Excursions in METAPOST

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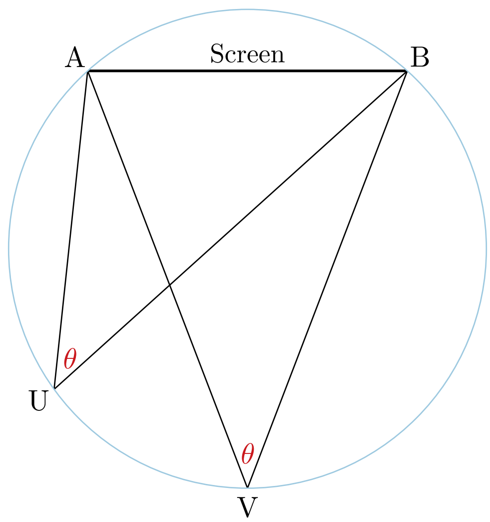
This example document includes geometric illustrations inspired by *Excursions in Geometry*, C. Stanley Ogilvy, OUP 1968. The illustrations are presented roughly in the same order as the book, with notes about how you can use METAPOST to produce similar. The section heading also approximately follow the book. You might like to read the PDF of this document side by side with the source code, so that you can see how each illustration is done. Each illustration is included as in-line METAPOST code, there are no external graphics files used.

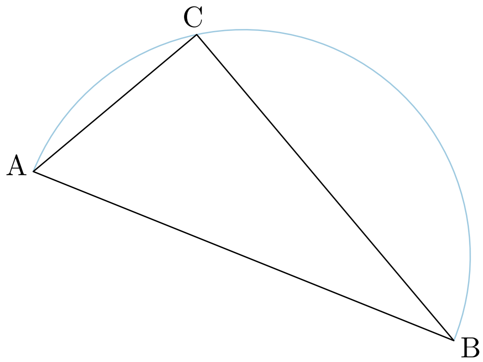
1 A bit of background

Ogilvie starts with a review of some circle theorems. In fact most of the book is about circles in one way or another.

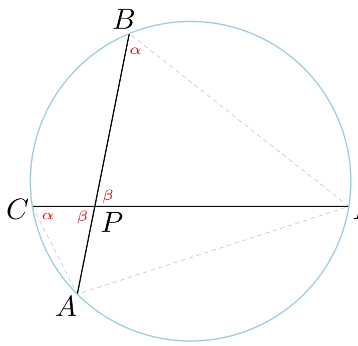
In this first diagram, you are given the width AB of the screen and the ideal viewing angle θ. The METAPOST code works out the rest from that, including a useful routine for a circle through three points.



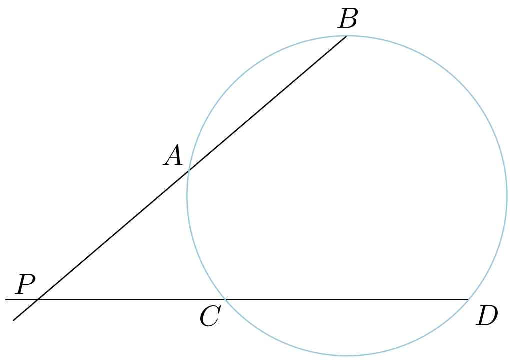
Here θ is half the angle measured by the intercepted arc, which gives us the useful corollary that any angle inscribed in a semicircle is a right angle, and conversely that if you can show that some angle ACB is a right angle, then the semicircle drawn with AB as a diameter must pass through C.



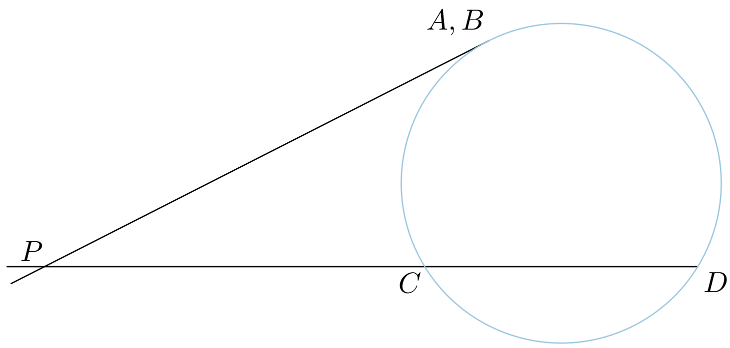
A basic theorem: If two chords intersect, the product of the lengths of the segments of the one equals the product of the lengths of the segments of the other. In all three cases below you have $PA/PD = PC/PB$ by similar triangles, hence $PA \cdot PB = PC \cdot PD$.



I.



II.



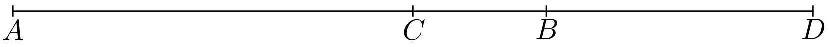
III.

This example also shows how to arrange sub-figures into one.

2 Harmonic division and Apollonian circles

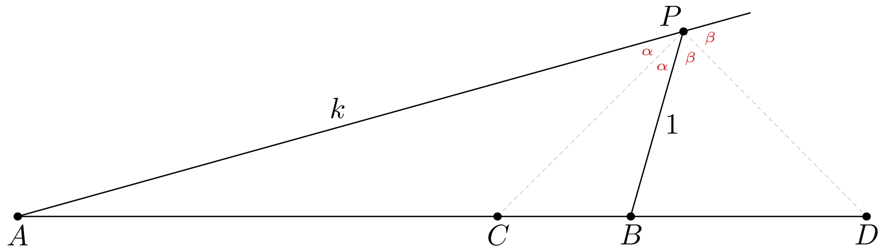
Can we find C and D on the line AB so that $AC/CB = AD/BD$?

Yes:

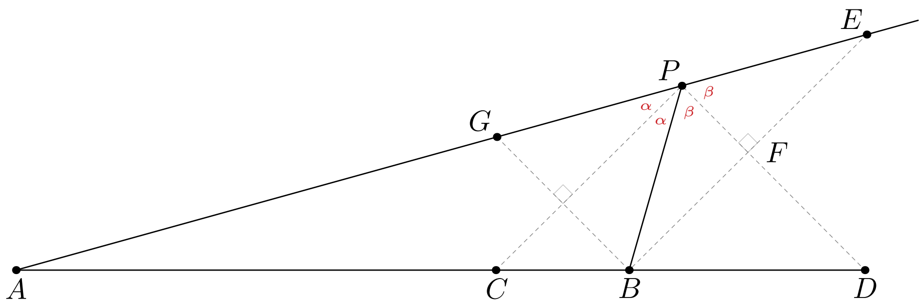


Theorem. The bisector of any angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

So given P, with $AP = k$ and $BP = 1$:



We have $AC/CB = AP/BP = k$ from the interior angles and $AD/BD = AP/BP = k$ from the exterior pair. The proof looks like this:



Since the lines PC and PD are bisectors, then we have $2\alpha + 2\beta = 180°$, hence $\alpha + \beta$ is a right angle, so if we draw BE parallel to PC it will be perpendicular to PD, and hence the two triangles PFE and PFB are congruent, so that $PE = PB$. But the parallel lines cut off proportional segments, so that

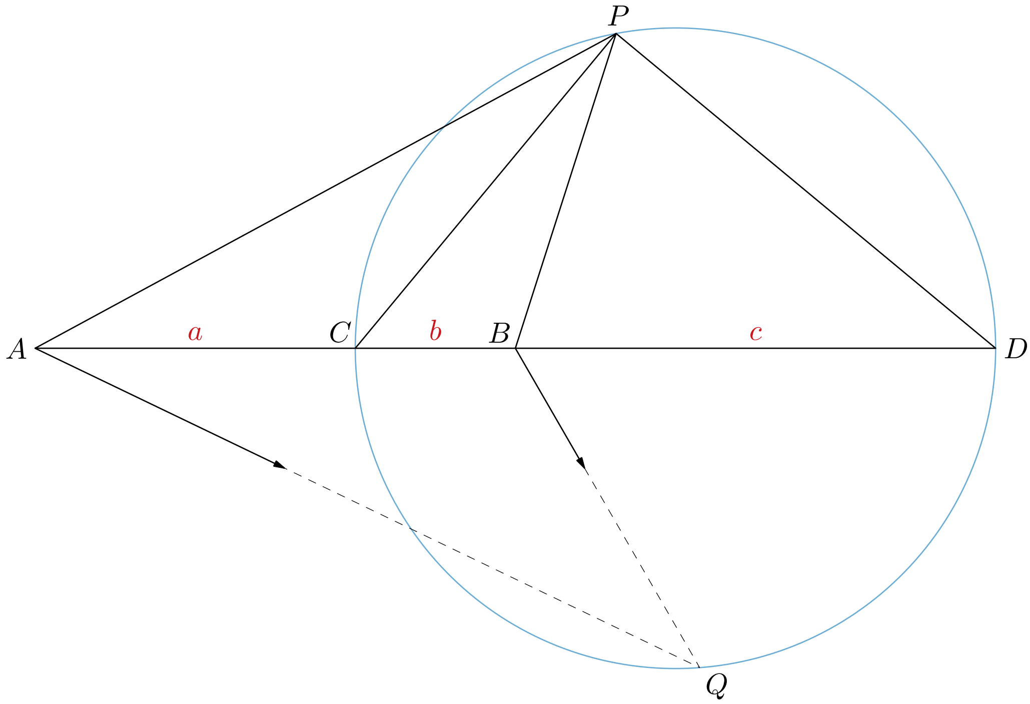
$$\frac{AC}{CB} = \frac{AP}{PE} = \frac{AP}{BP}$$

as required. And using the exterior angle

$$\frac{AD}{BD} = \frac{AP}{GP} = \frac{AP}{BP}$$

2.1 Applied Apollonian example

Given A, B, k, and course of the target, find intersection point Q.



But how can you find C and D in METAPOST given A, B, and k? Using a, b, and c for the lengths as shown above, then using the mediation syntax, point C is `(a/(a+b))[A, B]` and D is `((a+b+c)/(a+b))[A, B]`, so what are these fractions in terms of k? We know that $AC/CB = AD/BD = k$ so we have $a/b = (a+b+c)/c = k$ which we can develop as follows.

Starting with $a/b = k$, add 1 to each side, $a/b + b/b = k + 1$, hence $(a+b)/b = k+1$. We can then divide the first equation by the last to get $a/(a+b) = k/(k+1)$.

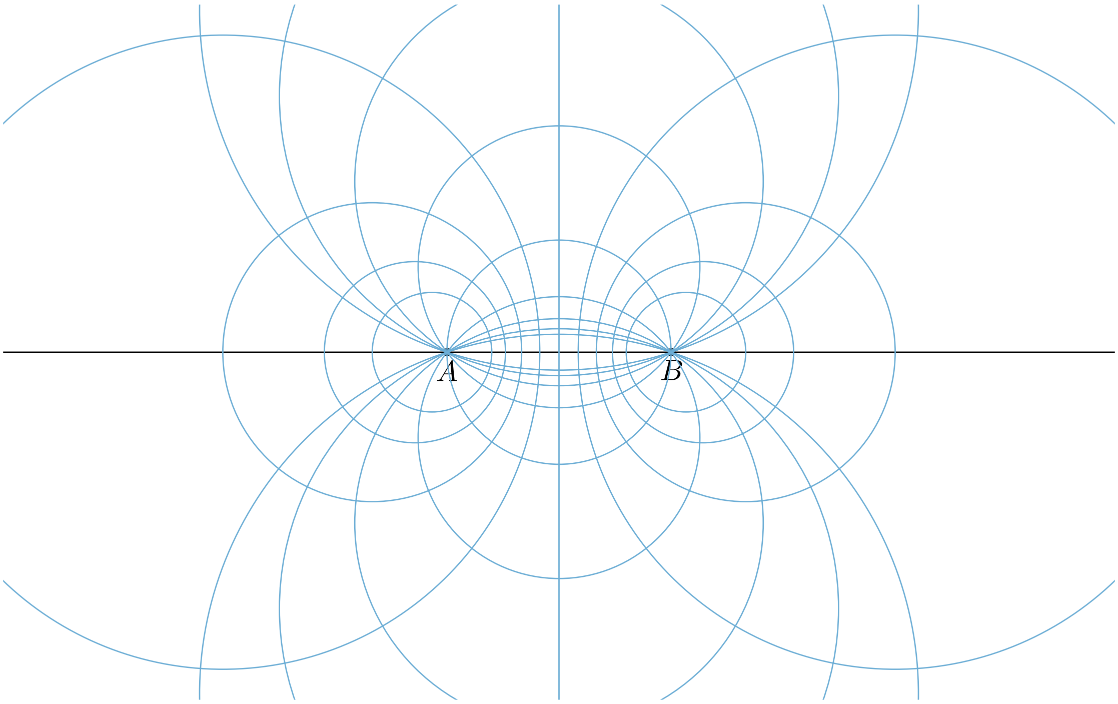
Similarly, starting with $(a+b+c)/c = k$, take 1 away from each side to get $(a+b+c)/c - c/c = k - 1$, hence $(a+b)/c = k - 1$. And again dividing the first by the last gives $(a+b+c)/(a+b) = k/(k-1)$.

So our METAPOST expressions for points C and D can be written like this:

```
C = (k/(k+1))[A, B]; D = (k/(k-1))[A, B];
```

If $k = 1$ you will get a "Division by zero" error from the second expression. This corresponds to D at infinity and C half way from A to B. The circle through these two points can be thought of as the perpendicular bisector of A and B. If you have $0 < k < 1$ then the circle will be round A not B. If $k < 0$ the positions of C and D will be swapped, and there is another error when $k = -1$.

Armed with these expressions, you can draw a family of non-intersection coaxial Apollonian circles.



The circles on the left are drawn with $0 < k < 1$ and those on the right with $k > 1$. The circles drawn with a given k is the mirror image of one drawn with $1/k$; so the smallest circle around A is drawn with $k = 1/4$ and the corresponding smallest circle around B has $k = 4$.