

Proofs without words I

Exercises in METAPOST

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Geometry and Algebra

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The Pythagorean theorem I



— adapted from the *Chou pei san ching*

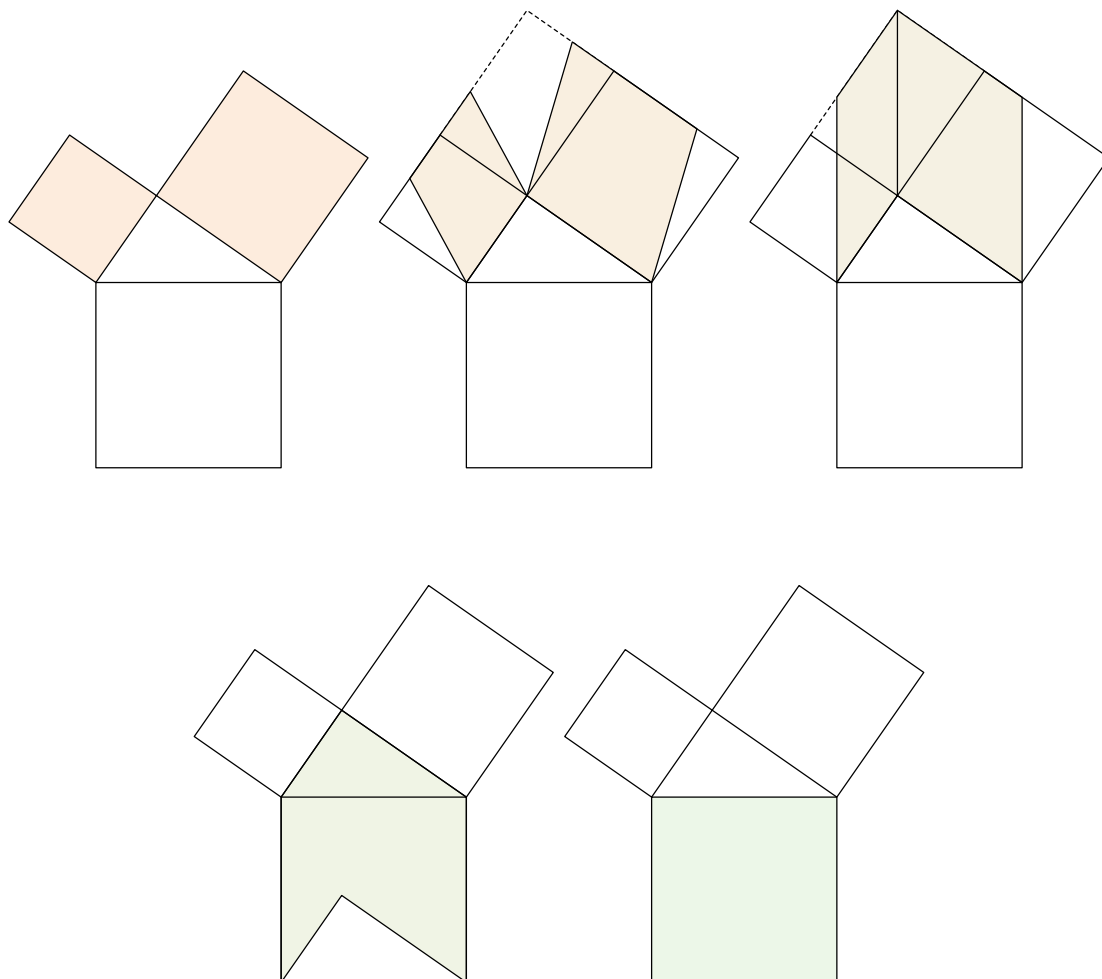
The Pythagorean theorem II



Behold!

— Bhāskara (12th century)

The Pythagorean theorem III



— based on Euclid's proof

The Pythagorean theorem IV



— H. E. Dudeney (1917)

The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a+b)^2$$

$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)

The Pythagorean theorem VI

$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2 + b^2 = c^2$$



— Michael Hardy

A Pythagorean theorem: $aa' = bb' + cc'$



$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$

$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$

$$\therefore aa' = a(x + y) = bb' + cc'.$$

— Enzo R. Gentile

The rolling circle squares itself



— Thomas Elsner

On trisecting an angle



— Rufus Isaacs

Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

— Eric Kincanon

Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

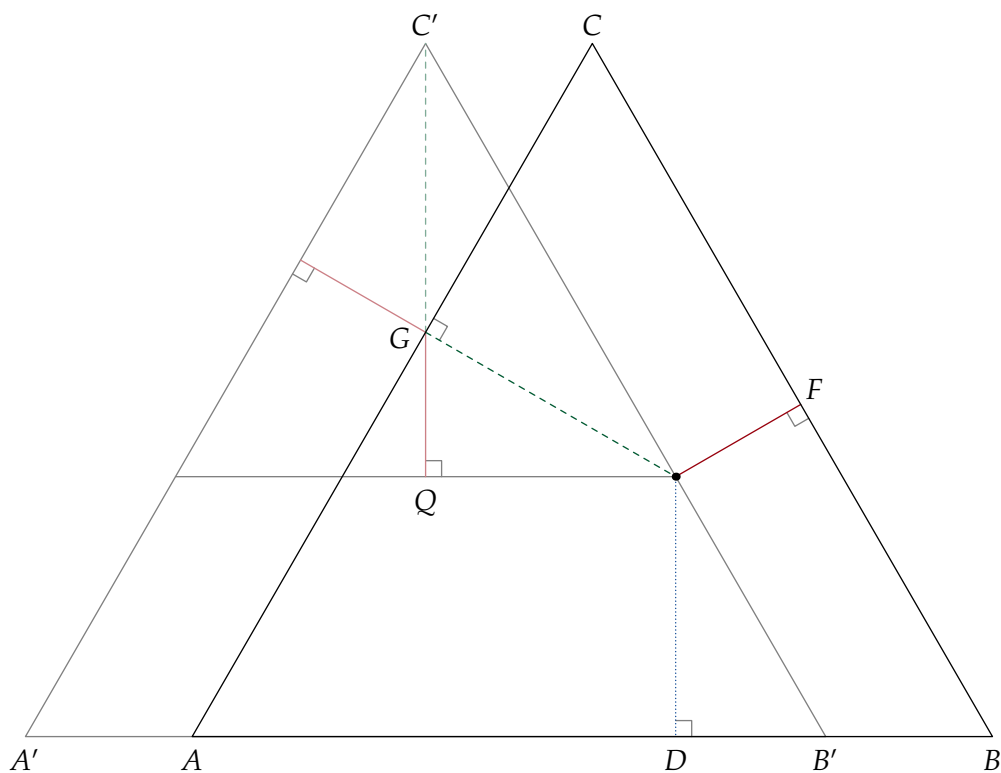
The vertex angles of a star sum to 180°



— Fouad Nakhli

Viviani's theorem I

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



This shows a particular example, with $C'GQ$ collinear, rather than the general case

— Samuel Wolf

Viviani's theorem II

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

A theorem about right angles

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse



— Roland H. Eddy

Area and the projection theorem of a right triangle



$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

Chords and tangents of equal length

If circle C_1 passes through the center O of circle C_2 , the length of the common chord \overline{PQ} is equal to the tangent segment \overline{PR} .



— Roland H. Eddy

Completing the square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

Algebraic areas I

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$



— Shirley Wakin

Algebraic areas II

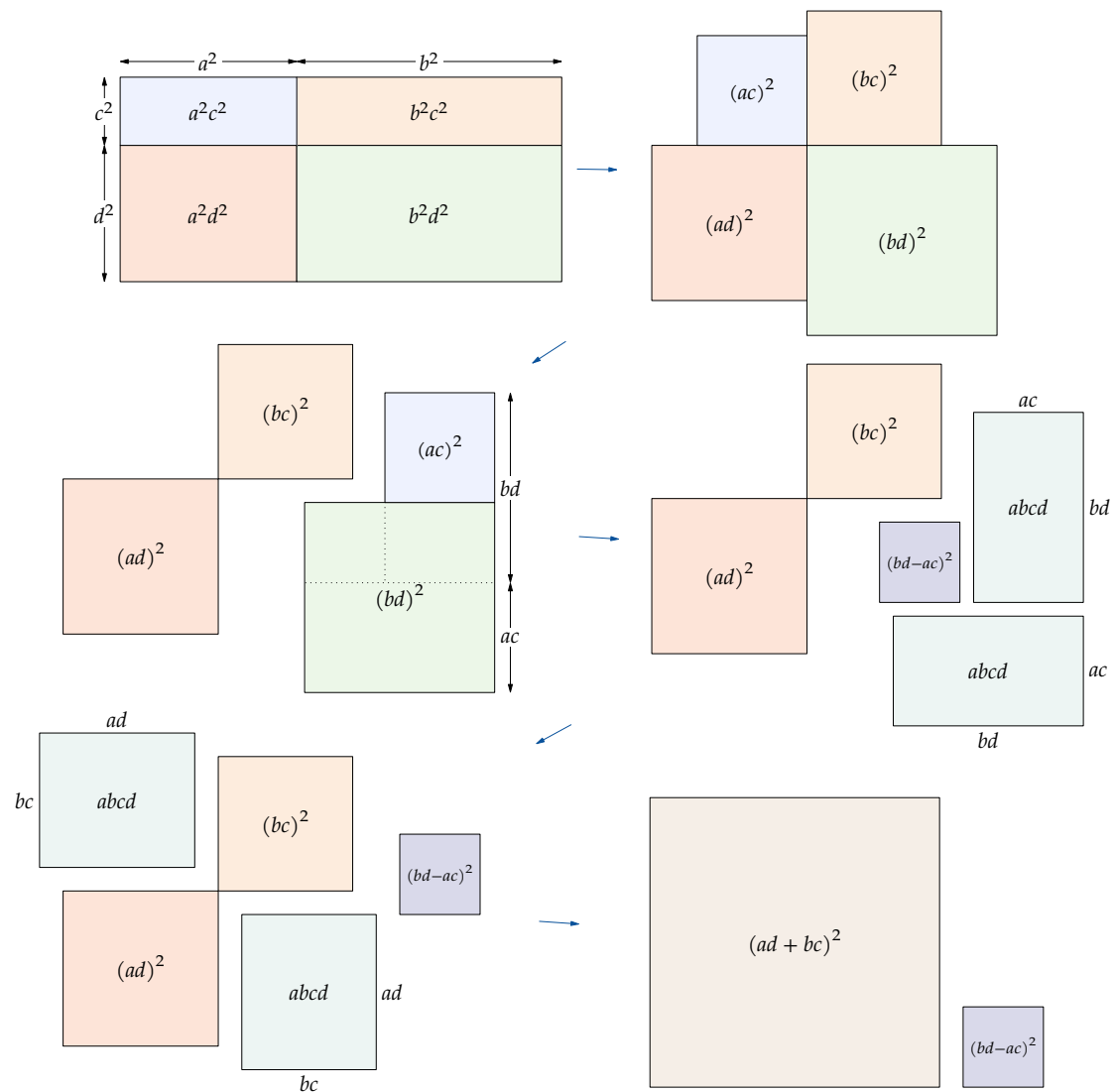
$$(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$



— Sam Pooley and K. Ann Drude

Sum of squares identity

$$(a^2 + b^2)(c^2 + d^2) = (ab + bc)^2 + (bd - ac)^2$$



— Diophantus of Alexandria

Polygonal numbers

The k^{th} n -gonal number is $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$



— Dave Logothetti

The volume of a frustum of a square pyramid



$$V_{P_1} = \frac{h}{b-a} \cdot V_{P_2} = \frac{h}{b-a} \cdot \frac{1}{3} (b^3 - a^3) = \frac{h}{3} (a^2 + ab + b^2)$$

— *The Moscow Papyrus*, c. 1850 BCE

The volume of a hemisphere via Cavalieri's Principle



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

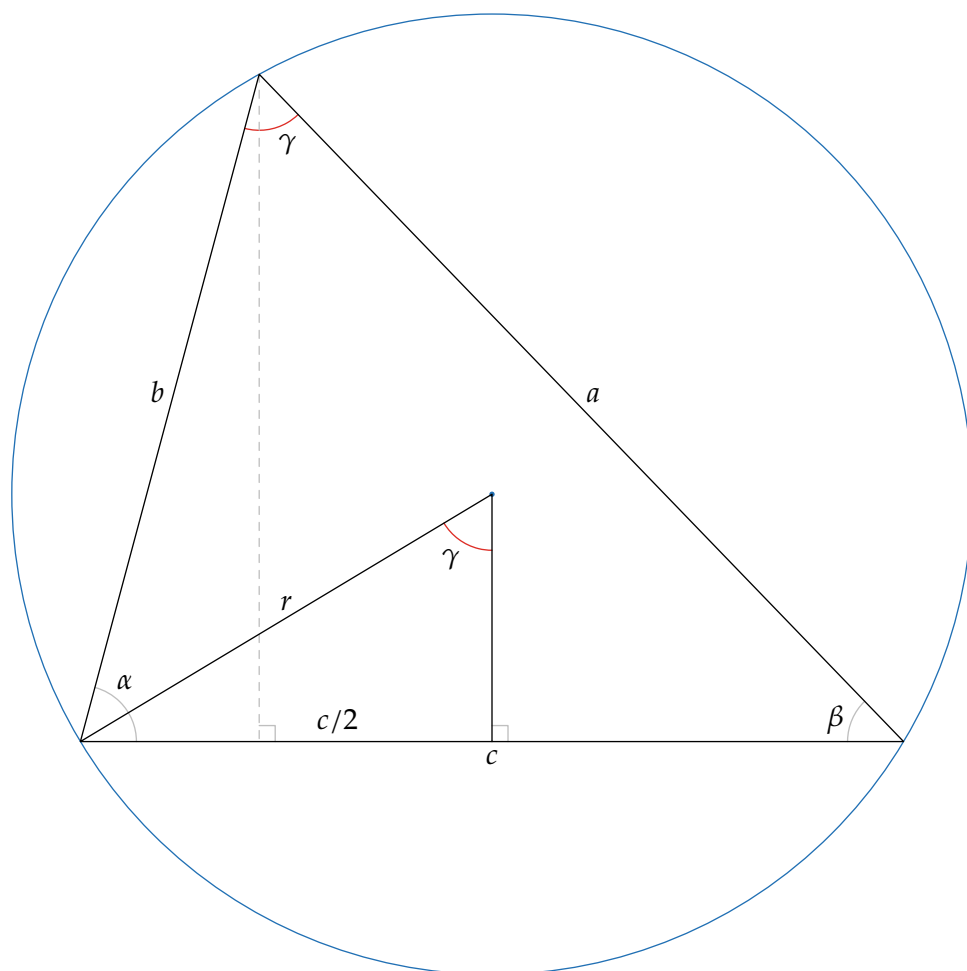
— Sidney H. Kung

Trigonometry, Calculus, & Analytic Geometry

Sine of the sum	29
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Sine of the sum

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \text{ for } \alpha + \beta < \pi$$



$$c = a \cos \beta + b \cos \alpha$$

$$r = 1/2 \implies \sin \gamma = \frac{c/2}{1/2} = c, \sin \alpha = a, \sin \beta = b$$

$$\sin(\alpha + \beta) = \sin(\pi - (\alpha + \beta)) = \sin \gamma = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

— Sidney H. Kung