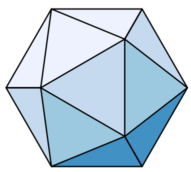
Proofs without words I

Exercises in METAPOST

Toby Thurston

March 2021 —



Contents

Geometry and Algebra **3**

Trigonometry, Calculus, & Analytic Geometry **28**

Inequalities **46**

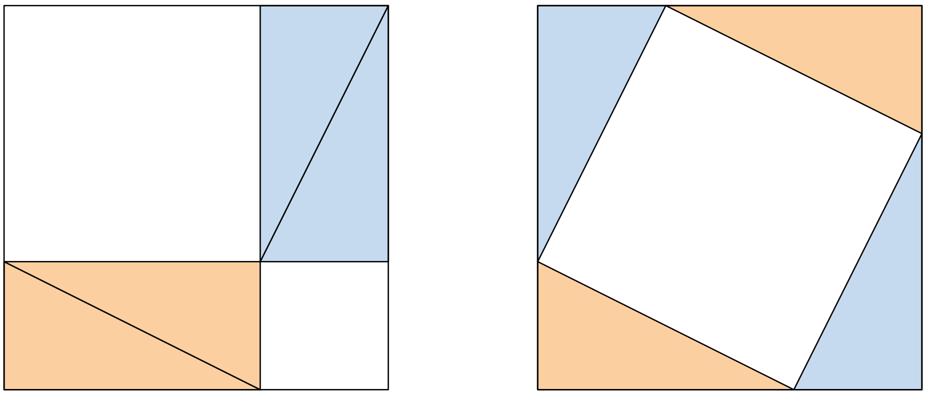
Integer sums **64**

Miscellaneous **101**

Geometry and Algebra

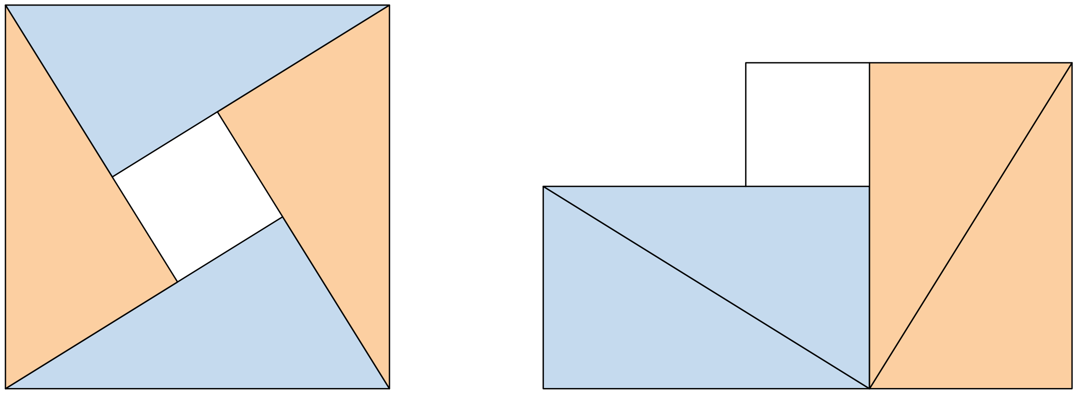
The Pythagorean theorem I . . . 4
The Pythagorean theorem II . . . 5
The Pythagorean theorem III . . . 6
The Pythagorean theorem IV . . . 7
The Pythagorean theorem V . . . 8
The Pythagorean theorem VI . . . 9
A Pythagorean theorem: $aa' = bb' + cc'$ . . . 10
The rolling circle squares itself . . . 11
On trisecting an angle . . . 12
Trisection in an infinite number of steps . . . 13
Trisection of a line segment . . . 14
The vertex angles of a star sum to 180° . . . 15
Viviani's theorem I . . . 16
Viviani's theorem II . . . 17
A theorem about right angles . . . 18
Area and the projection theorem of a right triangle . . . 19
Chords and tangents of equal length . . . 20
Completing the square . . . 21
Algebraic areas I . . . 22
Algebraic areas II . . . 23
Sum of squares identity . . . 24
Polygonal numbers . . . 25
The volume of a frustrum of a square pyramid . . . 26
The volume of a hemisphere via Cavalieri's Principle . . . 27

The Pythagorean theorem I



— adapted from the *Chou pei san ching*

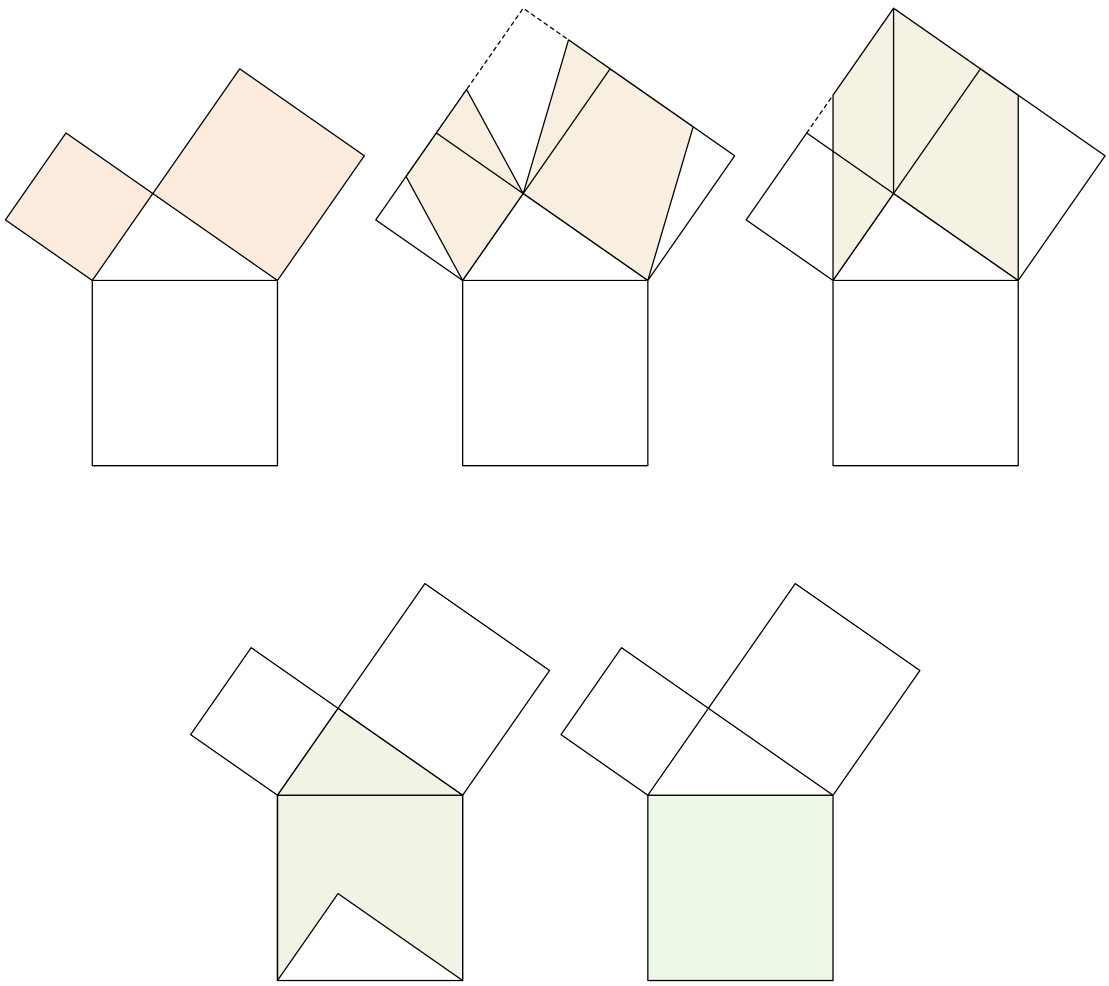
The Pythagorean theorem II



Behold!

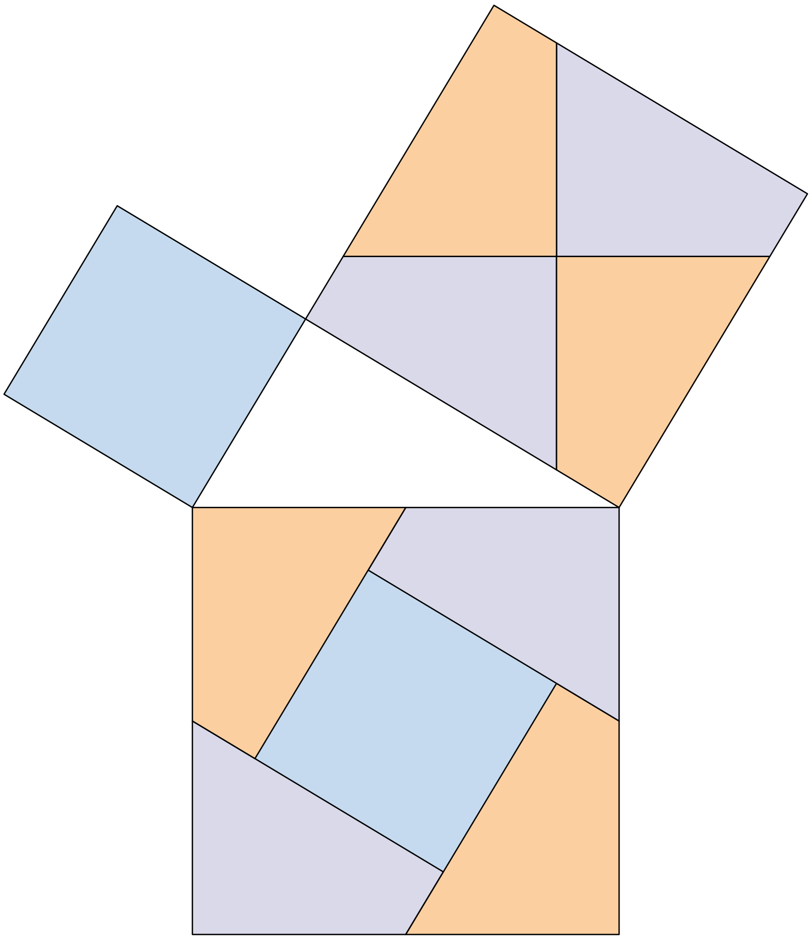
— Bhāskara (12th century)

The Pythagorean theorem III



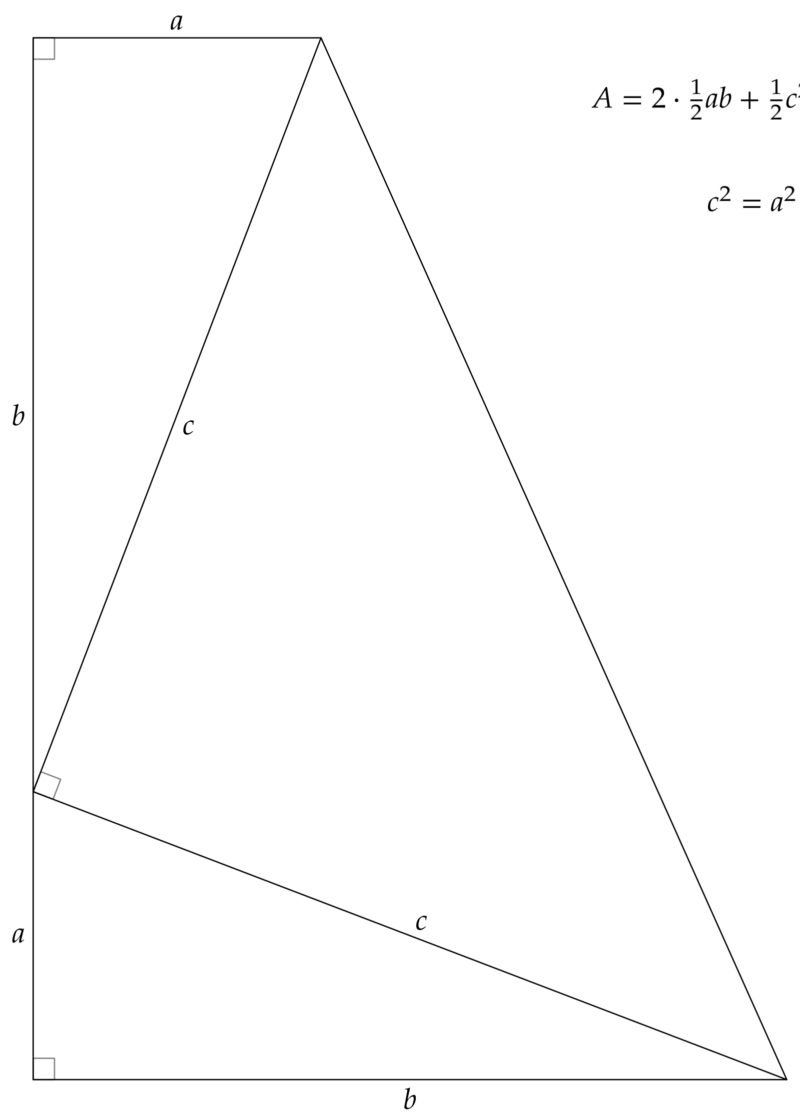
— based on Euclid's proof

The Pythagorean theorem IV



— H. E. Dudeney (1917)

The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a+b)^2$$

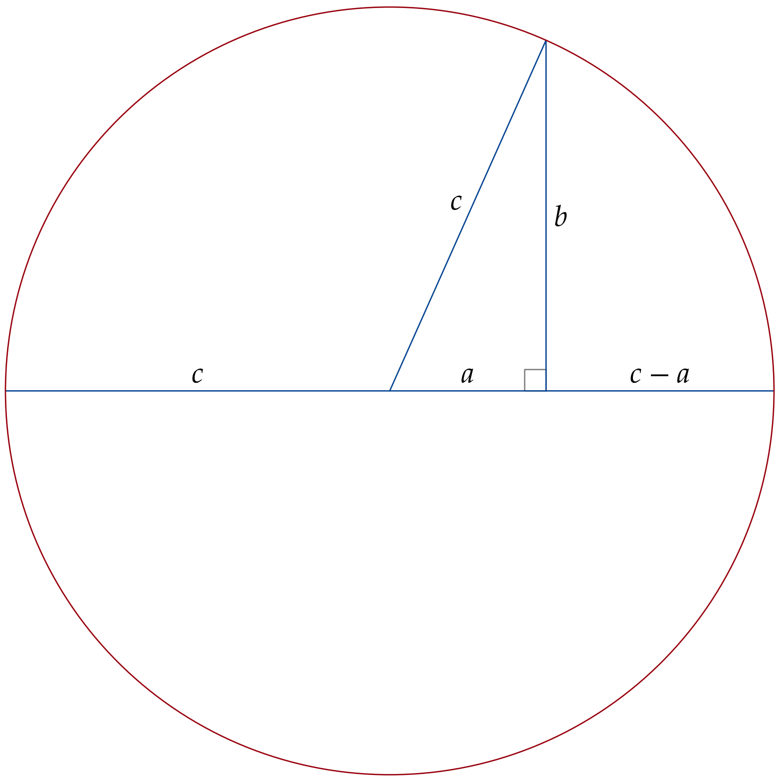
$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)

The Pythagorean theorem VI

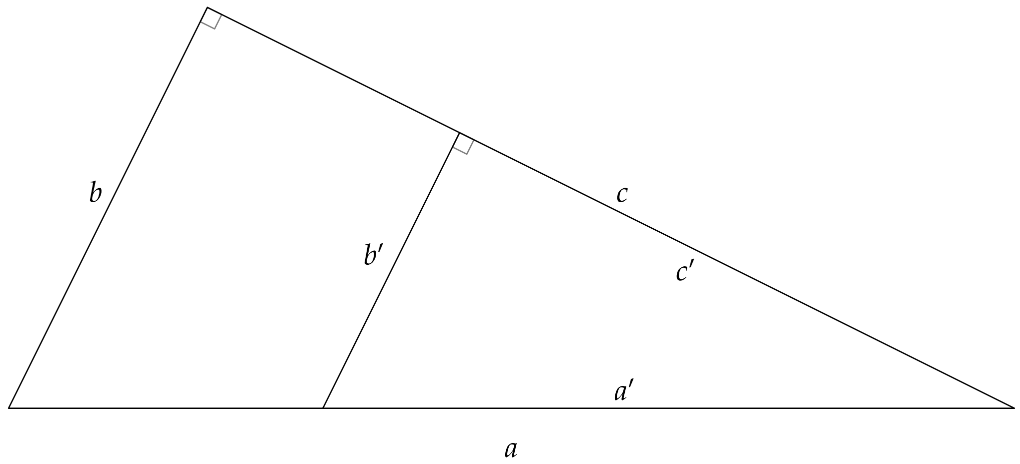
$$\frac{c+a}{b} = \frac{b}{c-a}$$

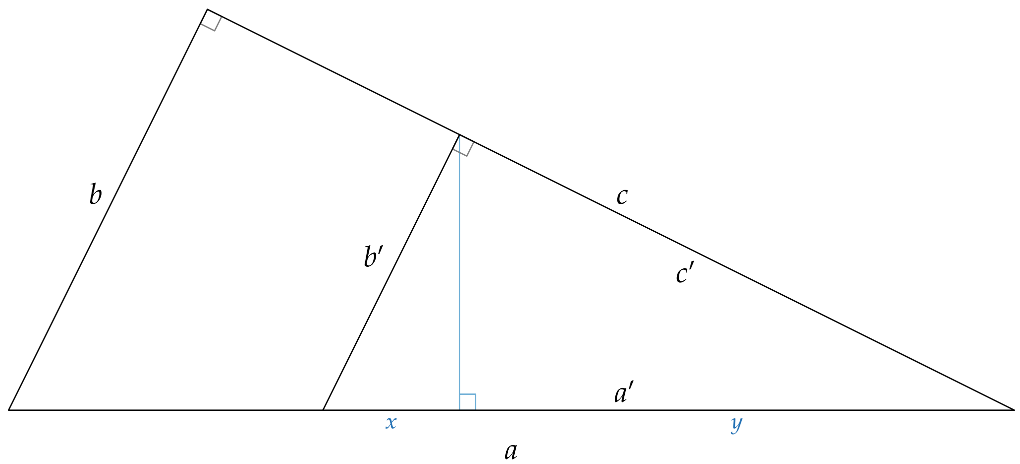
$$a^2 + b^2 = c^2$$



— Michael Hardy

A Pythagorean theorem: $aa' = bb' + cc'$





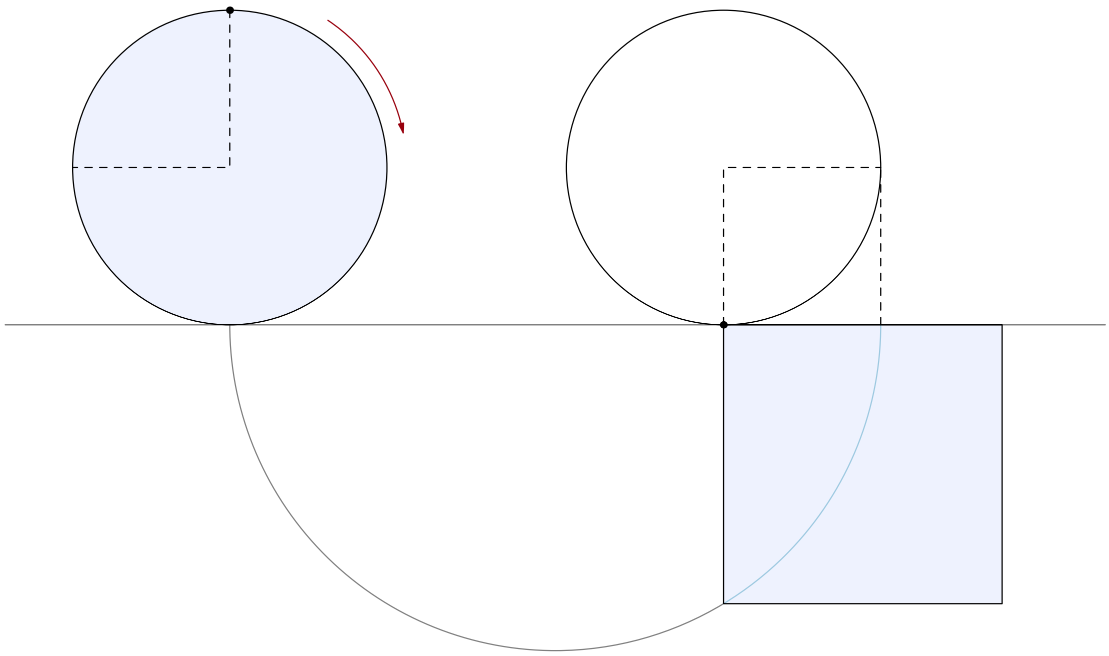
$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$

$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$

$$\therefore \quad aa' = a\,(x+y) = bb' + cc'.$$

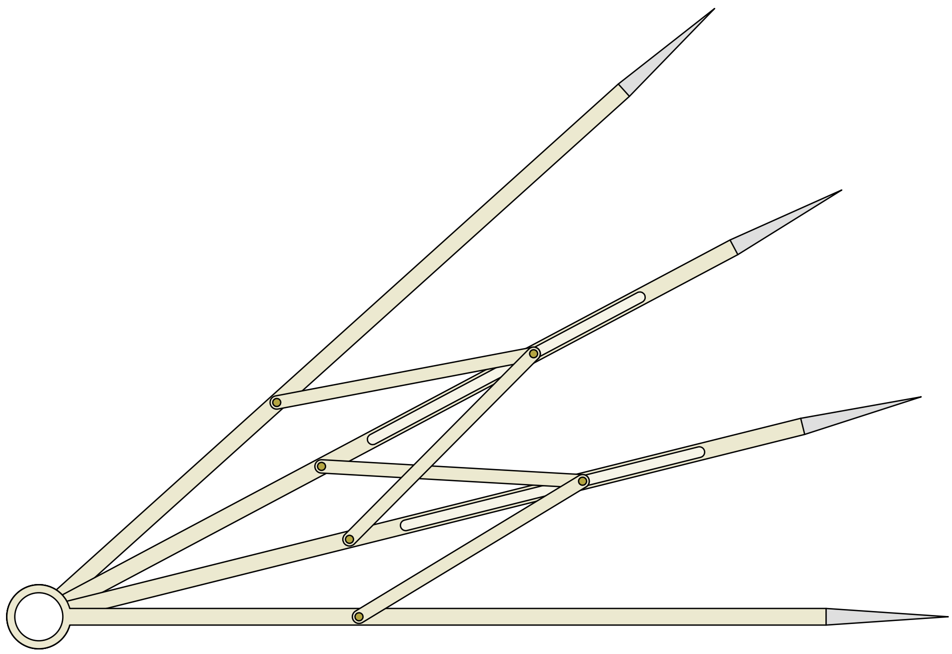
— Enzo R. Gentile

The rolling circle squares itself



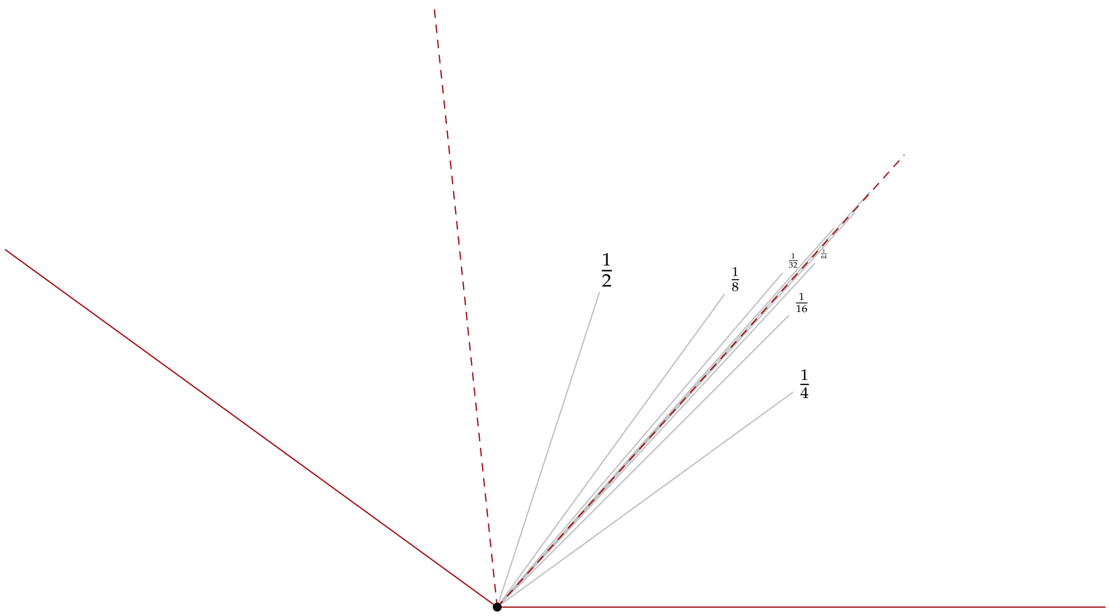
— Thomas Elsner

On trisecting an angle



— Rufus Isaacs

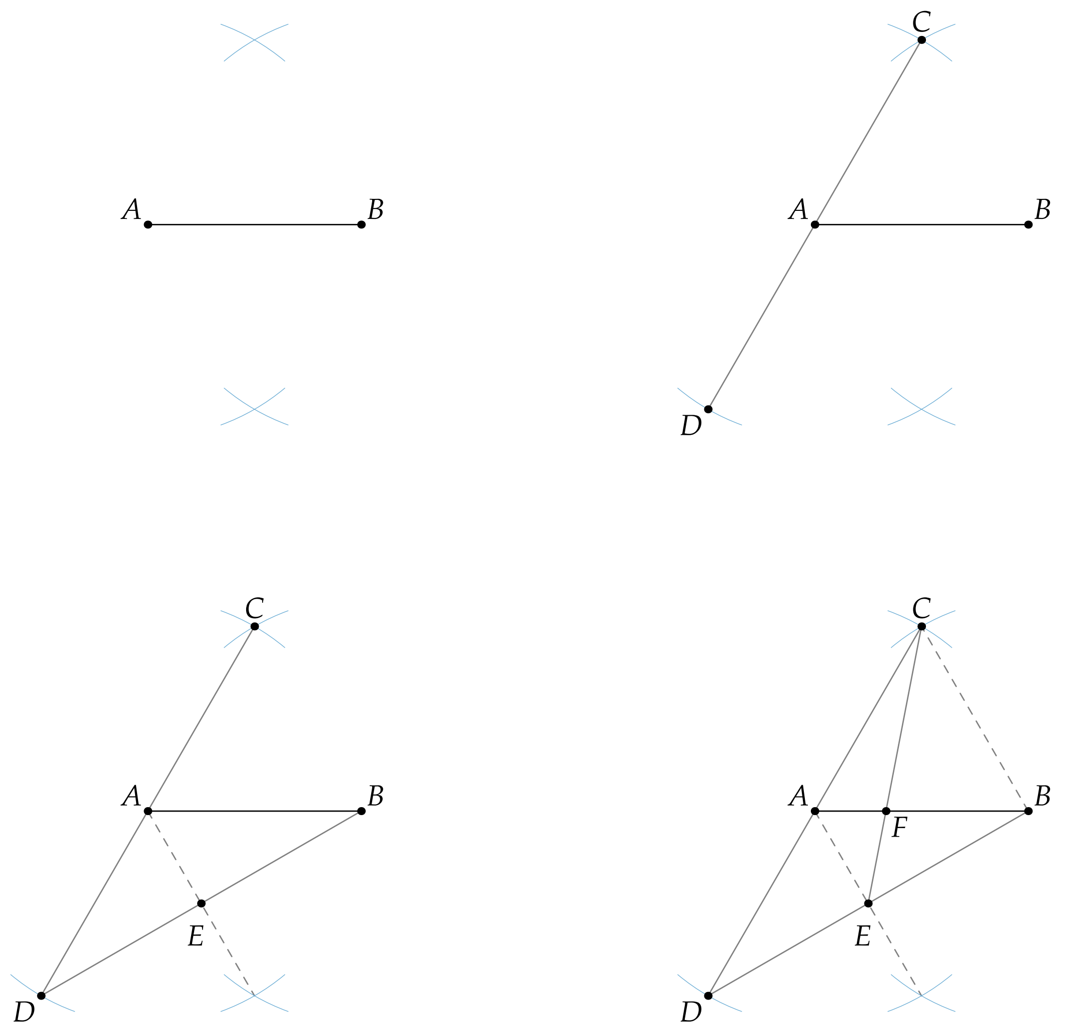
Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

— Eric Kincanon

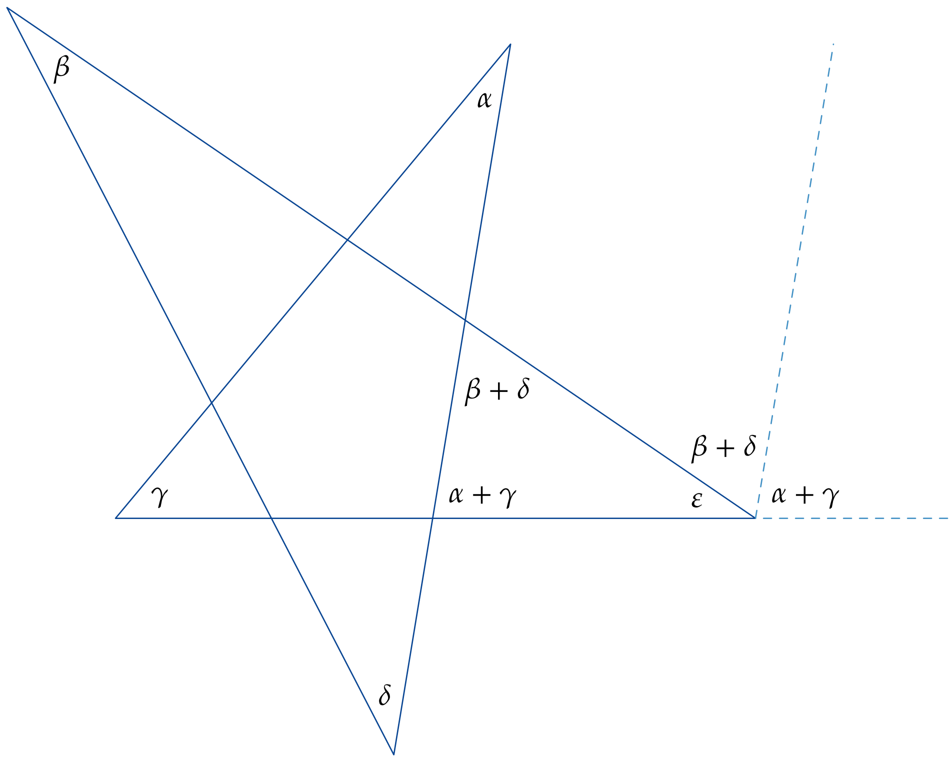
Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

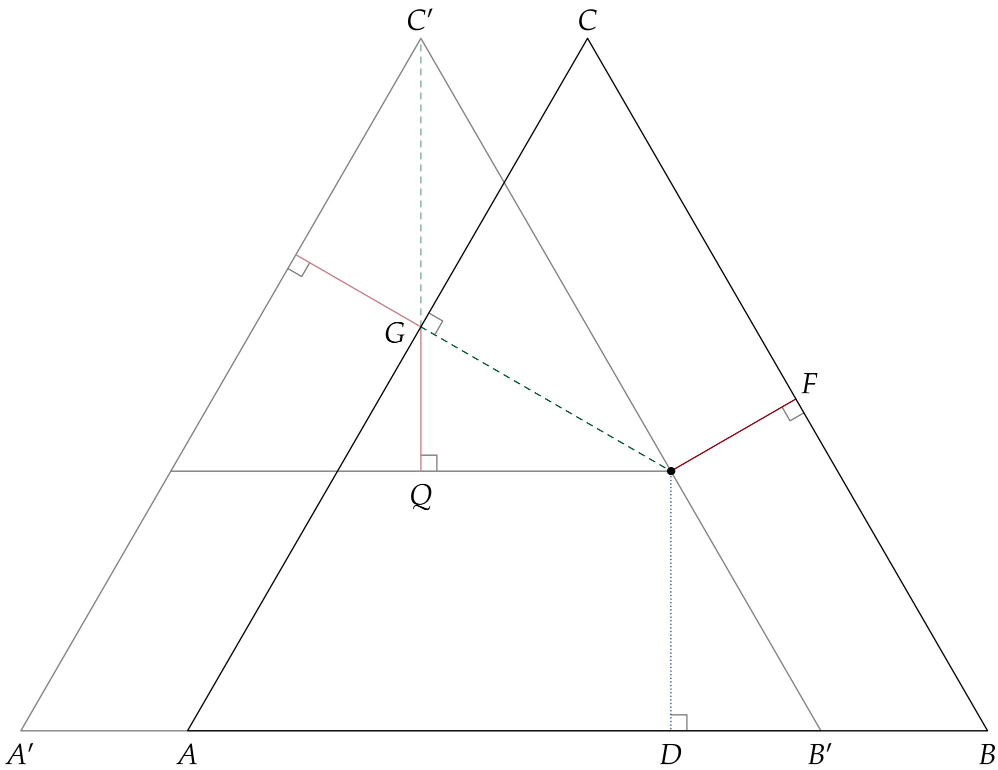
The vertex angles of a star sum to 180°



— Fouad Nakhli

Viviani's theorem I

The perpendiculars to the sides from a point on
the boundary or within an equilateral triangle
add up to the height of the triangle.

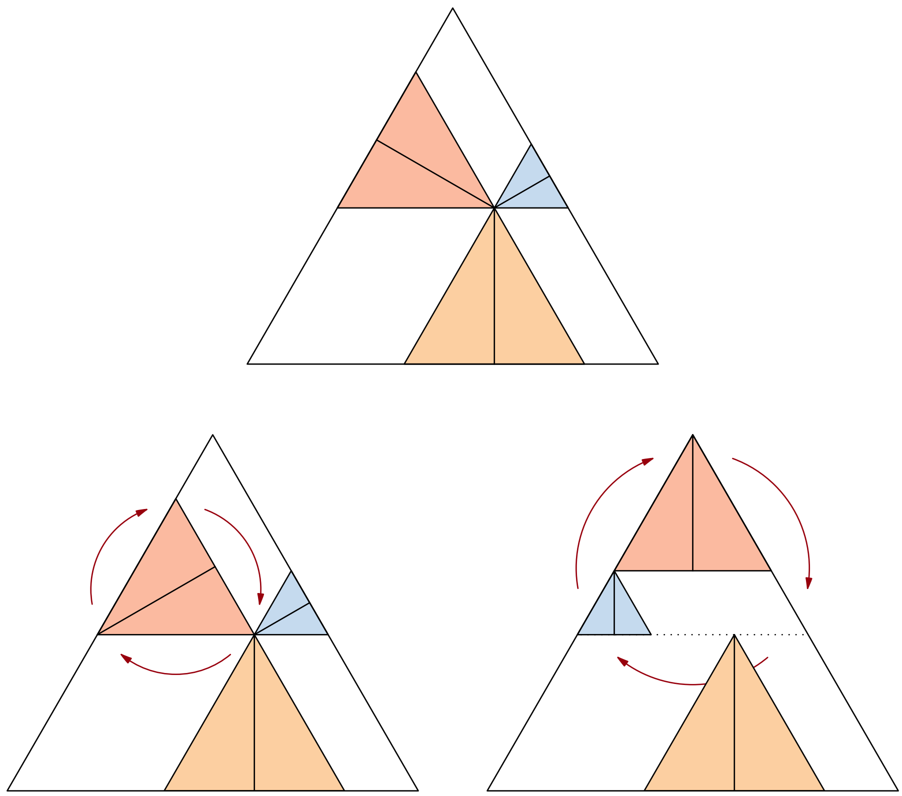


This shows a particular example, with $C'GQ$ collinear, rather than the general case

— Samuel Wolf

Viviani's theorem II

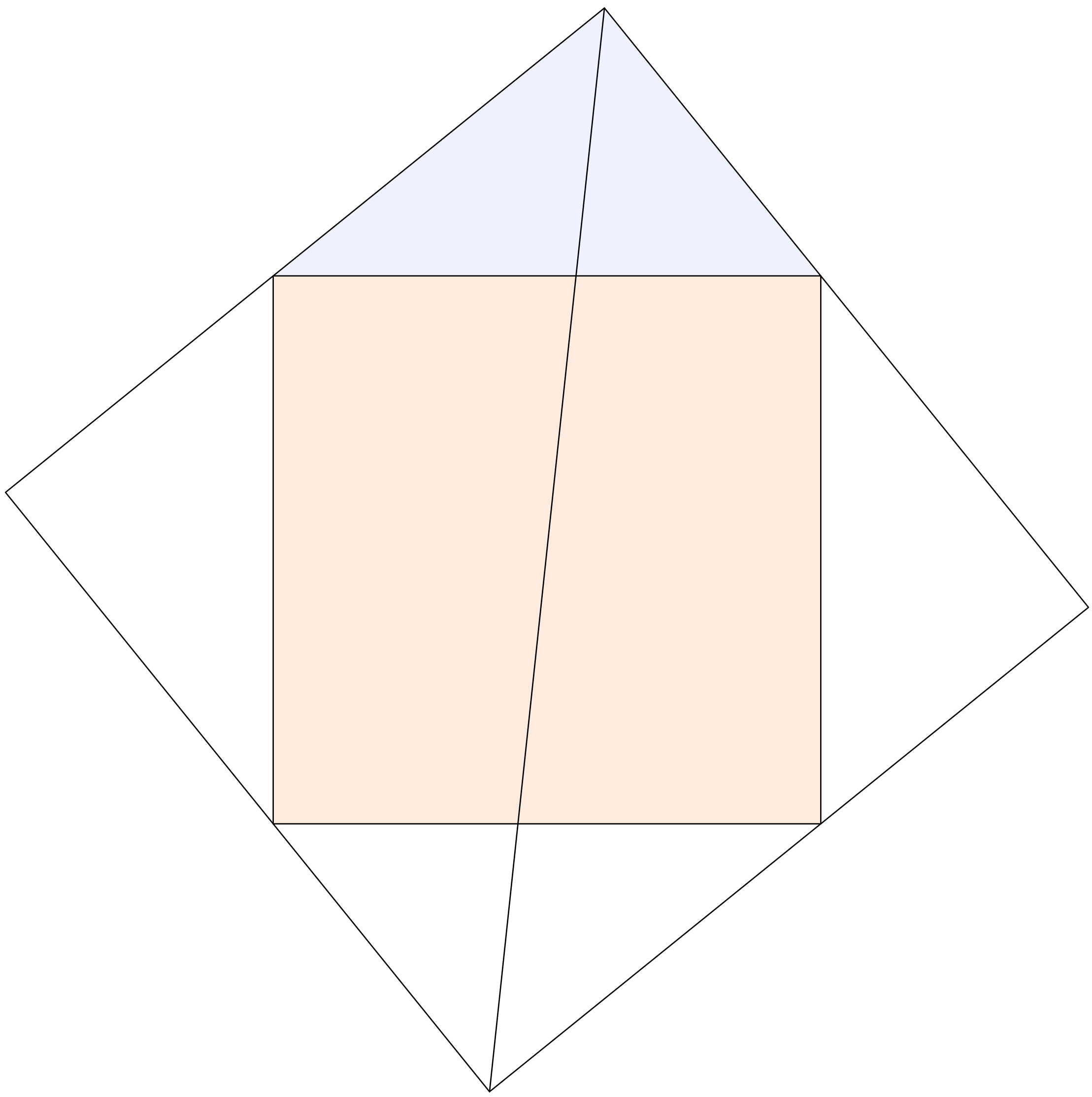
The perpendiculars to the sides from a point on
the boundary or within an equilateral triangle
add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

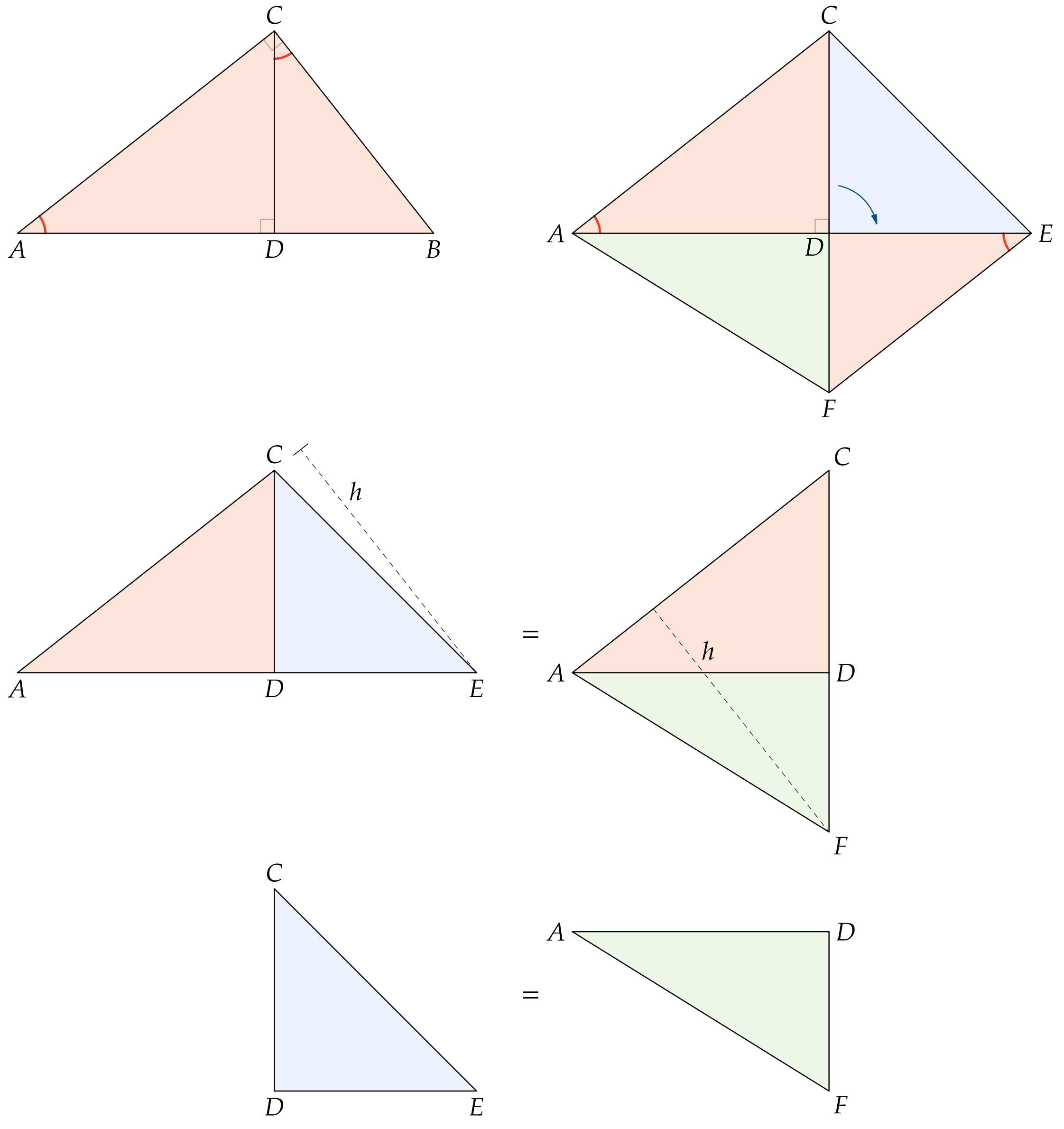
A theorem about right angles

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse



— Roland H. Eddy

Area and the projection theorem of a right triangle

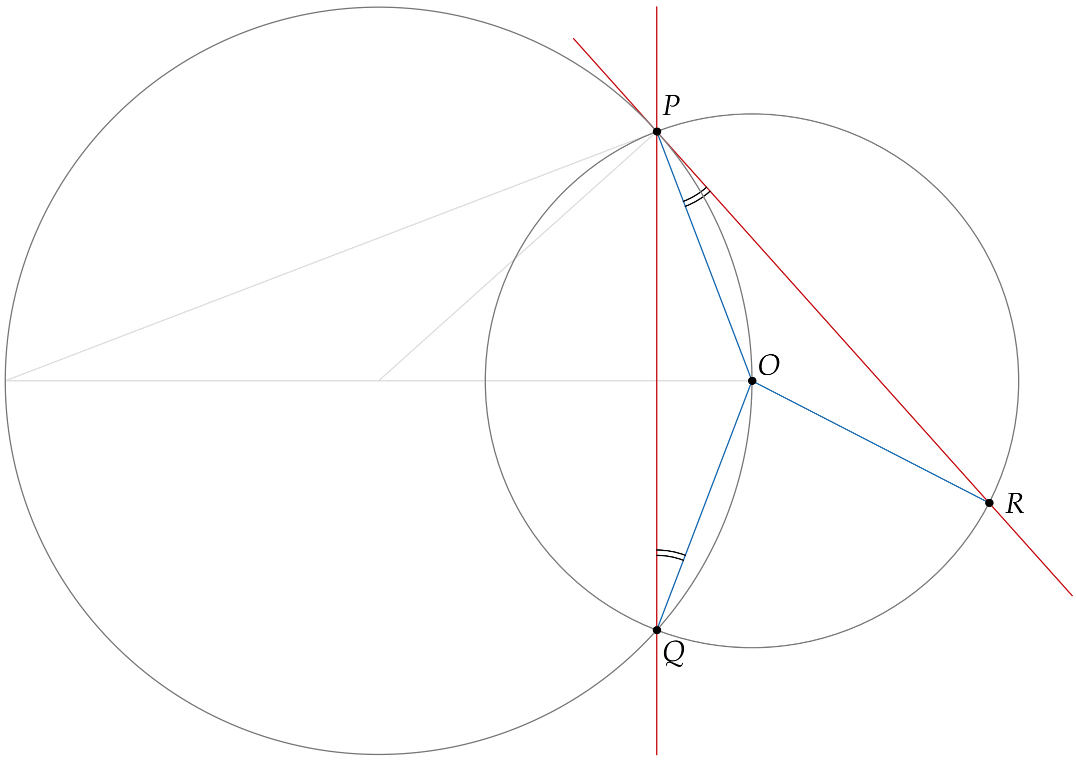


$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

Chords and tangents of equal length

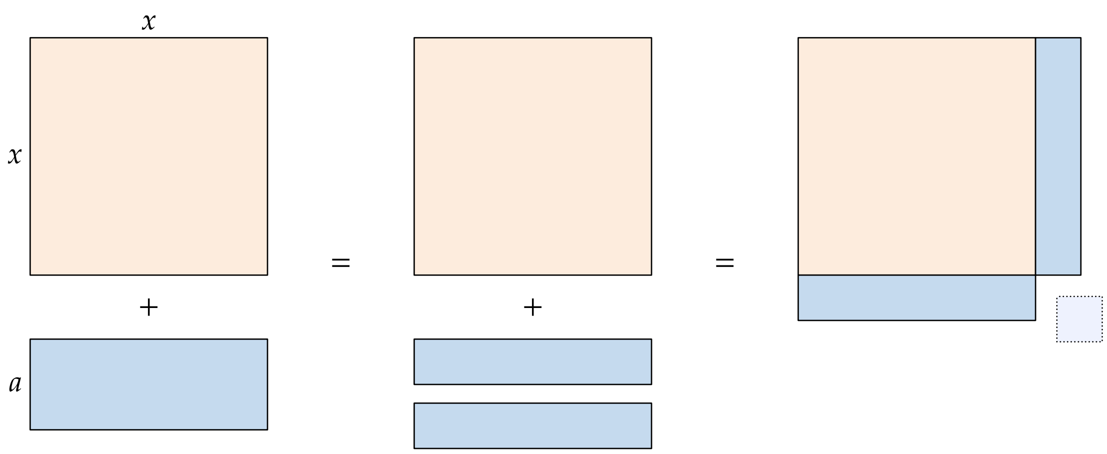
If circle C_1 passes through the center O of circle C_2, the length of the common chord $\overline{PQ}$ is equal to the tangent segment $\overline{PR}$.



— Roland H. Eddy

Completing the square

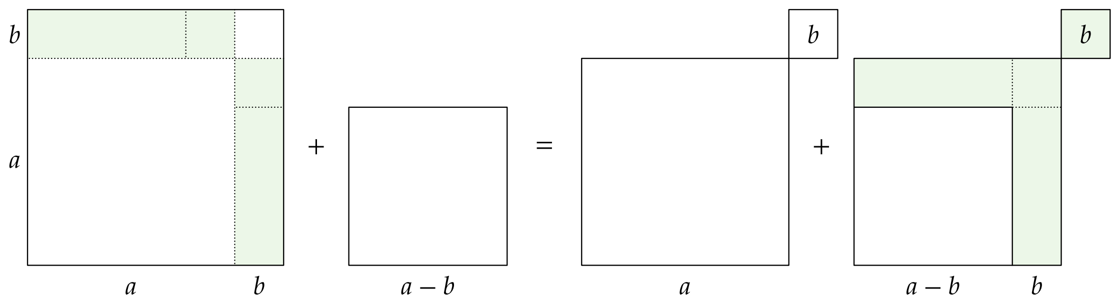
$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

Algebraic areas I

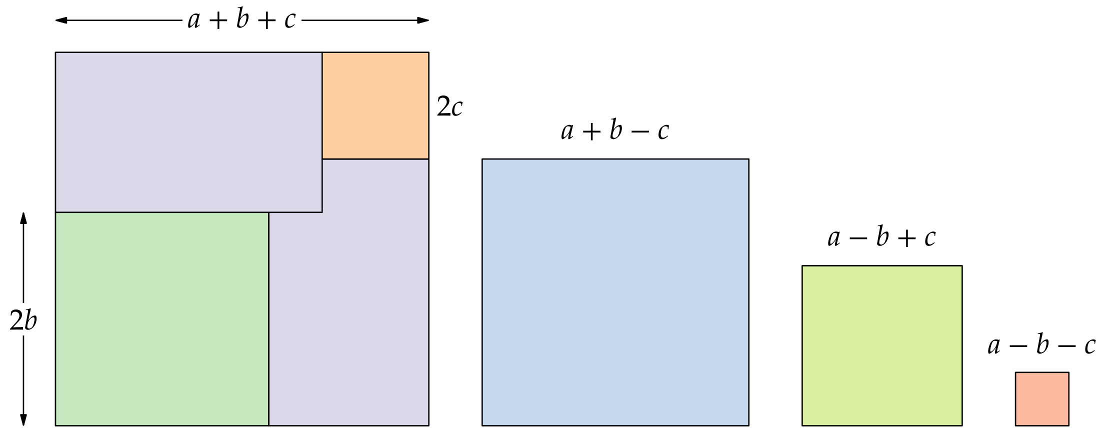
$$(a+b)^2 + (a-b)^2 = 2\left(a^2+b^2\right)$$

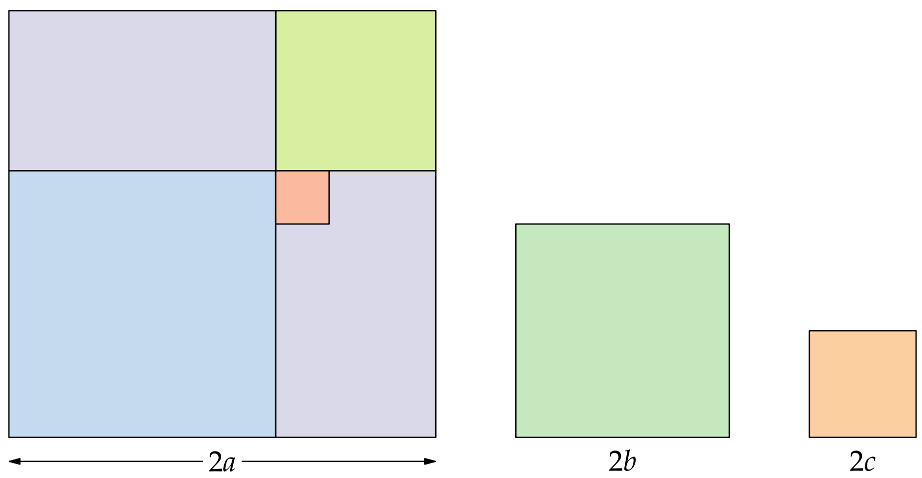


— Shirley Wakin

Algebraic areas II

$$(a+b+c)^2 + (a+b-c)^2 + (a-b+c)^2 + (a-b-c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$

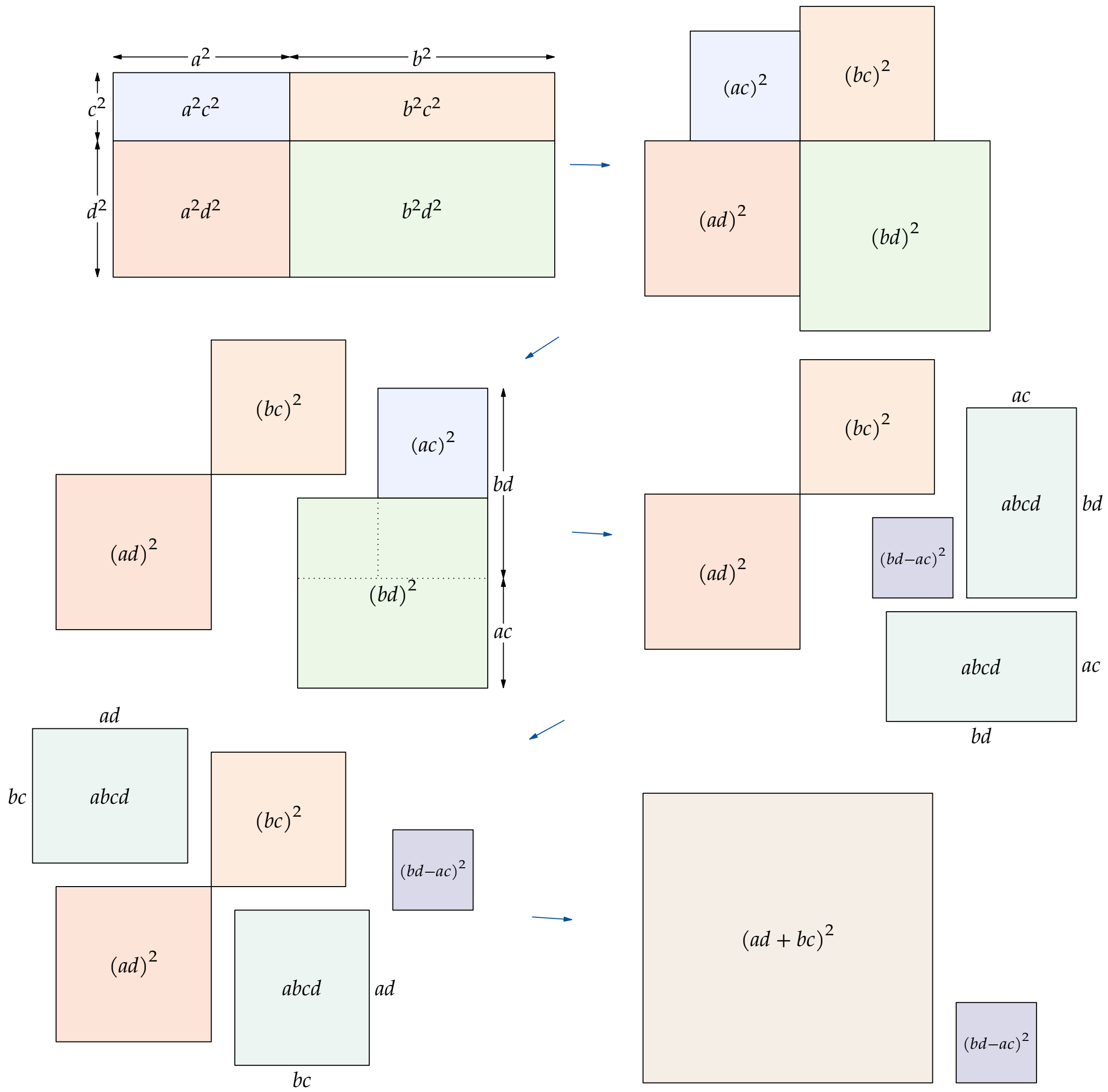




— Sam Pooley and K. Ann Drude

Sum of squares identity

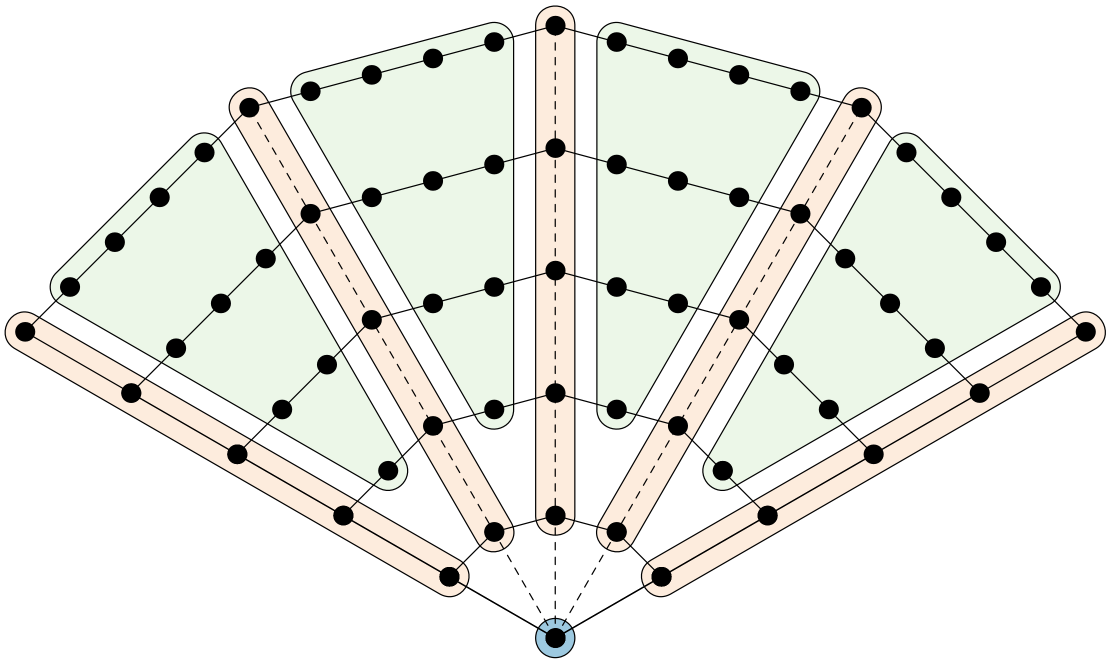
$$\left(a^2+b^2\right)\left(c^2+d^2\right) = (ab+bc)^2 + (bd-ac)^2$$



— Diophantus of Alexandria

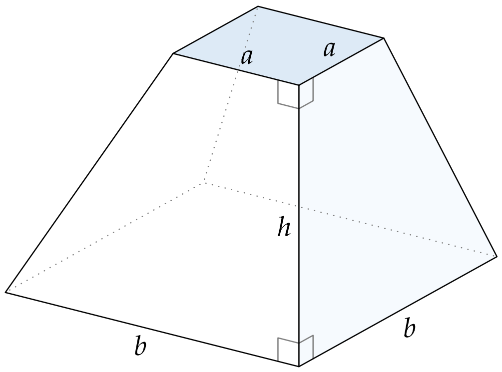
Polygonal numbers

The k^{th} n-gonal number is $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$

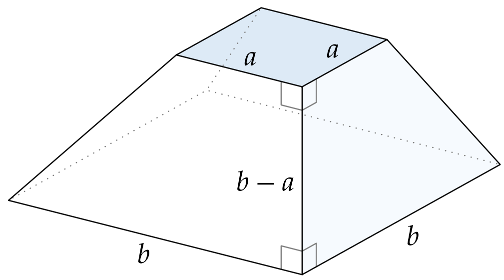


— Dave Logothetti

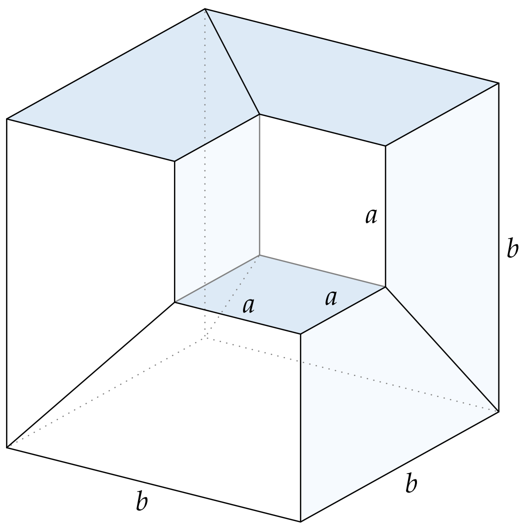
The volume of a frustrum of a square pyramid



P_1



P_2

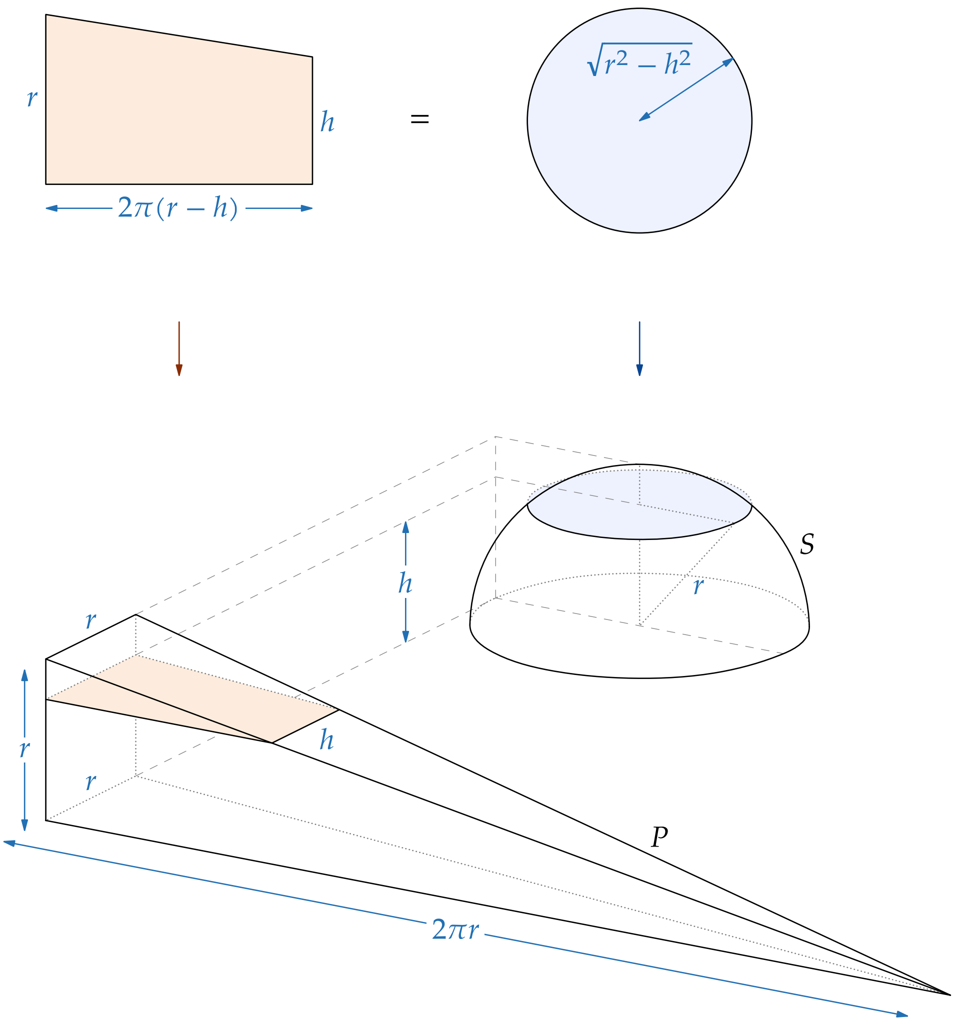


$3P_2$

$$V_{P_1} = \frac{h}{b-a} \cdot V_{P_2} = \frac{h}{b-a} \cdot \frac{1}{3}\left(b^3 - a^3\right) = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

— *The Moscow Papyrus*, c. 1850 BCE

The volume of a hemisphere via Cavalieri's Principle



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

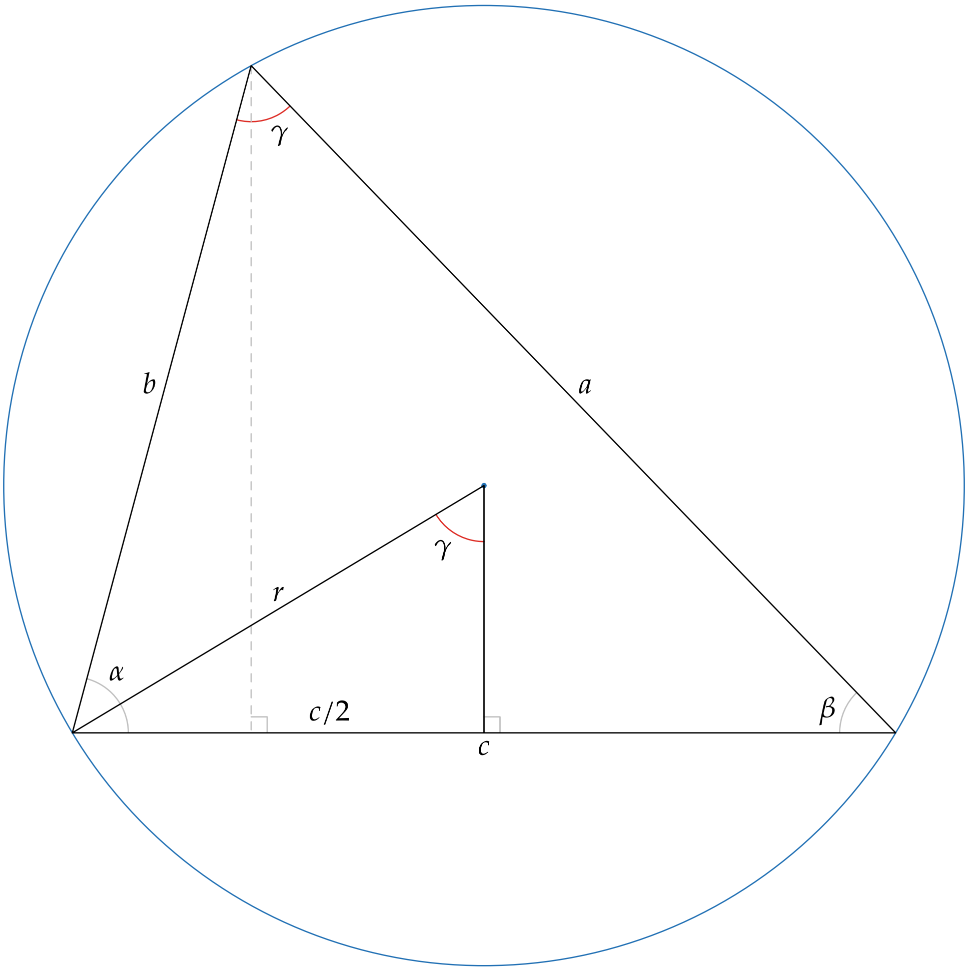
— Sidney H. Kung

Trigonometry, Calculus, & Analytic Geometry

Sine of the sum **29**
Area and difference formulas **30**
The law of cosines I **31**
The law of cosines II **32**
The law of cosines III (via Ptolemy's theorem) **33**
The double-angle formulae **34**
The half-angle tangent formulae **35**
Mollweide's equation **36**
Tangent, cotangent, secant, and cosecant **37**
Substitution to make a rational function of sine and cosine **38**
Sums of arctangents **39**
The distance between a point and a line **40**
The midpoint rule is better than the trapezoidal rule for concave functions **41**
Integration by parts **42**
The graphs of f and f^{-1} are reflections about the line $y = x$ **43**
The reflection property of the parabola **44**
Area under an arch of the cycloid **45**

Sine of the sum

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \text{ for } \alpha + \beta < \pi$$



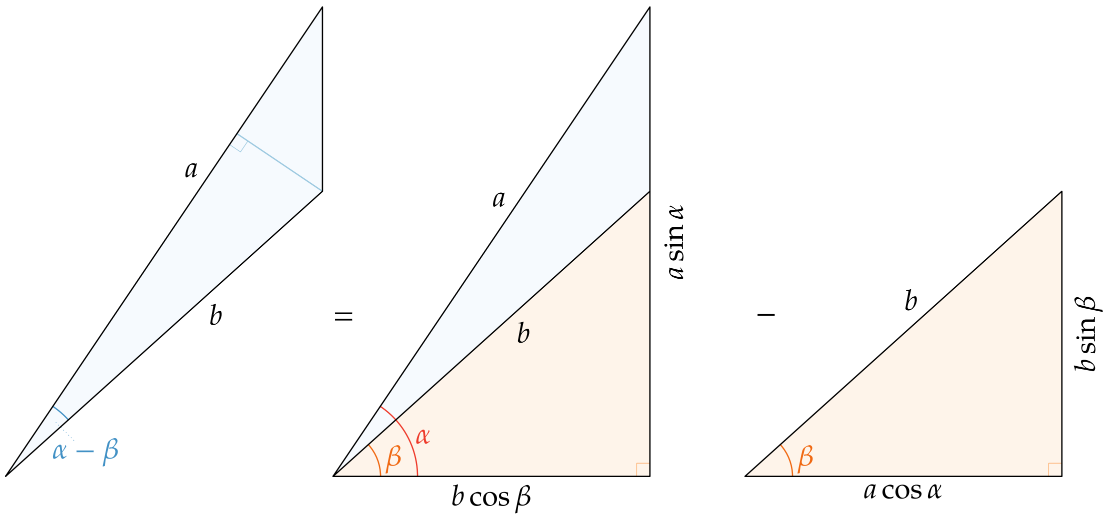
$$c = a\cos\beta + b\cos\alpha$$

$$r = 1/2 \quad \Longrightarrow \quad \sin\gamma = \frac{c/2}{1/2} = c, \ \sin\alpha = a, \ \sin\beta = b$$

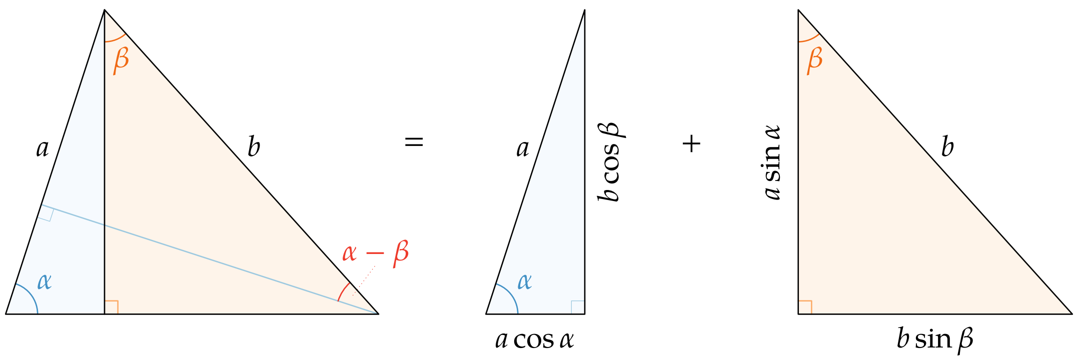
$$\sin(\alpha + \beta) = \sin(\pi - (\alpha + \beta)) = \sin\gamma = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

— Sidney H. Kung

Area and difference formulas



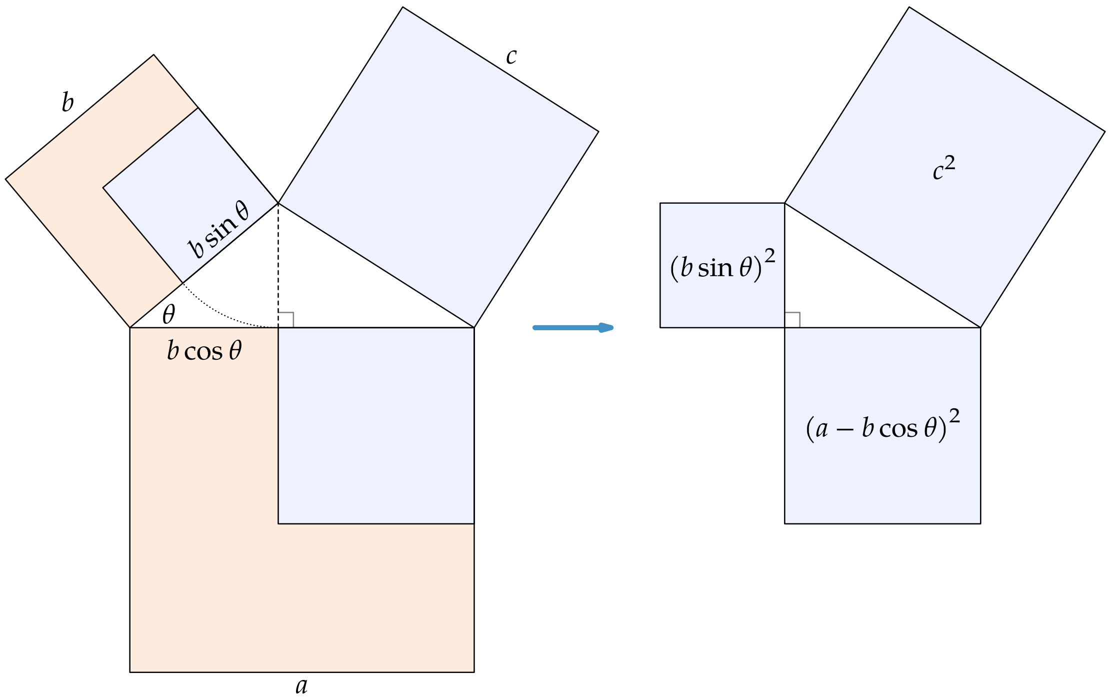
$$\tfrac{1}{2}\cdot a\cdot b\sin(\alpha-\beta)=\tfrac{1}{2}\cdot a\sin\alpha\cdot b\cos\beta-\tfrac{1}{2}\cdot a\cos\alpha\cdot b\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$



$$\tfrac{1}{2}\cdot a\cdot b\cos(\alpha-\beta)=\tfrac{1}{2}\cdot a\cos\alpha\cdot b\cos\beta+\tfrac{1}{2}\cdot a\sin\alpha\cdot b\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

— Sidney H. Kung

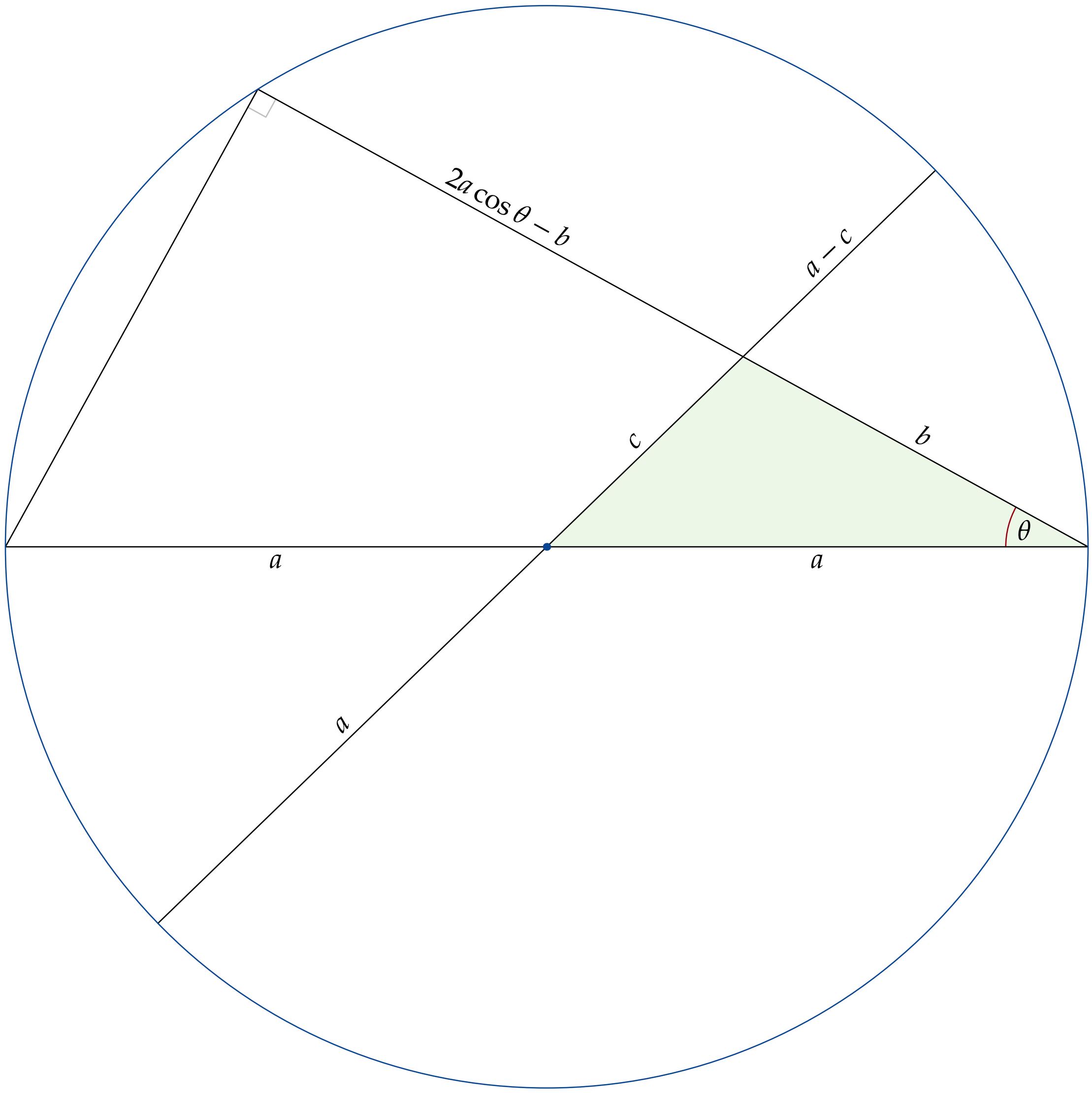
The law of cosines I



$$
\begin{aligned}
c^2 &= (b\sin\theta)^2 + (a - b\cos\theta)^2 \\
&= b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta \\
&= a^2 + b^2\left(\sin^2\theta + \cos^2\theta\right) - 2ab\cos\theta \\
&= a^2 + b^2 - 2ab\cos\theta
\end{aligned}
$$

— Timothy A. Sipka

The law of cosines II

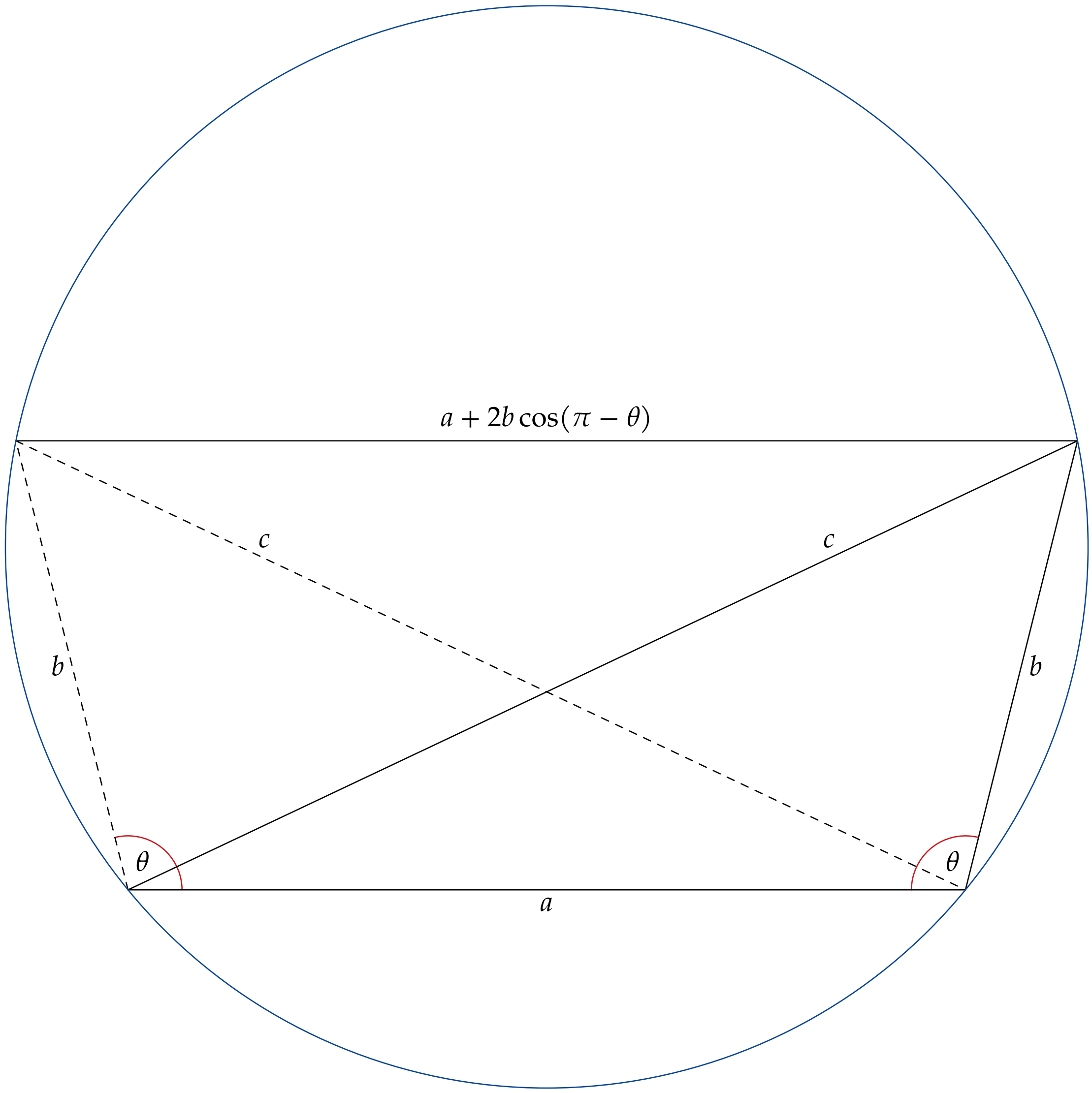


$$(2a\cos\theta - b)\cdot b = (a-c)\cdot(a+c)$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

— Sidney H. Kung

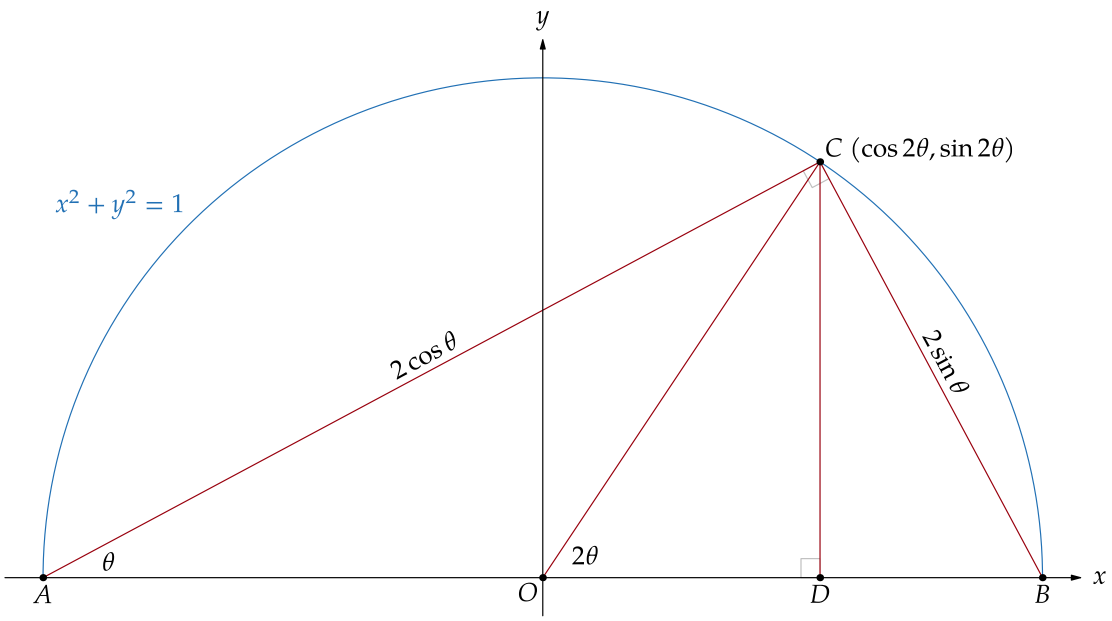
The law of cosines III (via Ptolemy's theorem)



$$c \cdot c = b \cdot b + \big(a + 2b\cos(\pi - \theta)\big) \cdot a$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

— Sidney H. Kung

The double-angle formulae



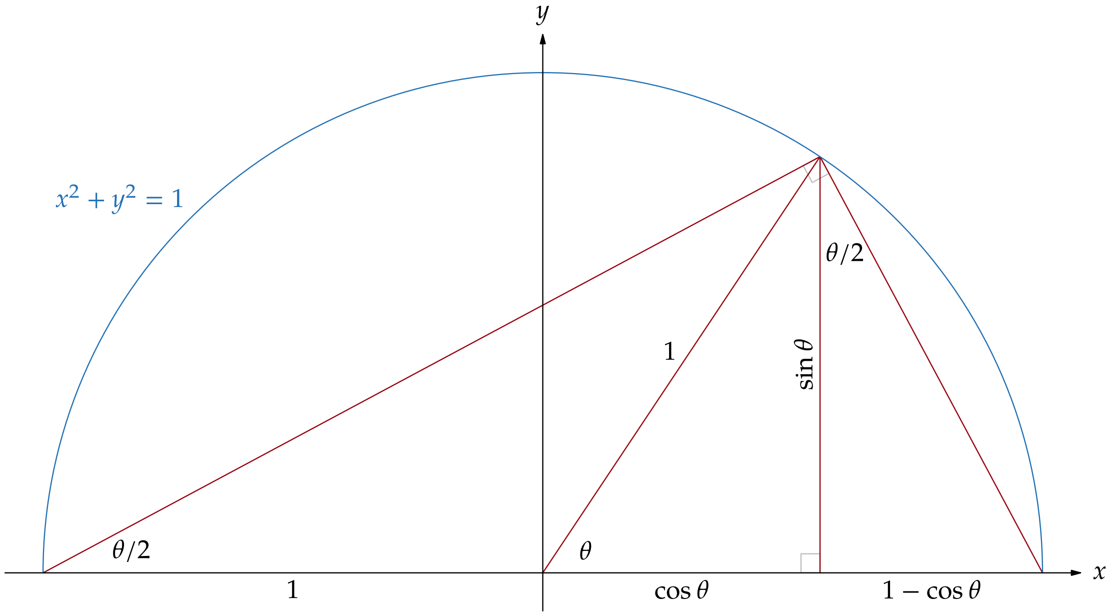
$$\triangle ACD \sim \triangle ABC$$

$$CD/AC = BC/AB$$
$$\sin 2\theta / 2\cos\theta = 2\sin\theta/2$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$AD/AC = AC/AB$$
$$(1+\cos 2\theta)/2\cos\theta = 2\cos\theta/2$$
$$\cos 2\theta = 2\cos^2\theta - 1$$

— Roger B. Nelsen

The half-angle tangent formulae

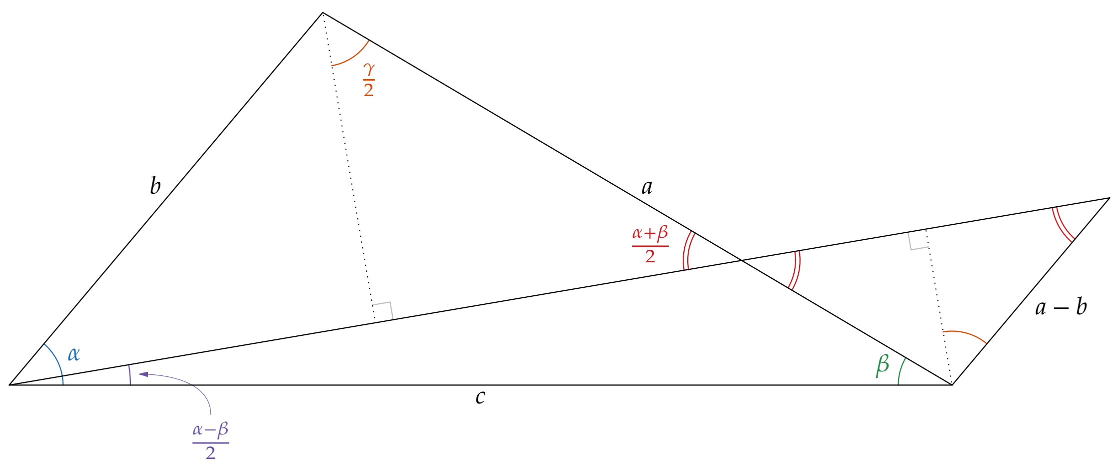


$$\tan\theta/2 = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

— R. J. Walker

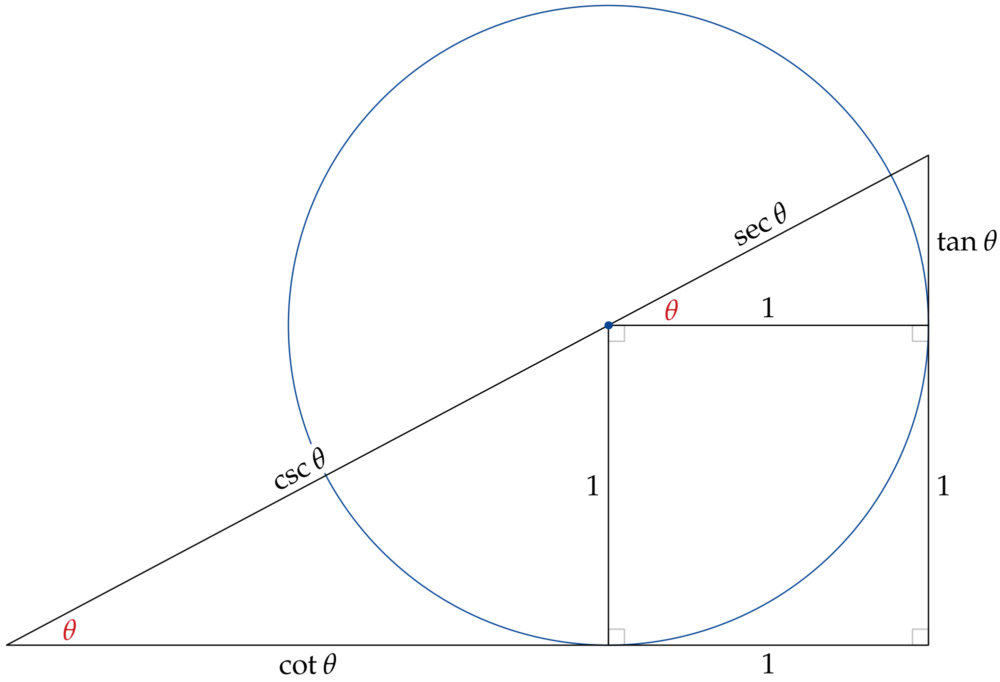
Mollweide's equation

$$(a-b)\cos\frac{\gamma}{2} = c\sin\left(\frac{\alpha-\beta}{2}\right)$$



— H. Arthur DeKleine

Tangent, cotangent, secant, and cosecant

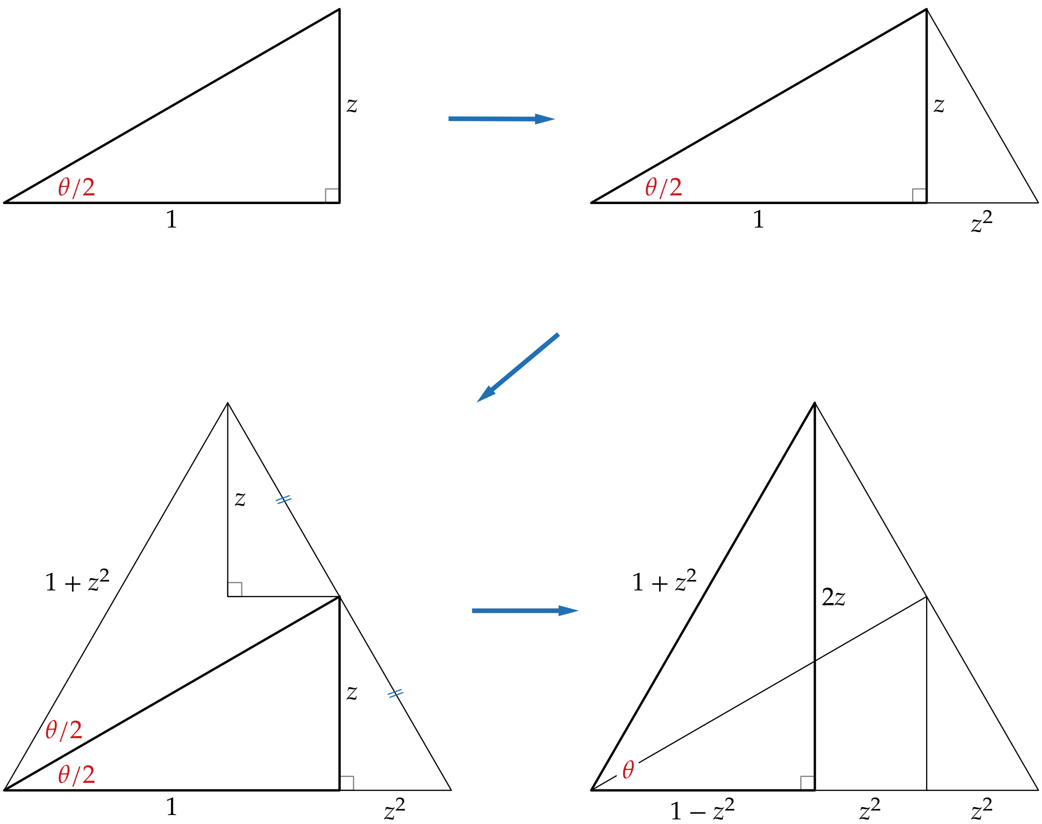


$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$
$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$

$$\text{also } \tan\theta = \frac{\tan\theta + 1}{\cot\theta + 1}$$

— William Romaine

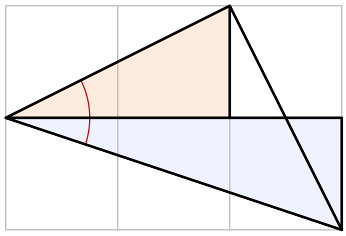
Substitution to make a rational function of sine and cosine



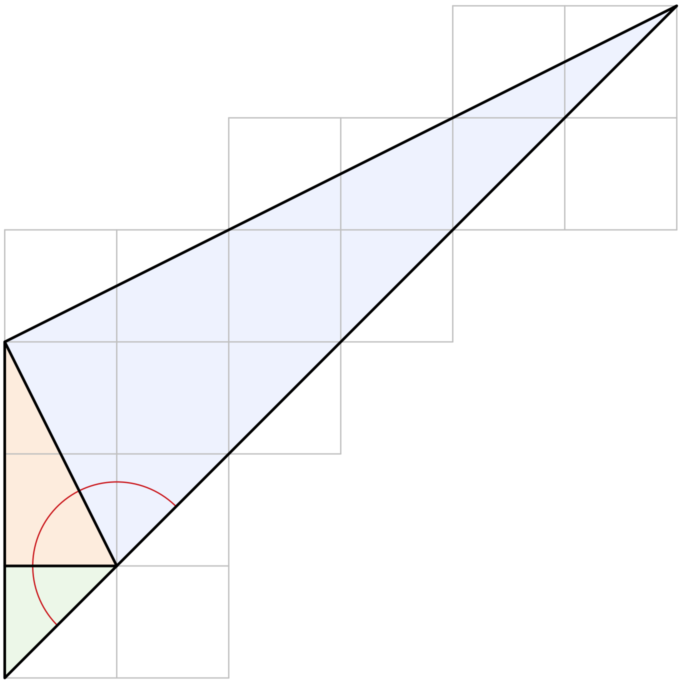
$$z = \tan(\theta/2) \implies \sin\theta = \frac{2z}{1+z^2} \quad \text{and} \quad \cos\theta = \frac{1-z^2}{1+z^2}$$

— Roger B. Nelsen

Sums of arctangents



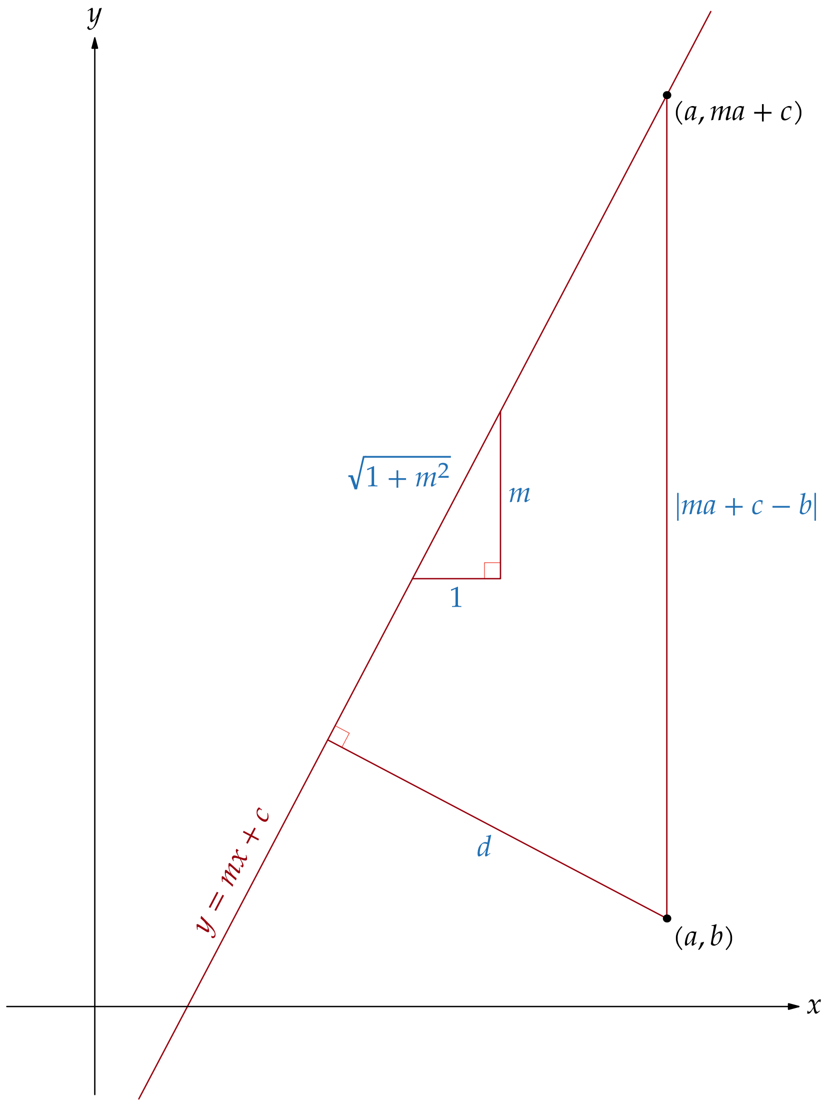
$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

— Edward M. Harris

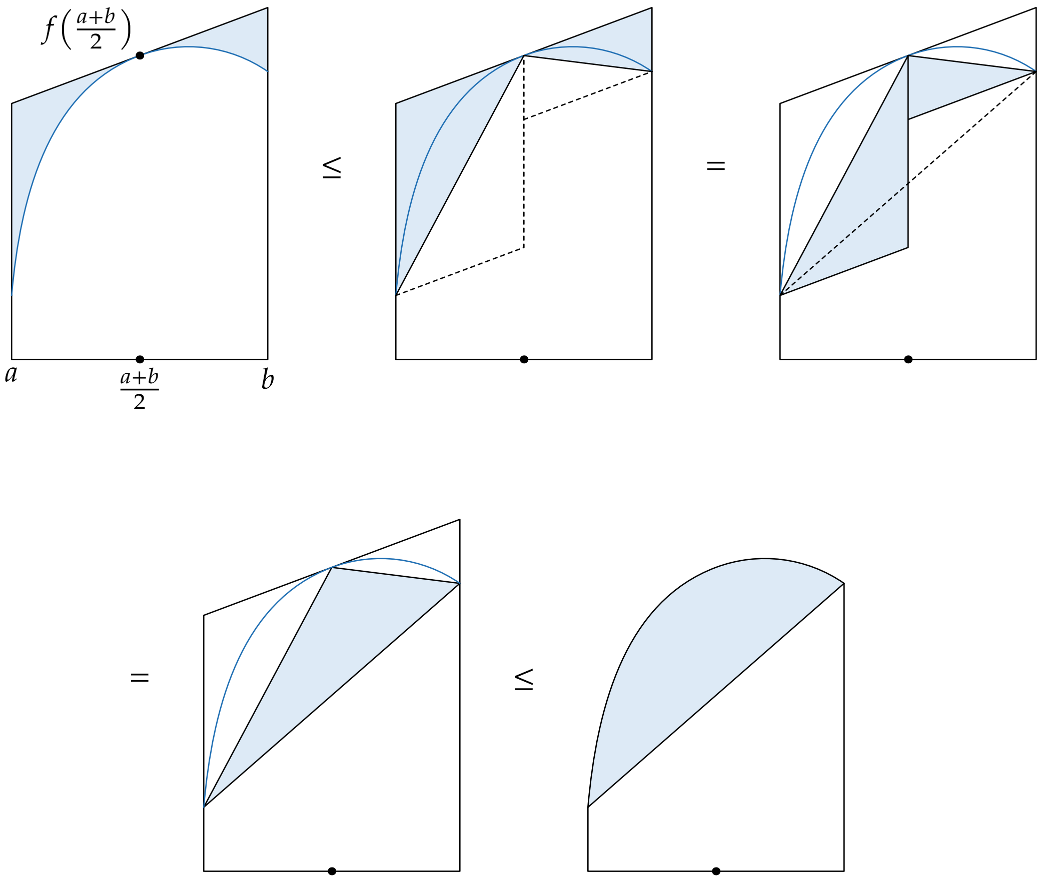
The distance between a point and a line



$$\frac{d}{1} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

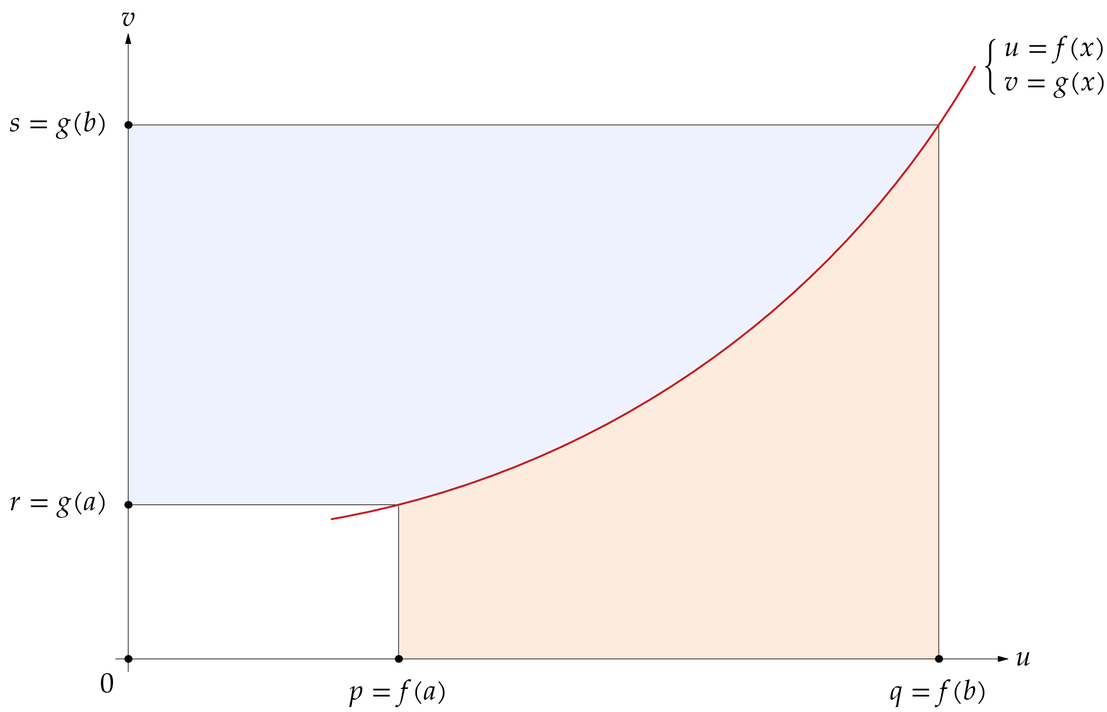
— R. L. Eisenman

The midpoint rule is better than the trapezoidal rule for concave functions



— Frank Burk

Integration by parts



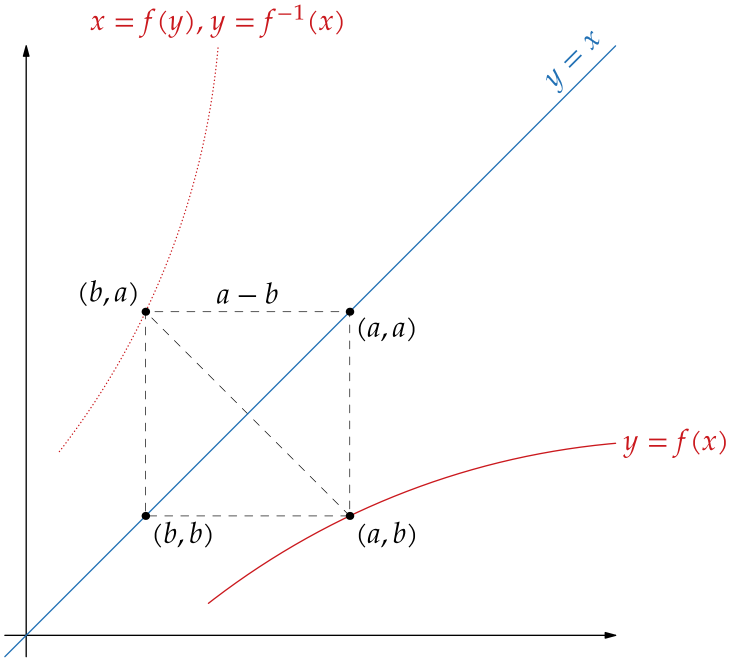
$$\text{Area} \quad + \text{Area} \quad = qs - pr$$

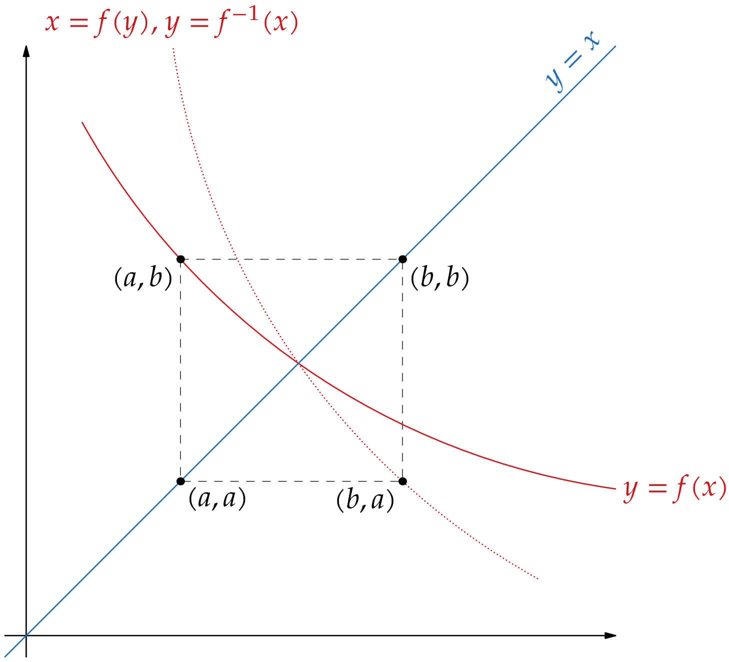
$$\int_r^s u\,dv + \int_p^q v\,du = uv\,\Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\,\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

— Richard Courant

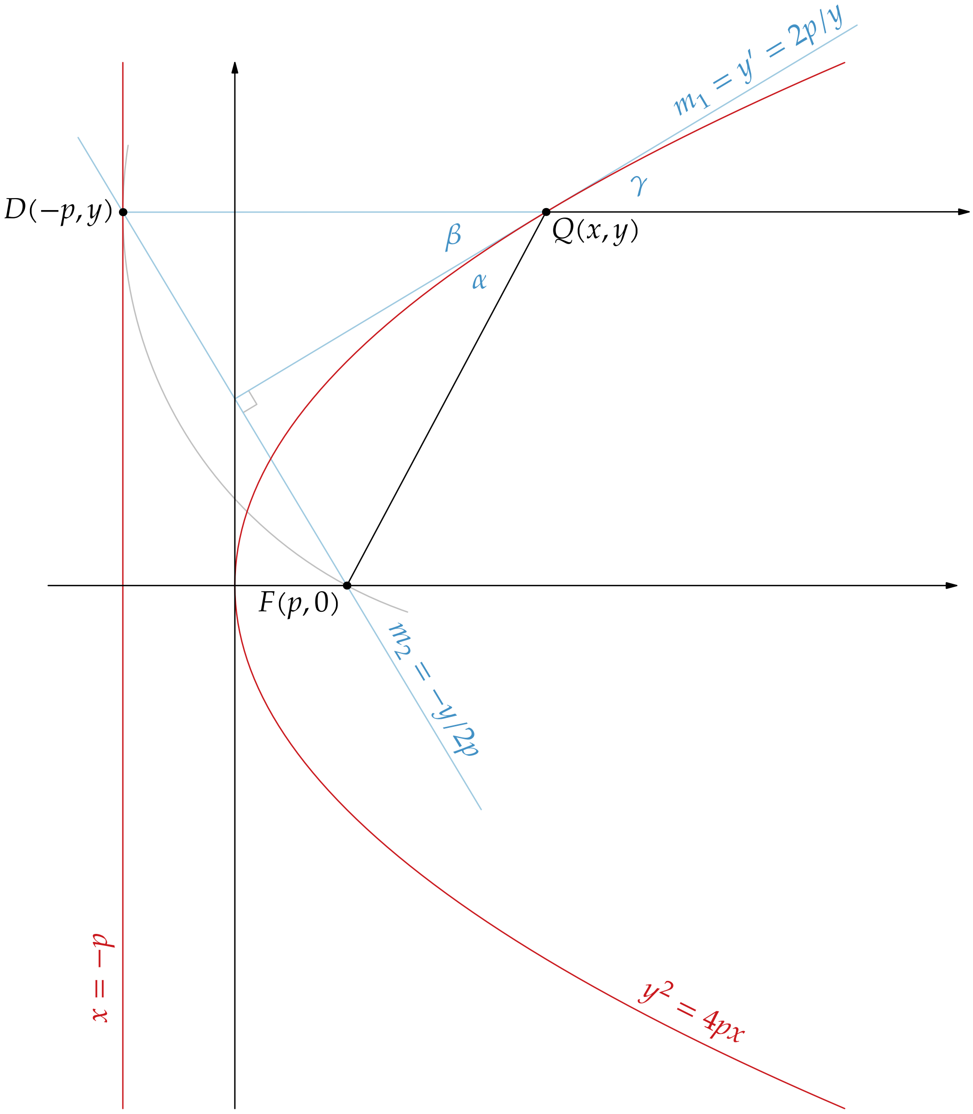
The graphs of f and f^{-1} are reflections about the line $y = x$





— Ayoub B. Ayoub

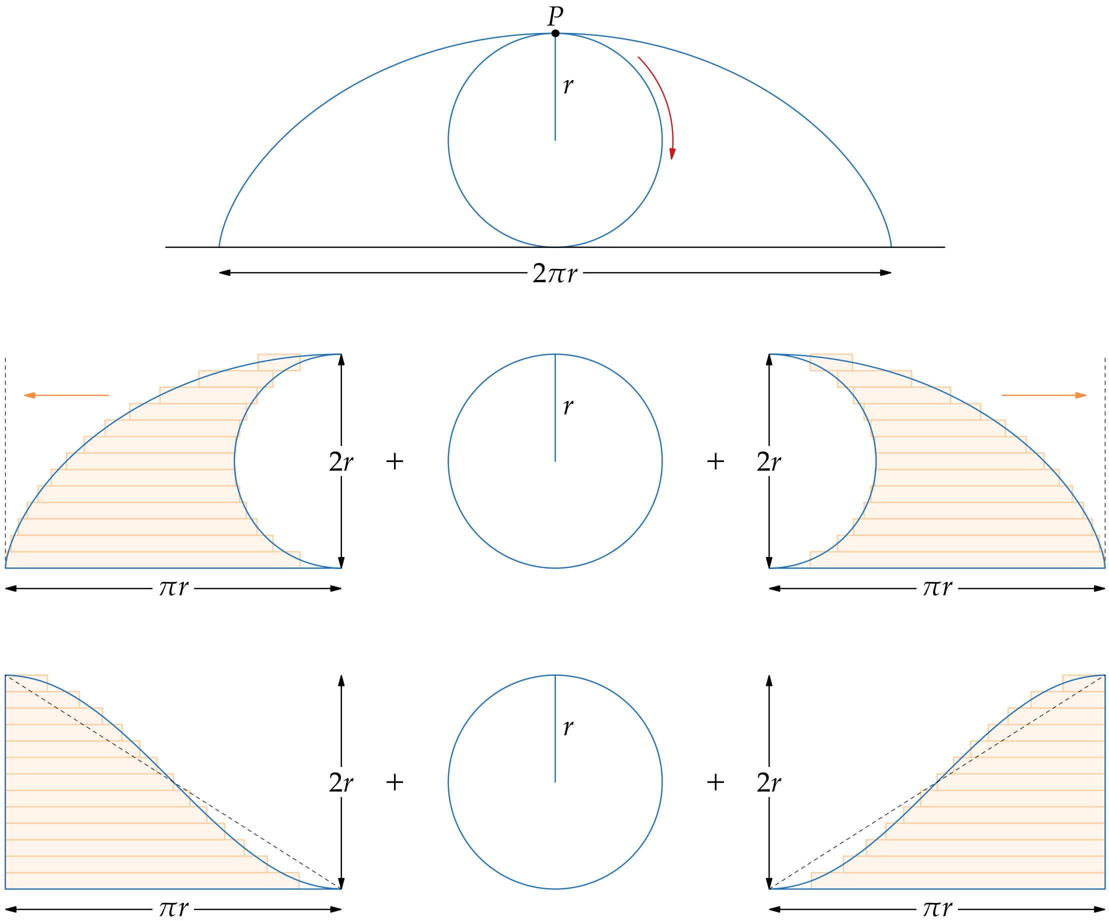
The reflection property of the parabola



$$QF = QD \quad \text{and} \quad m_1 \cdot m_2 = -1, \quad \text{therefore} \quad \alpha = \beta = \gamma$$

— Ayoub B. Ayoub

Area under an arch of the cycloid



$$\frac{1}{2}\pi r \cdot 2r \qquad + \qquad \pi r^2 \qquad + \qquad \frac{1}{2}\pi r \cdot 2r$$

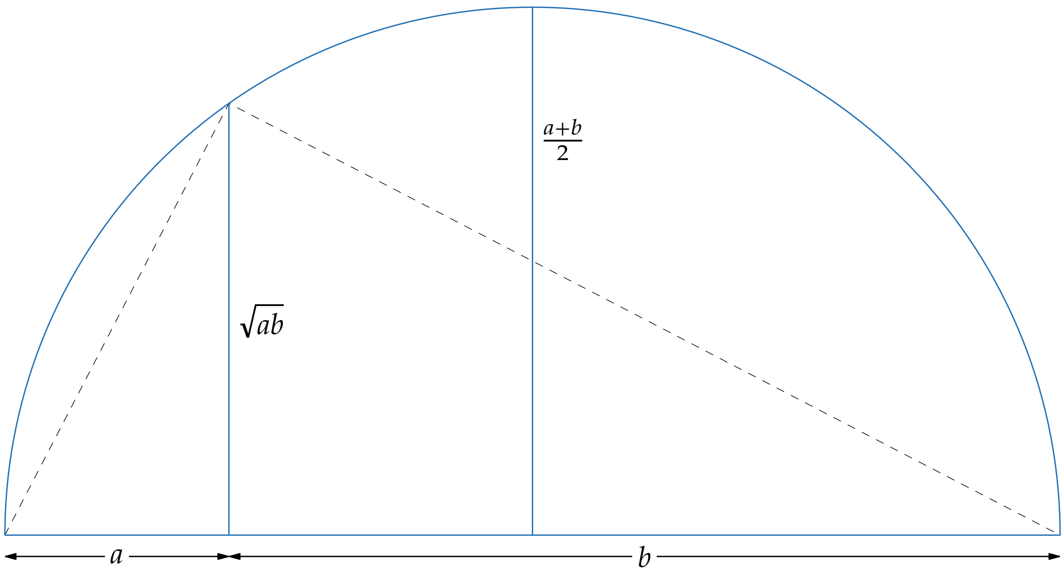
therefore $\quad A = 3\pi r^2$

— Richard M. Beekman

Inequalities

The arithmetic mean – geometric mean inequality I 47
The arithmetic mean – geometric mean inequality II 48
The arithmetic mean – geometric mean inequality III 49
Two extremum problems 50
The HM–GM–AM–QM inequalities I 51
The HM–GM–AM–QM inequalities II 52
The HM–GM–AM–QM inequalities III 53
Five means — and their means 54
$e^{\pi} > \pi^{e}$ 55
$A^B > B^A$ for $e \leq A < B$ 56
The mediant property 57
Regle des nombres moyens – two proofs 58
The sum of a positive number and its reciprocal is at least two 59
Aristarchus' inequalities 60
The Cauchy-Schwartz inequality 61
Bernoulli's inequality 62
Napier's inequality 63

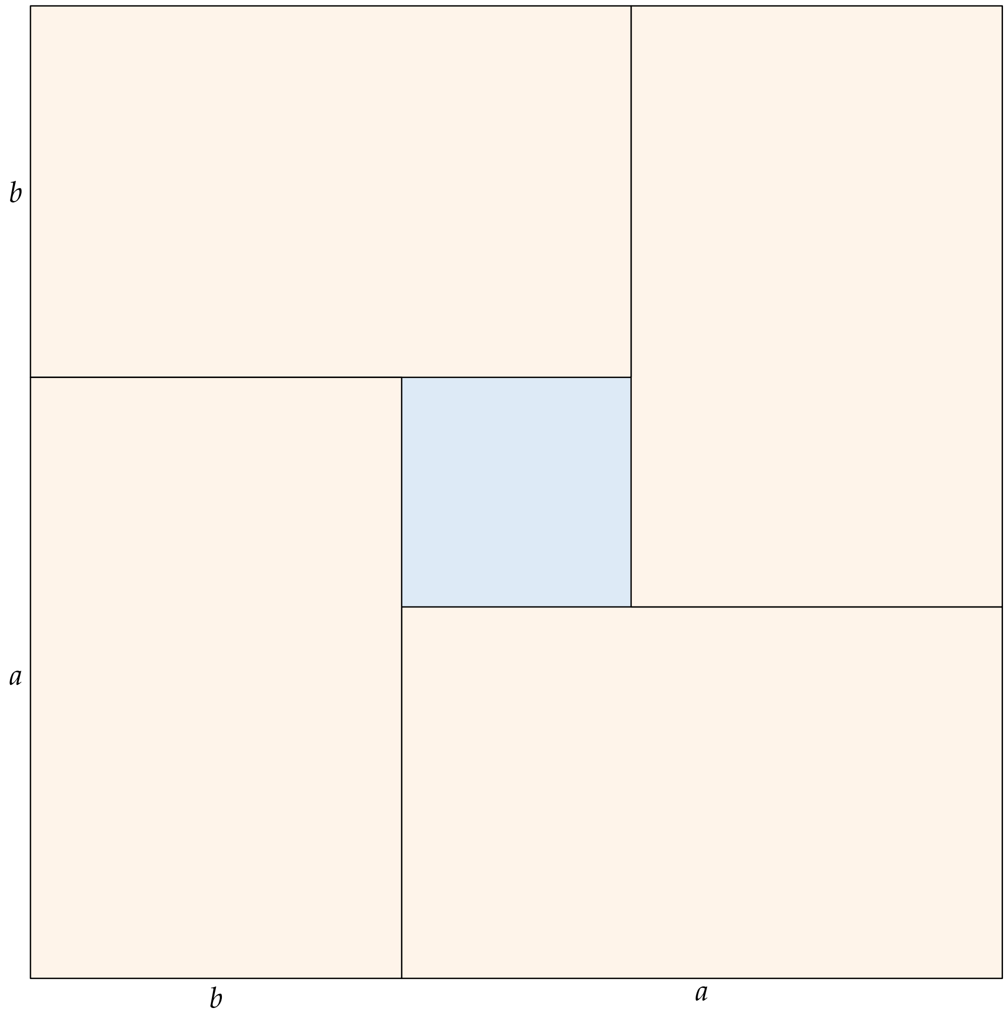
The arithmetic mean – geometric mean inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

— Charles D. Gallant

The arithmetic mean – geometric mean inequality II



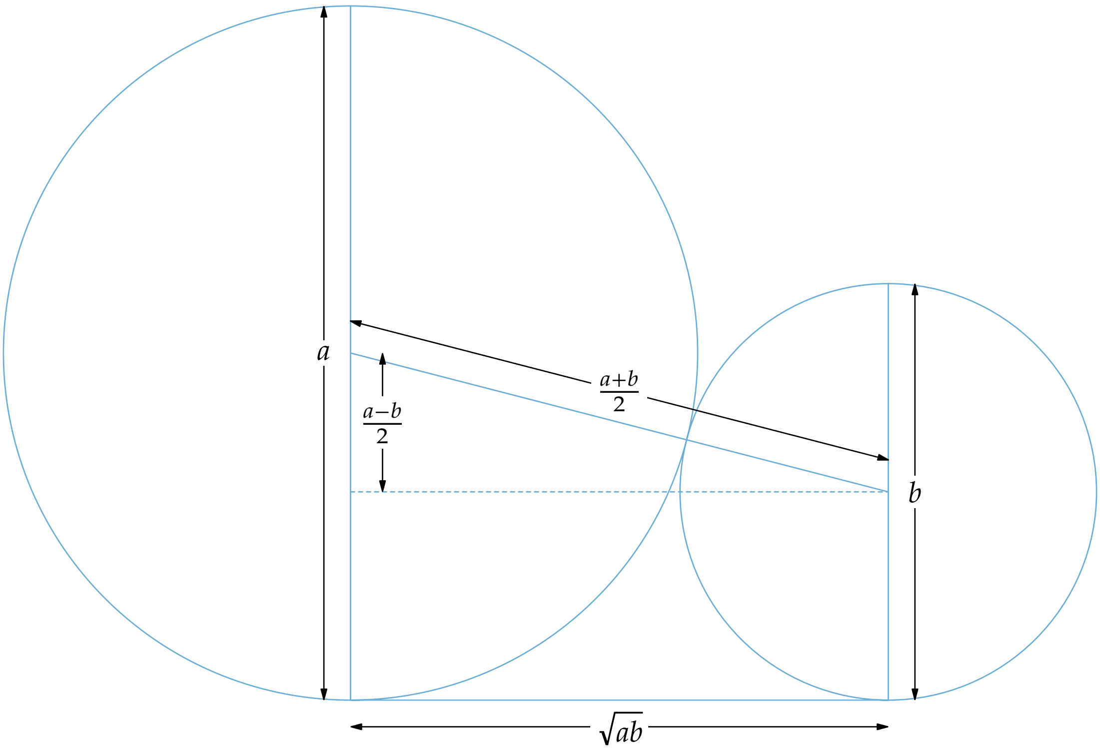
$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

— Doris Schattschneider

The arithmetic mean – geometric mean inequality III

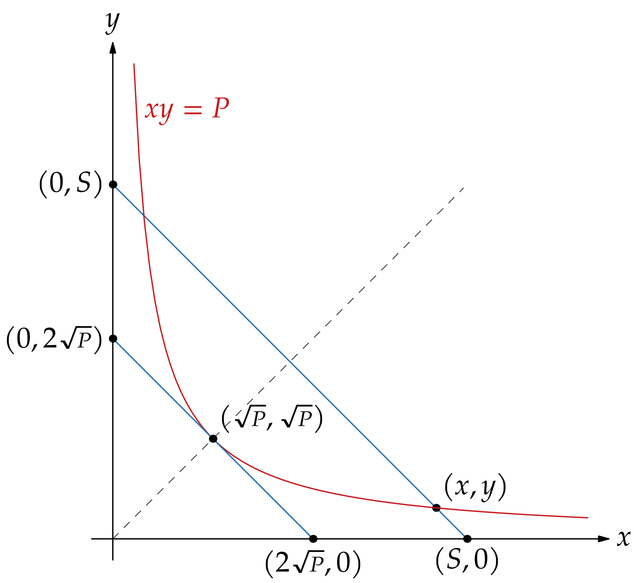
$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \text{with equality iff } a = b$$



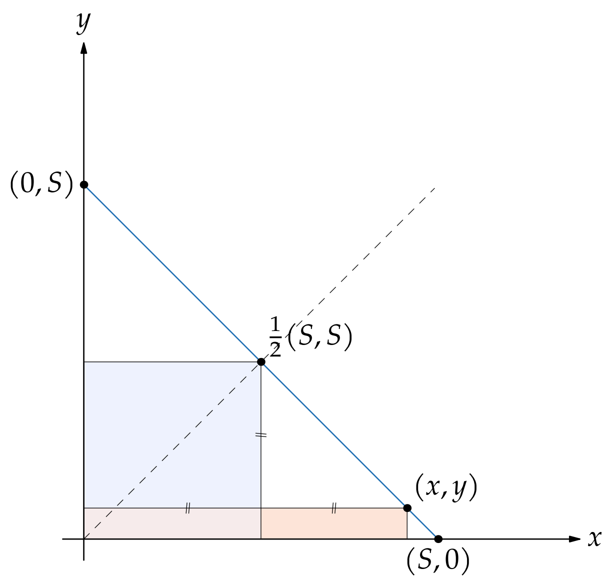
— Roland H. Eddy

Two extremum problems

For a given product, the sum of two positive numbers is minimal when the numbers are equal.

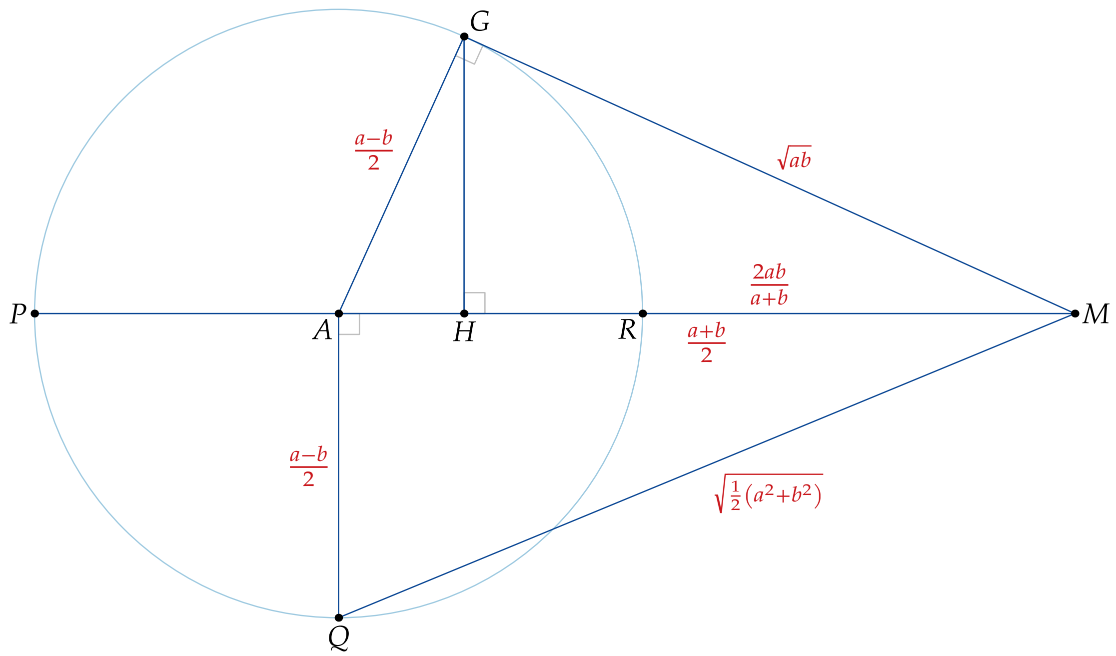


For a given sum, the product of two positive numbers is maximal when the numbers are equal.



— Paulo Montuchi and Warren Page

The HM–GM–AM–QM inequalities I



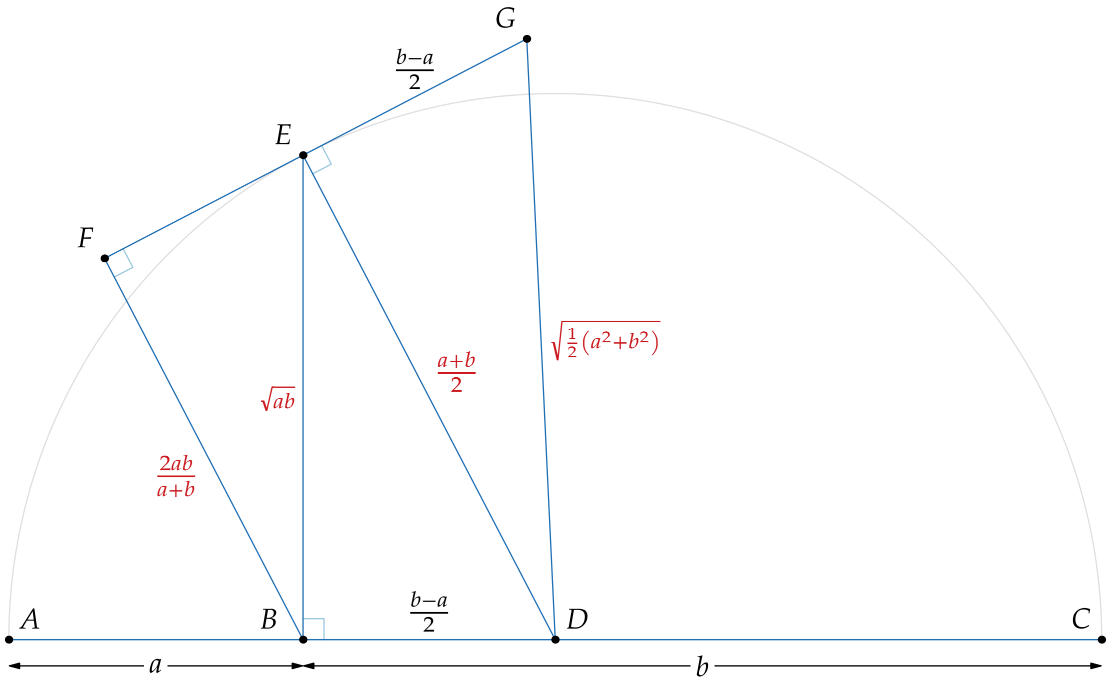
$$PM = a, \quad RM = b, \quad a > b > 0$$

$$HM < GM < AM < QM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{1}{2}\left(a^2+b^2\right)}$$

— Roger B. Nelsen

The HM–GM–AM–QM inequalities II



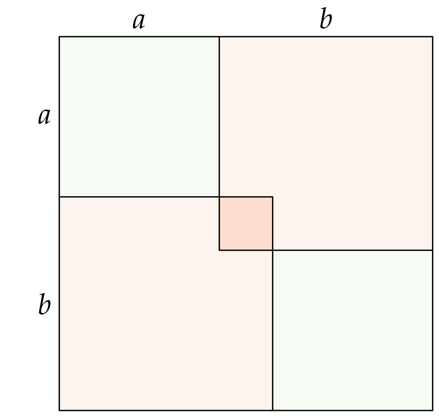
$$AB = a, \quad BC = b, \quad AD = DC = \frac{a+b}{2}$$

$$BE \perp AB, \quad DE = AD$$

$$FE \perp ED, \quad FB \parallel ED, \quad EG = BD = \frac{b-a}{2}$$

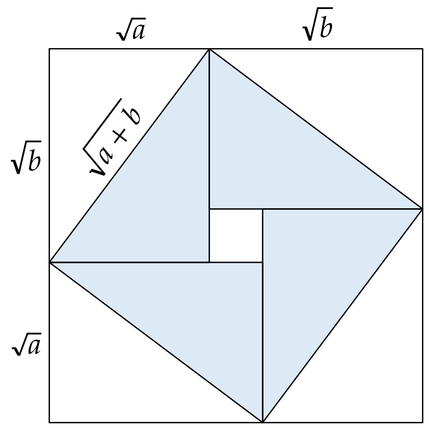
— Sidney H. Kung

The HM–GM–AM–QM inequalities III



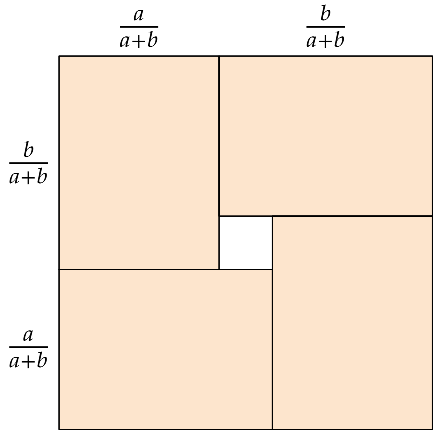
$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\tfrac{1}{2}\left(a^2+b^2\right)} \geq \frac{a+b}{2}$$



$$\left(\sqrt{a+b}\right)^2 \geq 4 \cdot \tfrac{1}{2}\sqrt{a}\sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

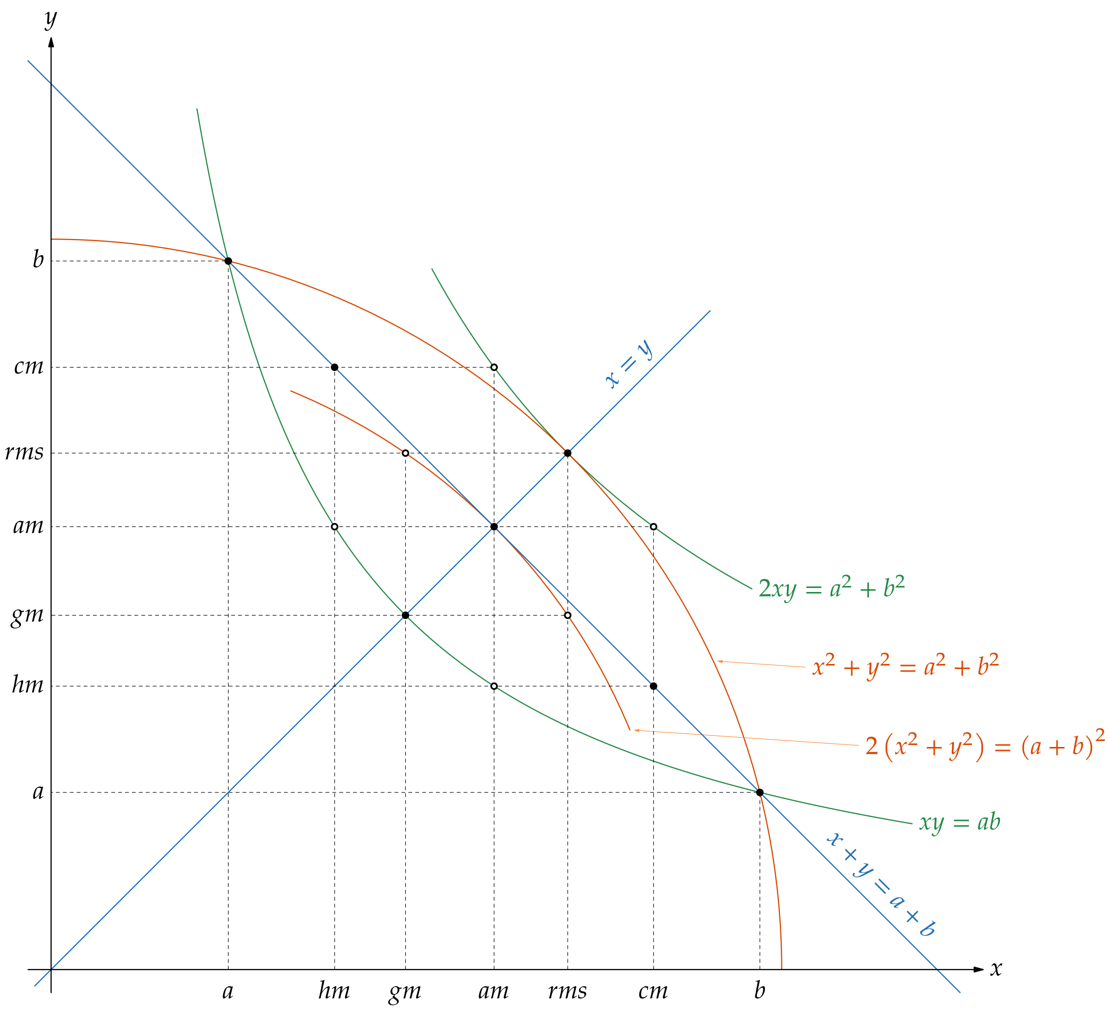


$$1 \geq 4 \cdot \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

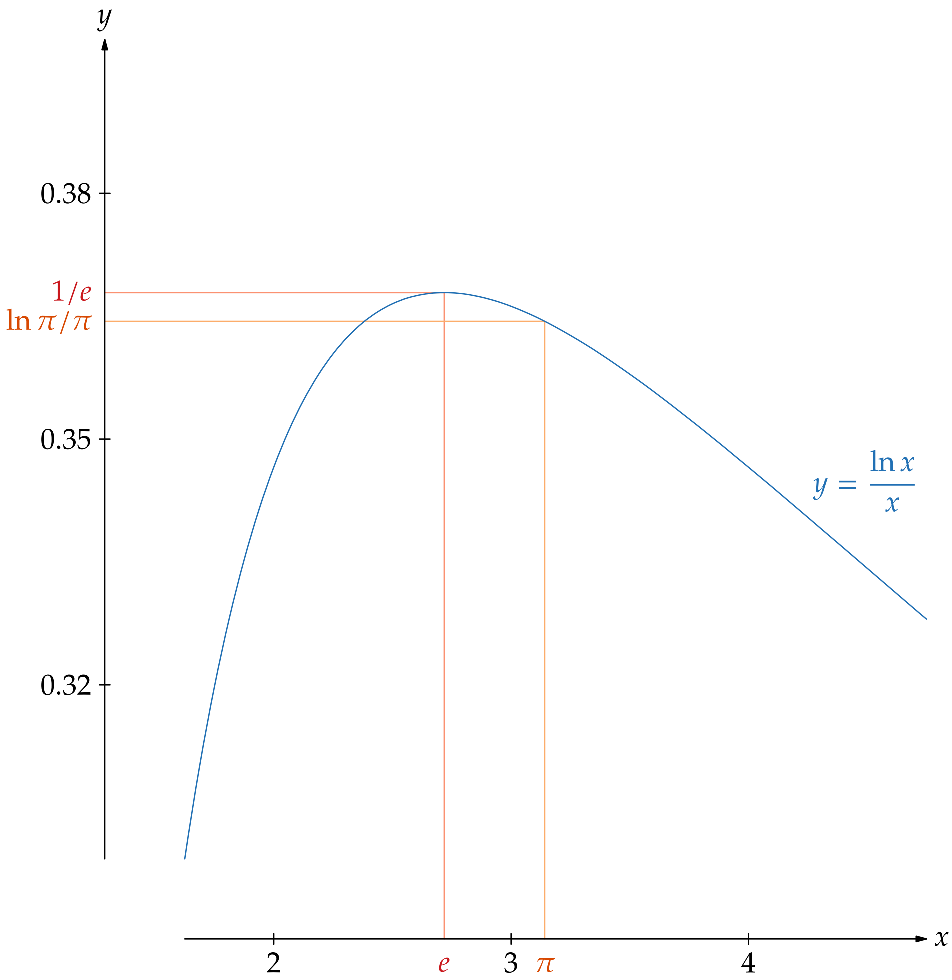
— Roger B. Nelsen

Five means — and their means



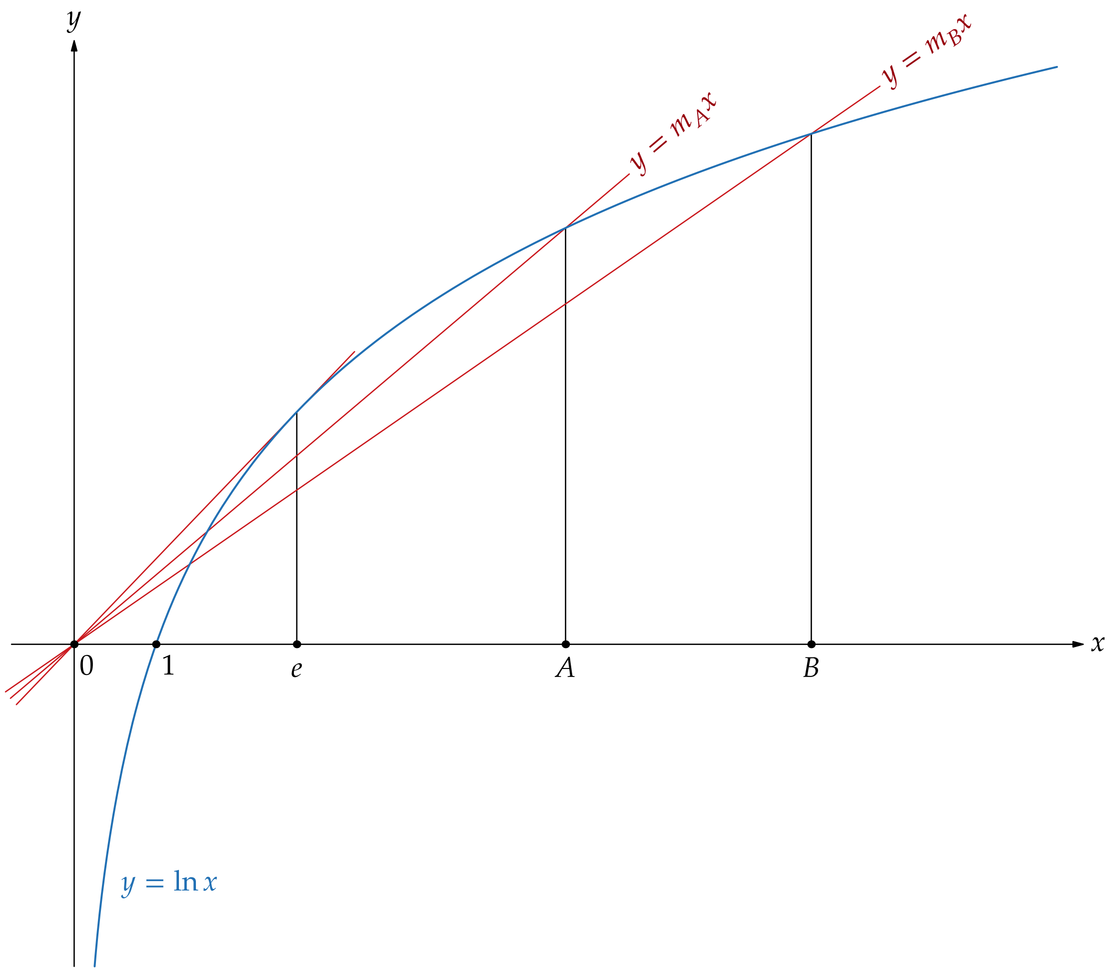
— Roger B. Nelsen

$e^{\pi} > \pi^{e}$



— Fouad Nakhli

$A^B > B^A$ **for** $e \leq A < B$

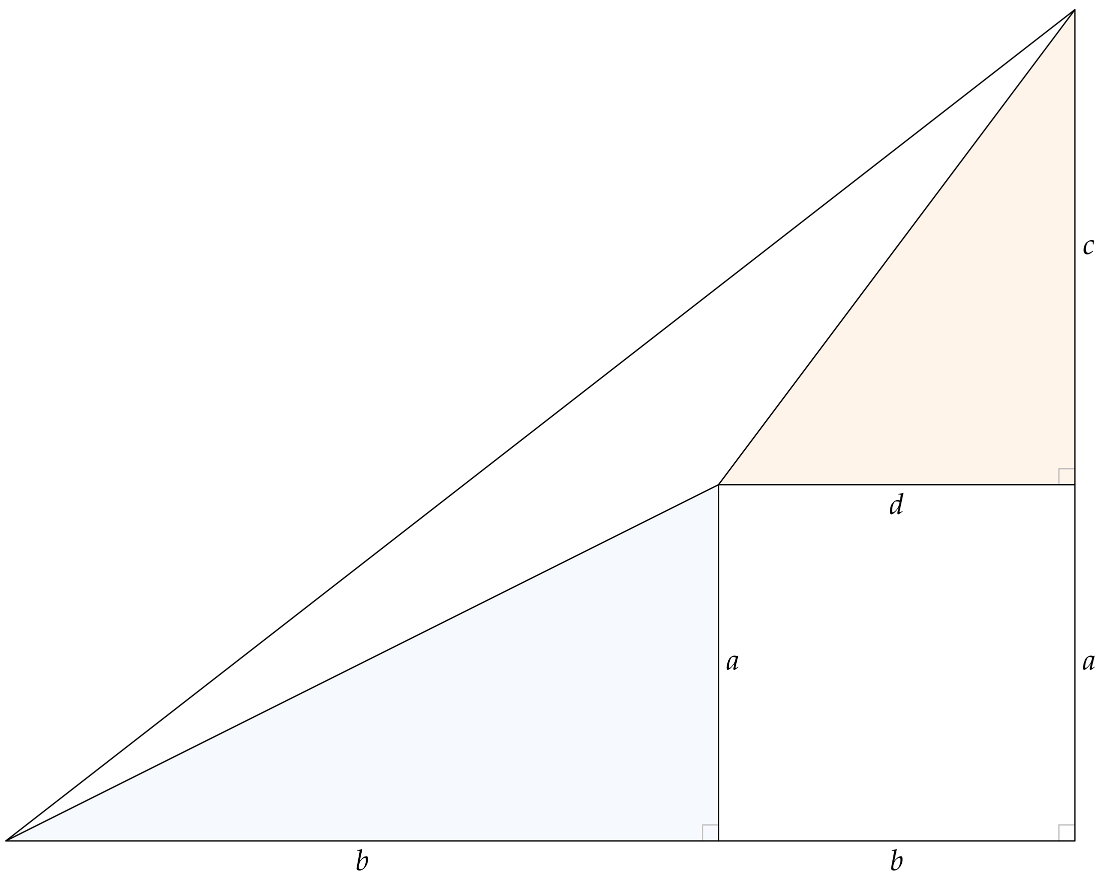


$$
\begin{aligned}
e \leq A < B \quad &\implies \quad m_A > m_B \\
&\implies \quad \frac{\ln A}{A} > \frac{\ln B}{B} \\
&\implies \quad A^B > B^A
\end{aligned}
$$

— Charles D. Gallant

The mediant property

$$\frac{a}{b} < \frac{c}{d} \quad \Rightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

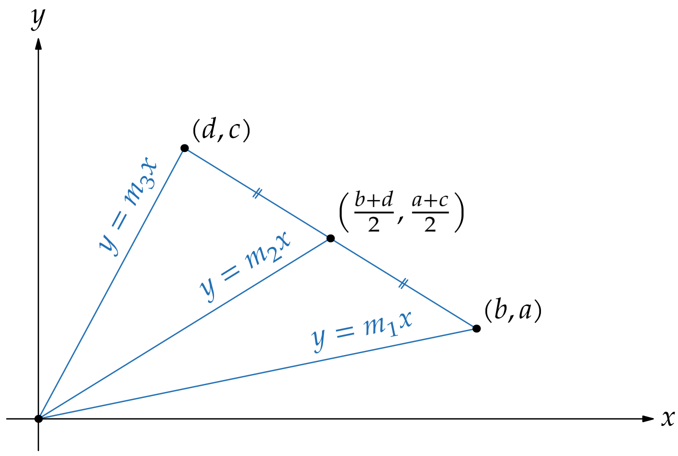


— Richard A. Gibbs

Regle des nombres moyens – two proofs

$$a, b, c, d > 0; \quad \frac{a}{b} < \frac{c}{d} \quad \Longrightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

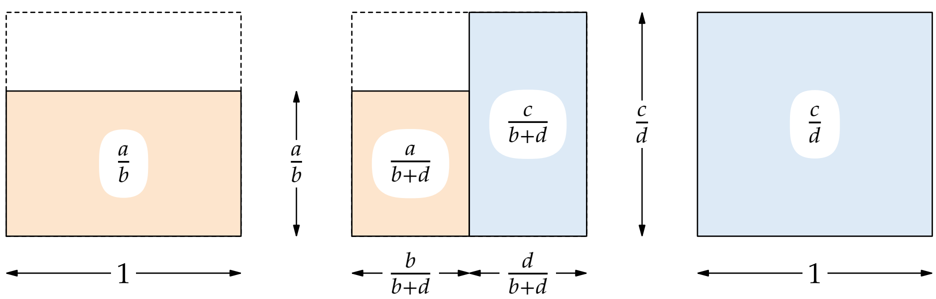
I.



$$m_1 < m_3 \quad \Longrightarrow \quad m_1 < m_2 < m_3$$

— Li Changming

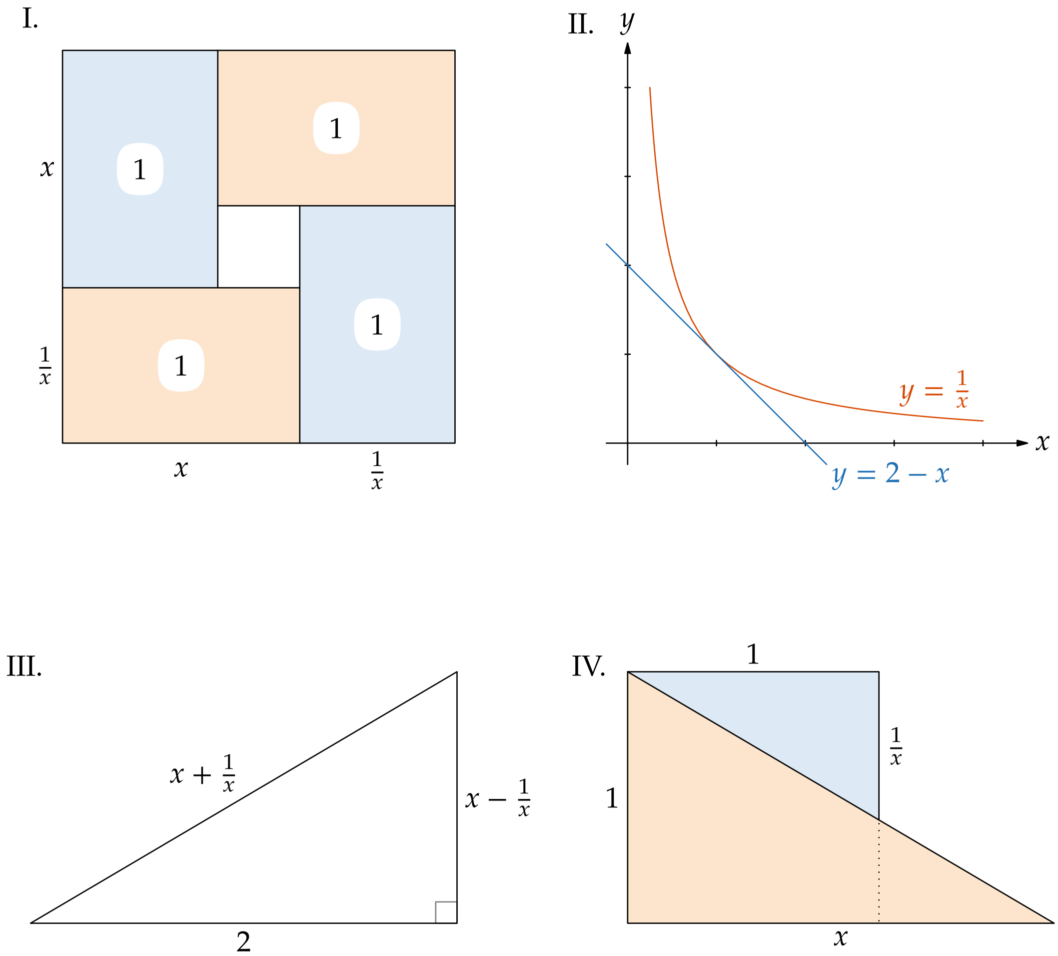
II.



$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

— Roger B. Nelsen

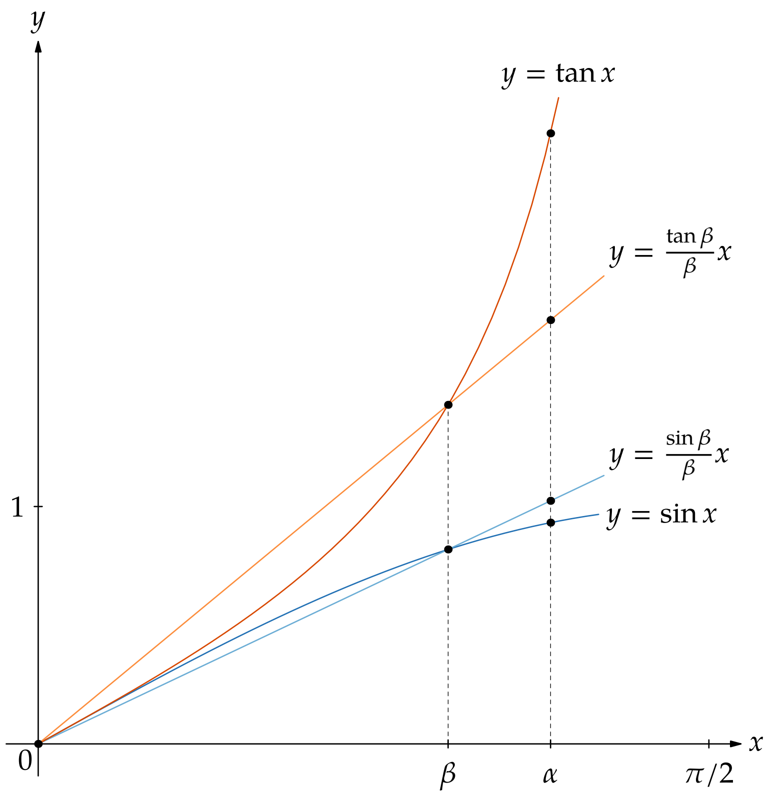
The sum of a positive number and its reciprocal is at least two



— Roger B. Nelsen

Aristarchus' inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \implies \frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$



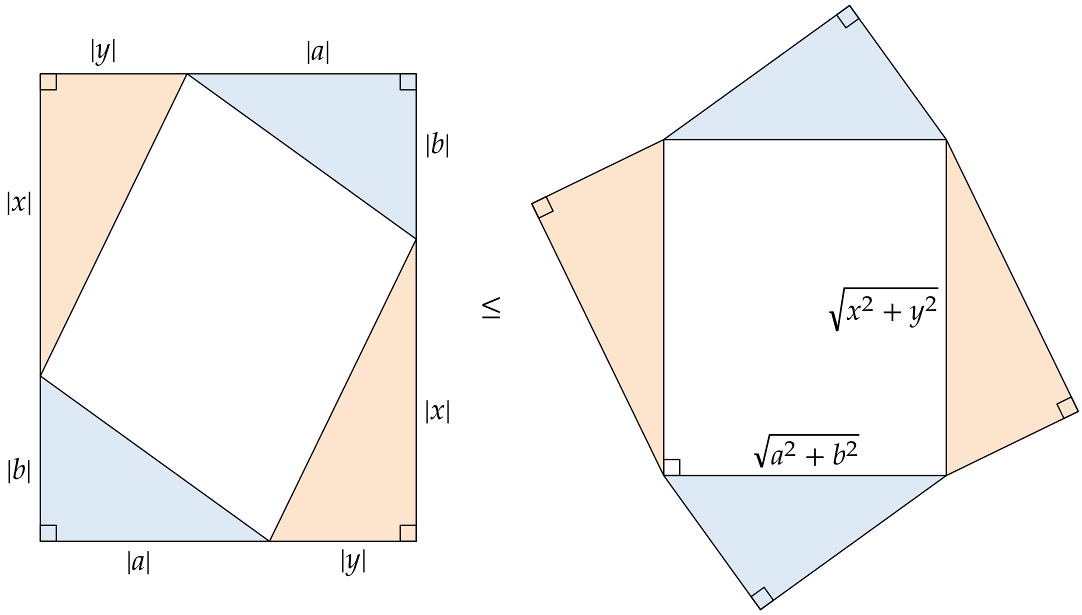
$$\sin\alpha < \tfrac{\sin\beta}{\beta}\alpha; \quad \tan\alpha > \tfrac{\tan\beta}{\beta}\alpha$$

$$\therefore \quad \frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$

— Roger B. Nelsen

The Cauchy-Schwartz inequality

$$|\langle a,b\rangle \cdot \langle x,y\rangle| \leq \|\langle a,b\rangle\|\,\|\langle x,y\rangle\|$$



$$(|a|+|y|)\,(|b|+|x|) \leq 2\left(\tfrac{1}{2}|a||b| + \tfrac{1}{2}|x||y|\right) + \sqrt{a^2+b^2}\sqrt{x^2+y^2}$$

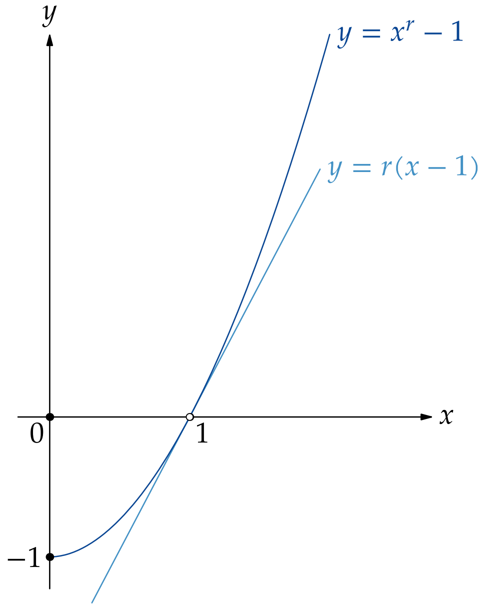
$$\therefore \quad |ax+by| \leq |a||x| + |b||y| \leq \sqrt{a^2+b^2}\sqrt{x^2+y^2}$$

— Roger B. Nelsen

Bernoulli's inequality

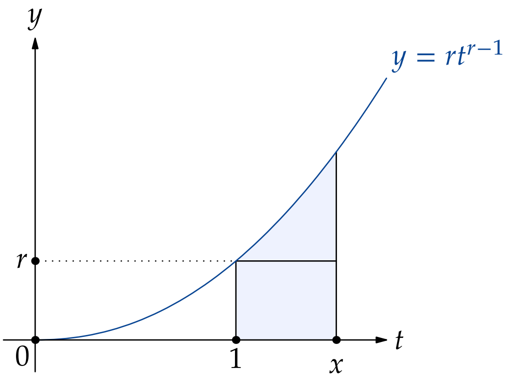
$$x > 0, x \neq 1, r > 1\colon x^r - 1 > r(x-1)$$

I. First semester calculus



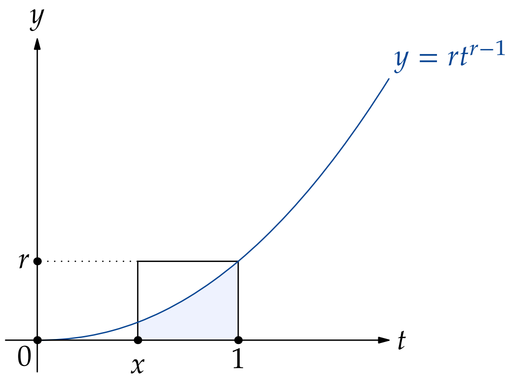
II. Second semester calculus

$x > 1$



$$x^r - 1 = \int_1^x rt^{r-1}\,dt > r(x-1)$$

$0 < x < 1$



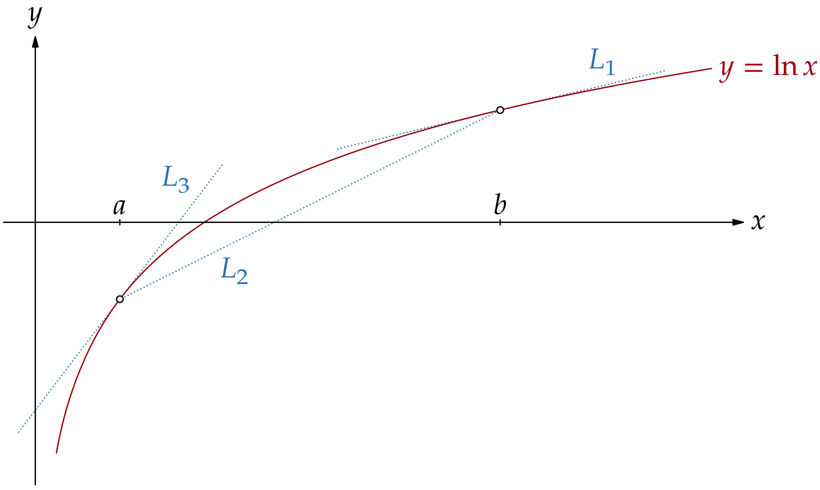
$$1 - x^r = \int_x^1 rt^{r-1}\,dt < r(1-x)$$

— Roger B. Nelsen

Napier's inequality

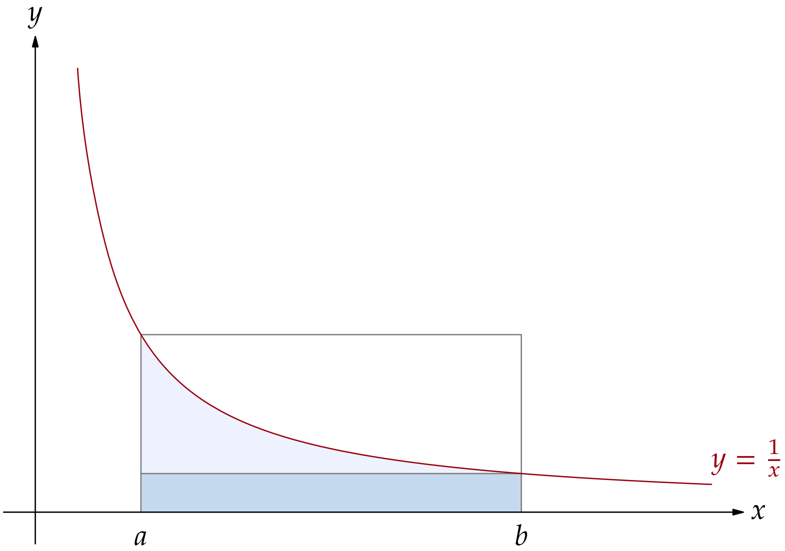
$$b > a > 0 \quad \text{implies} \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

I. First semester calculus



$$\mathrm{m}(L_1) < \mathrm{m}(L_2) < \mathrm{m}(L_3)$$

II. Second semester calculus



$$\tfrac{1}{b}(b-a) < \int_a^b \tfrac{1}{x}\,dx < \tfrac{1}{a}(b-a)$$

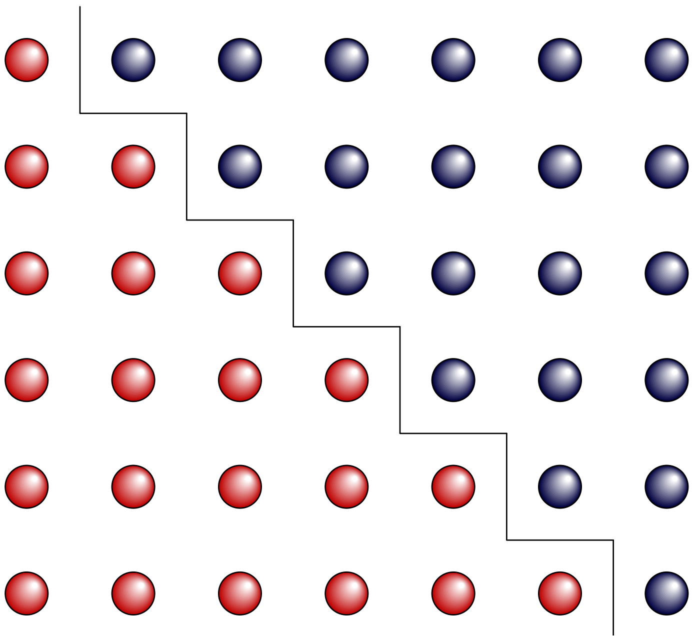
— Roger B. Nelsen

Integer sums

Sums of integers I 66
Sums of integers II 67
Sums of odd integers I 68
Sums of odd integers II 69
Sums of odd integers III 70
Squares and sums of integers I 71
Squares and sums of integers II 72
Arithmetic progressions with sum equal to square of number of terms . . 73
Sums of squares I 74
Sums of squares II 75
Sums of squares IV 76
Sums of squares V 77
Alternating sums of squares 78
Sums of squares of Fibonacci numbers 79
Sums of cubes I 80
Sums of cubes II 81
Sums of cubes III 82
Sums of cubes IV 83
Sums of cubes V 84
Sums of cubes VI 85
Sums of integers and sums of cubes 86
Sums of odd cubes are triangular numbers 87
Sums of fourth powers 88
k-th powers as sums of consecutive odd numbers 89
Sums of triangular numbers I 90
Sums of triangular numbers II 91
Sums of triangular numbers III 92
Sums of oblong numbers I 93
Sums of oblong numbers II 94
Sums of pentagonal numbers 95
On squares of positive integers 96
Consecutive sums of consecutive integers 97

Count the dots **98**
Identities for triangular numbers **99**
A triangular identity **100**

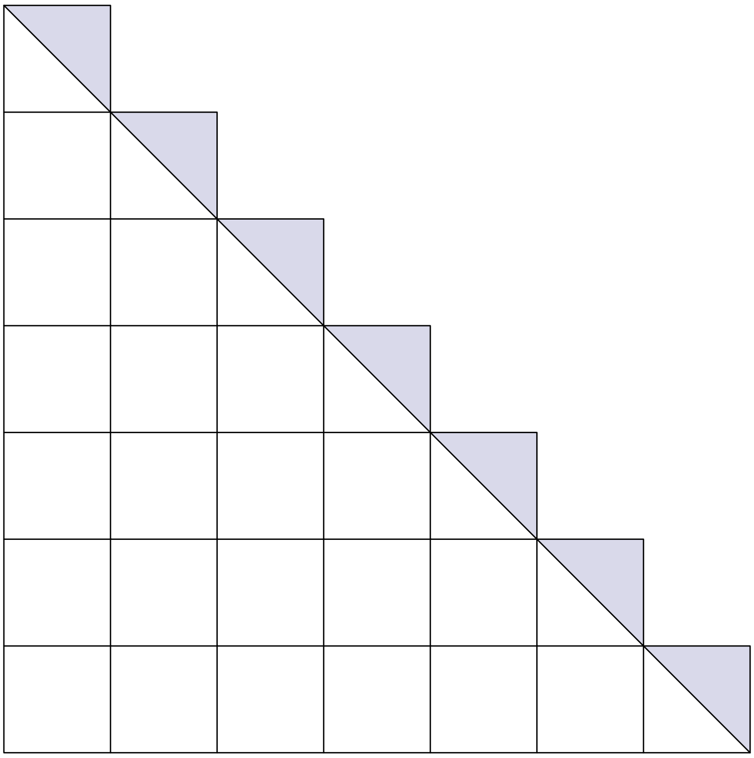
Sums of integers I



$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1)$$

— Ancient Greek

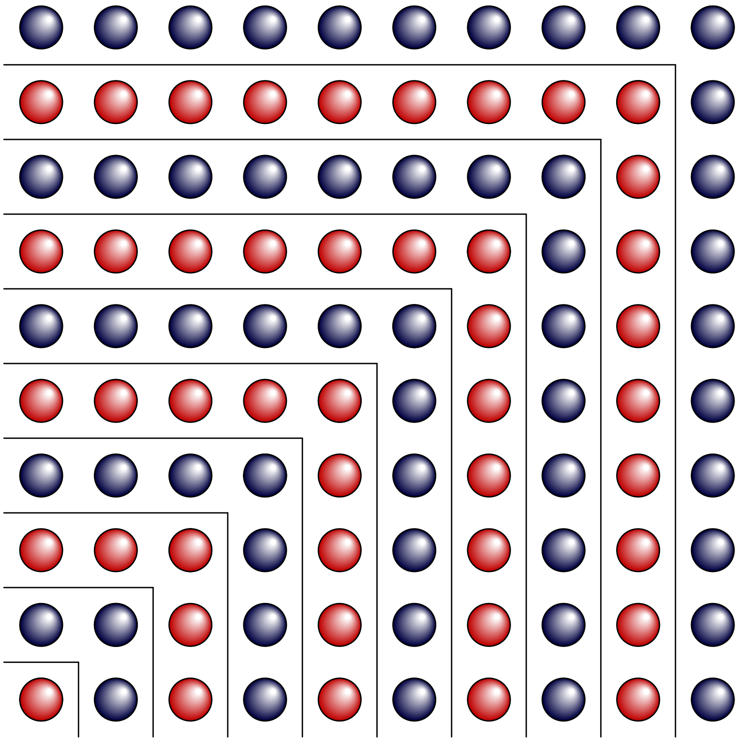
Sums of integers II



$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

— Ian Richards

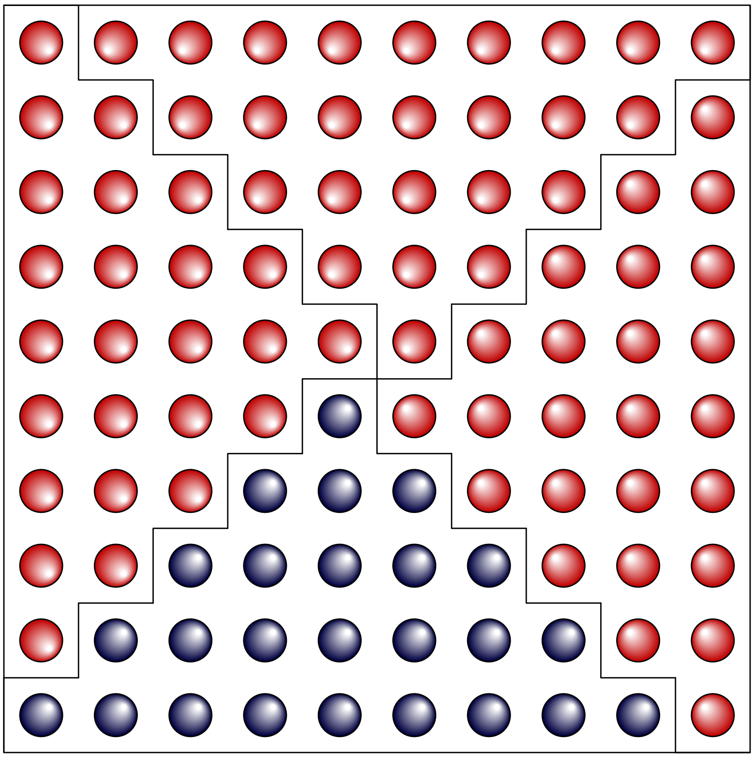
Sums of odd integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

— Nichomachus of Gerasa

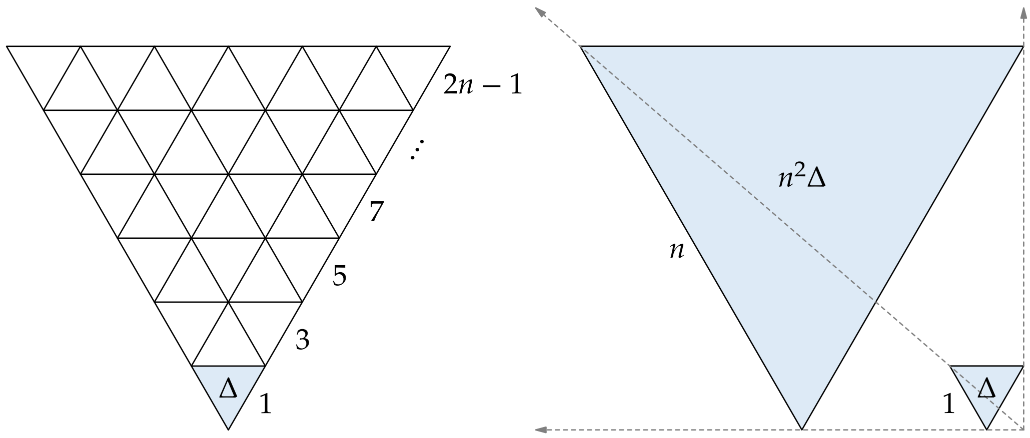
Sums of odd integers II



$$1 + 3 + \cdots + (2n-1) = \frac{1}{4}(2n)^2 = n^2$$

— Roger B. Nelsen

Sums of odd integers III

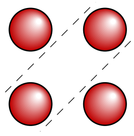


$$\Delta + 3 \cdot \Delta + \cdots + (2n-1) \cdot \Delta = A = n^2 \cdot \Delta$$

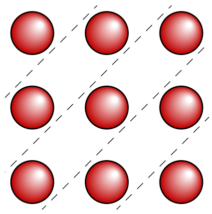
$$\sum_{i=1}^{n} (2i-1) = n^2$$

— Jenő Lehel

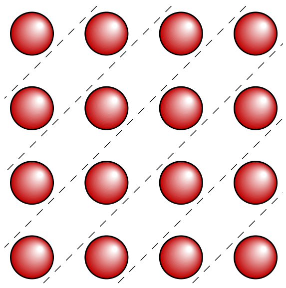
Squares and sums of integers I



$$1 + 2 + 1 = 2^2$$



$$1 + 2 + 3 + 2 + 1 = 3^2$$

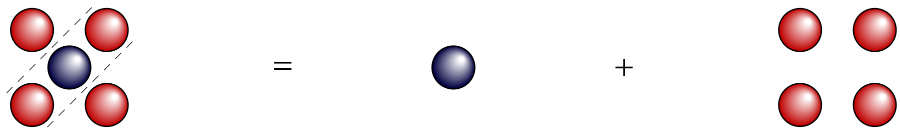


$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

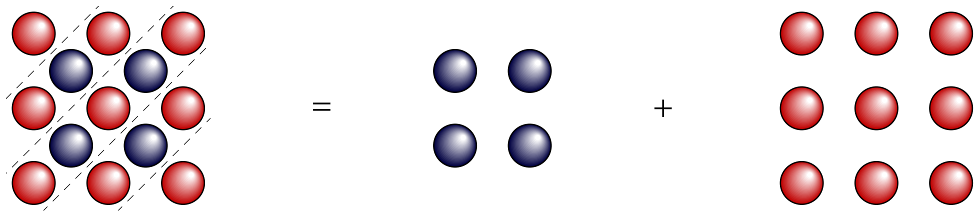
$$1 + 2 + \cdots + (n-1) + n + (n-1) + \cdots + 2 + 1 = n^2$$

— Ancient Greek

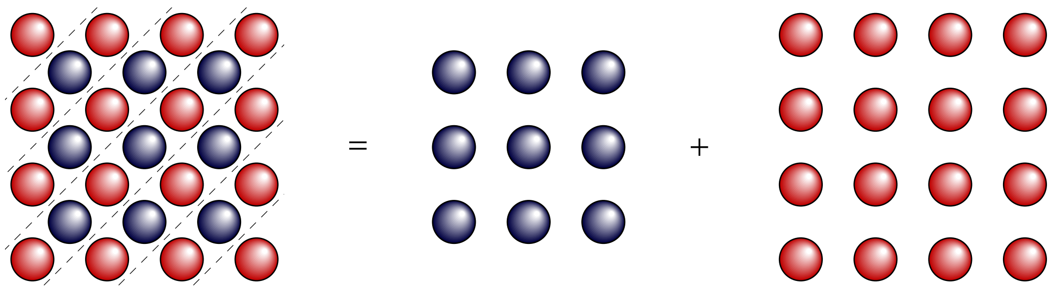
Squares and sums of integers II



$$1 + 3 + 1 = 1^2 + 2^2$$



$$1 + 3 + 5 + 3 + 1 = 2^2 + 3^2$$



$$1 + 3 + 5 + 7 + 5 + 3 + 1 = 3^2 + 4^2$$

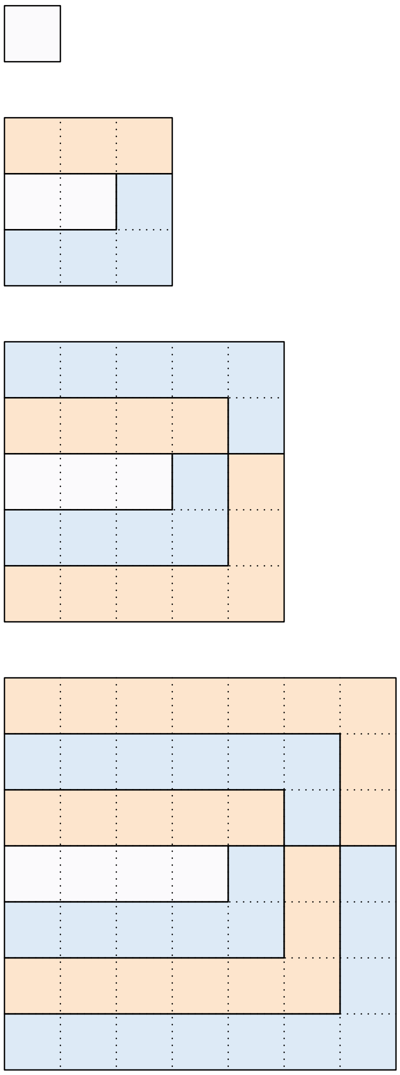
$$\vdots$$

$$1 + 3 + \cdots + (2n - 1) + (2n + 1) + (2n - 1) + \cdots + 3 + 1 = n^2 + (n + 1)^2$$

— Hee Sik Kim

Arithmetic progressions with sum equal to square of number of terms

$$\sum_{k=n}^{3n-2} k = (2n-1)^2;\ n = 1, 2, 3, \ldots$$



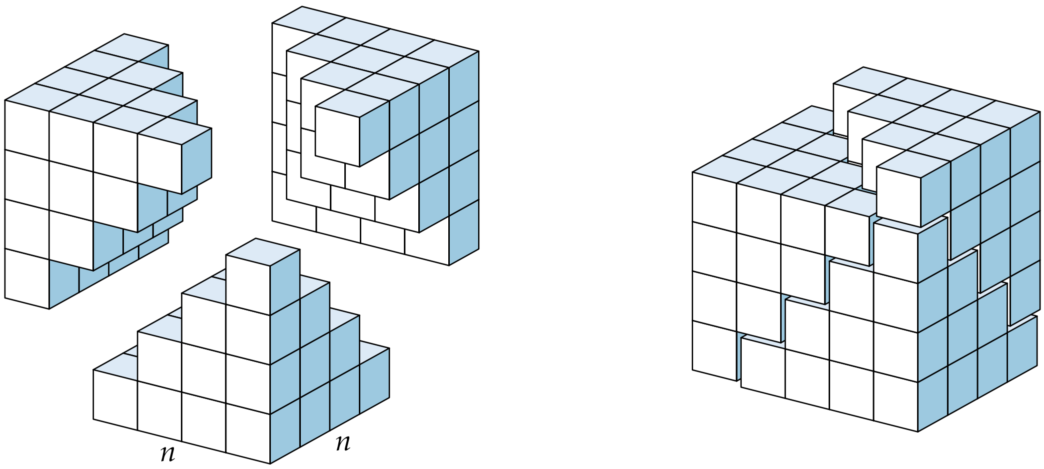
$n = 4$

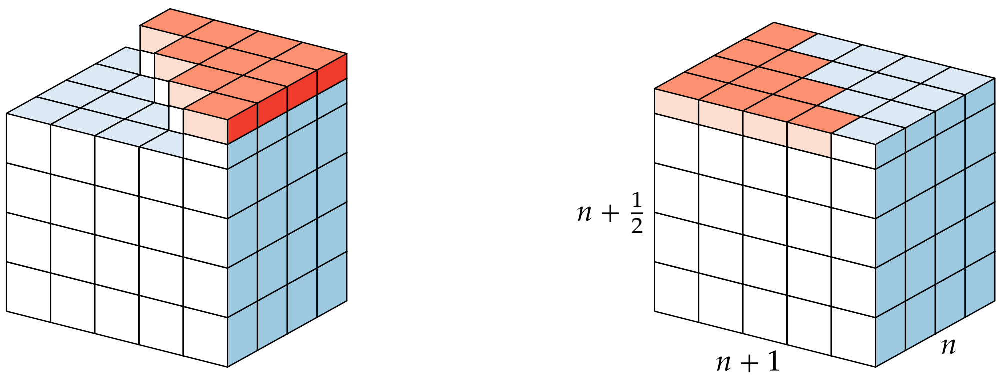
$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$

— James O. Chilaka

Sums of squares I

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3} n\,(n+1)\left(n + \frac{1}{2}\right)$$

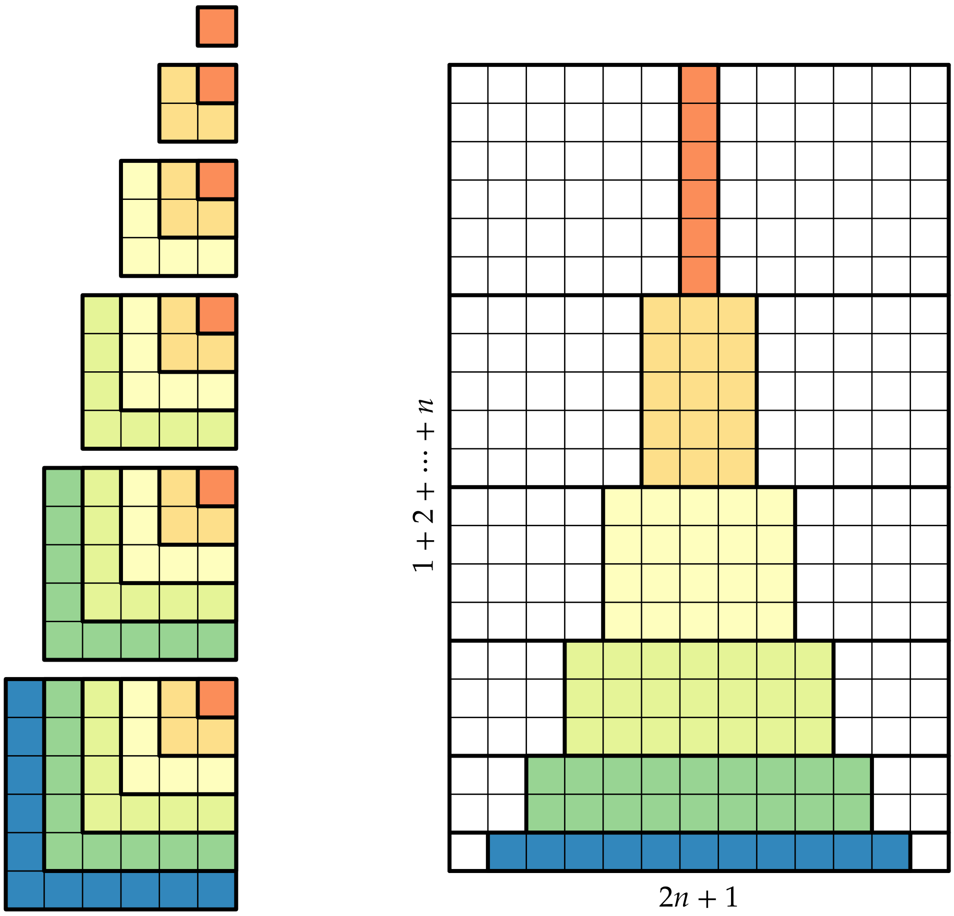




— Man-Keung Siu

Sums of squares II

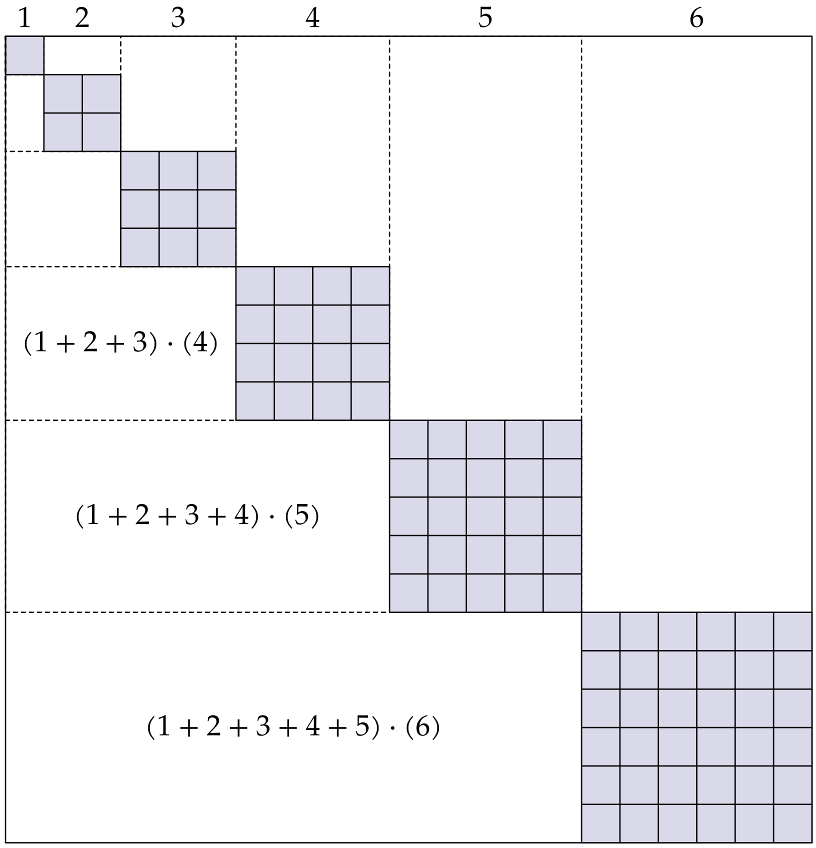
$$3\left(1^2+2^2+\cdots+n^2\right)=(2n+1)\,(1+2+\cdots+n)$$



— Dan Kalman

Sums of squares IV

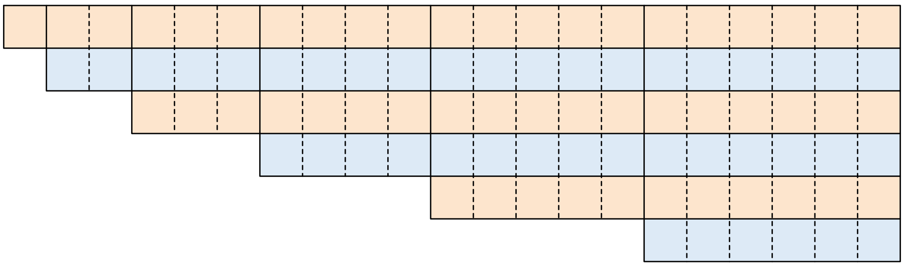
$$\sum_{k=1}^{n} k^2 = \left(\sum_{k=1}^{n} k\right)^2 - 2\sum_{k=1}^{n-1}\left((k+1)\sum_{i=1}^{k} i\right)$$



— James O.Chilaka

Sums of squares V

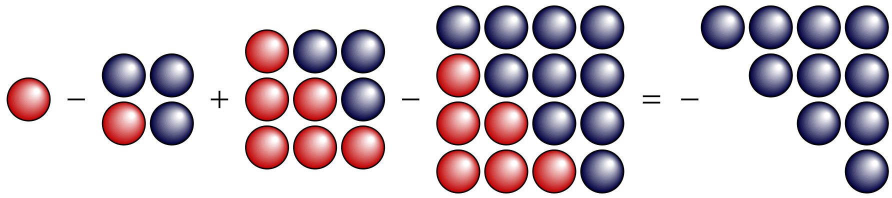
$$\sum_{i=1}^{n}\sum_{j=i}^{n} j = \sum_{i=1}^{n} i^2$$



— Pi-Chun Chuang

Alternating sums of squares

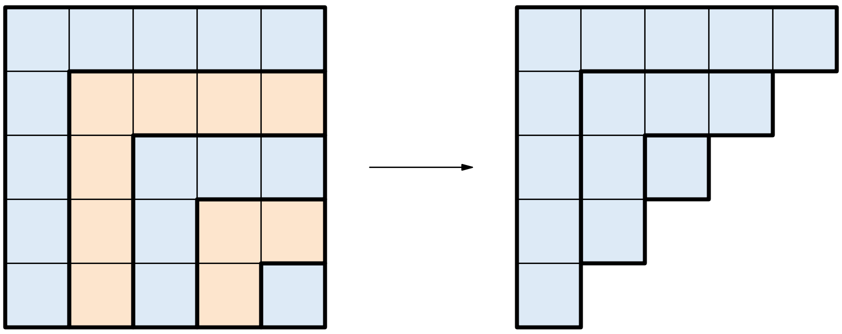
I.



$$\sum_{k=1}^{n} (-1)^{k+1} k^2 = (-1)^{n+1} T_n = (-1)^{n+1} \frac{n(n+1)}{2}$$

— Dave Logothetti

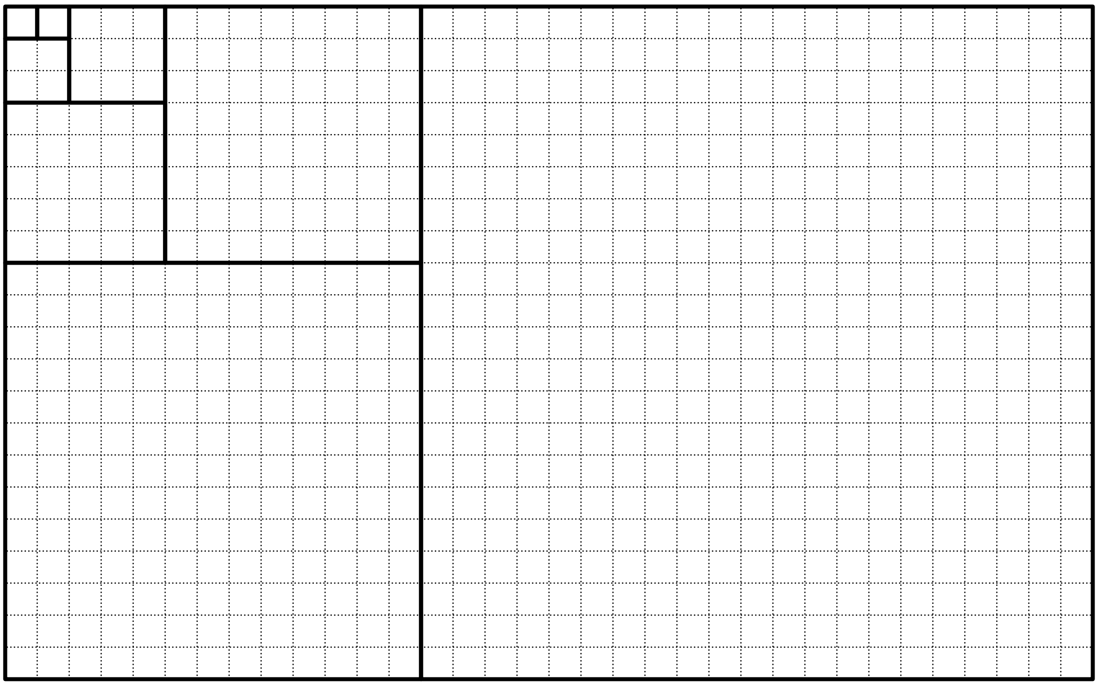
II.



$$n^2 - (n-1)^2 + \cdots + (-1)^{n-1} (1)^2 = \sum_{k=0}^{n} (-1)^k (n-k)^2 = \frac{n(n+1)}{2}$$

— Steven L. Snover

Sums of squares of Fibonacci numbers

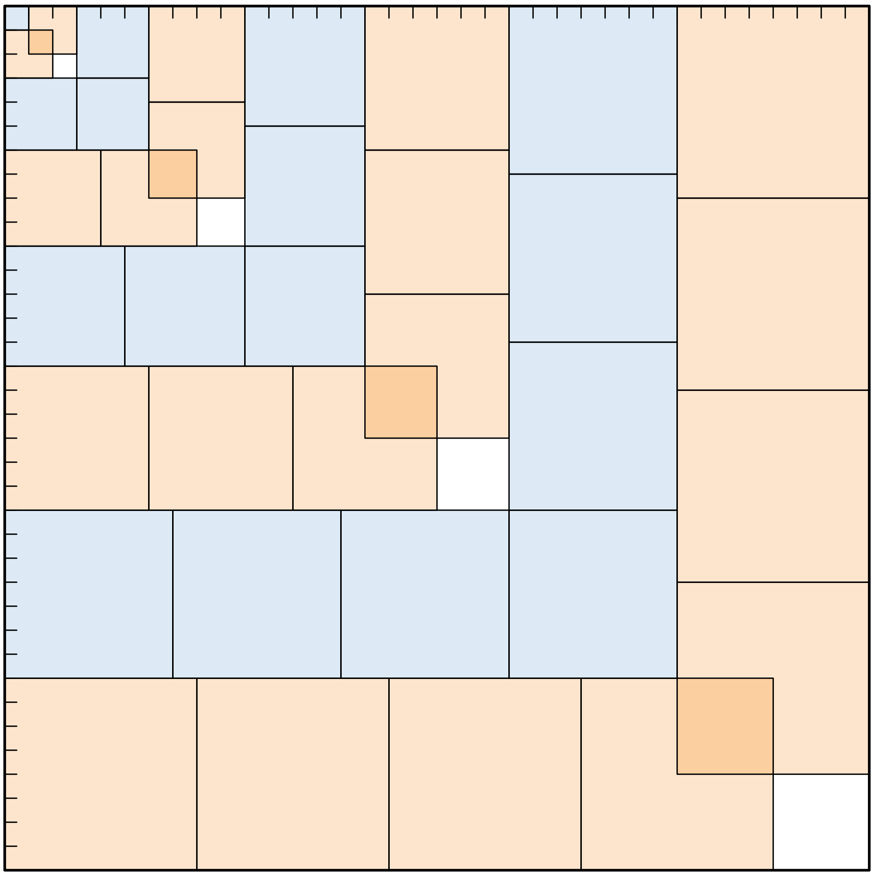


$$F_1 = F_2 = 1; F_{n+2} = F_{n+1} + F_n \quad \text{hence} \quad F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

— Alfred Brousseau

Sums of cubes I

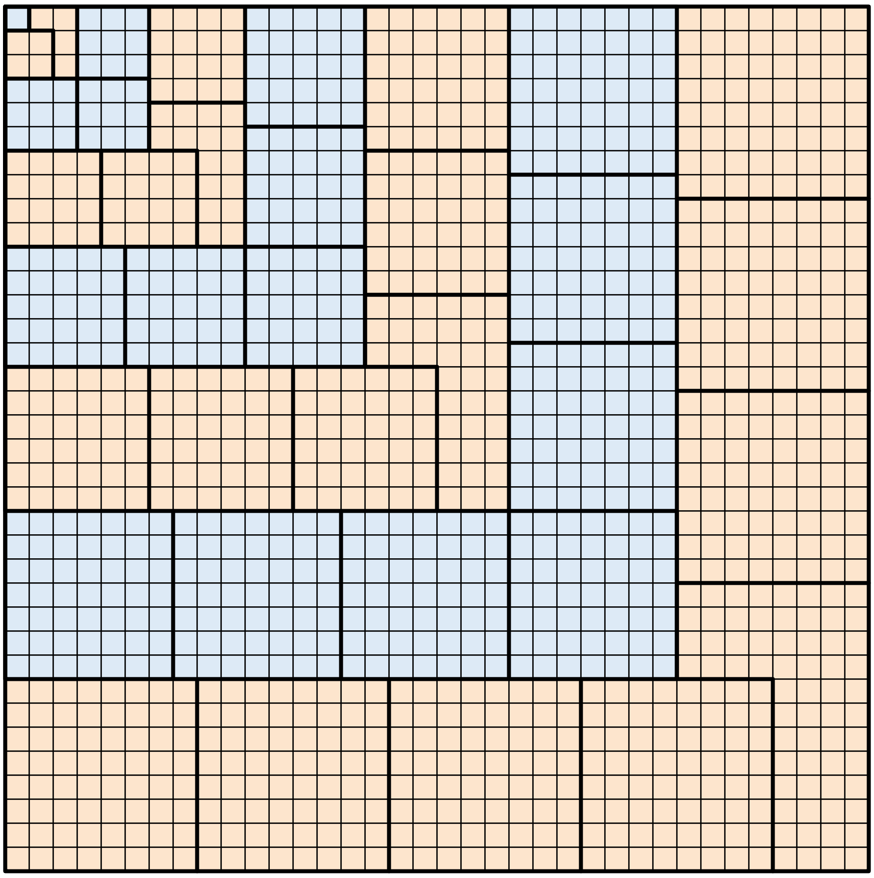
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— Solomon W.Golomb

Sums of cubes II

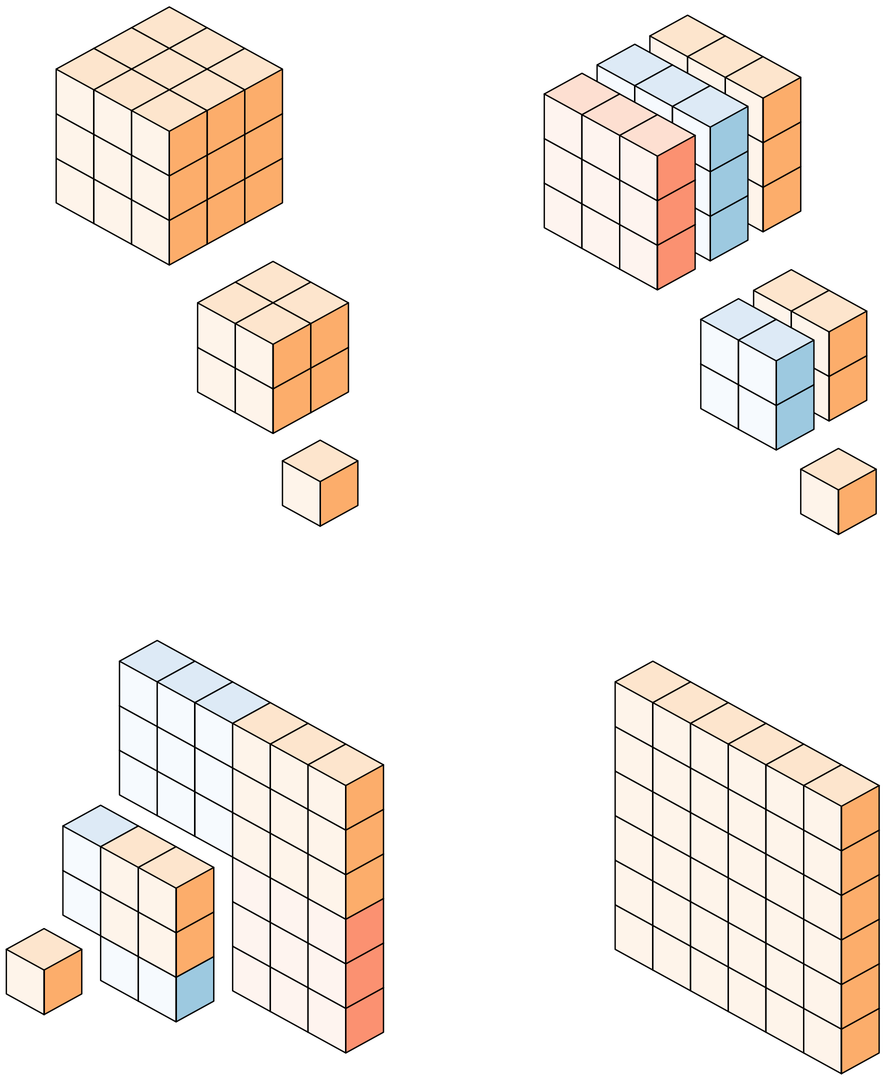
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— J. Barry Love

Sums of cubes III

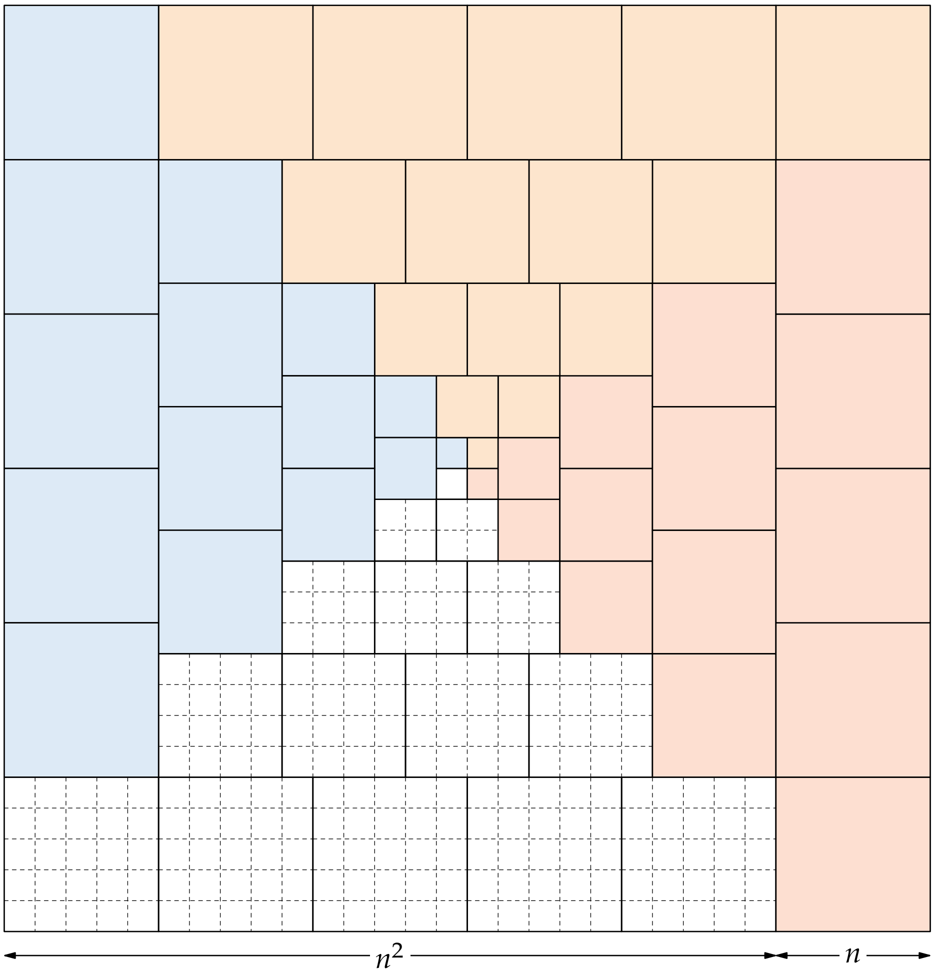
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



— Alan L. Fry

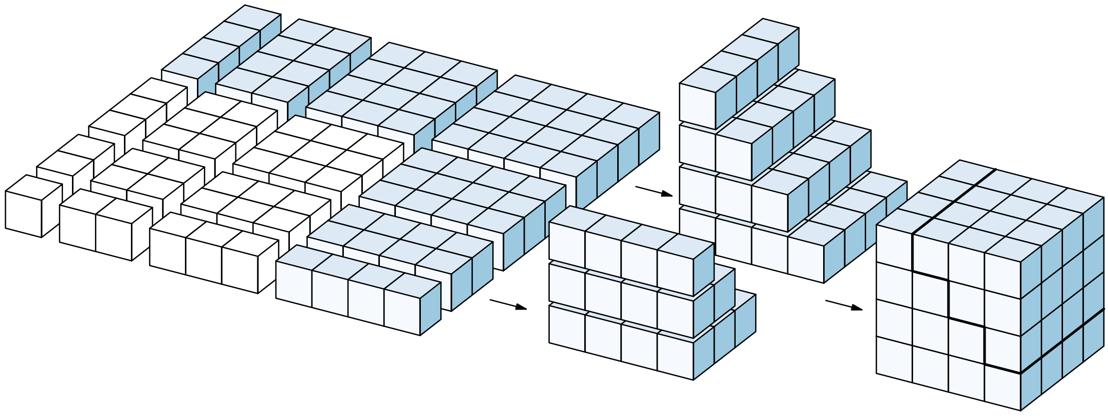
Sums of cubes IV

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}\left(n(n+1)\right)^2$$

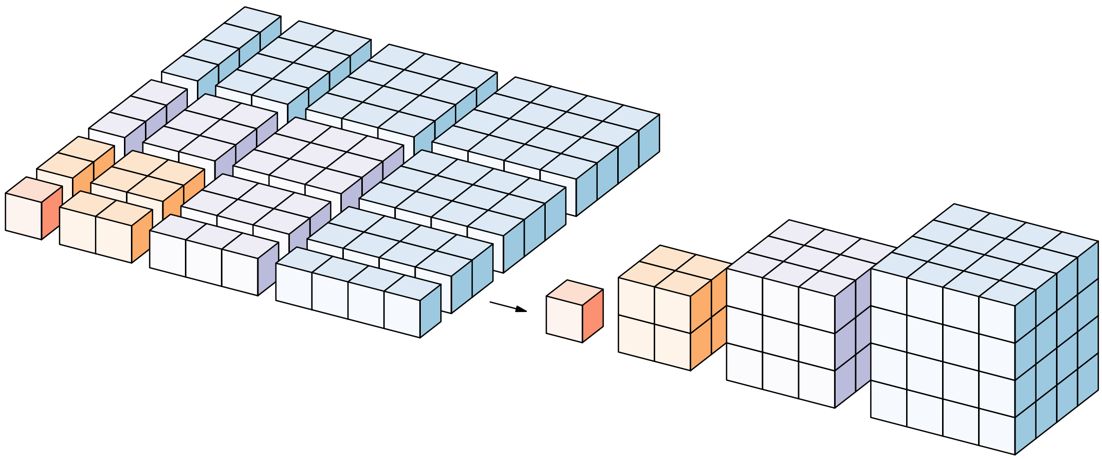


— Antonella Cupillari

Sums of cubes V



$$t_n = 1 + 2 + \cdots + n \quad \Rightarrow \quad t_n^2 - t_{n-1}^2 = n^3$$



$$t_n^2 = (1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$

— Roger Nelsen

Sums of cubes VI

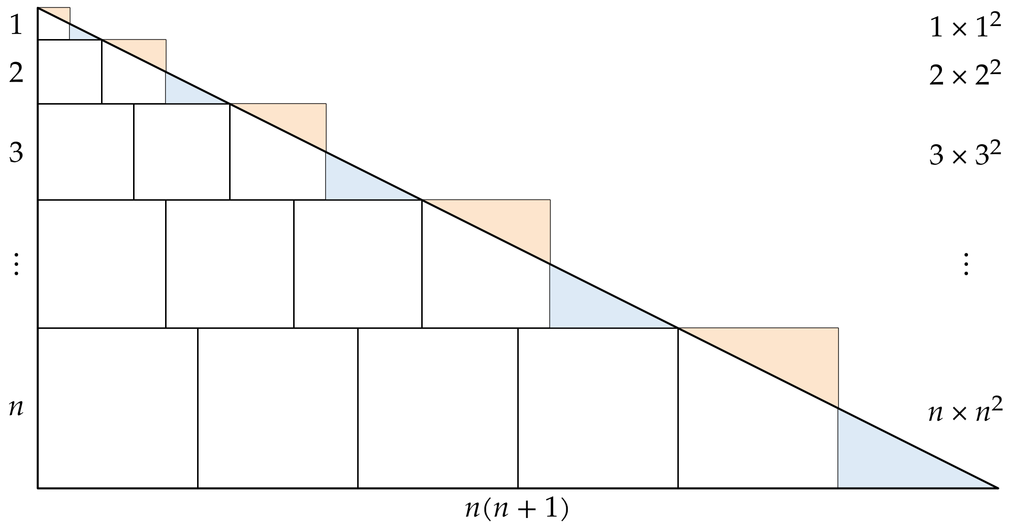
$$
\begin{array}{rccccc}
 & 1 & 2 & 3 & \cdots & n \\
+ & 2 & 4 & 6 & \cdots & 2n \\
+ & 3 & 6 & 9 & \cdots & 3n \\
+ & \vdots & \vdots & \vdots & \ddots & \vdots \\
+ & n & 2n & 3n & \cdots & n^2 \\
= & \multicolumn{5}{l}{\sum_{i=1}^{n} i + 2\sum_{i=1}^{n} i + \cdots + n\sum_{i=1}^{n} i} \\
= & \multicolumn{5}{l}{\left(\sum_{i=1}^{n} i\right)^2}
\end{array}
\qquad
\begin{array}{rccccc}
 & 1 & 2 & 3 & \cdots & n \\
+ & 2 & 4 & 6 & \cdots & 2n \\
+ & 3 & 6 & 9 & \cdots & 3n \\
+ & \vdots & \vdots & \vdots & \ddots & \vdots \\
+ & n & 2n & 3n & \cdots & n^2 \\
= & \multicolumn{5}{l}{1 + 2 \cdot 2^2 + 3 \cdot 3^2 + \cdots + n \cdot n^2} \\
= & \multicolumn{5}{l}{\sum_{i=1}^{n} i^3}
\end{array}
$$

— Farhood Pouryoussefi

Sums of integers and sums of cubes

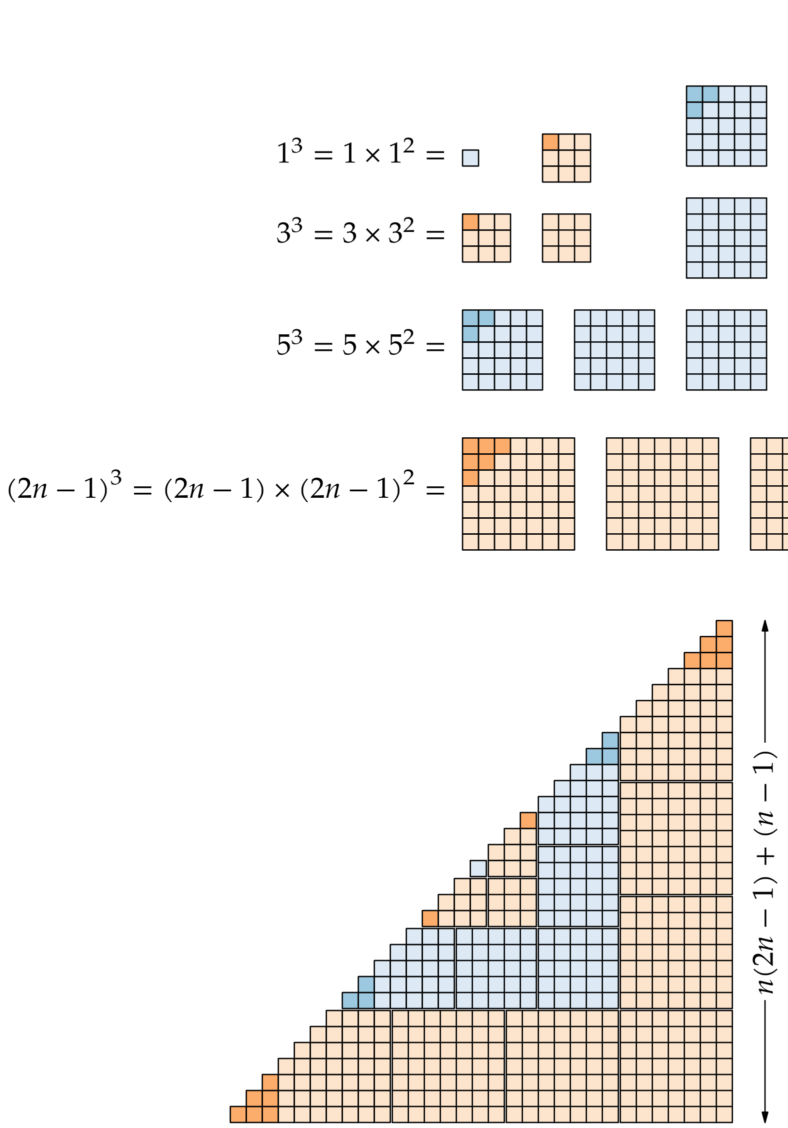
$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1)$$

$$1^3 + 2^3 + \cdots + n^3 = \left(\tfrac{1}{2}n(n+1)\right)^2$$



— Georg Schrage

Sums of odd cubes are triangular numbers

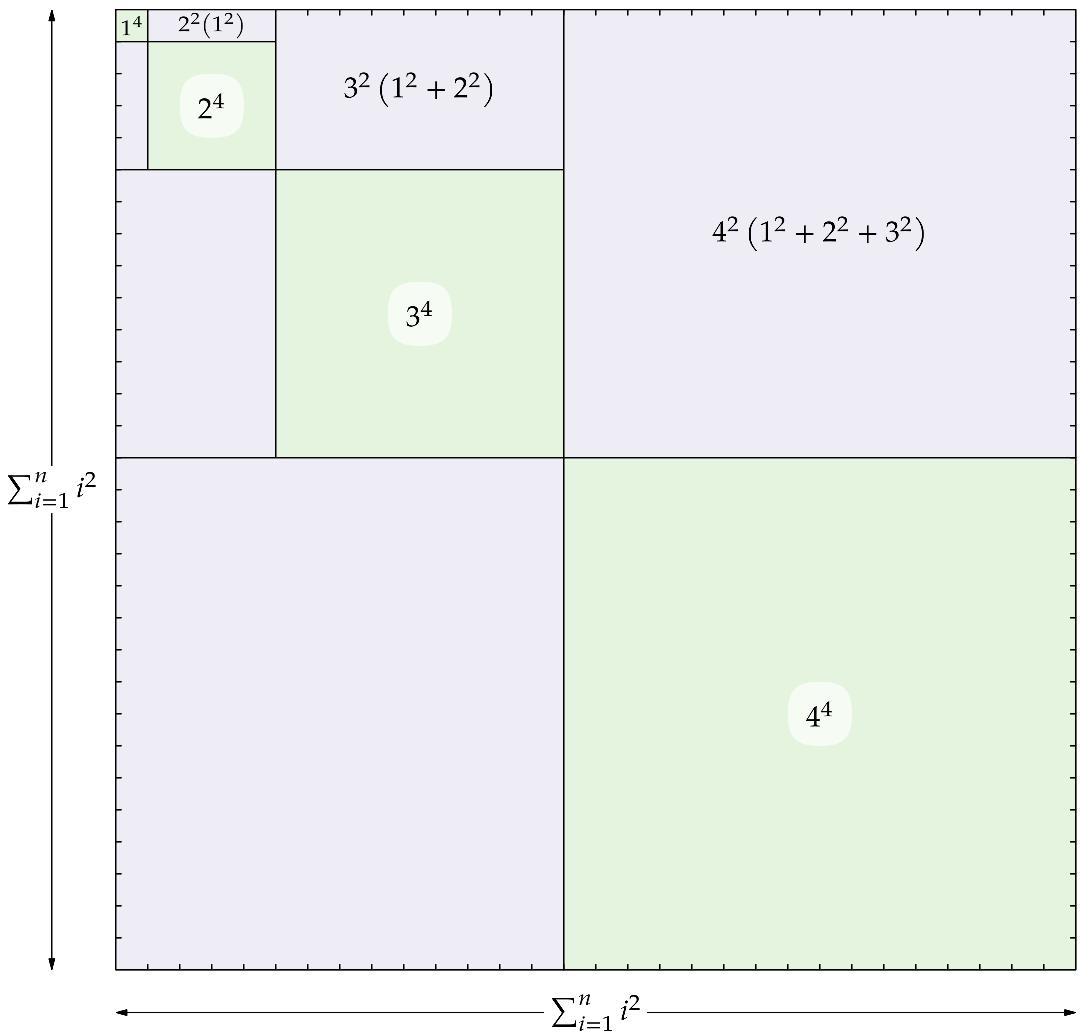


$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = 1 + 2 + 3 + \cdots + \left(2n^2 - 1\right) = n^2\left(2n^2 - 1\right)$$

— Monte J. Zerger

Sums of fourth powers

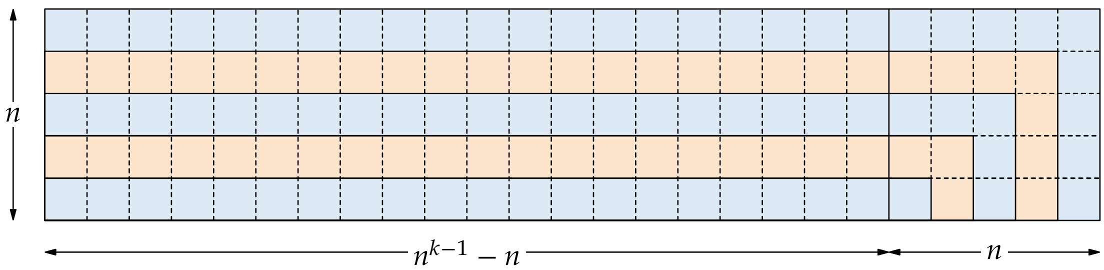
$$\sum_{i=1}^{n} i^4 = \left(\sum_{i=1}^{n} i^2\right)^2 - 2\left(\sum_{k=2}^{n}\left(k^2 \sum_{i=1}^{k-1} i^2\right)\right)$$



— Elizabeth M. Markham

k-th powers as sums of consecutive odd numbers

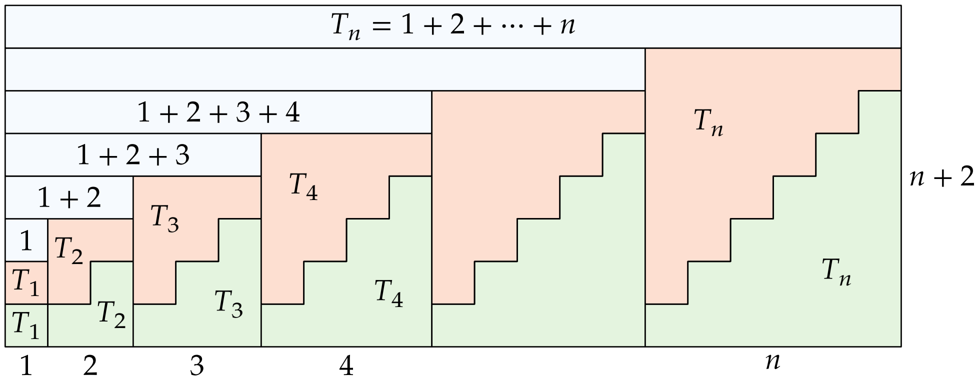
$$n^k = \left(n^{k-1} - n + 1\right) + \left(n^{k-1} - n + 3\right) + \cdots + \left(n^{k-1} - n + 2n - 1\right) \text{ for } k = 2, 3, \ldots$$



— N. Gopalakrishnan Nair

Sums of triangular numbers I

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



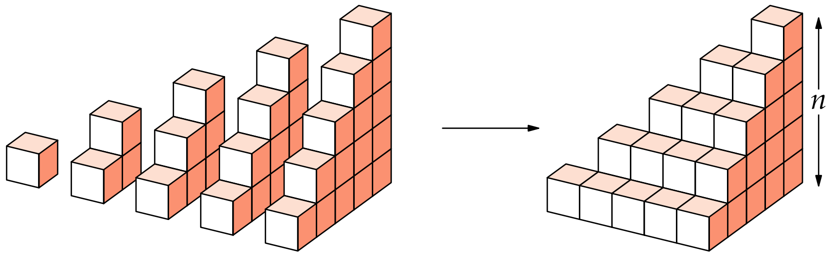
$$3\,(T_1 + T_2 + \cdots + T_n) = (n+2) \cdot T_n$$

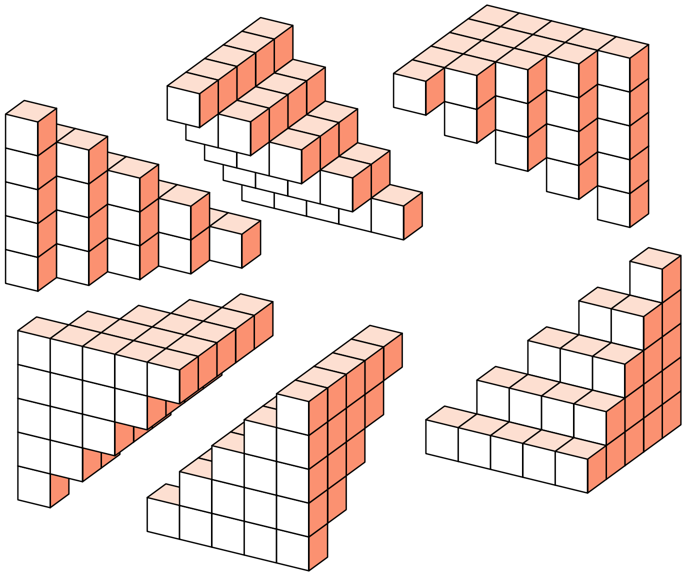
$$T_1 + T_2 + \cdots + T_n = \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

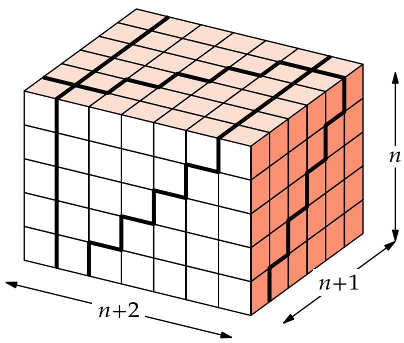
— Monte J. Zerger

Sums of triangular numbers II

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$







— Roger B. Nelsen

Sums of triangular numbers III

$$T_n = 1 + 2 + \cdots + n \quad \text{implies} \quad T_1 + T_2 + \cdots + T_n = \tfrac{1}{6} n(n+1)(n+2)$$

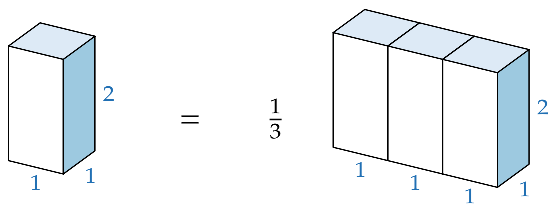
$$\begin{array}{c}
1 \\
1 \quad 2 \\
1 \quad 2 \quad 3 \\
1 \quad 2 \quad 3 \quad \cdot \\
1 \quad 2 \quad 3 \quad \cdot \quad \cdot \\
1 \quad 2 \quad 3 \quad \cdot \quad \cdot \quad n{-}1 \\
1 \quad 2 \quad 3 \quad \cdot \quad \cdot \quad n{-}1 \quad n
\end{array}
\;+\;
\begin{array}{c}
n \\
n{-}1 \quad n{-}1 \\
\cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
3 \quad 3 \quad 3 \quad 3 \quad 3 \\
2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}
\;+\;
\begin{array}{c}
1 \\
2 \quad 1 \\
3 \quad 2 \quad 1 \\
\cdot \quad 3 \quad 2 \quad 1 \\
\cdot \quad \cdot \quad 3 \quad 2 \quad 1 \\
n{-}1 \quad \cdot \quad \cdot \quad 3 \quad 2 \quad 1 \\
n \quad n{-}1 \quad \cdot \quad \cdot \quad 3 \quad 2 \quad 1
\end{array}$$

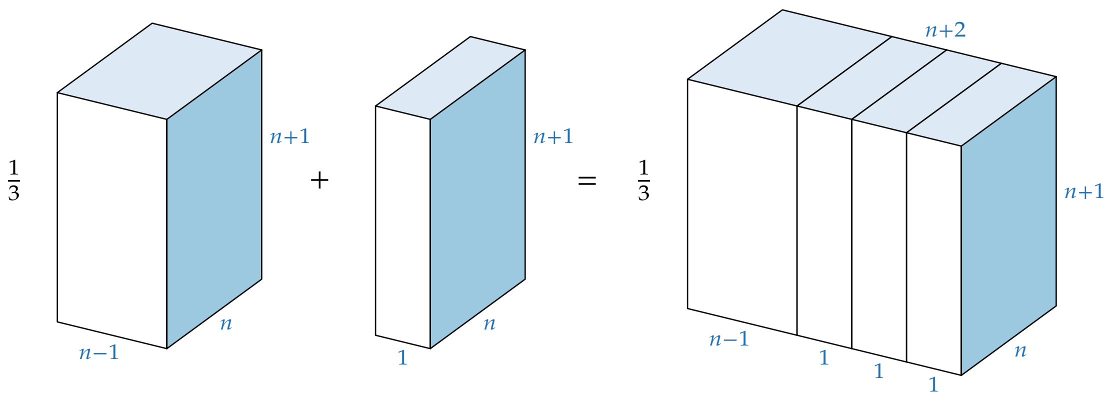
$$=\;
\begin{array}{c}
n{-}2 \\
n{-}2 \quad n{-}2 \\
n{-}2 \quad n{-}2 \quad n{-}2 \\
n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \\
n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \\
n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \\
n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2 \quad n{-}2
\end{array}$$

$$3\left(T_1 + T_2 + \cdots + T_n\right) = T_n \cdot (n+2)$$

Sums of oblong numbers I

$$(1\times 2) + (2\times 3) + (3\times 4) + \cdots + (n-1)n = \tfrac{1}{3}(n-1)n(n+1)$$

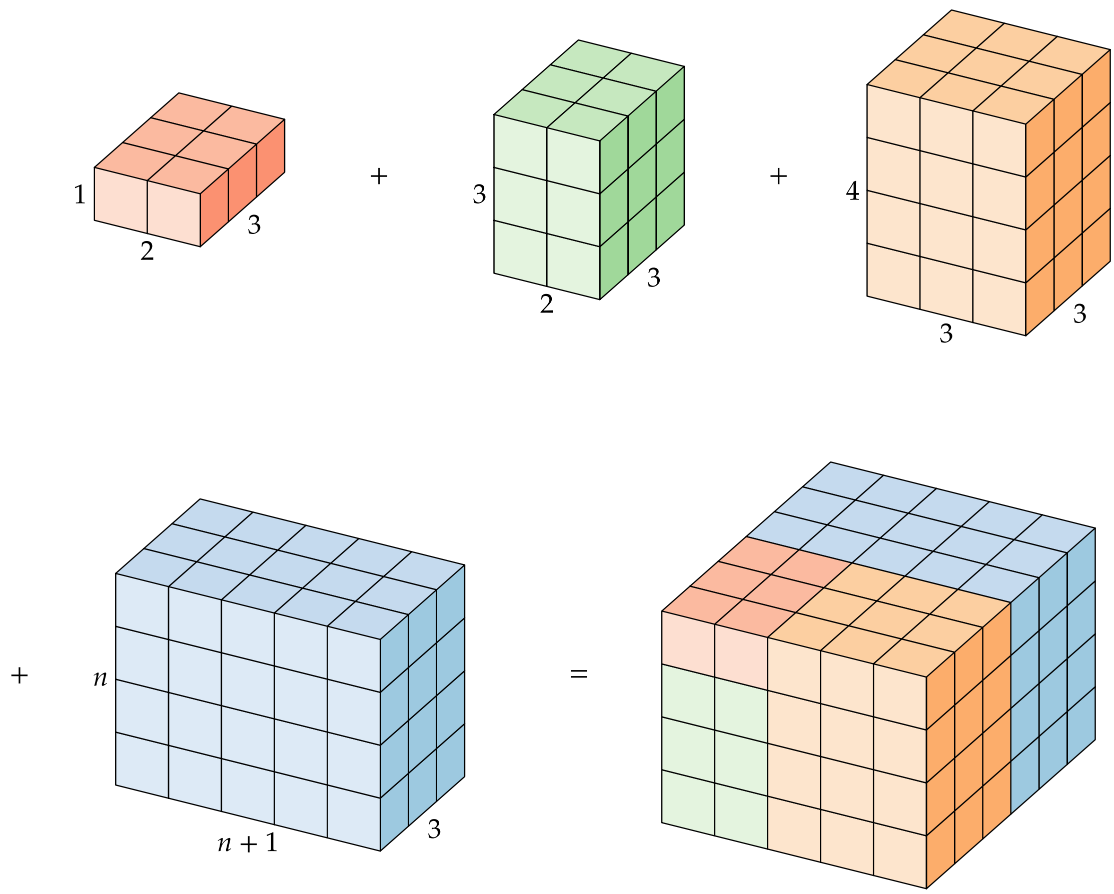




— T. C. Wu

Sums of oblong numbers II

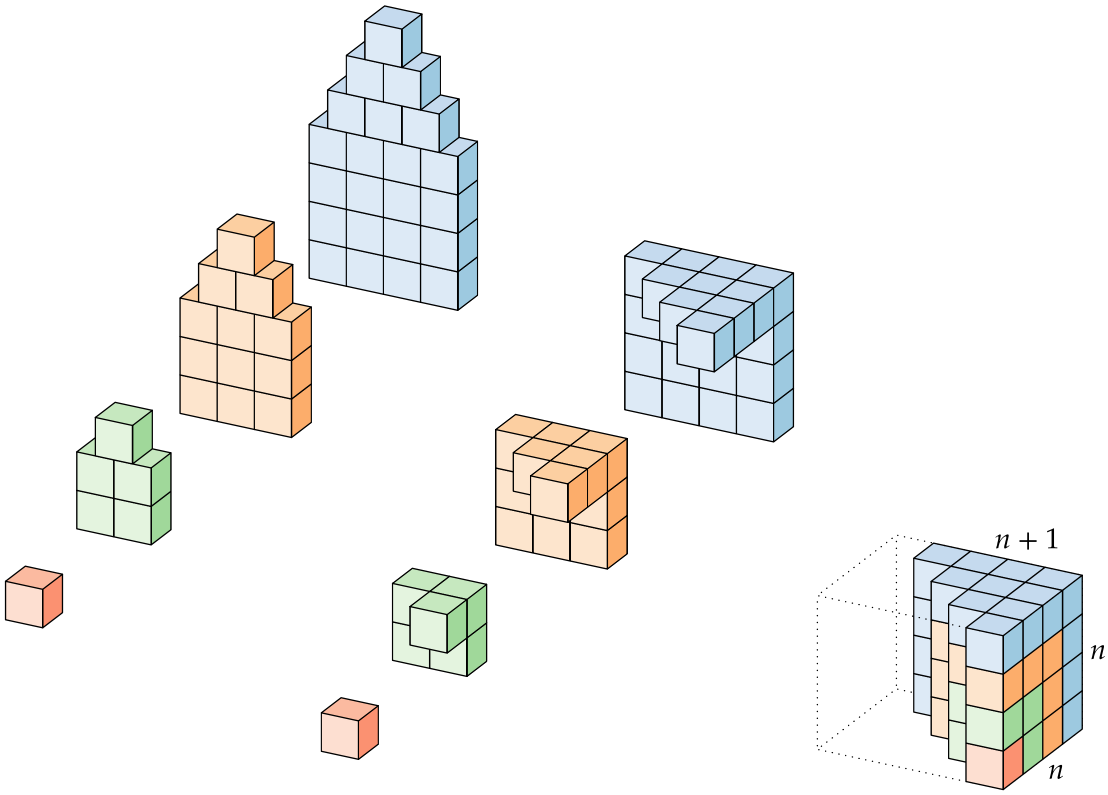
$$3(1\times 2+2\times 3+3\times 4+\cdots+n(n+1)) = n(n+1)(n+2)$$



— Sidney H. Kung

Sums of pentagonal numbers

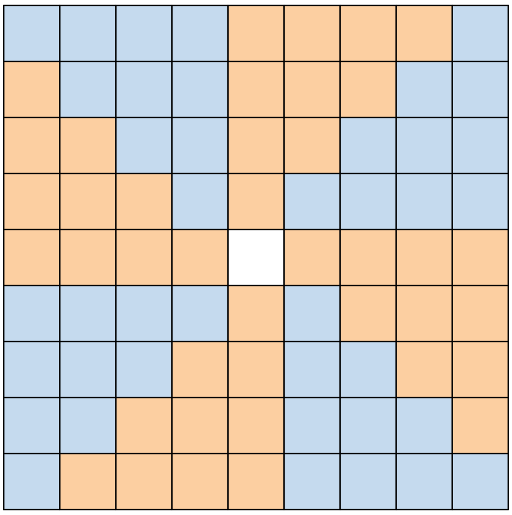
$$\frac{1\cdot 2}{2}+\frac{2\cdot 5}{2}+\frac{3\cdot 8}{2}+\cdots+\frac{n(3n-1)}{2}=\frac{n^2(n+1)}{2}$$



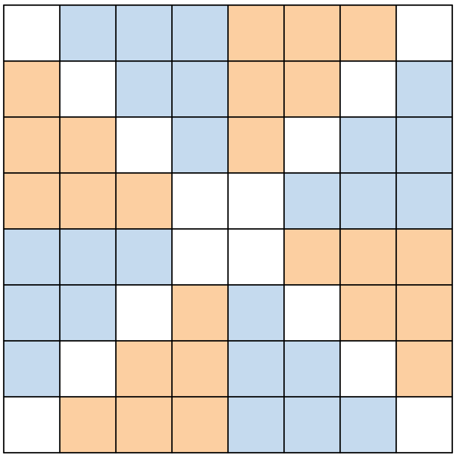
— William A. Miller

On squares of positive integers

$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$



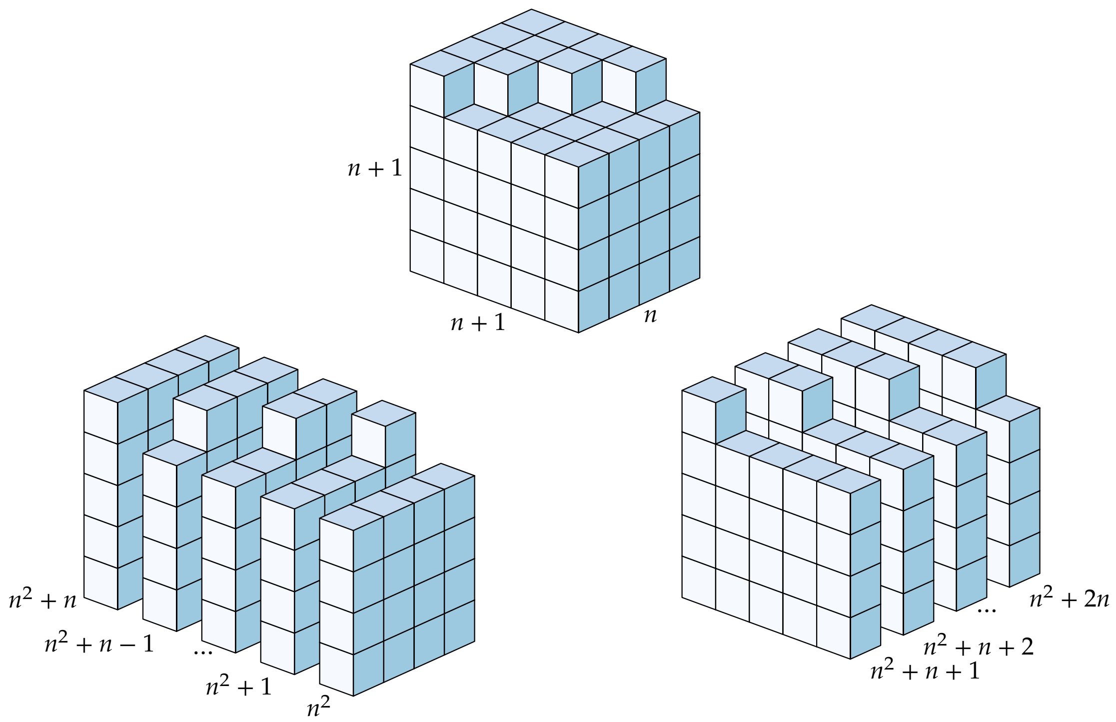
$$(2n+1)^2 = 8T_n + 1$$



$$(2n)^2 = 8T_{n-1} + 4n$$

— Edwin G. Landauer

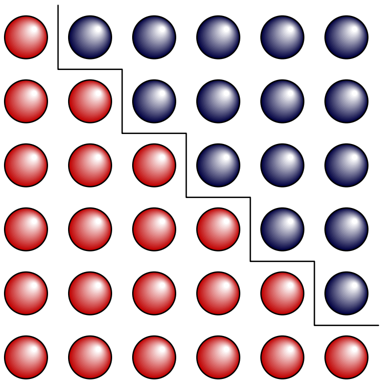
Consecutive sums of consecutive integers



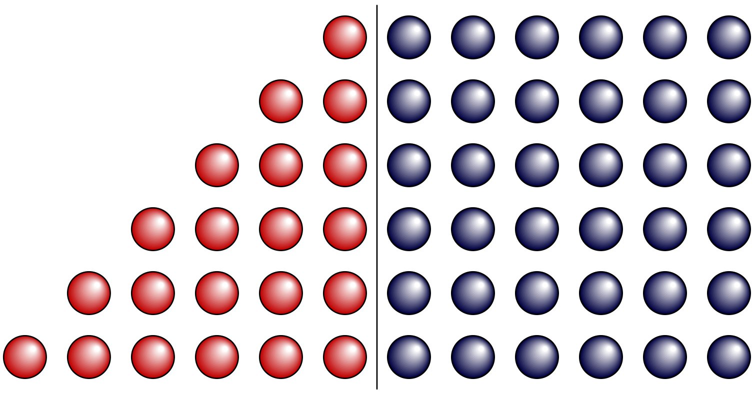
$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$
$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$
$$\vdots$$
$$n^2 + (n^2 + 1) + \cdots + (n^2 + n) = (n^2 + n + 1) + \cdots + (n^2 + 2n)$$

— Roger B. Nelsen

Count the dots



$$\sum_{k=1}^{n} k + \sum_{k=1}^{n-1} k = n^2$$

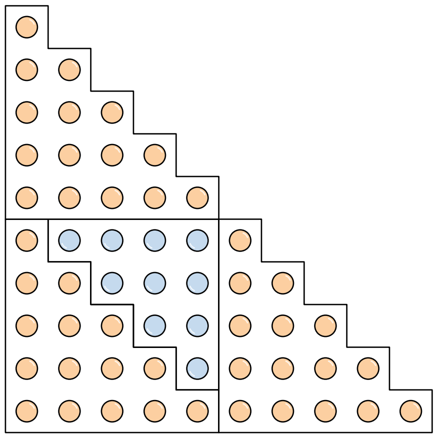


$$\sum_{k=1}^{n} k + n^2 = \sum_{k=n+1}^{2n} k$$

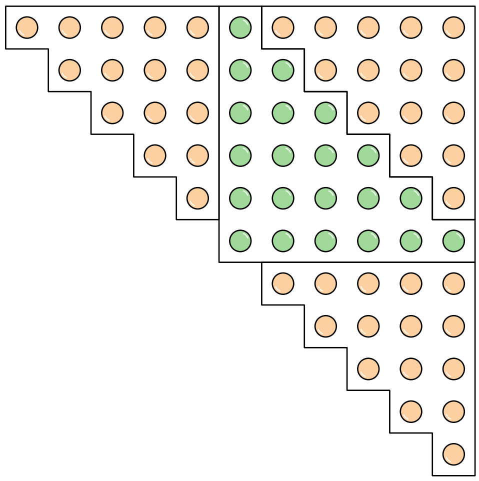
— Warren Page

Identities for triangular numbers

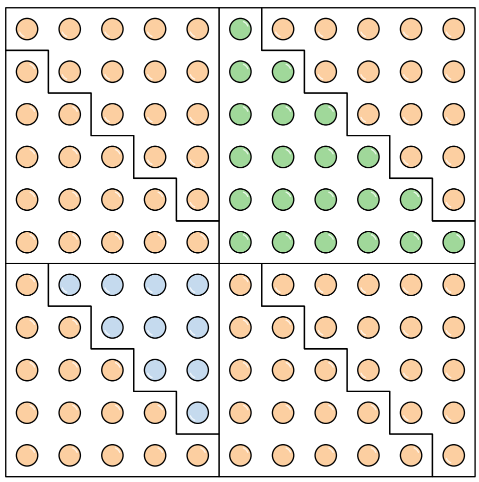
$$T_n = 1 + 2 + \cdots + n \Longrightarrow$$



$3T_n + T_{n-1} = T_{2n}$

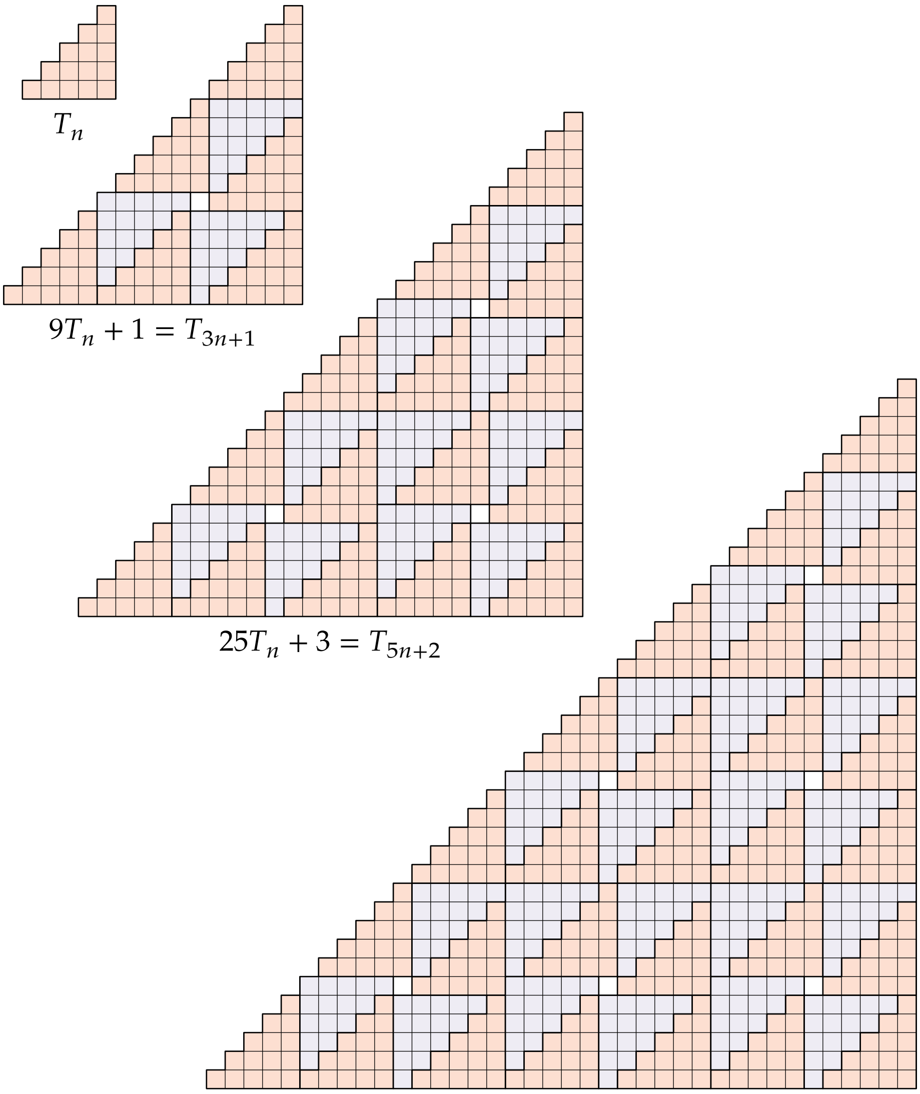


$3T_n + T_{n+1} = T_{2n+1}$



$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$

A triangular identity



T_n

$9T_n + 1 = T_{3n+1}$

$25T_n + 3 = T_{5n+2}$

$49T_n + 6 = T_{7n+3}$

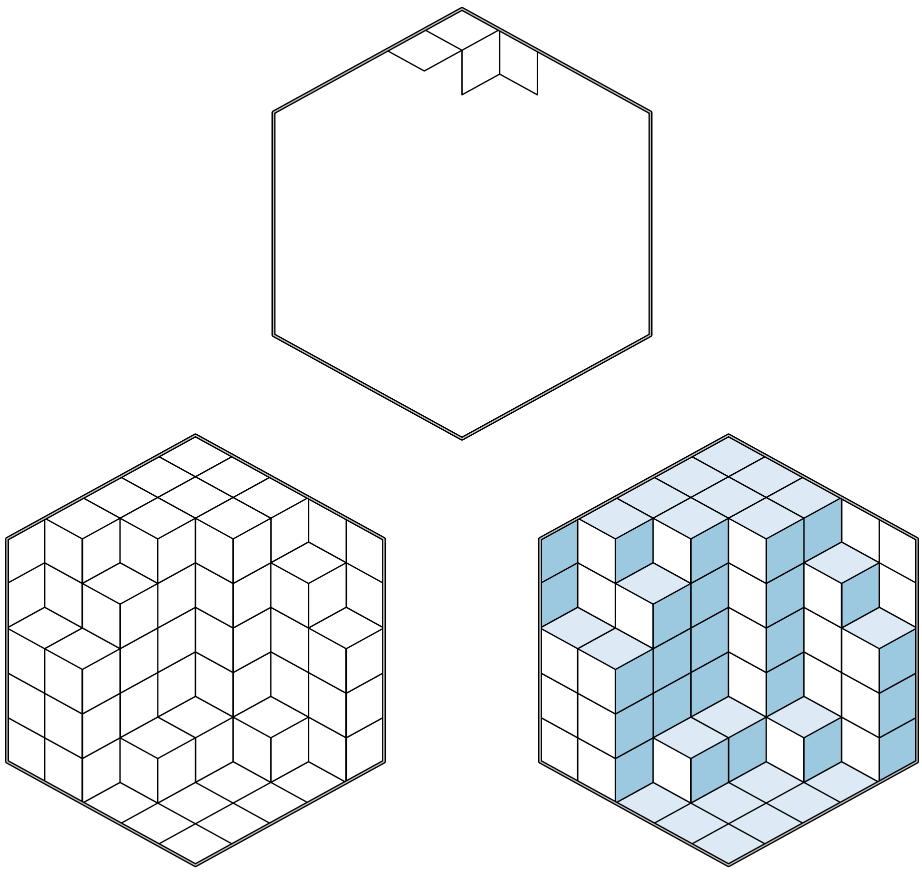
$$(2k+1)^2 T_n + T_k = T_{(2k+1)n+k}$$

— Roger B. Nelsen

Miscellaneous

The problem of the calissons **102**

The problem of the calissons



— Guy David and Carlos Tomei