Universidad Nacional de Ingeniería
Facultad de Ciencias
Escuela Profesional de Matemática — Ciclo 2022–II

Lista 2 - CM4F1

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1. Suppose $g : \mathbb{R}^2 \to \mathbb{R}^2$ has continuous first derivatives, and x^* is a fixed point of g such that:
$$\left|\frac{\partial g_j(x^*)}{\partial x_l}\right| < \frac{1}{5}$$
for $j = 1, 2$ and $l = 1, 2$.
Show that there exists an $\varepsilon > 0$ such that for all $x^{(0)} \in \overline{B}_\varepsilon(x^*)$ the sequence $\{x^{(k)}\}, x^{(k+1)} = g(x^{(k)})$ converges to x^* and satisfies:
$$\frac{\|x^{(k+1)} - x^*\|_\infty}{\|x^{(k)} - x^*\|_\infty} < \frac{1}{2}$$
for all k. Recall that $\overline{B}_\varepsilon(x^*) = \{x \in \mathbb{R}^2 / \|x - x^*\|_\infty \leq \varepsilon\}$.

2. Consider a 2D fixed point iteration of the form:
$$\begin{aligned} x_{k+1} &= f(x_k, y_k) \\ y_{k+1} &= g(x_k, y_k) \end{aligned}$$
Assume that the vector-value function $h(x, y) = (f(x, y), g(x, y))$ is continuously differentible, and the infinity norm of the Jacobian matrix is less than 1 at a unique fixed point (x_∞, y_∞).
Now consider the *nonlinear Gauss-Seidel* version of the iteration:
$$\begin{aligned} x_{k+1} &= f(x_k, y_k) \\ y_{k+1} &= g(x_{k+1}, y_k) \end{aligned}$$
Prove that the *nonlinear Gauss-Seidel* version is convergent, to the same fixed point, for initial conditions sufficiently close to the fixed point.

3. Starting with $(0, 0, 1)$, carry out an interation of Newton's method for nonlinear systems on:
$$\begin{aligned} xy - z^2 &= 1 \\ xyz - x^2 + y^2 &= 2 \\ e^x - e^y + z &= 3 \end{aligned}$$
Explain the results.

4. Use Newton's method to find the minimum value of the function:
$$f(x_1, x_2) = x_1^4 + x_1x_2 + (1 + x_2)^2$$
Experiment with various initial guesses and observe the pattern convergence.

5. Using Newton's method for nonlinear systems, solve for all roots of the following nonlinear system. Use graphs to estimate initial guesses:

   a) $x^2 + y^2 - 2x - 2y + 1 = 0, \quad x + y - 2xy = 0.$

   b) $x^2 + 2xy + y^2 - x + y - 4 = 0, \quad 5x^2 - 6xy + 5y^2 + 16x - 16y + 12 = 0.$

6. Use functional iteration to find solutions to the following nonlinear systems, accurate to within 10^{-5}, using the l_∞ norm:
$$\begin{aligned} x_1^2 + x_2 - 37 &= 0 \\ x_1 - x_2^2 - 5 &= 0 \\ x_1 + x_2 + x_3 - 3 &= 0 \end{aligned}$$

7. Use Broyden's method with $x^{(0)} = (-1, -2, 1)^T$ to compute $x^{(2)}$ for each of the following nonlinear systems:

$$\begin{aligned} x_1^3 + x_1^2 x_2 - x_1 x_3 + 6 &= 0 \\ e^{x_1} + e^{x_2} - x_3 &= 0 \\ x_2^2 - 2x_1 x_3 &= 4 \end{aligned}$$

8. Use Broyden's method to approximate solutions to the following nonlinear systems. Iterate until $\|x^{(k)} - x^{(k-1)}\|_\infty < 10^{-6}$.

$$\begin{aligned} 15x_1 + x_2^2 - 4x_3 &= 13 \\ x_1^2 + 10x_2 - x_3 &= 11 \\ x_2^3 - 25x_3 &= -22 \end{aligned}$$

9. Use the continuation method and Runge-Kutta method of order two on the following nonlinear systems:

$$\begin{aligned} 3x_1 - cos(x_2 x_3) - \frac{1}{2} &= 0 \\ 4x_1^2 - 625x_2^2 + 2x_2 - 1 &= 0 \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} &= 0 \end{aligned}$$

10. Use the continuation method and Runge-Kutta method of order two on the following nonlinear systems:

$$\begin{aligned} 15x_1 + x_2^2 - 4x_3 &= 13 \\ x_1^2 + 10x_2 - x_3 &= 11 \\ x_2^3 - 25x_3 &= -22 \end{aligned}$$

11. Use the continuation method and Runge-Kutta method of order four on the following nonlinear systems:

$$\begin{aligned} 3x_1 - cos(x_2 x_3) - \frac{1}{2} &= 0 \\ 4x_1^2 - 625x_2^2 + 2x_2 - 1 &= 0 \\ e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} &= 0 \end{aligned}$$

12. Use the continuation method and Runge-Kutta method of order four on the following nonlinear systems:

$$\begin{aligned} 15x_1 + x_2^2 - 4x_3 &= 13 \\ x_1^2 + 10x_2 - x_3 &= 11 \\ x_2^3 - 25x_3 &= -22 \end{aligned}$$

El profesor[1]
Lima, 23 de Noviembre del 2022.

[1] Hecho en LaTeX