$\begin{array}{ccccccc}
 & & & & n-2 & & \\
 & & & & n-2 & n-2 & \\
 & & & n-2 & n-2 & n-2 & \\
 = & & & n-2 & n-2 & n-2 & n-2 \\
 & & & n-2 & n-2 & n-2 & n-2 & n-2 \\
 & & & n-2 & n-2 & n-2 & n-2 & n-2 & n-2 \\
 & & & n-2 & n-2 & n-2 & n-2 & n-2 & n-2 & n-2 \\
 & & & n-2 & n-2 & n-2 & n-2 & n-2 & n-2 & n-2
 \end{array}$$

$$3(T_1 + T_2 + \cdots + T_n) = T_n \cdot (n+2)$$

## Sums of oblong numbers I

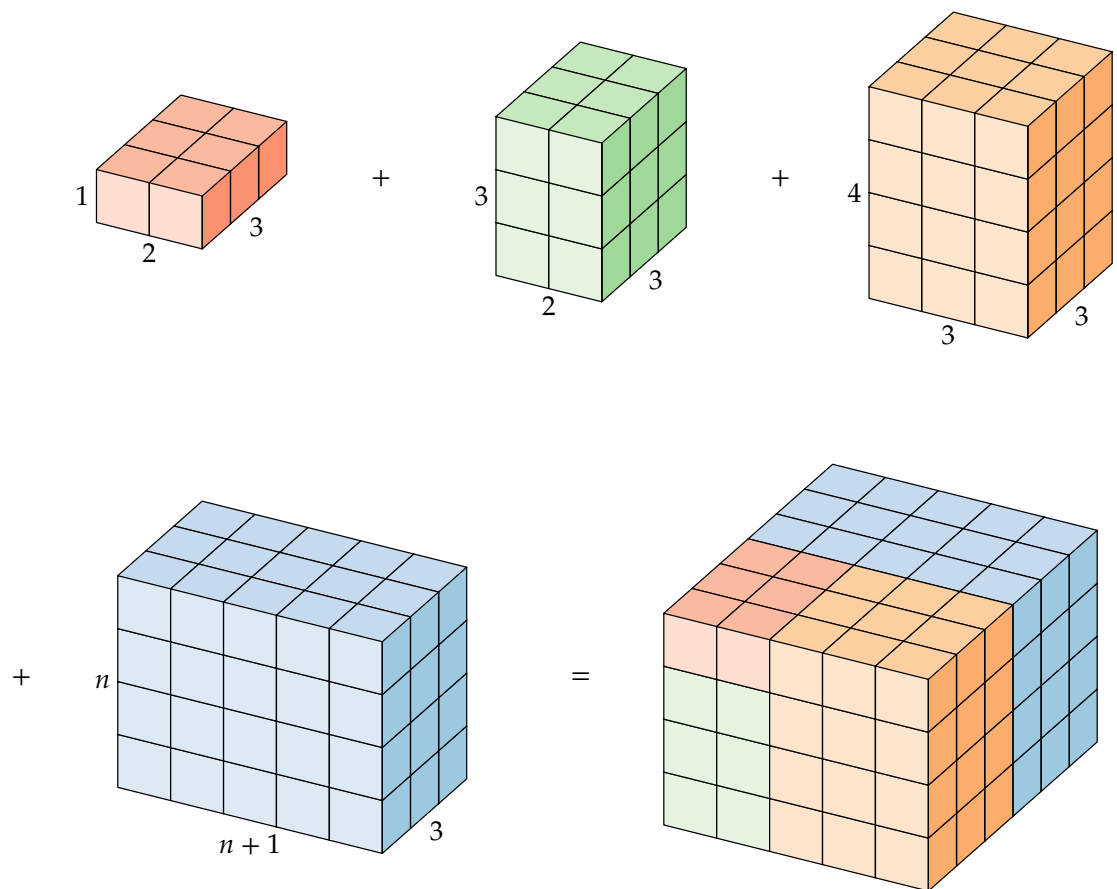
$$(1 \times 2) + (2 \times 3) + (3 \times 4) + \cdots + (n-1)n = \frac{1}{3}(n-1)n(n+1)$$



— T. C. Wu

## Sums of oblong numbers II

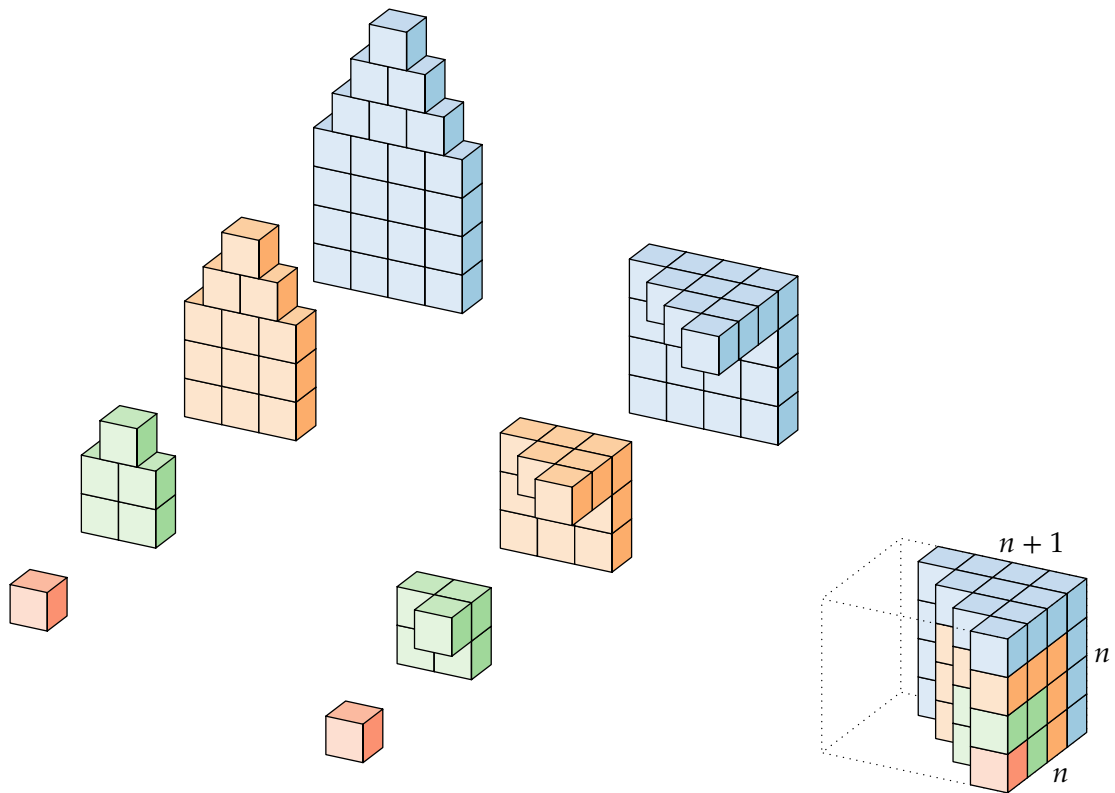
$$3(1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1)) = n(n+1)(n+2)$$



— Sidney H. Kung

## Sums of pentagonal numbers

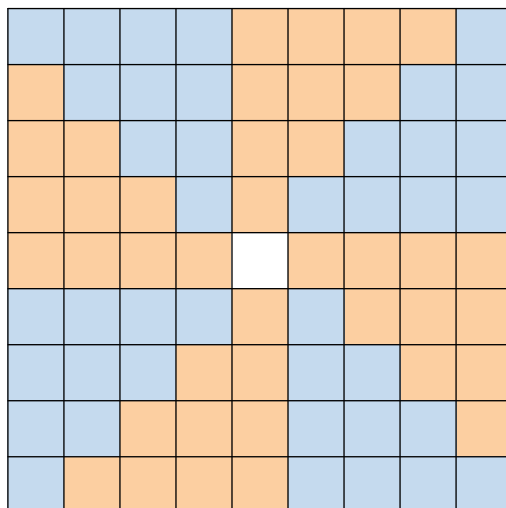
$$\frac{1 \cdot 2}{2} + \frac{2 \cdot 5}{2} + \frac{3 \cdot 8}{2} + \cdots + \frac{n(3n-1)}{2} = \frac{n^2(n+1)}{2}$$



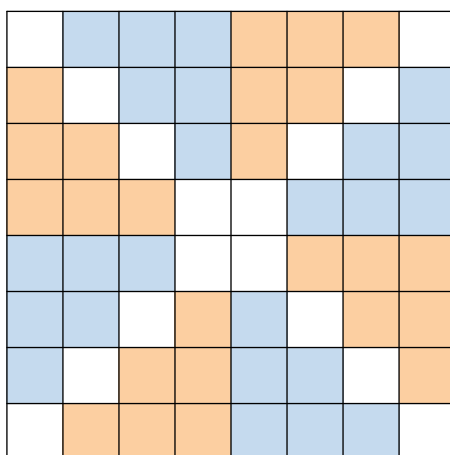
— William A. Miller

## On squares of positive integers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



$$(2n + 1)^2 = 8T_n + 1$$

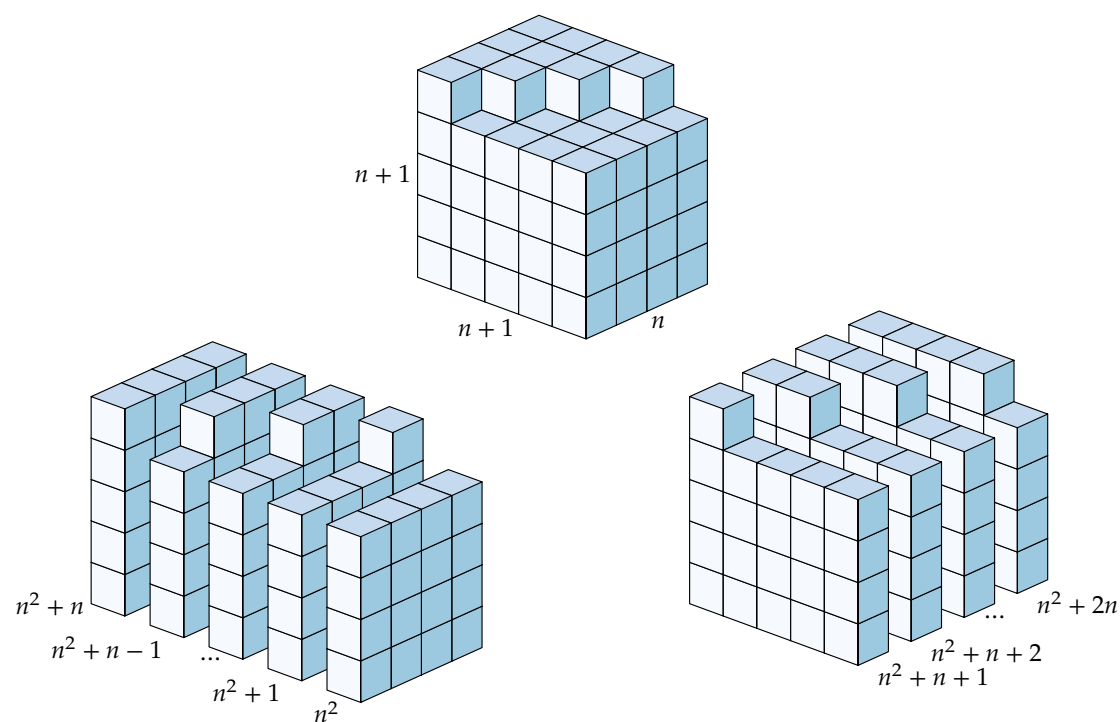


$$(2n)^2 = 8T_{n-1} + 4n$$

— Edwin G. Landauer



## Consecutive sums of consecutive integers



$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

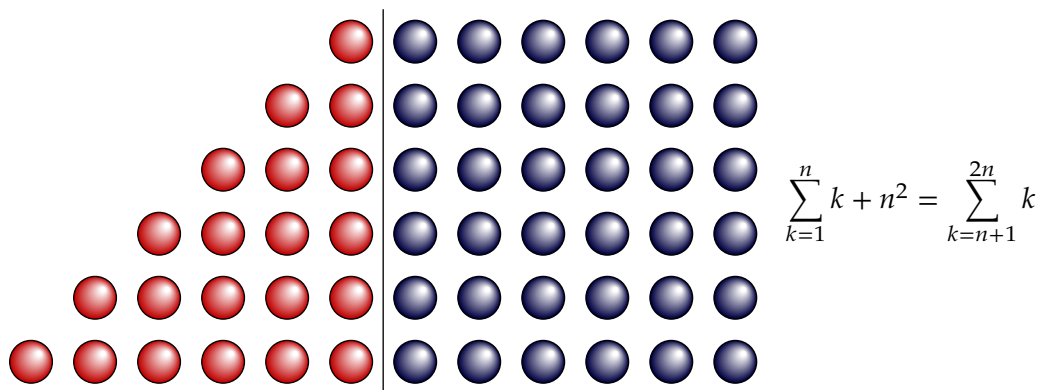
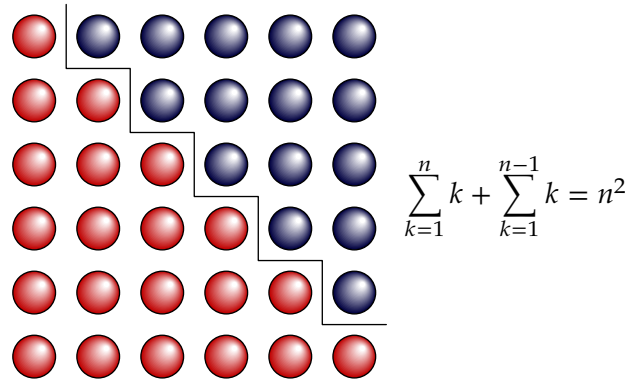
$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$

$\vdots$

$$n^2 + (n^2 + 1) + \cdots + (n^2 + n) = (n^2 + n + 1) + \cdots + (n^2 + 2n)$$

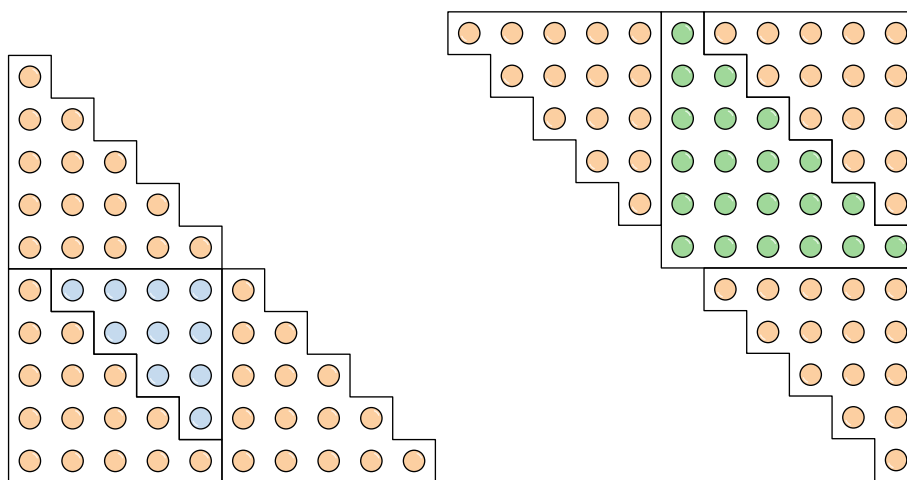
— Roger B. Nelsen

# Count the dots



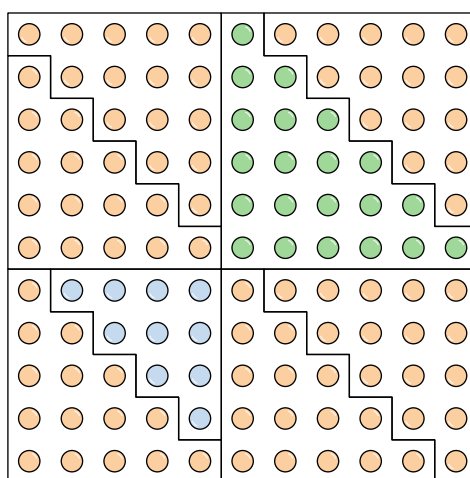
# Identities for triangular numbers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



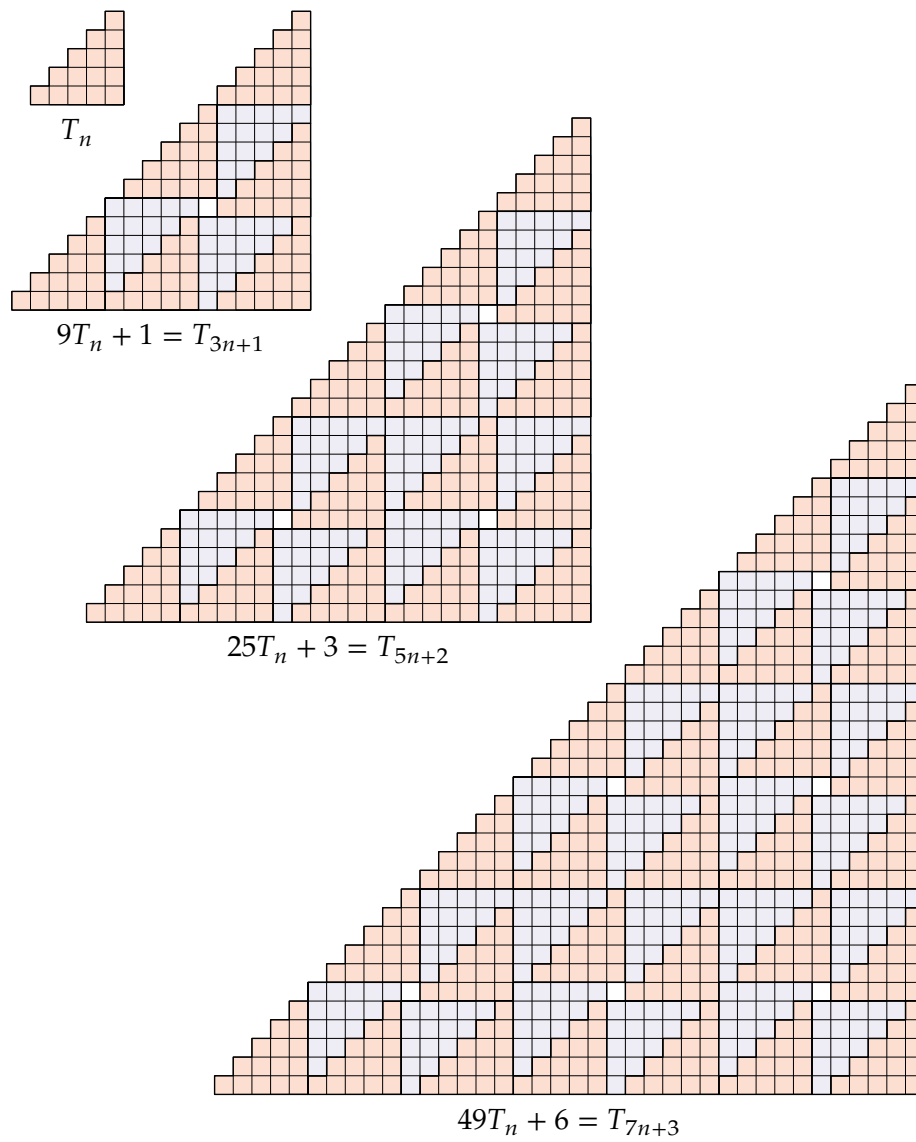
$$3T_n + T_{n-1} = T_{2n}$$

$$3T_n + T_{n+1} = T_{2n+1}$$



$$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$$

## A triangular identity

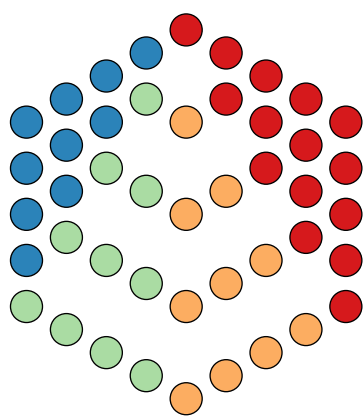


$$(2k + 1)^2 T_n + T_k = T_{(2k+1)n+k}$$

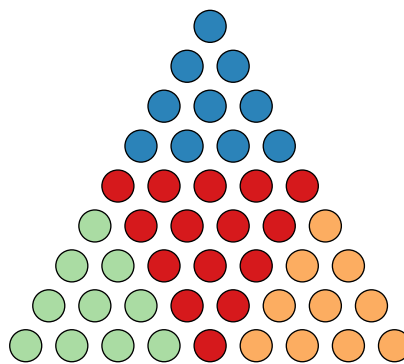
— Roger B. Nelsen

# Every hexagonal number is a triangular number

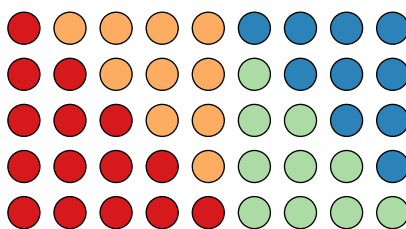
$$\left. \begin{array}{l} H_n = 1 + 5 + \cdots + (4n - 3) \\ T_n = 1 + 2 + \cdots + n \end{array} \right\} \Rightarrow H_n = 3T_{n-1} + T_n = T_{2n-1} = n(2n - 1)$$



$H_5$



$T_9$

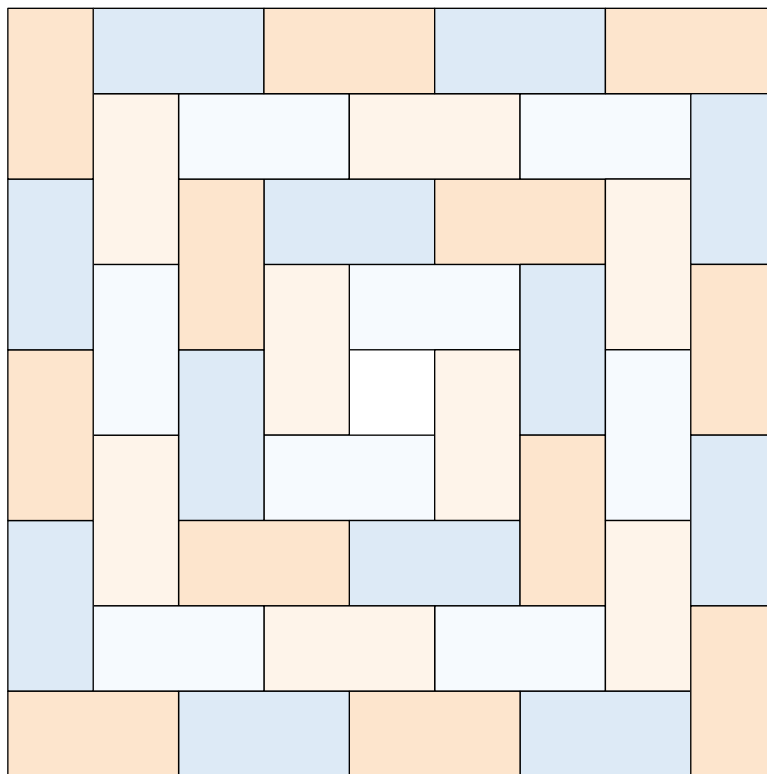


$5 \times 9$

**One domino = two squares : concentric squares**

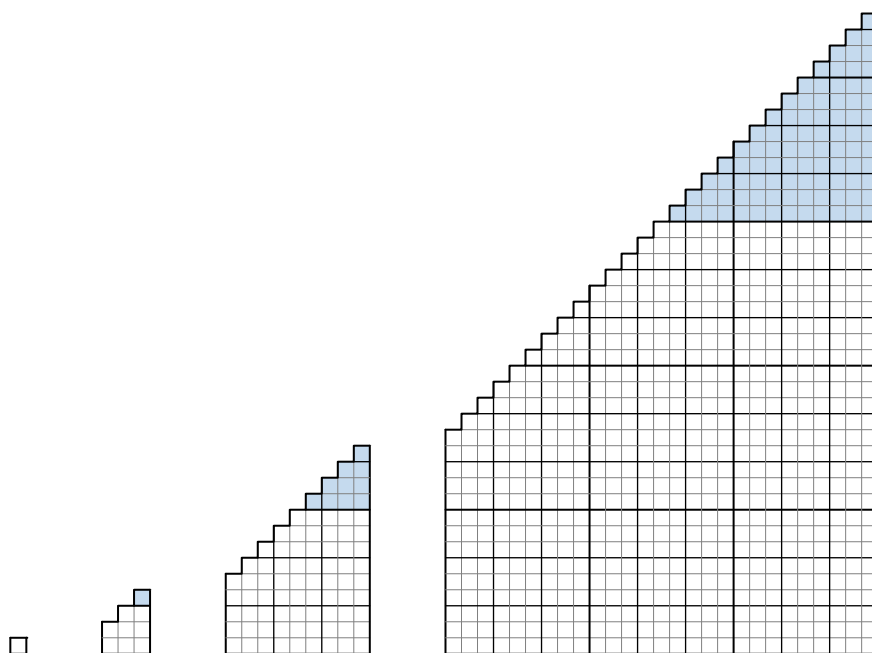
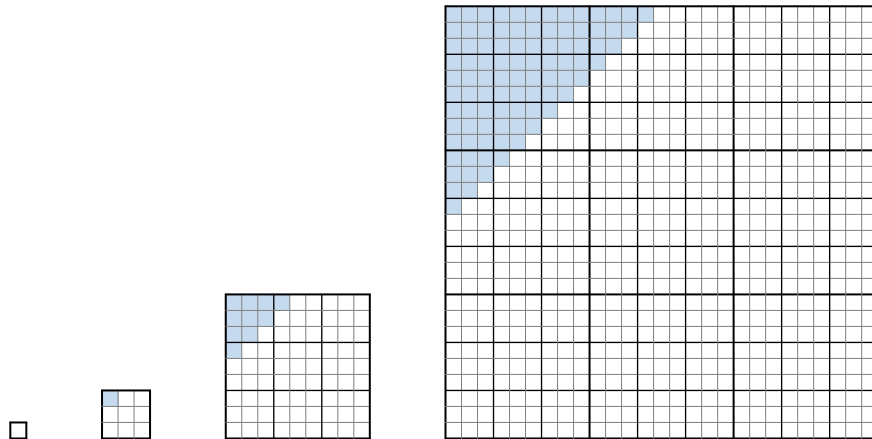
$$1 + 4 \times 2 + 8 \times 2 + 12 \times 2 + 16 \times 2 = 9^2$$

$$1 + 2 \sum_{k=1}^n 4k = (2n + 1)^2$$



— Shirley A. Wakin

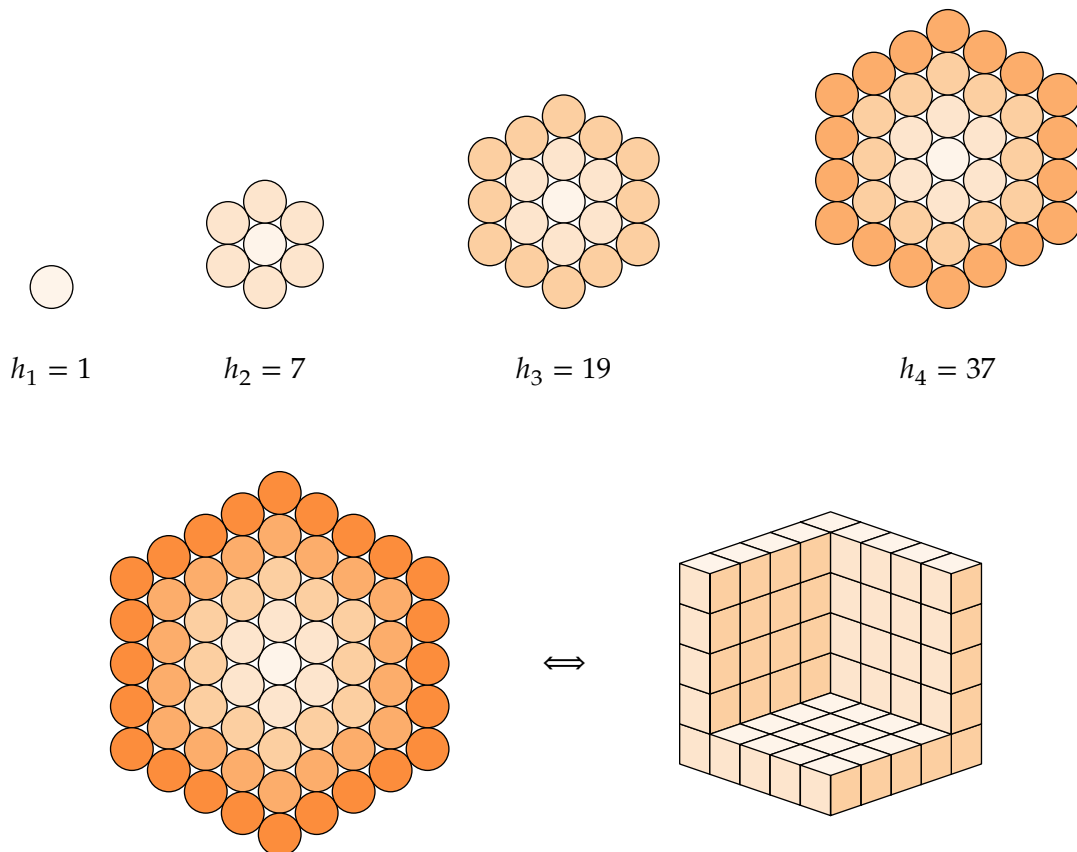
# Sums of consecutive powers of 9 are sums of consecutive integers



$$1 + 9 + \cdots + 9^n = 1 + 2 + 3 + \cdots + (1 + 3 + \cdots + 3^n)$$

— Roger B. Nelsen

# Sums of hex numbers are cubes



$$h_n = n^3 - (n - 1)^3$$

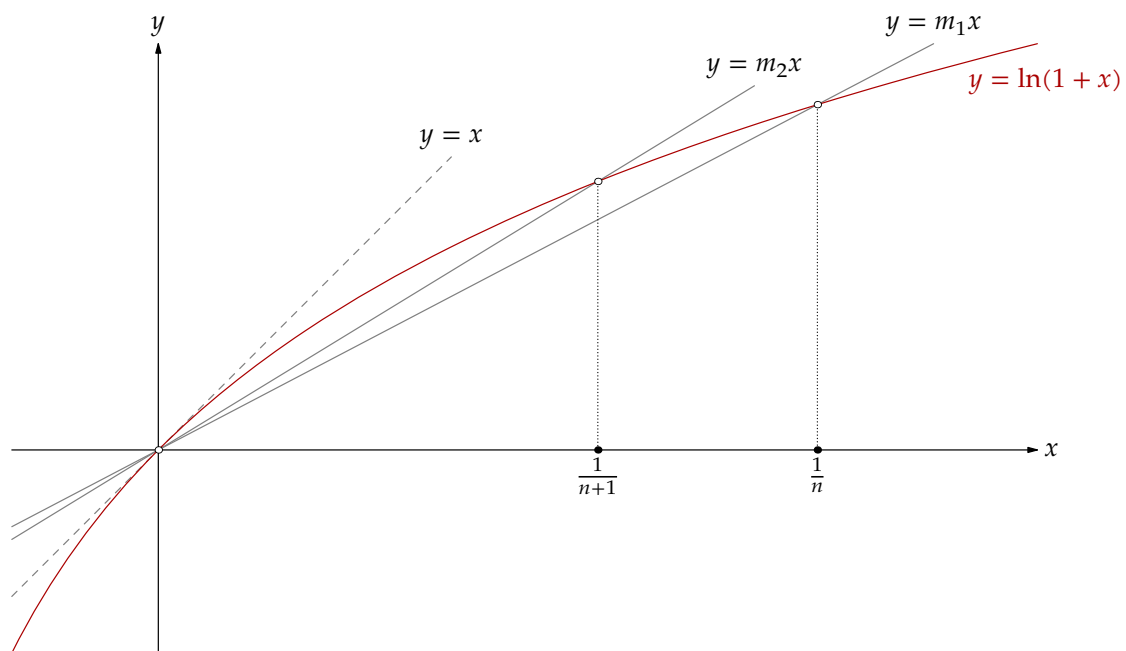
$$\therefore h_1 + h_2 + \cdots + h_n = n^3$$



# Sequences and series

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## A monotone sequence bounded by $e$



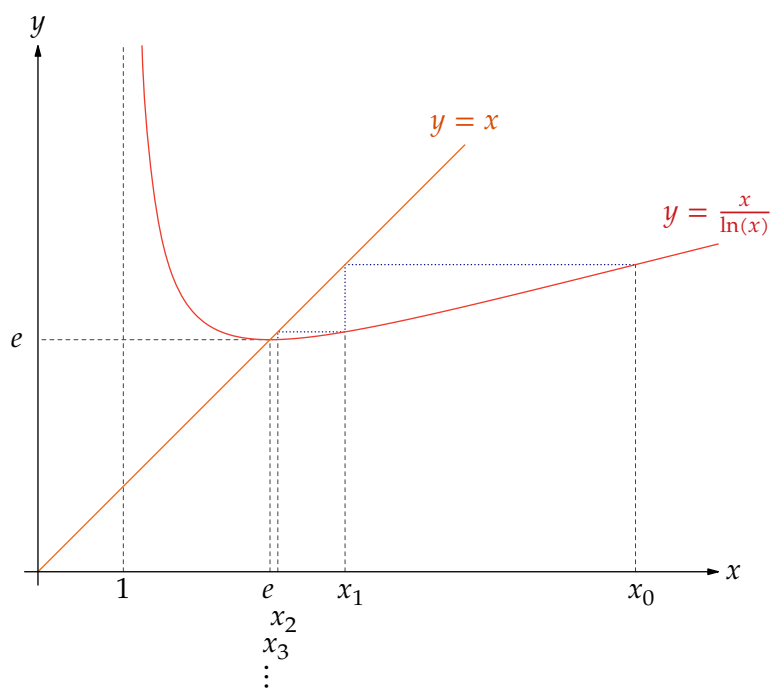
$$n \geq 1 \Rightarrow m_1 < m_2 < 1$$

$$\Rightarrow \frac{\ln(1 + 1/n)}{1/n} < \frac{\ln(1 + 1/(n+1))}{1/(n+1)} < 1$$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e$$

— Roger B. Nelsen

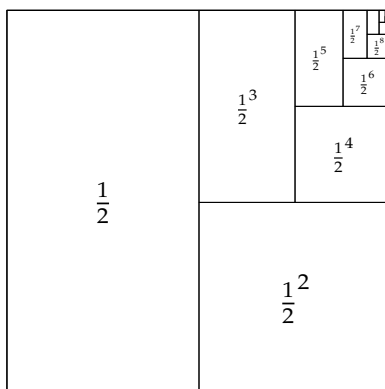
# A recursively defined sequence for $e$



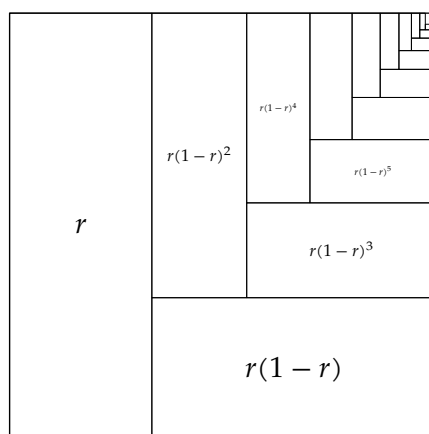
$$x_0 > 1 \ \& \ x_{n+1} = \frac{x_n}{\ln(x_n)} \implies \lim x_n = e$$

— Thomas P. Dence

## Geometric sums

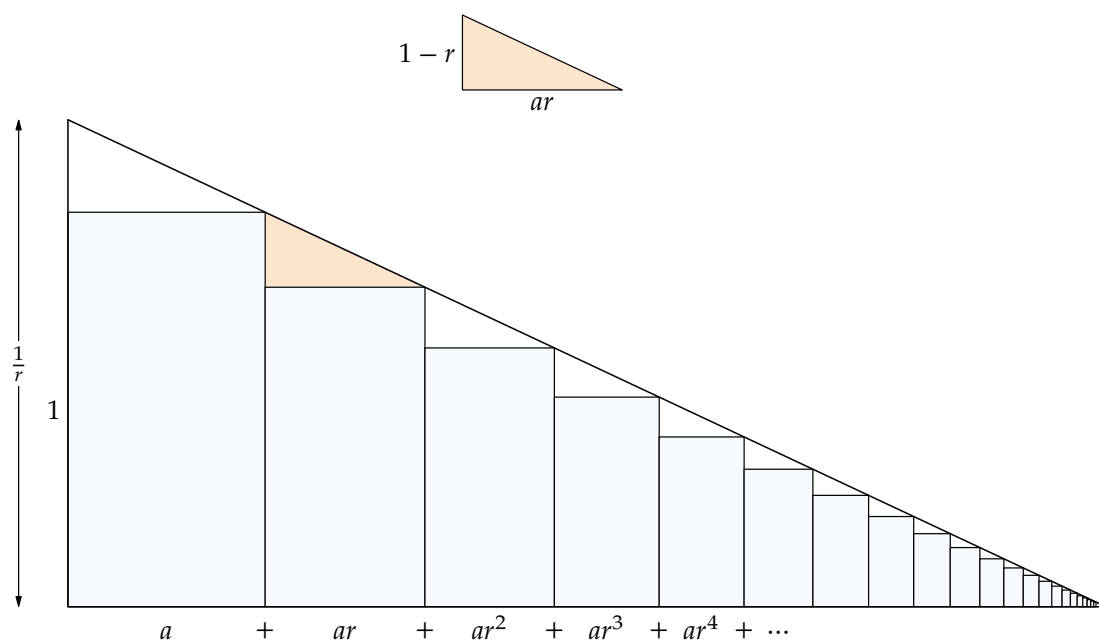


$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$



$$r + r(1-r) + r(1-r)^2 + \dots = 1$$

## Geometric series I



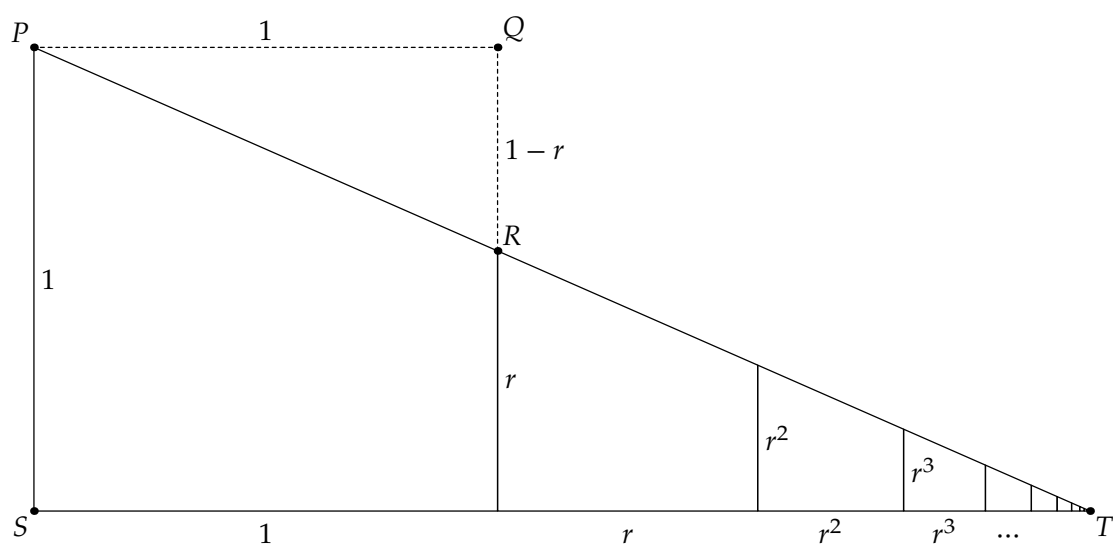
$$\frac{a + ar + ar^2 + ar^3 + ar^4 + \dots}{1/r} = \frac{ar}{1-r}$$

$$a + ar + ar^2 + ar^3 + ar^4 + \dots = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

— J. H. Webb

## Geometric series II

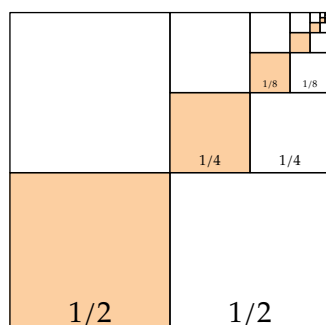


$$\triangle PQR \sim \triangle TSP$$

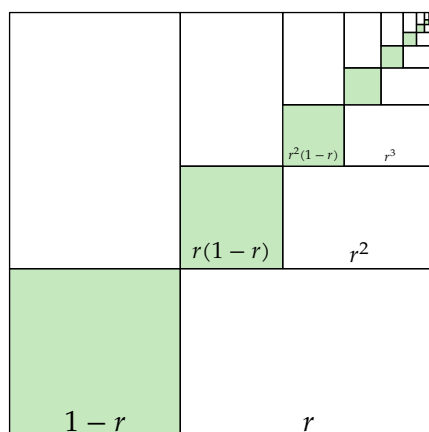
$$\therefore 1 + r + r^2 + r^3 + \dots = \frac{1}{1 - r}$$

— Benjamin G. Klein and Irl C. Bivens

## Geometric series III



$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{3}$$



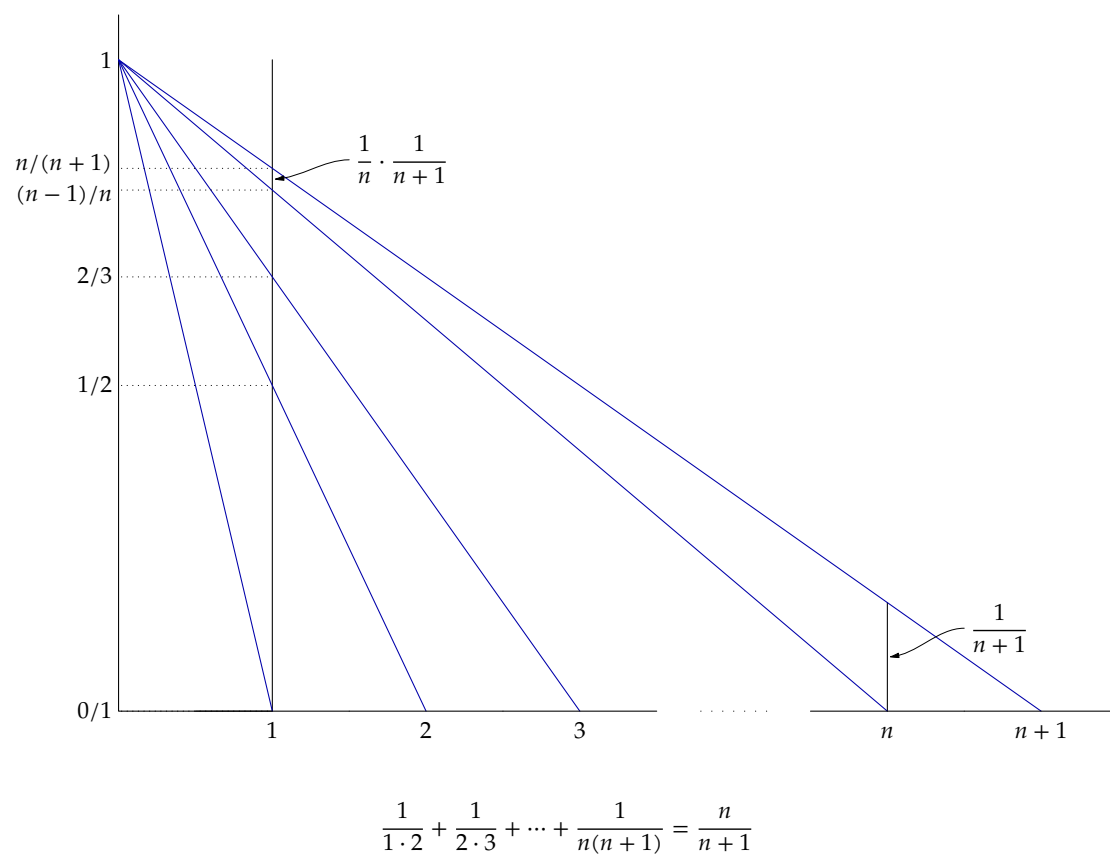
$$(1 - r)^2 + r^2(1 - r)^2 + r^4(1 - r)^2 + \dots = \frac{(1 - r)^2}{(1 - r)^2 + 2 \times r(1 - r)}$$

$$1 + r^2 + r^4 + \dots = \frac{1}{(1 - r)^2 + 2r(1 - r)} = \frac{1}{1 - r^2}$$

$$a + ar + ar^2 + \dots = \frac{a}{1 - r}$$

— Sunday A. Ajose

## Sum of reciprocals of successive integer products

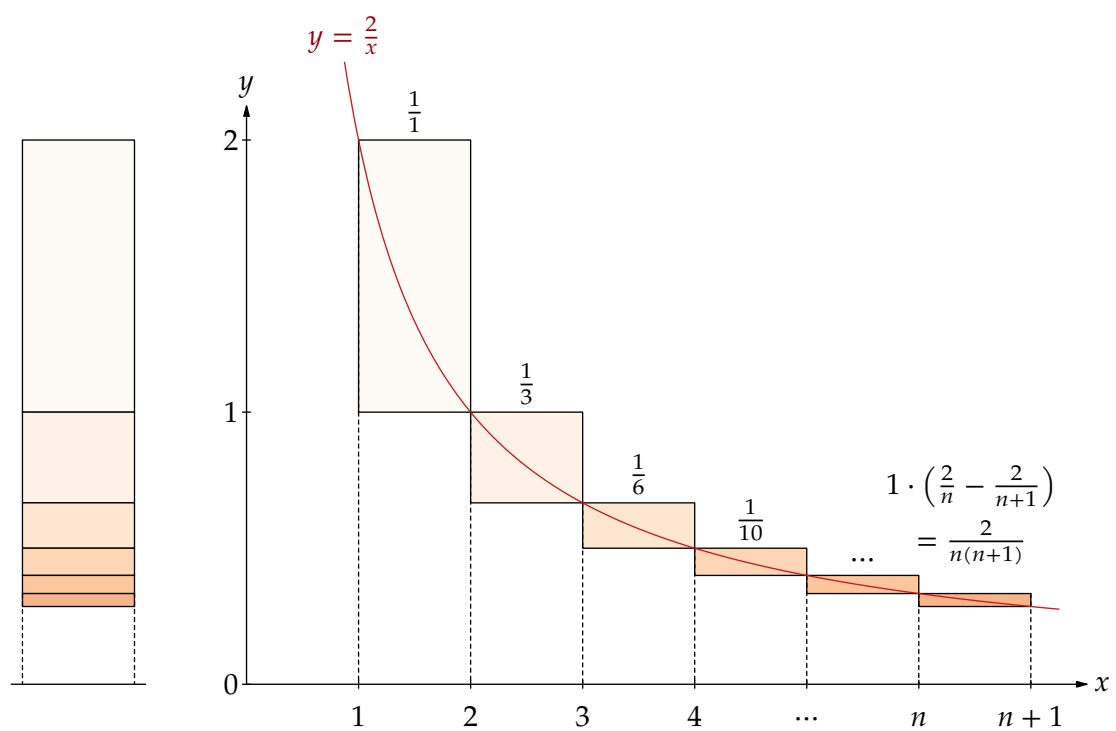


— Roman W. Wong



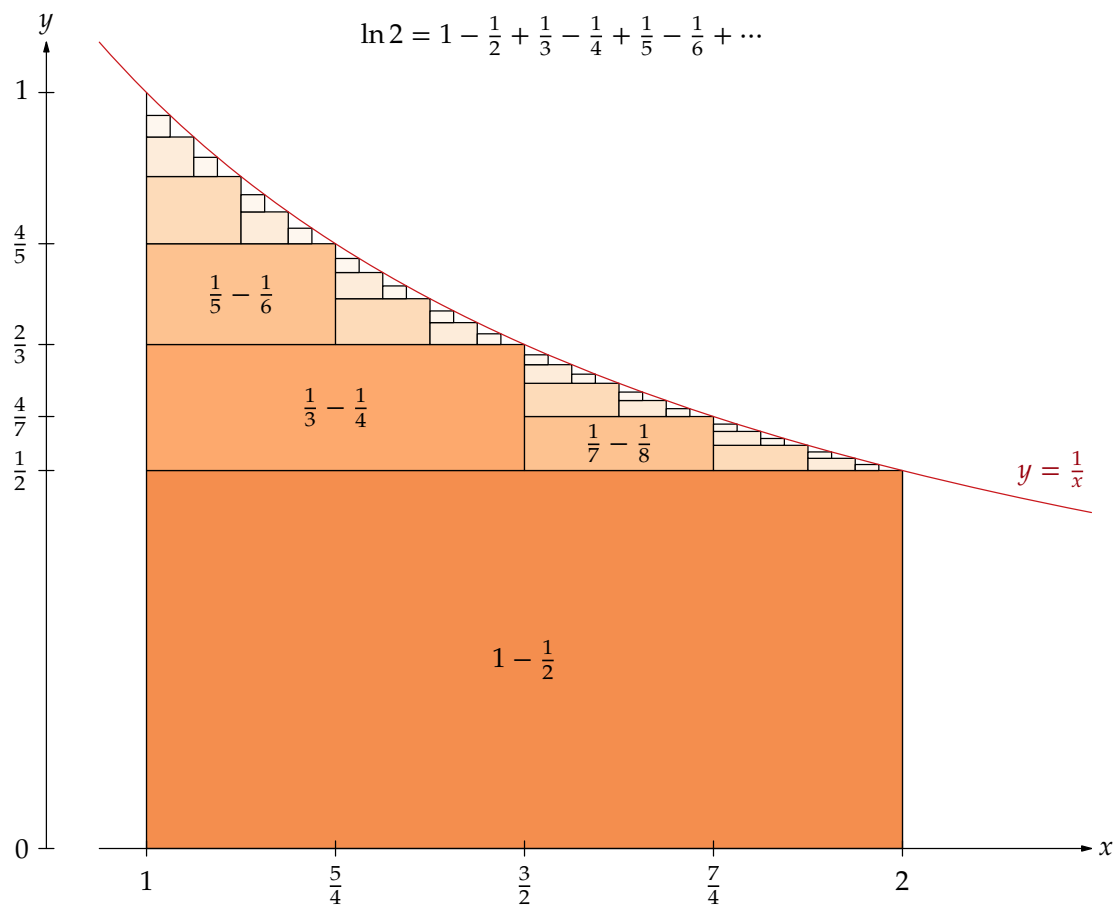
## Sum of reciprocals of triangular numbers

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{2}{n(n+1)} + \cdots = 2$$



— Roger B. Nelsen

## Alternating harmonic series



$$\frac{1}{2} \times \left( \frac{2}{3} - \frac{1}{2} \right) = \frac{1}{3} - \frac{1}{4}; \quad \frac{1}{4} \times \left( \frac{4}{5} - \frac{2}{3} \right) = \frac{1}{5} - \frac{1}{6}; \quad \frac{1}{4} \times \left( \frac{4}{7} - \frac{1}{2} \right) = \frac{1}{7} - \frac{1}{8}; \quad \dots$$

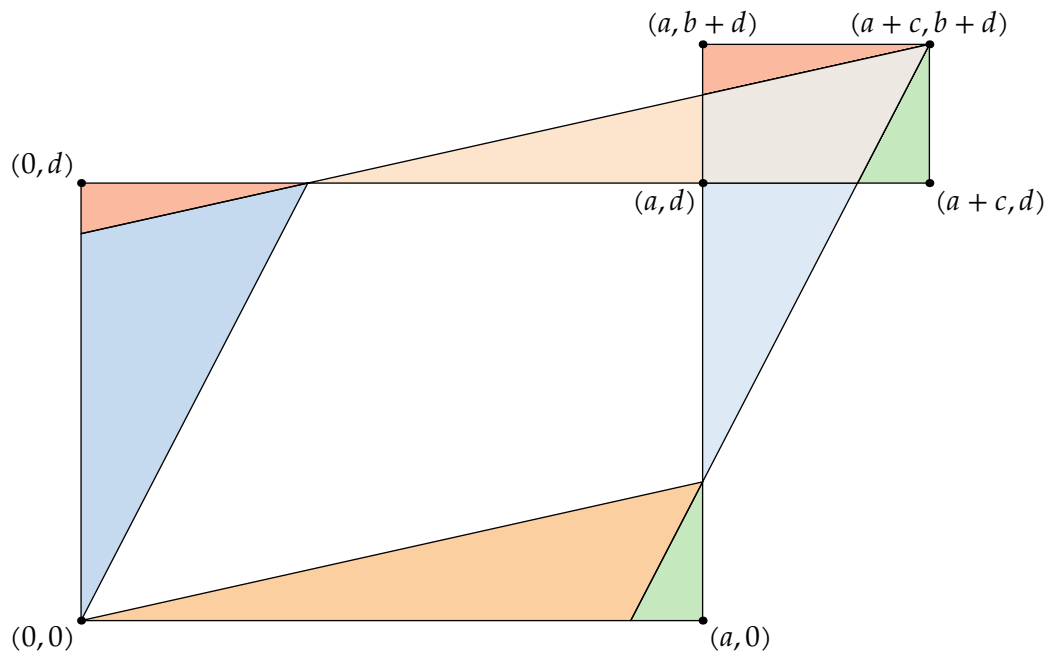
$$\ln 2 = \int_1^2 \frac{1}{x} dx = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

— Mark Finklestein

## Miscellaneous

A $2 \times 2$ determinant is the area of a parallelogram . . . . .	116
The problem of the calissons . . . . .	117

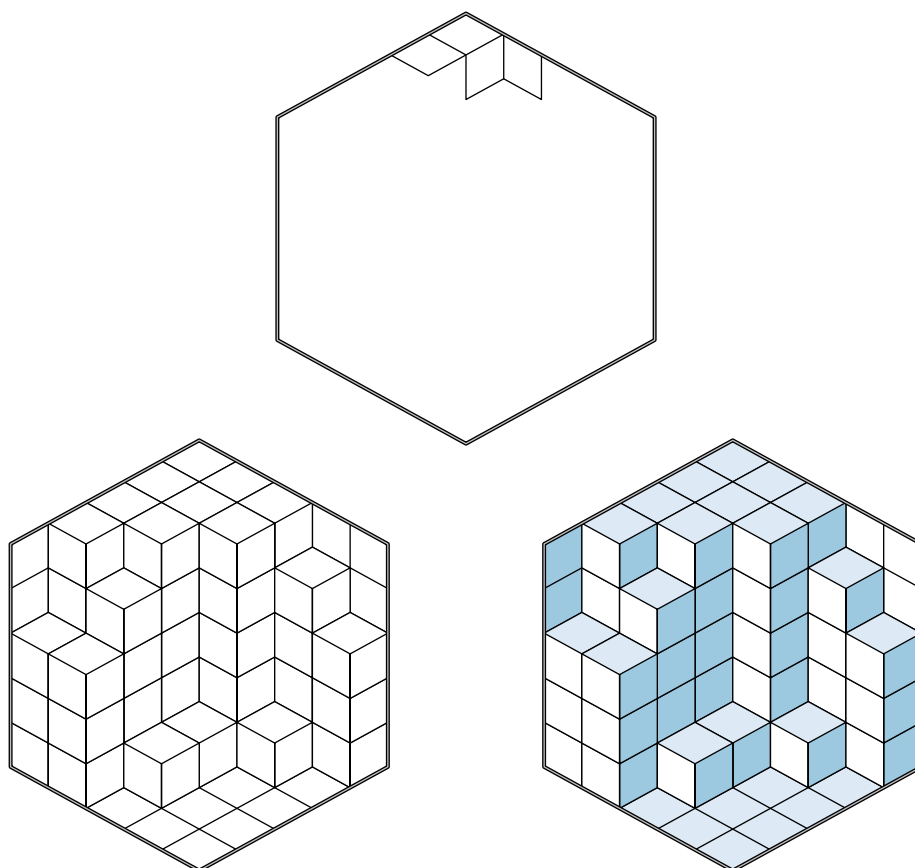
# **A $2 \times 2$ determinant is the area of a parallelogram**



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \boxed{\phantom{00}} - \boxed{\phantom{00}} = \text{parallelogram}$$

— Solomon W. Golomb

## The problem of the calissons



— Guy David and Carlos Tomei