An acoustic theory of special relativity

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Abstract

In medical ultrasound spatial intervals are defined from a measured pulse-echo time and an assumed constant speed of sound. This report demonstrates that in order to correctly predict the results of an ultrasound experiment physical models must be Lorentz invariant, with the speed of sound taking the traditional role of the speed of light. Ultrasound is therefore subject to relativistic considerations.

Sound does has a propagating medium. The formulation of the acoustic theory of relativity is therefore different to Einstein's formulation. However, it is very close to Poincaré's version. The role of the medium is investigated by formulating a theory of the acoustics of an ideal fluid that is measured with ultrasound. It is demonstrated that the generation and propagation of the sound obeys the same physical laws as Maxwell's electrodynamics. The acoustic analogue to the electric field is the Lamb vector and the acoustic analogue to the magnetic field is the vorticity.

1 Introduction

The privileged role of light is perhaps the most mysterious aspect of Einstein's special theory of relativity. What is it about this signal, as opposed to any other method of communication, that makes it so fundamental to the concepts of time and space? The answer, for Einstein, is that light has a constant speed that is independent of the motion of the light source [3]. This property enables the distance between an observer and a faraway object to be measured by determining the to-fro propagation time of a pulse of light. The measurement is local to the observer and all the observer requires to make the measurement is a light source, a receiver and a clock. Einstein's definition of measurement is then completed with his principle of relativity, which demands that a measuring device at rest with respect to an observer, such as a clock or a metre rule, gauge quantities that are independent of the observer's motion.

In Einstein's theory the observer does not need to know their speed relative to the speed of the light medium - the *æther*. The postulates demand that ætherial motion neither alters the speed of light nor alters the units of the measurements made by the observer. The observer can therefore consider themselves to be stationary and not consider the æther at all. This elimination of the æther, however, only emphasises the uniqueness of light, for it is the only signal that has a medium with no measurable mechanical properties. Satisfactory definitions of time and space seem to come at the expense of making light an even greater puzzle, more and more distant from the world that we can touch and hear.

The relativity theory of Poincaré is different. For Poincaré light does have a medium and all motions can be measured with respect to it; 'stationary' means stationary with respect to the æther [11]. Unlike Einstein's theory, the length of a measuring rule used by an observer is affected by the observers motion. Indeed, Poincaré *postulates* that it contracts from its length when stationary with respect to the æther, with the size of the contraction determined by the spatial Lorentz transformation [11]. It follows that the principle of relativity is also different; Poincaré assumes that there is no absolute reference from which to measure the speed of the æther. However, in its refusal of an absolute frame of reference, Poincaré's relativity postulate does enable a complete theory of special relativity to be derived. From it Poincaré determines the temporal Lorentz transformation from the spatial transformation. Poincaré's formulation of special relativity is not at odds with any experimental confirmation of Einstein's theory [10].

Light is not a privileged signal in Poincaré's theory but this flexibility is obtained only at the cost of admitting motion dependant deformations. A fundamental explanation for these is missing, however, and this gives Poincaré's theory an incomplete feel. When faced with a choice between the relativity theories of Einstein and Poincaré the community choose Einstein's. After all, if light really is different from any other signal then there is nothing further to add to Einstein's theory.

In this report we use sound to define time and space in the manner routinely used in medical ultrasound and other sonar-based technologies. It is demonstrated in section 2 that in order for ultrasound theory to agree with experiment the Lorentz transformations need to be applied to them. This is achieved by explicitly considering an acoustic analogue to the Michelson-Morely experiment. Since sound does have a mechanical medium through which it propagates and since ultrasound does not care about the speed of light it is the relativity theory of Poincaré that is recovered in acoustics. It is demonstrated in section 2 that Poincaré's motion dependent contraction results from the dependence of the sound speed on the bulk flow of the medium. The contraction is real in the sense that it will be reproducibly measured. However, since the contraction results from the measurement process there is no need to seek some fundamental interaction between material objects and their æther.

In section 3 it is demonstrated that the acoustics of an ideal fluid obey Maxwell's relations when time and space are defined acoustically. Therefore the generation and propagation of sound, when time and space are defined with sound, obey the same physical law as the generation and propagation of light, when time and space are defined with light.

2 The acoustic definition of time and space

In medical ultrasound distances are measured using the time it takes a pulse of sound to propagate from a transducer to a reflecting object and then to return again. If the sound is emitted from the transducer at a time, τ^-, and the sound returns at a time, τ^+, then the task is to find from these two numbers the spatio-temporal location, x, of the point of reflection.

What happens to the sound in between leaving the transducer and returning cannot be known by acoustic measurement. In this ignorance ultrasound practitioners assume that the time at which the echo occurred is the midpoint of τ^- and τ^+,

$$\tau(x) = \frac{\tau^+ + \tau^-}{2}. \tag{1a}$$

Other choices could certainly be made, but would imply a knowledge of the world beyond that learnt from τ^- and τ^+ alone. To measure distances from the times τ^- and τ^+ a sound speed, c, is required. Assuming, again in ignorance, that the sound returns at the same speed at which it left gives

$$\rho(x) = \frac{\tau^+ - \tau^-}{2}c. \tag{1b}$$

These are the definitions of time and space that are used in ultrasound. They are also identical to definitions used by Poincaré and Einstein, with the exception that the speed, c, is here the speed of sound rather than the speed of light.

Equation 1b requires an *a priori* knowledge of the sound speed for otherwise distances cannot be determined from temporal measurements. In diagnostic ultrasound scanners this speed is usually taken to be 1540 ms^{-1}. The constancy of the speed of sound is identical to Einstein's second postulate for special relativity [3], except that the sound speed takes the role of the speed of light. However, the speed of sound is here a constant not because of some physical law but because when using sound to make measurements there is no other choice but to assume the sound speeds constancy.

Ultrasound has inherited from fluid mechanics the principle that it is impossible to determine absolute uniform motions: an object at rest in a laminar flow is equivalent to an object moving in a stationary fluid. The velocity of an object may always be measured with respect to the bulk flow of the fluid, but the notion of a true speed with respect to some absolute reference is never invoked. This is the relativity postulate as envisaged by Poincaré.

2.1 Physics as measured with ultrasound

The measurement rules of equations 1a and 1b enable two properties of the world as measured by ultrasound to be stated immediately. The first is that an entity that moves away from the transducer at a speed that is faster than the speed of sound (with respect to the bulk flow of the medium) cannot be measured. This is not because such motions are impossible but because the sound will never catch up with the entity and so there will never be an echo to record.

The second is that ultrasound is not capable of measuring variations in the speed of sound. In order to measure distances (and therefore speeds) the speed of sound must be known *a priori*. Fluctuations in the sound speed cannot be known without further *a priori* knowledge of the medium. Fluctuations in the density can therefore play no role in determining the sound speed. It follows that the acoustic medium must be incompressible (in the relativistic sense[9;14?]) and that sound must propagate according to a linear wave equation. In section 3 it is demonstrated that this linear relation is identical to Maxwell's relation.

The ultrasound literature does not comply with these remarks. Currently, when describing the physics of ultrasound, a fluid medium is always described by a Galilean invariant theory such as Euler's equation or the Naiver-Stokes equation. This enables motions that are faster than the speed of sound to be measured and predicts that a sound pulse propagates according to a non-linear wave equation. Both of these predictions are impossible when the world is measured with sound. Currently ultrasound physics fails to recognise the distinction between two equally valid descriptions of the world - the world that is seen and the world that is heard. Curiously, ultrasound physics repeats the fallacy that the world must be seen to be believed.

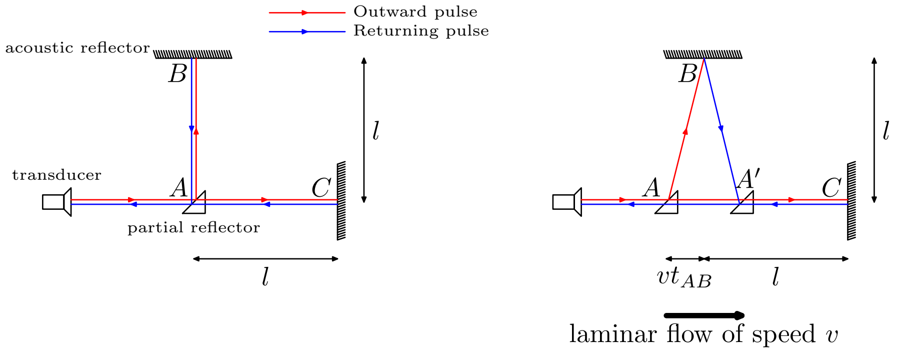
2.2 An acoustic Michelson-Morely experiment.

The discussion so far has been somewhat abstract. To make it concrete it is useful to discuss a simple pulse-echo experiment and compare the two viewpoints - the Galilean[1] world that is *seen*, compared with the world that is measured with ultrasound.

The first case to be considered is illustrated in Figure 1a. This apparatus is appropriate when the equipment is stationary with respect to the bulk flow of the medium. It is analogous to Michelson and Morely's famous experiment: a piezoelectric transducer replaces both the light source and the recorder while a medium that partially reflects sound replaces the semi-silvered mirror. The distance between A and B is denoted l and is the same as the distance between A and C. In the following the time it takes for the sound to propagate from A to B and back again is compared with the to-fro times between A and C.

If the apparatus of Figure 1a were not stationary with respect to the bulk flow of the medium then the experiment would fail. This is because the sound would not travel from A to B and return; the motion of the medium would drag the sound pulse with it. The setup illustrated in Figure 1b gives spirit of the Michelson-Morely experiment for the case when the apparatus is not stationary with respect to the bulk flow. In this case there are two separate partially reflecting surfaces. The time it takes the sound to propagate from A

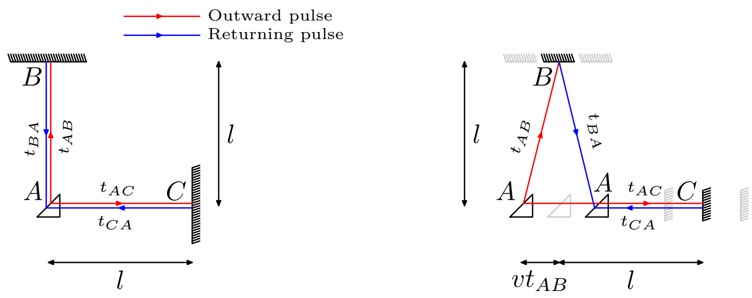
[1]Formally the 'Galilean' measurements are the distances and times that are measured with light signals in accordance to Einstein's method[3]. In ultrasound experiments, however, the Galilean approximation is entirely appropriate.



(a) Apparatus when bulk medium is stationary.

(b) Apparatus when the laminar flow of the bulk medium is v

Figure 1: A pulse-echo experiment when there is, and is not, a relative laminar flow past the apparatus.



(a) Observer stationary with respect to medium.

(b) Observer moves at speed v with respect to medium

Figure 2: Observed motion of apparatus for the setup of Figure 1a.

to B to A' is now compared with the time it takes the sound to go from A to C to A'.

When the to-fro times along the two arms are the same, irrespective of the flow of the bulk medium, the result is described as *null*. This is in accordance to the description of the Michelson-Morely result.

A Galilean interpretation

First we consider the case of the apparatus being stationary with respect to the bulk flow (Figure 1a). If the propagation of the sound pulse were observed by a Galilean observer that is also stationary with respect to the flow then they would observe the sound travelling according to Figure 2a. The time, t_{AB}, the time it takes for the sound to propagate from the A to B is the same as the time, t_{BA}, it takes the sound to propagate from B to A. It is given by l/c, where c is the speed of sound of the medium. This time interval is the same for the to and fro paths between A and C and so

$$t_{AB} = t_{BA} = t_{AC} = t_{CA} = l/c. \tag{2}$$

An observer for whom both the medium and apparatus flow past at a speed v will measure the same time intervals but will witness an altogether more complicated experiment. The acoustic paths that will be observed are illustrated in Figure 2b. When the sound travels between A and B the observer will record that the sound travels at an effective speed of

$$c_{\text{eff}}(v) = \sqrt{c^2 + v^2}. \tag{3}$$

This is due to the additional contribution to the speed given by the laminar flow. Additionally, the distance between A and B will be measured to be greater by $\sqrt{l^2 + v^2 t_{AB}^2}$. The increased distance and increased speed cancel so that

$$t_{AB} = t_{BA} = \frac{\sqrt{l^2 + v^2 t_{AB}^2}}{\sqrt{c^2 + v^2}} = \frac{l}{c}, \tag{4}$$

as before.

For the moving observer the bulk flow will also contribute to the effective speed of the pulse from A to C ($c_{\text{eff}} = c + v$) and hinder the return from C to A ($c_{\text{eff}} = c - v$). However, this is again exactly compensated by changes in the total distance that the moving observer measures. As is illustrated in Figure 2b, the effective distance from A to C is $l + vt$. When the sound travels from C to A the effective distance is $l - vt$. Therefore the measured times are

$$t_{AC} = \frac{l + vt_{AC}}{c + v} = t_{CA} = \frac{l - vt_{CA}}{c - v} = \frac{l}{c}. \tag{5}$$

Next, we must check that these timings still hold when the apparatus is moving with respect to the medium (Figure 1b). The equivalence of Figure 1b and Figure 2b demonstrate this. An observer that is stationary with respect to the apparatus (and moving with a speed v with respect to the medium) will record,

$$t_{AB} = t_{BA'} = \frac{\sqrt{l^2 + v^2 t_{AB}^2}}{\sqrt{c^2 + v^2}} = \frac{l}{c}, \tag{6}$$

and

$$t_{AC} = \frac{l + vt_{AC}}{c + v} = t_{CA'} = \frac{l - vt_{CA'}}{c - v} = \frac{l}{c}. \tag{7}$$

Equations 6 and 7 are exactly the same results as equations 4 and 5, respectively. If the observer is instead stationary with respect to medium then it is easy to see that equations 2 are repeated.

In summary, we find that the time it takes the sound to propagate from A to B and back again is identical to the time it takes the sound to propagate from A to C and back, irrespective of the speed of the observer with respect to the medium. The acoustic Michelson-Morely experiment, therefore, should yield a *null* result.

An acoustic interpretation

When distances are measured acoustically it is difficult to interpret experiments that are not stationary with respect to the medium. This is because the effective sound speed (such as is used in equation 3) is contradictory to the acoustic definition of time and space given in equation 1.

For concreteness let us again consider the experiment of Figure 1b, where the apparatus is stationary but is placed in a laminar flow of speed v. The ultrasound physicist has the recorded to-fro times between A, B, and A' and between A, C, and A', which are numerically equal to the value of $2l/c$ (equations 6 and 7). However, unlike the Galilean observer, they cannot directly measure the propagation of the sound through their apparatus. Instead the physicist must try to infer the sound path. To do so they must know the speed, v, of the apparatus with respect to the laminar flow. They can directly measure this by using the pulse echo technique and the definitions of equation 1. In order to *test* the predicted sound paths the ultrasound physicist must have further *a priori* knowledge. In this example we assume that the ultrasound physicist also knows the dimensions of their apparatus, as obtained from the manufacturer's specification, for example.

Knowledge of the apparatus causes the ultrasound physicist to predict the sound path illustrated in Figure 2b. However, the predicted path is subject to the rules of the measurement system and so the ultrasound physicist uses the ultrasound requirement that sound always propagates at a constant speed, c. For the propagation time between A, B and A' they predict (c.f. equation 4)

$$t_{AB} = t_{BA'} = \frac{\sqrt{l^2 + v^2 t_{BA'}^2}}{c} \implies t_{AB} = t_{BA'} = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{l}{c}. \tag{8}$$

For the sound pulse between A and C they predict

$$t_{AC} = \frac{l + v t_{AC}}{c} \implies t_{AC} = \frac{l}{c - v} \tag{9a}$$

and

$$t_{CA'} = \frac{l - v t_{CA'}}{c} \implies t_{CA'} = \frac{l}{c + v} \tag{9b}$$

rather than equation 5. The total to-fro time between A, C and A' is predicted therefore to be

$$t_{AC} + t_{CA'} = \frac{1}{1 - v^2/c^2} \frac{2l}{c}. \tag{10}$$

These predictions are of course wrong. Equation 8 and 10 do not agree with the experimentally measured intervals. The reassignment $c_{\text{eff}} \to c$ made by the ultrasound physicist has resulted in predicting time intervals for the sound to traverse between A, B and A' and between A, C and A' that are too large by a factor of γ and γ^2 respectively, where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{11}$$

is the Lorentz factor. Moreover, the to-fro time between A, B and A' is not predicted to equal the to-fro time between A, C and A'. This is in contradiction to the experimental result. This predicament faced by the ultrasound physicist is, of course, the same as that which faced Lorentz, Poincaré and Einstein at the beginning of the twentieth century.

The error is clear to the Galilean observer: the ultrasound physicist has been forced by the measurement process to set the effective speed of sound to equal c. To solve the problem the ultrasound physicist must compensate for the wrong sound speed by rescaling the temporal and spatial units used when modelling.

The ultrasound physicist, who cannot measure the sound speed, must work a little harder to come to this conclusion. The first explanation that they might try is to doubt the apparatus. If the distance between A, C and A' was actually a factor of γ^2 shorter than the manufacture claimed then the predicted time for that path would match the experimentally measured value. Likewise, all would be well if the distance between A, B and A' was shorter by a factor of γ. However, this explanation can be shown to be incorrect by counting the number of cycles of the sound wave that propagate through each arm of the apparatus. Counting the number of cycles that fit within the apparatus can be achieved by increasing the pulse length until the received signal starts to interfere with the emitted signal. The experimental result would be that $n = 2l/\left(cT\right)$ cycles fit both within the path A-B-A' and A-C-A', where T is the period of the sound wave. This result implies that the distance A-C-A' is not simply shorter than A-B-A' by a factor of γ, for then the number of cycles along each path would be different. Rather, it implies that *all* distances are shorter in the A-C-A' direction. That is, parallel to the flow the *unit of distance* is contracted by a factor of γ.

If the ultrasound physicist incorporates the number of pulses into equation 8 then they would predict that the period between A-B-A' is

$$T_{\text{us}} = \gamma \frac{l/n}{c} = \gamma T, \tag{12}$$

where T_{us} distinguishes the predicted period from the experimentally measured period T. This result can occur if the frequency of the ultrasound pulse is shifted by the factor of γ. However, this interpretation does not make sense physically because the frequency of the pulse is a function of the piezoelectric crystal in the transducer. The only alternative explanation is that the *unit of time* itself is scaled by the factor of γ.

This comparison can be summarised as follows: when modelling an acoustic experiment the unit of distance used in the model must be contracted by the Lorentz factor in order to agree with experimental results, and likewise the unit of time must be slow by the Lorentz factor. These are the results of Poincaré's special relativity Poincaré's postulated contraction in length is the manifestation of the dependence of the sound speed upon the flow of the medium. It exists because a signal that is used to measure distances must be assumed to be a constant, not because the signal is constant but because distances cannot be measured otherwise.

To finish this section we ask what changes to our reasoning must be made if the flow in the acoustic medium is not uniform. In this case the notion of a bulk flow of the medium is not well defined and we have no rule with which to define the 'stationary' frame of reference. We consider what happens if the stationary frame is arbitrarily assigned. To do so, we reconsider the above example for the case that the ob and so there is no stationary frame may be assigned

the laminar flow considered here is local, and is different from the far-away bulk flow of the medium.

That is, what if the flow carrying the

3 Acoustics when measured with ultrasound

Poincaré's relativity postulate does not eliminate the medium. To demonstrate the role of the medium we formulate the acoustics of an ideal fluid that is measured with ultrasound. It is shown in section 3.1 that the acoustics obey Maxwell's relations. The derivation is direct but the co-variant notation makes the comparison to conventional acoustics difficult. In section 3.2 Maxwell's relations are re-derived in the spirit of Lighthill's formulation of aeroacoustics. In doing so the acoustic analogues to the electric and magnetic field are obtained.

When measured with ultrasound the equations of motion of the ideal fluid must be Lorentz invariant. This condition is automatically fulfilled when the equations are obtained from the divergence of the energy-momentum tensor. The condition that the sound speed takes the role of the speed of light is enforced by simply equating these two speeds. This further requires that the energy density of the fluid, as measured acoustically, be a function of the pressure only (barotripic), for the sound speed cannot equal the speed of light otherwise [14].

3.1 The acoustics analogue to Maxwell's relation

The energy-momentum tensor of an ideal fluid is [6;14]

$$T^{ij} = (\epsilon + p)u^i u^j - g^{ij} p \tag{13}$$

where, $\epsilon \equiv \epsilon(p)$ is the barotropic total energy density, p is the pressure, g^{ij} is a diagonal metric tensor with $g^{00} = 1$ and $g^{ii} = -1$ for $i = 1, 2, 3$, and u is the velocity vector of the spacetime path, with the parametrisation chosen such that $u^2 = u^i u_i = 1$. That is, the units of length and time are chosen so that velocity of light is set to unity.

The speed of sound, c, given at constant entropy density, σ, is [6;14]

$$c^2 = \left.\frac{\partial p}{\partial \epsilon}\right|_{\sigma}. \tag{14}$$

This is the same as the non-relativistic expression except that the energy density has replaced the mass density. Setting the speed of sound to equal the speed of light (unity) and the integrating gives the equation of state first introduced by Taub [14],

$$\epsilon(p) = p. \tag{15}$$

Applying 15 to 13 simplifies the energy momentum tensor,

$$T^{ij} = p\left(2u^i u^j - g^{ij}\right) = \tfrac{1}{2}\left(A^i A^j - A^k A_k g^{ij}/2\right) \tag{16}$$

where the vector potential, A, satisfies

$$A^i = 2p^{1/2}u^i = 2\epsilon^{1/2}u^i. \tag{17}$$

The motivation for introducing the 4-vector A is that it represents a potential flow. To demonstrate this, we first note that the relativistic generalisation to the velocity potential, ψ, is defined [6] by

$$\partial_i \psi \equiv -\frac{\epsilon + p}{n} u_i = -\frac{2p}{n} u_i, \tag{18}$$

where $\partial_j \equiv \frac{\partial}{\partial x^j}$ and n is the proper particle number density of the fluid. Equation 15 has been used to obtain the second equality. To show that this is equal to the negative of the potential A, we use an argument of Taub [14]. The internal energy, ϵ, is equal to the sum of the rest mass and the thermodynamic internal energy [6;14], e,

$$\epsilon(p) = nm(1 + e(p)), \tag{19}$$

where m is the proper mass. From the isentropic thermodynamic relation $mde = -pd\left(\frac{1}{n}\right)$ it follows that

$$n d\epsilon = \epsilon dn - n^2 p d\left(\frac{1}{n}\right) = (\epsilon + p)\, dn. \tag{20}$$

Applying equation 15 and integrating we obtain

$$n = \sqrt{p}, \tag{21}$$

from which we find

$$A_i = 2\sqrt{p} u_i = \frac{\epsilon + p}{n} u_i = -\partial_i \psi. \tag{22}$$

In the absence of external fields, the equations of motion are obtained by setting the divergence of the energy momentum tensor (equation 16) to zero. By projecting the divergence of 16 along the timelike component we find

$$u_i \partial_j T^{ij} = \tfrac{1}{2} u_i A^i \partial_j A^j = 0. \tag{23}$$

Since, from 17, the vector A is parallel to u it follows that

$$\partial_j A^j = 0 \tag{24}$$

and so the vector potential A is conserved. The spacelike projection, $\partial_j T^{ij} - u^k u_i \partial_j T^{ij}$, gives in turn,

$$u_j\left(\partial^j A^k - \partial^k A^j\right) = 0. \tag{25}$$

The relativistic vorticity tensor is the exterior derivative (2 form) of the vector potential,

$$F^{jk} \equiv \partial^j A^k - \partial^k A^j \tag{26}$$

and so 25 implies that the vorticity is orthogonal to the velocity.

By taking the divergence of 26 and using 24 it follows that

$$\partial_i \partial^i A^j = \partial_i F^{ij} \tag{27}$$

Equation 27 is a wave equation and so we interpret the right-hand-side of 27 as an acoustic source, a 4-current, J. Therefore

$$\partial_i F^{ij} = J^j. \tag{28a}$$

Furthermore, from 26 we have

$$\epsilon_{ijkl} \partial^j F^{kl} = \epsilon_{ijkl} \partial^j \left(\partial^k A^l - \partial^l A^k \right) = 0, \tag{28b}$$

which follows due to the use of the repeated differential with the Levi-Civita permutation tensor, ϵ_{ijkl}. The two equations of 28 are Maxwell's relation and equation 24 has specified the Lorenz gauge.

As is well known, Maxwell's relations are invariant to a gauge transformation such as

$$A'_i = A_i - \partial_i \psi. \tag{29}$$

This transformation is equivalent to the addition of a potential flow to the equations. However, in equation 22 the vector potential was already interpreted as a potential flow. The gauge invariance is therefore the very same as the required invariance to the bulk flow of the medium. It is the manifestation of the Poincaré relativity postulate.

3.2 The acoustic analogues to the electric and magnetic fields

In classical electromagnetism the electric and magnetic fields are 3-dimensional vector fields that are (usually) measured in the laboratory frame. Such spatial vector quantities we denote in bold.

The most direct method of obtaining the acoustic analogues to the electric and magnetic fields is to project the vorticity tensor, F, into the laboratory frame[2;4]. The analogue to the electric field can then be defined to be the timelike component, and the analogue to the magnetic field the spacelike component. The directness of this method, however, comes at the cost of it bearing little resemblance to conventional acoustics.

To demonstrate the similarities and the differences between the ultrasound and the Galilean formulations of acoustics we re-derive Maxwell's relations using a relativistic version of Lighthill's formulation of aeroacoustics[7]. The analogues to the electric and magnetic field become clear in this process.

We start by projecting the temporal and spatial equations of motion, equations 24 and 25, into the laboratory frame. The result is

$$\nabla \cdot \mathbf{A} = -\partial_t \phi, \tag{30a}$$

$$\partial_t \mathbf{A} - \mathbf{v} \times (\nabla \times \mathbf{A}) = -\nabla \phi. \tag{30b}$$

ϕ and $\mathbf{A}$ are the temporal and spatial components of the vector potential A,

$$\phi \equiv 2\gamma\sqrt{p} \qquad \text{and} \qquad \mathbf{A} \equiv \phi \mathbf{v}, \tag{31}$$

where $\mathbf{v}$ is the velocity of the fluid as measured in the laboratory frame and $\gamma = (1 - \mathbf{v}^2)^{-1}$, as in 11.

The potential ϕ may be interpreted as the relativistic total enthalpy multiplied by the particle mass. To see this we first introduce the non-relativistic enthalpy, h, which is defined by

$$h \equiv e + p/(nm). \tag{32}$$

It then follows that

$$\phi = \gamma \frac{\epsilon + p}{n} = \gamma m \left(c^2 + h\right), \tag{33}$$

In the non-relativistic limit this becomes

$$\phi \to \left(1 + \tfrac{1}{2}\tfrac{v^2}{c^2}\right)\left(mc^2 + mh\right) = mc^2 + \tfrac{1}{2}mv^2 + mh \quad \text{as } v/c \to 0. \tag{34}$$

The term $h + \frac{1}{2}v^2$ is the usual expression of the total enthalpy. Equation 34 multiplies this by the particle mass, m, and adds the rest energy, mc^2, which is absent from all non-relativistic thermodynamics.

Equations 30a and 30b are the acoustically measured versions of the continuity and Euler equations. In the non-relativistic limit the equations reduce to Galilean invariant forms,

$$\nabla \cdot (\rho \mathbf{v}) = -\partial_t \rho, \tag{35a}$$

$$\partial_t \mathbf{v} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla \phi. \tag{35b}$$

Equation 35b is Euler's equation written in Crocco's form [5] and Equation 35a is very similar to the Galilean continuity equation. The difference between equation 35a and the Galilean form is that the mass density, ρ, in the Galilean form is replaced by the mass-scaled total enthalpy density, ϕ.

With the acoustic version of the continuity and Euler equations in hand, we may now apply the conventional formulations of acoustics. We proceed with Lighthill's acoustic analogy [5;7]. To do so we differentiate the continuity equation (equation 30a) with respect to time and subtract it from the spatial derivative of Euler's equation (equation 30b). A wave equation for the total enthalpy results

$$\left(\nabla^2 - \partial_t{}^2\right)\phi = \nabla \cdot (\mathbf{v} \times (\nabla \times \mathbf{A})) \equiv -\rho_q. \tag{36a}$$

Next, a wave equation for $\mathbf{A}$ is obtained by differentiating the continuity equation with respect to space and then adding the result to the temporal derivative of Euler's equation,

$$\left(\boldsymbol{\nabla}^2 - {\partial_t}^2\right) \mathbf{A} = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \partial_t \left(\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right) \equiv -\mathbf{J}. \tag{36b}$$

In keeping with Lighthill's analogy we interpret the right hand side of 36a and 36b as an acoustic source density, ρ_q, and acoustic current density, $\mathbf{J}$, respectively.

For comparison, had we carried out this procedure with the Gallilean continuity and Euler equation we would have obtained [5],

$$\left[D_t\left(\frac{1}{c^2}D_t\right) - \frac{1}{\rho}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla})\right]\phi = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot\left(\rho\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})\right) \tag{37a}$$

$$\left[D_t\left(\frac{1}{c^2}D_t\right) - \frac{1}{\rho}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla})\right]\mathbf{v} = \frac{1}{\rho}\boldsymbol{\nabla}\times\left(\rho\boldsymbol{\nabla}\times\mathbf{v}\right). \tag{37b}$$

These are Lighthill's equations expressed in terms of enthalpy and vorticity [5]. The operator $D_t = \partial_t + \mathbf{v}\cdot\boldsymbol{\nabla}$ on the left hand side of both equations of 37 describe a non-linear wave in homoentropic potential flow [5].

Equations 36a and 36b can be simplified by introducing

$$\mathbf{E} = -\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{38}$$

so that

$$\left(\boldsymbol{\nabla}^2 - {\partial_t}^2\right) \phi = -\boldsymbol{\nabla}\cdot\mathbf{E} \equiv -\rho_q. \tag{39a}$$

and

$$\left(\boldsymbol{\nabla}^2 - {\partial_t}^2\right) \mathbf{A} = -\boldsymbol{\nabla}\times\mathbf{B} + \mathbf{E} \equiv -\mathbf{J}. \tag{39b}$$

Equations 39 can now be recognised as Maxwell's equations written in terms of the potentials in the Lorenz gauge [2]. The vector $\mathbf{E}$ is known as the Lamb vector and is proportional to the Coriolis acceleration; it takes the role of the electric field in the analogy. The axial vector $\mathbf{B}$ is the spatial vorticity and takes the role of the magnetic field.

Writing out Maxwell's 4 equations explicitly gives

$$\boldsymbol{\nabla}\cdot\mathbf{E} = q, \tag{40a}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{J} + \partial_t E, \tag{40b}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\partial_t\mathbf{B}, \tag{40c}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{40d}$$

The acoustic interpretation of these equations are as follows:

1. Equation 40a is the definition of an acoustic source.
2. Equation 40b is the definition of the acoustic current.
3. Equation 40c is the Lorentz invariant version of the vorticity equation.
4. Equation 40d is an expression of Helmholtz theorem, which demands the conservation of vorticity.

4 Discussion

This report has demonstrated that when the acoustic definitions of time and space are used the world must be modelled in a Lorentz invariant fashion. Otherwise theoretical results do not match experiment. Ultrasound, as well as other sonar-based technologies, is therefore a relativistic subject where the speed of sound takes the role of the speed of light.

Light signals are not special in the relativity theory used in this report. They have been completely replaced by acoustic signals. This is possible because a propagating signal, whatever it may be, must be assumed to have a constant speed in order to measure distances with local temporal measurements. It is impossible to use sound, for example, to measure local variations in the speed of sound - even if such variations 'really occur'.

The æther does have a role in the relativity theory used in this report: sound does have a propagating medium. However, the æther is relativistic in the sense that it does not define a state of absolute rest. The speed of an object may be defined with respect to the bulk flow of the medium but the actual speed of the bulk flow is not determined.

The acoustics of a barotropic ideal fluid medium was explicitly considered. When the flow of the medium with respect to the transducer is measured acoustically it was shown that the energy momentum tensor may be written entirely as a function of the potential flow. It was then demonstrated that the acoustics of the medium obey Maxwell's relations. It was shown that Gauge invariance in Maxwell's relations may be interpreted as an invariance to the potential flow of the medium. Gauge invariance and a relativistic æther are the same condition.

By using a relativistic version of Lighthill's formulation of aeroacoustics it was demonstrated that the acoustic analogue to the electric field is the Lamb vector (proportional to the Coriolis acceleration), and that the acoustic analogue to the magnetic field is the vorticity, The spacetime vorticity tensor takes the role of the field tensor. An analogy in this form has long been suspected [8;13], however, to the author's knowledge this is the first time that the analogy has been completed. The key step, missing in previous attempts, is to note that acoustics must be formulated in terms of a Lorentz invariant fluid where *the speed of sound equals the speed of light.* It is only when this step is made that the analogy exists. The motivation for this step is obvious only when it is appreciated that the speed of sound may take the role of the speed of light in a relativistic theory.

In retrospect the derivation of the Maxwell relation not too surprising. Both acoustics as measured with ultrasound and electromagnetism as measured with light attempt to measure the properties of their propagating signal. Both, therefore, represent a similarly limited view of the world with the limitations manifesting themselves in the linearity of the equations.

In *Science and Hypothesis*[12] Poincaré declared that "Experiment is the sole source of truth" and in this report we agree. We must admit, however, that the measured dimensions of an object depends upon whether that object is measured with light or sound. This is because when time and space are measured with light the speed of sound is not constant, and when the time and space are measured with sound the speed of light is not constant. The two measurement systems cannot therefore be mixed. As a consequence every truth must be declared with a caveat that states the modality of the measurement. We emphasise that proceeding otherwise does not reduce the anthropocentric nature of the measurement, where results depend upon how the measurement was made; it merely prejudices one set of instruments over all others.

4.1 On the absence of non-linear propagation

In this report we have demonstrated that when time and space are measured acoustically the propagation of sound is linear. However, this is at odds with the understanding in medical ultrasound that the non-linear propagation of an acoustic pulse is not only measurable but also important. Our task is to explain how the non-linearity found in other measurement systems manifests itself in acoustic measurements. To do so it is useful to frame the discussion around the linear acoustically measured equations of 36 and their non-linear Galilean forms of 37.

The first point to note is that the non-linearity of the Galilean formulation of sound is entirely a matter of *convention*. It would in fact be more appropriate to rewrite equations 37 as linear wave equations with everything else interpreted as acoustic sources and currents. Then the sound is defined as the part of an acoustic disturbance that can propagate energy away to infinity; the rest of the disturbance being a 'local' source. This is, in fact, the usual final step of Lighthill's analogy. The reason it is rarely performed when the analogy is written in terms of the total enthalpy and vorticity is because the acoustic source terms become horribly complicated. The split of source and wave in 37 is convenient interpretatively, but is nevertheless rather ad-hoc, for it mixes local terms with those that can propagate indefinitely.

The influence of non-linear propagation on what can be measured acoustically is found by comparing the right hand sides of equations 36 and equations 37. It is seen that the only major difference between the two is the term $-\partial_t \left(\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right)$ on the right-hand-side of 36b. This term is part of the non-linear operator in equation 37b. It is, if you like, a ghost of the non-linear operator D_t on what can be measured acoustically. When measured with ultrasound it is interpreted as part of the current.

We note that an attempt to re-incorporate the ghost term back into some 'acoustically measured non-linear operator' would be ill-conceived for it would mean that the acoustic current is no longer conserved: the term $-\partial_t \left(\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right)$ most certainly is a current.

4.2 Other similar studies

Relativistic fluids where the sound speed equals the speed of light have been studied many times before as theoretical curiosities [9;14]. For example, Pekeris observed that Hick's spherical vortex conserves angular momentum if and only if the sound speed equals the speed of light [9]. The importance of such fluids, however, has not to the author's knowledge been recognised. Such fluids represent *what can be measured* when distances are obtained by echo-location.

Another interesting analogy between acoustics and electromagnetism is 'acoustic analogue gravity' literature (see Barceló, Liberati and Visser [1] for a review). The approach constructs an *acoustic* metric that describes the acoustics of sound carried in bulk flow. While the description of space and time in this formulation is Euclidean, the acoustic metric turns out to be pseudo-Euclidean, and therefore obeys the Lorentz transformation. This results because sound carried away in bulk flow faster than the speed of sound will never reach us. The speed of sound is therefore a limiting velocity in transformations. The analogue gravity literature then goes on to study the gravitational implications of the acoustic metric.

While the motivations behind the two approaches is similar, in particular the demand that acoustics must be obey a Lorentz invariant metric, the approach given here and the approach of analogue gravity are fundamentally different. Analogue gravity does not consider the measurement process and so operates within a world characterised by two metrics, the Lorentz invariant acoustic metric and the Galilean invariant spacetime metric. Ultrasound measurement demands that the world be described by a Lorentz invariant metric. There is no other way to describe space and time acoustically. In analogue gravity the acoustic metric is Lorentz-invariant, but is not the same as the metric used here. In analogue gravity the metric is a function of the bulk flow, whereas we argue that this is impossible: the sound speed must be an a priori constant in order to say anything about the world. Nevertheless, the success of analogue gravity is very encouraging.

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