Journal of Banking & Finance 33 (2009) 1242–1254

Contents lists available at ScienceDirect

Journal of Banking & Finance

journal homepage: www.elsevier.com/locate/jbf

Portfolio performance evaluation with generalized Sharpe ratios: Beyond the mean and variance

Valeri Zakamouline *, Steen Koekebakker

University of Agder, Service Box 422, 4604 Kristiansand, Norway

ARTICLE INFO

Article history:
Received 22 August 2008
Accepted 18 January 2009
Available online 4 February 2009

JEL classification:
G11

Keywords:
Sharpe ratio
Skewness
Kurtosis
Portfolio performance evaluation

ABSTRACT

This paper presents a theoretically sound portfolio performance measure that takes into account higher moments of distribution. This measure is motivated by a study of the investor's preferences to higher moments of distribution within Expected Utility Theory and an approximation analysis of the optimal capital allocation problem. We show that this performance measure justifies the notion of the Generalized Sharpe Ratio (GSR) introduced by Hodges (1998). We present two methods of practical estimation of the GSR: nonparametric and parametric. For the implementation of the parametric method we derive a closed-form solution for the GSR where the higher moments are calibrated to the normal inverse Gaussian distribution. We illustrate how the GSR can mitigate the shortcomings of the Sharpe ratio in resolution of Sharpe ratio paradoxes and reveal the real performance of portfolios with manipulated Sharpe ratios. We also demonstrate the use of this measure in the performance evaluation of hedge funds.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

The Sharpe ratio is a commonly used measure of portfolio performance. However, because it is based on the mean-variance theory, it is valid only for either normally distributed returns or quadratic preferences. In other words, the Sharpe ratio is a meaningful measure of portfolio performance when the risk can be adequately measured by standard deviation. When return distributions are non-normal, the Sharpe ratio can lead to misleading conclusions and unsatisfactory paradoxes, see, for example, Hodges (1998) and Bernardo and Ledoit (2000). For instance, it is well-known that the distribution of hedge fund returns deviates significantly from normality (see, for example, Brooks and Kat, 2002; Agarwal and Naik, 2004; Malkiel and Saha, 2005). Therefore, performance evaluation of hedge funds using the Sharpe ratio seems to be dubious. Moreover, recently a number of papers have shown that the Sharpe ratio is prone to manipulation (see, for example, Leland, 1999; Spurgin, 2001; Goetzmann et al., 2002; Ingersoll et al., 2007). Manipulation of the Sharpe ratio consists largely in selling the upside return potential, thus creating a distribution with high left-tail risk.

The literature on performance evaluation that takes into account higher moments of distribution is a vast one. Motivated by a common interpretation of the Sharpe ratio as a *reward-to-risk* ratio, many researchers replace the standard deviation in the Sharpe ratio by an alternative risk measure. For example, Sortino and Price (1994) replace standard deviation by downside deviation. The examples of the use of the risk measures on the basis of value-at-risk (VaR) include: Dowd (2000) (standard VaR), Favre and Galeano (2002) (modified for skewness and kurtosis VaR), and Rachev et al. (2007) (conditional VaR). Other researchers replace both the risk premium and the standard deviation in the Sharpe ratio with alternative measures of reward and risk. Some examples of this approach are: Stutzer (2000) introduced the Stutzer index which is based on the behavioral hypothesis that investors aim to minimize the probability of returns being below a given threshold. The Omega ratio was introduced by Shadwick and Keating (2002). This measure is expressed as the ratio of the gains with respect to some threshold to the loss with respect to the same threshold. Kaplan and Knowles (2004) introduced the Kappa measure which generalizes the Sortino and Omega ratios. The list can be made much longer and we apologize that not all alternative performance measures[1] can be mentioned here. However, whereas the Sharpe ratio is, in principle, based on Expected Utility Theory which is the cornerstone of modern finance, most of the alternative

* Corresponding author. Tel.: +47 3814 1039; fax: +47 3814 1027.
E-mail addresses: Valeri.Zakamouline@uia.no (V. Zakamouline), Steen.Koekebakker@uia.no (S. Koekebakker).

[1] For a brief review of different portfolio performance measures, as well as different reward and risk measures used in performance evaluation, the interested reader can consult Eling and Schuhmacher (2007) and especially Farinelli et al. (2008).

0378-4266/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.jbankfin.2009.01.005

performance measures lack a solid theoretical underpinning.[2] In addition, most of these measures take into account only downside risk, the upside return potential (for example, positive skewness) is not appreciated.

The main purpose of this paper is to present a theoretically sound portfolio performance measure that takes into account higher moments of the distribution of returns. First, we study the investor's preferences to higher moments of distribution. This is done by considering the optimal capital allocation problem in the standard Expected Utility Theory framework. Our analysis motivates for the introduction of the notion of the "relative preference" to a higher moment of distribution. The relative preference to a moment of distribution is the relation between the absolute preference to a moment of distribution and the absolute preference to the second moment of distribution (the Arrow–Pratt measure of absolute risk aversion) raised to some power. We demonstrate that the investor's preferences to the moments of distribution might be completely described by the Arrow–Pratt measure of absolute risk aversion and the relative preferences to higher moments of distribution.

The main advantage of relative preferences over absolute preferences is that they allow a simple comparison of degrees of preferences to higher moments among different investors. The traditional utility functions most often used in financial economics represent variants of hyperbolic absolute risk aversion (HARA) utility. For HARA utility we derive a formula for the degree of the investor's relative preference to a higher moment of distribution. We demonstrate that if the investor has HARA utility, then the relative preferences to moments of distribution do not depend on the investor's wealth. We also show that an investor with constant relative risk aversion (CRRA) exhibits stronger preferences to higher moments of distribution than an investor with constant absolute risk aversion (CARA). However, as a CRRA investor's coefficient of relative risk aversion increases, the relative preferences to higher moments of distribution of a CRRA investor decrease and converge to the relative preferences of a CARA investor.

Next, we attack the problem of measuring portfolio performance. We start by introducing a definition of a performance measure. For us a performance measure is not just some arbitrary reward-to-risk ratio. In our definition a performance measure is a score, attached to each financial asset, which is related to the level of expected utility provided by the asset. That is, the higher the performance measure of an asset, the higher level of expected utility the asset provides. We believe that this is the most natural definition of a performance measure within the Expected Utility Theory framework. We show that generally there exists a multitude of equivalent performance measures. Moreover, if each investor has a unique utility function, then each investor generally has an individual performance measure. This is due to the fact that each investor might have individual preferences to higher moments of distribution. Nevertheless, if all investors have the same HARA utility, then there exists a common performance measure which is independent of the investors' wealths and is suitable for all investors.

To illustrate the role played by the relative preferences to higher moments of distribution in the performance measurement, we provide an approximation analysis of the optimal capital allocation problem and derive a formula for the Sharpe ratio adjusted for skewness[3] of distribution. This performance measure is denoted as the Adjusted for Skewness Sharpe Ratio (*ASSR*). The *ASSR* preserves the standard Sharpe ratio for zero skewness. Depending on the value and the sign of skewness, the value of the *ASSR* increases when skewness is positive and increases. By contrast, the value of the *ASSR* decreases when skewness is negative and increases (in absolute value). In the formula for the *ASSR* the adjustment for skewness depends on the degree of the investor's relative preference to skewness (third moment of distribution). Unfortunately, the formula for the *ASSR* is only a result of approximation analysis and, therefore, the precision of this formula for practical applications is rather limited. Moreover, this formula accounts only for the first three moments of distribution. Nevertheless, we indicate that the *ASSR* justifies the notion of the Generalized Sharpe ratio (GSR) introduced by Hodges (1998). The GSR seems to be the ultimate generalization of the Sharpe ratio that accounts for all moments of distribution.

It is natural to use the GSR as a ranking statistic in the comparison of performances of different risky assets and portfolios. However, the value of the GSR is generally not unique for all investors, but depends on the investor's preferences to all moments of distribution. To avoid the ambiguity in performance evaluation we further suppose that a sensible performance measure should satisfy the following properties: (1) the measure's value should depend on the return distribution rather than the portfolio's dollar value; (2) the measure should be consistent with the standard financial market equilibrium; (3) the measure should imply the representative investor's risk preferences that are consistent with empirical data. The first two requirements imply that the representative investor has the power utility. The third requirement implies that the value of the relative risk aversion coefficient should be rather large to be consistent with the observed equity premium. It turns out that these three requirements are enough to clear up the ambiguity.

In the paper we also present two methods of practical estimation of the GSR: nonparametric and parametric. The nonparametric estimation of the GSR consists in using the empirical probability distribution of the risky asset in the solution of the optimal capital allocation problem. The GSR computed using the nonparametric method, therefore, accounts for all moments of the empirical probability distribution. However, the nonparametric method relies heavily on numerical methods. Thus, the computation of the GSR using the nonparametric method might be cumbersome. The parametric estimation of the GSR requires making an assumption about the underlying probability distribution of the risky asset. Our choice here is the normal inverse Gaussian (NIG) distribution which allows to define the values of skewness and kurtosis. The parameters of the NIG probability density can easily be identified from the first four moments of distribution. Using the NIG distribution we derive a closed-form solution for the GSR which we denote as the Adjusted for Skewness and Kurtosis Sharpe ratio (ASKSR) since this performance measure accounts for the first four moments of distribution. We demonstrate that the ASKSR reduces to the standard Sharpe ratio when the NIG distribution reduces to the normal distribution. This is yet another justification of the notion of the GSR.

[2] It is possible, however, to justify the usage of some alternative performance measures by assuming that the investor has some non-traditional/non-expected utility function. For example, the usage of downside deviation is closely related to the usage of a more general lower partial moment as a risk measure, see the papers of Fishburn (1977) and Bawa (1978). These authors proposed the mean-lower partial moment model for portfolio selection, in which risk is measured in terms of deviations below some pre-specified target rate of return. These authors also show that the usage of the mean - lower partial moment objective corresponds to a specific utility function of the investor. Moreover, Haley and McGee (2006) demonstrated that such apparently various portfolio selection criteria as the Sharpe ratio, the safety first rule, and the Stutzer index can be obtained from a single behavioral assumption, namely that investors seek the portfolio that minimizes the probability of realizing a return below some pre-determined target or benchmark level.

[3] In the literature there is also another approach to account for the skewness preferences, namely using the general equilibrium model. For example, Kraus and Litzenberger (1976) developed a three moment CAPM to account for the skewness of return distributions. However, this approach requires making some additional, and probably rather restrictive, assumptions in the model. For a recent development in this approach the interested reader can consult Post et al. (2008).

Lastly, we illustrate how the GSR can be used for the purpose of portfolio performance evaluation. In particular, we show how this measure can mitigate the shortcomings of the Sharpe ratio in resolving some Sharpe ratio paradoxes and revealing the real performance of portfolios with manipulated Sharpe ratios. We also demonstrate how this measure can be applied for the comparison of hedge funds performances. Our empirical study shows that the values of the GSRs computed using both nonparametric and parametric methods are very close and the rank correlation between the nonparametric GSR ranking and the parametric GSR ranking is very high.

The rest of the paper is organized as follows. In Section 2 we present the general results on the investor's preferences to higher moments of distribution and portfolio performance measurement. In Section 3, using the approximation analysis, we derive the formula for the *ASSR* and show the connection between the *ASSR* and the GSR. In Section 4 we present the nonparametric and parametric methods of estimation of the GSR. In Section 5 we demonstrate the use of the GSR for practical purposes. Section 6 concludes the paper.

2. Investor's relative preferences to moments of distribution and performance measurement

2.1. The set up

All the results in this paper are obtained by considering the optimal capital allocation problem in the standard Expected Utility Theory framework. In particular, we consider an investor who wants to allocate his wealth between a risk-free and a risky asset. We assume that the return of the risky asset over a small time interval Δt is given by

$$x = \mu_x + \sigma_x \varepsilon = \mu \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon,$$

where μ and σ are, respectively, the mean and standard deviation of the risky asset return per unit of time, and ε is some normalized stochastic variable such that $E[\varepsilon] = 0$ and $\mathrm{Var}[\varepsilon] = 1$. The return on the risk-free asset over the same time interval equals

$$r_f = r\Delta t,$$

where r is the risk-free interest rate per unit of time.

We further assume that the investor has a wealth of w and invests a in the risky asset and, consequently, $w - a$ in the risk-free asset. Thus, the investor's wealth after Δt is

$$\tilde{w} = a(x - r_f) + w(1 + r_f).$$

The investor's expected utility

$$E[U(\tilde{w})] = E[U(a(x - r_f) + w(1 + r_f))], \tag{1}$$

for given utility function $U(\cdot)$. We suppose that $U(\cdot)$ is increasing, concave, and everywhere differentiable function. The investor's objective is to choose a to maximize the expected utility

$$E[U^*(\tilde{w})] = \max_a E[U(\tilde{w})]. \tag{2}$$

We assume that the solution to the investor's objective (2) exists and unique and denote by a^* the optimal amount invested in the risky asset. To shorten the subsequent notation, we denote $w_r = w(1 + r_f)$.

2.2. Investor's relative preferences to moments of distribution

To proceed to the study of the investor's preferences to the moments of the distribution of the risky asset return, we apply Taylor series expansion for $U(\tilde{w})$ around w_r. This yields

$$E[U(\tilde{w})] = U(w_r) + \sum_{n=1}^{\infty} \frac{1}{n!} U^{(n)}(w_r) a^n E[(x - r_f)^n], \tag{3}$$

where $U^{(n)}$ denotes the nth derivative. Denote by m_n the nth moment of distribution of x around r_f. That is, $m_n = E[(x - r_f)^n]$. Observe that m_n is proportional to the nth central moment of the distribution of x, $E[(x - E[x])^n]$. Combining (3) and (2) gives

$$E[U^*(\tilde{w})] = \max_a U(w_r) + \sum_{n=1}^{\infty} \frac{1}{n!} U^{(n)}(w_r) a^n m_n. \tag{4}$$

The first-order condition for the optimality[4] of a, with subsequent division of the left- and right-hand side of the resulting equation by $U^{(1)}$, yields the following equation

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} a^{n-1} \frac{U^{(n)}(w_r)}{U^{(1)}(w_r)} m_n = 0. \tag{5}$$

It is well-known (recall, for example, the analysis of Arrow, 1971) that the quantity $\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)}$ shows the investor's absolute preference to the second moment of distribution, m_2. Similarly, the quantity $\frac{U^{(n)}(w_r)}{U^{(1)}(w_r)}$ shows the investor's absolute preference to the nth moment of distribution, m_n. It is straightforward to observe that Eq. (5) can be regarded as a polynomial equation which solution is some function of its coefficients. Hence, the solution for optimal a in (5) may be given by

$$a^* = g\left(m_1, \ldots, \frac{U^{(n)}(w_r)}{U^{(1)}(w_r)} m_n, \ldots\right),$$

where $g(\ldots)$ is some function where each moment of distribution, m_n, is multiplied by the investor's absolute preference to the moment, $\frac{U^{(n)}(w_r)}{U^{(1)}(w_r)}$. However, a close investigation reveals that the investor's preferences to the moments of distribution are strongly interrelated. It is possible to describe the investor's preferences to the moments of distribution in terms of relative preferences. See the following theorem.

Theorem 1. *The solution for the optimal amount that should be invested in the risky asset (in the optimization problem (2)) has the following form*

$$a^* = \frac{1}{\gamma} f(m_1, \ldots, b_n m_n, \ldots), \tag{6}$$

where γ is the Arrow–Pratt measure of absolute risk aversion

$$\gamma = -\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)}, \tag{7}$$

b_n is given by

$$b_n = \frac{\frac{U^{(n)}(w_r)}{U^{(1)}(w_r)}}{\left(\frac{U^{(2)}(w_r)}{U^{(1)}(w_r)}\right)^{n-1}} = (-1)^{n-1} \frac{\frac{U^{(n)}(w_r)}{U^{(1)}(w_r)}}{\gamma^{n-1}}, \tag{8}$$

and $f(\ldots)$ is some function where each moment of distribution m_n is multiplied by b_n. The expression for the investor's maximum expected utility has the following form

$$E[U^*(\tilde{w})] = U(w_r) + \frac{U^{(1)}(w_r)}{\gamma} \sum_{n=1}^{\infty} \frac{(-1)^{1-n}}{n!} f^n(m_1, \ldots, b_n m_n, \ldots) b_n m_n. \tag{9}$$

The proof is given in the Appendix.

[4] Find the first derivative of the investor's expected utility with respect to a and set it to zero.

Remark 1. Observe that in Eqs. (6) and (9) each moment of distribution m_n is always multiplied[5] by b_n. Therefore we denote b_n as is the investor's relative preference to the nth moment of distribution. The term "relative" appears because the investor's absolute preference to a moment of distribution is divided by the investor's absolute preference to the second moment of distribution raised to some power.

Remark 2. Note that in this theorem we do not derive the solution, but only present the general form of the solution. In the subsequent sections we will derive some particular solutions to the optimal capital allocation problem under some simplifying assumptions.

The traditional utility functions most often used in financial economics (quadratic utility, CARA utility, CRRA utility, even linear utility) represent variants of HARA utility which can be given by

$$U(w) = \frac{\rho}{1-\rho}\left(\frac{\lambda w}{\rho} + \phi\right)^{1-\rho}. \tag{10}$$

When $\rho = -1$, HARA utility reduces to quadratic utility. When $\phi = 1$ and $\rho \to \infty$, HARA utility converges to CARA utility where $\lambda > 0$ is a measure of the investor's absolute risk aversion. When $\phi = 0$, we obtain CRRA utility where $\rho > 0$ is the measure of the investor's relative risk aversion.

Besides being very versatile, HARA utility possesses some other properties that makes this utility especially suitable for financial applications. Cass and Stiglitz (1970) show that two-fund separation holds if all investors have the same HARA utility. More precisely, in this case the allocation of wealth in risky asset i relative to risky asset j does not depend on the investor's wealth. The following theorem establishes one more particular property of this utility.

Theorem 2. *If the investor has HARA utility, then the investor's relative preference to the nth moment of distribution, where $n \geqslant 3$ (assuming $U^{(n)} \neq 0$), is given by*

$$b_n = \frac{\prod_{k=1}^{n-2}(\rho + k)}{\rho^{n-2}}. \tag{11}$$

In particular, for a HARA investor the relative preference to the nth moment of distribution does not depend on the investor's wealth.

The proof is given in the Appendix.

Observe that for quadratic utility $b_n = 0$ for $n \geqslant 3$. That is, the investor with quadratic preferences is indifferent to the higher moments of distribution above the second one. For CRRA utility the relative preference to the nth moment of distribution is given directly by Eq. (11). Here note that the relative preference to the nth moment of distribution depends on the coefficient of the relative risk aversion, ρ. The lower the coefficient of relative risk aversion, the higher the preference to each moment of distribution. $b_n \to \infty$ as $\rho \to 0$. For example, the investor's relative preference to the skewness of distribution is given by

$$b_3 = \frac{\rho + 1}{\rho}. \tag{12}$$

The lower ρ, the more the investor appreciates positive skewness and the more the investor dislikes negative skewness. This will be illustrated in more details in the subsequent section. Finally, since we obtain CARA utility (from HARA utility) by letting $\rho \to \infty$, for CARA utility $b_n = 1$. Consequently, a CRRA investor has stronger relative preferences to higher moments of distribution than a CARA investor (in other words, for a CRRA investor $b_n > 1$). However, as a CRRA investor's coefficient of relative risk aversion increases, the relative preferences to higher moments of distribution decrease and converge to the relative preferences of a CARA investor.

[5] By definition of b_n we have $b_1 = 1$ for any utility function and $b_2 = 1$ for any non-linear utility function.

2.3. Performance measurement

In this subsection we discuss the performance measurement. To formalize the discussion we give the following definition of a performance measure.

Definition 1 (*Performance measure*). By a performance measure we mean a score (number/value) attached to each financial asset. A performance measure is related to the level of maximum expected utility provided by the asset. That is, the higher the performance measure of an asset, the higher level of maximum expected utility the asset provides.

Remark 3. We believe that this is the most natural definition of a performance measure within any utility theory. However, there is another valid approach to define a performance measure, namely, the axiomatization approach. For example, Meucci (2005) introduces an index of satisfaction, which is largely equivalent to a performance measure, and discusses a construction of coherent indexes of satisfaction. De Giorgi (2005) presents an axiomatic construction of coherent reward and risk measures. Such measures can be used to define sensible reward-to-risk performance measures.

First of all, note the following property of a performance measure:

Theorem 3. *A positive linear transformation of a performance measure produces an equivalent performance measure.*

Proof. Suppose that PM_A is the performance measure of asset A and PM_B is the performance measure of asset B. Define performance measures $PM'_A = cPM_A + d$ and $PM'_B = cPM_B + d$ for any real c and d such that $c > 0$. Then PM'_A and PM'_B are equivalent to PM_A and PM_B in the sense that they produce equal ranking of the assets. That is, if $PM_A > PM_B$, then also $PM'_A > PM'_B$. □

Theorem 4. *In the optimal capital allocation framework presented in this paper, an equivalent performance measure might be given by*

$$PM = \sum_{n=1}^{\infty} \frac{(-1)^{1-n}}{n!} f^n(m_1, \ldots, b_n m_n, \ldots) b_n m_n. \tag{13}$$

Moreover, if all investors have the same HARA utility, then this performance measure is independent of an investor's wealth and is suitable for all investors.

Proof. According to Definition 1, a performance measure of an asset is related to the level of the investor's maximum expected utility associated with the investment in this asset. According to Theorem 1, the maximum expected utility of an investor is given by (9) which can be written as

$$E[U^*(\tilde{w})] = U(w_r) + \frac{U^{(1)}(w_r)}{\gamma} PM. \tag{14}$$

The factor $\frac{U^{(1)}(w_r)}{\gamma}$ is unique for each investor since it depends of the investor's wealth and the absolute risk aversion. However, this factor is positive since $U^{(1)}(w_r) > 0$ and for any risk averse investor $\gamma > 0$. Therefore, for any investor PM can be obtained as a positive linear transformation of $E[U^*(\tilde{w})]$. The investor's preferences in PM are expressed only by b_n. According to Theorem 2, for a HARA investor b_n does not depend on the investor's wealth. Consequently, for a HARA investor PM is independent of the investor's wealth. Finally, if all investors have the same HARA utility (by this we mean the same coefficients ρ, λ, and ϕ in (10)), the value of PM is the same for all investors. □

Remark 4. Note that generally the value of PM is individual for every investor. In particular, if every investor has a unique set of parameters of HARA utility, the performance measure PM is an individual performance measure as it reflects the investor's individual preferences to higher moments of distribution.

3. Adjusted for skewness Sharpe ratio and generalized Sharpe ratios

3.1. Adjusted for skewness Sharpe ratio

Observe that generally one cannot arrive at a closed-form solution to the investor's optimal capital allocation problem. However, in some cases one can derive an approximate solution. One famous example is the mean-variance approximation where in the infinite Taylor series (3) one keeps only the terms with the first two derivatives of the utility function. It is well-known that in this case the solution for optimal a is given by

$$a^* \approx \frac{\mu - r}{\gamma\sigma^2} = \frac{SR}{\gamma\sigma\sqrt{\Delta t}}, \tag{15}$$

which gives the following expression for the investor's maximum expected utility

$$E[U^*(\tilde{w})] \approx U(w_r) + \frac{U^{(1)}(w_r)}{\gamma}\frac{1}{2}SR^2, \tag{16}$$

where SR is the standard Sharpe ratio

$$SR = \frac{\mu - r}{\sigma}\sqrt{\Delta t}. \tag{17}$$

If in the infinite Taylor series (3) we keep only the terms with the first three derivatives of the utility function, then under some additional conditions we can also arrive at a simple approximate solution to the investor's problem (2). See the following theorem.

Theorem 5. *If in the infinite Taylor series (3) we keep the terms up to $\Delta t^{\frac{3}{2}}$ and disregard the terms with higher powers of Δt, then the solution for optimal a is given by*

$$a^* \approx \frac{SR}{\gamma\sigma\sqrt{\Delta t}}\left(1 + b_3\frac{Skew}{2}SR\right), \tag{18}$$

which gives the following expression for the investor's maximum expected utility

$$E[U^*(\tilde{w})] \approx U(w_r) + \frac{U^{(1)}(w_r)}{\gamma}\frac{1}{2}SR^2\left(1 + b_3\frac{Skew}{3}SR\right), \tag{19}$$

where Skew is the skewness of the distribution of x defined by

$$Skew = \frac{E[(x - E[x])^3]}{E[(x - E[x])^2]^{\frac{3}{2}}},$$

and b_3 is the investor's relative preference to the skewness of distribution.

The proof is given in the Appendix.

Remark 5. The mean-variance approximation of the expected utility can be justified by assuming that Δt is very small. If we increase Δt and make it "rather" small, to improve the approximation we need to take into account the skewness of distribution. Generally, the longer Δt the more terms we need in order to provide a good approximation of the expected utility by means of a Taylor series.

Remark 6. It is worth noting that the results of Theorem 5 are, in fact, particular cases of the general results presented in Theorem 1 (as well as the results of the mean-variance approximation analysis (15) and (16)). That is, when we take into account only the first three moments of distribution, Theorem 5 presents an approximate solution for $f(m_1, \ldots, b_n m_n, \ldots)$ in Eq. (6), and an approximate solution for PM in Theorem 4, which also enters Eq. (9).

Theorem 6. *In the mean-variance-skewness framework of Theorem 5, the investor's individual performance measure can be given by*

$$ASSR = SR\sqrt{1 + b_3\frac{Skew}{3}SR}, \tag{20}$$

where ASSR stands for Adjusted for Skewness Sharpe Ratio, under condition that ASSR is a positive real number.

Proof. According to Theorem 4, an equivalent performance measure can be given by $PM = \frac{1}{2}ASSR^2$ (see Eq. (19)). According to Theorem 3, $ASSR^2$ is also an equivalent performance measure. According to Definition 1, a performance measure of an asset is related to the level of the investor's maximum expected utility associated with the investment in this asset. Now observe that if $ASSR$ is a positive real number, then the higher the value of $ASSR$, the higher the value of $ASSR^2$. That is, the value of $ASSR$ also reflects the level of maximum expected utility. Therefore, it also can be used as a performance measure. The main reason for introducing this particular performance measure is because this measure is comparable with the Sharpe ratio. Observe that when either the skewness of distribution is zero or the investor is indifferent to skewness ($b_3 = 0$ as for quadratic utility), the adjusted for skewness Sharpe ratio reduces to the standard Sharpe ratio. Finally note that in contrast to the Sharpe ratio where the investor's risk preferences apparently disappear, to compute the $ASSR$ one needs to determine the value of b_3. This means that the performance measure $ASSR$ is *not unique for all investors, but rather an individual performance measure*. That is, in principle, investors with different skewness preferences might rank differently the same set of risky assets. We remind the reader that for CARA utility $b_3 = 1$, and for CRRA utility b_3 is given by (12). In particular, for logarithmic utility $b_3 = 2$ (since for logarithmic utility $\rho = 1$). □

Remark 7. Note that the $ASSR$ can be perceived as the SR times the Skewness Adjustment Factor SAF $= \sqrt{1 + b_3\frac{Skew}{3}SR}$. Here observe that the SAF depends on the value of the SR. The higher the SR, the larger the adjustment for skewness of distribution (it is easy to show that $\frac{\partial SAF}{\partial SR} > 0$). In other words, if the value of the SR is rather low, the adjustment for skewness is rather unimportant. Note in addition that the SR depends on the investment horizon Δt (see formula (17)). The longer the investment horizon Δt, the higher the SR. Consequently, the SAF increases with the investment horizon.

Remark 8. Note that according to our definition of a performance measure, in the mean-variance framework without the short sale restriction the appropriate performance measure should be given by $|SR|$, not SR. This is because in the mean-variance framework an equivalent performance measure[6] is given by SR^2. It is always

[6] Treynor and Black (1973) were the first to introduce the term "Sharpe ratio", but they defined it actually as SR^2, that is, as the square of the measure that is used nowadays.

non-negative even if $SR < 0$ that can happen when $E[x] < r$. Note that in case $E[x] < r$ it is optimal to sell short the risky asset and the investor attains higher expected utility as compared with the policy where the investor avoids the risky asset. In the presence of the short sale restriction, the value of the SR must be set to zero if the computed value of the SR is negative.

Recall that b_3 denotes the investor's relative preference to the skewness of distribution. It turns out that the value of this parameter has a deep economic intuition. Therefore, we would like to elaborate on this intuition. First, we demonstrate that the value of the parameter b_3 in the *ASSR* can tell us whether the investor has increasing or decreasing (absolute) risk aversion. Indeed, in a seminal paper Pratt (1964) shows that the condition for decreasing risk aversion is

$$U^{(3)}U^{(1)} > (U^{(2)})^2,$$

which could be restated as $b_3 > 1$. Consequently, the value of the parameter b_3 in relation to 1 shows whether the investor has decreasing or increasing risk aversion. Researchers seem to agree that it is most natural for the investor to have decreasing risk aversion. This means that the greater the investor's wealth, the lesser the investor's absolute risk aversion. In other words, an increase in the investor's wealth should not result in a lesser amount invested in the risky asset. In addition, observe that the value of the parameter b_3 shows how fast the investor's risk tolerance changes when the investor's wealth changes. Indeed,

$$\frac{d}{dw}\left(-\frac{U^{(1)}}{U^{(2)}}\right) = \frac{U^{(1)}U^{(3)} - \left(U^{(2)}\right)^2}{\left(U^{(2)}\right)^2} = \frac{\frac{U^{(3)}}{U^{(1)}}}{\left(\frac{U^{(2)}}{U^{(1)}}\right)^2} - 1 = b_3 - 1.$$

That is, the value of the parameter b_3 is related to the first derivative of the investor's risk tolerance with respect to wealth. The higher the speed of change of the investor's risk tolerance, the larger the investor's preference for positive skewness (and dislike of negative skewness).

Theorem 7. *Under the conditions of Theorem 5 the investor's maximum expected utility can be written as*

$$E[U^*(\tilde{w})] \approx U\left(\frac{1}{2\gamma}ASSR^2 + w(1+r_f)\right). \tag{21}$$

The proof is given in the Appendix.

Eq. (21) implies that for CARA utility $U(w) = -e^{-\lambda w}$ where $\lambda \equiv \gamma$, the maximum expected utility is (approximately) given by

$$E[U^*(\tilde{w})] \approx -e^{-\frac{1}{2}ASSR^2 - \lambda w(1+r_f)}. \tag{22}$$

Similarly, Eq. (21) implies that for power utility $U(w) = \frac{w^{1-\rho}}{1-\rho}$ where $\gamma = \frac{\rho}{w(1+r_f)}$, the maximum expected utility is given by

$$E[U^*(\tilde{w})] \approx \frac{\left(w(1+r_f)\left(1+\frac{1}{2\rho}ASSR^2\right)\right)^{1-\rho}}{1-\rho}. \tag{23}$$

For logarithmic utility where $\gamma = \frac{1}{w(1+r_f)}$, the maximum expected utility is given by

$$E[U^*(\tilde{w})] \approx \log\left(w(1+r_f)\left(1+\frac{1}{2}ASSR^2\right)\right). \tag{24}$$

3.2. Generalized Sharpe ratios

The shortcomings of the Sharpe ratio can, for instance, be illustrated by paradoxes. Consider the following example presented by Hodges (1998). The probability distributions of two assets, A and B, in percentage of excess return over some horizon, are given in Table 1. The assets A and B are identical except that asset A has 1% chance for excess return of 35%, whereas asset B has 1% chance for excess return of 45%. Clearly, asset B should be preferred to asset A by the principle of first-order stochastic dominance. However, asset A has a Sharpe ratio of 0.500, whereas asset B has a Sharpe ratio of 0.493. According to the Sharpe ratio, asset A is better than asset B.

Table 1
Probability distributions of two assets.

Probability	0.01	0.04	0.25	0.40	0.25	0.04	0.01
Excess return asset A	−25	−15	−5	5	15	25	35
Excess return asset B	−25	−15	−5	5	15	25	45

Source: Hodges (1998).

To resolve the paradox, Hodges introduced the notion of the Generalized Sharpe Ratio[7] (GSR). In particular, Hodges points out that for *normally distributed* risky asset returns and the investor with negative exponential utility who has zero initial wealth, the maximum investor's expected utility is given by

$$E[U^*(\tilde{w})] = -e^{-\frac{1}{2}SR^2}.$$

Therefore, the standard Sharpe ratio can be computed using

$$\frac{1}{2}SR^2 = -\log(-E[U^*(\tilde{w})]).$$

Using this identity Hodges *conjectures* that for any distribution of the risky asset returns and the investor with zero wealth, the maximum investor's expected utility is given by

$$E[U^*(\tilde{w})] = -e^{-\frac{1}{2}\mathrm{GSR}^2}. \tag{25}$$

Consequently, the GSR can be computed using

$$\frac{1}{2}\mathrm{GSR}^2 = -\log(-E[U^*(\tilde{w})]). \tag{26}$$

Having implemented the numerical computations of the GSRs for assets A and B, Hodges shows that the GSR of asset B is greater than the GSR of asset A.

Definition 2 (*Generalized Sharpe Ratio*). The GSR is a performance measure that converges to the standard Sharpe ratio when the probability distribution of the risky asset converges to normal probability distribution.

Remark 9. Recall the definition of a performance measure. It is worth noting that the GSR is computed after the solution of the optimal capital allocation/optimal portfolio choice problem and the computation of the maximum expected utility of the investor. Observe that the GSR presumably accounts for all moments of the distribution of risky asset returns.

Note that the *ASSR* justifies, to a large extent, the notion of the GSR (just compare the conjecture of Hodges (25) and our result (22) when $w = 0$). The *ASSR* can be interpreted as a particular form of the GSR in the case where the investor is indifferent to all moments of distribution higher than the third one. The GSR was originally introduced for an investor with negative exponential utility. But the idea of a generalized Sharpe ratio can be applied to other utility functions as well. The result in (23) suggests that the GSR for power utility can be computed as

$$\frac{1}{2}\mathrm{GSR}^2 = \rho\left(\frac{\left(E[U^*(\tilde{w})](1-\rho)\right)^{\frac{1}{1-\rho}}}{w(1+r_f)} - 1\right). \tag{27}$$

[7] This idea was further developed and used by, for example, Madan and McPhail (2000), Cherny (2003) and Ziemba (2003).

Similarly, the result in (24) implies that the GSR for the logarithmic utility can be computed using

$$\frac{1}{2}\mathrm{GSR}^2 = \frac{e^{E[U^*(\tilde{w})]}}{w(1+r_f)} - 1. \tag{28}$$

3.3. Feasibility of using the GSR in portfolio performance evaluation

In this section we have derived the formula for the *ASSR*. Unfortunately, this formula is only a result of approximation analysis and, therefore, the precision of this formula for practical applications is rather limited. Moreover, this formula takes into account only the skewness of distribution. For practical purposes it is also important to take into account other higher moments of distribution, for example, kurtosis. However, the results of this section are very important for practical applications as they justify the notion of the GSR.

It is natural to use the GSR for pragmatic purposes as a measure of comparison of performances of different risky assets and portfolios. However, as we have seen from the analysis presented in the preceding sections, it turns out that generally the values of the PM and the GSR are not unique and depend on the investor's individual preferences to skewness and other higher moments of distribution as specified by the investor's utility function. It seems that we stumble upon ambiguity here. Nevertheless, it turns out that this ambiguity can be removed under reasonable assumptions.

First of all, if we want a portfolio performance measure to reflect sensible risk preferences, we need to impose the condition of non-increasing risk aversion. This condition says, in particular, that the investor's relative preference to skewness should be equal to or greater than 1, that is, $b_3 \geqslant 1$. Observe that even if we do not know the value of the skewness preference parameter of a representative investor, a priori we can claim that the use of the GSR with $b_3 = 1$ (that is, the use of CARA utility in the computation of the GSR) mitigates the shortcomings[8] of the Sharpe ratio (where $b_3 = 0$) when the return distributions are non-normal. Consequently, the condition of non-increasing risk aversion can, to some extent, clear up the ambiguity and justify the pragmatic use of the GSR.

Second, if we want a portfolio performance measure to be consistent with a market in equilibrium, the investor's utility function must be restricted to have the CRRA form. There are two justifications for this conclusion (see a similar discussion in Ingersoll et al., 2007). Since utility is defined over wealth or consumption, whereas all performance measures are defined over returns, a performance measure that treats returns identically is consistent with Expected Utility Theory only if the utility of returns is independent of wealth. This is true amongst additively separable utility functions only for CRRA utility. Additionally, as shown by Rubinstein (1976) and He and Leland (1993), if the market portfolio's rate of return is independent and identically distributed over time and markets are perfect, then the representative investor must have CRRA utility.

The risk preferences of an investor with CRRA utility are specified by a single parameter, namely, the coefficient of relative risk aversion. But what is the relative risk aversion coefficient for a representative investor with CRRA utility? Several studies provide such estimates. Merton (1980) gives an estimates of ρ to be 1.89. Friend and Blume (1975) and Constantinides (1990) estimate ρ to be 2. Arrow (1971) argues on theoretical grounds that ρ should be approximately one. Kydland and Prescott (1982) provide empirical estimates of ρ somewhere in between one and two. However, in a very influential study by Mehra and Prescott (1985), the authors argue that ρ must be rather high (to be consistent with the observed equity premium) and give an estimate of around 30. Now observe that for CRRA investors for whom the value of the risk aversion coefficient is sufficiently greater than 1 (that is, for whom $\rho \gg 1$) the relative preferences to all higher moments of distribution $b_n \approx 1, n \geqslant 3$, see formula (11). Consequently, when ρ is rather high, the relative preferences to all moments of distribution of CRRA investors are very similar to those of CARA investors.

To summarize, it seems reasonable to compute the GSR using the representative investor with CRRA utility for whom the value of the relative risk aversion coefficient is rather high. In this case we can safely assume that the values of the relative preferences to all higher moments of distribution are close to 1. The latter also means that, for the purpose of computational convenience,[9] instead of CRRA utility we can use CARA utility in the calculation of the GSR. This is also well justified by the results of Theorem 4. According to this theorem, for an investor with HARA utility a possible performance measure is given by (13) which is independent of the investor's wealth and suitable for any HARA investor. Both CARA and CRRA utilities are particular forms of HARA utility. When ρ increases, CRRA utility converges to CARA utility. Therefore, when ρ increases, an equivalent performance measure of a CRRA investor converges to an equivalent performance measure of a CARA investor.

4. Estimation of the generalized Sharpe ratio

4.1. Nonparametric estimation

The nonparametric estimation of the GSR consists in using the empirical probability distribution of x in the solution of the optimal capital allocation problem (2). Since we employ negative exponential utility function and the investor with zero initial wealth, the optimal capital allocation problem that must be solved is

$$E[U^*(\tilde{w})] = \max_a E[-e^{-\lambda a(x-r_f)}]. \tag{29}$$

This problem can be solved only numerically where one must implement a robust numerical optimization method. Having computed the investor's maximum expected utility $E[U^*(\tilde{w})]$, the GSR is computed using (see (26))

$$GSR = \sqrt{-2\log(-E[U^*(\tilde{w})])}. \tag{30}$$

4.2. Parametric estimation and explicit solution for adjusted for skewness and kurtosis Sharpe ratio

The implementation of the parametric estimation of the GSR requires making an assumption about the underlying probability distribution of the risky asset. Here we need a probability distribution that can match the observed higher moments. In particular, in a suitable probability distribution it should be possible to define the values of skewness and kurtosis.

There are several probability distributions that can suit our purpose. Some examples are: a class of *stable* distributions which generalize the normal distribution (see, for example, Samorodnitsky and Taqqu, 1994 or Uchaikin and Zolotarev, 1999), the *variance gamma* (VG) distribution introduced by Madan and Seneta (1990), and the *normal inverse Gaussian* (NIG) distribution (see,

[8] The Sharpe ratio criterion violates the first-order stochastic dominance principle because the quadratic utility function has a "satiation" point above which the utility decreases. Increasing risk aversion is closely related to the satiation effect.

[9] Note that CRRA utility is not defined for negative wealth. This might create problems for implementation of a numerical optimization method which is required for nonparametric estimation of the GSR. Using parametric approach to the estimation of the GSR, analytical solutions can be obtained much more easily for CARA utility. See the subsequent section.

for example, Barndorff-Nielsen, 1995; Barndorff-Nielsen, 1998). Our choice here is the NIG distribution. The reasons for this choice are: (1) the NIG distribution has an explicit expression for the probability density function (for example, there are no explicit formulas for general stable densities); (2) distributions of risky asset returns can often be fitted extremely well by the NIG distribution (see Barndorff-Nielsen, 1998 and references therein); (3) for the NIG distribution we have explicit formulas for finding the parameters of the distribution via the values of the first four moments of the distribution, see explanation below (for example, to find the values of the parameters of the VG distribution via the values of the first four moments, one has to employ a numerical algorithm, see Madan and McPhail, 2000).

4.2.1. A short primer on NIG distribution

A random variable X follows the NIG distribution with parameter vector $(\alpha, \beta, \eta, \delta)$ if its probability density function is

$$f(x;\alpha,\beta,\eta,\delta) = \frac{\delta\alpha e^{\delta\varphi+\beta(x-\eta)}}{\pi\sqrt{\delta^2+(x-\eta)^2}} K_1\left(\alpha\sqrt{\delta^2+(x-\eta)^2}\right), \tag{31}$$

where

$$\varphi = \sqrt{\alpha^2-\beta^2}$$

and K_1 is the modified Bessel function of the third kind with index 1. η and δ are ordinary parameters of location and scale whereas α and β determine the shape of the density. In particular, β determines the degree of skewness. For symmetrical densities $\beta = 0$. The conditions for a viable NIG density are: $\delta > 0$, $\alpha > 0$, and $\frac{|\beta|}{\alpha} < 1$. The mean, variance, skewness and kurtosis of X are

$$\begin{aligned} &\mu_x = E[X] = \eta + \delta\tfrac{\beta}{\varphi}, \qquad \sigma_x^2 = \mathrm{Var}[X] = \delta\tfrac{\alpha^2}{\varphi^3}, \\ &S = \mathrm{Skew}[X] = 3\tfrac{\beta}{\alpha\sqrt{\delta\varphi}}, \quad K = \mathrm{Kurt}[X] = 3 + \tfrac{3}{\delta\varphi}\left(1+4\left(\tfrac{\beta}{\alpha}\right)^2\right). \end{aligned} \tag{32}$$

The Eq. (32) can be solved explicitly for the parameters of the NIG distribution. After tedious but straightforward calculations one can obtain (see also Karlis, 2002)

$$\begin{aligned} &\alpha = \tfrac{3\sqrt{3K-4S^2-9}}{\sigma_x^2(3K-5S^2-9)}, \qquad \beta = \tfrac{3S}{\sigma_x(3K-5S^2-9)}, \\ &\eta = \mu_x - \tfrac{3S\sigma_x}{3K-4S^2-9}, \quad \delta = 3\sigma_x\tfrac{\sqrt{3K-5S^2-9}}{3K-4S^2-9}. \end{aligned} \tag{33}$$

Note that to get meaningful parameters of the NIG distribution the following condition must be satisfied

$$K > 3 + \frac{5}{3}S^2. \tag{34}$$

The moment generating function of the NIG distribution is

$$\begin{aligned} M_X(u) = E[e^{uX}] &= \int_{-\infty}^{\infty} e^{ux} f(x;\alpha,\beta,\eta,\delta)dx \\ &= e^{u\eta+\delta\left(\varphi-\sqrt{\alpha^2-(\beta+u)^2}\right)}. \end{aligned} \tag{35}$$

The normal distribution $N(\mu,\sigma^2)$ appears as a limiting case of the NIG distribution when $\beta = 0, \alpha \to \infty, \delta \to \infty$, and $\frac{\delta}{\alpha} = \sigma_x^2$ (see Barndorff-Nielsen, 1995).

4.2.2. Adjusted for skewness and kurtosis Sharpe ratio

As in the nonparametric approach, the computation of the GSR starts with finding the solution of (29) which in this case becomes

$$E[U^*(\tilde{w})] = \max_a \int_{-\infty}^{\infty} -e^{-\lambda a(x-r_f)} f(x;\alpha,\beta,\eta,\delta)dx. \tag{36}$$

Theorem 8. *If the probability distribution of the risky asset follows the NIG distribution, then the solution for optimal a is given by*

$$a^* = \frac{1}{\lambda}\left(\beta + \frac{\alpha(\eta-r_f)}{\sqrt{\delta^2+(\eta-r_f)^2}}\right), \tag{37}$$

which gives the following expression for the GSR

$$\mathrm{ASKSR} = \sqrt{2\left(\lambda a^*(\eta-r_f) - \delta\left(\varphi - \sqrt{\alpha^2-(\beta-\lambda a^*)^2}\right)\right)}, \tag{38}$$

where ASKSR stands for the Adjusted for Skewness and Kurtosis Sharpe ratio.

The proof is given in the Appendix.

Remark 10. We use the term ASKSR because the ASKSR represents a particular form of the GSR. Whereas the GSR presumably accounts for all moments of distribution, the ASKSR accounts only for the first four moments.

Remark 11. In the proof of the theorem we also demonstrate that the ASKSR reduces to the standard Sharpe ratio when the NIG distribution reduces to the normal distribution. This is yet another justification of the notion of the GSR. Note also that the ASKSR does not depend on the value of the investor's absolute risk aversion λ, as this parameter disappears in the formula for the ASKSR.

The implementation of the parametric estimation of the GSR starts with the estimation of the first four moments of distribution. Then the parameters of the NIG distribution can be identified from the moments using (33). Finally, the ASKSR is computed using (38) where a^* is given by (37).

5. Performance evaluation using the GSR

In this section we use the GSR in performance evaluation. The purpose here is not to present a very detailed and comprehensive study, but rather to illustrate the usage of the GSR. In particular, we show how this measure can mitigate the shortcomings of the Sharpe ratio in resolving some Sharpe ratio paradoxes and revealing the real performance of portfolios with manipulated Sharpe ratios. We also demonstrate how this measure can be applied for the comparison of hedge funds performances. In each case we compute the following three performance measures: (1) SR – the standard Sharpe ratio; (2) GSR – the generalized Sharpe ratio computed using the nonparametric estimation method; (3) ASKSR – the adjusted for skewness and kurtosis Sharpe ratio which is the GSR computed using the parametric estimation method.

Both the nonparametric and parametric methods of the estimation of the GSR have some advantages and disadvantages. The main advantage of the nonparametric method is that it uses empirical probability distribution without making any assumptions on the underlying probability density function. The computed GSR, therefore, accounts for all moments of real probability distribution. However, the nonparametric method relies heavily on numerical methods. Thus, the computation of the GSR using the nonparametric method might be cumbersome. The estimation of the GSR (ASKSR) using the parametric method is very fast and simple.[10] However, it relies on the particular underlying probability density function and accounts only for the first four moments of distribution. Nevertheless, our empirical study shows that the values of the GSR and ASKSR are very close and the rank correlation between the GSR ranking and the ASKSR ranking is very high.

Observe that in our optimal capital allocation framework there

[10] In some cases condition (34) is not satisfied by an empirical probability distribution. Similar limitation also exists for the VG distribution, see Madan and McPhail (2000). Thus, in this case the only possibility is to use the nonparametric estimation method.

is no short sale restriction and the values of both the GSR and ASKSR are always non-negative, see Remark 8. That is, looking at the value of, for example, the GSR we cannot tell whether it is optimal to buy and hold or sell short the risky asset. It is the sign of a^* that shows whether it is optimal to buy and hold the risky asset (when the sign is positive) or sell it short (when the sign is negative). In the presence of the short sale restriction the values of the GSR and ASKSR must be set to zero if $a^* < 0$.

5.1. Resolving the Sharpe ratio paradox

Table 2 presents the returns statistics and performances of the two assets in the Sharpe ratio paradox of Hodges (1998) (see also Table 1). Whereas the Sharpe ratio of asset A is higher than the Sharpe ratio of asset B, both the GSRs of asset A are lower than the corresponding GSRs of asset B. This demonstrates that the value of the GSR helps mitigate the shortcomings of the Sharpe ratio in the resolution of the Sharpe ratio paradoxes.

5.2. Estimating performances of portfolios with manipulated Sharpe ratios

Recently a number of papers have shown that the standard Sharpe ratio is prone to manipulation, see, for example, Leland (1999), Spurgin (2001), Goetzmann et al. (2002) and Ingersoll et al. (2007). In particular, Leland (1999) and Spurgin (2001) show that managers can increase the Sharpe ratio by selling off the upper end of the return distribution. Goetzmann et al. (2002) identify a class of strategies that maximize the Sharpe ratio without requiring any manager skills. They show how to achieve the maximum Sharpe ratio by either selling out-of-the-money call options or selling both out-of-the-money call and put options on the underlying stock portfolio.

In this subsection we study the performance of a portfolio, described in Goetzmann et al. (2002), with a manipulated Sharpe ratio. The benchmark portfolio is a stock which price process follows the geometric Brownian motion

$$P(t+\Delta t) = P(t)\exp\left((\mu - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}z\right),$$

where z is a standard normal variable. The model parameters are the following: $\mu = 0.15, r = 0.05, \sigma = 0.15$, and $\Delta t = 1$. The manipulation strategy consists in holding the stock and selling 2.58 put options with strike $0.88P(t)$ and selling 0.77 call options with strike $1.12P(t)$. We simulate the future values of the benchmark portfolio and the portfolio with short positions in options by generating 1,000,000 stock prices and compute the returns statistics as well as the different ratios, see Table 3. Fig. 1 illustrates the empirical probability distribution of the portfolio with the manipulated

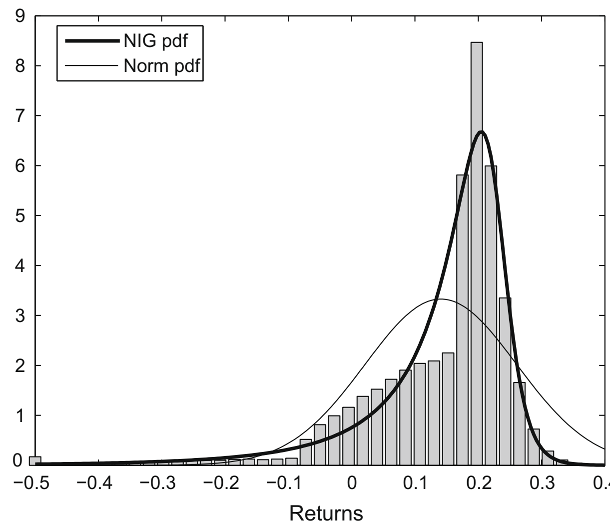

Fig. 1. Empirical probability distribution of the portfolio consisting of the stock and short call and put options on the stock with fitted NIG and normal distributions.

Sharpe ratio. It is easy to see that this strategy produces a return distribution which is far from the normal probability distribution. In particular, the return distribution has low standard deviation but large negative skewness. The Sharpe ratio of this strategy is higher than the Sharpe ratio of the underlying stock. However, if we look at the values of the GSRs, it is easy to note that the strategy with the manipulated Sharpe ratio shows worse performances than the benchmark strategy.

5.3. Evaluating performances of hedge funds

It is well-known that the distribution of hedge fund returns deviates significantly from normality. These deviations have been widely described in the literature, see, for example, Brooks and Kat (2002), Agarwal and Naik (2004) and Malkiel and Saha (2005). In particular, the hedge funds returns often exhibit high negative skewness and positive excess kurtosis. This can be explained by various reasons, for example, an extensive use of options and option-like dynamic trading strategies. Consequently, the ranking of hedge funds using the standard Sharpe ratio is dubious since it only takes into account the first two moments of distribution. That is, the significant left-tail risk in the distribution is ignored.

In this subsection we evaluate the performances of the CS/Tremont hedge fund indexes. The CS/Tremont indexes are based on the TASS database which tracks around 2600 hedge funds. Using a subset of around 650 funds CS/Tremont calculates 13 indexes (in addition to the main index) which track every major style of hedge fund management. Our sample consists of monthly returns of the CS/Tremont indexes from April 1994 to September 2007. We exclude the dedicated short bias index as it produces a negative Sharpe ratio. Table 5 reports the summary statistics and different performance measures. As we can see from the table, if we change the performance measure from the SR to either the ASKSR

Table 2
Returns statistics and performances of the two assets in the Sharpe ratio paradox of Hodges (1998).

Asset	Mean excess return	Std	Skew	Kurt	SR	ASKSR	GSR
A	0.050	0.100	0.000	3.400	0.500	0.498	0.498
B	0.051	0.103	0.305	4.487	0.493	0.499	0.499

Table 3
Returns statistics and performances of the benchmark strategy and the strategy with the manipulated Sharpe ratio.

Strategy	Mean	Std	Skew	Kurt	SR	ASKSR	GSR
Only stock	0.162	0.177	0.456	3.342	0.631	0.672	0.672
Stock, puts and calls	0.139	0.120	−2.358	12.355	0.743	0.598	0.603

Table 4
Spearman's rank correlations between different ratios in the ranking of the hedge fund indexes.

	SR	ASKSR	GSR
SR	1.0000		
ASKSR	0.7143	1.0000	
GSR	0.8077	0.9780	1.0000

or the GSR, many indexes exhibit shift in their ranking, but a few indexes are stable in their ranking. The rankings of the best index and the three worst indexes remain the same irrespective of the performance measure. For intermediate indexes, there is a major revision of ranking if we compare the values of the SR and either the ASKSR or the GSR. The maximum move in ranking is an upgrade by 4 and a downgrade by 5. The rank correlation between the different ratios is presented in Table 4. From this table we see that there is a high correlation between the ASKSR and the GSR rankings and a notably lower correlation between the SR and either the ASKSR or the GSR ranking.

6. Summary

In this paper we presented the study of the investor's preferences to higher moments of distribution. We introduced the notion of the "relative preferences" to higher moments of distribution. These relative preferences allow a simple comparison of degrees of preferences to higher moments of distribution among different investors. We demonstrated that for a HARA investor the relative preferences to moments of distribution do not depend on the investor's wealth. We showed that a CRRA investor exhibits stronger preferences to higher moments of distribution than a CARA investor.

We introduced a definition of a performance measure which is related to the level of expected utility provided by a financial asset. We demonstrated that if all investors have the same HARA utility, then there exists a common performance measure which is independent of the investors' wealths and is suitable for all investors. To illustrate the role played by relative preferences to higher moments of distribution and the computation of a performance measure, we provided an approximation analysis of the optimal capital allocation problem and derived the formula for the adjusted for skewness Sharpe ratio. We indicated that this performance measure justifies the notion of the GSR introduced by Hodges (1998) which presumably accounts for all moments of distribution.

In the paper we also presented two methods of practical estimation of the GSR: nonparametric and parametric. The nonparametric estimation of the GSR consists in using the empirical probability distribution of the risky asset in the solution of the optimal capital allocation problem. For the implementation of the parametric method we derived the closed-form solution for the GSR where the higher moments are calibrated to the normal inverse Gaussian distribution. We showed how the GSR can mitigate the shortcomings of the Sharpe ratio in resolution of Sharpe ratio paradoxes and reveal the real performance of portfolios with manipulated Sharpe ratios. We also demonstrated the use of this measure in the performance evaluation of hedge funds.

Acknowledgements

The authors would like to thank Fred Espen Benth, David Feldman, Kathy Walsh, William T. Ziemba, and the anonymous referees for their insightful comments on the earlier drafts of the article. The article has also benefited from comments by seminar participants at the University of Agder, the Norwegian University of Science and Technology, the HIA 2007 Workshop on Stochastic Dynamic Systems, the European Financial Management Association 2007 Annual Conference, and the (International) Financial Management Association 2008 European Conference. Any remaining errors are the authors' responsibility.

Appendix

To shorten the length of the paper, in this appendix we only present brief proofs of theorems. Detailed proofs can be obtained from the corresponding author.

A.1. Proof of Theorem 1

We rewrite Eq. (5) as

$$\sum_{n=1}^{\infty} \frac{(-1)^{1-n}}{(n-1)!}(a\gamma)^{n-1} b_n m_n = 0,$$

where b_n is given by (8) and γ is given by (7). Now it is easy to see that the solution for optimal a has the form given by (6). Next we insert (6) in the equation for the investor's maximum expected utility (4) and obtain

$$\begin{aligned} E[U^*(\tilde{w})] &= U(w_r) + \sum_{n=1}^{\infty} \frac{1}{n!} U^{(n)}(w_r) \frac{1}{\gamma^n} f^n(m_1, \ldots, b_n m_n, \ldots) m_n \\ &= U(w_r) + \sum_{n=1}^{\infty} \frac{(-1)^{1-n}}{n!} \frac{U^{(1)}(w_r)}{\gamma} (-1)^{n-1} \\ &\quad \times \frac{\frac{U^{(n)}(w_r)}{U^{(1)}(w_r)}}{\gamma^{n-1}} f^n(m_1, \ldots, b_n m_n, \ldots) m_n. \end{aligned}$$

The last line of the equation above can be written as (9).

Table 5
Hedge fund returns statistics and performances. Data as of September 31, 2007. The means and standard deviations are annualized. Ratios are calculated using the rolling 90 day T-bill rate. Numbers in the brackets show the rank of a fund using a particular ratio.

Hedge fund index	Mean	Std	Skew	Kurt	SR	ASKSR	GSR
CS/Tremont hedge fund index	0.115	0.074	0.147	5.734	1.042(07)	0.968(03)	0.976(03)
Convertible arbitrage	0.089	0.046	−1.371	6.285	1.094(06)	0.890(05)	0.909(06)
Emerging markets	0.105	0.155	−0.805	8.346	0.429(12)	0.396(12)	0.398(12)
Equity market neutral	0.098	0.028	0.320	3.485	2.100(01)	2.209(01)	2.337(01)
Event driven	0.116	0.055	−3.522	28.174	1.416(03)	0.829(08)	0.895(07)
Distressed	0.130	0.061	−3.049	23.518	1.501(02)	0.888(06)	0.961(04)
Multi-strategy	0.109	0.059	−2.520	19.410	1.185(05)	0.783(09)	0.829(09)
Risk arbitrage	0.077	0.041	−1.088	9.112	0.938(09)	0.757(10)	0.773(10)
Fixed income arbitrage	0.065	0.036	−3.133	20.471	0.737(11)	0.550(11)	0.562(11)
Global macro	0.145	0.103	0.057	6.555	1.029(08)	0.926(04)	0.940(05)
Long/short equity	0.127	0.098	0.195	7.226	0.903(10)	0.831(07)	0.840(08)
Managed futures	0.067	0.121	0.019	3.116	0.235(13)	0.236(13)	0.236(13)
Multi-strategy	0.095	0.043	−1.186	6.145	1.331(04)	1.052(02)	1.082(02)

A.2. Proof of Theorem 2

To derive the formula for b_n we need to compute the first, second, and nth derivative of HARA utility given by (10). Straightforward computations give

$$U^{(1)} = \lambda\left(\frac{\lambda w}{\rho}+\beta\right)^{-\rho}, \quad U^{(2)} = -\lambda^2\left(\frac{\lambda w}{\rho}+\beta\right)^{-\rho-1}.$$
$$U^{(n)} = (-1)^{n-1}\lambda^n\frac{\prod_{k=1}^{n-2}(\rho+k)}{\rho^{n-2}}\left(\frac{\lambda w}{\rho}+\beta\right)^{-\rho-n+1}, \quad n \geqslant 3.$$

Using these formulas we arrive at the expression for b_n given by (11). Finally note that b_n does not depend on w.

A.3. Proof of Theorem 5

If in the infinite Taylor series (3) we keep the terms with first three derivatives of the utility function, then the investor's expected utility becomes

$$E[U(\tilde{w})] \approx U(w_r) + U^{(1)}(w_r)aE[(x-r_f)] + \frac{1}{2}U^{(2)}(w_r)a^2E[(x-r_f)^2] + \frac{1}{6}U^{(3)}(w_r)a^3E[(x-r_f)^3]. \tag{39}$$

Note that up to the leading terms with $\Delta t^{\frac{3}{2}}$ we have the following:

$$E[(x-r_f)] = (\mu-r)\Delta t,$$
$$E[(x-r_f)^2] = E[(x-\mu\Delta t+\mu\Delta t-r\Delta t)^2] \approx E[(x-E(x))^2] = \left(\sigma\sqrt{\Delta t}\right)^2E[\varepsilon^2],$$
$$E[(x-r_f)^3] = E[(x-\mu\Delta t+\mu\Delta t-r\Delta t)^3] \approx E[(x-E(x))^3] = \left(\sigma\sqrt{\Delta t}\right)^3E[\varepsilon^3].$$

To shorten the subsequent notation let

$$p = E[(x-r_f)], \quad v = E[(x-E(x))^2], \quad s = E[(x-E(x))^3].$$

Observe that p is the expected excess returns on the risky asset, v and s are the second and third central moments of the return distribution. Recall the investor's objective function (2). The first-order condition (5) for the optimality of a becomes

$$\frac{1}{2}\frac{U^{(3)}(w_r)}{U^{(1)}(w_r)}a^2s + \frac{U^{(2)}(w_r)}{U^{(1)}(w_r)}av + p = 0.$$

Using the notions of γ and b_3, the equation above can be rewritten as

$$\frac{1}{2}a^2\gamma^2b_3s - a\gamma v + p = 0.$$

This is a quadratic equation that has two solutions. The solutions are given by

$$a_{1,2} = \frac{v \pm \sqrt{v^2-2b_3sp}}{\gamma b_3 s}. \tag{40}$$

In this form, it is difficult to gain any intuition about the nature of the optimal solution. It is also difficult to pick the correct solution from the two ones. To progress further we again rely on an approximation analysis to find a unique solution for a. Consider the following term in (40)

$$\sqrt{v^2-2b_3sp}. \tag{41}$$

Observe that v^2 is of order Δt^2, but $2b_3sp$ is of order $\Delta t^{\frac{5}{2}}$. That is, $2b_3sp \ll v^2$ and, hence, we can apply a Taylor series expansion to the function (41). We will use three terms around v^2 to account for skewness. This gives us

$$\sqrt{v^2-2b_3sp} \approx \sqrt{v^2} - \frac{1}{2}\frac{2b_3sp}{\sqrt{v^2}} - \frac{1}{2}\frac{1}{4}\frac{(2b_3sp)^2}{(v^2)^{\frac{3}{2}}} = v - \frac{b_3sp}{v} - \frac{(b_3sp)^2}{2v^3}.$$

Then the solutions for optimal a are

$$a_{1,2} = \frac{v \pm \left(v - \frac{b_3sp}{v} - \frac{(b_3sp)^2}{2v^3}\right)}{\gamma b_3 s}.$$

Observe that the correct solution for a should reduce to the solution (15) when skewness is zero, that is, to $a = \frac{p}{\gamma v}$. Thus, we arrive at the following solution for the optimal amount that should be invested in the risky asset

$$a = \frac{p}{\gamma v} + \frac{sb_3p^2}{2\gamma v^3} = \frac{p}{\gamma v}\left(1 + \frac{b_3sp}{2v^2}\right). \tag{42}$$

Recall that the skewness is defined by

$$Skew[x] = \frac{E[(x-E[x])^3]}{E[(x-E[x])^2]^{\frac{3}{2}}} = \frac{s}{v^{\frac{3}{2}}}. \tag{43}$$

Now rewrite (42) as

$$a = \frac{\mu-r}{\gamma\sigma^2}\left(1 + b_3\frac{Skew}{2}\left(\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}\right).$$

Using the expression for the Sharpe ratio (17) the solution for a can be rewritten as (18).

Finally, we express the (approximate) maximum expected utility in terms of the Sharpe ratio adjusted for the skewness preference. Recall that the investor's expected utility is given by (39), the optimal amount invested in the risky asset is given by (18), the skewness is defined by (43), the Sharpe ratio is given by (17), and that $U^{(2)}(w_r) = -\gamma U^{(1)}(w_r)$, and $U^{(3)}(w_r) = b_3\gamma^2U^{(1)}(w_r)$. This gives the expression for the maximum expected utility as

$$E[U^*(\tilde{w})] \approx U(w_r) + \frac{U^{(1)}(w_r)}{\gamma}SR^2\left(1 + b_3\frac{Skew}{2}SR\right) - \frac{U^{(1)}(w_r)}{\gamma}\frac{1}{2}SR^2\left(1 + b_3\frac{Skew}{2}SR\right)^2 + \frac{U^{(1)}(w_r)}{\gamma}\frac{b_3Skew}{6}SR^3\left(1 + b_3\frac{Skew}{2}SR\right)^3.$$

Again, we want to keep only the leading terms up to $\Delta t^{\frac{3}{2}}$. Since $Skew = E[\varepsilon^3]$ does not depend on Δt and SR is of order $\Delta t^{\frac{1}{2}}$, in the resulting expression we only collect the terms up to SR^3. This yields

$$E[U^*(\tilde{w})] \approx U(w_r) + \frac{U^{(1)}(w_r)}{\gamma}\frac{1}{2}SR^2\left(1 + b_3\frac{Skew}{3}SR\right).$$

A.4. Proof of Theorem 7

Start with

$$E[U^*(\tilde{w})] = U\left(\frac{1}{2\gamma}ASSR^2 + w(1+r_f)\right)$$

and apply Taylor series expansion for $E[U^*(\tilde{w})]$ around w_r. This gives

$$E[U^*(\tilde{w})] = U(w_r) + U^{(1)}(w_r)\frac{ASSR^2}{2\gamma} + \sum_{n=2}^{\infty}\frac{1}{n!}U^{(n)}(w_r)\left(\frac{ASSR^2}{2\gamma}\right)^n.$$

Observe that $ASSR^2$ is of leading order $\Delta t, (ASSR^2)^2$ is of leading order Δt^2, etc. If we keep only the terms up to $\Delta t^{\frac{3}{2}}$, then we arrive to

$$E[U^*(\tilde{w})] \approx U(w_r) + U^{(1)}(w_r)\frac{ASSR^2}{2\gamma},$$

which is the same as (19).

A.5. Proof of Theorem 8

First of all, rewrite (36) as

$$\max_a E[U(\tilde{w})] = -e^{\lambda a r_f}\int_{-\infty}^{\infty} e^{-\lambda a x} f(x;\alpha,\beta,\eta,\delta)dx. \quad (44)$$

Observe now that the integral in (44) corresponds to the computation of the moment generating function of a NIG variable (see (35)) where u is replaced by $-\lambda a$. Consequently,

$$\max_a E[U(\tilde{w})] = \max_a -e^{-\lambda a(\eta - r_f)+\delta\left(\varphi-\sqrt{\alpha^2-(\beta-\lambda a)^2}\right)}. \quad (45)$$

The maximization problem (45) is equivalent to

$$\max_a \lambda a(\eta - r_f) - \delta\left(\varphi - \sqrt{\alpha^2-(\beta-\lambda a)^2}\right).$$

The first-order condition for the optimality of a, after some rearrangement, gives the following quadratic equation in a (or $a\lambda$)

$$(a\lambda)^2 - 2a\lambda\beta + \left(\beta^2 - \frac{\alpha^2(\eta-r_f)^2}{\delta^2+(\eta-r_f)^2}\right) = 0.$$

The quadratic equation above has two solutions

$$a_{1,2} = \frac{1}{\lambda}\left(\beta \pm \sqrt{\beta^2 - \left(\beta^2 - \frac{\alpha^2(\eta-r_f)^2}{\delta^2+(\eta-r_f)^2}\right)}\right) = \frac{1}{\lambda}\left(\beta \pm \frac{\alpha(\eta-r_f)}{\sqrt{\delta^2+(\eta-r_f)^2}}\right). \quad (46)$$

Recall that the normal distribution appears as a limiting case of the NIG distribution when $\beta = 0, \alpha \to \infty, \delta \to \infty$, and $\frac{\delta}{\alpha} = \sigma_x^2$. Thus, if the probability distribution is normal, then we have to arrive at the well-known solution $a^* = \frac{1}{\lambda}\frac{\mu_x - r_f}{\sigma_x^2}$. Consequently, the correct solution for the optimal value of a is given by

$$a^* = \frac{1}{\lambda}\left(\beta + \frac{\alpha(\eta-r_f)}{\sqrt{\delta^2+(\eta-r_f)^2}}\right) = \frac{1}{\lambda}\left(\beta + \frac{\eta - r_f}{\sqrt{\frac{\delta^2}{\alpha^2}+\frac{(\eta-r_f)^2}{\alpha^2}}}\right). \quad (47)$$

The maximum expected utility is then given by

$$E[U^*(\tilde{w})] = -e^{-\lambda a^*(\eta-r_f)+\delta\left(\varphi-\sqrt{\alpha^2-(\beta-\lambda a^*)^2}\right)} = -e^{-\frac{1}{2}\mathrm{ASKSR}^2}.$$

Therefore, the solution for ASKSR is given by

$$\mathrm{ASKSR} = \sqrt{2\left(\lambda a^*(\eta-r_f) - \delta\left(\varphi - \sqrt{\alpha^2-(\beta-\lambda a^*)^2}\right)\right)}. \quad (48)$$

It remains to demonstrate that the ASKSR reduces to the SR when the NIG distribution reduces to the normal distribution. Consider the following two terms in (48)

$$I_1 = \lambda a^*(\eta - r_f), \quad I_2 = \delta\left(\varphi - \sqrt{\alpha^2-(\beta-\lambda a^*)^2}\right),$$

such that the ASKSR can be written as $\mathrm{ASKSR} = \sqrt{2(I_1 - I_2)}$. The first term reduces to (see the solution for a^* given by (47) when $\beta = 0$ such that $\eta = \mu_x$)

$$I_1 = \frac{(\mu_x - r_f)^2}{\sqrt{\frac{\delta^2}{\alpha^2}+\frac{(\mu_x-r_f)^2}{\alpha^2}}} = \frac{(\mu_x-r_f)^2}{\sigma_x^2} = SR^2,$$

since $\frac{\delta^2}{\alpha^2} \to \sigma_x^4$ and $\frac{(\mu_x-r_f)^2}{\alpha^2} \to 0$. The second term reduces to

$$I_2 = \delta\left(\alpha - \sqrt{\alpha^2 - \frac{\alpha^2(\mu_x-r_f)^2}{\delta^2+(\mu_x-r_f)^2}}\right) = \delta\alpha - \frac{\delta^2\alpha}{\sqrt{\delta^2+(\mu_x-r_f)^2}} = \frac{\delta\sqrt{\delta^2+(\mu_x-r_f)^2}-\delta^2}{\sqrt{\frac{\delta^2}{\alpha^2}+\frac{(\mu_x-r_f)^2}{\alpha^2}}} = \frac{\delta^2\left(1+\frac{(\mu_x-r_f)^2}{\delta^2}\right)^{\frac{1}{2}}-\delta^2}{\sqrt{\frac{\delta^2}{\alpha^2}+\frac{(\mu_x-r_f)^2}{\alpha^2}}}.$$

Observe now that the denominator in I_2 converges to σ_x^2 as the NIG distribution converges to the normal distribution. To determine what value the numerator in I_2 converges to, recall that for any function $h, h(1+y) = h(1) + h'(1)y + \mathcal{O}(y^2)$ as $y \to 0$. Therefore, the numerator in I_2 converges to $\left(\text{assume now that } h(1+y) = (1+y)^{\frac{1}{2}} \text{ and } y = \frac{(\mu_x-r_f)^2}{\delta^2}\right)$

$$\delta^2\left(1+\frac{(\mu_x-r_f)^2}{\delta^2}\right)^{\frac{1}{2}} - \delta^2 = \delta^2\left(1+\frac{1}{2}\frac{(\mu_x-r_f)^2}{\delta^2}\right) - \delta^2 = \frac{1}{2}(\mu_x-r_f)^2.$$

Consequently, we have found that I_2 converges to $\frac{1}{2}\frac{(\mu_x-r_f)^2}{\sigma_x^2} = \frac{1}{2}SR^2$. Finally, the ASKSR converges to (assuming $SR > 0$)

$$\mathrm{ASKSR} = \sqrt{2(I_1 - I_2)} = \sqrt{SR^2} = SR.$$

References

Agarwal, V., Naik, N.Y., 2004. Risk and portfolio decisions involving hedge funds. Review of Financial Studies 17 (1), 63–98.

Arrow, K.J., 1971. Essays in Theory of Risk-Bearing. North-Holland, Amsterdam.

Barndorff-Nielsen, O.E., 1995. Normal inverse Gaussian distributions and stochastic volatility modelling. Scandinavian Journal of Statistics 24, 1–13.

Barndorff-Nielsen, O.E., 1998. Processes of normal inverse Gaussian type. Finance and Stochastics 2, 41–68.

Bawa, V.S., 1978. Safety-first, stochastic dominance, and optimal portfolio choice. Journal of Financial and Quantitative Analysis 13 (2), 255–271.

Bernardo, A.E., Ledoit, O., 2000. Gain, loss and asset pricing. Journal of Political Economy 108 (1), 144–172.

Brooks, C., Kat, H.M., 2002. The statistical properties of hedge fund index returns and their implications for investors. Journal of Alternative Investments 5 (2), 26–44.

Cass, D., Stiglitz, J.E., 1970. The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds. Journal of Economic Theory 2, 122–160.

Cherny, A., 2003. Generalized Sharpe ratios and asset pricing in incomplete markets. European Finance Review 7 (2), 191–233.

Constantinides, G., 1990. Habit formation: A resolution of the equity premium puzzle. Journal of Political Economy 98, 519–543.

De Giorgi, E., 2005. Reward-risk portfolio selection and stochastic dominance. Journal of Banking and Finance 29 (4), 895–926.

Dowd, K., 2000. Adjusting for risk: An improved Sharpe ratio. International Review of Economics and Finance 9, 209–222.

Eling, M., Schuhmacher, F., 2007. Does the choice of performance measure influence the evaluation of hedge funds? Journal of Banking and Finance 31 (9), 2247–2632.

Farinelli, S., Ferreira, M., Rossello, D., Thoeny, M., Tibiletti, L., 2008. Beyond Sharpe ratio: Optimal asset allocation using different performance ratios. Journal of Banking and Finance 32 (10), 2057–2063.

Favre, L., Galeano, J.-A., 2002. Mean-modified value-at-risk optimization with hedge funds. Journal of Alternative Investments 5, 21–25.

Fishburn, P.C., 1977. Mean-risk analysis with risk associated with below-target returns. American Economic Review 67 (2), 116–126.

Friend, I., Blume, M.E., 1975. The demand for risky assets. American Economic Review 65, 900–922.

Goetzmann, W., Ingersoll, J., Spiegel, M., Welch, I., 2002. Sharpening Sharpe ratios. Working Paper, NBER.

Haley, M.R., McGee, M.K., 2006. Tilting safety first and the Sharpe portfolio. Finance Research Letters 3 (3), 173–180.

He, H., Leland, H., 1993. On equilibrium asset price processes. Review of Financial Studies 6, 593–617.

Hodges, S., 1998. A generalization of the Sharpe ratio and its applications to valuation bounds and risk measures. Working Paper, Financial Options Research Centre, University of Warwick.

Ingersoll, J., Spiegel, M., Goetzmann, W., 2007. Portfolio performance manipulation and manipulation-proof performance measures. Review of Financial Studies 20 (5), 1503–1546.

Kaplan, P.D., Knowles, J.A., 2004. Kappa: A generalized downside risk-adjusted performance measure. Journal of Performance Measurement 8 (3), 42–54.
Karlis, D., 2002. An EM type algorithm for maximum likelihood estimation of the normal inverse Gaussian distribution. Statistics and Probability Letters 57, 43–52.
Kraus, A., Litzenberger, R.H., 1976. Skewness preference and the valuation of risk assets. Journal of Finance 31 (4), 1085–1100.
Kydland, F.E., Prescott, E.C., 1982. Time to build and aggregate fluctuations. Econometrica 50, 1345–1370.
Leland, H.E., 1999. Beyond mean-variance: Risk and performance measurement in a nonsymmetrical world. Financial Analysts Journal 1, 27–36.
Madan, D.B., McPhail, G.S., 2000. Investing in skews. Journal of Risk Finance 2 (1), 10–18.
Madan, D.B., Seneta, G.E., 1990. The variance gamma (VG) model for share market returns. Journal of Business 63 (4), 511–524.
Malkiel, B.G., Saha, A., 2005. Hedge funds: Risk and return. Financial Analysts Journal 61 (6), 80–88.
Mehra, R., Prescott, E.C., 1985. The equity premium: A puzzle. Journal of Monetary Economics 15, 145–161.
Merton, R.C., 1980. On estimating the expected return on the market: An exploratory investigation. Journal of Financial Economics 8, 321–361.
Meucci, A., 2005. Risk and Asset Allocation. Springer.
Post, T., van Vliet, P., Levy, H., 2008. Risk aversion and skewness preference. Journal of Banking and Finance 32 (7), 1178–1187.
Pratt, J.W., 1964. Risk aversion in the small and in the large. Econometrica 32, 122–136.
Rachev, S., Jasic, T., Stoyanov, S., Fabozzi, F.J., 2007. Momentum strategies based on reward-risk stock selection criteria. Journal of Banking and Finance 31 (8), 2325–2346.
Rubinstein, M., 1976. The valuation of uncertain income streams and the pricing of options. Bell Journal of Economics 7, 407–425.
Samorodnitsky, G., Taqqu, M.S., 1994. Stable Non-Gaussian Random Processes. Chapman and Hall, New York.
Shadwick, W.F., Keating, C., 2002. A universal performance measure. Journal of Performance Measurement 6 (3), 59–84.
Sortino, F.A., Price, L.N., 1994. Performance measurement in a downside risk framework. Journal of Investing 3 (3), 59–65.
Spurgin, R.B., 2001. How to game your Sharpe ratio. Journal of Alternative Investments 4 (3), 38–46.
Stutzer, M., 2000. A portfolio performance index. Financial Analysts Journal 56 (3), 52–61.
Treynor, J.L., Black, F., 1973. How to use security analysis to improve portfolio selection. Journal of Business 46 (1), 66–86.
Uchaikin, V.V., Zolotarev, V.M., 1999. Chance and Stability: Stable Distributions and Their Applications, VSP, Utrecht, The Netherlands.
Ziemba, W.T., 2003. The Stochastic Programming Approach to Asset, Liability, and Wealth Management. Research Foundation of CFA Institute.