Problem 1

$$min\|w\|^2 \; s.t.: \; y_i \left(w^T x_i + \theta\right) \geq 1, \forall i \in \{1, \; ..., \; n\} \tag{1}$$

a)

Formulate the Lagrangian:

$$\Lambda\left(w, \theta, \alpha\right) = \frac{1}{2}\|w\|^2 + \sum_{i=1}^{n} \alpha_i \left[y_i \left(w^T x_i + \theta\right) - 1\right] \tag{2}$$

b)

Partial Derivatives

$$\frac{\Lambda\left(w, \theta, \alpha\right)}{\partial w} = w + \sum_{i=1}^{n} \alpha_i y_i x_i \stackrel{!}{=} 0 \Rightarrow w = -\sum_{i=1}^{n} \alpha_i y_i x_i \tag{3}$$

$$\frac{\Lambda\left(w, \theta, \alpha\right)}{\partial \theta} = \sum_{i=1}^{n} \alpha_i y_i \stackrel{!}{=} 0 \tag{4}$$

Substitute the results from the partial derivatives (3) and (4) into the primal (2) to get the dual problem:

$$\begin{aligned}
\Lambda\left(w, \theta, \alpha\right) &= \frac{1}{2}\|w\|^2 + \sum_{i=1}^{n} \alpha_i \left[y_i \left(w^T x_i + \theta\right) - 1\right] \\
&= \frac{1}{2}\|\sum_{i=1}^{n} \alpha_i y_i x_i\|^2 - \sum_{k=1}^{n} \alpha_k \left[y_k \left(\left(\sum_{l=1}^{n} \alpha_l y_l x_l\right)^T x_k + \theta\right) - 1\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j - \sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_k \alpha_l y_k y_l x_k^T x_l + \sum_{k}^{n} \alpha_k y_k \theta + \sum_{k}^{n} \alpha_k \\
&= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j + \theta \underbrace{\sum_{k}^{n} \alpha_k y_k}_{=0} + \sum_{k}^{n} \alpha_k \\
&= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum_{k}^{n} \alpha_k \stackrel{!}{=} L\left(\alpha\right)
\end{aligned} \tag{5}$$

The dual problem can therefore be expressed as:

$$\begin{aligned} max - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum_{k}^{n} \alpha_k \\ s.t. \sum_{i=1}^{n} \alpha_i y_i x_i \stackrel{!}{=} 0 \; and \; \alpha_i \geq 0 \; \forall i \in \{1, \; ..., \; n\} \end{aligned} \tag{6}$$

Which is now a standard optimization problem in α.

c)

Primal - kernelized:

$$\begin{aligned} min \frac{1}{2} \|w\|^2 + \sum_{i=1}^{n} \alpha_i \left[y_i \left(w^T x_i + \theta \right) - 1 \right] \\ = min \; k \left(w^T w \right) + \sum_{i=1}^{n} \alpha_i \left[y_i \left(w^T x_i + \theta \right) - 1 \right] \end{aligned} \tag{7}$$

Dual - kernelized:

$$\begin{aligned} max \; - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum_{k}^{n} \alpha_k \\ max \; - \frac{1}{2} k \left(w^T w \right) + \sum_{k}^{n} \alpha_k \end{aligned} \tag{8}$$

Problem 2

Reformulate the dual problem (6)

$$\begin{aligned} min \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j - \sum_{k}^{n} \alpha_k \\ s.t. \sum_{i=1}^{n} \alpha_i y_i x_i \stackrel{!}{=} 0 \; and \; \alpha_i \geq 0 \; \forall i \in \{1, \; ..., \; n\} \end{aligned} \tag{9}$$

According to this we can formulate

1. the quadratic form as:

$$\begin{aligned} x^T P x &\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j \\ &= \alpha^T \begin{bmatrix} y_1 y_1 x_1^T x_1 & \dots & y_1 y_n x_1^T x_n \\ \vdots & \ddots & \vdots \\ y_n y_1 x_n^T x_1 & \dots & y_n y_n x_n^T x_n \end{bmatrix} \alpha \\ &= \alpha^T \left[y y^T * X X^T X \right] \alpha \end{aligned} \tag{10}$$

   where $*$ denotes element-wise multiplication.

2. the scalar-product as:

$$q^T x \Rightarrow -\sum_{k}^{n} \alpha_k = -\mathbb{1}^T \alpha \tag{11}$$

   where $\mathbb{1}$ denotes the vector of ones

3. the first constraint:

$$Gx \preceq h \Rightarrow -\alpha \preceq 0 \tag{12}$$

4. the second constraint:

$$Ax = b \Rightarrow y^T \alpha = \vec{0} \tag{13}$$

To sum up:

1. $P = \left[y y^T * X X^T X \right]$
2. $q = -\mathbb{1}$
3. $G = -1$
4. $h = 0$
5. $A = y^T$
6. $b = \vec{0}$