

APPENDIX

Proposition 2. For any solution (x_l, x_u) with $\beta \leq \gamma$, $x_l \in X$ and $x_u \in X$, there always exists another solution (x_l, x'_u) having $C(x_l, x'_u) \leq C(x_l, x_u)$, where $x'_u = \min X_c + 2^\beta$.

Proof. According to $\beta = \lceil \log(\max X_c - \min X_c + 1) \rceil$ in Formula 8, we have

$$\begin{aligned} \log(\max X_c - \min X_c + 1) &\leq \beta \\ \max X_c - \min X_c + 1 &\leq 2^\beta \\ \max X_c &\leq \min X_c + 2^\beta - 1 \\ \max X_c &< x'_u. \end{aligned}$$

(1) For $x_u > x'_u$, it follows $\max X_c < x'_u < x_u = \min X_u$. Since there is no value between $\max X_c$ and $\min X_u$ in X , according to Definitions 2 and 4, we have $\min X'_u = \min X_u$.

(2) For $x_u \leq x'_u$, referring to Definition 4, we have $\max X'_u \geq \max X_u$.

Combining the above two cases, we can conclude that

$$\min X'_u \geq \min X_u.$$

For $n_u = |X_u|$ and $n'_u = |X'_u|$ introduced after Definition 4, it follows $n_u \geq n'_u$.

Let $n_\Delta = |X_u \setminus X'_u|$ be the size of the increment, having $n_\Delta = n_u - n'_u \geq 0$.

Given the same x_l and the corresponding identical X_l, n_l , we could get the difference C_Δ between $C(x_l, x'_u)$ and $C(x_l, x_u)$ defined in Formula 5,

$$\begin{aligned} C_\Delta &= C(x_l, x'_u) - C(x_l, x_u) \\ &= n_l(\lceil \log(\max X_l - x_{\min} + 1) \rceil + 1) \\ &\quad + n'_u(\lceil \log(x_{\max} - \min X'_u + 1) \rceil + 1) \\ &\quad + (n - n_l - n'_u)\lceil \log(\max X'_c - \min X'_c + 1) \rceil \\ &\quad - n_l(\lceil \log(\max X_l - x_{\min} + 1) \rceil + 1) \\ &\quad - n_u(\lceil \log(x_{\max} - \min X_u + 1) \rceil + 1) \\ &\quad - (n - n_l - n_u)\lceil \log(\max X_c - \min X_c + 1) \rceil. \end{aligned} \quad (12)$$

The same x_l also infers $\min X'_c = \min X_c$. Together with $n_u = n'_u + n_\Delta$, we have

$$C_\Delta = C_1 - C_2, \quad (13)$$

where

$$\begin{aligned} C_1 &= (n - n_l - n_u)\lceil \log(\max X'_c - \min X_c + 1) \rceil \\ &\quad + n_\Delta \lceil \log(\max X'_c - \min X_c + 1) \rceil \\ &\quad + n'_u(\lceil \log(x_{\max} - \min X'_u + 1) \rceil + 1) \end{aligned}$$

and

$$\begin{aligned} C_2 &= (n - n_l - n_u)\lceil \log(\max X_c - \min X_c + 1) \rceil \\ &\quad - n_\Delta(\lceil \log(x_{\max} - \min X_u + 1) \rceil + 1) \\ &\quad - n'_u(\lceil \log(x_{\max} - \min X_u + 1) \rceil + 1) \\ &= (n - n_l - n_u)\beta - n_\Delta(\gamma + 1) - n'_u(\gamma + 1). \end{aligned}$$

(i) Referring to Definition 2, we have $\max X'_c < x'_u = \min X_c + 2^\beta$. It follows

$$\begin{aligned} \log(\max X'_c - \min X_c + 1) &\leq \log(2^\beta) \\ \lceil \log(\max X'_c - \min X_c + 1) \rceil &\leq \beta. \end{aligned}$$

(ii) With the aforesaid proved $\min X'_u \geq \min X_u$, we infer

$$\lceil \log(x_{\max} - \min X'_u + 1) \rceil \leq \lceil \log(x_{\max} - \min X_u + 1) \rceil = \gamma.$$

Applying the above two conditions, we further derive

$$\begin{aligned} C_\Delta &\leq (n - n_l - n_u)\beta + n_\Delta\beta + n'_u(\gamma + 1) \\ &\quad - (n - n_l - n_u)\beta - n_\Delta(\gamma + 1) - n'_u(\gamma + 1) \\ &= n_\Delta(\beta - \gamma - 1) \leq 0. \end{aligned}$$

Given $\beta \leq \gamma$ and $n_\Delta \geq 0$, the conclusion is proved. \square

Proposition 3. For any solution (x_l, x_u) with $\beta > \gamma$, $x_l \in X$ and $x_u \in X$, there always exists another solution (x_l, x'_u) having $C(x_l, x'_u) \leq C(x_l, x_u)$, where $x'_u = x_{\max} - 2^\gamma + 1$.

Proof. According to $\gamma = \lceil \log(x_{\max} - \min X_u + 1) \rceil$ in Formula 9, we have

$$\begin{aligned} \log(x_{\max} - \min X_u + 1) &\leq \gamma \\ x_{\max} - 2^\gamma + 1 &\leq \min X_u \\ x'_u &\leq \min X_u = x_u. \end{aligned}$$

(1) For $x'_u = x_u = \min X_u$, it is exactly the (x_l, x_u) solution, having $\min X'_u = x'_u = \min X_u$, $\max X'_c = \max X_c$.

(2) For $\max X_c < x'_u < x_u = \min X_u$, since there is no value between $\max X_c$ and $\min X_u$ in X , according to Definitions 2 and 4, we have $\min X'_u = \min X_u$, $\max X'_c = \max X_c$ as well.

(3) For $x'_u \leq \max X_c < x_u$, referring to Definitions 2 and 4, it follows $\max X'_c < \min X'_u \leq \max X_c < \min X_u$.

Combining the above three cases, we can infer that

$$\begin{aligned} \min X'_u &\leq \min X_u \\ \max X'_c &\leq \max X_c. \end{aligned}$$

For $n_u = |X_u|$ and $n'_u = |X'_u|$ introduced after Definition 4, it follows $n'_u \geq n_u$. Let $n_\Delta = |X'_u \setminus X_u|$ be the size of the increment, having $n_\Delta = n'_u - n_u \geq 0$.

Given the same x_l and the corresponding identical X_l, n_l , we could get the difference C_Δ between $C(x_l, x'_u)$ and $C(x_l, x_u)$ defined in Formula 5,

$$\begin{aligned} C_\Delta &= C(x_l, x'_u) - C(x_l, x_u) \\ &= n_l(\lceil \log(\max X_l - x_{\min} + 1) \rceil + 1) \\ &\quad + n'_u(\lceil \log(x_{\max} - \min X'_u + 1) \rceil + 1) \\ &\quad + (n - n_l - n'_u)\lceil \log(\max X'_c - \min X'_c + 1) \rceil \\ &\quad - n_l(\lceil \log(\max X_l - x_{\min} + 1) \rceil + 1) \\ &\quad - n_u(\lceil \log(x_{\max} - \min X_u + 1) \rceil + 1) \\ &\quad - (n - n_l - n_u)\lceil \log(\max X_c - \min X_c + 1) \rceil. \end{aligned}$$

The same x_l also infers $\min X'_c = \min X_c$. Together with $n'_u = n_u + n_\Delta$, we have

$$C_\Delta = C_1 - C_2,$$

where

$$\begin{aligned} C_1 &= (n - n_l - n'_u) \lceil \log(\max X'_c - \min X_c + 1) \rceil \\ &\quad + n_\Delta (\lceil \log(x_{\max} - \min X'_u + 1) \rceil + 1) \\ &\quad + n_u (\lceil \log(x_{\max} - \min X'_u + 1) \rceil + 1) \end{aligned}$$

and

$$\begin{aligned} C_2 &= (n - n_l - n'_u) \lceil \log(\max X_c - \min X_c + 1) \rceil \\ &\quad - n_\Delta \lceil \log(\max X_c - \min X_c + 1) \rceil \\ &\quad - n_u (\lceil \log(x_{\max} - \min X_u + 1) \rceil + 1) \\ &= (n - n_l - n'_u) \beta - n_\Delta \beta - n_u (\gamma + 1). \end{aligned}$$

(i) Referring to Definition 2, we have $x'_u = x_{\max} - 2^\gamma + 1 \leq \min X'_u$. It follows

$$\begin{aligned} \log(x_{\max} - \min X'_u + 1) &\leq \log(2^\gamma) \\ \lceil \log(x_{\max} - \min X'_u + 1) \rceil &\leq \gamma. \end{aligned}$$

(ii) With the aforesaid proved $\max X'_c \leq \max X_c$, we infer

$$\lceil \log(\max X'_c - \min X_c + 1) \rceil \leq \lceil \log(\max X_c - \min X_c + 1) \rceil = \beta.$$

Applying the above two conditions, we further derive

$$\begin{aligned} C_\Delta &\leq (n - n_l - n'_u) \beta + n_\Delta (\gamma + 1) + n_u (\gamma + 1) \\ &\quad - (n - n_l - n'_u) \beta - n_\Delta \beta - n_u (\gamma + 1) \\ &= n_\Delta (\gamma + 1 - \beta) \leq 0. \end{aligned}$$

Given $\beta > \gamma$ and $n_\Delta \geq 0$, the conclusion is proved. \square