An acoustic theory of special relativity

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Abstract

In medical ultrasound spatial intervals are determined from a measured pulse-echo time and an assumed constant speed of sound. This report demonstrates that in order to correctly predict the results of an ultrasound experiment physical models must be Lorentz invariant, with the speed of sound taking the traditional role of the speed of light. The models used in ultrasound are therefore subject to relativistic considerations.

Sound does has a propagating medium. The acoustic theory of relativity presented here is therefore different to Einstein's formulation, but is very close to Poincaré's version. The role of the medium is investigated by considering an ideal fluid that is measured by the acoustic pulse-echo technique. It is demonstrated that the generation and the propagation of the sound in such a system obeys the same physical laws as Maxwell's electrodynamics. The acoustic analogue to the electric field is the Lamb vector and the acoustic analogue to the magnetic field is the vorticity.

1 Introduction

The privileged role of light is perhaps the most mysterious aspect of Einstein's special theory of relativity. What is it about this signal, as opposed to any other method of communication, that makes it so fundamental to the concepts of time and space? The answer, for Einstein, is that light has a constant speed that is independent of the motion of the light source [4]. This property enables the distance between an observer and a faraway object to be measured by determining the to-fro propagation time of a pulse of light. The measurement is local to the observer and all the observer requires to make the measurement is a light source, a receiver and a clock. Einstein's definition of measurement is then completed with his principle of relativity, which demands that a measuring device at rest with respect to an observer, such as a clock or a metre rule, gauge quantities that are independent of the observer's motion [4;13].

In Einstein's theory the observer does not need to know their speed relative to the speed of the light medium - the *æther*. The postulates demand that ætherial motion neither alters the speed of light nor alters the units of the measurements made by the observer. The observer can therefore consider herself to be stationary and not consider the æther at all. This elimination of the æther, however, only emphasises the uniqueness of light, for it is the only signal that has a medium with no measurable mechanical properties. Satisfactory definitions of time and space seem to come at the expense of making light an even greater puzzle, more and more distant from the world that we can touch and hear.

The relativity theory of Poincaré is different. For Poincaré, light does have a medium and all motions can be measured with respect to it; 'stationary' means stationary with respect to the æther [13;17;18]. Unlike Einstein's theory, the length of a measuring rule used by an observer is affected by the observers motion. Indeed, Poincaré *postulates* that it contracts from its length measured when stationary with respect to the æther, with the size of the contraction determined by the spatial Lorentz transformation [13;16]. It follows that the principle of relativity is also different; Poincaré assumes that there is no absolute reference from which to measure the speed of the æther [16]. However, Poincaré's formulation of special relativity is not at odds with any experimental confirmation of Einstein's theory [12]. This is because Poincaré, like Einstein, uses the Lorentz transformations to switch between the spatial-temporal measurements of different observers, and because both theories are invariant in the quadratic form; light does have a constant speed in Poincaré's theory of relativity [12]. Like Einstein, Poincaré recognises that the constancy of the speed of light is postulate. In 1898 [14] he notes that when an astronomer measures the speed of light,

> He has begun by supposing that light has a constant velocity, and in particular that its velocity is the same in all directions. That is a postulate without which no measurement of this velocity could be attempted.

However, Poincaré's theory does not depend upon this postulate, for Poincaré uses the Lorentz length contraction instead [13].

Light is not a privileged signal in Poincaré's theory but this flexibility is obtained only at the cost of admitting motion dependant deformations. A fundamental explanation for these is missing, however, and this gives Poincaré's theory an incomplete feel. Einstein's theory is, of course, just as incomplete as it does not answer why light should enter, through the concepts of time and space, every physical force. With regard to this question Poincaré [16] notes that:

> Either there is nothing in the world that is not of electromagnetic origin, or this part [the speed of light], which is common to all physical phenomena, is only an appearance, something stemming from our methods of measurement.

Modern physical theories do not agree with the first of these options. The consequences of the second, however, are seldom addressed. In any case, when faced with a choice between the relativity theories of Einstein and Poincaré, the community choose Einstein's.

In this report we use sound to define time and space in the manner routinely used in medical ultrasound and other sonar-based technologies. It is demonstrated in section 2 that in order for ultrasound theory to agree with experiment the Lorentz transformations need to be applied. This is achieved by explicitly considering an acoustic analogue to the Michelson-Morely experiment. Since sound does have a mechanical medium through which it propagates and since ultrasound does not care about the speed of light it is the relativity theory of Poincaré that is recovered in acoustics. It is demonstrated in section 2 that Poincaré's motion dependent contraction results from the dependence of the sound speed on the bulk flow of the medium. The contraction is real in the sense that it is measured. However, since the contraction results from the measurement process there is no need to seek some fundamental interaction between material objects and their æther.

In section 3 it is demonstrated that when time and space are defined with sound, the acoustics of an ideal fluid obey the same relations as Maxwell's formulation of electrodynamics. Therefore the generation and propagation of sound, when time and space are defined with sound, obey the same physical law as the generation and propagation of light, when time and space are defined with light.

2 The acoustic definition of time and space

In medical ultrasound distances are measured using the time it takes a pulse of sound to propagate from a transducer to a reflecting object and then to return again. If the sound is emitted from the transducer at a time, τ^-, and the sound returns at a time, τ^+, then the task is to find from these two numbers the spatio-temporal location, x, of the point of reflection.

What happens to the sound in between leaving the transducer and returning cannot be known by acoustic measurement. In this ignorance ultrasound practitioners assume that the time at which the echo occurred is the midpoint of τ^- and τ^+,

$$\tau(x) = \frac{\tau^+ + \tau^-}{2}. \tag{1a}$$

Other choices could certainly be made, but would imply a knowledge of the world beyond that learnt from τ^- and τ^+ alone. To measure distances from the times τ^- and τ^+ a sound speed, c, is required. Assuming, again in ignorance, that the sound returns at the same speed at which it left gives

$$\rho(x) = \frac{\tau^+ - \tau^-}{2} c. \tag{1b}$$

These are the definitions of time and space that are used in ultrasound. They are also identical to definitions used by Poincaré [13;17] and Einstein [2;4] with the exception that the speed, c, is here the speed of sound rather than the speed of light.

Equation 1b requires an *a priori* knowledge of the sound speed for otherwise distances cannot be determined from temporal measurements. In diagnostic ultrasound scanners this speed is usually taken to be 1540 ms^{-1}. The constancy of the speed of sound is identical to Einstein's second postulate for special relativity [4], except that the sound speed takes the role of the speed of light. However, the speed of sound is here a constant not because of some physical law but because when using sound to make measurements there is no other choice but to assume the sound speeds constancy. As discussed in the introduction, this conforms more to Poincaré's view of the light postulate than to Einstein's.

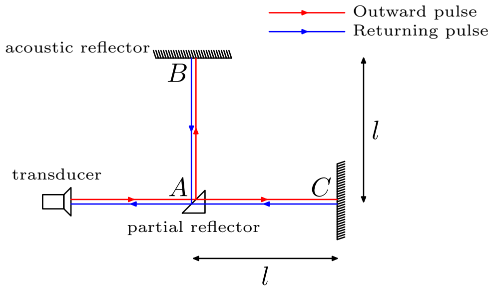
Ultrasound has inherited from fluid mechanics the principle that it is impossible to determine absolute uniform motions: an object at rest in a laminar flow is equivalent to an object moving uniformly in a stationary fluid. The velocity of an object within a fluid, or even parts of the fluid itself, may always be measured with respect to the bulk flow of the fluid. The notion of a true speed with respect to some absolute reference is never invoked. This is the relativity postulate as envisaged by Poincaré.

2.1 Physics as measured with ultrasound

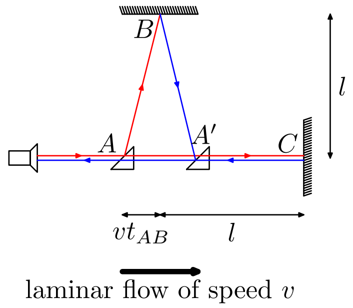
The measurement rules of equations 1a and 1b enable two properties of the world as measured by ultrasound to be stated immediately. The first is that an entity that moves away from the transducer at a speed that is faster than the speed of sound (with respect to the bulk flow of the medium) cannot be measured. This is not because such motions are impossible but because the sound will never catch up with the entity and so there will never be an echo to record.

The second is that ultrasound is not capable of measuring variations in the speed of sound. Since the sound speed must be known before any distance can be measured, changes in the sound speed cannot be measured. They may only be determined with additional *a priori* knowledge of the structure of the medium. Therefore, in ultrasound theory, fluctuations in the density cannot alter the sound speed. It follows that the acoustic medium must be incompressible (in the relativistic sense [10;11;20]) and that sound must propagate according to a linear wave equation. In section 3 it is demonstrated that this linear relation is identical to Maxwell's relation.

The ultrasound literature does not comply with these remarks. Currently, when modelling an ultrasound experiment, a fluid medium is always described by a Galilean invariant theory such as Euler's equation or the Naiver-Stokes equation. The resulting model is then capable of predicting motions that are faster than the speed of sound and predicts that a sound pulse propagates according to a non-linear wave equation. Both of these predictions are impossible when the world is measured with sound. The ultrasound literature fails to recognise the distinction between two equally valid descriptions of the world - the world that is seen and the world that is heard. Curiously, ultrasound physics repeats the fallacy that the world must be seen to be believed.



(a) Apparatus when bulk medium is stationary.



(b) Apparatus when the laminar flow of the bulk medium is v

Figure 1: A pulse-echo experiment when there is, and is not, a relative laminar flow past the apparatus.

2.2 An acoustic Michelson-Morely experiment.

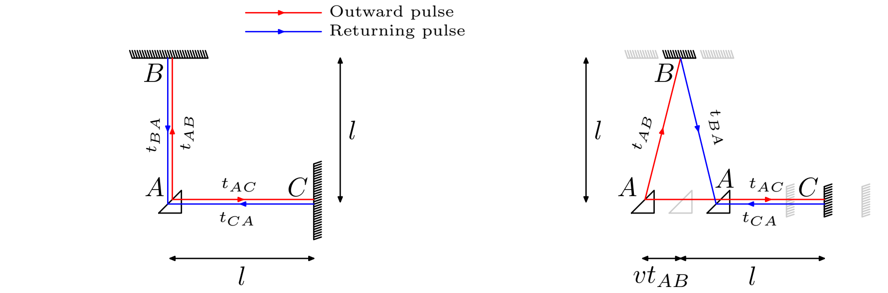
The discussion so far has been somewhat abstract. To make it concrete it is useful to discuss a simple pulse-echo experiment and compare the two view-points - the Galilean[1] world that is *seen*, with the world that is measured with ultrasound.

The first case to be considered is illustrated in Figure 1a. This apparatus is appropriate when the equipment is stationary with respect to the bulk flow of the medium. It is analogous to Michelson and Morely's famous experiment: a piezoelectric transducer replaces both the light source and the receiver while a medium that partially reflects sound replaces the semi-silvered mirror. The distance between A and B is denoted l and is the same as the distance between A and C. In the following the time it takes for the sound to propagate from A to B and back again is compared with the to-fro times between A and C.

If the apparatus of Figure 1a were not stationary with respect to the bulk flow of the medium then the experiment would fail. This is because the sound would not travel from A to B and return; the motion of the medium would drag the sound pulse with it. The setup illustrated in Figure 1b gives spirit of the Michelson-Morely experiment for the case when the apparatus is not stationary with respect to the bulk flow. In this case there are two separate partially reflecting surfaces. The time it takes the sound to propagate from A to B to A' is now compared with the time it takes the sound to go from A to C to A'.

When the to-fro times along the two arms are the same, irrespective of the flow of the bulk medium, the result is described as *null*. This is in accordance to the description of the Michelson-Morely result.

[1] Formally the 'Galilean' measurements are the distances and times that are measured with light signals in accordance to Einstein's method [4]. In ultrasound experiments, however, the Galilean approximation is entirely appropriate.



(a) Observer stationary with respect to medium.

(b) Observer moves at speed v with respect to medium

Figure 2: Observed motion of apparatus for the setup of Figure 1a.

A Galilean interpretation

First we consider the case of the apparatus being stationary with respect to the bulk flow (Figure 1a). If the propagation of the sound pulse were observed by a Galilean observer that is also stationary with respect to the flow then she would observe the sound travelling according to Figure 2a. The time, t_{AB}, it takes for the sound to propagate from A to B is the same as the time, t_{BA}, it takes the sound to propagate from B to A. It is given by l/c, where c is the speed of sound of the medium. This time interval is the same for the to and fro paths between A and C,

$$t_{AB} = t_{BA} = t_{AC} = t_{CA} = l/c. \tag{2}$$

An observer for whom both the medium and apparatus flow past at a speed, v, will measure the same time intervals but will witness an altogether more complicated experiment. The acoustic paths that will be observed are illustrated in Figure 2b. When the sound travels between A and B the observer will record that the sound travels at an effective speed of

$$c_{\text{eff}}(v) = \sqrt{c^2 + v^2}. \tag{3}$$

This is due to the contribution of the laminar flow. Additionally, the measured distance between A and B will be greater by $\sqrt{l^2 + v^2 t_{AB}^2}$. The increased distance and increased speed cancel so that

$$t_{AB} = t_{BA} = \frac{\sqrt{l^2 + v^2 t_{AB}^2}}{\sqrt{c^2 + v^2}} = \frac{l}{c}, \tag{4}$$

as before.

For the moving observer the bulk flow will also contribute to the effective speed of the pulse from A to C ($c_{\text{eff}} = c + v$) and hinder the return from C to A ($c_{\text{eff}} = c - v$). However, this is again exactly compensated by changes in the total distance that the moving observer measures. As is illustrated in

Figure 2b, the total distance from A to C is $l + vt$. When the sound travels from C to A the total distance is $l - vt$. Therefore the measured times are

$$t_{AC} = \frac{l + vt_{AC}}{c + v} = t_{CA} = \frac{l - vt_{CA}}{c - v} = \frac{l}{c}. \tag{5}$$

Next, we must check that these timings still hold when the apparatus is moving with respect to the medium (Figure 1b). The equivalence of Figure 1b and Figure 2b demonstrate this. An observer that is stationary with respect to the apparatus (and moving with a speed, v, with respect to the medium) will record,

$$t_{AB} = t_{BA'} = \frac{\sqrt{l^2 + v^2 t_{AB}^2}}{\sqrt{c^2 + v^2}} = \frac{l}{c}, \tag{6}$$

and

$$t_{AC} = \frac{l + vt_{AC}}{c + v} = t_{CA'} = \frac{l - vt_{CA'}}{c - v} = \frac{l}{c}. \tag{7}$$

Equations 6 and 7 are exactly the same results as equations 4 and 5, respectively. If the observer is instead stationary with respect to medium then it is easy to see that equations 2 are repeated.

In summary, we find that the time it takes the sound to propagate from A to B and back again is identical to the time it takes the sound to propagate from A to C and back, irrespective of the speed of the observer with respect to the medium. The acoustic Michelson-Morely experiment, therefore, should yield a *null* result.

An acoustic interpretation

Unlike the Galilean observer, the ultrasound physicist cannot directly measure the propagation of sound. The sound path of a pulse-echo experiment must be inferred afterwards from the measurements and the definitions of equation 1. In order to predict a sound path the ultrasound physicist must have further *a priori* knowledge, which we assume here to be the dimensions of the apparatus.

Let us again consider the experiment of Figure 1a, where all the apparatus is stationary with respect to the bulk flow of the fluid. The sound path illustrated in Figure 2a is the simplest through the apparatus and we assume that this path is predicted. To test this prediction the ultrasound physicist measures the to-fro times between A and B and between A and C. Both of these times are equal to $2l/c$ (equation 2), which is consistent with the known lengths, l. The predicted sound paths are to this extent confirmed.

Let us now suppose that the same setup is measured by a transducer that moves uniformly at a speed, v, with respect to the medium. Again with knowledge of the apparatus, we assume that the ultrasound physicist predicts the simplest path. This is the path illustrated in Figure 2b and is the same path that is measured by the moving Galilean observer. A difference from the Galilean case arises, however, because the predicted path is subject to the rules of the measurement system and, for the ultrasound physicist, sound always propagates at a constant speed, c. For the propagation time between A and B the ultrasound physicist therefore predicts (c.f. equation 4)

$$t_{AB} = t_{BA} = \frac{\sqrt{l^2 + v^2 t_{BA}^2}}{c} \implies t_{AB} = t_{BA} = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{l}{c}. \tag{8}$$

For the sound pulse between A and C they predict

$$t_{AC} = \frac{l + v t_{AC}}{c} \implies t_{AC} = \frac{l}{c - v} \tag{9a}$$

and

$$t_{CA} = \frac{l - v t_{CA}}{c} \implies t_{CA} = \frac{l}{c + v} \tag{9b}$$

rather than equation 5. Therefore the total to-fro time between A and C is predicted to be

$$t_{AC} + t_{CA'} = \frac{1}{1 - v^2/c^2} \frac{2l}{c}. \tag{10}$$

These predictions are of course wrong. Equation 8 and 10 do not agree with the experimentally measured intervals. The reassignment $c_{\text{eff}} \to c$ made by the ultrasound physicist has resulted in predicting time intervals for the sound to traverse between A and B and between A and C that are too large by a factor of γ and γ^2 respectively, where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{11}$$

is the Lorentz factor. Moreover, the to-fro time between A and B is not predicted to equal the to-fro time between A and C, in contradiction to the experimental result. This predicament faced by the ultrasound physicist is, of course, the same as that which faced Lorentz, Poincaré and Einstein at the beginning of the twentieth century.

The error is clear to the Galilean observer: the ultrasound physicist has been forced by the measurement process to set the effective speed of sound to equal c. To solve the problem the ultrasound physicist must compensate for the wrong sound speed by rescaling the temporal and spatial units used when modelling.

The ultrasound physicist, who cannot measure variations in the sound speed, must work a little harder to come to this conclusion. The first explanation that they might try is to doubt the apparatus. If the distance between A and C was actually a factor of γ^2 shorter than the manufacture claimed then the predicted time for that path would match the experimentally measured value. Likewise, all would be well if the distance between A and B was shorter by a factor of γ. However, this explanation can be shown to be incorrect by counting the number of cycles of the sound wave that propagate through each arm of the apparatus. This can be done straight-forwardly with ultrasound, the pulse length is simply increased until the received signal starts to interfere with the emitted signal. The experimental result would be that $n = 2l/(cT)$ cycles fit both between A and B and between A and C, where T is the period of the sound wave. This result implies that the distance between A and C is not simply shorter than between A and B, for then the number of cycles along each path would be different. Rather, it implies that *all* distances are shorter in the A-C direction, including the wavelength of the sound wave. That is, parallel to the motion the *unit of distance* is contracted by a factor of γ.

If the ultrasound physicist incorporates the number of pulses into equation 8 then they would predict that the period between A and B is

$$T_{\mathrm{us}} = \gamma \frac{l/n}{c} = \gamma T, \tag{12}$$

where T_{us} distinguishes the predicted period from the experimentally measured period T. That is to say, the *unit of time* used in the model must be scaled by a factor of γ in order to agree with experiment.

As before, the equivalence of Figure 1b and Figure 2b guarantee that the same conclusion would be drawn when the apparatus is not stationary with respect to the bulk flow.

The comparison between the Galilean and experimental observer can be summarised as follows: when modelling an acoustic experiment the unit of distance used in the model must be contracted by the Lorentz factor in order to agree with experimental results, and likewise the unit of time must be reduced by the Lorentz factor. These are the results of Poincaré's special relativity. Poincaré's postulated contraction in length is the manifestation of the dependence of the sound speed upon the flow of the medium. It exists because the speed of a signal that is used to measure distances must be assumed to be a constant, not because the speed is constant but because distances cannot be measured otherwise.

3 Acoustics when measured with ultrasound

Poincaré's relativity postulate does not eliminate the medium. To demonstrate the role of the medium we formulate the acoustics of an ideal fluid that is measured with ultrasound, where the motions of the fluid are understood to be local perturbations to the bulk flow. It is shown in section 3.1 that the acoustics obey the same law as Maxwell's equations of electromagnetism. The derivation is direct but the co-variant notation makes the comparison to conventional acoustics difficult. In section 3.2 Maxwell's relations are re-derived in the spirit of Lighthill's formulation of aeroacoustics. In doing so the acoustic analogues to the electric and magnetic field are obtained.

A model that is to be compared to ultrasound measurements must be Lorentz invariant. This condition is automatically fulfilled when the equations of fluid motion obtained from the divergence of the energy-momentum tensor of an ideal fluid. The condition that the sound speed takes the role of the speed of light is enforced by simply equating these two speeds. This further requires that the energy density of the fluid, as measured acoustically, be a function of the pressure only (barotripic), for the sound speed cannot equal the speed of light otherwise[20].

3.1 The acoustics analogue to Maxwell's relation

The energy-momentum tensor of an ideal fluid is [7;20]

$$T^{ij} = (\epsilon + p)u^i u^j - g^{ij} p \tag{13}$$

where, $\epsilon \equiv \epsilon(p)$ is the barotropic total energy density, p is the pressure, g^{ij} is a diagonal metric tensor with $g^{00} = 1$ and $g^{ii} = -1$ for $i = 1, 2, 3$, and u is the velocity vector of the spacetime path, with the parametrisation chosen such that $u^2 = u^i u_i = 1$. That is, the units of length and time are chosen so that velocity of light is set to unity.

The speed of sound, c, given at constant entropy density, σ, is [7;20]

$$c^2 = \left.\frac{\partial p}{\partial \epsilon}\right|_\sigma . \tag{14}$$

This is the same as the non-relativistic expression except that the energy density has replaced the mass density. The speed of sound to equal the speed of light (unity) if

$$\epsilon(p) = p. \tag{15}$$

This equation of state was first introduced by Taub [20].

Applying 15 to 13 simplifies the energy momentum tensor,

$$T^{ij} = p\left(2u^i u^j - g^{ij}\right) = \frac{1}{2\Xi_0^2}\left(A^i A^j - A^k A_k g^{ij}/2\right) \tag{16}$$

where the vector potential, A, satisfies

$$A^i = 2\tfrac{1}{\Xi_0} p^{1/2} u^i = 2\tfrac{1}{\Xi_0} \epsilon^{1/2} u^i. \tag{17}$$

The constant scale-factor, Ξ_0, is determined from the ambient proper number density of the fluid, n_0, and the ambient pressure, p_0, as follows,

$$\Xi_0 = \frac{n_0}{\sqrt{p_0}}. \tag{18}$$

The motivation for introducing the 4-vector A is that it represents a potential flow. To demonstrate this, we first note that the relativistic generalisation to the velocity potential, ψ, is defined [7] by

$$\partial_i \psi \equiv -\frac{\epsilon + p}{n} u_i = -\frac{2p}{n} u_i, \tag{19}$$

where $\partial_j \equiv \frac{\partial}{\partial x^j}$ and n is the proper particle number density of the fluid. Equation 15 has been used to obtain the second equality. To show that this is equal to the negative of the potential A, we use a thermodynamic argument given by Taub [20]. The internal energy density, ϵ, is equal to the sum of the rest mass and the internal energy per particle [7;20], e,

$$\epsilon(p) = nm(1 + e(p)), \tag{20}$$

where m is the proper mass. From the isentropic thermodynamic relation $mde = -pd\left(\frac{1}{n}\right)$ it follows that

$$nd\epsilon = \epsilon dn - n^2 pd\left(\frac{1}{n}\right) = (\epsilon + p)\, dn. \tag{21}$$

Applying equation 15 and integrating we obtain

$$n = \Xi_0 \sqrt{p}, \tag{22}$$

where Ξ_0 is the constant introduced in 18. With the aid of equation 15 it follows that

$$A_i = 2\tfrac{1}{\Xi_0}\sqrt{p}u_i = \frac{\epsilon + p}{n}u_i = -\partial_i \psi, \tag{23}$$

as asserted.

In the absence of external fields, the equations of motion are obtained by setting the divergence of the energy momentum tensor (equation 16) to zero. By projecting the divergence of 16 along the timelike component we find

$$u_i \partial_j T^{ij} = \tfrac{1}{2} u_i A^i \partial_j A^j = 0. \tag{24}$$

Since, from 17, the vector A is parallel to u it follows that

$$\partial_j A^j = 0 \tag{25}$$

and so the vector potential A is conserved. The spacelike projection, $\partial_j T^{kj} - u^k u_i \partial_j T^{ij}$, gives in turn,

$$u_j \left(\partial^j A^k - \partial^k A^j\right) = 0. \tag{26}$$

The relativistic vorticity tensor is the exterior derivative of the vector potential,

$$F^{jk} \equiv \partial^j A^k - \partial^k A^j \tag{27}$$

and so 26 implies that the vorticity tensor is orthogonal to the velocity.

By taking the divergence of 27 and using 25 it follows that

$$\partial_i \partial^i A^j = \partial_i F^{ij}. \tag{28}$$

Equation 28 is a wave equation and so we interpret the right-hand-side of 28 as an acoustic source, a 4-current, J. Therefore

$$\partial_i F^{ij} = J^j. \tag{29a}$$

Furthermore, from 27 we have

$$\epsilon_{ijkl} \partial^j F^{kl} = \epsilon_{ijkl} \partial^j \left(\partial^k A^l - \partial^l A^k\right) = 0, \tag{29b}$$

which follows due to the use of the repeated differential with the Levi-Civita permutation tensor, ϵ_{ijkl}. The two equations of 29 are Maxwell's relation and equation 25 has specified the Lorenz gauge.

As is well known, Maxwell's relations are invariant to a gauge transformation such as

$$A'_i = A_i - \partial_i \psi. \tag{30}$$

This transformation is equivalent to the addition of a potential flow to the equations. However, in equation 23 the vector potential was already interpreted as a potential flow. The gauge invariance is therefore the very same as the required invariance to the bulk flow of the medium. It is the manifestation of the Poincaré relativity postulate.

3.2 The acoustic analogues to the electric and magnetic fields

In classical electromagnetism the electric and magnetic fields are 3-dimensional vector fields that are (usually) measured in the laboratory frame. Such spatial vector quantities are denoted in bold in this section.

The most direct method of obtaining the acoustic analogues to the electric and magnetic fields is to project the vorticity tensor, F, into the laboratory frame [3;5]. The analogue to the electric field can then be defined to be the timelike component, and the analogue to the magnetic field the spacelike component [5]. The directness of this method, however, comes at the cost of it bearing little resemblance to conventional acoustics.

To demonstrate the similarities and the differences between the ultrasonic and the Galilean formulations of acoustics we re-derive Maxwell's relations using a relativistic version of Lighthill's formulation of aeroacoustics [8]. The analogues to the electric and magnetic field become clear in this process. To aid the comparison, in this section we revert to S.I. units and so the speed of sound will again be denoted c.

We start by projecting the temporal and spatial equations of motion, equations 25 and 26, into the laboratory frame. The result is

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\tfrac{1}{c^2}\partial_t \phi, \tag{31a}$$

$$\partial_t \mathbf{A} - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) = -\boldsymbol{\nabla}\phi. \tag{31b}$$

$\partial_t \equiv \frac{\partial}{\partial t}$ and $\boldsymbol{\nabla}$ is the spatial vector derivative; ϕ/c and $\mathbf{A}$ are the temporal and spatial components of the vector potential A, such that

$$\phi \equiv 2\gamma \tfrac{1}{\Xi_0}\sqrt{p} \qquad \text{and} \qquad \mathbf{A} \equiv \tfrac{1}{c^2}\phi \mathbf{v}, \tag{32}$$

where $\mathbf{v}$ is the velocity of the fluid as measured in the laboratory frame and $\gamma = (1 - \mathbf{v}^2/c^2)^{-1}$, as in 11.

The potential ϕ may be interpreted as the relativistic total enthalpy multiplied by the particle mass. To see this we first introduce the non-relativistic enthalpy, h, which is defined by

$$h \equiv e + p/(nm). \tag{33}$$

It then follows that

$$\phi = \gamma \frac{\epsilon + p}{n} = \gamma m \left(c^2 + h\right). \tag{34}$$

In the non-relativistic limit this becomes

$$\phi \rightarrow \left(1 + \tfrac{1}{2}\tfrac{v^2}{c^2}\right)\left(mc^2 + mh\right) = mc^2 + \tfrac{1}{2}mv^2 + mh \quad \text{as } v/c \rightarrow 0. \tag{35}$$

The term $h + \frac{1}{2}v^2$ is the usual expression of the total enthalpy. Equation 35 multiplies this by the particle mass, m, and adds the rest energy, mc^2, which is absent from all non-relativistic thermodynamics.

Equations 31a and 31b are the acoustically measured versions of the continuity and Euler equations. In the non-relativistic limit the equations reduce to Galilean invariant forms,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{36a}$$

$$\partial_t \mathbf{v} - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = -\boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + h\right). \tag{36b}$$

Equation 36b is the incompressible version of Euler's equation written in Crocco's form [6] and Equation 36a is the continuity equation of an incompressible fluid.

With the continuity and Euler equation that are valid for acoustic measurements now available, we may apply them to the conventional formulations of acoustics. We proceed with Lighthill's acoustic analogy [6;8]. To do so we differentiate the continuity equation (equation 31a) with respect to time and subtract it from the spatial derivative of Euler's equation (equation 31b). A wave equation for the total enthalpy results

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}{\partial_t}^2\right)\phi = \boldsymbol{\nabla} \cdot \left(\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right). \tag{37a}$$

Next, a wave equation for $\mathbf{A}$ is obtained by differentiating the continuity equation with respect to space and then subtracting the result from the temporal derivative of Euler's equation,

$$\left(\boldsymbol{\nabla}^2 - \tfrac{1}{c^2}{\partial_t}^2\right)\mathbf{A} = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \tfrac{1}{c^2}\partial_t\left(\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right). \tag{37b}$$

For comparison, had we carried out this procedure with the Galilean invariant continuity and Euler equation (not the incompressible versions) we would have obtained [6],

$$\left[D_t\left(\frac{1}{c^2}D_t\right) - \frac{1}{\rho}\boldsymbol{\nabla} \cdot (\rho\boldsymbol{\nabla})\right]\left(\tfrac{1}{2}v^2 + h\right) = -\frac{1}{\rho}\boldsymbol{\nabla} \cdot \left(\rho\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})\right) \tag{38a}$$

$$\left[D_t\left(\frac{1}{c^2}D_t\right) - \frac{1}{\rho}\boldsymbol{\nabla} \cdot (\rho\boldsymbol{\nabla})\right]\mathbf{v} = \frac{1}{\rho}\boldsymbol{\nabla} \times \left(\rho\boldsymbol{\nabla} \times \mathbf{v}\right), \tag{38b}$$

where $D_t \equiv \partial_t + \mathbf{v} \cdot \boldsymbol{\nabla}$. The equations of 38 express Lighthill's analogy in terms of enthalpy and vorticity [6]. The left hand side of both describe a non-linear wave in homoentropic potential flow [6].

In keeping with Lighthill's acoustic analogy, we interpret the right hand side of 37a and 37b as the fluctuations generated by the acoustic sources. If the magnitude of the fluctuations is proportional to the density of the acoustic sources, ρ_q, then we may define the constant of proportionality, ξ_0, so that

$$\boldsymbol{\nabla} \cdot \left(\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right) \equiv -\frac{\rho_q}{\xi_0}. \tag{39a}$$

Likewise, we may define an acoustic current, $\mathbf{J} = \rho_q \mathbf{v}$, by

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \tfrac{1}{c^2}\partial_t \left(\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right) \equiv \mu_0 \mathbf{J}, \tag{39b}$$

where μ_0 is again a constant of proportionality. If the acoustic current is conserved then it follows that the two constants are related:

$$c^2 = \frac{1}{\xi_0 \mu_0}. \tag{40}$$

In the rest of this report we assume this to be the case.

This section is completed by noting that equations 39a and 39b can be simplified by introducing

$$\mathbf{E} = -\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{41}$$

so that

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}{\partial_t}^2\right)\phi = -\boldsymbol{\nabla} \cdot \mathbf{E} \equiv -\frac{\rho_q}{\xi_0} \tag{42a}$$

and

$$\left(\boldsymbol{\nabla}^2 - \tfrac{1}{c^2}{\partial_t}^2\right)\mathbf{A} = -\boldsymbol{\nabla} \times \mathbf{B} + \tfrac{1}{c^2}\partial_t \mathbf{E} \equiv -\mu_0 \mathbf{J}. \tag{42b}$$

The equations of 42 are the same as Maxwell's equations of electromagnetism when written in terms of the potentials in the Lorenz gauge [3] (equation 31a). The vector $\mathbf{E}$ is known as the Lamb vector and is proportional to the Coriolis acceleration; it takes the role of the electric field in the analogy. The axial vector $\mathbf{B}$ is the spatial vorticity and takes the role of the magnetic field. The constants ξ_0 and μ_0 are, respectively, the analogues of the permitivity and permeability of free space.

Writing out Maxwell's 4 equations explicitly gives

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho_q}{\xi_0}, \tag{43a}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \tfrac{1}{c^2}\partial_t E, \tag{43b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{43c}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{43d}$$

The acoustic interpretation of these equations are as follows:

1. Equation 43a is the definition of an acoustic source.

2. Equation 43b is the definition of the acoustic current.

3. Equation 43c is the Lorentz invariant version of the vorticity equation.

4. Equation 43d is an expression of Helmholtz theorem, which demands the conservation of vorticity.

4 Discussion

In the physics community there is a widespread misconception that relativistic corrections only become important when predicting motions that are close to the speed of light. This is due, perhaps, to the explicit reference to light made in Einstein's postulates of special relativity, or due to Einstein's elimination of the æther. There is an impression that the postulated properties of light, its constancy of speed and lack of medium, are unique and enable light alone to be used to define distances.

In this report it has been demonstrated that when sound is used to measure distance then relativistic corrections must used to correctly predict motions that are close to the speed of sound. Ultrasound, as well as other sonar-based technologies, is therefore a relativistic subject where the speed of sound takes the role of the speed of light. Furthermore, it has been demonstrated that when distances are measured by acoustic pulse-echo, the generation and propagation of sound in an ideal homentropic fluid obeys Maxwell's equations, the physical law that describes the generation and propagation of light. This fundamentally relativistic theory is completely unaltered when the speed of sound take the role of the speed of light.

The implication is that relativistic theories are not dependant upon the propagating signal used to make the measurement. This is because to measure distances from the to-fro propagation time of a signal, a constant propagation speed must be assumed. Moreover, the constancy of the speed can never be experimentally refuted: you cannot use light signals to measure fluctuations in the speed of light; you cannot use sound signals to measure fluctuations in the speed of sound; how a signal actually traverses its medium can never be determined without knowing beforehand the world that is to be measured. The Lorentz transformations arise from the difference between the assumed speed of sound that is required to measure distance, and the effective speed of the sound that is altered by motion of the fluid medium. It is wrong to say that motion faster than the speed of light is impossible, just as it wrong to say that motion faster than the speed of sound is impossible. All that can be said is that motions that are faster than the propagating signal cannot be measured.

The æther does have a role in the relativity theory used in this report: sound does have a propagating medium. However, the æther is relativistic in the sense that it does not define a state of absolute rest. Moreover, it was shown that gauge invariance in Maxwell's relations may be interpreted as an invariance to the potential flow of the medium. This global gauge invariance is the same condition as the requirement that the æther be invariant to uniform motions. The relativity theory used in this report is therefore that of Poincaré, rather than Einstein. Indeed, as was noted in the introduction, the central argument of this report, that the constancy of the speed of light and the resulting Lorentz transformations result entirely from the method of measurement, is suggested by Poincaré in 1906 [16]. Unfortunately, however, Poincaré's contribution to the theory of relativity has been largely ignored by the wider community.

4.1 On the absence of non-linear propagation

In this report we have demonstrated that when time and space are measured acoustically the propagation of sound is linear. However, this is at odds with the understanding in medical ultrasound that the non-linear propagation of an acoustic pulse is not only measurable but also important. Our task is to explain how the non-linearity found in other measurement systems manifests itself in acoustic measurements. To do so it is useful to frame the discussion around the linear equations of 37, which are appropriate when distances are measured acoustically, and their non-linear Galilean forms of 38.

The first point to note is that the non-linearity of the Galilean formulation of sound is entirely a matter of *convention.* It would in fact be more appropriate to rewrite equations 38 as linear wave equations with everything else interpreted as acoustic sources and currents. Then the sound is defined as the part of an acoustic disturbance that can propagate energy away to infinity; the rest of the disturbance being a 'local' source. This is, in fact, the usual final step of Lighthill's analogy. The reason it is rarely performed when the analogy is written in terms of the total enthalpy and vorticity is because the acoustic source terms become horribly complicated. The split of source and wave in 38 is convenient interpretatively, but is nevertheless rather ad-hoc, for it mixes local terms with those that can propagate indefinitely.

The influence of non-linear propagation on what can be measured acoustically is found by comparing the right hand sides of equations 37 and equations 38. It is seen that the only major difference between the two is the term $-\frac{1}{c^2}\partial_t\left(\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right)$ on the right-hand-side of 37b. This term is part of the non-linear operator in equation 38b. It is, if you like, a ghost of the non-linear operator D_t on what can be measured acoustically. When measured with ultrasound it is interpreted as part of the current.

We note that an attempt to re-incorporate the ghost term back into some 'acoustically measured non-linear operator' would be ill-conceived for it would mean that the acoustic current is no longer conserved: the term $-\frac{1}{c^2}\partial_t\left(\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right)$ most certainly is a current.

4.2 Other similar studies

By using a relativistic version of Lighthill's formulation of aeroacoustics it was demonstrated that the acoustic analogue to the electric field is the Lamb vector (proportional to the Coriolis acceleration), and that the acoustic analogue to the magnetic field is the vorticity. An analogy in this form has been presented before by both Marmanis [9] and Sridhar [9;19]. However, both these attempts were constructed from Galilean fluid mechanics and so the analogy was only partial. To the author's knowledge, the derivation in this report is the first time that the analogy has been completed. The key step, missing in previous attempts, is to note that acoustics must be formulated in terms of a Lorentz invariant fluid where *the speed of sound equals the speed of light.* It is only when this step is made that the analogy exists. The motivation for this step is obvious only when it is appreciated that the speed of sound may take the role of the speed of light in a relativistic theory.

Relativistic fluids where the sound speed equals the speed of light have been studied many times before as theoretical curiosities [10;11;20]. For example, Pekeris found that Hick's spherical vortex conserves angular momentum if and only if the sound speed equals the speed of light [11]. The importance of such fluids, however, has not to the author's knowledge been recognised. Such fluids represent *what can be measured* when distances are obtained by echo-location.

An alternative analogy between acoustics and special relativity is found in the 'acoustic analogue gravity' literature (see Barceló, Liberati and Visser [1] for a review). An *acoustic* metric is constructed that describes sound carried in bulk flow. While the description of space and time in this formulation is Euclidean, the acoustic metric turns out to be pseudo-Euclidean, and therefore obeys the Lorentz transformation. This results because sound carried away by a supersonic flow will never reach us and so the speed of sound is a limiting velocity in transformations. The analogue gravity literature then goes on to study the gravitational implications of the acoustic metric. The acoustic metric, albeit Lorentzian, is not the same as Minkowski's metric used here, but is a function of the bulk flow. Analogue gravity does not consider the measurement process and operates within a world characterised by two metrics, the Lorentz invariant acoustic metric and the Galilean invariant spacetime metric. The correspondence of analogue gravity with relativity theory is therefore partial. The acoustic analogue to special relativity presented here is complete, except that the speed of sound takes the role of the speed of light.

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