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Bringing it Back

- ► First used by Carl Friedrich Gauss but went unnoticed.
- ▶ James Cooley and John Tukey independently discovered the Cooley-Tukey algorithm in 1965.
- ▶ Improves the $O(N^2)$ running time it takes to perform DFT to $O(N \log_2 N)$ (regarding complex addition and multiplication).

		n=1000	$n=10^6$	$n=10^9$
\triangleright	n^2	10^{6}	10^{12}	10^{18}
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A degree
$$N-1$$
 polynomial is given by
$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}.$$

Other representations:

- ► Coefficient vector: A vector $(a_0, a_1, \dots, a_{N-1})$ in a space of monomials.
- ▶ **Roots:** Given roots of a polynomial, we can write $P(x) = k \prod_{n=0}^{N-1} (x r_n)$ for some constant k.
- ▶ **Samples:** We can also take N distinct pairs of $(x_n, y_n = P(x_n))$ to uniquely determine the polynomial P(x) by Lagrange polynomial interpolation.

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There are three main things we want to do with polynomials, namely:

- **Evaluation:** Evalutae the polynomial at a point in its domain.
- ► **Addition:** Add two polynomials together.
- Multiplication: Multiply two polynomials together. Note that polynomial multiplication in the coefficient vector form actually corresponds to the finite discrete convolution of two vectors in their inner product space.

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Running-time for each operation on degree $N-1$ polynomials				
	Coeffcients	Roots	Samples	
Evaluation	O(N)	O(N)	$O(N^2)$	
Addition	O(N)	N/A	O(N)	
Multiplication	$O(N^2)$	O(N)	O(N)	

Objective: convert coefficients to samples efficiently as to avoid $\mathcal{O}(N^2)$ running-time.

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We can sample $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{N-1}x^{N-1}$ by doing

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{N-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix}.$$

This calculates N samples at N distinct points in the set $X = \{x_0, \dots, x_{N-1}\}$ with a running-time of $O(N^2)$...

To evaluate our polynomial P(x) at the point x, we can define the following:

►
$$P_{\text{even}}(x) = \sum_{k=0}^{[(N-1)/2]} a_{2k} x^k$$

► $P_{\text{odd}}(x) = \sum_{k=0}^{[(N-1)/2]} a_{2k+1} x^k$

► $P(x) = P_{\text{even}}(x^2) + x \cdot P_{\text{odd}}(x^2)$

Do this for every x we want to sample... N of them

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Do this for every x we want to sample... N of them.

Let T measure the running-time of calculating P(x) for every x in X, then we have that

$$T(N, |X|) = 2T(\frac{N}{2}, |X|) + O(|X|)$$

$$= 2\left[2T(\frac{N}{4}, |X|) + O(|X|)\right] + O(|X|)$$

$$\vdots$$

$$= 2^{\log_2 N} T(1, |\mathbf{X}|) + \sum_{l=0}^{\log_2 N} 2^l O(|X|)$$

$$= \mathbf{N}^2 + O(|X| \log_2 N).$$

 $O(N^2)$ again!

Root cause? This algorithm chokes when it has to evaluate monomials for every element in the set.

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1\} &= \{e^{2\pi i \frac{1}{1}}\}, N = 1 \\
\{-1, 1\} &= \{e^{2\pi i \frac{1}{2}}, e^{2\pi i \frac{2}{2}}\}, N = 2 \\
\{i, -1, -i, 1\} &= \{e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{2}{4}}, e^{2\pi i \frac{3}{4}}, e^{2\pi i \frac{4}{4}}\}, N = 4 \\
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Eventually end up with the Nth roots of unity $\{e^{2\pi i \frac{k}{N}}\}_{k=0}^{N-1}$.

Key property: any such root computes to 1 when raised to the Nth power.

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$$\begin{pmatrix} 1 & (e^{2\pi i\frac{0}{N}})^1 & (e^{2\pi i\frac{0}{N}})^2 & \cdots & (e^{2\pi i\frac{0}{N}})^{N-1} \\ 1 & (e^{2\pi i\frac{1}{N}})^1 & (e^{2\pi i\frac{1}{N}})^1 & \cdots & (e^{2\pi i\frac{1}{N}})^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (e^{2\pi i\frac{N-1}{N}})^1 & (e^{2\pi i\frac{N-1}{N}})^2 & \cdots & (e^{2\pi i\frac{N-1}{N}})^{N-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}.$$

Now recall again the discrete Fourier coefficients

$$A_n = \frac{1}{N} \sum_{k=0}^{N-1} F(k) (e^{-2\pi i \frac{n}{N}})^k \text{ for } n \text{ running from } 0 \text{ to } N-1.$$

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Let V be the DFT matrix as described earlier, then

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Then
$$\vec{a} = V^{-1}\vec{y}$$
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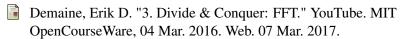
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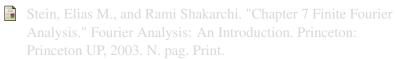
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