An acoustic analogue to the special theory of relativity

Tom H. Shorrock

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Abstract

Abstract goes here

1 Introduction

The privileged role of light is perhaps the most mysterious aspect of Einstein's special theory of relativity. What is it about this signal, as opposed to any other means of communication, that makes it so fundamental to the concepts of time and space? The answer, for Einstein, was the constancy of the speed of light, and using this fact with the relativity postulate he eliminated the various ætherial explanations of electromagnetism that were present in the nineteenth century. This triumph, however, only emphasises the uniqueness of light, for it is the only signal that has no measurable medium, or *æther*. The central role of light in the measurement process remains unsettling. How lucky we are to be born with eyes! Satisfactory definitions of time and space seem to come only at the expense of making light an even greater puzzle, more and more distant from the world that we can touch and hear.

The relativity theory of Poincaré, on the other hand, neither postulated a fundamental speed for light nor eliminated the æther. Instead, in addition to the relativity postulate, Poincaré postulated a contraction to an object when it moves with respect to the æther. The most jarring aspects of Einstein's theory are absent from Poincaré's theory but only at the cost of introducing motion dependant deformations. Without a physical explanation for these contractions this cure, for many, is worse than the illness.

In this report we use sound to define time and space in the manner routinely used in medical ultrasound and other sonar-based technologies. It is demonstrated in section 2 that Poincaré's relativity is recovered if the relativity postulate is also assumed. Furthermore, in section 3 it is demonstrated that the acoustics of an ideal fluid, when measured with ultrasound, obey Maxwell's relation. In short, the generation and propagation of sound, when time and space are defined with sound, obeys the same physics as the generation and propagation of light, when time and space are defined with light.

Poincaré's motion dependent contraction makes sense in the context of ultrasound if it is taken very literally. In *Science and Hypothesis*[18] Poincaré declared that "Experiment is the sole source of truth". It is easily demonstrated (section 2) that the dimensions of an object that is measured with ultrasound is dependant upon its motion with respect to the bulk flow of the medium. Poincaré's contraction follows by demanding that these experimental results be accepted as true. It follows that every truth must be followed by a caveat that explains how time and space were measured in a given instance. This is because the measured dimensions of an object moving with respect to the medium will depend upon whether it is measured with light or sound. However, proceeding otherwise does not reduce the anthropocentric nature of measurement, it merely prejudices one sense over all others.

Finally in section 8 the radial pulsations of a micron sized bubble are studied as a specific example. The resonances of such bubbles are important in medical ultrasound because they radiate sound and are used to improve contrast. At high pressures the surface of the bubble is predicted to collapse and rebound at faster than the speed of sound. The loss of temporal ordering to such events mean that without a relativistic treatment the ultrasound measurements are impossible to predict.

2 The acoustic definition of time and space

In medical ultrasound distances are measured using the time it takes a pulse of sound to propagate from a transducer to a reflecting object and then to return again. If the sound is emitted from the transducer at a time, τ^-, and the sound returns at a time, τ^+, then the task is to find from these two numbers the spatio-temporal location, x, of the point of reflection.

What happens to the sound in between leaving the transducer and returning cannot be known by acoustic measurement. In this ignorance ultrasound practitioners assume that the time at which the echo occurred is the midpoint of τ^- and τ^+,

$$\tau(x) = \frac{\tau^+ + \tau^-}{2}. \tag{1a}$$

Other choices could certainly be made, but would imply a knowledge of the world beyond that learnt from τ^- and τ^+ alone. To measure distances from the times τ^- and τ^+ a sound speed, c, is required. Assuming, again in ignorance, that the sound returns at the same speed at which it left gives

$$\rho(x) = \frac{\tau^+ - \tau^-}{2} c. \tag{1b}$$

These are the definitions of time and space that are used in ultrasound.

Equation 1b requires an *a priori* knowledge of the sound speed for otherwise distances could not be determined from temporal measurements. In diagnostic ultrasound scanners it is usually taken to be 1540 ms^{-1}.

To determine the speed of sound of a given medium an independent notion of length is required. This practically would be found with a set of callipers. The average speed of sound is then found from equation 1b, using the times measured and the predetermined (*a priori*) distance. The point is that in order to use equation 1b something must be taken for granted; it is impossible to measure simultaneously both speeds and distances from temporal measurements.

When using equation 1b to measure distances the sound speed must be constant. This is because there is no way to measure its variations. This constancy of the speed of sound is identical to Einstein's second postulate for special relativity [4], except that the sound speed takes the role of the speed of light. Likewise, Einstein used equations 1 for the definition of time and space, except that the constant c was understood to be the speed of light rather than the speed of sound. In the relativity literature equations 1 go by the names of radar-time and radar-distance and their application to the example of the "twin paradox" is discussed by Dolby and Gull [2].

In this thesis we assume that the dynamics of entities is invariant to inertial motions even when measured acoustically (Einstein's first postulate). There has never been a counter example to this symmetry and we do not expect ultrasound to provide one. It then follows that measurements in ultrasound are subject to the considerations of special relativity, where the speed of sound rather than the speed of light takes the role of the limiting velocity.

The surface of a bubble that has been induced to pulsate by an ultrasound wave may collapse at a significant fraction of the speed of sound [16]. The pulsations of a bubble as measured by ultrasound are therefore expected to disagree with the same pulsations measured by an optical techniques. To investigate this, we now derive and analyse a version of the Keller-Miksis model [10] - a commonly used model for a pulsating bubble - that is consistent with how ultrasound measures spatio-temporal locations.

3 Maxwell

4 Introduction

In chapter **??** we argued that ultrasound is a relativistic theory, where the sound speed takes the role of the speed of light. This was argued in terms of the ultrasound measurement process, that a constant sound speed must be known before anything can be said about distance. We also assumed physical invariance between inertial observers on the belief that ultrasound should not change this.

Acoustics was then formulated for a ideal and isentropic fluid. Consistency with the measurement process required the fluid to be relativistic, in the sense that is invariant to Lorentz transformations. To enforce the sound speed to take the role of the speed of light, these two constants were equated. By enforcing the conservation of the energy momentum tensor (or equivalently, by Noether's theorem, translational invariance) we found the temporal and spatial equations of motion,

$$\nabla \cdot A = 0 \tag{62}$$

and

$$v \cdot (\nabla \wedge A) = 0, \tag{63}$$

respectively, where c.g.s. units have been used. From 62 we found that it follows that sound waves and acoustic sources are described by Maxwell's relation,

$$\nabla F = J, \tag{66}$$

where

$$F = \nabla \wedge A \tag{64}$$

is the spacetime vortiticy, and J is the acoustic source in the wave equation $\nabla^2 A = J$. The potential $A \equiv 2\sqrt{p}v$, where v is the velocity of a fluid particle.

The consequences of this correspondence were not explored, however, and we tie this loose end here. In section 5 and find that the role of magnetism is played by the spatial vorticity of the fluid, and that the electric field corresponds to the Coriolis acceleration. The space-time vorticity tensor assumes the role of the electromagnetic field tensor. This completes similar but partial attempts by others to construct an analogy between acoustics and electromagnetism. In particular, both Marmanis[14] and Srihar[22] have suggested and analogy between vorticity and magnetism, and the Coriolis acceleration (or Lamb vector) with the electric field. However, both authors constructed their analogy in the incompressible Galilean frame, and therefore could not write down a wave equation (Maxwell's relation).

The equivalence between electromagnetism and acoustics (as measured with ultrasound) poses several issues of interpretation:

1. On the one hand sound is a longitudinal wave, the molecules vibrate parallel to the perturbation. On the other hand, sound is transverse in the Coriolis acceleration and vorticity. In what sense are these two views consistent?
2. Non-linear propagation is known to be important in ultrasound, however, in our formulation it is impossible. How can this be.

Solutions to both these problems are suggested in section 6.

5 The acoustic analogues to the electric and magnetic fields

In classical electromagnetism the electric and magnetic fields are 3-dimensional vector fields that are measured (usually) in the laboratory frame. Such spatial vector quantities we denote in bold.

The most direct method of obtaining the analogues is to project the vorticity bivector, F, into the laboratory frame [3;5]. The analogue to the electric field can then be defined to be the timelike component, and the analogue to the magnetic field the spacelike component. The directness of this method, however, comes at the cost of it bearing little resemblance to conventional acoustics.

To demonstrate the similarities and the differences of ultrasound formulation of acoustics to the Gallilean formulation, we re-derive Maxwell's relation using an argument very similar to Lighthill's acoustic analogy [13] of aeroacoustics. The analogues to the electric and magnetic field become clear in this process.

We start by projecting the temporal and spatial equations of motion, equations 62 and 63 into the laboratory frame. The result is

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\partial_t \phi, \tag{2a}$$

$$\partial_t \mathbf{A} - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) = -\boldsymbol{\nabla} \phi. \tag{2b}$$

ϕ and $\mathbf{A}$ are the temporal and spatial components of the vector potential A. As we found in section 8.2, ϕ may be interpreted as the total enthalpy density, and $\mathbf{A} = \phi \mathbf{v}$. $\mathbf{v}$ is the three dimensional velocity.

Equations 2a and 2b are the acoustically measured versions of the continuity and Euler equations. In the non-relativistic limit the equations reduce to Galilean invariant forms,

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = -\partial_t \rho, \tag{3a}$$

$$\partial_t \mathbf{v} - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = -\boldsymbol{\nabla} \phi, \tag{3b}$$

with equation 3b being Euler's equation written in Crocco's form [8]. The difference between the two sets of equations is that the mass density, ρ, in the Gallilean forms is replaced by the total enthalpy density ϕ. Replacing the mass with an energy is typical of Lorentz invariant theories.

With the continuity and Euler equations in hand, we may now apply the conventional formulations of acoustics. We proceed with Lighthill's acoustic analogy [8;13].

To do so we differentiate the continuity equation (equation 2a) with respect to time and subtract it from the spatial derivative of Euler's equation (equation 2b). A wave equation for the total enthalpy results

$$\left(\boldsymbol{\nabla}^2 - \partial_t{}^2\right)\phi = \boldsymbol{\nabla}\cdot\left(\mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right)\right) \equiv -\rho_q. \tag{4a}$$

Next, a wave equation for $\mathbf{A}$ is obtained by by differentiating the continuity equation with respect to space and then adding the result to the temporal derivative of Euler's equation,

$$\left(\boldsymbol{\nabla}^2 - \partial_t{}^2\right)\mathbf{A} = -\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right) - \partial_t\left(\mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right)\right) \equiv -\mathbf{J}. \tag{4b}$$

In keeping with Lighthill's analogy we interpret the right hand side of 4a and 4b as an acoustic source density, ρ_q, and acoustic current density, $\mathbf{J}$, respectively.

For comparison, had we carried out this procedure with the Gallilean continuity and Euler equation we would have obtained [8],

$$\left[D_t\left(\frac{1}{c^2}D_t\right) - \frac{1}{\rho}\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{\nabla}\right)\right]\phi = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot\left(\rho\mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{v}\right)\right) \tag{5a}$$

$$\left[D_t\left(\frac{1}{c^2}D_t\right) - \frac{1}{\rho}\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{\nabla}\right)\right]\mathbf{v} = \frac{1}{\rho}\boldsymbol{\nabla}\times\left(\rho\boldsymbol{\nabla}\times\mathbf{v}\right). \tag{5b}$$

These are Lighthill's equations expressed in terms of enthalpy and vorticity [8]. The operator $D_t = \partial_t + \mathbf{v}\cdot\boldsymbol{\nabla}$. The left hand side of both equations 5 describe a non-linear wave in homoentropic potential flow [8].

Equations 4a and 4b can be simplified by introducing

$$\mathbf{E} = -\mathbf{v}\times\left(\boldsymbol{\nabla}\times\mathbf{A}\right) \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{6}$$

so that

$$\left(\boldsymbol{\nabla}^2 - \partial_t{}^2\right)\phi = -\boldsymbol{\nabla}\cdot\mathbf{E} \equiv -\rho_q. \tag{7a}$$

and

$$\left(\boldsymbol{\nabla}^2 - \partial_t{}^2\right)\mathbf{A} = -\boldsymbol{\nabla}\times\mathbf{B} + \mathbf{E} \equiv -\mathbf{J}. \tag{7b}$$

Equations 7 can now be recognised as Maxwell's equations written in terms of the potentials in the Lorenz gauge [3]. The vector $\mathbf{E}$ is the Coriolis acceleration, and takes the role of the electric field in the analogy. The axial vector $\mathbf{B}$ is the spatial vorticity and takes the role of the magnetic field.

Writing out Maxwell's 4 equations explicitly gives

$$\nabla \cdot \mathbf{E} = q, \tag{8a}$$
$$\nabla \times \mathbf{B} = \mathbf{J} + \partial_t E, \tag{8b}$$
$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{8c}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{8d}$$

The acoustic interpretation of these equations are as follows:

1. Equation 8a is the definition of an acoustic source.
2. Equation 8b is the definition of the acoustic current.
3. Equation 8c is the Lorentz invariant version of the vorticity equation.
4. Equation 8d is an expression of Helmholtz theorem, which demands the conservation of vorticity.

5.1 The acoustic sources

It is interesting to note the dimensionality of the acoustic source. To do so it is necessary to reintroduce SI units so that 8a and 8b become

$$\frac{\rho_q}{\epsilon_0} = -\nabla \cdot (\mathbf{v}/c \times (\nabla \times \mathbf{A})) \tag{9a}$$

and

$$\frac{\mathbf{J}}{c^2 \epsilon_0} = \nabla \times \left(\nabla \times \frac{\mathbf{A}}{c} \right) + \frac{1}{c^2} \partial_t \left(\mathbf{v} \times \nabla \times \mathbf{A} \right). \tag{9b}$$

where ϵ_0 is the acoustic analogue to the electric permittivity. The units of the left hand side of 9a are

$$\left[\frac{\rho_q}{\epsilon_0} \right] = \frac{\text{Energy}}{\text{distance}^2}. \tag{10}$$

The analogue units in electromagnetism are

$$\left[\left. \frac{\rho_q}{\epsilon_0} \right|_{\text{em}} \right] = \frac{\text{Energy}}{\text{distance}^2 \times \text{charge}}. \tag{11}$$

It follows that the acoustic analogue to charge is *dimensionless*, and the analogue to permittivity has the units

$$[\epsilon_0] = \frac{1}{\text{distance} \times \text{Energy}}. \tag{12}$$

We do not, however, have an adequate interpretation of the acoustic permittivity. We suppress the issue by returning to c.g.s. units, where $c = \epsilon_0 = 1$.

6 The interpretation of a sound pulse

The equivalence of the formulation of ultrasound measured acoustics and electromagnetism raises a number of issues of interpretation. The first of these is the linearity of the theory. Nonlinear propagation of the sound pulse is well known to be important in ultrasound physics, and yet it vanishes altogether in our formulation. This issue is considered in section 6.1.

The second is that a sound pulse is a longitudinal wave, the compressions and rarefactions of the density are in the direction of propagation, whereas Maxwell's relations describe a pair of self interacting transverse waves (the Coriolis acceleration and the vorticity). The compatibility of these two equations is discussed in section 6.2

6.1 On the absence of non-linear propagation

The linearity of the acoustic field is entirely appropriate for ultrasound. To apply a non-linear wave equation you need to know the spatial variations of the speed of sound. This is seen in the left hand side of equations 5. As argued in chapter **??**, such knowledge is impossible when using sound to define the concept of distance: the sound speed used by ultrasound must be a spatially invariant constant to be able to measure anything at all. What is to be explained, therefore, is not why ultrasound measured acoustics is linear - this is surely correct - but to explain how the non-linearity found in other measurement systems manifests itself in acoustic measurements. To do so it is useful to frame the discussion around the linear acoustically measured equations of 4 and their non-linear Galilean forms 5.

The first point to note is that the non-linearity of the Galilean formulation of sound is entirely a matter of *convention.* It would in fact be more appropriate to rewrite equations 5 as linear wave equations with everything else interpreted as acoustic sources and currents. Then the sound is defined as the part of an acoustic disturbance that can propagate energy away to infinity, the rest of the disturbance being a 'local' source. This is, in fact, the usual final step of Lighthill's analogy. The reason it is rarely performed when the analogy is written in terms of the total enthalpy and vorticity is because the acoustic source terms become horribly complicated. The split of source and wave in 5 is convenient interpretatively, but is nevertheless rather ad-hoc, for it mixes local terms with those that can propagate indefinitely.

The influence of non-linear propagation on what can be measured acoustically is found by comparing the right hand sides of equations 4 and equations 5. It is seen that the only major difference between the two is the term $-\partial_t (\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}))$ on the right-hand-side of 4b. This term is, if you like, the ghost of the non-linear operator D_t on what can be measured acoustically. When measured with ultrasound it is interpreted as part of the current.

We note that an attempt to re-incorporate the ghost term back into some 'acoustically measured non-linear operator' would be ill-conceived, for it would mean that the acoustic current is no longer conserved: the term $-\partial_t (\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}))$ most certainly is a current.

6.2 Interpreting the transversivity of the sound pulse

An acoustic plain wave is a perturbation of the fluid particles in the direction of propagation of the pulse. It is also a transverse wave in terms of the Coriolis and vorticity fields. We now propose a method of squaring these seemingly contradictory views.

We consider a segment of the plain wave to be a narrow tube, so that the entire plain wave is formed by adding together many such tubes. Within the tube a pertabation in pressure propagates the sound wave longitudinally. Outside of the tube no such pertabation exists.

Both within and outside of the tube there is no vorticity and so there is no propagation of the transverse wave. On the interface of the tube, however, there is shearing of the fluid: the molecules within tube move as the wave passes, the molecules outside do not. This shearing induces a *vortex sheet* on the interface[8]. The vorticity is confined to the sheet and has a strength equal to the average speed of the molecules on either side of the interface. From the acoustic analogue Maxwell's equations, this vorticity then induces a Coriolis acceleration orthogonal to the interface. A longitudinal wave therefore induces a transverse wave on its boundary.

There remains a difficulty with this interpretation, however. A transverse wave should have exactly two helicity states. The interpretation given to the first seems, to the author at least, entirely plausible. The second is more difficult, however. The construction of a transverse wave with the Coriolis and vorticity vectors interchanged does not seem obvious. Rather than speculate, we leave this question unanswered.

7 Discussion

This chapter has explicitly formulated the (exact) analogy between ultrasound measured acoustics and electromagnetism. This ties the loose ends of chapter **??**, where the analogy was noticed but not explored. It has been found that the acoustic analogue to the electric field is the Coriolis acceleration, and that the acoustic analogue to the magnetic field is the vorticity, The spacetime vorticity bivector takes the role of the field tensor. The analogy in this form has long been suspected[14;22], however, to the authors knowledge this is the first time that the analogy has been completed. The key step, missing in previous attempts, is to note that acoustics must be formulated in terms of a Lorentz invariant fluid where *the speed of sound equals the speed of light.* It is only when this step is made that the analogy exists.

Relativistic fluids where the sound speed equals the speed of light have been studied many times before as theoretical curiosities [17;23]. For example, Pekeris observed that Hick's spherical vortex conserves angular momentum if and only if the sound speed equals the speed of light [17]. The importance of such fluids, however, has not to the authors knowledge been recognised. Such fluids represent *what can be measured* when distances are obtained by echo-location. Acoustics as measured with ultrasound is therefore *identical* to electromagnetism (as measured with light). Retrospectively this correspondence is not too surprising. For both acoustics as measured with ultrasound and electromagnetism as measured with light attempt to measure the properties of their propagating signal. Both, therefore, represent a similarly limited view of the world, the limitations manifesting themselves in the linearity of the equations.

Another interesting analogy between acoustics and electromagnetism is 'acoustic analogue gravity' literature (see Barceló, Liberati and Visser [1] for a review). The approach constructs an *acoustic* metric that describes the acoustics of sound carried in bulk flow. While the description of space and time in this formulation is Euclidean, the acoustic metric turns out to be pseudo-Euclidean, and therefore obeys the Lorentz transformation. This results because sound carried away in bulk flow faster than the speed of sound will never reach us. The speed of sound is therefore a limiting velocity in transformations. The analogue gravity literature then goes on to study the gravitational implications of the acoustic metric.

While the motivations behind the two approaches is similar, in particular the demand that acoustics must be obey a Lorentz invariant metric, the approach given here and the approach of analogue gravity are fundamentally different. Analogue gravity does not consider the measurement process and so operates within a world characterised by two metrics, the Lorentz invariant acoustic metric and the Galilean invariant spacetime metric. Ultrasound measurement demands that the world be described by a Lorentz invariant metric. There is no other way to describe space and time acoustically. In analogue gravity the acoustic metric is Lorentz-invariant, but is not the same as the metric used here. In analogue gravity the metric is a function of the bulk flow, whereas we argue that this is impossible: the sound speed must be an a priori constant in order to say anything about the world. Nevertheless, the success of analogue gravity is very encouraging. Acoustically observed microbubbles could offer an interesting experimental model in this field.

7.1 The acoustic Lagrangian

Finally, let us use the acoustic analogy to set out some directions for future work. The common mathematical description between acoustics and electromagnetism can be enforced by deriving both from a Lagrangian of the same form. The electromagnetic Lagrangian density is[3;12],

$$Ł = \frac{1}{2} F \cdot F - A \cdot J, \tag{13}$$

and the acoustic Lagrangian is same so long as the symbols F, A and J take their acoustical meaning. Note that the acoustical Lagrangian is not the same as the Lagrangian of the ideal fluid. It describes directly the sound and the acoustic sources, rather than the fundamental motions of the fluid particles.

The symmetries of 13 can be found by minimising the action,

$$S = \int \left| d^4 x \right| Ł(A, \nabla \wedge A; x). \tag{14}$$

In the absence of acoustic sources ($J = 0$) this results in the Euler-Lagrange equation

$$\partial_A Ł + \nabla \cdot \left(\nabla_{\nabla \wedge A} Ł \right) = 0. \tag{15}$$

Demanding translational invariance of the action and assuming that the Euler-Lagrange equations are satisfied results (via Noether's theorem) in the conservation of the acoustic energy momentum tensor[3;12]

$$T\left(a\right) = -\tfrac{1}{2} F a F. \tag{16}$$

Demanding rotational invariance and applying 15 results in the conservation of the angular momentum. The adjoint to the angular momentum tensor is the most useful and is[3;12]

$$\bar{\mathcal{J}}(n) = A \wedge (F \cdot n) + \bar{T}(n) \wedge x. \tag{17}$$

The second term, $\bar{T}(n) \wedge x$, is the orbital angular momentum of the sound pulse as moves through space. The first,

$$S(n) = A \wedge (F \cdot n)\,, \tag{18}$$

is the intrinsic spin of the acoustic field.

The acoustic energy momentum tensor can be applied immediately to derive the force law for an acoustic source when in the presence of an external vorticity or Coriolis field. Remembering Maxwell's relation, $\nabla \cdot F = J$ (equation 66) we obtain

$$f = \tilde{T}(\tilde{\nabla}) = -\tfrac{1}{2} \left(\tilde{F} \tilde{\nabla} F + F \nabla F \right) = -\tfrac{1}{2} \left(-JF + FJ \right) = J \cdot F. \tag{19}$$

If the acoustic source has a constant rest mass, m_0, and moves at a speed u, then, by writing $J = qu$, where q is the acoustic source, we obtain

$$f = m_0 \dot{u} = qu \cdot F, \tag{20}$$

Equation 20 is the Lorentz force law.

To write equation 20 in its more conventional form we project into the laboratory frame,

$$J \cdot F = \langle (\rho + \mathbf{J})\, \gamma_0 \,(\mathbf{E} + I\mathbf{B}) \rangle = -\,(J \cdot E + q\mathbf{E} + q\mathbf{u} \times \mathbf{B})\, \gamma_0. \tag{21}$$

The timelike $q\mathbf{u} \cdot \mathbf{E}$ is the work done. The remaining spatial component is

$$\mathbf{f} = -q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})\,, \tag{22}$$

which is the Lorentz force law in its usual form.

Equation 20 describes how a turbulent source moves in the presence of external flows when measured with ultrasound. It is an incredibly powerful result. Its use, however, requires individual acoustic sources to be tracked though space. This is generally beyond the capability of ultrasound, although this is changing with the increasing availability of 3 dimensional probes.

7.2 Helicity and Spin

The existence of the intrinsic acoustic spin raises further interpretative issues. In the absence of satisfactory explanation for these, however, we resist the temptation to speculate. All we say on the matter is to note that helicity of the sound wave, the projection of the spin on the direction of the momentum, is identical to the hydro dynamical helicity introduced by Moffatt [15]. This is interesting because the integral of the hydrodynamic helicity is a topological invariant that measures the degree of knottedness of the flow [15].

To see this, we introduce the helicity,

$$\mathcal{H} \equiv \hat{\mathbf{P}} \cdot S(\gamma_0) = \hat{\mathbf{P}} \cdot\, (\phi\mathbf{E} - \mathbf{A} \times \mathbf{E}) = \frac{|\mathbf{E}|^2}{|\mathbf{E} \times \mathbf{B}|}\mathbf{A} \cdot \mathbf{B}. \tag{23}$$

where $\hat{\mathbf{P}}$ is the unit Poynting vector,

$$\hat{\mathbf{P}} = \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|}. \tag{24}$$

and $S(\gamma_0)$ is the timelike component of the Spin as measured in the laboratory frame,

$$S(\gamma_0) = \tfrac{1}{2}\langle A\gamma_0\gamma_0\,(F\gamma_0 - \gamma_0 F)\rangle_2 \tag{25}$$

$$= \phi\mathbf{E} - \mathbf{A} \times \mathbf{E}. \tag{26}$$

For a sound wave $|\mathbf{E}| = c\,|\mathbf{B}|$ and so the helicity simplifies to

$$\mathcal{H} = \mathbf{A} \cdot \mathbf{B}. \tag{27}$$

The term $\mathbf{A} \cdot \mathbf{B}$ on the right hand side is the same as the relativistic generalisation[1] to the *hydrodynamic helicity* per unit volume defined by Moffatt[15]. It is also readily interpretable. $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ measures a rotation, and the rotation is projected about $\mathbf{A}$. The helicity therefore measures the degree to which the fluid streamlines rotate about themselves, that is, the degree to which the streamlines are helical[15;19]. The contribution to $\mathbf{A} \cdot \mathbf{B} dV \approx \mathbf{A}_0 \cdot \mathbf{B}_0 dV$ is positive or negative depending on the orientation of the helix[15].

In this way Moffatt identifies the component of the spin parallel to the momentum with the *orbital angular momentum* of a fluid streamline about its mean trajectory. The equivalence between the two helicities (acoustic and hydrodynamic) suggests that this interpretation of the spin is valid.

Finally we note that the hydrodynamic helicity was introduced by Moffatt because

$$I = \int |dV| \, \mathcal{H} = \int |dV| \, \mathbf{A} \cdot \mathbf{B} \tag{28}$$

is an invariant that defines the degree of knottness of the fluid. In the case of two vortex rings, for example, $\mathcal{H} = 2\alpha\kappa_1\kappa_2$ where κ_i are the circulations around the two rings. If they are unlinked $\alpha = 0$, whereas if they are singularly linked $\alpha = \pm 1$. The helicity in a volume, the component of the spin parallel to the momentum, is therefore quantised in accordance with the topology of the streamlines.

The author does not have an interpretation for this fascinating result and so does not comment further. We note, however, that there has been considerable effort in applying the hydrodynamic helicity electromagnetism, with implications to both the quantisation of the spin and to the quantisation of the charge. We refer the interested reader to the literature[20;24;25].

8 The acoustically-measured bubble

8.1 The original formulation

The Keller-Miksis model[10] assumes that a gas bubble is located within a stationary and vorticity free fluid medium. The fluid particles are described by the velocity potential, ψ. The bubble is assumed to remain spherical, with a time dependent radius, $a \equiv a(t)$, and assumed to remain at the origin. From the spherical symmetry of the model only radial components of the velocity need to be considered.

[1] The relativistic generalisation is accomplished by replacing $\mathbf{v}$ in Moffatt's definition with $\mathbf{A}$. Notice, furthermore, that Moffatt did not normalise the Helicity, that is $\mathcal{H}_{\text{Moffatt}} = p \cdot S$ rather than $\frac{1}{|p|} p \cdot S$, where p is the momentum. When we make the comparison to $\mathcal{H} = \epsilon_0 A \cdot B$, therefore, we implicitly divide $\mathcal{H}_{\text{Moffatt}}$ by a unit momentum, so that dimensionally all is correct.

Keller and Miksis retain perturbations in density only up to first order, from which it follows that variations in the sound speed, c, are neglected and that the velocity potential obeys the linear wave equation,

$$\left(\partial_r{}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0. \tag{29}$$

The notation $\partial_r \equiv \frac{\partial}{\partial r}$ is used to denote the radial derivative while both $\partial_t \equiv \frac{\partial}{\partial t}$ and the over dot notation will be employed to denote the differential with respect to time.

The solution to 29 is

$$\psi = \frac{1}{r}\left(f_1\left(t - \tfrac{r}{c}\right) + f_2\left(t + \tfrac{r}{c}\right)\right), \tag{30}$$

where r is the radial distance from the centre of the bubble and f_1 and f_2 are functions to be determined.

Since equation 29 is second order two boundary conditions are required. The first is that the radial velocity of the fluid, v, is equal to the velocity of the bubble wall. That is

$$v \equiv \dot{a}(t) = v \text{ at } r = a. \tag{31a}$$

The second boundary condition is that the pressure in the liquid adjacent to the surface of bubble, $p(a,t)$, must equal the pressure on the bubble wall, p_b,

$$p(a,t) = p_b. \tag{31b}$$

We will consider explicit expression for p_b section 8.5 and in section 8.6.

To apply these boundary conditions Keller and Miksis eliminated the spatial derivate of ψ by using its definition,

$$\partial_r\psi = v, \tag{32a}$$

and used Bernoulli's equation to eliminate the temporal derivatives,

$$\partial_t\psi = -\frac{1}{2}v^2 - h. \tag{32b}$$

h is the enthalpy of the fluid.

The completes the specification of the model that was setup and solved by Keller and Miksis[10].

8.2 Alterations required when making measurements with ultrasound

Equations that describe measurements made with ultrasound require, via special relativity, Lorentz invariance. Equations 32a and 32b do not have this property and so do not apply to acoustical measurement. To fix 32a and 32b the relativistic generalisation to the velocity potential [11] must be used,

$$\nabla\psi = -\frac{wu}{nc} \equiv -A. \tag{33}$$

Here w is the heat function per proper volume and n is the particle number per proper volume. The velocity u and the vector potential, A, are spacetime vectors and ∇ is the spacetime derivative. The heat per proper volume can be written in terms of the total energy density, such that [3;11]

$$w = \epsilon + p = nmc^2 + nme + p, \tag{34}$$

where e is the thermodynamic energy and m is the proper mass. Spacetime vectors are sometimes referred to as 4-vectors, although this terminology will not be used here. The spacetime velocity, u, is parameterised so that $u^2 = c^2$ where c is the speed of sound. The symbol u has been used to avoid confusion with the radial component, v, of the spatial velocity (a 3-vector).

The temporal and spatial projections of 33 in the laboratory frame are

$$\partial_r\psi = \frac{\gamma wv}{nc} \equiv \frac{\phi v}{c} \tag{35a}$$

and

$$\partial_t\psi = -\frac{\gamma wc}{n} \equiv -\phi c \tag{35b}$$

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ is the Lorentz factor and the potential

$$\phi \equiv \frac{\gamma w}{n} \tag{36}$$

has been introduced for convenience. The heat function per particle is related to the enthalpy by the proper mass,

$$w/n = \left(mc^2 + me + p/n\right) = m\left(c^2 + h\right). \tag{37}$$

The thermodynamic relation $h = e + p/(nm)$ has been used in the second equality.

The potential ϕ is the relativistic generalisation to the total enthalpy (multiplied by the proper mass). In the non-relativistic it becomes,

$$\phi \to \left(1 + \tfrac{1}{2}\tfrac{v^2}{c^2}\right)\left(mc^2 + mh\right) = mc^2 + \tfrac{1}{2}mv^2 + mh \quad \text{as } v/c \to 0. \tag{38}$$

The right hand side is the energy contributed by the rest mass plus the standard non-relativistic expression for the total enthalpy (multiplied by the mass), as claimed.

By replacing equations 32a and 32b with equations 35a and 35b the derivation of Keller and Miksis can be used without further alteration. The resulting equation will be Lorentz invariant and therefore satisfy the constraints imposed by using pulse-echo to define the spatio-temporal locations of the echo's source.

8.3 The derivation of the acoustically-measured-Keller-Miksis equation

Differentiating equation 30 with respect to time obtains

$$r\partial_t\psi = f_1' + f_2', \tag{39}$$

where the prime denotes differentiation with respect to the argument, while differentiating with respect to the radius gives,

$$r^2 c\nabla\psi = r\left(f_2' - f_1'\right) - c\left(f_1 + f_2\right). \tag{40}$$

Equations 39 and 40, evaluated at $r = a$, are combined to eliminate f_1',

$$a^2\left(\partial_t\psi + c\partial_r\psi\right) - 2af_2' + c\left(f_1 + f_2\right) = 0. \tag{41}$$

By using equations 35a and 35b in 41 (rather than 32a and 32b) we obtain

$$a^2\left(\phi\dot{a} - \phi c\right) - 2af_2' + c\left(f_1 + f_2\right) = 0. \tag{42}$$

Differentiating 42 with respect to time and reusing 39 and 35b gives

$$a\frac{\phi}{c^2}\ddot{a} + 2\dot{a}^2\frac{\phi}{c^2} - \phi\left(1 + \frac{\dot{a}}{c}\right) - a\frac{\dot{\phi}}{c}\left(1 - \frac{\dot{a}}{c}\right) - \frac{2}{c^2}f_2''\left(1 + \frac{\dot{a}}{c}\right) = 0 \tag{43}$$

The driving acoustic pressure comes from the converging acoustic wave, f_2. To evaluate this term, following Keller and Miksis, we assume that the incident wave is planar and decompose it into spherical harmonics. By assumption, however, the bubble pulsates purely radially and so only the zeroth harmonic interacts with the bubble. The potential of the incoming wave, $\psi_i(r,t)$, is accordingly $\psi_i(r,t) = \frac{1}{r}\left(f_2(t + r/c) + f_3(t - r/c)\right)$. Requiring that the potential is finite at the bubble's centre implies that $f_2 = -f_3$ and so

$$\psi_i(r,t) = \frac{f_2(t + r/c) - f_2(t - r/c)}{r}. \tag{44}$$

The bubble is small in comparison to the wavelength and so the velocity potential of the fluid near the origin satisfies

$$\psi_i(a,t) = \frac{f_2(t + a/c) - f_2(t - a/c)}{a} \approx \frac{2}{c}f_2'(\tau). \tag{45}$$

The differential on the right-hand-side of 45 is a function of the proper time, τ, of the bubble. This is because the equation holds only in a frame of reference where the bubble is stationary and at the origin. Differentiating 45 with respect to the proper time we obtain

$$\frac{d}{d\tau}\psi_i \equiv \gamma\left(\partial_t + v\partial_r\right)\psi_i = u\cdot\nabla\psi = \frac{2}{c}f''. \tag{46}$$

Using equation 33 this gives

$$\frac{2}{c}f'' = -\frac{wc}{n} = -\frac{\phi_i c}{\gamma}, \tag{47}$$

where ϕ_i is the incident potential. Substituting 47 into 43 and using equation 36 in the form $\phi = \gamma m \left(c^2 + h\right)$ we obtain

$$\begin{aligned} a\ddot{a}\gamma\left(1+\frac{h}{c^2}\right)\left(1-\frac{\dot{a}}{c}\gamma^2\left(1-\frac{\dot{a}}{c}\right)\right) + 2\dot{a}^2\gamma\left(1+\frac{h}{c^2}\right) \\ - \left[\gamma\left(c^2+h\right)-\left(c^2+h_i\right)\right]\left(1+\frac{\dot{a}}{c}\right) - \gamma\dot{h}\left(\frac{a}{c}\right)\left(1-\frac{\dot{a}}{c}\right) = 0 \end{aligned} \tag{48}$$

Equation 48 is the final answer. The enthalpy contains the pressure terms, and with them the boundary condition of equation 31b

The non-relativistic limit (with $\gamma \approx 1 - \frac{\dot{a}^2}{2c^2}$), equation 48 reduces to

$$a\ddot{a}\left(1-\frac{\dot{a}}{c}\right) + \frac{3}{2}\dot{a}^2\left(1-\frac{1}{3}\frac{\dot{a}}{c}\right) - (h-h_i)\left(1+\frac{\dot{a}}{c}\right) - \dot{h}\left(\frac{a}{c}\right) = 0, \tag{49}$$

which is the original Keller-Miksis equation [7].

The acoustically-measured-Keller-Miksis model therefore reduces to the original when the maximum speed of the bubble wall is small in comparison to the sound speed. In the non-relativistic approximation made to obtain equation 49 discards terms of order M^2, where $M = \left|\frac{\dot{a}}{c}\right|$ is the Mach number of the bubble wall. Therefore, the original Keller-Miksis model approximates the acoustically measured motion when M^2 is small. I.e. up to about $M = 0.4$.

8.4 The applicability of the equation

In the original derivation Keller and Miksis linearised the density. The consequence of this is that fluctuations in the sound speed are ignored. This enabled our starting point, the linear wave equation of 29, to be derived. The discarded second order terms become important when M^2 is not small. The approximation holds, therefore, up to about Mach 0.4. Interestingly, the optical and acoustical observer agree over the range of speeds to which the original Keller-Miksis equation is applicable.

When the measurements are made with ultrasound, however, no approximation is made to obtain equation 29. The reason is that ultrasound is not capable of measuring variations in the speed of sound. Second order fluctuations in the density can therefore play no role when ultrasound is used to describe the world. It is instructive to see this argument borne out in the equations that describe an acoustically measured fluid. The equations of motion must be Lorentz invariant and this condition is automatically fulfilled when the equations are obtained from the divergence of the energy-momentum tensor of the ideal fluid. In the derivation we also need to use the hitherto unused condition that the sound speed takes the role of the speed of light. This condition is imposed by equating the two speeds. This further requires that the energy density for the fluid, as measured acoustically, be a function of the pressure only (barotripic), for the sound speed cannot equal the speed of light otherwise [23].

The energy-momentum tensor of an ideal fluid is [3]

$$T(a) = wa \cdot uu - ap = (\epsilon + p)a \cdot uu - ap, \tag{50}$$

where, $\epsilon \equiv \epsilon(p)$ is the barotropic total energy density, p is the pressure and $u = u(\tau)$ is the velocity vector of the spacetime path, with the parameterisation chosen such that $u^2 = 1$. Natural units have been chosen in this section, and so that the speed of light is unity.

The speed of sound, c, given at constant entropy density, σ, is [11;23]

$$c^2 = \left.\frac{\partial p}{\partial \epsilon}\right|_{\sigma}. \tag{51}$$

This is the same as the non-relativistic expression except that the energy density has replaced the mass density. Setting the speed of sound to equal the speed of light (unity) and the integrating gives,

$$\epsilon = p' - p_0 + \epsilon_0 \tag{52}$$

where p' is a pressure that fluctuates with position, so that $dp = dp'$, and p_0 and ϵ_0 are the ambient pressure and mean energy density, respectively. The thermodynamic pressure is therefore

$$p \equiv p' - p_0 + \epsilon_0. \tag{53}$$

and the fluid obeys the equation of state,

$$\epsilon(p) = p. \tag{54}$$

At infinity $p' = p_0$ and so $\epsilon_\infty = p_\infty = \epsilon_0 \neq p_0$.

Applying 54 simplifies the energy momentum tensor considerably,

$$T(a) = p\,(2a \cdot uu - a) \equiv \frac{1}{4}AaA, \tag{55}$$

where vector potential, A, introduced on the right hand side satisfies

$$A = 2p^{1/2}u = 2\epsilon^{1/2}u. \tag{56}$$

The vector A is the same as was introduced in 33, i.e.

$$\nabla\psi = -\frac{wu}{n} = -2\sqrt{p}u. \tag{57}$$

To demonstrate this second equality we use an argument of Taub [23]. Using the isentropic thermodynamic relation $mde = -pd\left(\frac{1}{n}\right)$ with the relation for the internal energy, $\epsilon = nm(1 + e(p))$, it follows that

$$nd\epsilon = \epsilon dn - n^2 pd\left(\frac{1}{n}\right) = (\epsilon + p)\, dn. \tag{58}$$

Applying equation 54 and integrating we obtain

$$n = \sqrt{p}, \tag{59}$$

from which we find

$$A = 2\sqrt{p}u = \frac{2\epsilon}{n}u = \frac{wu}{n} = -\nabla\psi, \tag{60}$$

as claimed.

The divergence of the energy momentum tensor (equation 55) vanishes in the absence of external fields. Therefore, by projecting the divergence of 55 along the timelike component we find

$$u \cdot \tilde{T}(\tilde{\nabla}) = \tfrac{1}{2} u \cdot A \nabla \cdot A = 0. \tag{61}$$

The check denotes the scope of the derivative. Since, from 56, A is parallel to u it follows that

$$\nabla \cdot A = 0 \tag{62}$$

and so the vector potential A is conserved. The spacelike projection, $\tilde{T}(\tilde{\nabla}) - uu \cdot \tilde{T}(\tilde{\nabla})$, gives in turn,

$$u \cdot (\nabla \wedge A) = 0. \tag{63}$$

The relativistic vorticity bivector is

$$F = \nabla \wedge A, \tag{64}$$

and so 63 implies that the vorticity is orthogonal to the velocity.

By taking the divergence of 62 and using the vector identity $\nabla\nabla \cdot A = \nabla^2 A - \nabla \cdot (\nabla \wedge A)$ we find,

$$\nabla^2 A = \nabla \cdot F = \nabla F. \tag{65}$$

The second equality follows because $\nabla F = \nabla \cdot F + \nabla \wedge F$ and because the operator identity $\nabla \wedge \nabla = 0$ causes $\nabla \wedge F$ to vanish. Equation 65 is a wave equation and so interpreting the right-hand-side of 65 as an acoustic source current, J, we obtain

$$\nabla F = J. \tag{66}$$

Interestingly equation 66 is Maxwell's equation and equation 62 has specified that we are working in the Lorenz gauge. We explore this observation more in chapter **??**.

When deriving the Keller-Miksis equation potential flow in the fluid was assumed. This implies that the vorticity tensor vanishes, for $F = \nabla \wedge A = -\nabla \wedge \nabla\psi = 0$. Therefore the acoustic sources vanish and the wave equation $\nabla^2 A = 0$ is satisfied. This was the starting equation in the derivation of the Keller-Miksis model, equation 29. Notice that other than potential flow, no approximation has been made to derive 29. When using ultrasound, sound must be measured to propagate linearly. This is simply because it is impossible for ultrasound to measure fluctuations in the speed of its propagating signal. The acoustically-measured-Keller-Miksis equation is exact.

8.5 The pressure on the surface of the bubble

When introducing the original Keller-Miksis model in section 8.1 the pressure on the surface of the bubble was not specified (equation 31b). In this section and the next we fix this boundary condition for the cases with and without viscosity.

The gas within the bubble may simply be modelled a polytropic exponent, κ. The pressure within the bubble is then $p_e\left(\frac{a}{a_e}\right)^{-3\kappa}$ where p_e is the pressure of the gas within the bubble at equilibrium, and a_e is the radius of the bubble at equilibrium[7]. The contribution of the vapour pressure has been neglected for simplicity. This within the bubble pressure exceeds the pressure of the fluid at the surface, $p(a,t)$, due to the contributions of the surface tension, σ. The pressure boundary condition is then

$$p_b(a) = p_e\left(\frac{a}{a_e}\right)^{-3\kappa} - 2\frac{\sigma}{a}. \tag{67}$$

To write the boundary condition in terms of the enthalpy we use the thermodynamic relation $dh = \frac{1}{mn}dp$. Substituting in 59 gives

$$h(a) = \int_{p_\infty}^{p} \frac{1}{mn}dp = 2\left(\sqrt{p} - \sqrt{p_\infty}\right) = 2\frac{p_\infty}{n_\infty}\left(\sqrt{p/p_\infty} - 1\right) \tag{68}$$

where $n_\infty = \sqrt{p_\infty} = \sqrt{\epsilon_\infty}$ has been used in the second equality. Expanding p from 53 gives

$$h(a) = 2\frac{p_\infty}{n_\infty}\left(\sqrt{p - p_0 + p_\infty/p_\infty} - 1\right) = 2\frac{p_\infty}{n_\infty}\left(\sqrt{1 + p - p_0 p_\infty} - 1\right) \tag{69}$$

In the non-relativistic limit $p_\infty = \epsilon_\infty \approx n_\infty mc^2 \approx 1$ GPa[7]. This is much larger than any pressures reached in ultrasound and so we may write, simply, that

$$h(a) \approx \frac{p - p_0}{mn_\infty} \tag{70}$$

The acoustically-measured-Keller-Miksis equation is now complete. We emphasise that for an free polytropic gas bubble pulsating in ideal vortex-free fluid, the model is exact.

8.6 Acoustically-measured viscosity

Viscous fluids have so far been excluded from our analysis. Viscosity usually plays a minor role in medical applications - an ideal fluid does a fairly good job for propagation. However, the viscosity does play an important role in dampening the oscillations of the bubble. In ultrasound contrast physics, therefore, the viscosity is usually incorporated into the pressure boundary condition of equation 31b. To model the viscous dampening it is assumed that the fluid is Newtonian; only the *dynamic viscosity*, η, is considered. Then the stress tensor measured with ultrasound is[11],

$$\sigma(a) = ap - c\eta\left(a\cdot\nabla u + \nabla u\cdot a - a\cdot uu\cdot\nabla u - uu\cdot\nabla u\cdot a - \tfrac{2}{3}\nabla\cdot u\,(a - a\cdot uv)\right). \tag{71}$$

The derivative with respect to the vector a yields the trace[6],

$$\mathrm{Tr}\ \sigma(a) \equiv \partial_a\cdot\sigma(a) = 4p. \tag{72}$$

We wish to evaluate the radial component of the stress tensor, $\sigma_{rr} \equiv \hat{r}\cdot\sigma(\hat{r})$, where $\hat{r}$ is the radial unit vector. This is because in our spherically symmetric model it is this component that dampens the oscillation. From 71 this is evaluated to be

$$\sigma_{rr} = -p + 2\eta\left[\partial_r v + \tfrac{\gamma^2 v}{c}\tfrac{cu\cdot\nabla}{\gamma}\left(\tfrac{\gamma v}{c}\right) - \tfrac{1}{3}c\nabla\cdot u\left(1 + \tfrac{\gamma^2 v^2}{c^2}\right)\right], \tag{73}$$

where v is the radial component of the velocity. To keep the expressions short the inner products $u\cdot\nabla = \frac{\gamma}{c}\left(\partial_t + v\partial_r\right)$ and $\nabla\cdot u = \frac{1}{c}\left(\partial_t\gamma + \partial_r(\gamma v)\right)$ have not been expanded.

Equation 73 can be simplified by noting that $\left(1 + \frac{\gamma^2 v^2}{c^2}\right) = \left(1 - \gamma^2\left(1 - \frac{v^2}{c^2}\right) + \gamma^2\right) = \gamma^2$. Using a similar trick we find,

$$\frac{\gamma^2 v}{c}\frac{cu\cdot\nabla}{\gamma}\left(\frac{\gamma v}{c}\right) = -\left(1-\gamma^2\right)\frac{cu\cdot\nabla}{\gamma}\gamma + \frac{\gamma^3 v}{c^2}\frac{cu\cdot\nabla}{\gamma}v = \gamma cu\cdot\nabla\gamma. \tag{74}$$

The relation $d\gamma = \frac{\gamma^3}{c^2}vdv$ has been used to obtain the second equality. Equation 73 then simplifies to

$$\sigma_{rr} = -p + 2\eta\left[\partial_r v + \gamma cu\cdot\nabla\gamma - \tfrac{1}{3}\gamma^2 c\nabla\cdot u\right]. \tag{75}$$

To make further progress we re-evaluate the trace using 75 and the other components of the diagonal of the stress tensor,

$$\sigma_{tt} = p - 2\eta\left[\partial_t\gamma - \gamma c u\cdot\nabla\gamma - \tfrac{1}{3}\left(1-\gamma^2\right)c\nabla\cdot u\right] \tag{76}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = -p + 2\eta\left[\tfrac{v}{r} - \tfrac{1}{3}c\nabla\cdot u\right], \tag{77}$$

and find that

$$\mathrm{Tr}\ \sigma(a) = 4p - 2\eta\left[2\tfrac{v}{r} + \partial_r v - \partial_r(\gamma v)\right]. \tag{78}$$

Equating 72 with 78 implies that

$$\partial_r v = -2\frac{v}{r} + \partial_r(\gamma v) \tag{79}$$

and so

$$\sigma_{rr} = -p - 4\eta\left[\frac{v}{r} - \tfrac{1}{2}\left(1-\tfrac{1}{3}\gamma^2\right)\partial_r(\gamma v) - \tfrac{6\gamma^2}{5c^2}v\partial_t v\right]. \tag{80}$$

In the non-relativistic limit this equals

$$\sigma_{rr} = -p - 4\eta\left[\frac{v}{r} - \tfrac{1}{3}\partial_r v\right], \tag{81}$$

which is the standard expression for the radial stress exerted on a bubble [7].

The spatial derivative $\partial_r(\gamma v)$ in 80 may be evaluated from the relation $\partial_r(\phi v) = -\partial_t\phi$, obtained from the spatial projection of 62 in the laboratory frame. However, the resulting equation is complicated to evaluate and contributes of order c^-2 compared to the first term of 75, which of itself is small if the viscosity of the fluid is low. Therefore, for analytic simplicity, we neglect the small terms on the right of 80 and write

$$\sigma_{rr} \approx -p - 4\eta\frac{v}{r}. \tag{82}$$

Equation 82 is of adequate accuracy for our purposes and is used in the numerical studies that follow. It has the additional virtue in that it is identical for both acoustical and optical measurement.

The pressure boundary condition is then

$$p_b(a) = p_e\left(\frac{a}{a_e}\right)^{-3\kappa} - 2\frac{\sigma}{a} - 4\eta\frac{\dot{a}}{r}. \tag{83}$$

9 Analysis of the equation

We finish by briefly examining the non-linear response of the two models to an incident sound pulse. The strongly non-linear response of bubbles is important in medical applications because it is a property not shared by the surrounding tissue, and therefore provides a means of identifying the bubble.

To start, we compare the response of the two models for 3 different pressures: 100 kPa, 300 kPa and 500 kPa. The equilibrium radius of the bubble is chosen to be 2 µm, which, like the pressures, is typical for diagnostic ultrasound applications. The pressure applied is a sinusoidal with a frequency of 500 kHz. The pulse consists of 10 cycles and the first and last quarter of the pulse is tempered with a cosine function. The radial and velocity response of the bubble is plotted in Figure **??**.

For the low incident pressure of 100 kPa the radial response of the bubble is essentially identical in the two models (Figure 1a). This is as would be expected, for as Figure 1b shows, at this pressure the velocity of the bubble wall is always a small fraction of the sound speed.

At the higher pressure of 100 kPathe original Keller-Miksis model predicts that the bubble wall collapses at very high velocities, even surpassing the speed of sound on some occasions. It is obvious that ultrasound measurements cannot measure the speed of a bubble wall undergoing supersonic collapse, and so the predictions of the original Keller-Miksis equation is contrary to what is measured acoustically. Figure 1d illustrates what ultrasound would measure when a bubble responds to the pulse. The acoustically measured velocity is always slower than the sound speed. The radial response as measured by ultrasound is predicted to be different to the response as measured optically, as is shown in Figure 1c.

At the yet higher pressure of 500 kPa there is a small surprise. As before the original Keller-Miksis model predicts that the bubble wall will collapse at speeds that cannot be measured acoustically, and as before the acoustically-observed-Keller-Miksis model assigns the spatial and temporal locations according to the pulse-echo definitions. The surprise, however, is how well it does in comparison to Figure 1c. In Figure 1e the predicted radial response of the bubble looks very similar with both models; the maximal radii are in good agreement, as are the times at which the bubble reaches its minimal radii. All this despite the large differences in the predicted velocity Figure 1e.

A clue as to why the radial response at some pressures looks very similar for both models (Figure 1a and Figure 1e), while at other pressures it looks very different (Figure 1c) - apparently without any obvious correlation with the speed of the bubble wall - is found in the predicted scattering cross section of the two models.

9.1 The scattering cross section

The scattering cross section, σ is found from the ratio of the emitted acoustic power to the incident intensity,[21]

$$\sigma(\omega) = 4\pi r^2 \oint \frac{\left(p(r,t)a(t)\right)^2}{p_i(r,t)^2} dt, \tag{84}$$

where p is the emitted pressure, p_i is the incident pressure (a plain wave of frequency ω, and a is the bubble radius. In the acoustic far field the pressure emitted by the bubble[9] is

$$p(r,t) = \frac{a^2\ddot{a} + 2a\dot{a}^2}{r^2}, \tag{85}$$

and so the r^2 dependence of 84 and 85 cancels. The scattering cross section may be normalised by dividing out the area of the bubble at equilibrium, $4\pi a_e^2$.

The scattering cross section is only well defined for an incident planar wave. The period of the emitted wave can be different from period of the incident wave. The integral is carried out over the time period where both incident and emitted waves are stable - the closed integral sign being an mnemonic of this. This occurs when both the imaging and emitted wave oscillate an integer number of times within the period. However, this can make the scattering cross section hard to evaluate, for such a period may not exist (the ratio of the periodicity of the incident and emitted waves may be irrational), or else may be very long, and hence hard to find numerically.

Since the temporal and spatial dependence of the scattering cross section are integrated out, the scattering cross section is not expected to be dependant upon the measurement process. On the other hand, however, the scattering cross section is dependent upon the bubble wall's measured radius, velocity and acceleration in 85 - and therefore the two models will give different answers when M^2 is not small. We note, however, that the original Keller-Miksis model, unlike the acoustically measured version, never claimed to be accurate in presence of high velocities. The scattering cross section should therefore be computed using the acoustically measured theory.

To find the scattering cross section numerically a finite incident sinusoid must be used to drive the oscillation, and the incident number of cycles must be sufficient for the transient response to dampen. In this chapter, we use 750 cycles. To evaluate whether the response has stabilised a 12 cycle section of the radial response (i.e. from cycle 738 to 750) is chosen as a reference and the cross-correlation of the this 12-cycle segment is evaluated with previous 12-cycle segments. When the average of the cross-correlation coefficients is to within 0.1 % of the average of the autocorrelation coefficients, we consider that the bubble response is sufficiently stable and that the ratio of the periodicities of the incident and driving wave are sufficiently close to being integer. The scattering cross section is then evaluated over the segment. This procedure will fail if

1. the transient response has not been sufficiently damped.

2. the ratio of the periodicities of the incident and driving wave wave cannot be expressed in 12 cycles (i.e. the super-harmonic ratio is not 1/6, 1/3,1/2, 2/3, 1, 2, 3/2, 3, 6).

If the procedure fails then then the search is abandoned and the scattering cross section is not given. While it is possible for more super-harmonics to be searched for, this process cannot go on forever, for irrational super-harmonics will never be found.

The normalised scatting cross section of a two micron bubble as a function of the incident frequency is plotted in Figure 2. Below the figures it is plotted where the scattering cross section could not be evaluated.

At 10 kPa the bubble's response is nearly linear when evaluated with both models, and Figure 2a exhibits a large resonance peak near to 2 MHz- which is familiar from linear studies on bubble response [7]. At the higher pressure of 100 kPa, drawn in Figure 2b, the fundamental resonance occurs at lower frequency, the bubble also responds when pulsated at the first harmonic and at fractions of the fundamental. The response of the two models at 100 kPa is essentially identical, as was seen for the same pressure in Figure 1a and Figure 1b.

At the higher pressures shown in Figure 2 differences do emerge between the responses predicted by the two models. Considering Figure 2a first, we find that the fundamental again occurs at a lower frequency. Above this resonance the scattering cross section predicted by the two models is essentially identical. Near the resonance it becomes hard to evaluate the scattering cross-section, and there is a large drop-out in returned values. Such dropouts occur when unusual harmonics are present (or developing) within the bubble's response. Below the resonance the scattering cross section for both models becomes more stable (with few dropouts) but the scattering cross section evaluated using the acoustically-measured model is systematically lower than for when the scattering cross section is evaluated from the original model. These observations are repeated in Figure 2b, except that the discrepancy seems to begin at a harmonic of the fundamental.

Figure 2 may now be used to give an explanation the observations of Figure **??**. The frequency used to pulsate the bubbles in Figure **??** was 1/2 MHz. In Figure 2a 1/2 MHz is seen to occur in the region of the graph near resonance where the scatting cross section is hard to evaluate. This suggests that the differences observed in the response of Figure 1c are due to differences in the harmonic response of the two equations. In Figure 2b, 1/2 MHz is well below the dropout region, suggesting that the harmonics have little role. This would explain why the radial response appeared so similar. The difference in the scattering cross

(a) 100 kPa (b) 100 kPa (c) 300 kPa (d) 300 kPa (e) 500 kPa (f) 500 kPa

Figure 1: The calculated response of a two micron bubble to a 1/2 MHz wave at various pressures as measured optically (using the Keller-Miksis model) and acoustically (using the acoustically-measured-Keller-Miksis model) The radial response is shown in the figures on the left, the velocity response on the right.

(a) 10 kPa (b) 100 kPa

(a) 300 kPa (b) 500 kPa

Figure 2: The calculated normalised scattering cross section as a function of frequency evaluated at various pressures as measured optically (using the Keller-Miksis model) and acoustically (using the acoustically-measured-Keller-Miksis model). The bubble has an equilibrium radius of 2 µm. Below the graphs the frequencies at which the scattering cross section could not be evaluated is plotted. The small vertical axis of this plot is meaningless, it is used to help convey the density of points.

section between the two models in Figure 2b would therefore seem to result from the differing velocity and acceleration that is measured by the two models, which manifests itself in the scattering cross via the $\dot{a}$ and $\ddot{a}$ terms in equation 85.

10 Discussion

In this chapter we have derived for the first time the pulsations of a bubble as they would be measured with ultrasound. The model is based upon the Keller-Miksis model, but is exact. Indeed, the original Keller-Miksis equation can be obtained by approximating the acoustically-measured version - valid for when velocity of the bubble wall is small in comparison to the sound speed.

The derivation followed from noting that for ultrasound to measure distances (using pulse-echo) the sound speed in the medium must be known *a priori*, with no possibility of measuring variations in this sound speed. This has two consequences:

1. The sound speed must be measured to be a constant.
2. The sound must be measured to be propagating linearly.

If invariance to inertial translations is also assumed, then the constancy of the sound speed implies that ultrasound is subject to the considerations of special relativity, with the sound speed taking the role of the speed of light. This invariance has been assumed here.

To describe the propagation of the sound we considered an ideal fluid medium. When measured acoustically the relativistic (Lorentz invariant) description of the ideal fluid should be used, with the additional constraint that the speed of sound equal the speed of light. We have shown that the sound does indeed propagate linearly in this case. Indeed, we have shown it obeys Maxwell's relations.

Finally, the response of the acoustically-measured-Keller-Miksis equation to an acoustic wave has been simulated, and the results compared to the original Keller-Miksis equation. The radial pulsations observed by the two models is similar when the harmonic response of the bubble is not strong - otherwise the pulsations become quite different. The velocity response for the two models diverges when the bubble wall speed is high. The acoustically-measured-Keller-Miksis equation correctly maintains that ultrasound cannot measure supersonic bubble collapse, and predicts what is measured acoustically when such collapses do occur.

When the bubble wall moves at speeds close or exceeding the sound speed, differences are found between the scattering cross section obtained from the two models. The acoustically-measured-Keller-Miksis model claims accuracy in this regime. The original does not. The scattering cross section predictions therefore provide a means of testing the acoustically-measured-Keller-Miksis equation directly.

References

[1] Carlos Barceló, Stefano Liberati, and Matt Visser. Analogue gravity. *Living Reviews in Relativity*, 8:12, 2005. arXiv:gr-qc/0505065v2.

[2] Carl E. Dolby and Stephen F. Gull. On radar time and the twin 'paradox'. *American Journal of Physics*, 69(12):1257–1261, October 2001.

[3] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press, 2003.

[4] A. Einstein. Zur elektrodynamik bewegter körper. *Annalen der Physik*, 17: 891–921, 1905. English translation available in *The Priciple of Relativity*, Dover, 1952.

[5] D. Hestenes. Spacetime physics with geometric algebra. *American Journal of Physics*, 71(7):691–714, July 2003.

[6] David Hestenes and Garret Sobczyk. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*. D. Reidel Publishing Company, Kluwer Academic Publishers Group, 1984.

[7] L Hoff. *Acoustic Characterisation of Contrast Agents for Medical Ultrasound Imaging*. Kluser Academic Publishers, 2001.

[8] M. S. Howe. *Acoustics of Fluid-Structure Interactions*. Cambridge Monographs on Mechanics. Cambridge University Press, 1998.

[9] M. S. Howe. *Theory of Vortex Sound*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2003.

[10] Joseph B. Keller and Michael Miksis. Bubble oscillations of large amplitude. *Journal of the Acoustical Society of America*, 68(2):628–633, Aug 1980.

[11] L.D. Landau and E.M. Lifshitz. *Fluid Mechanics*. Number 6 in Course of Theoretical Physics. Elsevier Butterworth Heinemann, 2nd edition, 1987.

[12] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. A multivector derivative approach to lagrangian field theory. *Foundations of Physics*, 23(10):1295–1327, 1993.

[13] M. J. Lighthill. On sound generated aerodynamically. i. general theory. *Proc. R. Soc. Lond. A*, 211(1107):564–587, 20 March 1952.

[14] Haralabos Marmanis. *Analogy between the Electromagnetic and Hydrodynamic Equations: Appliaton to Turbulence*. PhD thesis, Brown University, 2000.

[15] H. K. Moffatt. The degree of knottedness in tangled vortex lines. *Journal of Fluid Mechanics*, 35(1):117–129, 1969.

[16] E.A. Neppiras. Acoustic cavitation. *Physics Reports*, 61(3):159–251, 1980.

[17] C. L. Pekeris. Relativistic axially symmetric flows of a perfect fluid. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 355(1680):53–60, Jun. 30 1977.

[18] Henri Poincaré. *Science and Hypothesis*. English translation The Walter Scott Translation Co. Ltd. 1905, 1902.

[19] Antonio F. Rañada. Topological electromagnetism. *J. Phys. A: Math. Gen.*, 25:1621, 1992. doi: DOI: 10.1088/0305-4470/25/6/020.

[20] Antonio F. Rañada and José Trueba. Topological electromagnetism with hidden nonlinearity. In Myron W. Evans, editor, *Modern Nonlinear Optics, Part III*, volume 119 of *Advances in Chemical Physics*, pages 197–253. John Wiley & Sons, Inc., 2nd edition, 2002.

[21] V.A. Shutilov. *Fundalmental Physics of Ultrasound*. Gordaon and Breach Science Publishers, 1988.

[22] S. Sridhar. Turbulent transport of a tracer: An electromagnetic formulation. *Physical Review E*, 58(1):522–525, July 1998.

[23] A. H. Taub. Relativistic fluid mechanics. *Annual Reviews in Fluid Mechanics*, 10(301–332), 1978.

[24] José Trueba and Antonio F. Rañada. The electromagnetic helicity. *The European Journal of Physics*, 17:141–144, 1996.

[25] José Trueba and Antonio F. Rañada. Helicity in classical electrodynamics and its topological quantization. *APEIRON*, 7(1-2):83–88, January-April 2000.