

Proofs without words I

Exercises in METAPOST

Toby Thurston

March 2021 —



Contents

Geometry and Algebra

The Pythagorean theorem I	4
The Pythagorean theorem II	5
The Pythagorean theorem III	6
The Pythagorean theorem IV	7
The Pythagorean theorem V	8
The Pythagorean theorem VI	9
A Pythagorean theorem: $aa' = bb' + cc'$	10
The rolling circle squares itself	11
On trisecting an angle	12
Trisection in an infinite number of steps	13
Trisection of a line segment	14
The vertex angles of a star sum to 180°	15
Viviani's theorem I	16
Viviani's theorem II	17
A theorem about right angles	18
Area and the projection theorem of a right triangle	19
Chords and tangents of equal length	20
Completing the square	21
Algebraic areas I	22
Algebraic areas II	23
Sum of squares identity	24
Polygonal numbers	25
The volume of a frustrum of a square pyramid	26
The volume of a hemisphere via Cavalieri's Principle	27

The Pythagorean theorem I



— adapted from the *Chou pei san ching*

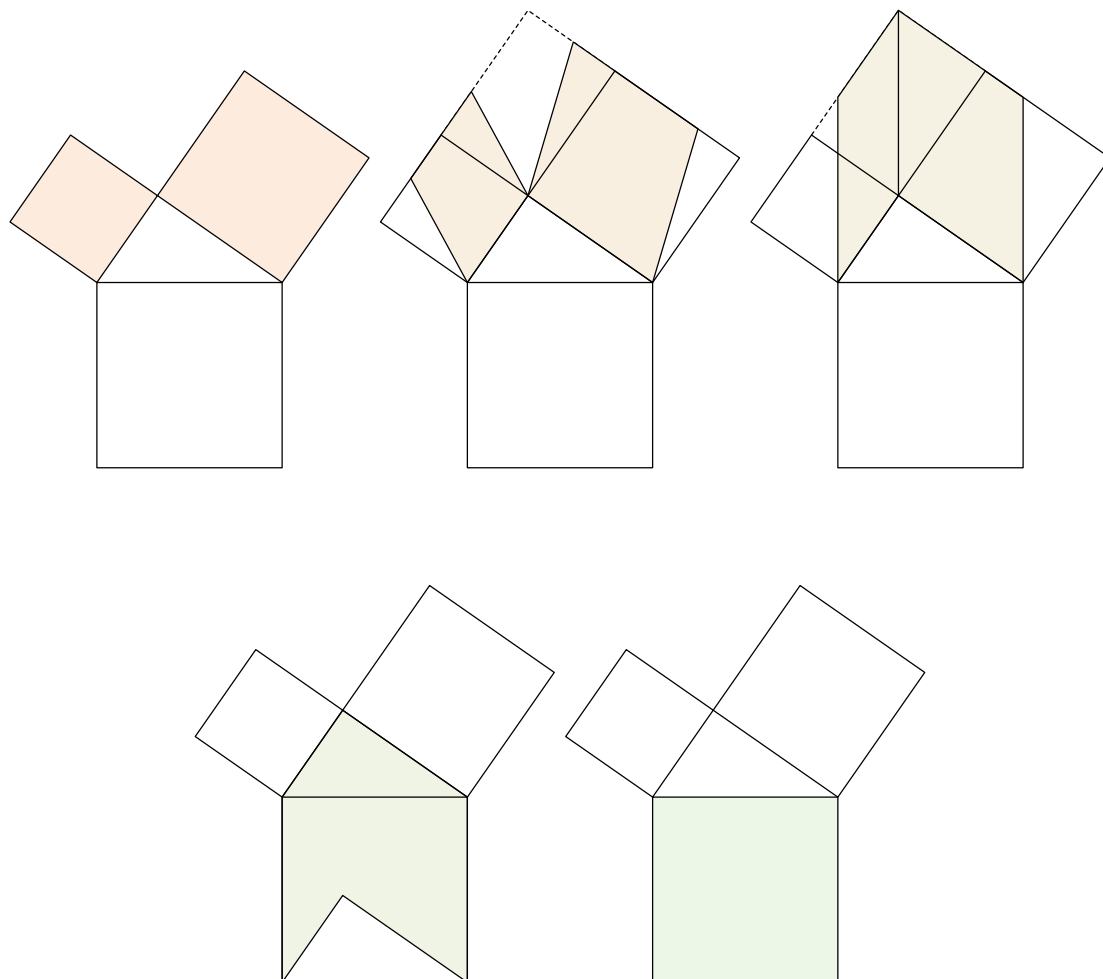
The Pythagorean theorem II



Behold!

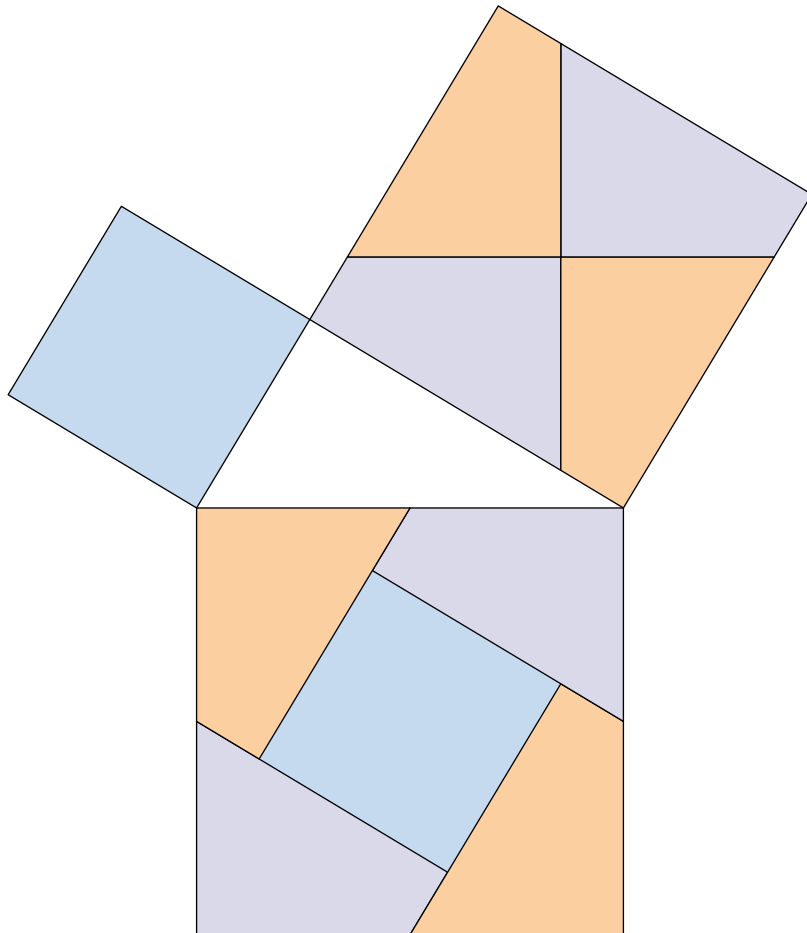
— Bhāskara (12th century)

The Pythagorean theorem III



— based on Euclid's proof

The Pythagorean theorem IV



— H. E. Dudeney (1917)

The Pythagorean theorem V



$$A = 2 \cdot \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a+b)^2$$

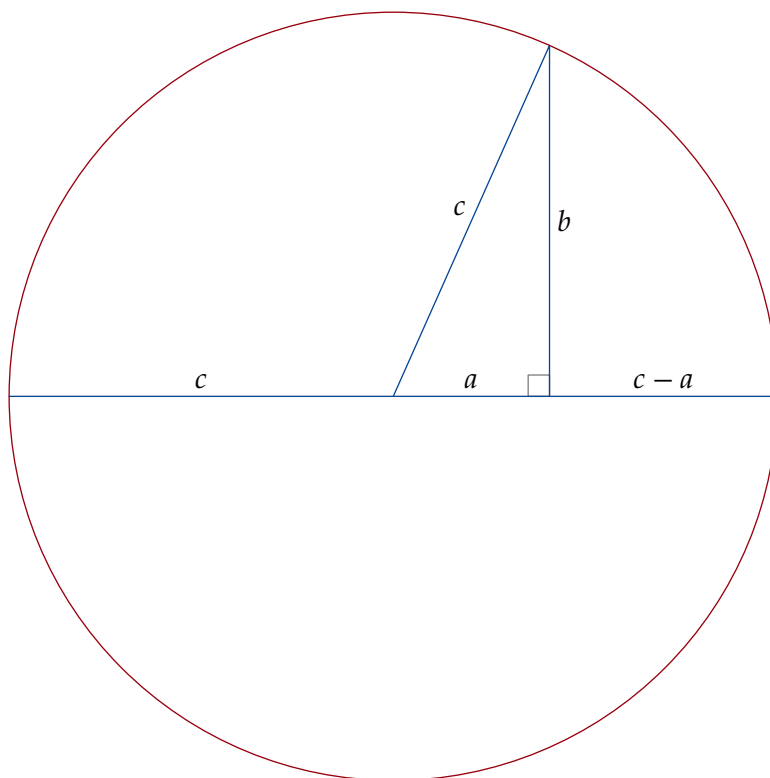
$$c^2 = a^2 + b^2$$

— James A. Garfield (1876)

The Pythagorean theorem VI

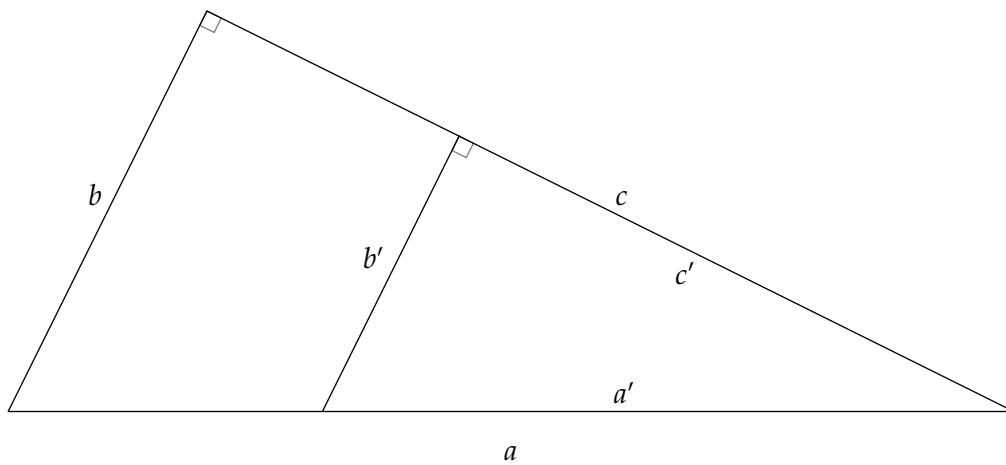
$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2 + b^2 = c^2$$



— Michael Hardy

A Pythagorean theorem: $aa' = bb' + cc'$



$$\frac{x}{b'} = \frac{b}{a} \implies \frac{x}{b} = \frac{b'}{a} \implies ax = bb';$$

$$\frac{y}{c'} = \frac{c}{a} \implies \frac{y}{c} = \frac{c'}{a} \implies ay = cc';$$

$$\therefore aa' = a(x + y) = bb' + cc'.$$

— Enzo R. Gentile

The rolling circle squares itself



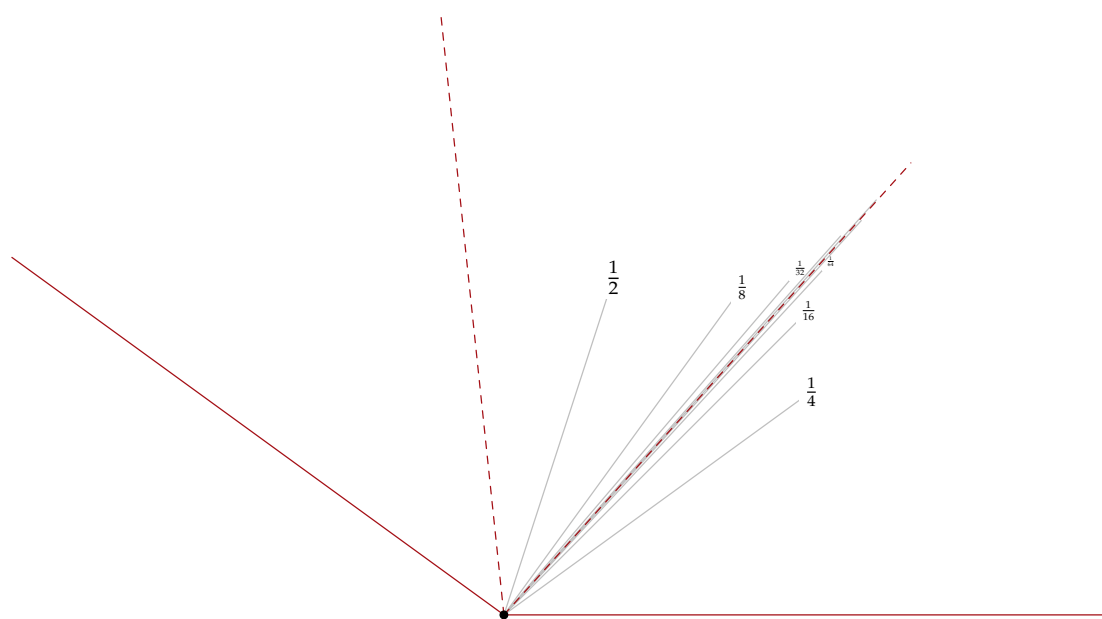
— Thomas Elsner

On trisecting an angle



— Rufus Isaacs

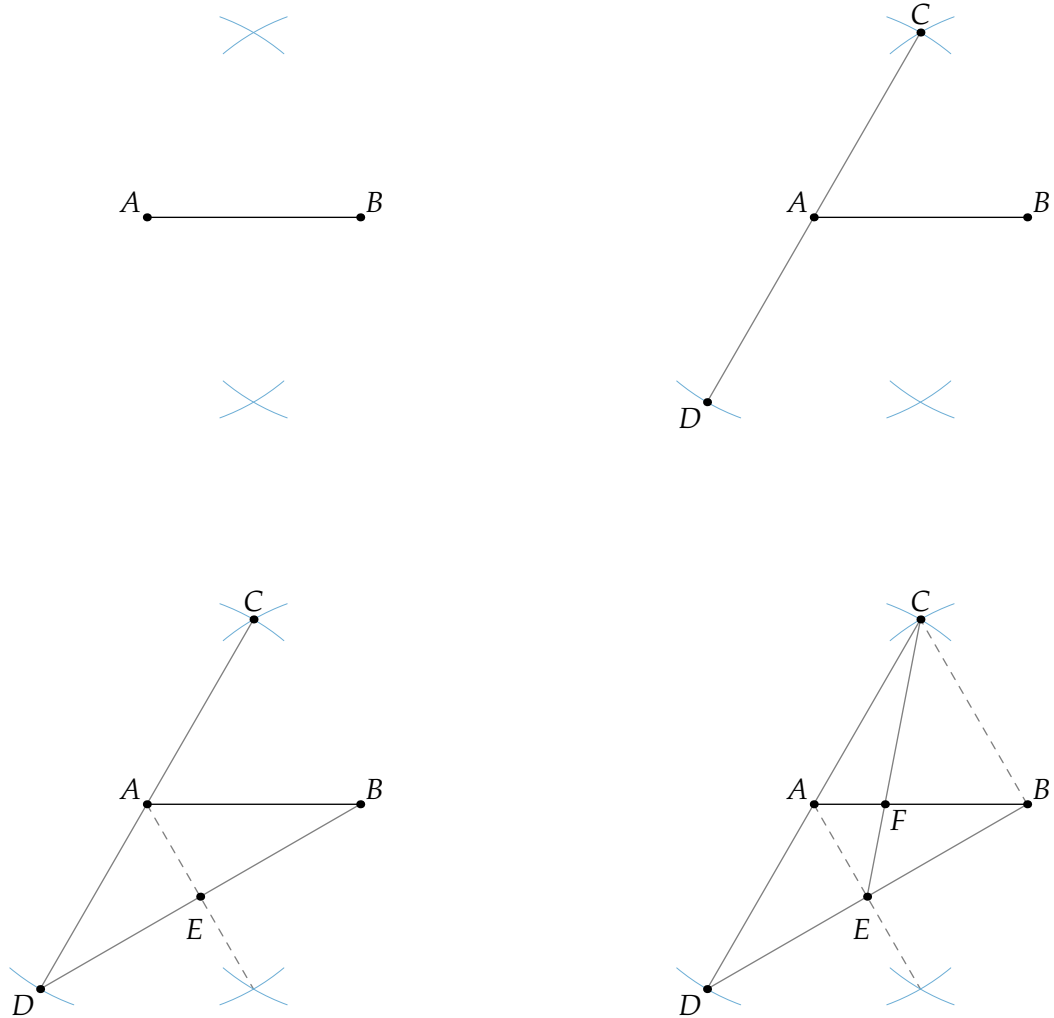
Trisection in an infinite number of steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

— Eric Kincanon

Trisection of a line segment



$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

— Scott Cobel

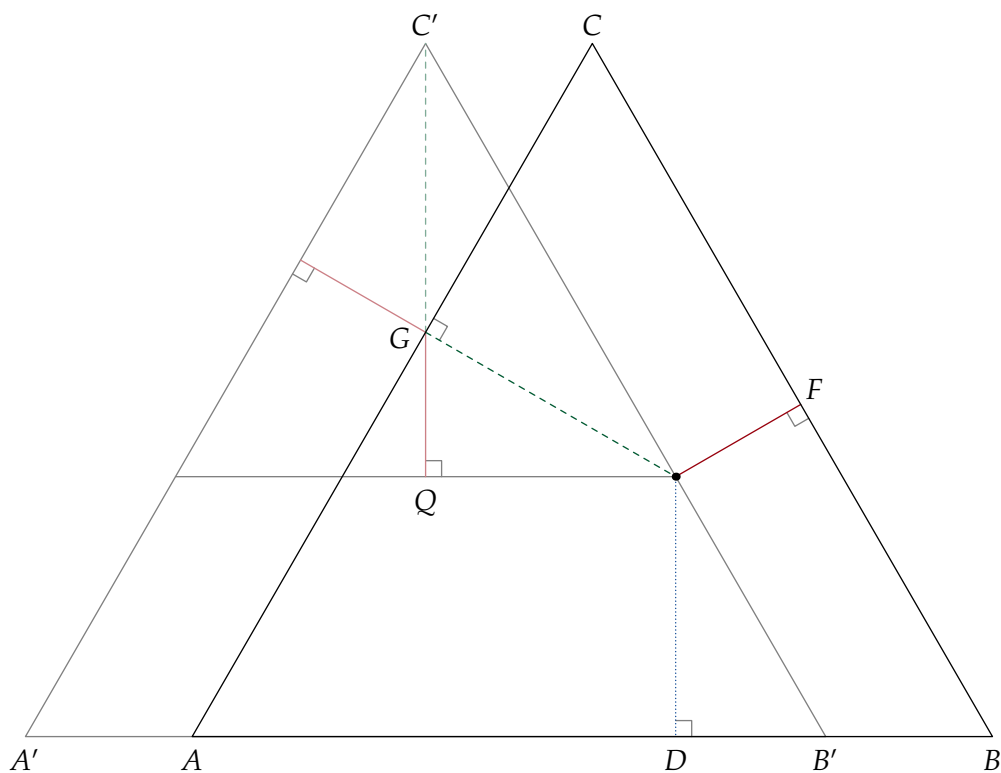
The vertex angles of a star sum to 180°



— Fouad Nakhli

Viviani's theorem I

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



This shows a particular example, with $C'GQ$ collinear, rather than the general case

— Samuel Wolf

Viviani's theorem II

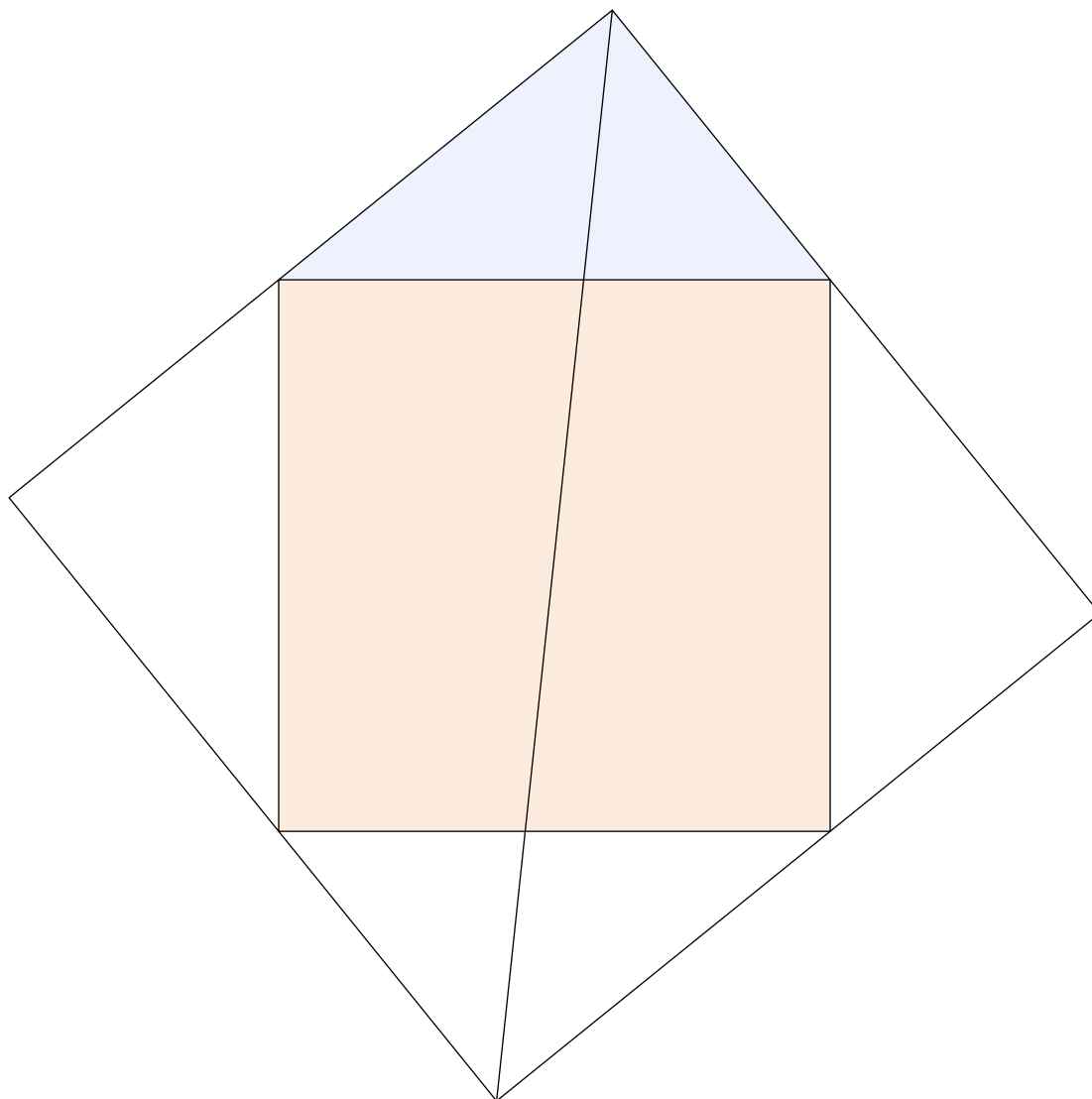
The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



— Ken-Ichiroh Kawasaki

A theorem about right angles

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse



— Roland H. Eddy

Area and the projection theorem of a right triangle



$$CD^2 = AD \cdot DB$$

— Sidney H. Kung

Chords and tangents of equal length

If circle C_1 passes through the center O of circle C_2 , the length of the common chord \overline{PQ} is equal to the tangent segment \overline{PR} .



— Roland H. Eddy

Completing the square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



— Charles D. Gallant

Algebraic areas I

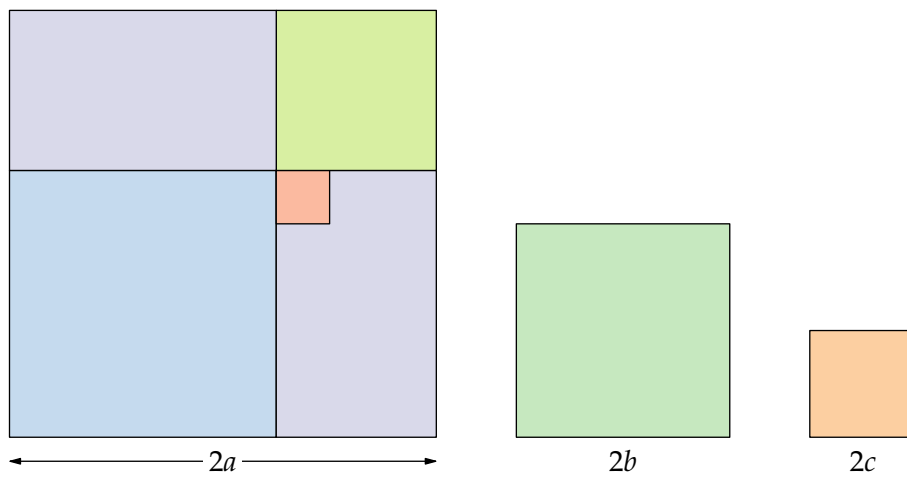
$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$



— Shirley Wakin

Algebraic areas II

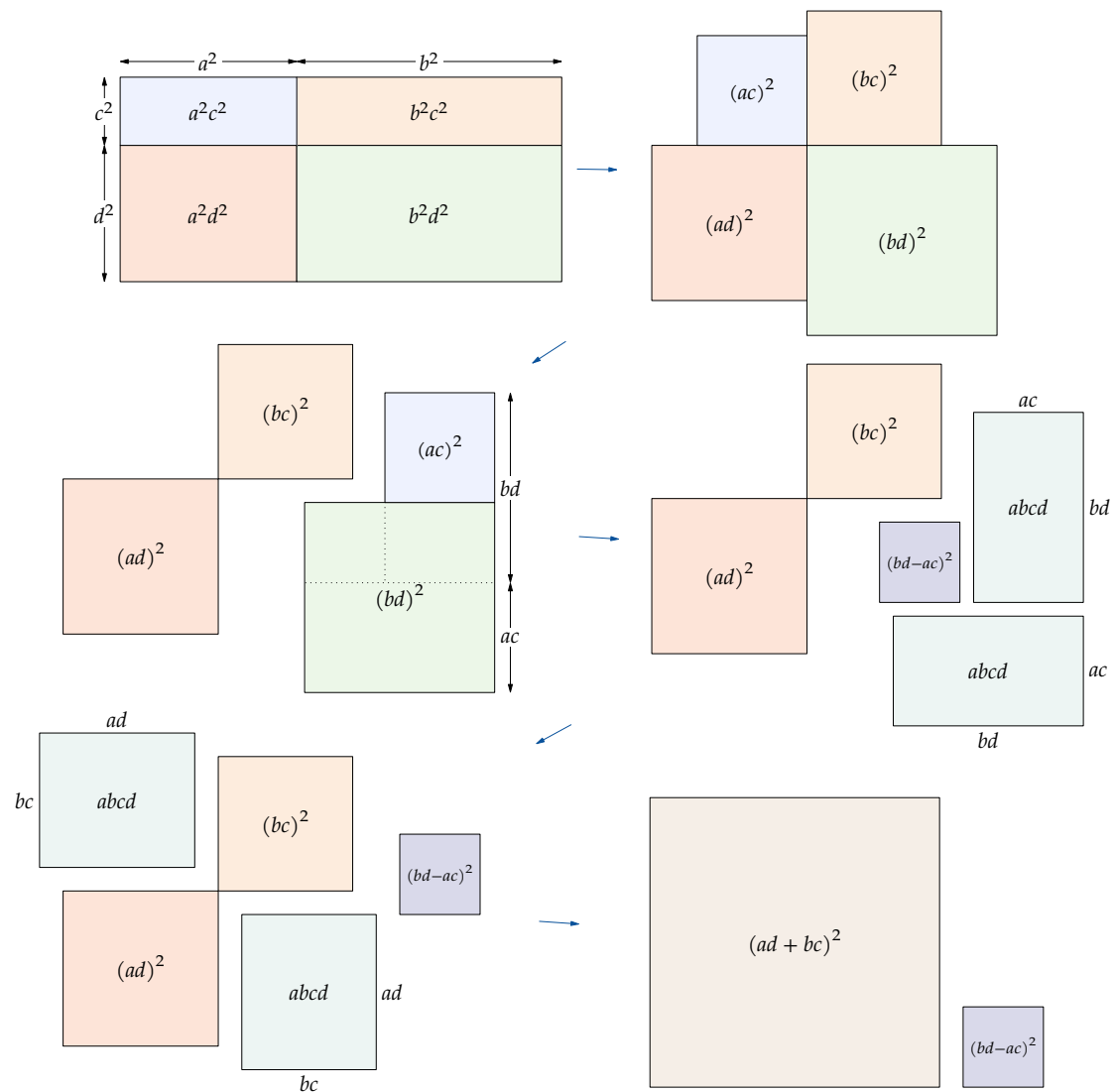
$$(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = (2a)^2 + (2b)^2 + (2c)^2$$



— Sam Pooley and K. Ann Drude

Sum of squares identity

$$(a^2 + b^2)(c^2 + d^2) = (ab + bc)^2 + (bd - ac)^2$$



— Diophantus of Alexandria

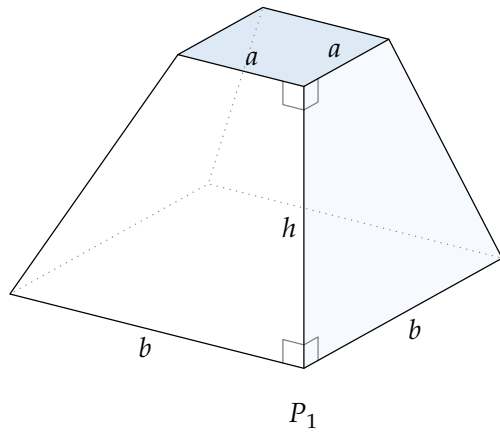
Polygonal numbers

The k^{th} n -gonal number is $1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$



— Dave Logothetti

The volume of a frustum of a square pyramid



$$V_{P_1} = \frac{h}{b-a} \cdot V_{P_2} = \frac{h}{b-a} \cdot \frac{1}{3} (b^3 - a^3) = \frac{h}{3} (a^2 + ab + b^2)$$

— *The Moscow Papyrus*, c. 1850 BCE

The volume of a hemisphere via Cavalieri's Principle



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

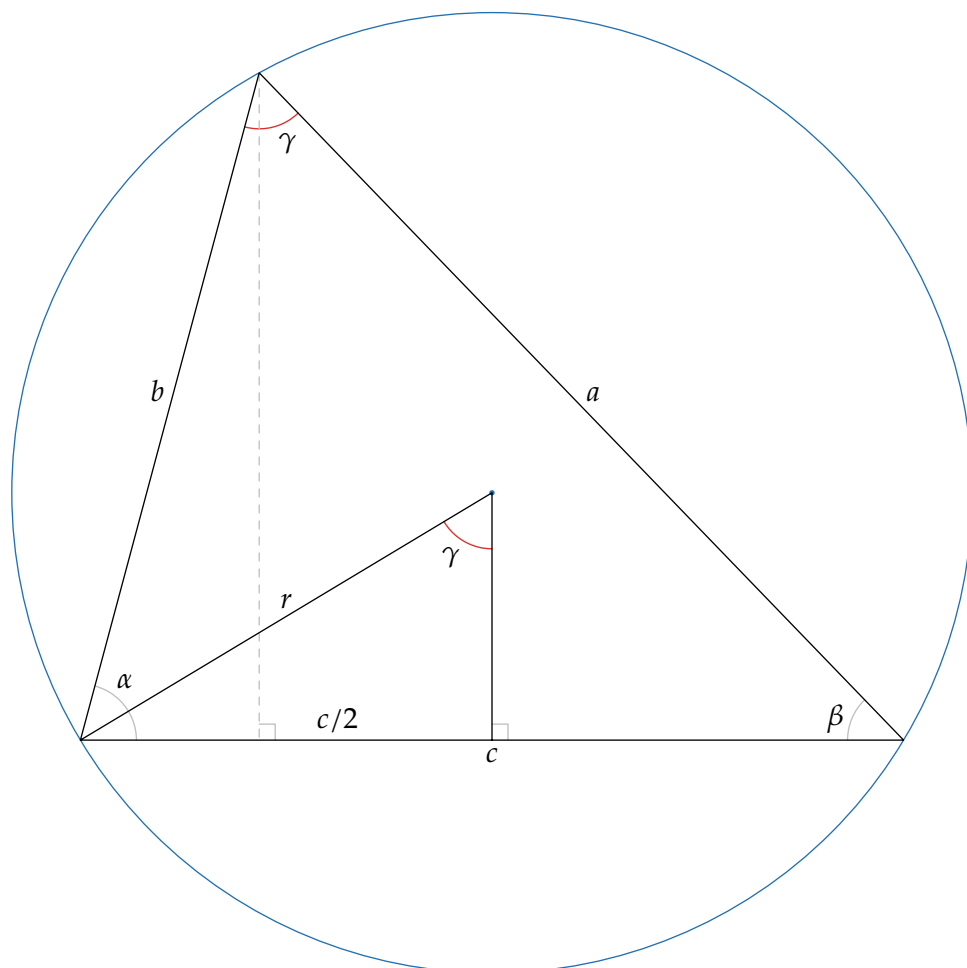
— Sidney H. Kung

Trigonometry, Calculus, & Analytic Geometry

Sine of the sum	29
Area and difference formulas	30
The law of cosines I	31
The law of cosines II	32
The law of cosines III (via Ptolemy's theorem)	33
The double-angle formulae	34
The half-angle tangent formulae	35
Mollweide's equation	36
Tangent, cotangent, secant, and cosecant	37
Substitution to make a rational function of sine and cosine	38
Sums of arctangents	39
The distance between a point and a line	40
The midpoint rule is better than the trapezoidal rule for concave functions	41
Integration by parts	42
The graphs of f and f^{-1} are reflections about the line $y = x$	43
The reflection property of the parabola	44
Area under an arch of the cycloid	45

Sine of the sum

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \text{ for } \alpha + \beta < \pi$$



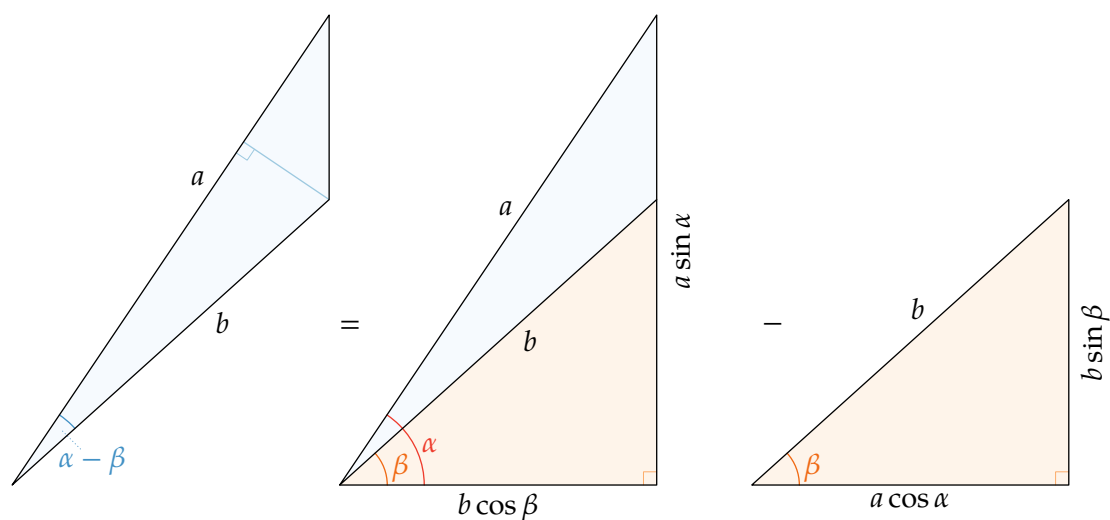
$$c = a \cos \beta + b \cos \alpha$$

$$r = 1/2 \implies \sin \gamma = \frac{c/2}{1/2} = c, \sin \alpha = a, \sin \beta = b$$

$$\sin(\alpha + \beta) = \sin(\pi - (\alpha + \beta)) = \sin \gamma = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

— Sidney H. Kung

Area and difference formulas



$$\frac{1}{2} \cdot a \cdot b \sin(\alpha - \beta) = \frac{1}{2} \cdot a \sin \alpha \cdot b \cos \beta - \frac{1}{2} \cdot a \cos \alpha \cdot b \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$\frac{1}{2} \cdot a \cdot b \cos(\alpha - \beta) = \frac{1}{2} \cdot a \cos \alpha \cdot b \cos \beta + \frac{1}{2} \cdot a \sin \alpha \cdot b \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

— Sidney H. Kung

The law of cosines I



$$\begin{aligned}
 c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\
 &= b^2 \sin^2 \theta + a^2 - 2ab \cos \theta + b^2 \cos^2 \theta \\
 &= a^2 + b^2 (\sin^2 \theta + \cos^2 \theta) - 2ab \cos \theta \\
 &= a^2 + b^2 - 2ab \cos \theta
 \end{aligned}$$

— Timothy A. Sipka

The law of cosines II

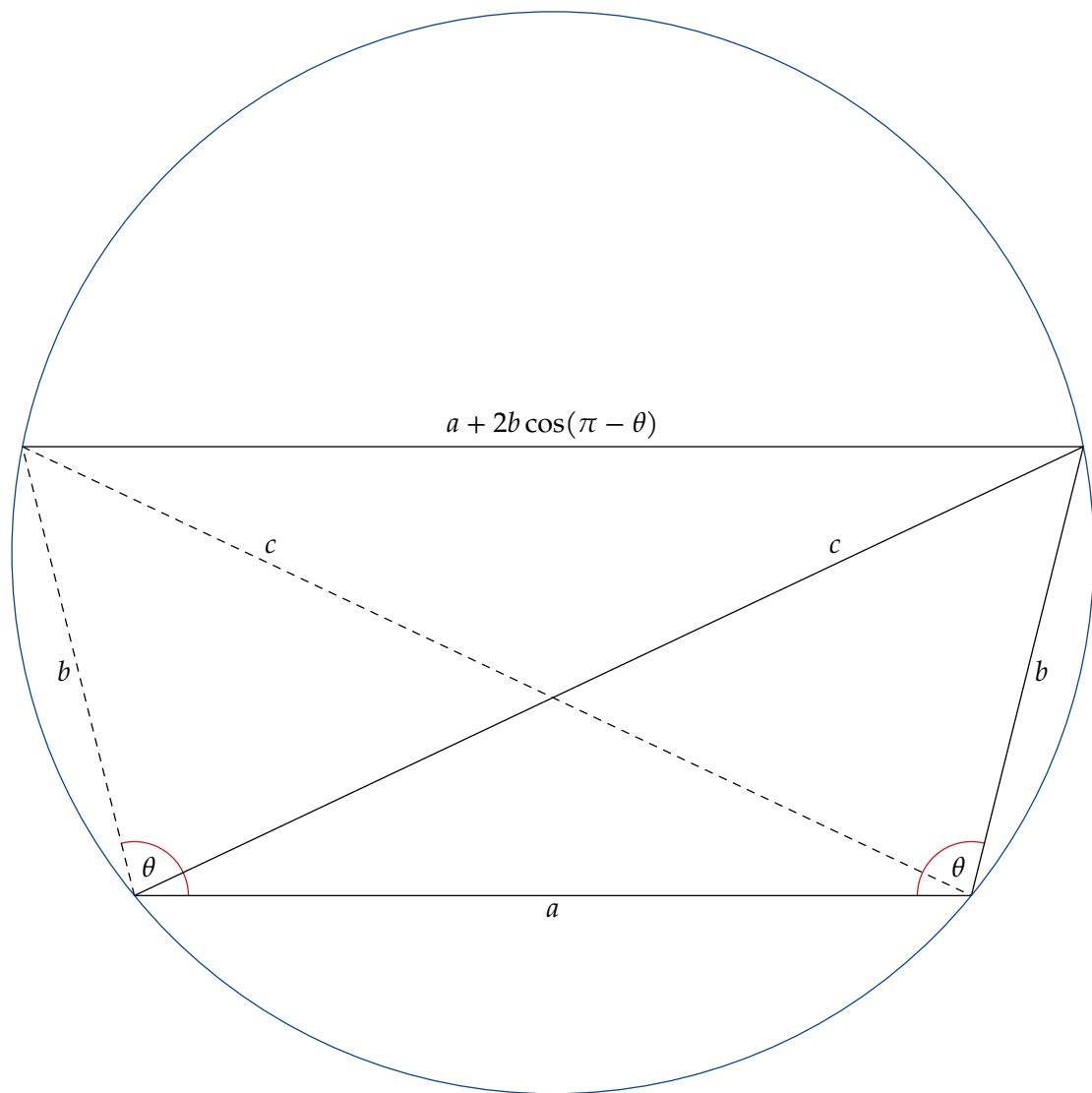


$$(2a \cos \theta - b) \cdot b = (a - c) \cdot (a + c)$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

— Sidney H. Kung

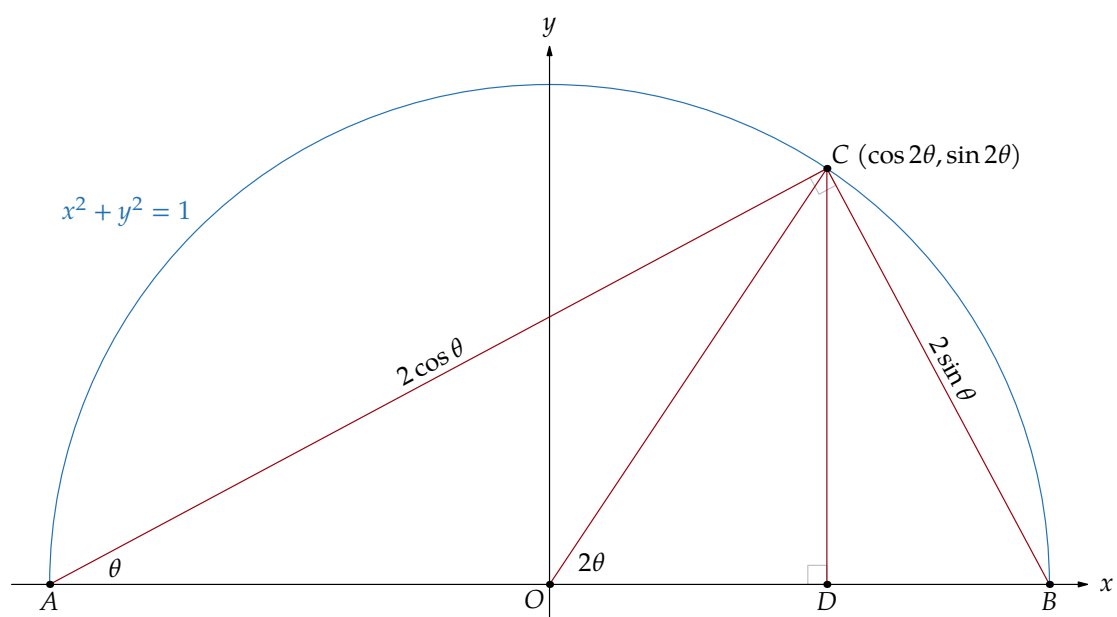
The law of cosines III (via Ptolemy's theorem)



$$\begin{aligned}c \cdot c &= b \cdot b + (a + 2b \cos(\pi - \theta)) \cdot a \\c^2 &= a^2 + b^2 - 2ab \cos \theta\end{aligned}$$

— Sidney H. Kung

The double-angle formulae



$$\triangle ACD \sim \triangle ABC$$

$$CD/AC = BC/AB$$

$$\sin 2\theta / 2 \cos \theta = 2 \sin \theta / 2$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$AD/AC = AC/AB$$

$$(1 + \cos 2\theta) / 2 \cos \theta = 2 \cos \theta / 2$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

— Roger B. Nelsen

The half-angle tangent formulae



$$\tan \theta/2 = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

— R. J. Walker

Mollweide's equation

$$(a - b) \cos \frac{\gamma}{2} = c \sin \left(\frac{\alpha - \beta}{2} \right)$$



— H. Arthur DeKleine

Tangent, cotangent, secant, and cosecant



$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$(\tan \theta + 1)^2 + (\cot \theta + 1)^2 = (\sec \theta + \csc \theta)^2$$

$$\text{also } \tan \theta = \frac{\tan \theta + 1}{\cot \theta + 1}$$

— William Romaine

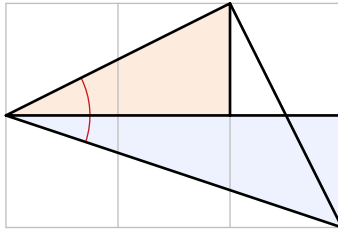
Substitution to make a rational function of sine and cosine



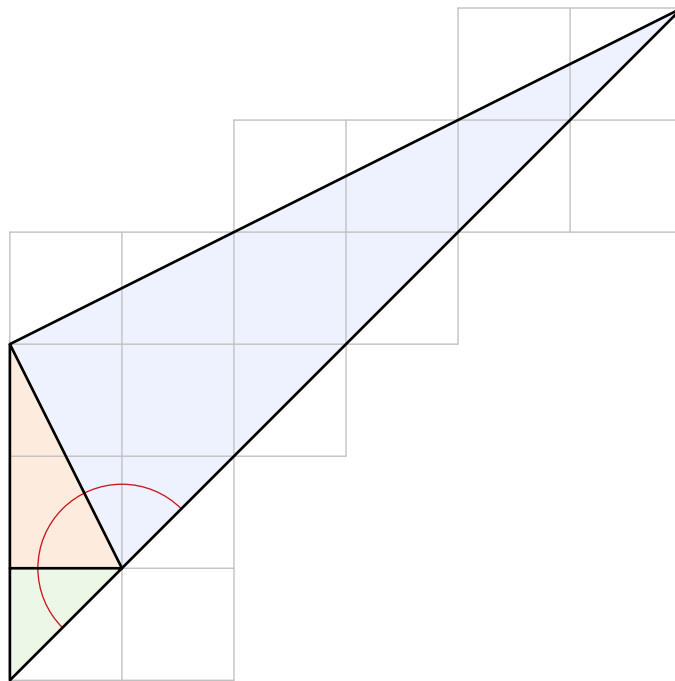
$$z = \tan(\theta/2) \implies \sin \theta = \frac{2z}{1+z^2} \quad \text{and} \quad \cos \theta = \frac{1-z^2}{1+z^2}$$

— Roger B. Nelsen

Sums of arctangents



$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

— Edward M. Harris

The distance between a point and a line



$$\frac{d}{1} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

— R. L. Eisenman

The midpoint rule is better than the trapezoidal rule for concave functions



— Frank Burk

Integration by parts



$$\text{Area } \text{blue} + \text{Area } \text{orange} = qs - pr$$

$$\int_r^s u \, dv + \int_p^q v \, du = uv \Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) \, dx$$

— Richard Courant

The graphs of f and f^{-1} are reflections about the line $y = x$



— Ayoub B. Ayoub

Area under an arch of the cycloid



$$\frac{1}{2}\pi r \cdot 2r \quad + \quad \pi r^2 \quad + \quad \frac{1}{2}\pi r \cdot 2r$$

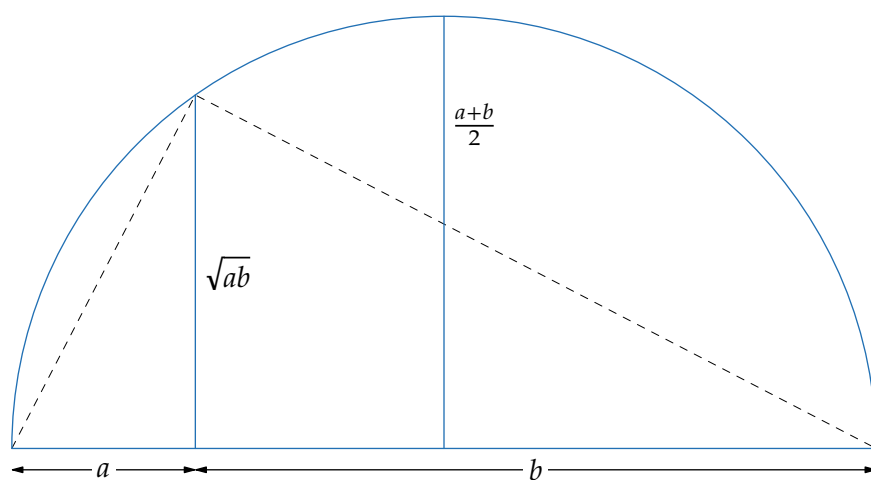
therefore $A = 3\pi r^2$

— Richard M. Beekman

Inequalities

The arithmetic mean – geometric mean inequality I	47
The arithmetic mean – geometric mean inequality II	48
The arithmetic mean – geometric mean inequality III	49

The arithmetic mean – geometric mean inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

— Charles D. Gallant

The arithmetic mean – geometric mean inequality II

— Doris Schattschneider

The arithmetic mean – geometric mean inequality III

— Roland H. Eddy