

# OVERVIEW

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## The Fast Fourier Transform

- ▶ First used by Carl Friedrich Gauss but went unnoticed.
- ▶ James Cooley and John Tukey independently discovered the Cooley-Tukey algorithm in 1965.
- ▶ Improves the  $O(N^2)$  running time it takes to perform DFT to  $O(N \log_2 N)$  (regarding complex addition and multiplication).

	n=1000	n=10 <sup>6</sup>	n=10 <sup>9</sup>
▶ $n^2$	10 <sup>6</sup>	10 <sup>12</sup>	10 <sup>18</sup>
$n \log_2 n$	~10 <sup>4</sup>	~2 · 10 <sup>7</sup>	~3 · 10 <sup>10</sup>

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A degree  $N - 1$  polynomial is given by

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}.$$

Other representations:

- ▶ **Coefficient vector:** A vector  $(a_0, a_1, \cdots, a_{N-1})$  in a space of monomials.
- ▶ **Roots:** Given roots of a polynomial, we can write  $P(x) = k \prod_{n=0}^{N-1} (x - r_n)$  for some constant  $k$ .
- ▶ **Samples:** We can also take  $N$  distinct pairs of  $(x_n, y_n = P(x_n))$  to uniquely determine the polynomial  $P(x)$  by Lagrange polynomial interpolation.

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There are three main things we want to do with polynomials, namely:

- ▶ **Evaluation:** Evaluate the polynomial at a point in its domain.
- ▶ **Addition:** Add two polynomials together.
- ▶ **Multiplication:** Multiply two polynomials together. Note that polynomial multiplication in the coefficient vector form actually corresponds to the finite discrete convolution of two vectors in their inner product space.

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Running-time for each operation on degree $N - 1$ polynomials			
	Coefficients	Roots	Samples
Evaluation	$O(N)$	$O(N)$	$O(N^2)$
Addition	$O(N)$	N/A	$O(N)$
Multiplication	$O(N^2)$	$O(N)$	$O(N)$

Objective: convert coefficients to samples efficiently as to avoid  $O(N^2)$  running-time.

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We can sample  $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}$  by doing

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{N-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix}.$$

This calculates  $N$  samples at  $N$  distinct points in the set  $X = \{x_0, \cdots, x_{N-1}\}$  with a running-time of  $O(N^2)$ ...

Divide and conquer! (this is where we assume  $N$  is a power of 2)

To evaluate our polynomial  $P(x)$  at the point  $x$ , we can define the following:

$$\blacktriangleright P_{\text{even}}(x) = \sum_{k=0}^{[(N-1)/2]} a_{2k} x^k$$

$$\blacktriangleright P_{\text{odd}}(x) = \sum_{k=0}^{[(N-1)/2]} a_{2k+1} x^k$$

$$\blacktriangleright P(x) = P_{\text{even}}(x^2) + x \cdot P_{\text{odd}}(x^2)$$

Do this for every  $x$  we want to sample...  $N$  of them.



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Do this for every  $x$  we want to sample...  $N$  of them.

Let  $T$  measure the running-time of calculating  $P(x)$  for every  $x$  in  $X$ , then we have that

$$\begin{aligned}
 T(N, |X|) &= 2T\left(\frac{N}{2}, |X|\right) + O(|X|) \\
 &= 2 \left[ 2T\left(\frac{N}{4}, |X|\right) + O(|X|) \right] + O(|X|) \\
 &\vdots \\
 &= 2^{\log_2 N} T(1, |X|) + \sum_{l=0}^{\log_2 N} 2^l O(|X|) \\
 &= N^2 + O(|X| \log_2 N).
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$O(N^2)$  again!

Root cause? This algorithm chokes when it has to evaluate monomials **for every element in the set**.

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What if there is a set that shrinks in size whenever we square each element?

$$\begin{aligned}\{1\} &= \{e^{2\pi i \frac{1}{1}}\}, N = 1 \\ \{-1, 1\} &= \{e^{2\pi i \frac{1}{2}}, e^{2\pi i \frac{2}{2}}\}, N = 2 \\ \{i, -1, -i, 1\} &= \{e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{2}{4}}, e^{2\pi i \frac{3}{4}}, e^{2\pi i \frac{4}{4}}\}, N = 4 \\ &\vdots\end{aligned}$$

Eventually end up with **the Nth roots of unity**  $\{e^{2\pi i \frac{k}{N}}\}_{k=0}^{N-1}$ .

Key property: any such root computes to 1 when raised to the Nth power.

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Hello,  $O(N \log_2 N)$ !

But what does converting polynomial forms have to do with FFT?

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$$\begin{pmatrix} 1 & (e^{2\pi i \frac{0}{N}})^1 & (e^{2\pi i \frac{0}{N}})^2 & \dots & (e^{2\pi i \frac{0}{N}})^{N-1} \\ 1 & (e^{2\pi i \frac{1}{N}})^1 & (e^{2\pi i \frac{1}{N}})^1 & \dots & (e^{2\pi i \frac{1}{N}})^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (e^{2\pi i \frac{N-1}{N}})^1 & (e^{2\pi i \frac{N-1}{N}})^2 & \dots & (e^{2\pi i \frac{N-1}{N}})^{N-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}.$$

Now recall again the discrete Fourier coefficients

$$A_n = \frac{1}{N} \sum_{k=0}^{N-1} F(k) (e^{-2\pi i \frac{n}{N}})^k \text{ for } n \text{ running from } 0 \text{ to } N-1.$$

Replace  $a_k$  with  $\frac{1}{N}F(k)$  and take the conjugate of the DFT matrix then we have Cooley and Tukey's radix-2 FFT.

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After multiplication/convolution, we need to convert samples back into coefficients.

Luckily, this is easy to do!

Let  $V$  be the DFT matrix as described earlier, then

$$V^{-1} = \frac{1}{N} \bar{V},$$

which can be proven by showing that

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$$V^{-1} = \frac{1}{N} \bar{V},$$

which can be proven by showing that

$$V\bar{V} = NI.$$

Then  $\vec{a} = V^{-1}\vec{y}$ .

After multiplication/convolution, we need to convert samples back into coefficients.

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## References





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