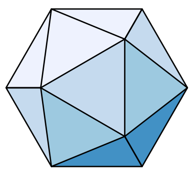
Proofs without words I

Exercises in METAPOST

Toby Thurston

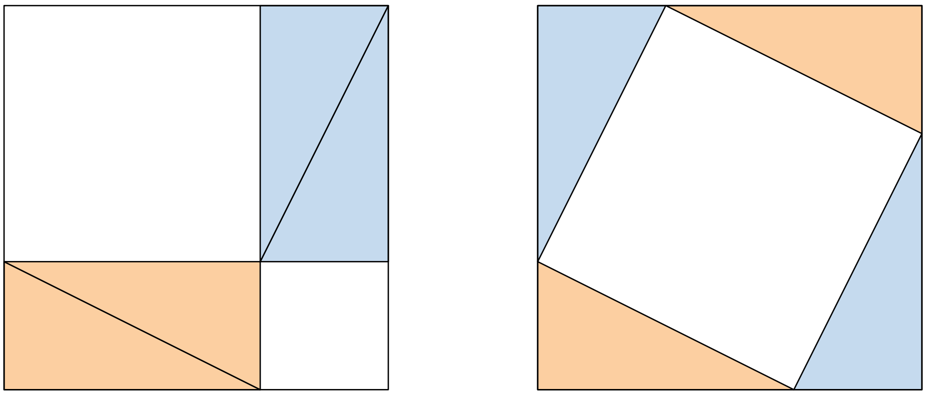
March 2021 —



Contents

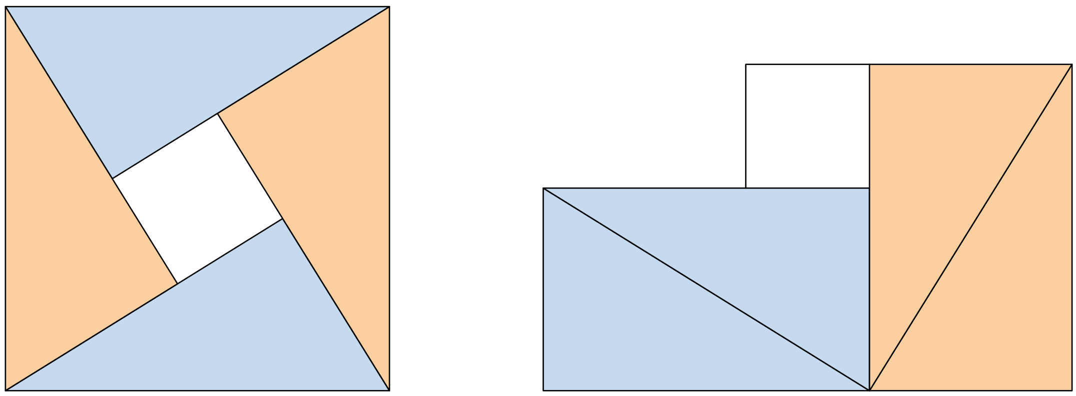
Geometry and Algebra

The Pythagorean theorem I



— adapted from the *Chou pei san ching*

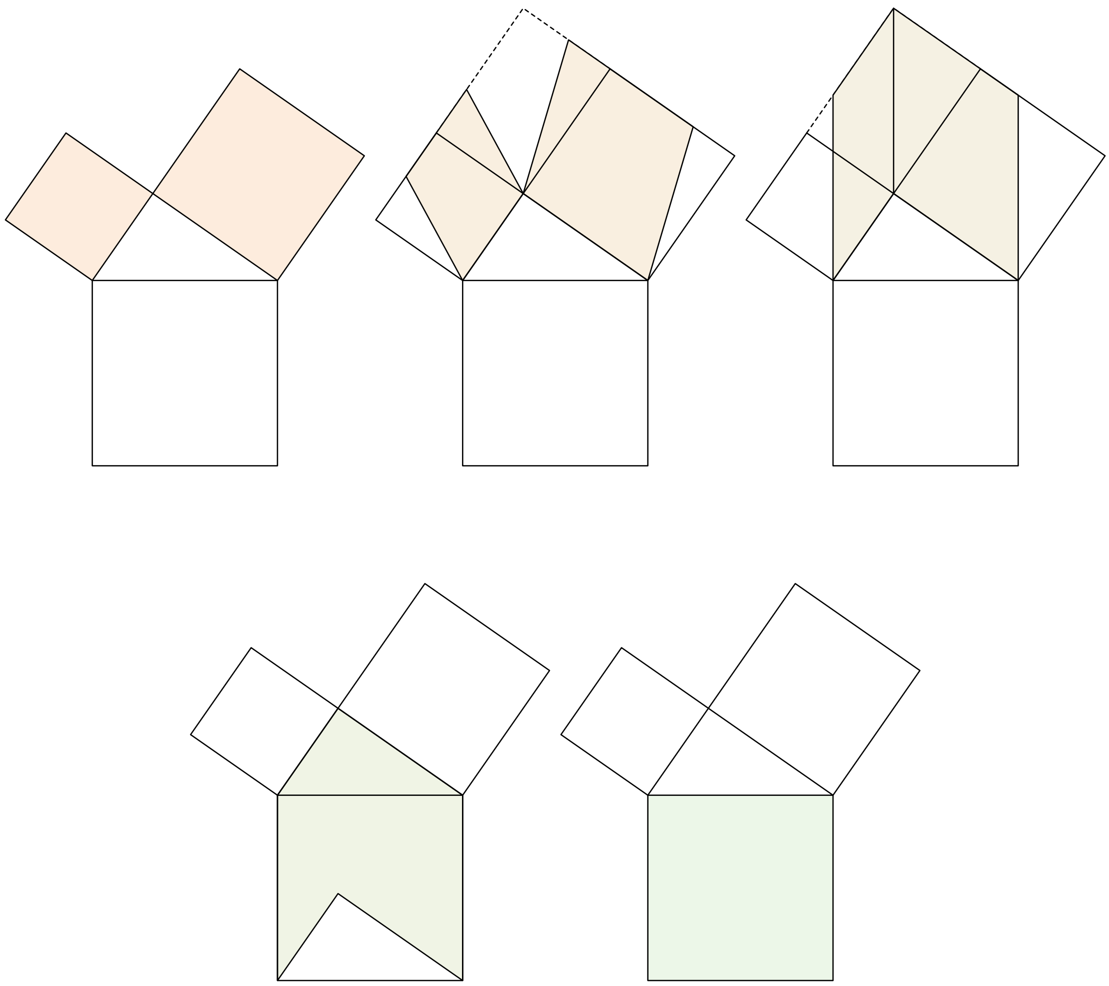
The Pythagorean theorem II



Behold!

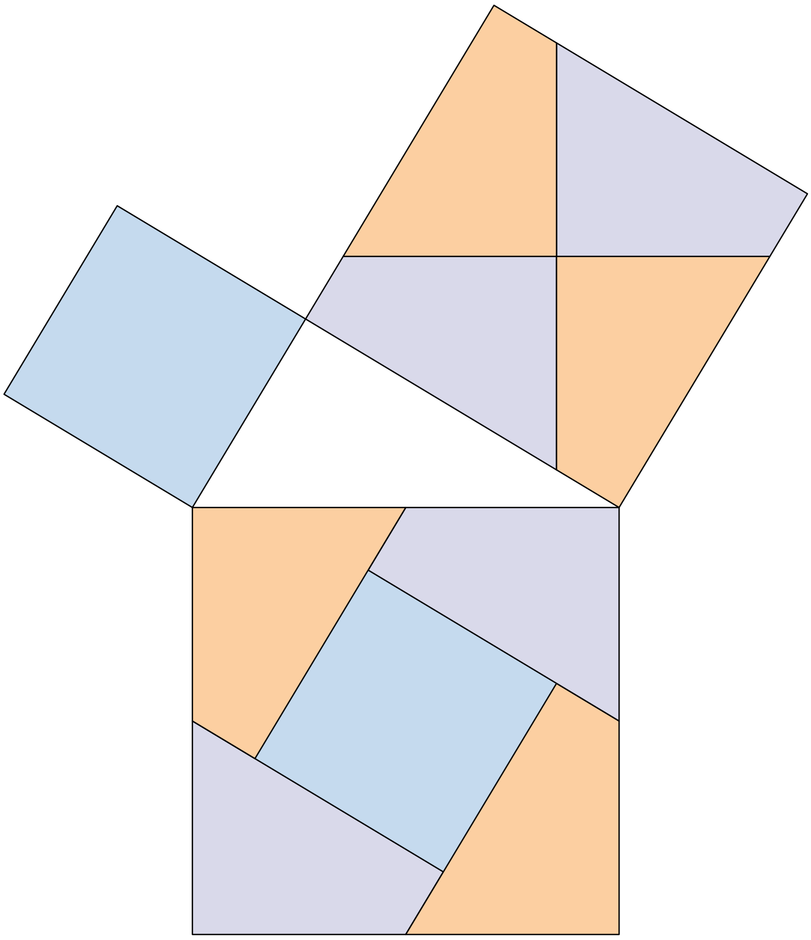
— Bhāskara (12th century)

The Pythagorean theorem III



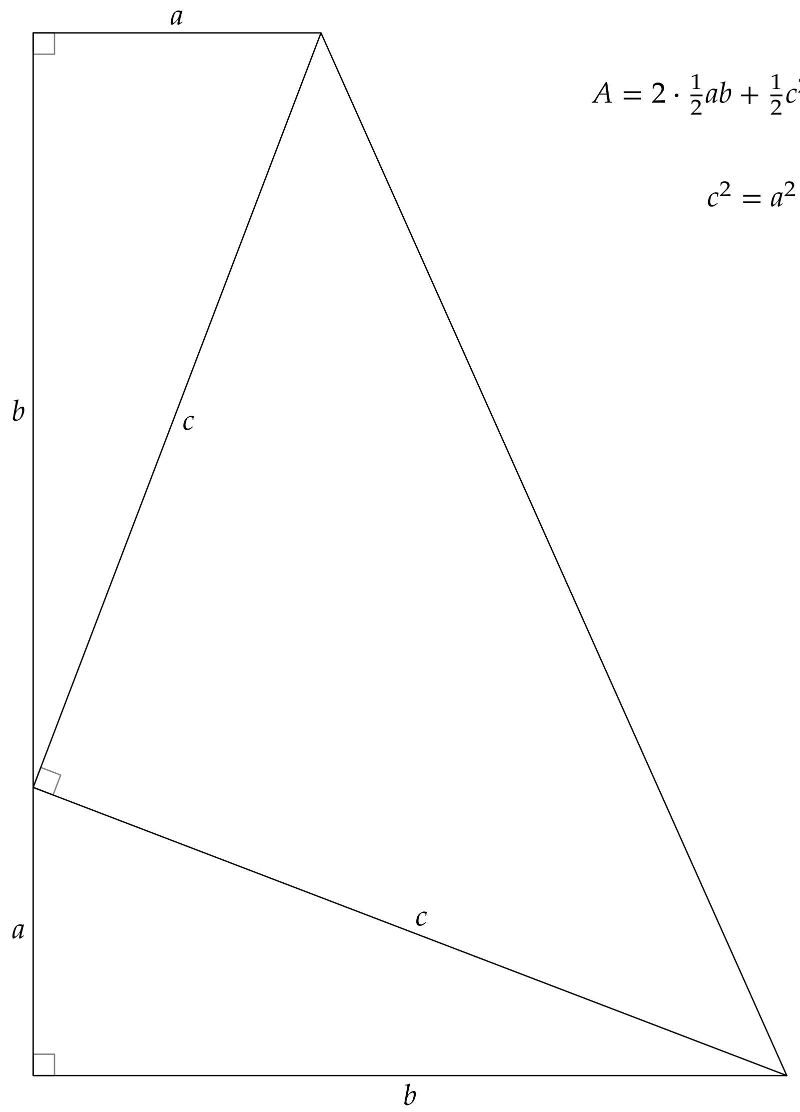
— based on Euclid's proof

The Pythagorean theorem IV



— H. E. Dudeney (1917)

The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}\left(a+b\right)^2$$

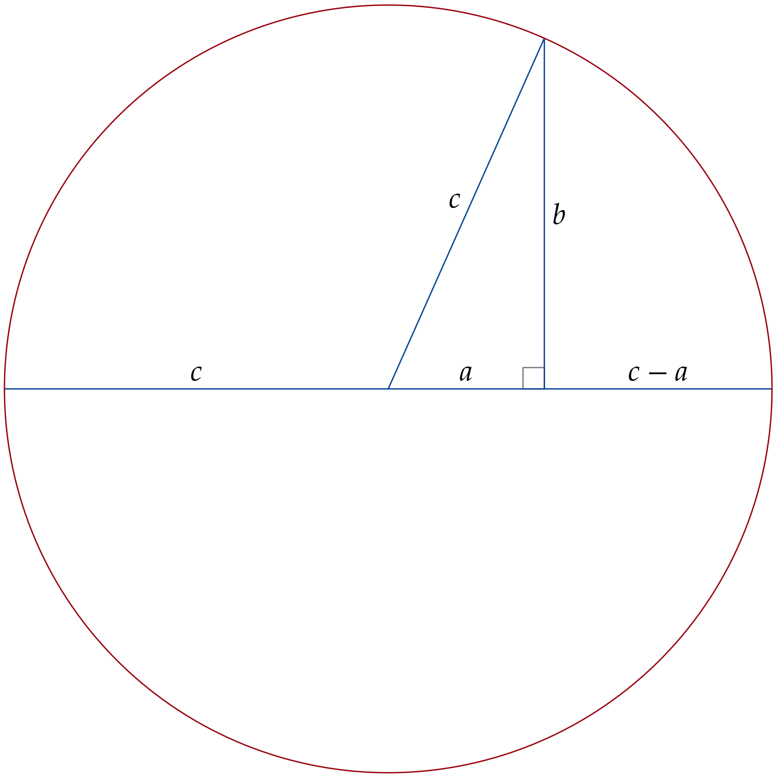
$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)

The Pythagorean theorem VI

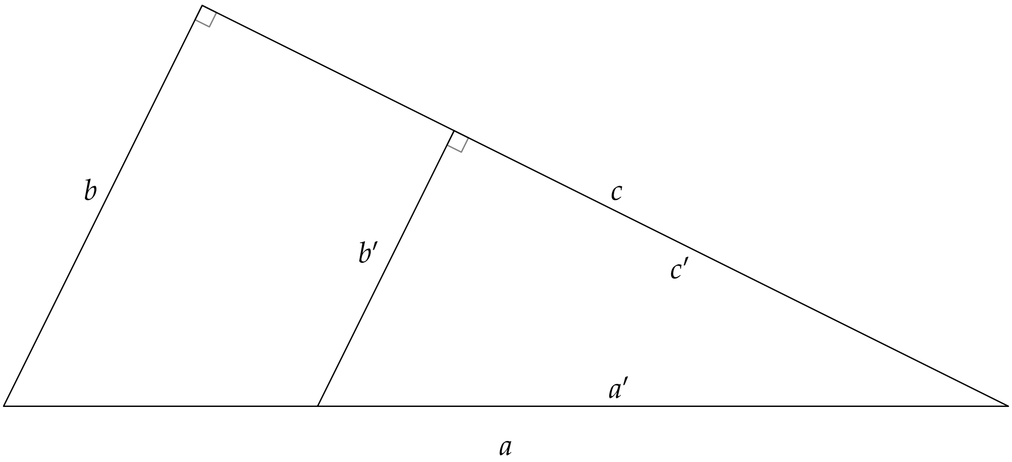
$$\frac{c+a}{b} = \frac{b}{c-a}$$

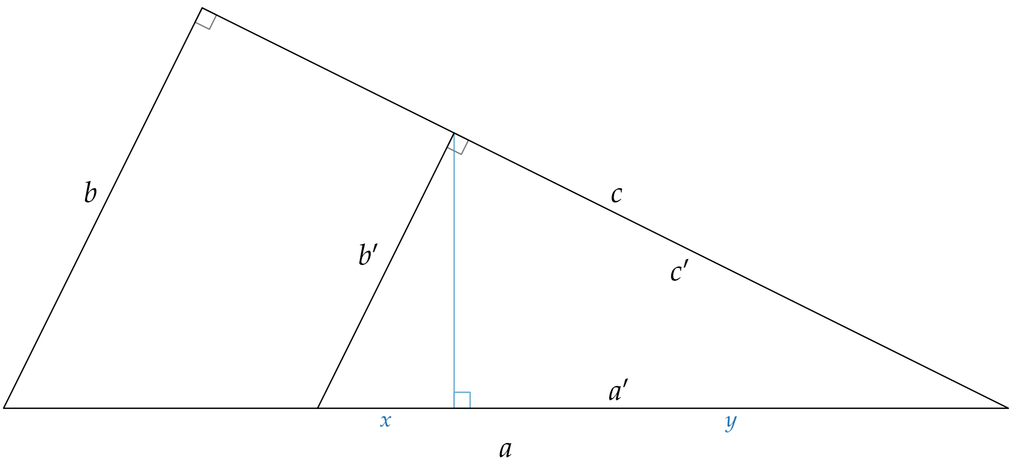
$$a^2 + b^2 = c^2$$



— Michael Hardy

A Pythagorean theorem: $aa^{'} = bb^{'} + cc^{'}$

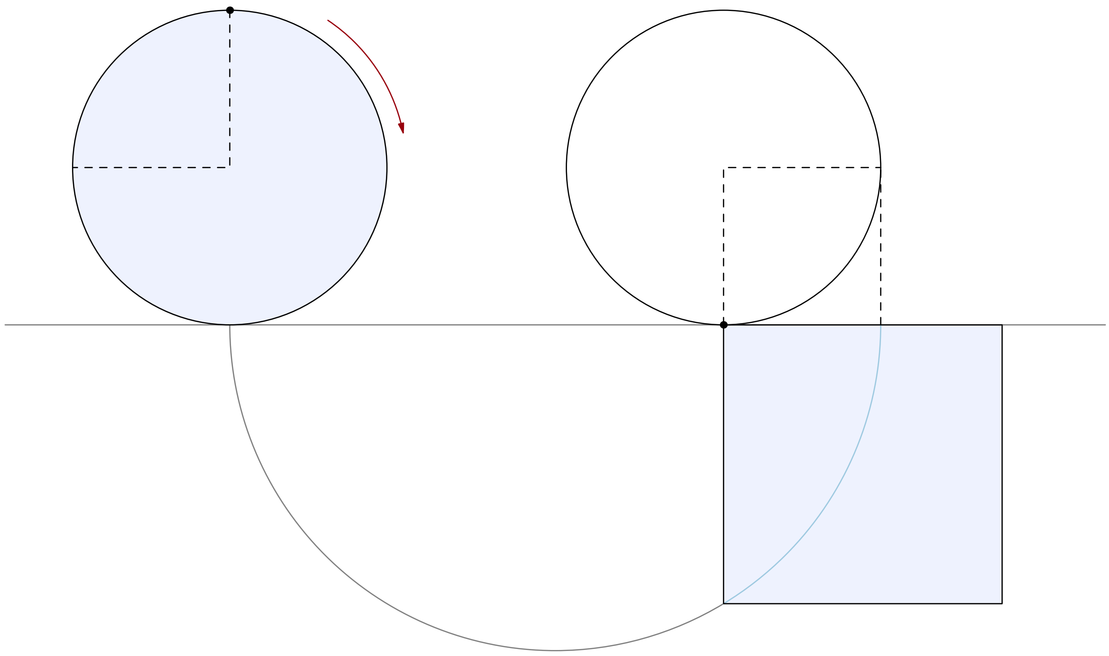




$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$
$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$
$$\therefore \quad aa' = a\,(x+y) = bb' + cc'.$$

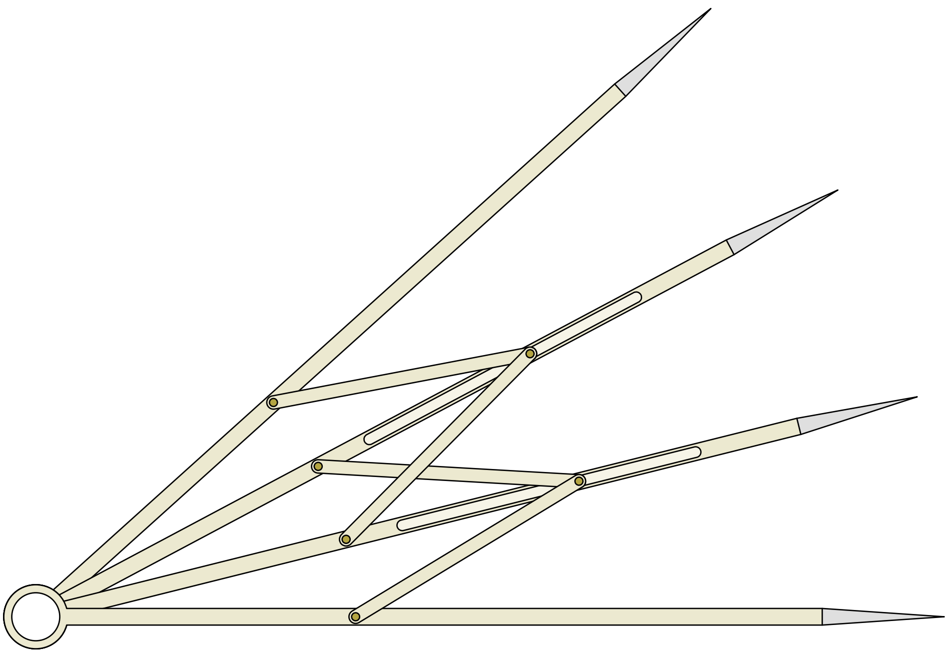
— Enzo R. Gentile

The rolling circle squares itself



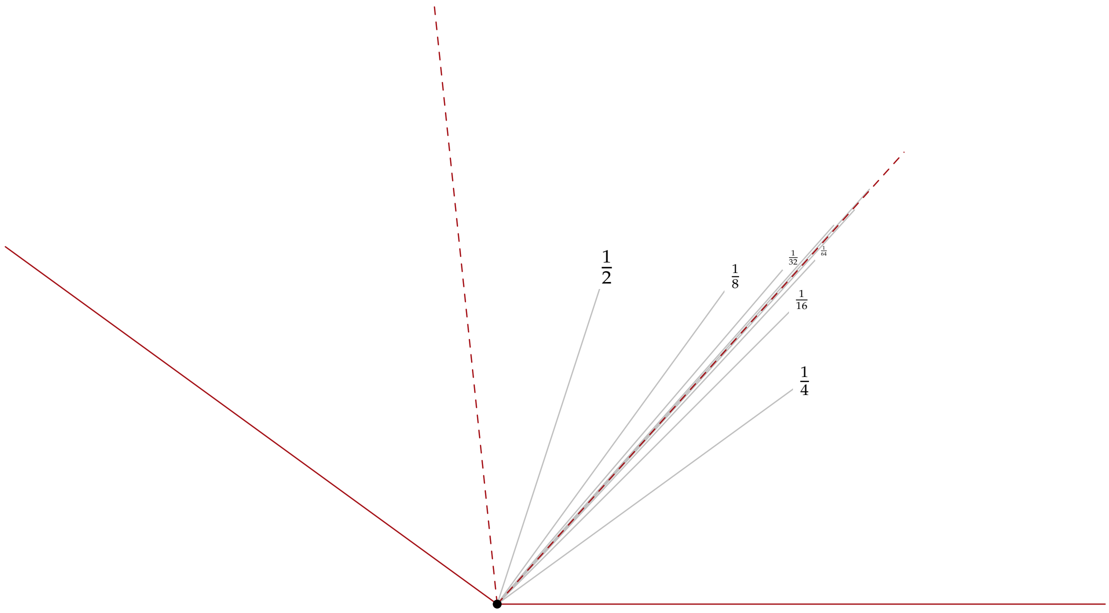
— Thomas Elsner

On trisecting an angle



— Rufus Isaacs

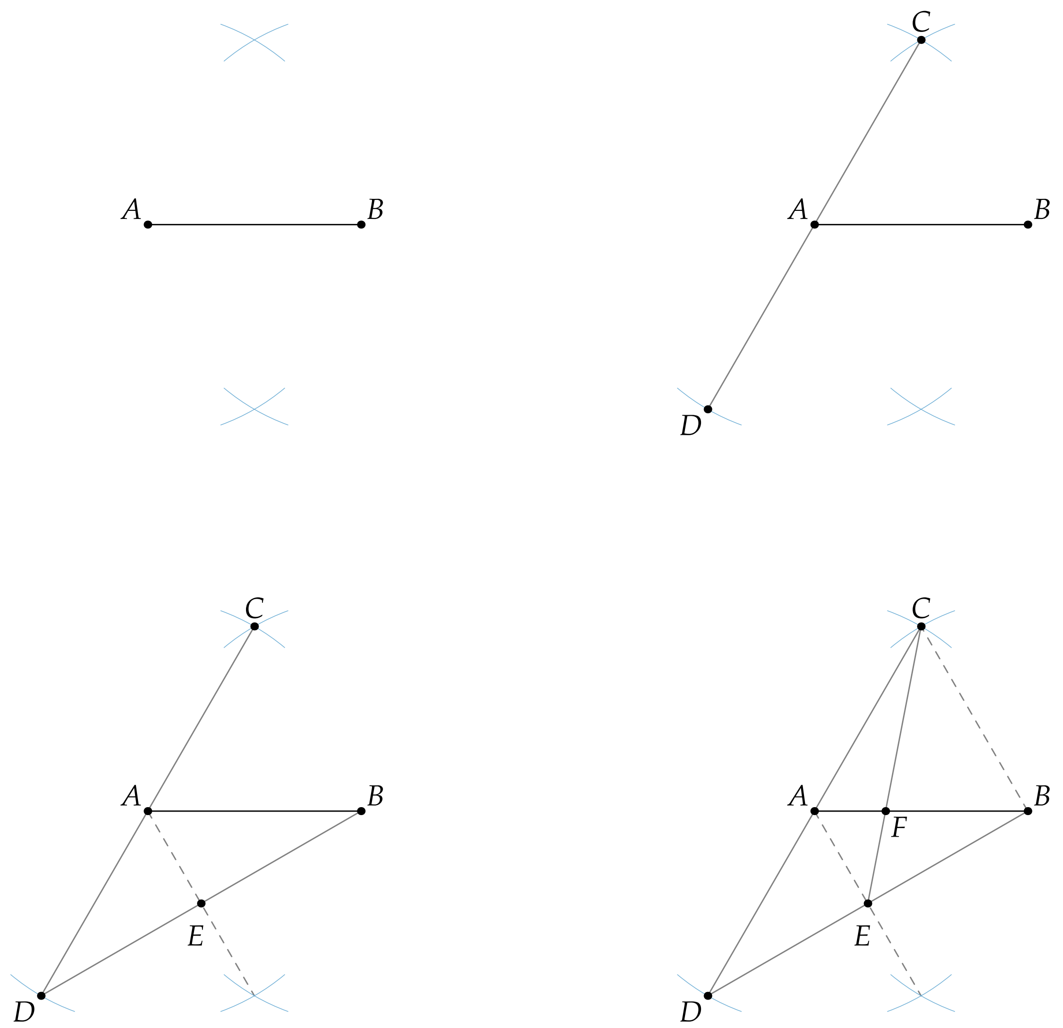
Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

— Eric Kincanon

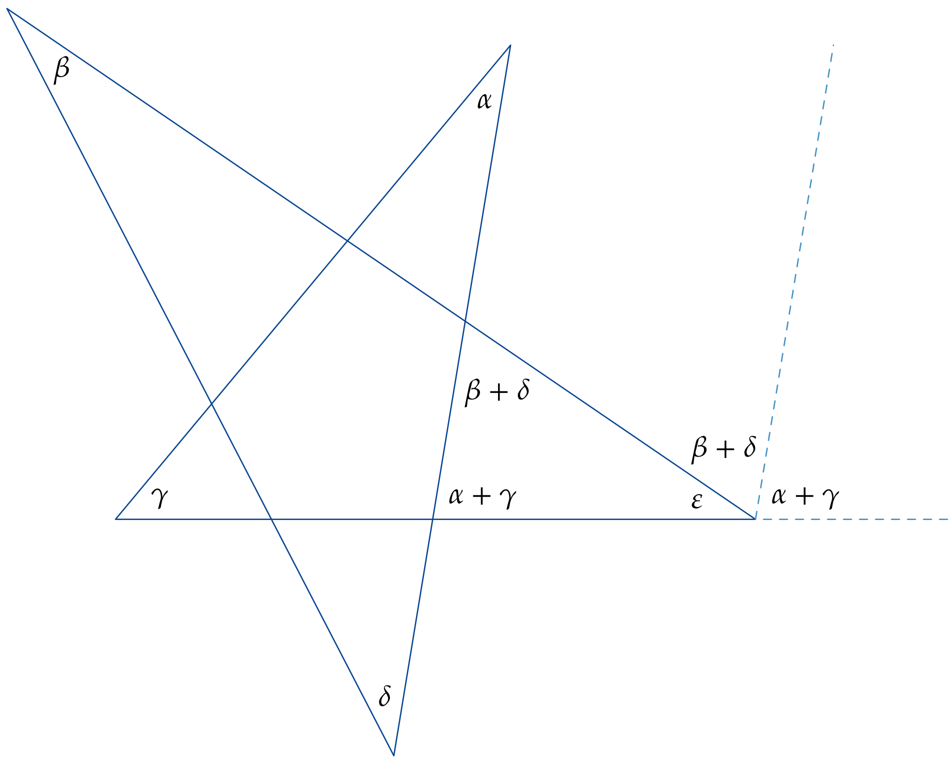
Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

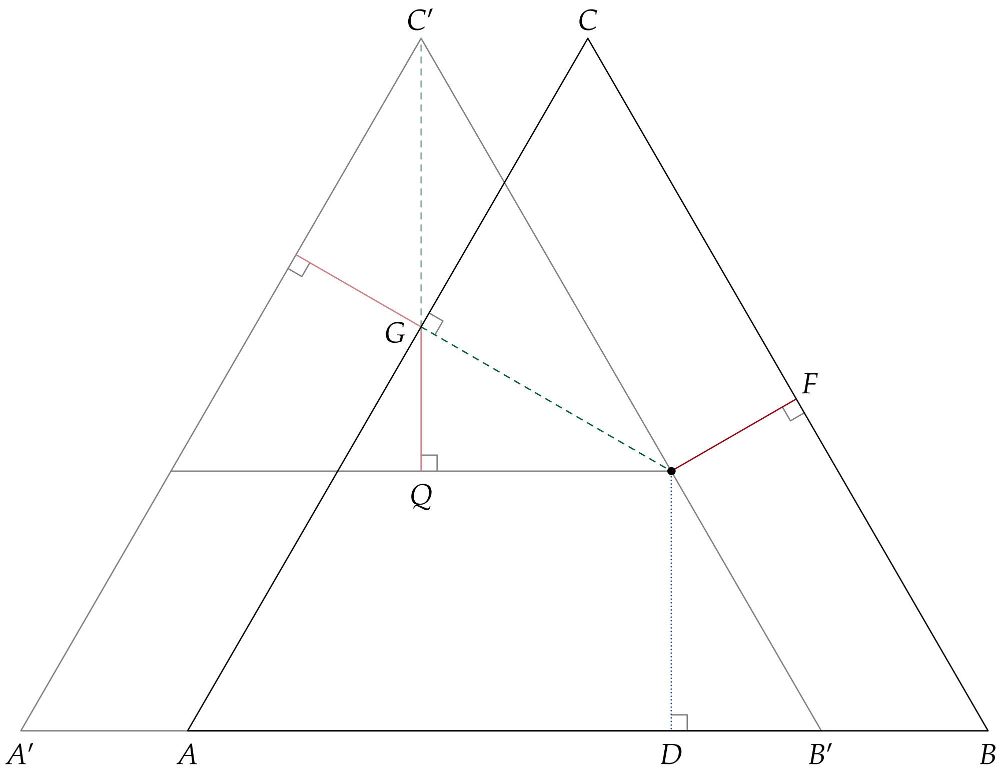
The vertex angles of a star sum to 180°



— Fouad Nakhli

Viviani's theorem I

The perpendiculars to the sides from a point on
the boundary or within an equilateral triangle
add up to the height of the triangle.

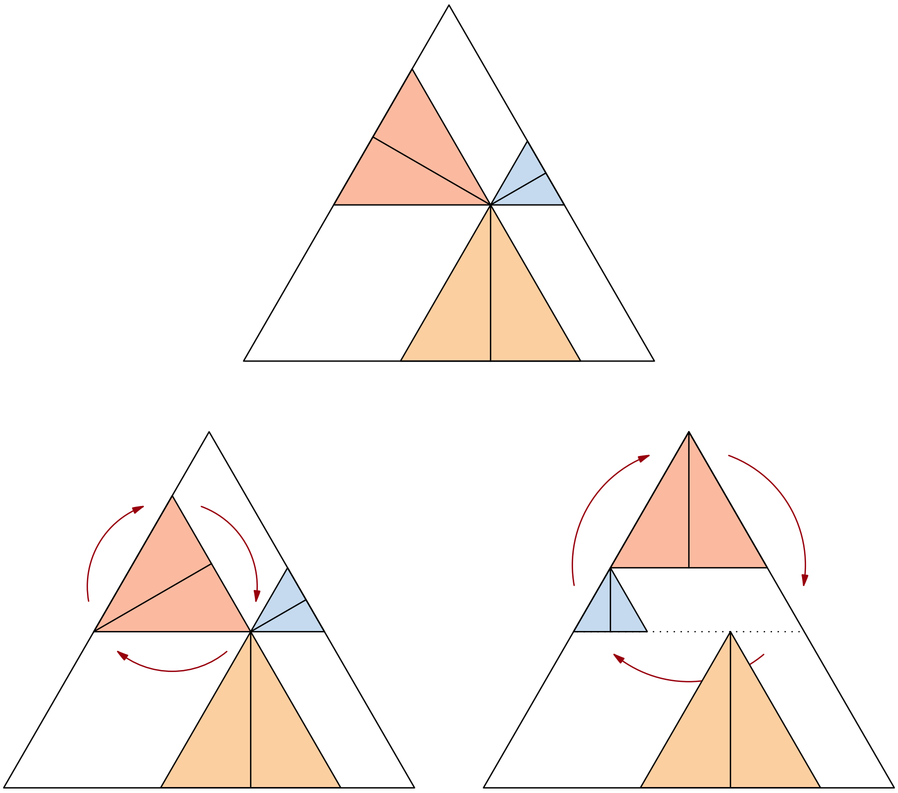


This shows a particular example, with $C'GQ$ collinear, rather than the general case

— Samuel Wolf

Viviani's theorem II

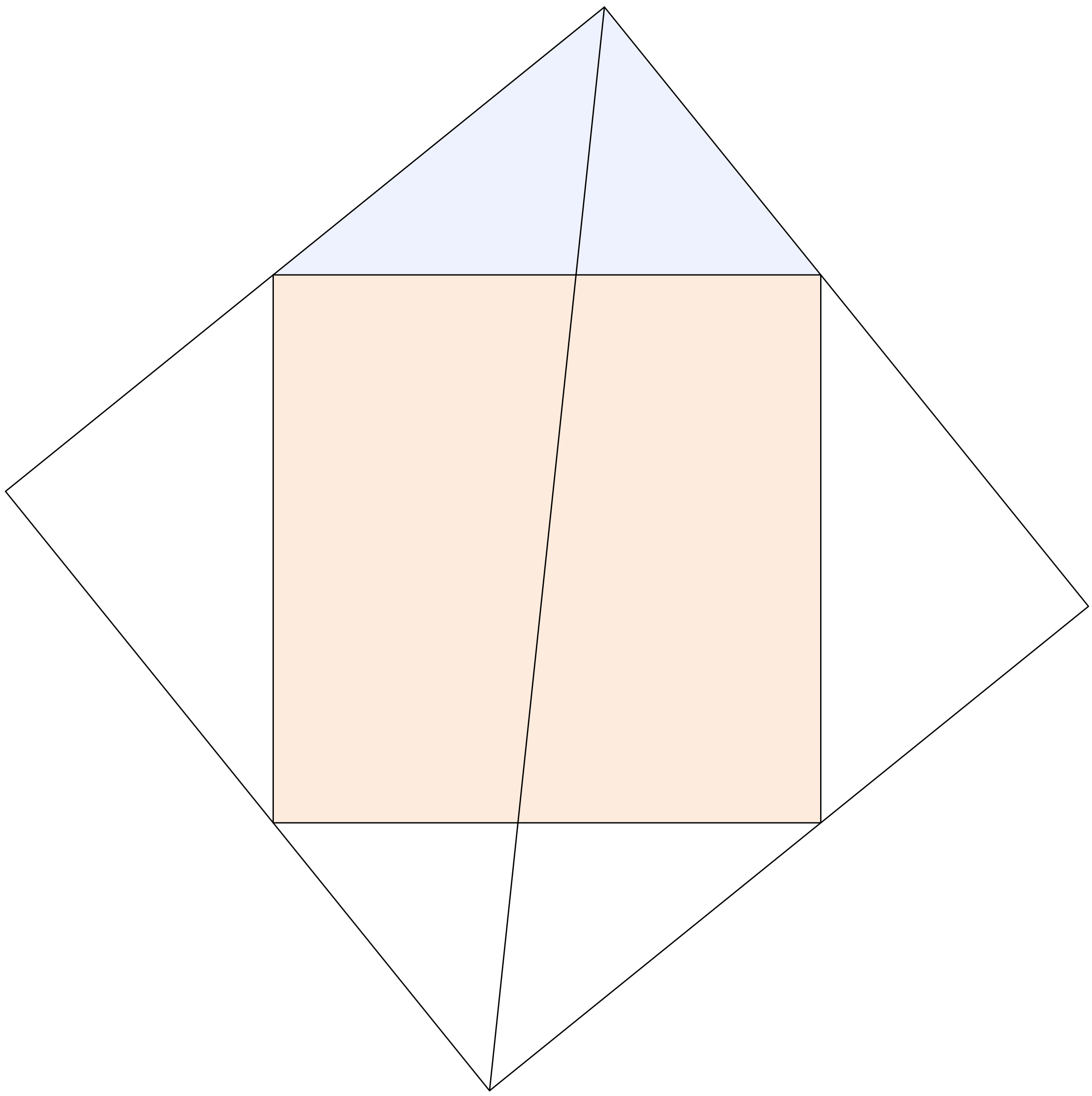
The perpendiculars to the sides from a point on
the boundary or within an equilateral triangle
add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

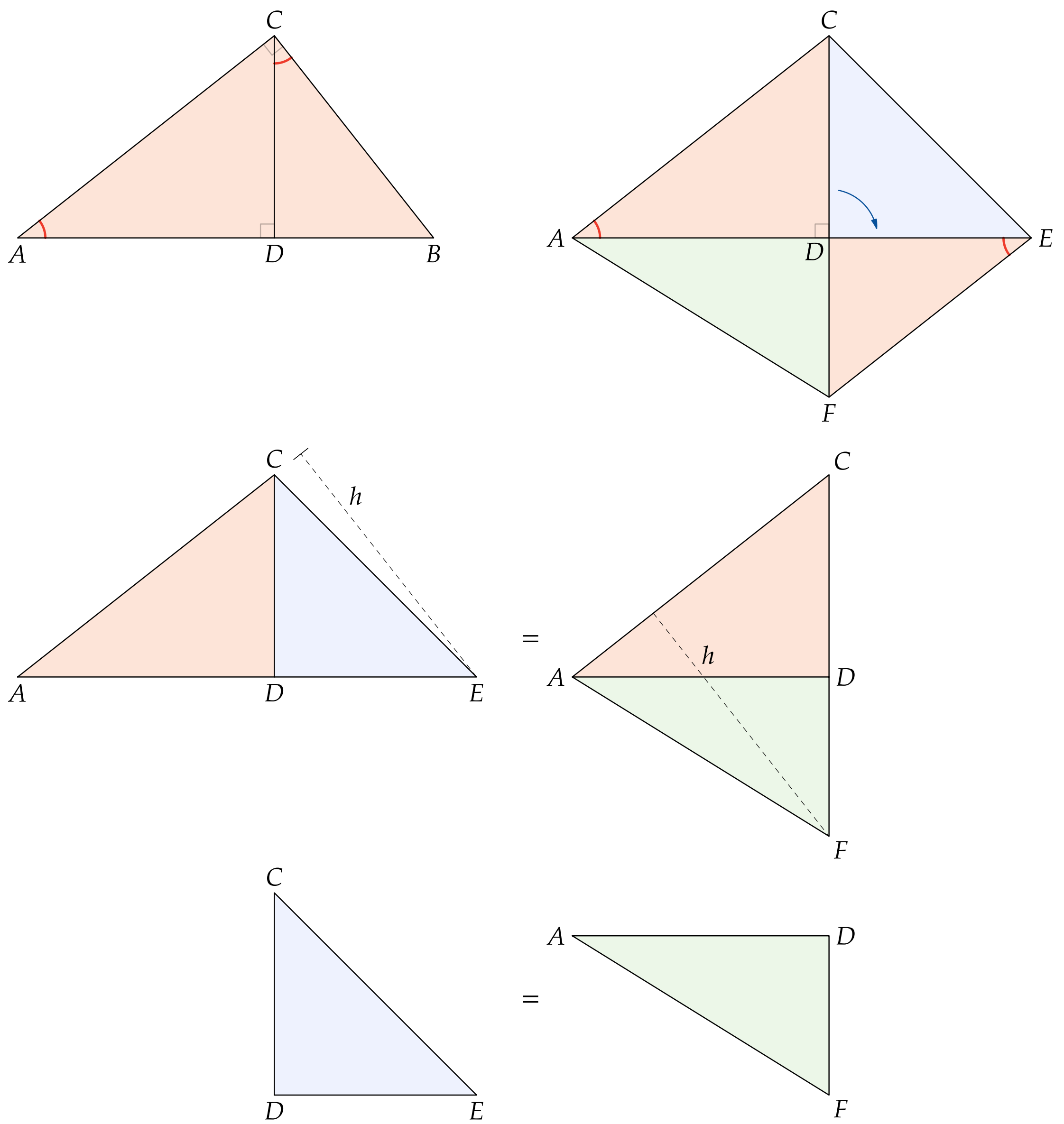
A theorem about right angles

The internal bisector of the right angle of a right
triangle bisects the square on the hypotenuse



— Roland H. Eddy

Area and the projection theorem of a right triangle

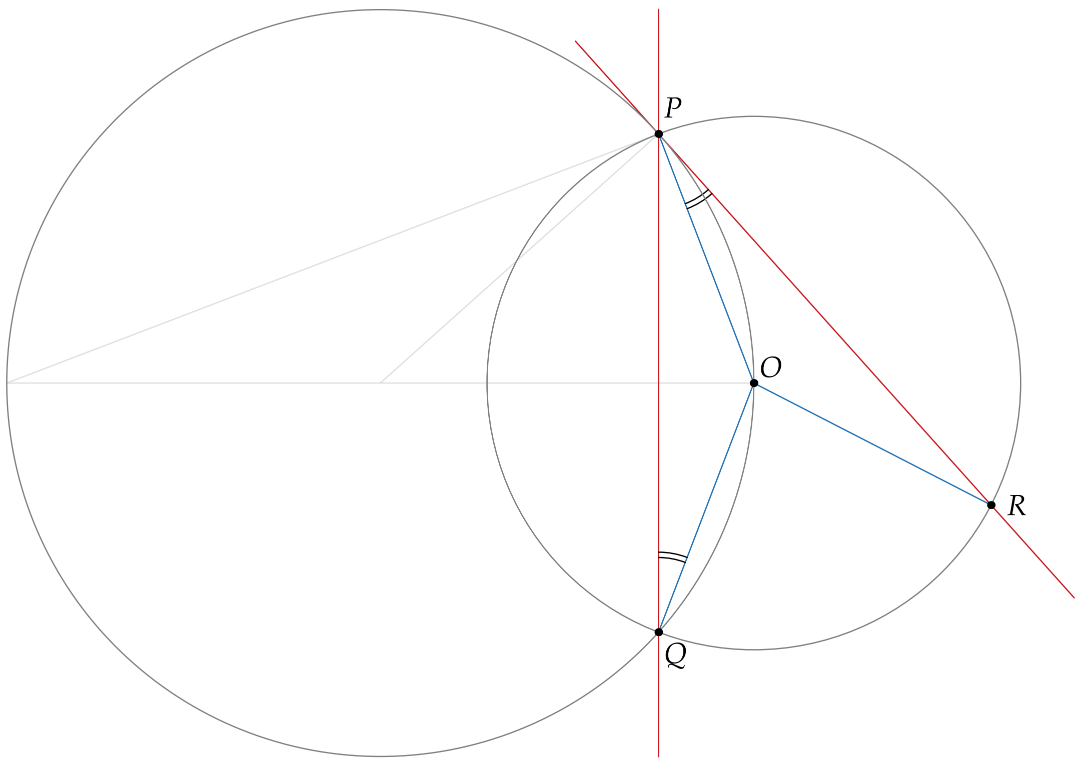


$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

Chords and tangents of equal length

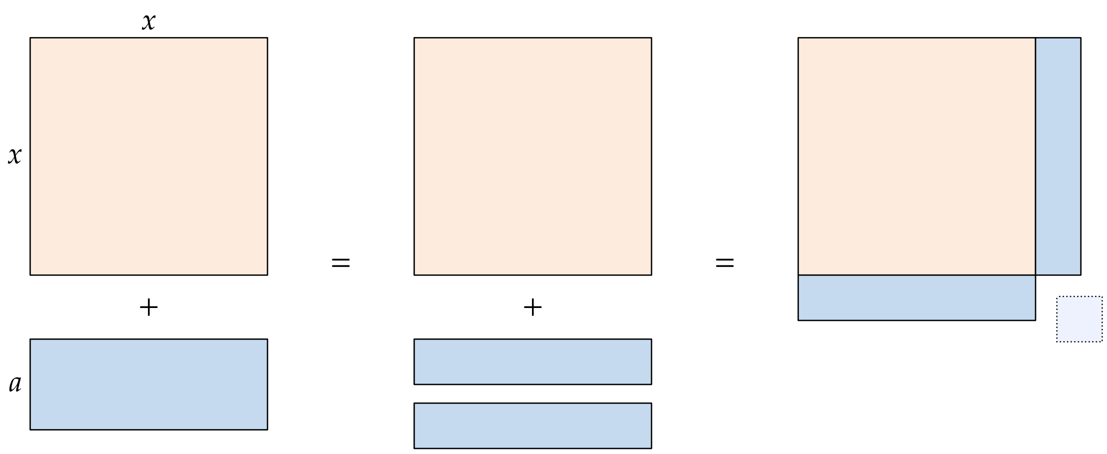
If circle C_1 passes through the center O of circle C_2, the length of the common chord $\overline{PQ}$ is equal to the tangent segment $\overline{PR}$.



— Roland H. Eddy

Completing the square

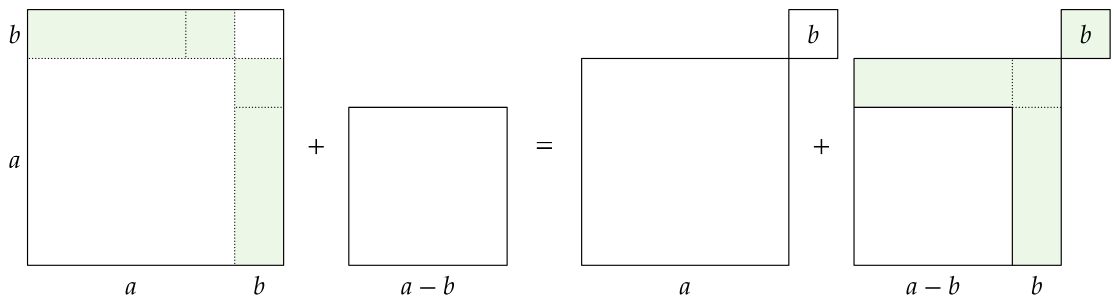
$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

Algebraic areas I

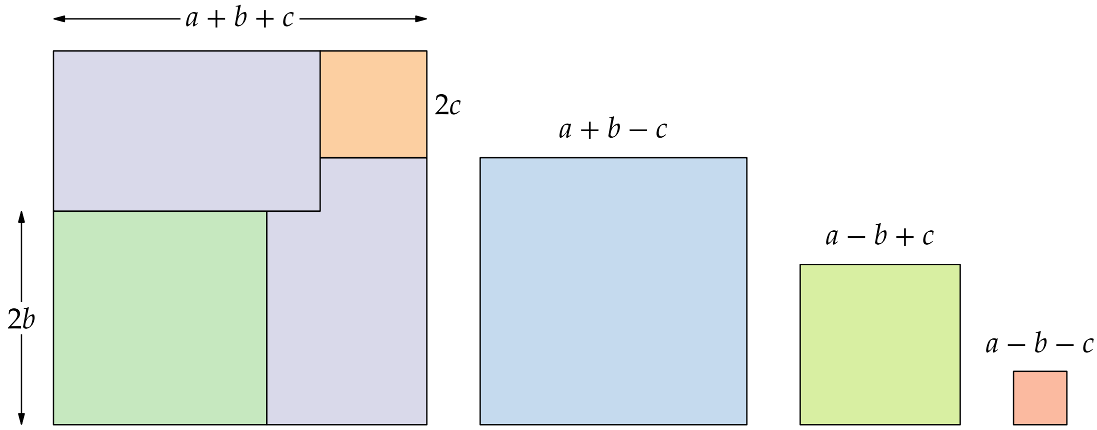
$$(a+b)^2 + (a-b)^2 = 2\left(a^2+b^2\right)$$

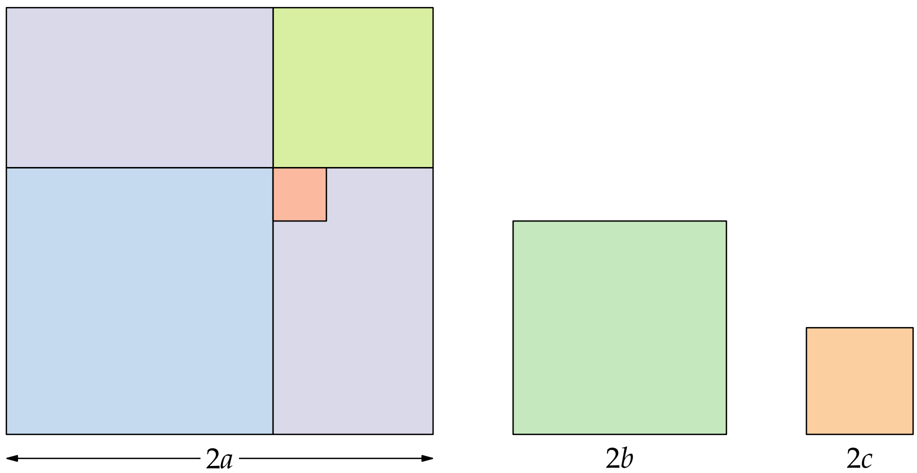


— Shirley Wakin

Algebraic areas II

$$(a+b+c)^2 + (a+b-c)^2 + (a-b+c)^2 + (a-b-c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$

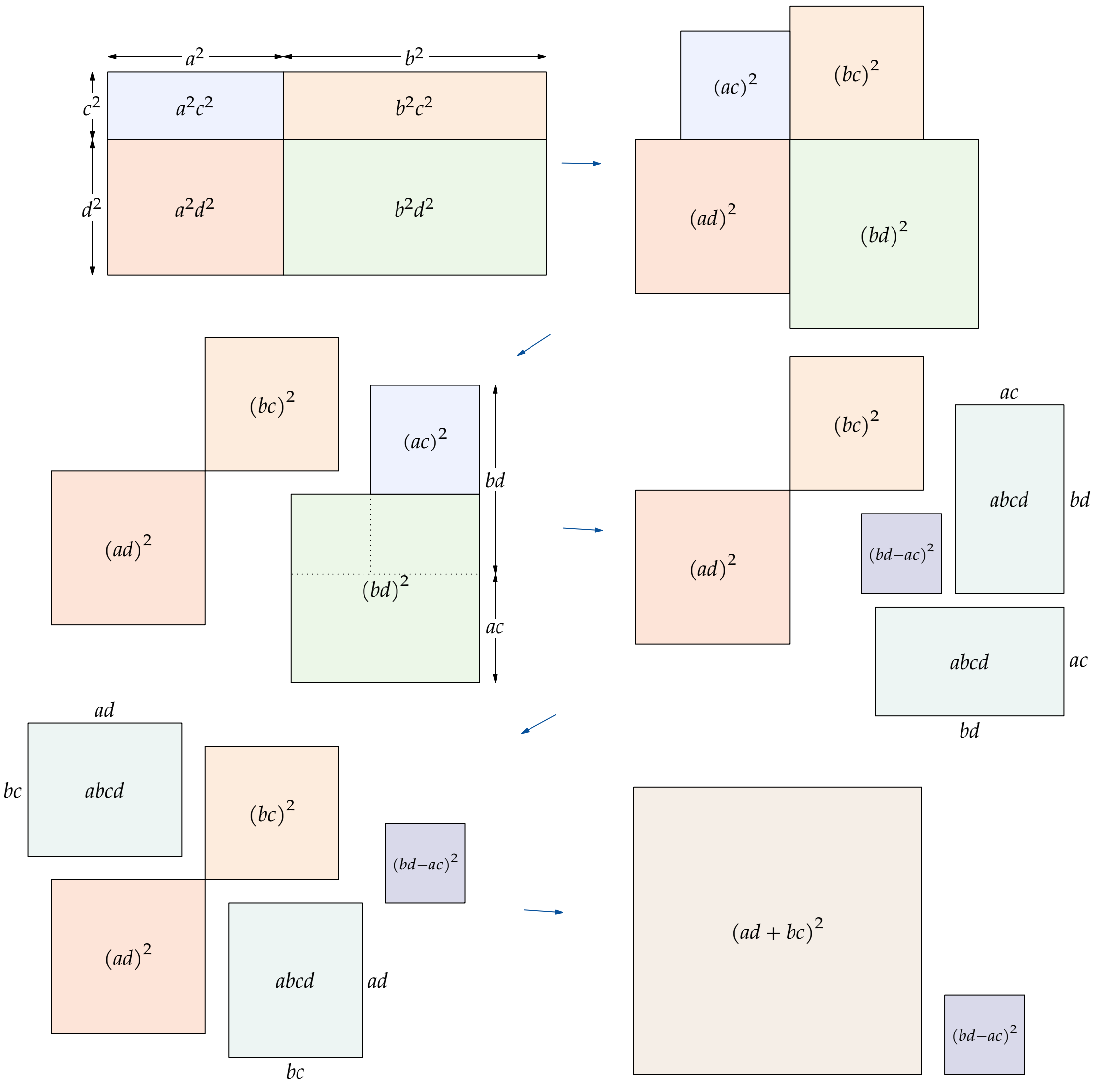




— Sam Pooley and K. Ann Drude

Sum of squares identity

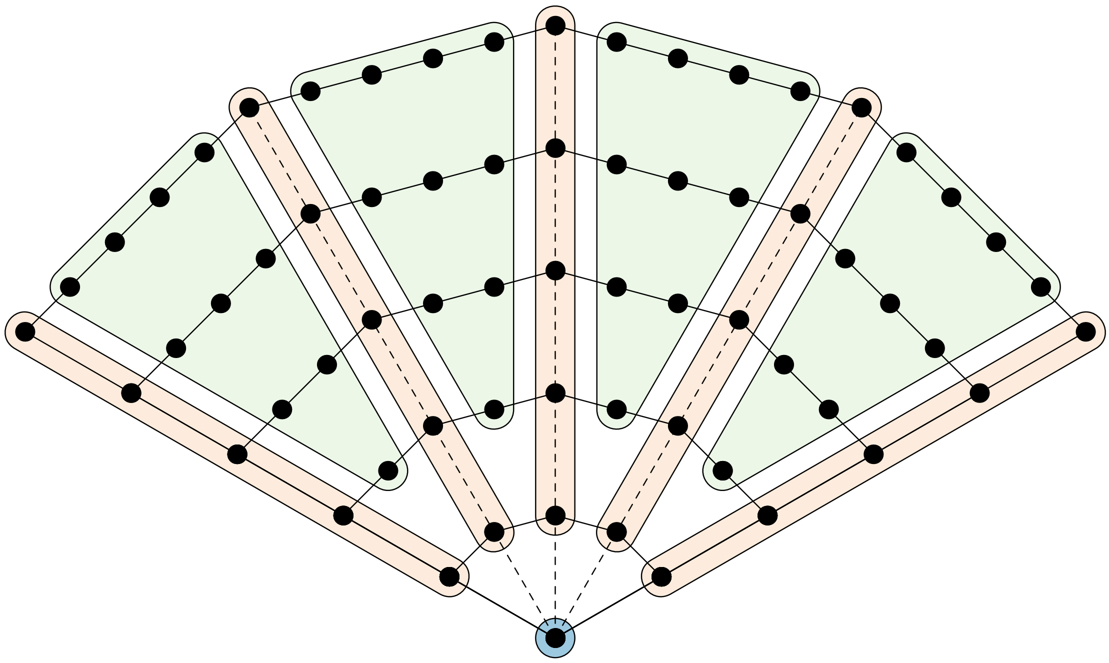
$$\left(a^2+b^2\right)\left(c^2+d^2\right) = (ab+bc)^2 + (bd-ac)^2$$



— Diophantus of Alexandria

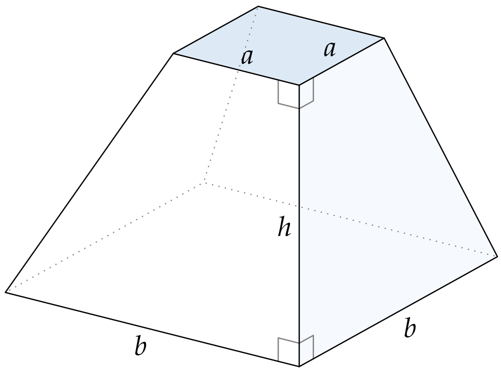
Polygonal numbers

The k^{th} n-gonal number is $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$

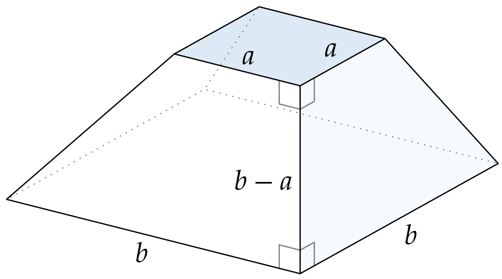


— Dave Logothetti

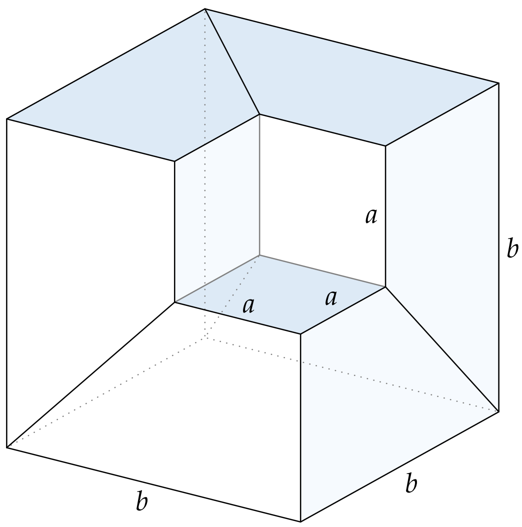
The volume of a frustrum of a square pyramid



P_1



P_2

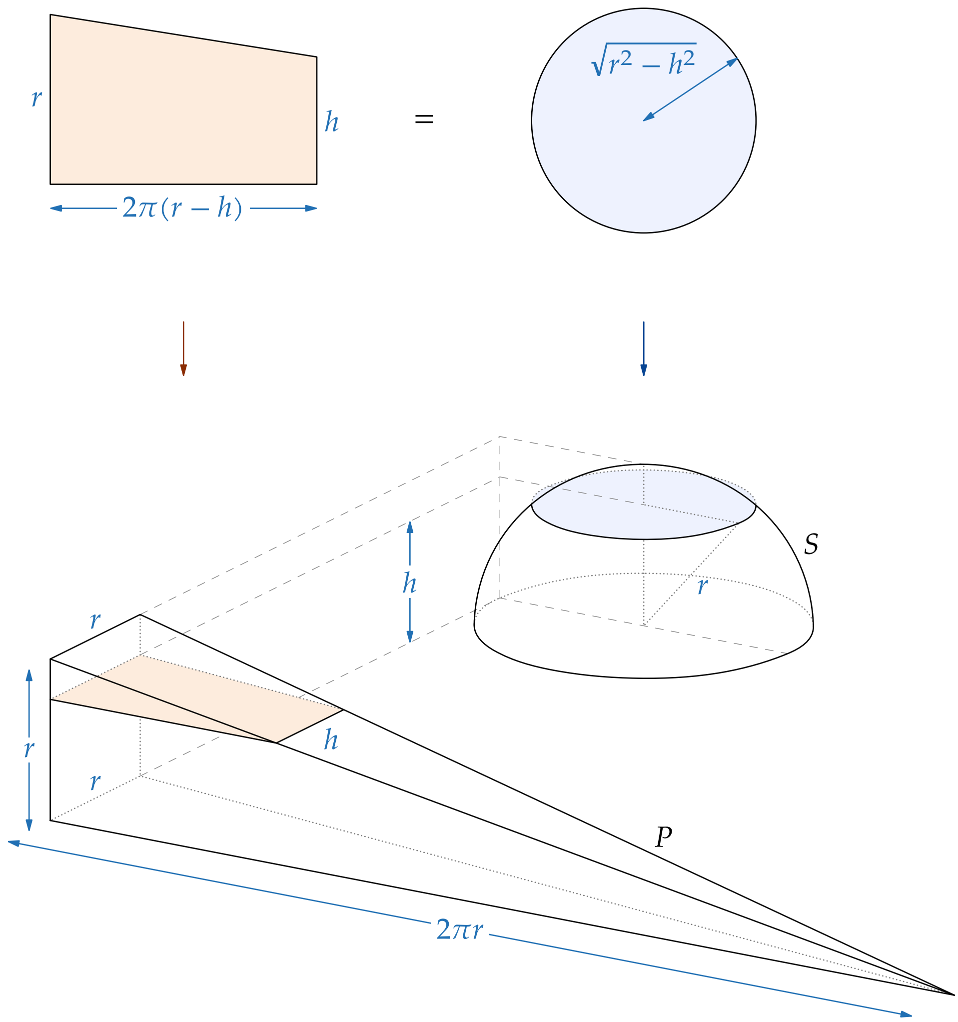


$3P_2$

$$V_{P_1} = \frac{h}{b-a} \cdot V_{P_2} = \frac{h}{b-a} \cdot \frac{1}{3}\left(b^3 - a^3\right) = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

— *The Moscow Papyrus*, c. 1850 BCE

The volume of a hemisphere via Cavalieri's Principle



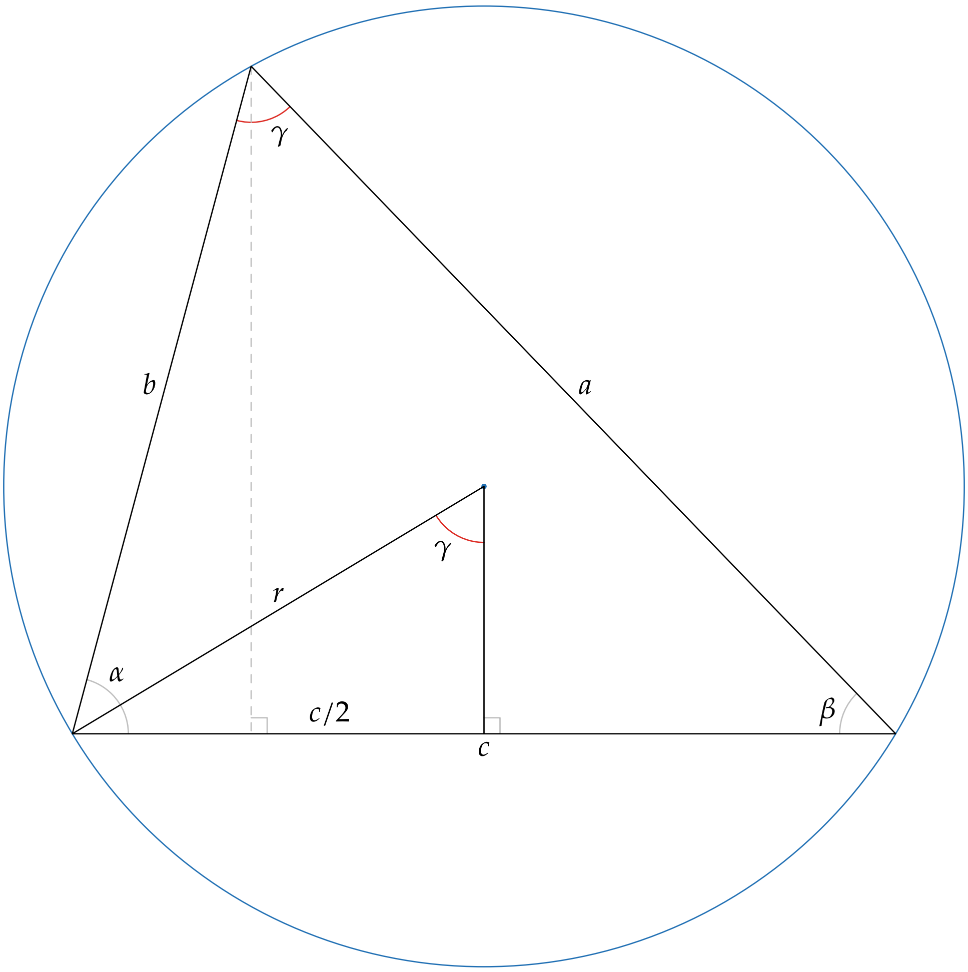
$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

— Sidney H. Kung

Trigonometry, Calculus, & Analytic Geometry

Sine of the sum

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \text{ for } \alpha+\beta<\pi$$



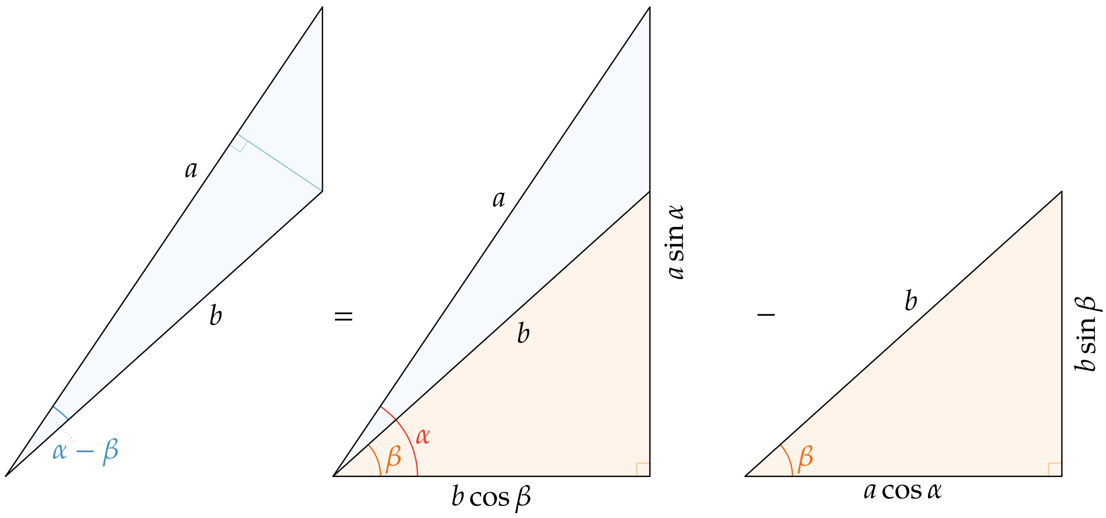
$$c = a\cos\beta + b\cos\alpha$$

$$r = 1/2 \quad\Longrightarrow\quad \sin\gamma = \frac{c/2}{1/2} = c,\ \sin\alpha = a,\ \sin\beta = b$$

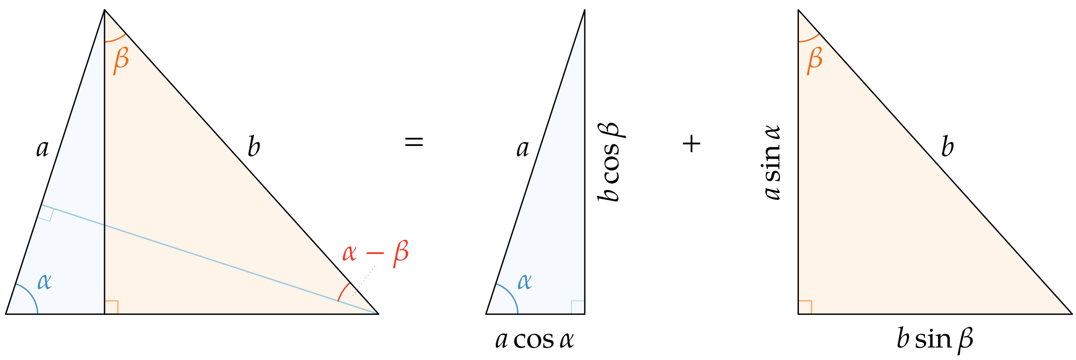
$$\sin(\alpha+\beta) = \sin(\pi-(\alpha+\beta)) = \sin\gamma = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

— Sidney H. Kung

Area and difference formulas



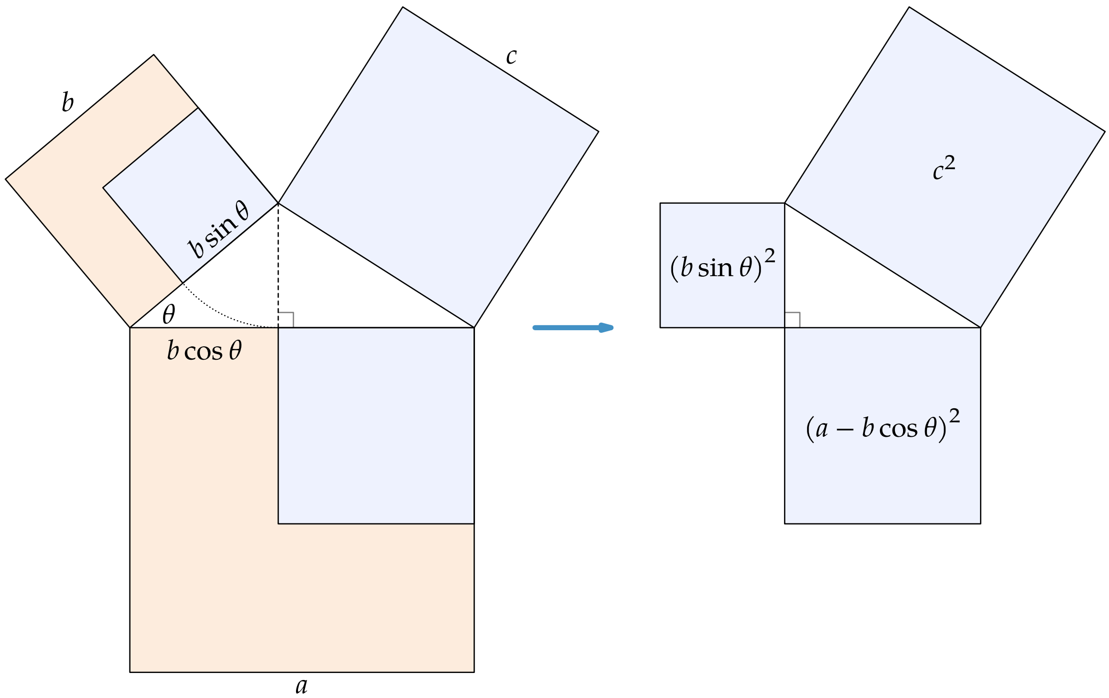
$$\tfrac{1}{2} \cdot a \cdot b \sin(\alpha - \beta) = \tfrac{1}{2} \cdot a \sin \alpha \cdot b \cos \beta - \tfrac{1}{2} \cdot a \cos \alpha \cdot b \sin \beta$$
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\tfrac{1}{2} \cdot a \cdot b \cos(\alpha - \beta) = \tfrac{1}{2} \cdot a \cos \alpha \cdot b \cos \beta + \tfrac{1}{2} \cdot a \sin \alpha \cdot b \sin \beta$$
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

— Sidney H. Kung

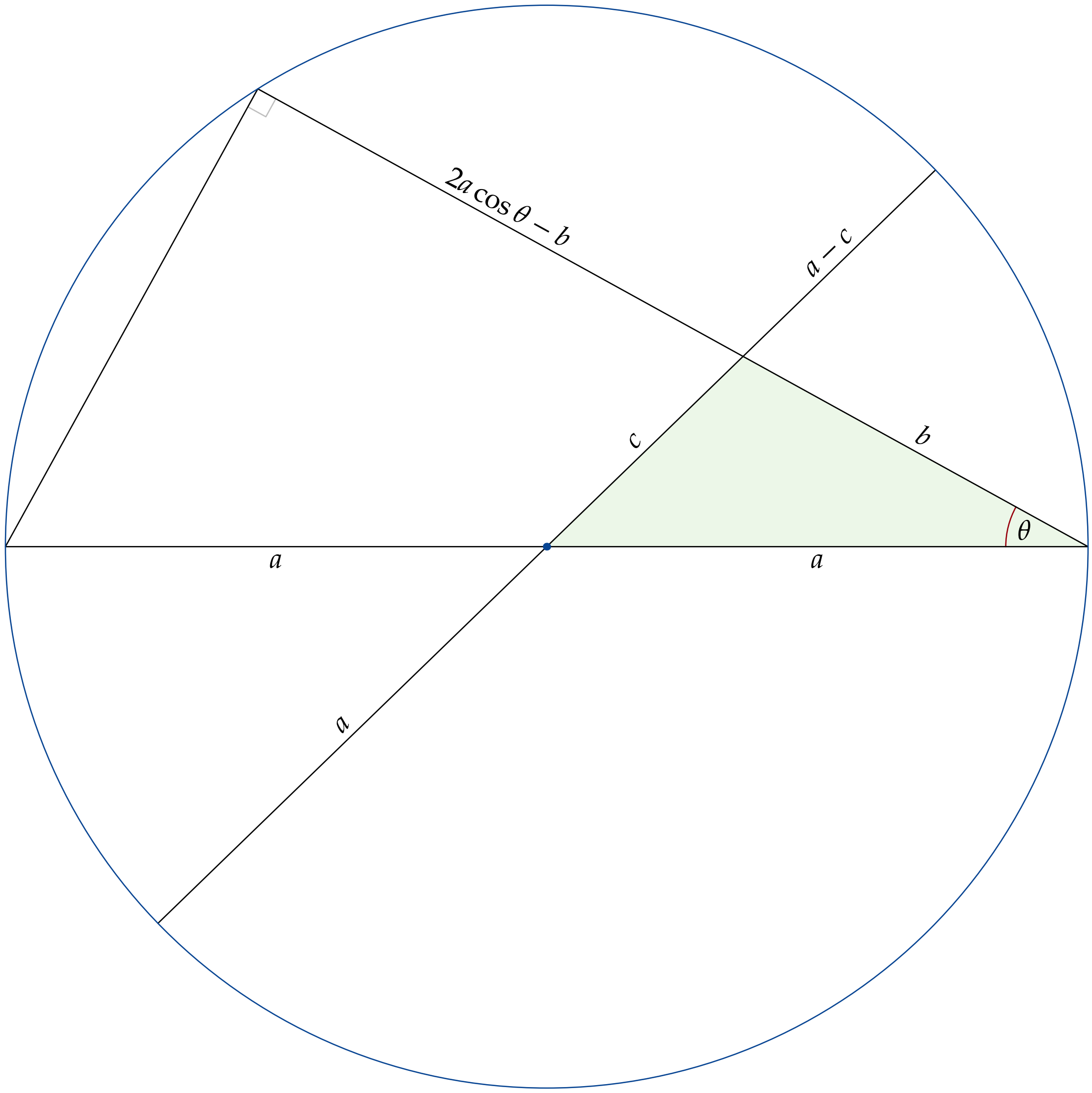
The law of cosines I



$$\begin{aligned}
c^2 &= (b\sin\theta)^2 + (a - b\cos\theta)^2 \\
&= b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta \\
&= a^2 + b^2\left(\sin^2\theta + \cos^2\theta\right) - 2ab\cos\theta \\
&= a^2 + b^2 - 2ab\cos\theta
\end{aligned}$$

— Timothy A. Sipka

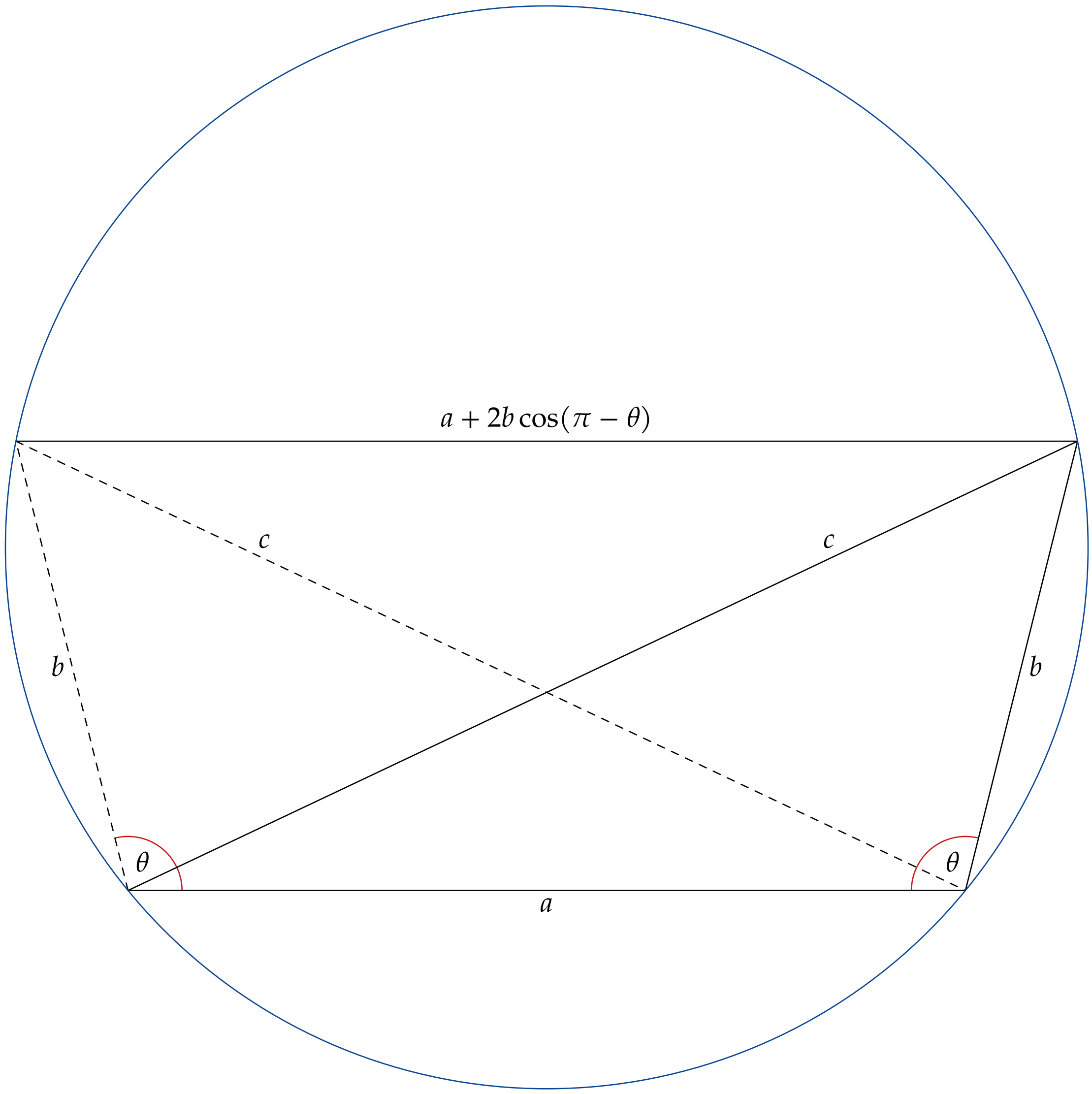
The law of cosines II



$$(2a\cos\theta - b)\cdot b = (a-c)\cdot(a+c)$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

— Sidney H. Kung

The law of cosines III (via Ptolemy's theorem)

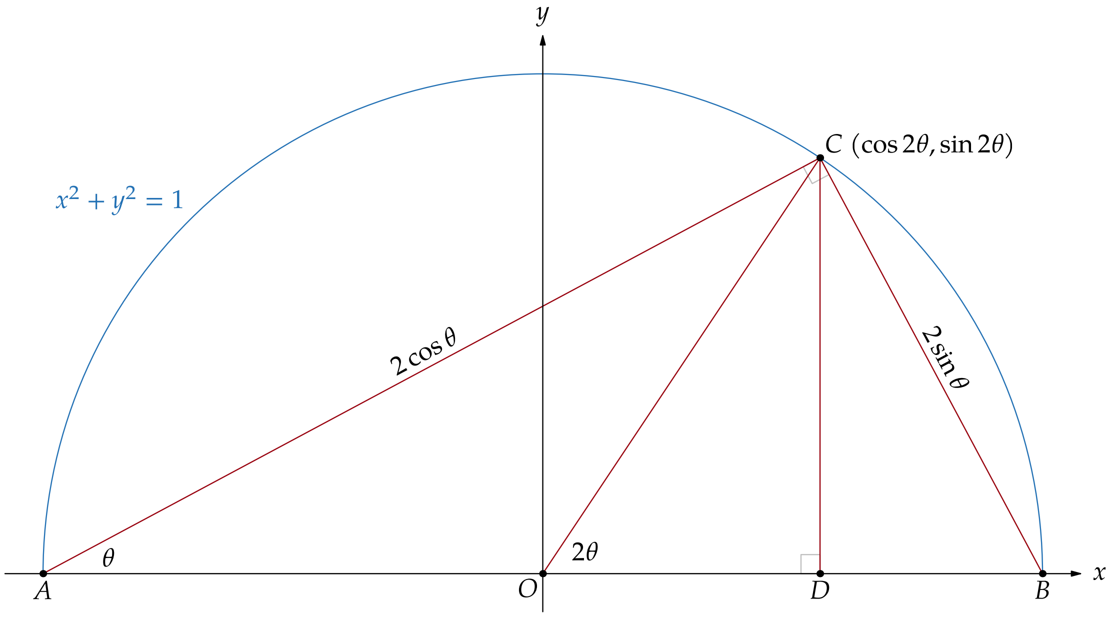


$$c \cdot c = b \cdot b + \Big(a + 2b\cos(\pi - \theta)\Big) \cdot a$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

— Sidney H. Kung

The double-angle formulae



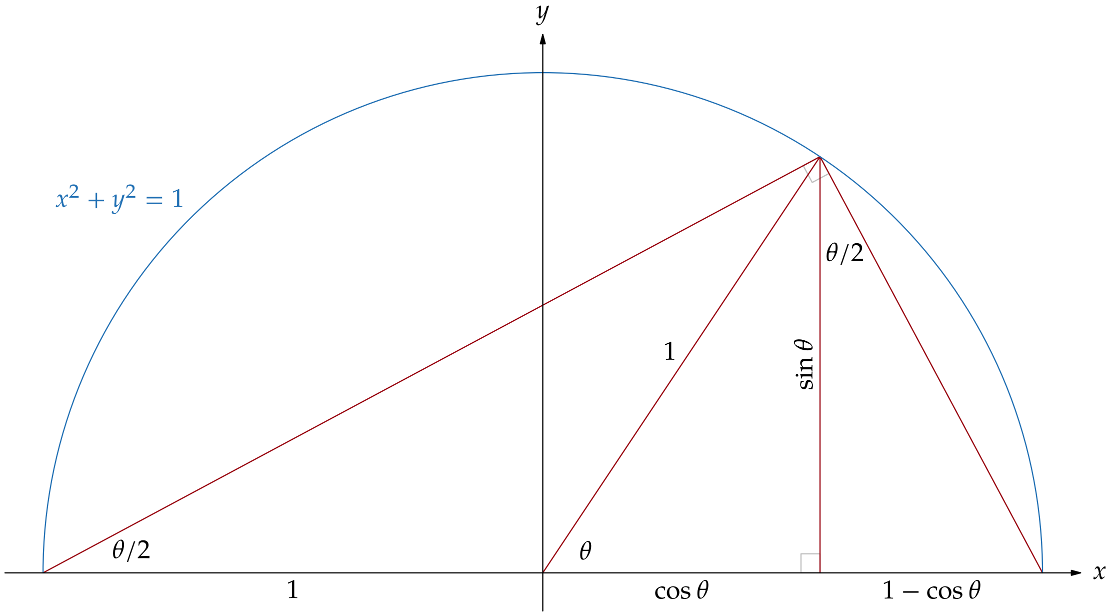
$$\triangle ACD \sim \triangle ABC$$

$$CD \big/ AC = BC \big/ AB$$
$$\sin 2\theta / 2\cos\theta = 2\sin\theta / 2$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$AD \big/ AC = AC \big/ AB$$
$$(1 + \cos 2\theta)/2\cos\theta = 2\cos\theta/2$$
$$\cos 2\theta = 2\cos^2\theta - 1$$

— Roger B. Nelsen

The half-angle tangent formulae

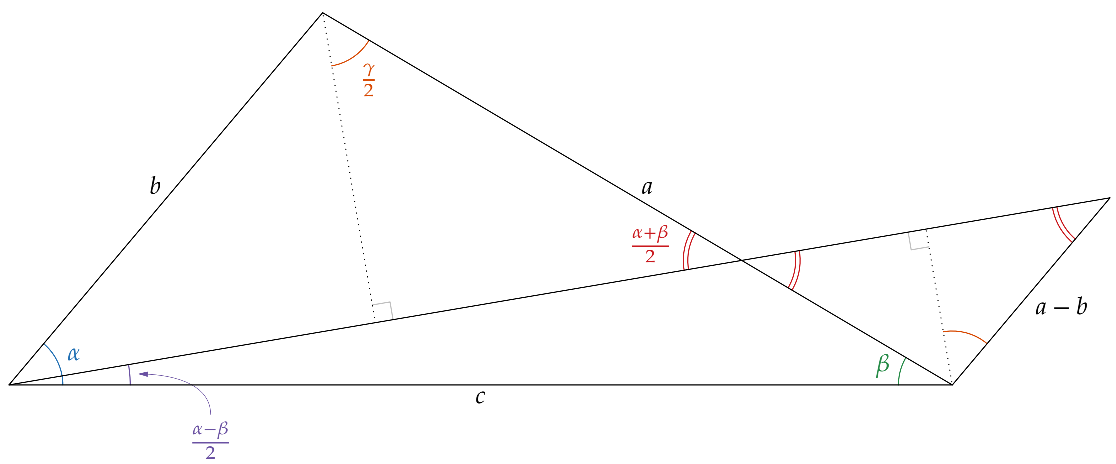


$$\tan\theta/2 = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

— R. J. Walker

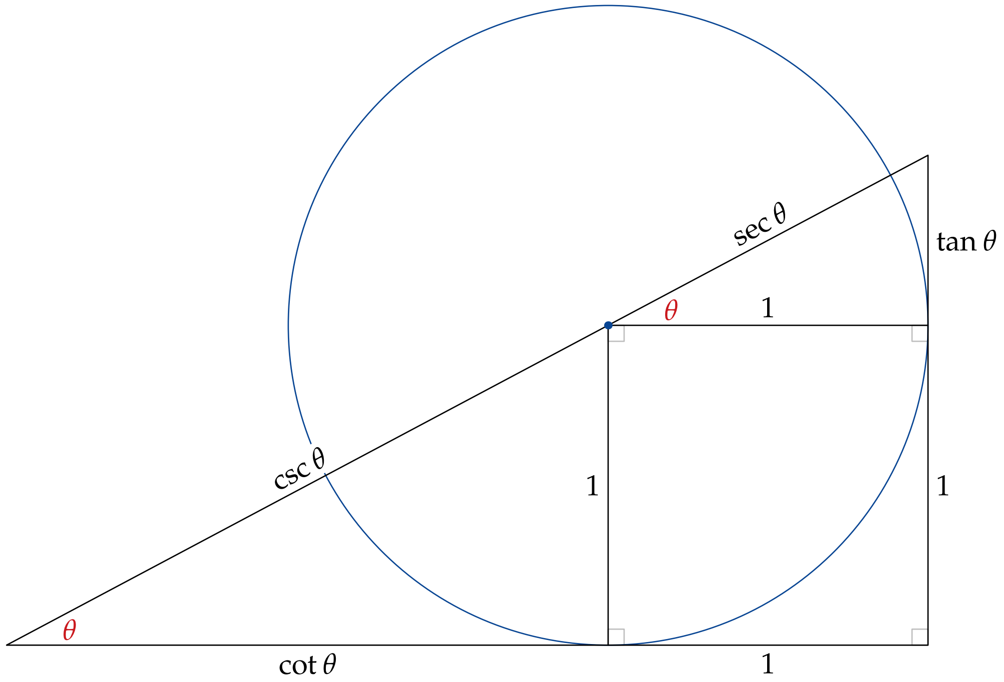
Mollweide's equation

$$(a-b)\cos\frac{\gamma}{2} = c\sin\left(\frac{\alpha-\beta}{2}\right)$$



— H. Arthur DeKleine

Tangent, cotangent, secant, and cosecant

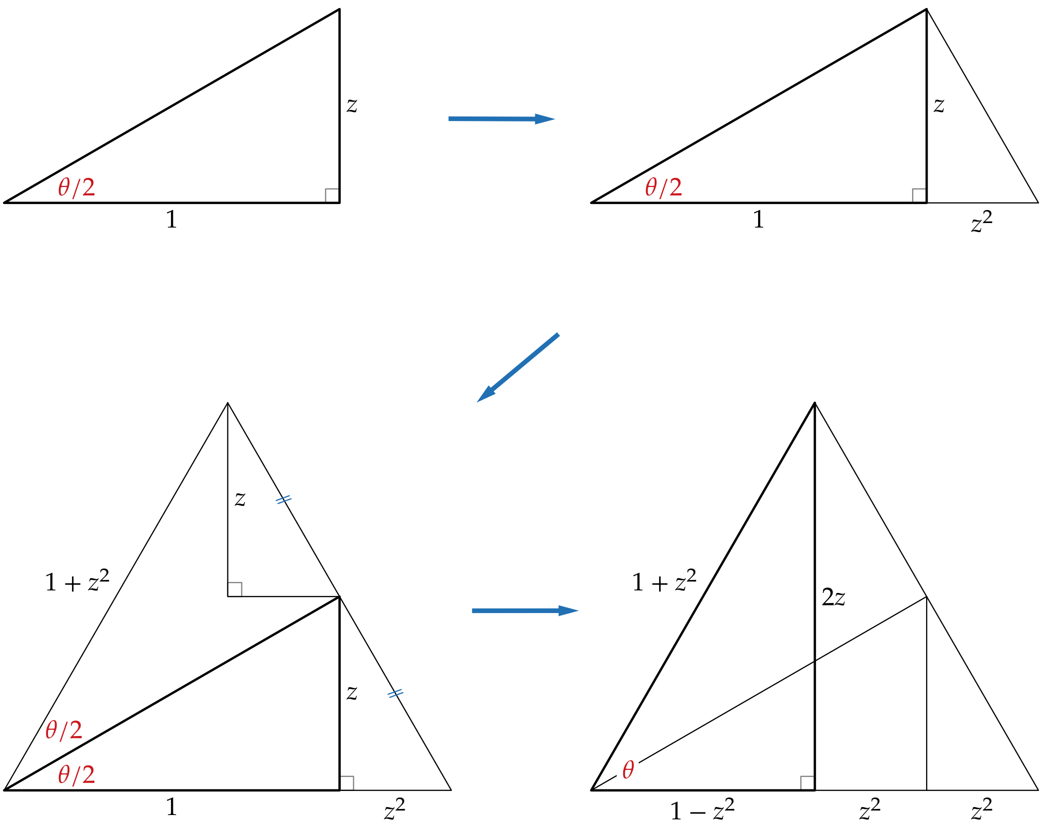


$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$
$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$

$$\text{also } \tan\theta = \frac{\tan\theta + 1}{\cot\theta + 1}$$

— William Romaine

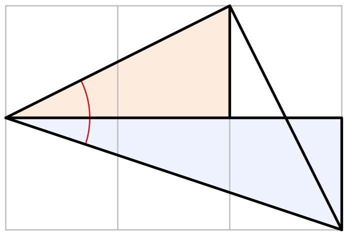
Substitution to make a rational function of sine and cosine



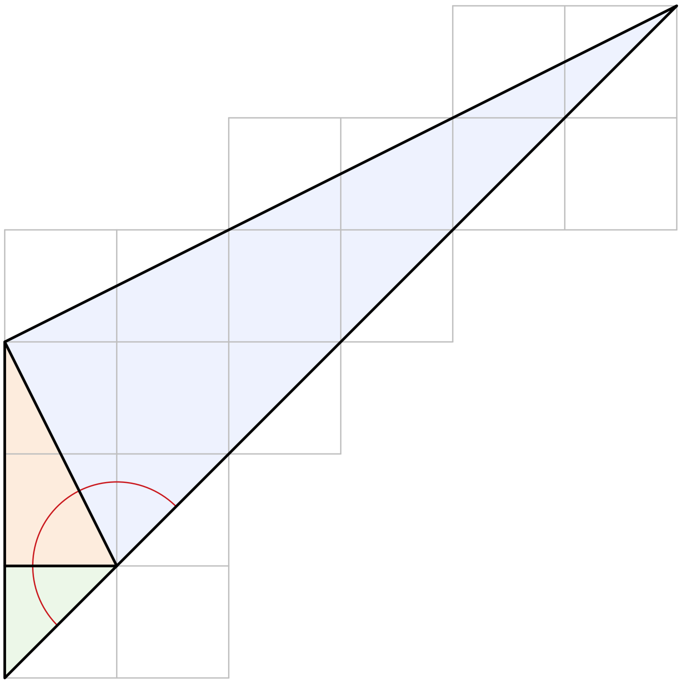
$$z = \tan(\theta/2) \implies \sin\theta = \frac{2z}{1+z^2} \quad \text{and} \quad \cos\theta = \frac{1-z^2}{1+z^2}$$

— Roger B. Nelsen

Sums of arctangents



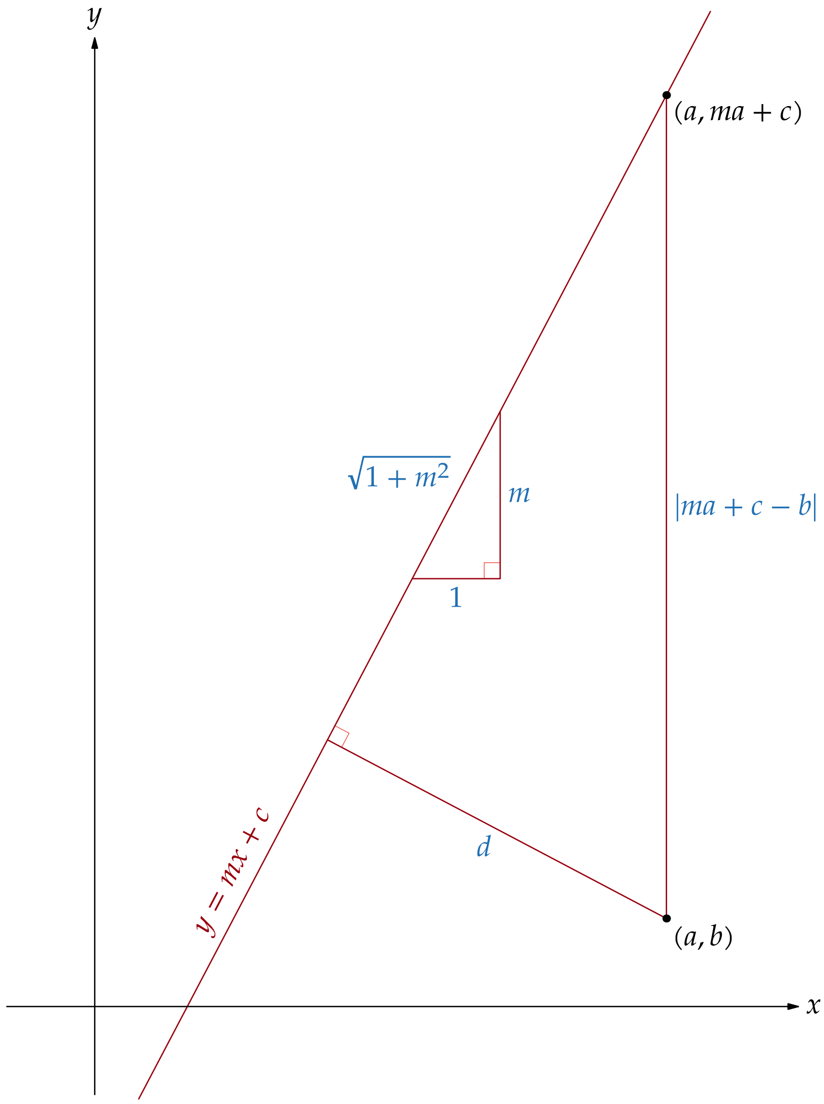
$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

— Edward M. Harris

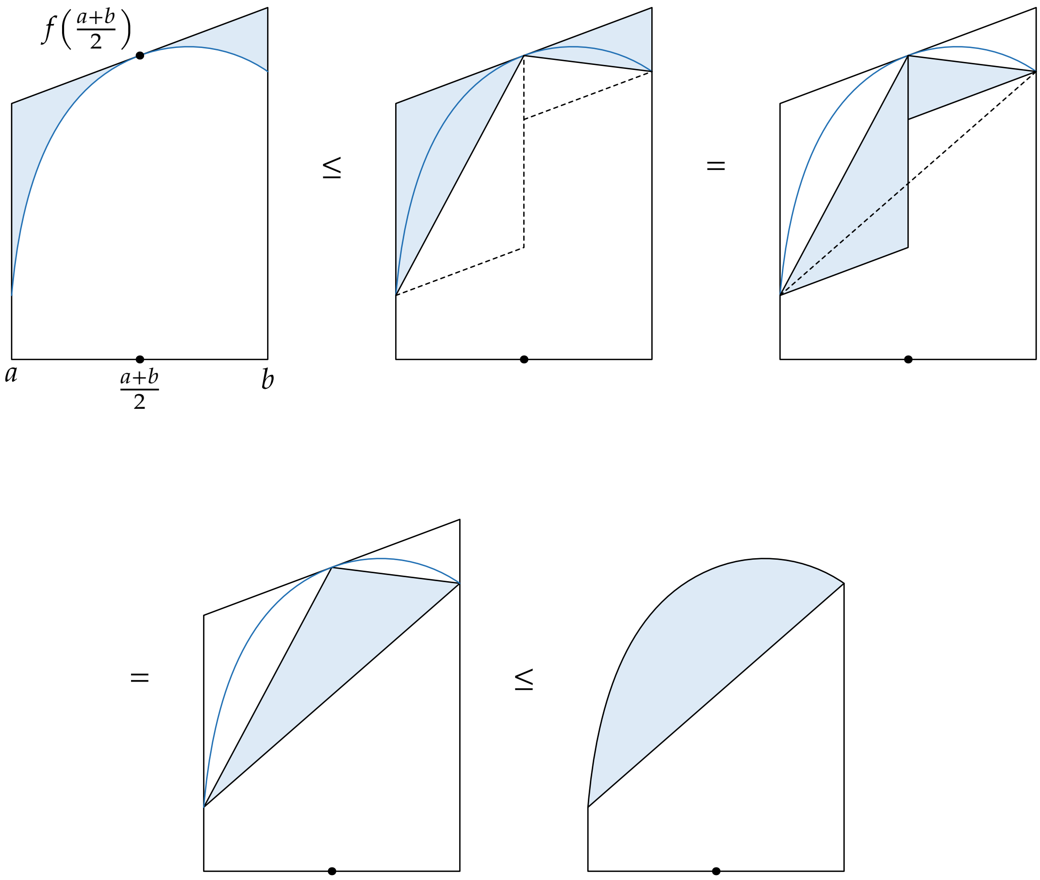
The distance between a point and a line



$$\frac{d}{1} = \frac{|ma+c-b|}{\sqrt{1+m^2}}$$

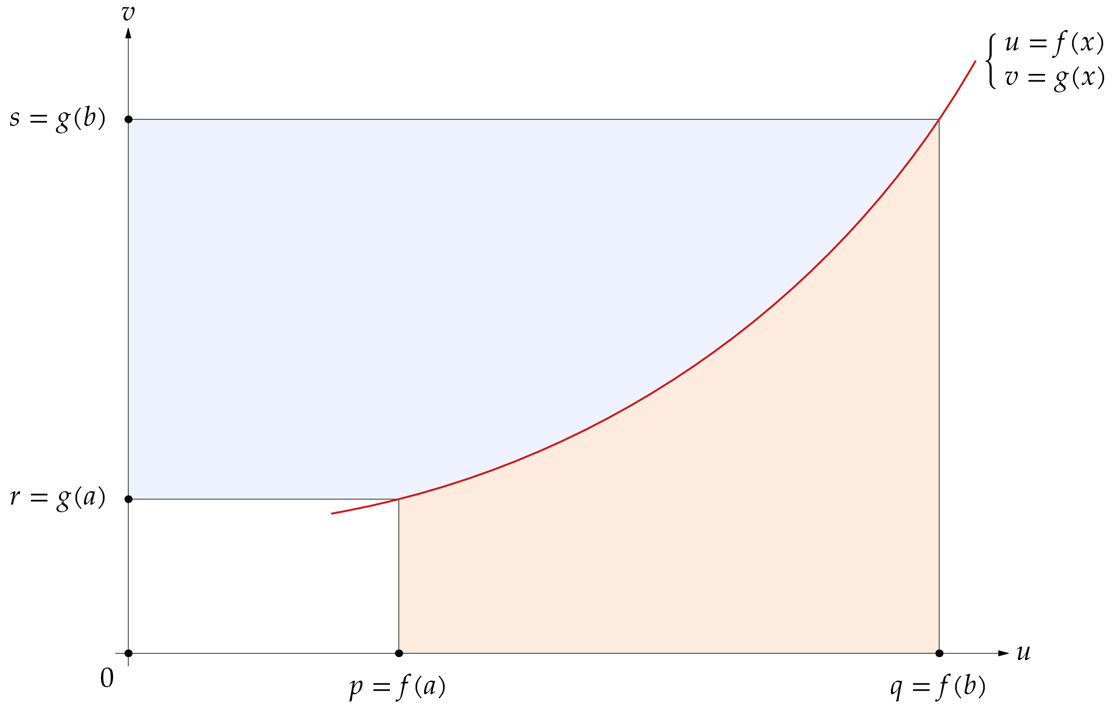
— R. L. Eisenman

The midpoint rule is better than the trapezoidal rule for concave functions



— Frank Burk

Integration by parts



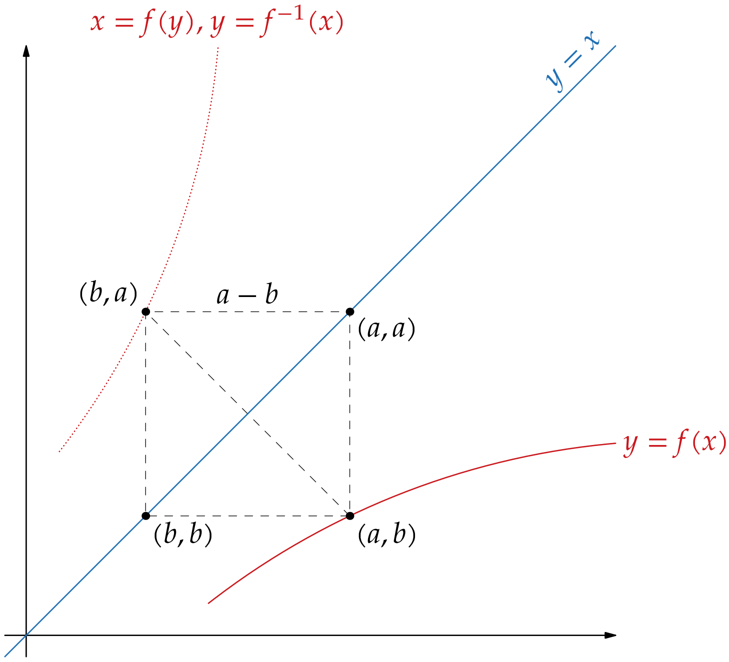
Area [blue] + Area [orange] $= qs - pr$

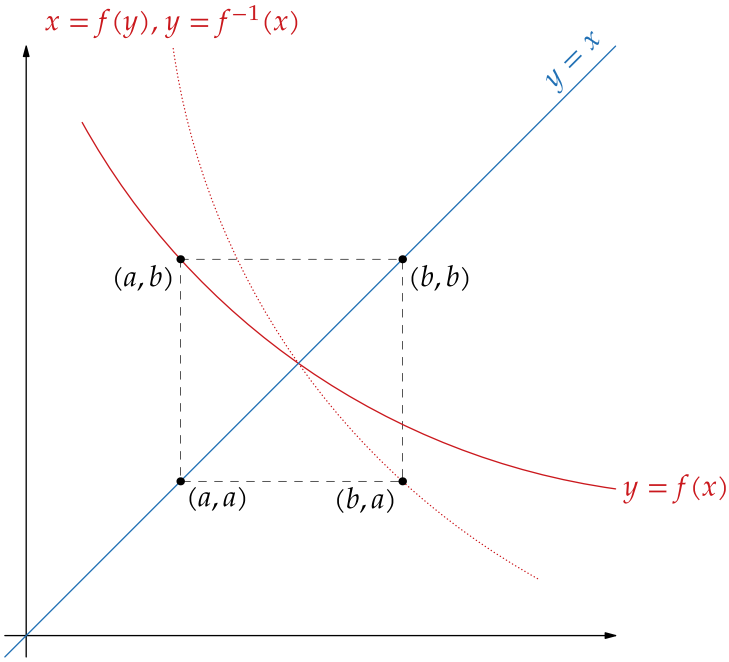
$$\int_r^s u\,dv + \int_p^q v\,du = uv\Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

— Richard Courant

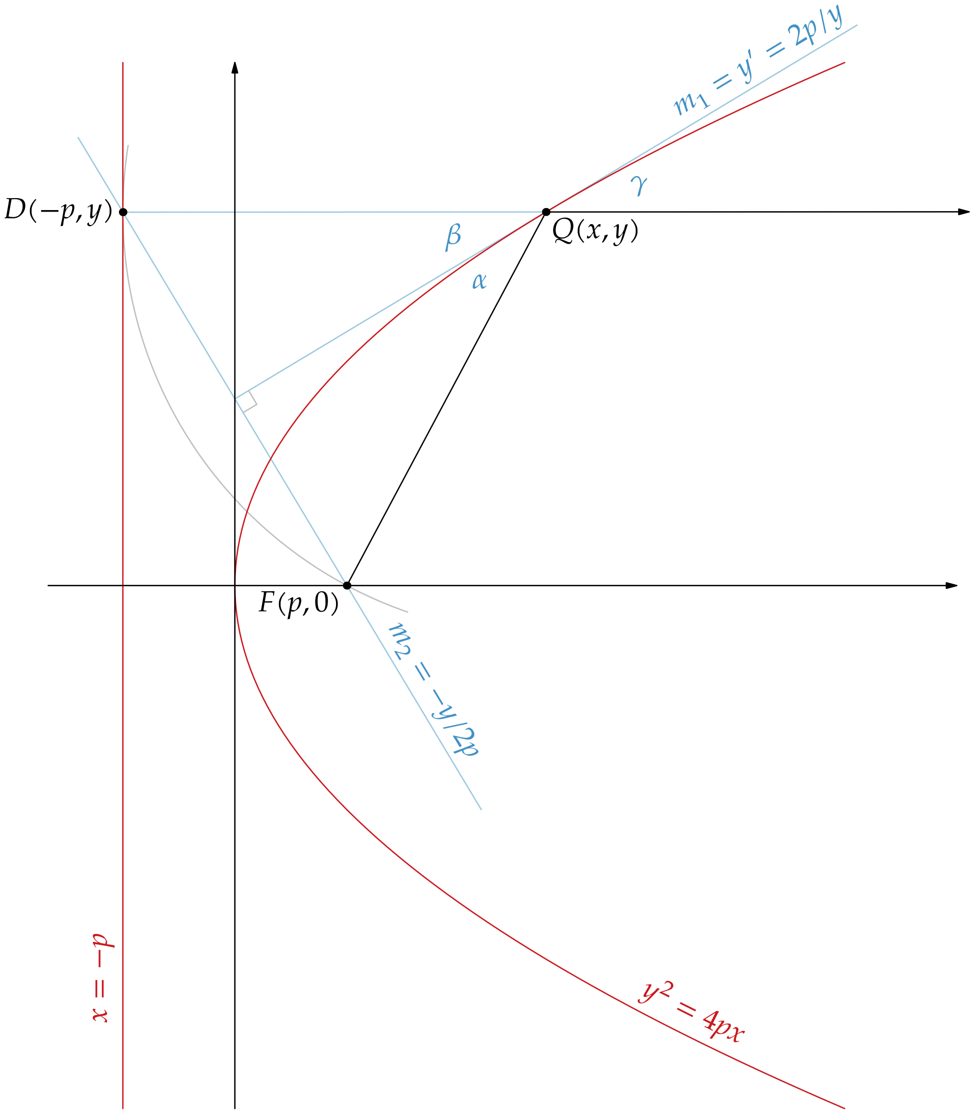
The graphs of f and f^{-1} are reflections about the line $y = x$





— Ayoub B. Ayoub

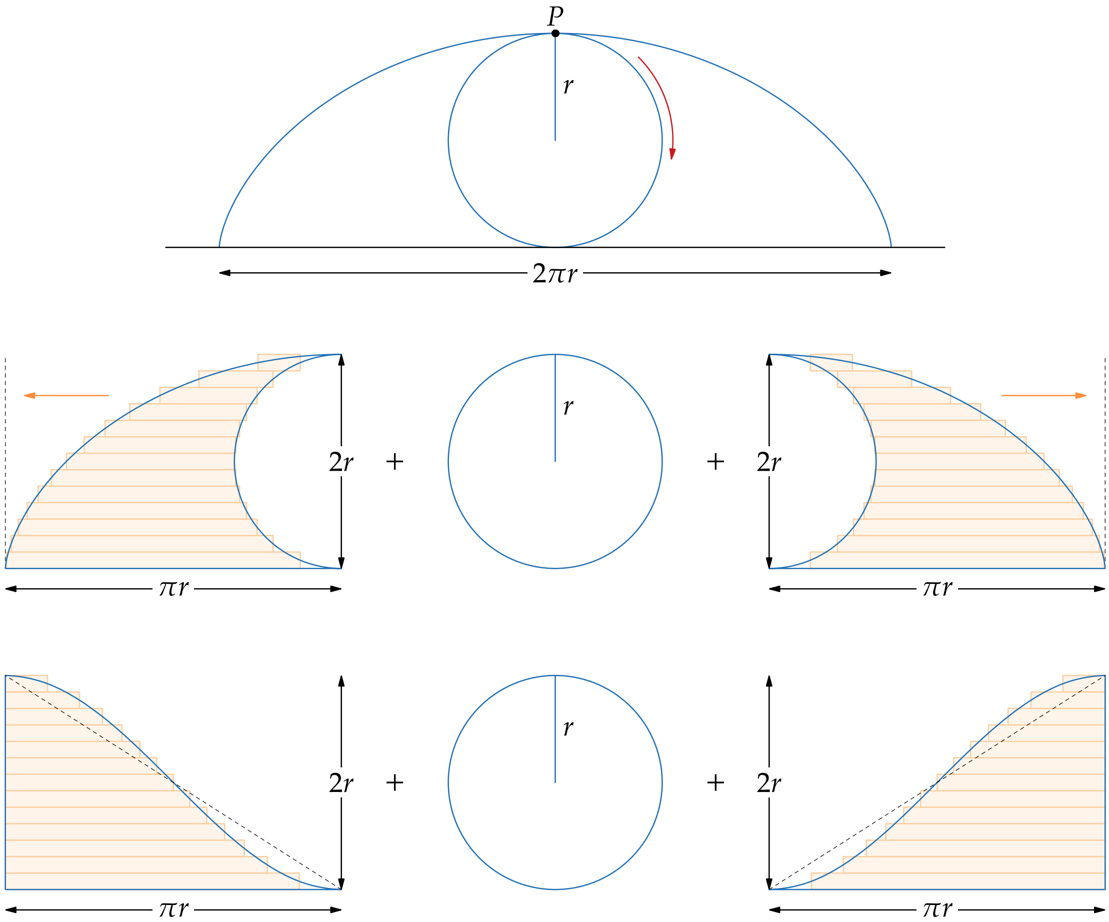
The reflection property of the parabola



$QF = QD$ and $m_1 \cdot m_2 = -1$, therefore $\alpha = \beta = \gamma$

— Ayoub B. Ayoub

Area under an arch of the cycloid



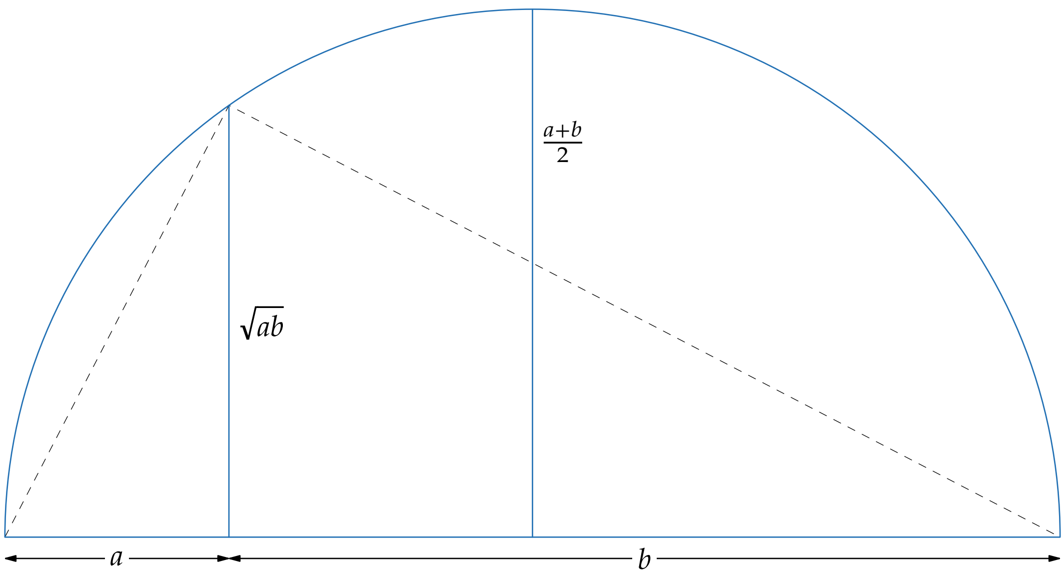
$$\frac{1}{2}\pi r \cdot 2r \qquad + \qquad \pi r^2 \qquad + \qquad \frac{1}{2}\pi r \cdot 2r$$

therefore $\quad A = 3\pi r^2$

— Richard M. Beekman

Inequalities

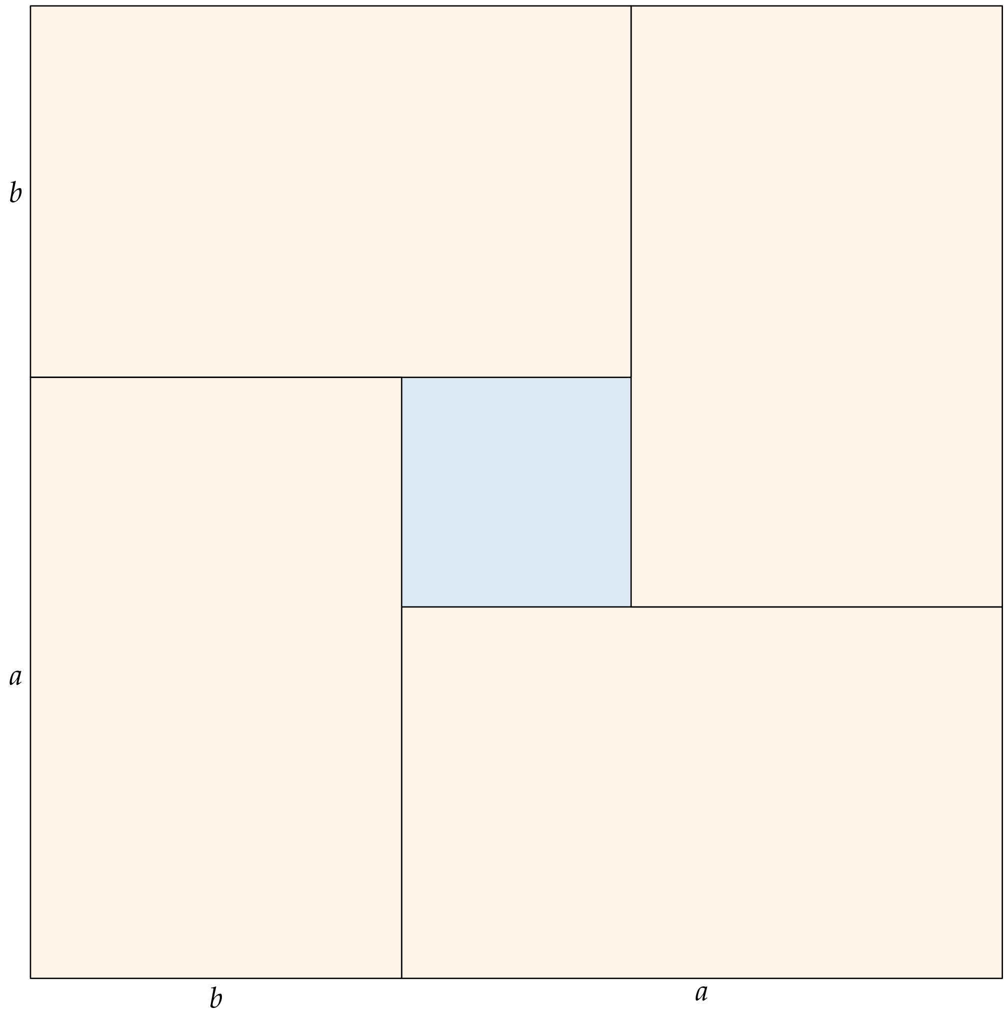
The arithmetic mean – geometric mean inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

— Charles D. Gallant

The arithmetic mean – geometric mean inequality II



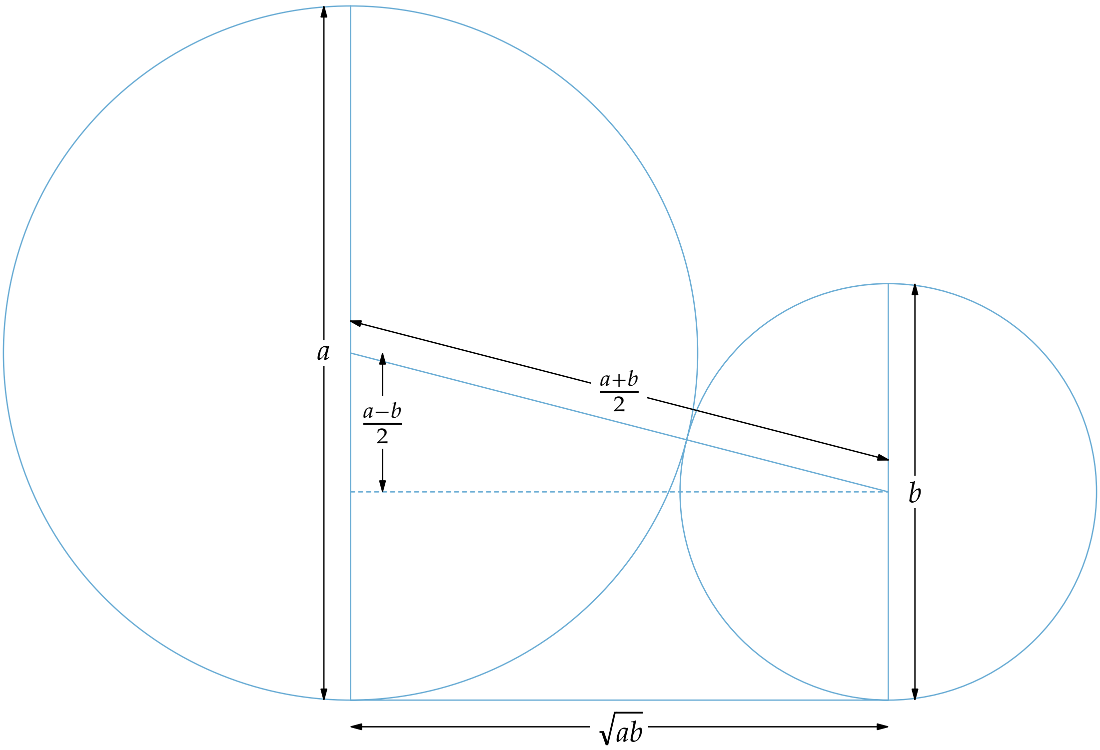
$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

— Doris Schattschneider

The arithmetic mean – geometric mean inequality III

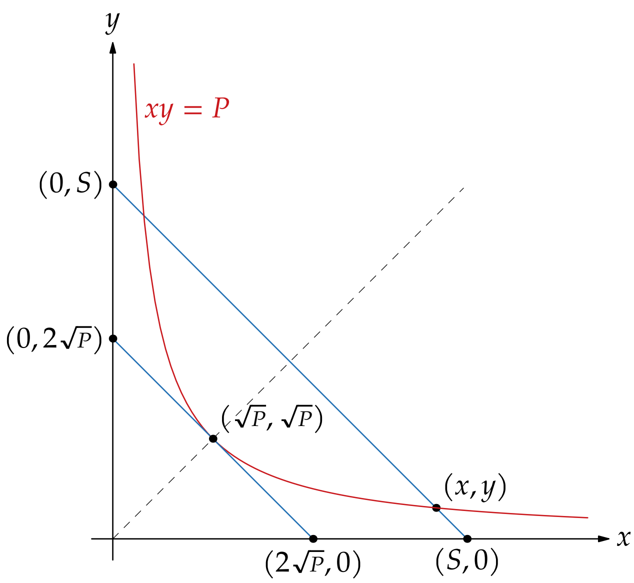
$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \text{with equality iff } a = b$$



— Roland H. Eddy

Two extremum problems

For a given product, the sum of two positive numbers
is minimal when the numbers are equal.



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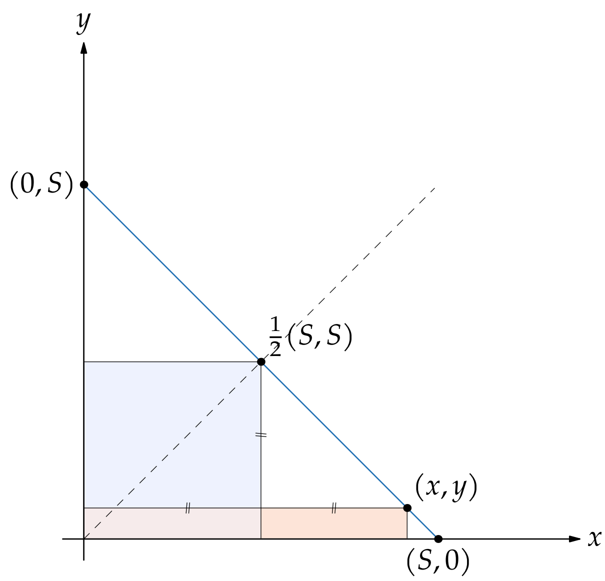
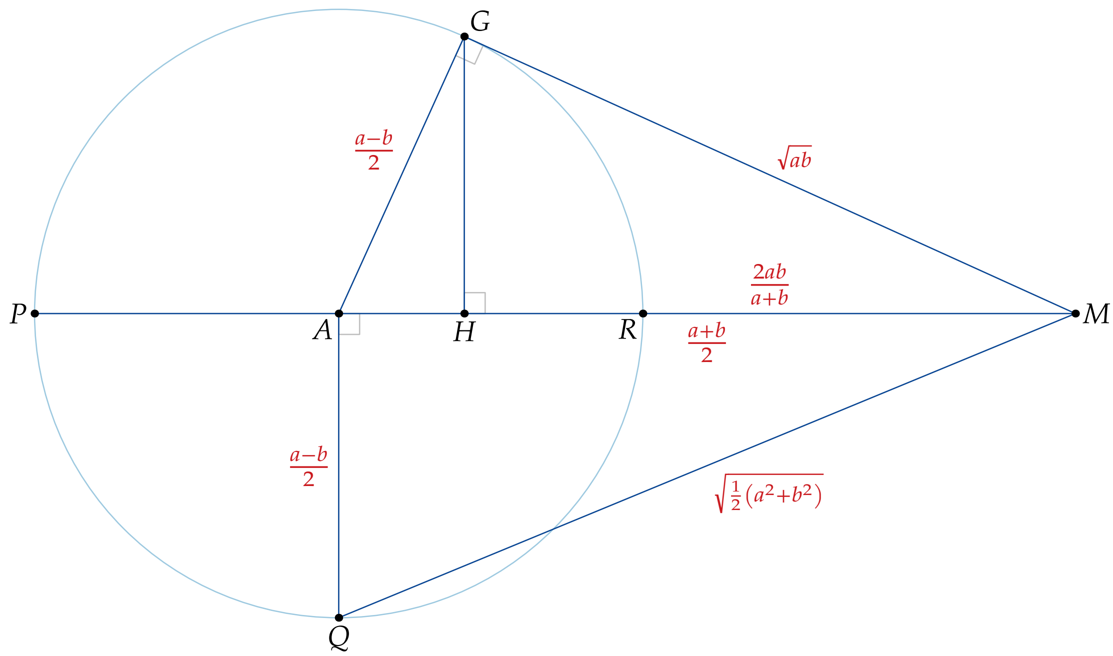


— Paulo Montuchi and Warren Page

The HM–GM–AM–QM inequalities I



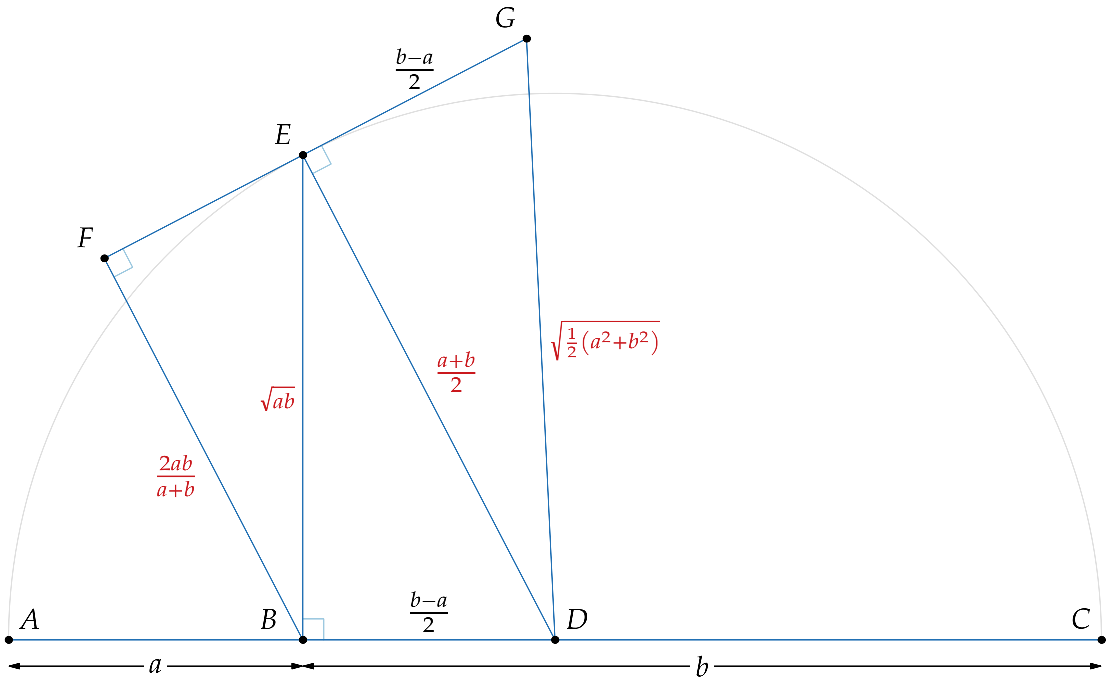
$$PM = a, \quad RM = b, \quad a > b > 0$$

$$HM < GM < AM < QM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{1}{2}\left(a^2 + b^2\right)}$$

— Roger B. Nelsen

The HM–GM–AM–QM inequalities II



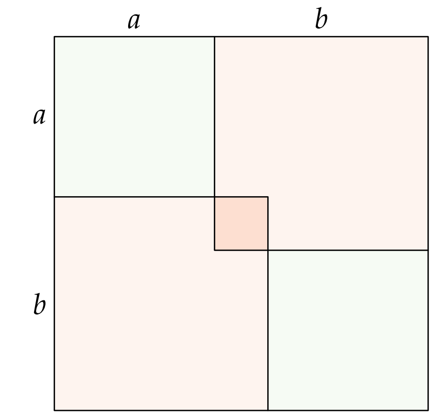
$$AB = a, \quad BC = b, \quad AD = DC = \tfrac{a+b}{2}$$

$$BE \perp AB, \quad DE = AD$$

$$FE \perp ED, \quad FB \parallel ED, \quad EG = BD = \tfrac{b-a}{2}$$

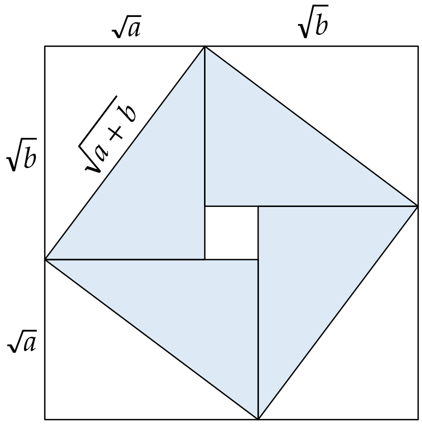
— Sidney H. Kung

The HM–GM–AM–QM inequalities III



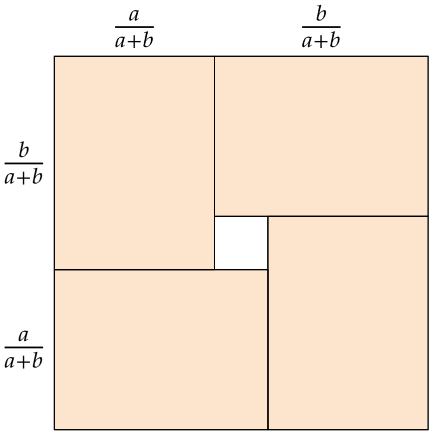
$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\tfrac{1}{2}\left(a^2+b^2\right)} \geq \frac{a+b}{2}$$



$$\left(\sqrt{a+b}\right)^2 \geq 4 \cdot \tfrac{1}{2}\sqrt{a}\sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

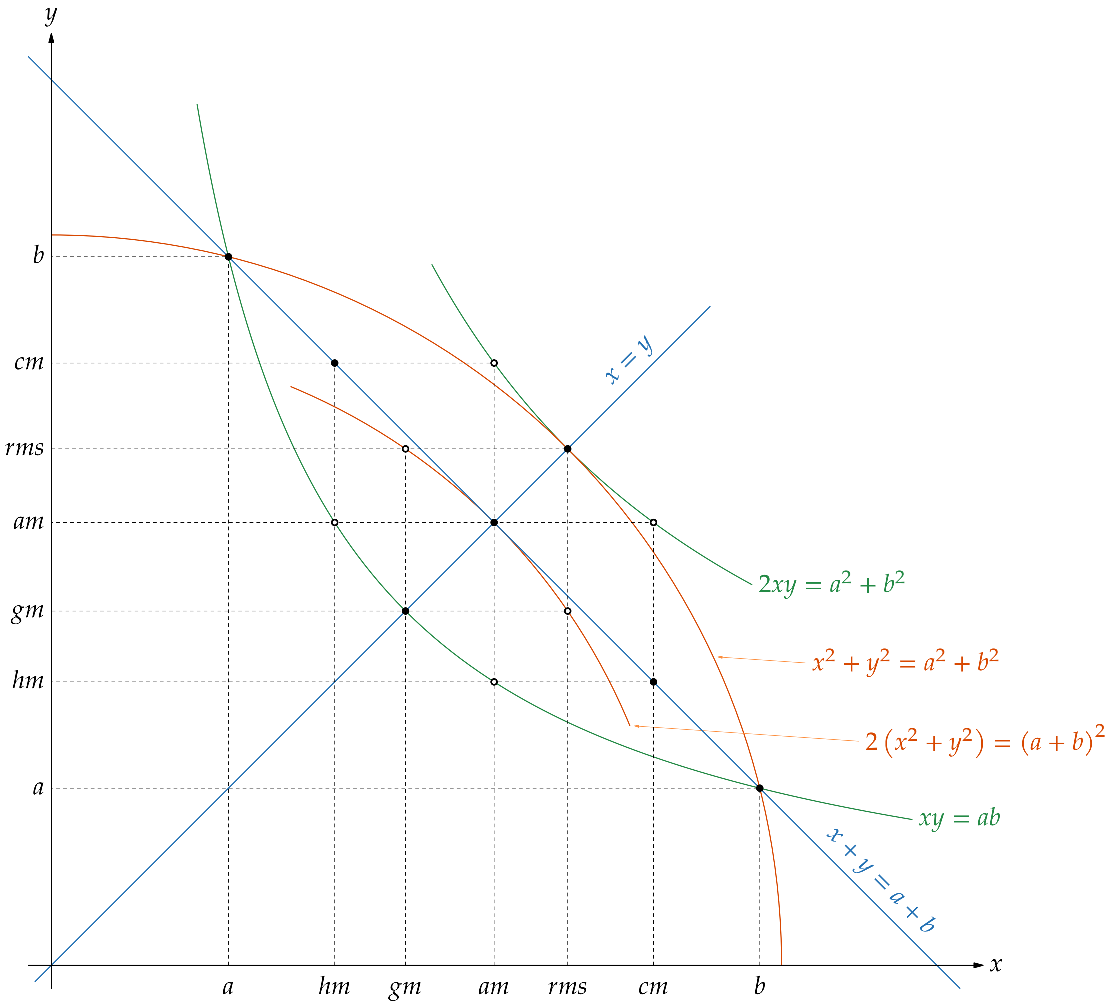


$$1 \geq 4 \cdot \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

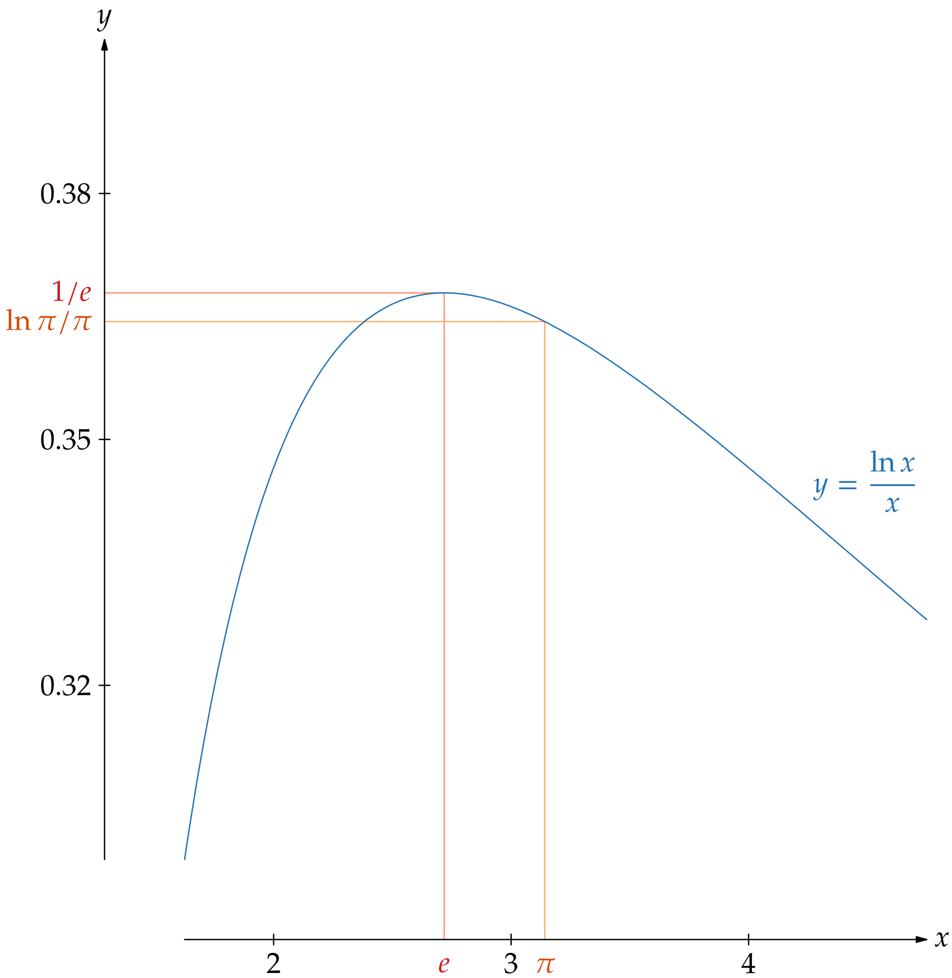
— Roger B. Nelsen

Five means — and their means



— Roger B. Nelsen

$e^{\pi} > \pi^{e}$



— Fouad Nakhli