Pairs Trading

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Outline

Outline

Cointegration

- Cointegration is a very interesting property that can be exploited in finance for trading.
- Idea: While it may be difficult to predict individual stocks, it may be easier to predict relative behavior of stocks.
- Illustrative example: A drunk man is wandering the streets (random walk) with a dog. Both paths of man and dog are nonstationary and difficult to predict, but the distance between them is mean-reverting and stationary.

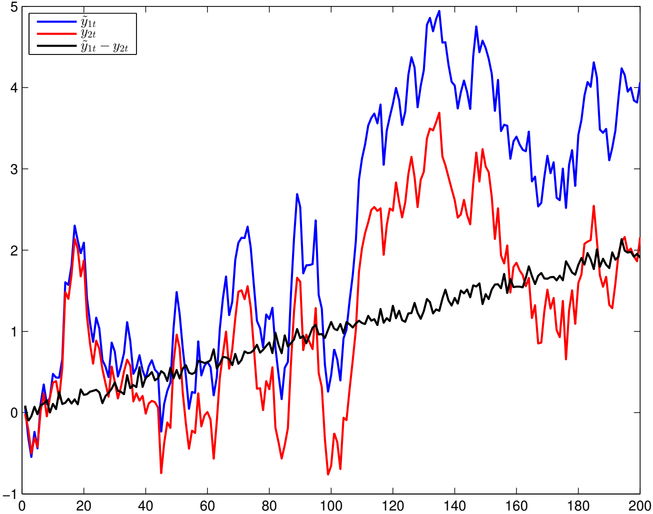
Correlation vs. cointegration

- Everybody is familiar with the concept of correlation between two random variables:
  - correlation is high when they co-move
  - correlation is zero when they move independently
- So what is cointegration?
  - cointegration is high when two quantities move together or remain close to each other
  - cointegration is inexistent if the two quantities do not stay together
- Clear? 😉 You can see why this concept may be difficult to grasp at first, but the truth is that it's easy.[1]
- In the financial context:
  - Cointegration of (log-)prices y_t refers to **long-term** co-movements.
  - Correlation of (log-)returns $\Delta y_t = y_t - y_{t-1}$ characterizes **short-term** co-movements in (log-)prices y_t.

[1] Y. Feng and D. P. Palomar, *A Signal Processing Perspective on Financial Engineering*. Foundations and Trends in Signal Processing, Now Publishers, 2016.

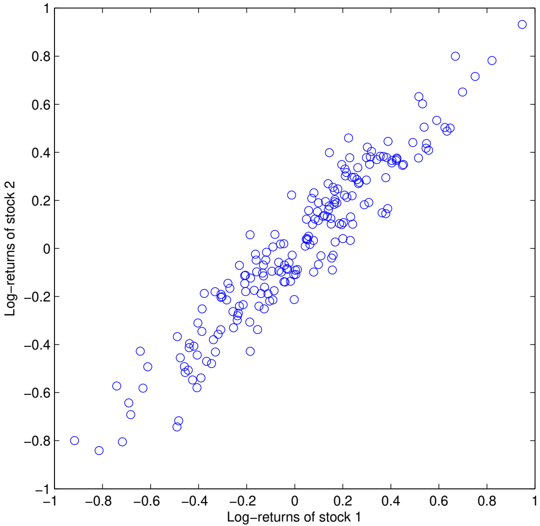
Correlation vs. cointegration

- Example of high correlation with no cointegration:



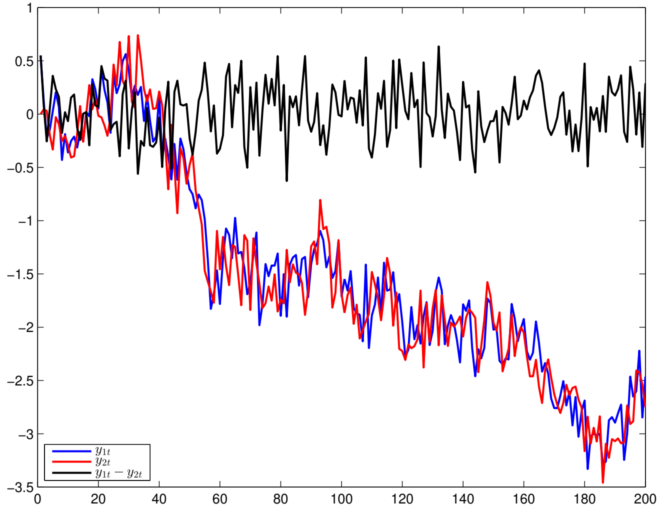
Correlation vs. cointegration

- Indeed the returns are highly correlated, see scatter plot:



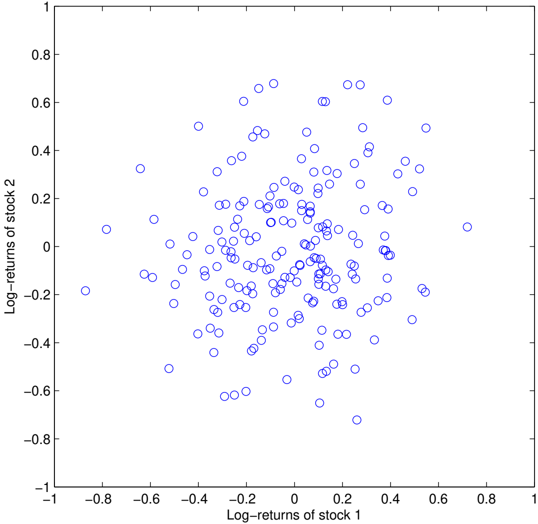
Correlation vs. cointegration

- Opposite example of high cointegration with no correlation:



Correlation vs. cointegration

- Indeed the returns are not correlated, see scatter plot:



Cointegration

- A time series is called integrated of order p, denoted as $I(p)$, if the time series obtained
  - by differencing the time series p times is weakly stationary,
  - while by differencing the time series $p-1$ times is not weakly stationary.
- Example: stock log-prices y_t are integrated of order $I(1)$ because
  - log-prices are not stationary
  - but log-returns $y_t - y_{t-1}$ are stationary (at least for some period of time).
- A multivariate time series is said to be cointegrated if it has at least one linear combination being integrated of a lower order, e.g., $\mathbf{y}_t$ is not stationary but $\mathbf{w}^T\mathbf{y}_t$ is stationary for some weights $\mathbf{w}$.

Cointegration

- Consider the following two nonstationary time series (e.g., log-prices of stocks):

$$y_{1t} = \gamma x_t + w_{1t}$$
$$y_{2t} = x_t + w_{2t}$$

  with a stochastic common trend defined as a random walk:

$$x_t = x_{t-1} + w_t$$

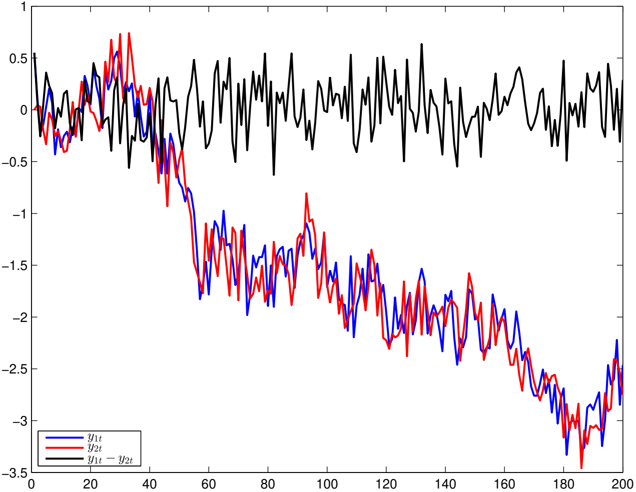
  where w_{1t}, w_{2t}, w_t are i.i.d. residual terms mutually independent.
- The coefficient γ is the secret ingredient here.
- If γ is known, then we can define the so-called "spread"

$$z_t = y_{1t} - \gamma y_{2t} = w_{1t} - \gamma w_{2t}$$

  which is stationary and **mean reverting**.
- Interestingly, the differences (i.e., log-returns) Δy_{1t} and Δy_{2t} can have an arbitrarily small correlation:
  $\rho = 1/\left(\sqrt{1 + 2\sigma_1^2/\sigma^2}\sqrt{1 + 2\sigma_2^2/\sigma^2}\right)$.

Cointegration

- The log-prices y_{1t} and y_{2t} are cointegrated and the spread $z_t = y_{1t} - \gamma y_{2t}$ is stationary (assume $\gamma = 1$):



Outline

Basic Idea of Pairs Trading

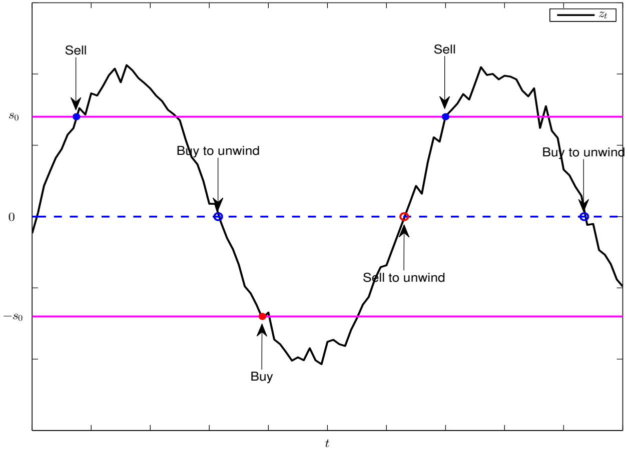
- Recall that if two time series are cointegrated, then in the long term they remain close to each other.
- In other words, the spread $z_t = y_{1t} - \gamma y_{2t}$ is **mean reverting**.
- This mean-reverting property of the spread can be exploited for trading and it is commonly referred to as "pairs trading" or "statistical arbitrage".
- The idea behind pairs trading is to
    - short-sell the relatively overvalued stocks and buy the relatively undervalued stocks,
    - unwind the position when they are relatively fairly valued.

Trading the spread

- Suppose the spread $z_t = y_{1t} - \gamma y_{2t}$ is mean-reverting with zero mean.
- Stat-arb trading:
  - if spread is low ($z_t < -s_0$), then stock 1 is undervalued and stock 2 overvalued:
    - buy the spread (i.e., buy stock 1 and short-sell stock 2)
    - unwind the positions when it reverts to zero after i time steps $z_{t+i} = 0$
  - if spread is high ($z_t > s_0$), then stock 1 is overvalued and stock 2 undervalued:
    - short-sell the spread (i.e., short-sell stock 1 and buy stock 2)
    - unwind the positions when it reverts to zero after i time steps $z_{t+i} = 0$
- The profit, say, from buying low and unwinding at zero is $z_{t+i} - z_t = s_0$. So easy!
- Indeed $z_{t+i} - z_t = -\gamma(y_{2,t+i} - y_{2t}) + (y_{1,t+i} - y_{1t})$, so the whole process is like having used a portfolio with weigths $\mathbf{w} = \begin{bmatrix} 1 \\ -\gamma \end{bmatrix}$.
- Recall that the return of a portfolio $\mathbf{w}$ is $\mathbf{w}^T \Delta \mathbf{y}_t$.

Trading the spread

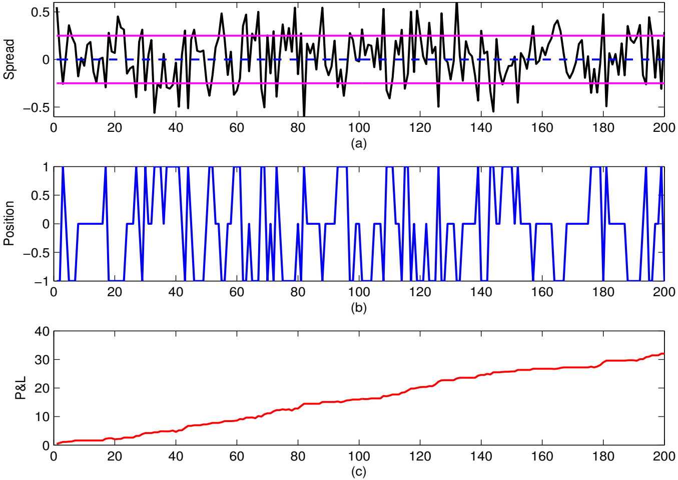
- Illustration on how to trade the spread $z_t = y_{1t} - \gamma y_{2t}$:[2]



[2] G. Vidyamurthy, *Pairs Trading: Quantitative Methods and Analysis*. John Wiley & Sons, 2004.

Pairs trading or statistical arbitrage

- Statistical arbitrage can be used in practice with profits:[3]



[3] M. Avellaneda and J.-H. Lee, "Statistical arbitrage in the US equities market," *Quantitative Finance*, vol. 10, no. 7, pp. 761–782, 2010.

But how to discover cointegrated pairs and γ?

- One interesting approach is based on a VECM modeling of the universe of stocks: From the parameter $\boldsymbol{\beta}$ contained in the low-rank matrix $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}^T$ one can extract a cointegration subspace. After that, one can design some portfolio within that cointegration subspace.[4]
- A simpler approach to discover pairs is by brute force, i.e., try exhaustively different combinations of pairs of stocks and see if they are cointegrated.
- But, given a potential pair, how do we obtain the "secret" γ?
- Easy! Just a simple LS regression!
- Recall that
  - γ is needed to form the spread to be traded (i.e., portfolio)
  - the spread mean μ is needed to determine the thresholds for entering a trade and unwind later the position.

[4] Z. Zhao and D. P. Palomar, "Mean-reverting portfolio with budget constraint," *IEEE Trans. Signal Process.*, vol. 66, no. 9, pp. 2342–2357, 2018.

Outline

Design of a pairs trading strategy

- We first focus on pairs trading (i.e., statistical arbitrage between two stocks) as the example to introduce the main steps of statistical arbitrage.
- In practice, pairs trading contains three main steps[5]:
  - **Pairs selection**: identify stock pairs that could potentially be cointegrated.
  - **Cointegration test**: test whether the identified stock pairs are indeed cointegrated or not.
  - **Trading strategy design**: study the spread dynamics and design proper trading rules.

[5] G. Vidyamurthy, *Pairs Trading: Quantitative Methods and Analysis*. John Wiley & Sons, 2004.

Outline

Pairs selection: normalized price distance

- Normalized price distance[6] (as a rough proxy to measure cointegration):
$$\text{NPD} \triangleq \sum_{t=1}^{T} (\tilde{p}_{1t} - \tilde{p}_{2t})^2$$
where the normalized price $\tilde{p}_{1t}$ of stock 1 is given by $\tilde{p}_{1t} = p_{1t}/p_{10}$. The normalized prices of stock 2 defined similarly.
- One can easily (i.e., cheaply) compute the NPD for all the possible combination of pairs and select some pairs with smallest NPD as the potentially cointegrated pairs.
- Later one can use a more refined measure of cointegration (more computationally demanding).

[6] E. Gatev, W. N. Goetzmann, and K. G. Rouwenhorst, "Pairs trading: Performance of a relative-value arbitrage rule," *Review of Financial Studies*, vol. 19, no. 3, pp. 797–827, 2006.

Outline

Least Squares (LS) regression

- If the spread z_t is stationary, it can be written as[7]

$$z_t = y_{1t} - \gamma y_{2t} = \mu + \epsilon_t$$

  where
  - μ represents the equilibrium value and
  - ϵ_t is a zero-mean residual.
- Equivalently, it can be written as

$$y_{1t} = \mu + \gamma y_{2t} + \epsilon_t$$

  which now has the typical form of linear regression.
- Least squares (LS) regression over T observations:

$$\underset{\mu,\gamma}{\text{minimize}} \quad \sum_{t=1}^{T} \left(y_{1t} - (\mu + \gamma y_{2t})\right)^2$$

[7] G. Vidyamurthy, *Pairs Trading: Quantitative Methods and Analysis*. John Wiley & Sons, 2004.

Cointegration test

- LS regression is used to estimate the parameters μ and γ, obtaining the estimates $\hat{\mu}$ and $\hat{\gamma}$.
- If y_{1t} and y_{2t} are $I(1)$ and are cointegrated, then the estimates converge to the true values as the number of observations goes to infinity[8].
- Using the estimated parameters $\hat{\mu}$ and $\hat{\gamma}$, we can compute the residuals

$$\hat{\epsilon}_t = y_{1t} - \hat{\gamma} y_{2t} - \hat{\mu}.$$

- Then, one has to decide whether the spread is stationary, i.e., ϵ_t is stationary. In practice, the estimated residuals are used $\hat{\epsilon}_t$
- There are many well-defined mathematical tests for the stationarity of $\hat{\epsilon}_t$, e.g., augmented Dicky-Fuller (ADF) test, Johansen test, etc.

[8] R. F. Engle and C. W. J. Granger, "Co-integration and error correction: Representation, estimation, and testing," *Econometrica: Journal of the Econometric Society*, pp. 251–276, 1987.

Outline

Optimum threshold

- Once some identified pairs have passed the cointegration test, one still needs to decide the entry and exit thresholds to open and unwind the positions, respectively.
- For the sake of concreteness, we focus on studying the entry threshold:
  - open positions when the spread diverges from its long-term mean by s_0
  - unwind the position when it reverts to its mean
- Thus, the key problem now is how to design the value of s_0 such that the total profit is maximized.
- Total profit:

$$\text{profit of each trade} \times \text{number of trades}$$

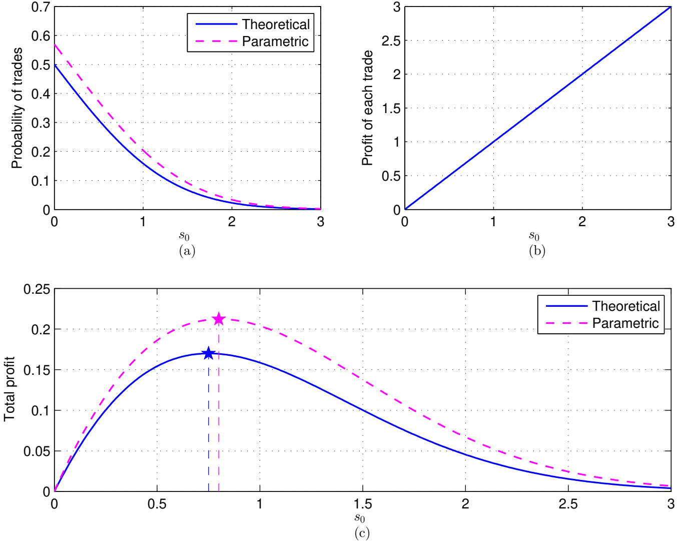
  - profit of each trade is s_0
  - number of trades is related to the zero crossings, which can be analized theoretically as well as empirically.
- We focus now on estimating the number of trades.

Optimum threshold s_0: Parametric approach

- Suppose the spread follows a standard Normal distribution.
- The probability that the spread deviates above from the mean by s_0 or more is
$$1 - \Phi(s_0)$$
where $\Phi(\cdot)$ is the c.d.f. of the standard Normal distribution.
- For a path with T days, the number of tradable events is
$$T(1 - \Phi(s_0)).$$
- For each trade, the profit is s_0 and then the total profit is $s_0 T(1 - \Phi(s_0))$.
- Then the optimal threshold is $s_0^\star = \arg\max_{s_0} \{s_0 T(1 - \Phi(s_0))\}$.
- In practice, one cannot know the true distribution but can estimate the distribution parameters.
- Then one can compute the total profit based on estimated distribution.

Optimum threshold s_0: Parametric approach

- Optimal threshold $s_0^\star$ maximizes the total profit:



Optimum threshold s_0: Non-parametric approach

- Suppose the observed sample path has length T: $z_1, z_2, \ldots, z_T$.
- We consider J discretized threshold values as $s_0 \in \{s_{01}, s_{02}, \ldots, s_{0J}\}$ and the empirical trading frequency for the threshold s_{0j} is

$$\bar{f}_j = \frac{\sum_{t=1}^{T} \mathbf{1}_{\{z_t > s_{0j}\}}}{T}.$$

- The empirical values $\bar{f}_j$ may not be a smoothed enough and the resulted profit function may not be accurate enough.
- Smooth the trading frequency function by regularization:

$$\underset{\mathbf{f}}{\text{minimize}} \quad \sum_{j=1}^{J} (\bar{f}_j - f_j)^2 + \lambda \sum_{j=1}^{J-1} (f_j - f_{j+1})^2$$

Optimum threshold s_0: Non-parametric approach

- The problem can be rewritten as

$$\underset{\mathbf{f}}{\text{minimize}} \quad \|\bar{\mathbf{f}} - \mathbf{f}\|_2^2 + \lambda \|\mathbf{D}\mathbf{f}\|_2^2$$

where

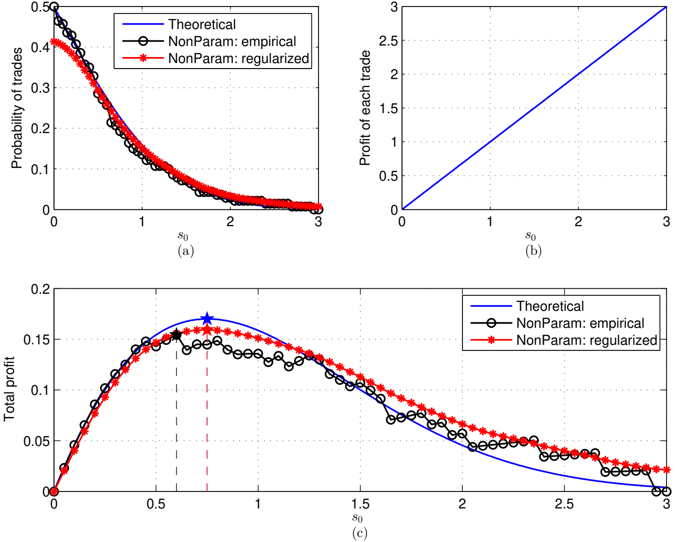
$$\mathbf{D} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(J-1)\times J}.$$

- Setting the derivative of the objective w.r.t. $\mathbf{f}$ to zero yields the optimal solution $\mathbf{f}^\star = (\mathbf{I} + \lambda \mathbf{D}^T\mathbf{D})^{-1}\bar{\mathbf{f}}$.
- The optimal threshold is the one maximizes the total profit:

$$s_0^\star = \underset{s_{0j}\in\{s_{01}, s_{02}, \ldots, s_{0J}\}}{\arg\max} \{s_{0j} f_j\}.$$

Optimum threshold s_0: Non-parametric approach

- Optimal threshold $s_0^\star$ maximizes the total profit:



Outline

LS regression for pairs trading

- If the spread z_t is stationary, it can be written as

$$z_t = y_{1t} - \gamma y_{2t} = \mu + \epsilon_t$$

  where
  - μ represents the equilibrium value and
  - ϵ_t is a zero-mean residual.
- Equivalently, it can be written as

$$y_{1t} = \mu + \gamma y_{2t} + \epsilon_t$$

  which now has the typical form for linear regression.
- Least squares (LS) regression over T observations:

$$\underset{\mu,\gamma}{\text{minimize}} \quad \sum_{t=1}^{T} \left(y_{1t} - (\mu + \gamma y_{2t})\right)^2$$

- By stacking the T observations in the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$, we can finally write:

$$\underset{\mu,\gamma}{\text{minimize}} \quad \left\|\mathbf{y}_1 - (\mu\mathbf{1} + \gamma\mathbf{y}_2)\right\|^2$$

LS regression for pairs trading

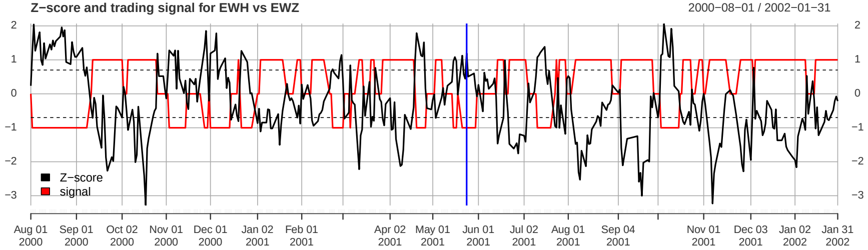
- Using the estimated parameters $\hat{\mu}$ and $\hat{\gamma}$, we can compute the residuals $\hat{\epsilon}_t = y_{1t} - \hat{\mu} - \hat{\gamma} y_{2t}$.
- Then, one has to decide whether the cointegration is acceptable or not so move to the trading part.
- There are many well-defined mathematical tests for the stationarity of $\hat{\epsilon}_t$, e.g., augmented Dicky-Fuller (ADF) test, Johansen test, etc.
- Total profit:

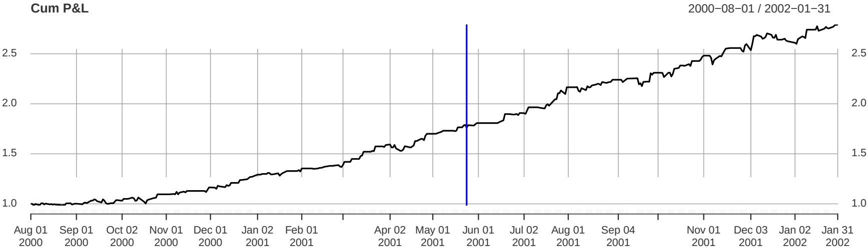
$$\text{profit of each trade} \times \text{number of trades}$$

  - profit of each trade is s_0
  - number of trades is related to the zero crossings, which can be analized theoretically as well as empirically.
- Ideally, we want residuals with large amplitude (variance) as well as a strong mean reversion because they directly affect the profit.

LS regression for pairs trading

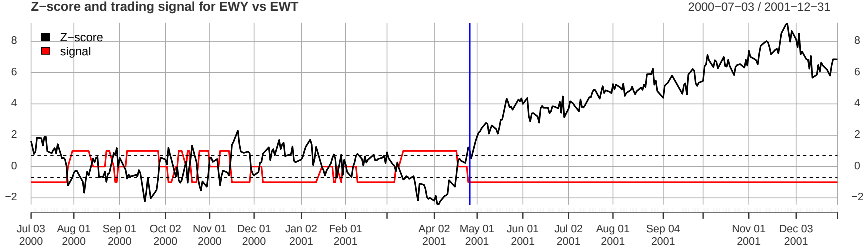
- One good case: 😃

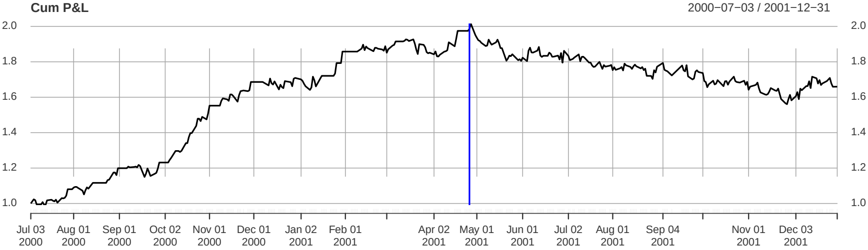




LS regression for pairs trading

- But also a bad case: 😞





LS regression for pairs trading

- The problem with the LS regression is that it assumes that μ and γ are constant.
- In practice, they can change with time, resulting in a spread that drifts from equilibrim never to revert back with huge potential losses.
- Thus, in practice, μ and γ are time-varying and have to be tracked.
- How to track time-varying parameters?
- Of course... Kalman!!!
- Well, you can also try a rolling regression or exponential smoothing, but Kalman works better.

Kalman for pairs trading

- Recall the previous static relationship for cointegrated series y_{1t} and y_{2t}:

$$y_{1t} = \mu + \gamma y_{2t} + \epsilon_t$$

- Let's make it time-varying:

$$y_{1t} = \mu_t + \gamma_t y_{2t} + \epsilon_t$$

- Let's further assume that the parameters μ_t and γ_t change slowly over time:

$$\mu_{t+1} = \mu_t + \eta_{1t}$$
$$\gamma_{t+1} = \gamma_t + \eta_{2t}$$

- Obviously, this fits nicely the Kalman framework!

Interlude: The Kalman filter

- Kalman filter consist of two equations that model the time-varying hidden state $\mathbf{x}_t$ and the observations $\mathbf{y}_t$:

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{T}_t \mathbf{x}_t + \boldsymbol{\eta}_t \\ \mathbf{y}_t &= \mathbf{Z}_t \mathbf{x}_t + \boldsymbol{\epsilon}_t \end{aligned}$$

- The observation equation $\mathbf{y}_t = \mathbf{Z}_t \mathbf{x}_t + \boldsymbol{\epsilon}_t$ relates the observation $\mathbf{y}_t$ to the hidden state $\mathbf{x}_t$ as a linear relationship, where $\mathbf{Z}_t$ is the time-varying observation matrix and $\boldsymbol{\epsilon}_t$ is a zero-mean Gaussian error $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ with covariance matrix $\mathbf{R}$.
- The state transition equation $\mathbf{x}_{t+1} = \mathbf{T}_t \mathbf{x}_t + \boldsymbol{\eta}_t$ expresses the transition of the hidden state from $\mathbf{x}_t$ to $\mathbf{x}_{t+1}$ as a linear relationship, where $\mathbf{T}_t$ is the time-varying transition matrix and $\boldsymbol{\eta}_t$ is a zero-mean Gaussian error $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ with covariance matrix $\mathbf{Q}$.
- The Kalman filter is extremely versatile in modeling a variety of real-life processes.[9]

[9] J. Durbin and S. J. Koopman, *Time Series Analysis by State Space Methods*, 2nd Ed. Oxford University Press, 2012.

Kalman for pairs trading

- Kalman filter (state transition equation and observation equation):

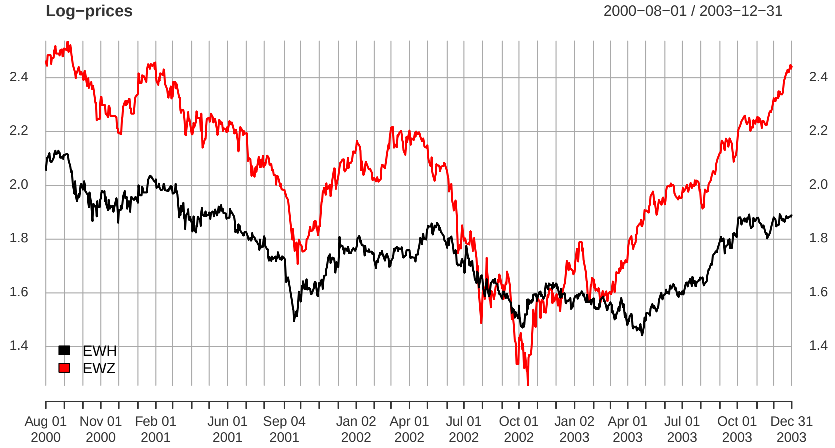
$$\mathbf{x}_{t+1} = \mathbf{T}\mathbf{x}_t + \boldsymbol{\eta}_t$$
$$y_{1t} = \mathbf{Z}_t\mathbf{x}_t + \epsilon_t$$

  where
  - $\mathbf{x}_t \triangleq \begin{bmatrix} \mu_t \\ \gamma_t \end{bmatrix}$ is the hidden state
  - $\mathbf{T} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the state transition matrix
  - $\boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ is the i.i.d. state transition noise with $\mathbf{Q} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$
  - $\mathbf{Z}_t \triangleq \begin{bmatrix} 1 & y_{2t} \end{bmatrix}$ is the observation coefficient matrix
  - $\epsilon_t \sim \mathcal{N}\left(0, \sigma_\epsilon^2\right)$ is the i.i.d. observation noise
- Note that this is a time-varying Kalman filter since $\mathbf{Z}_t$ is time-varying.

- Parameters σ_1^2, σ_2^2, σ_ϵ^2 can be estimated using the EM algorithm using historical data for calibration.

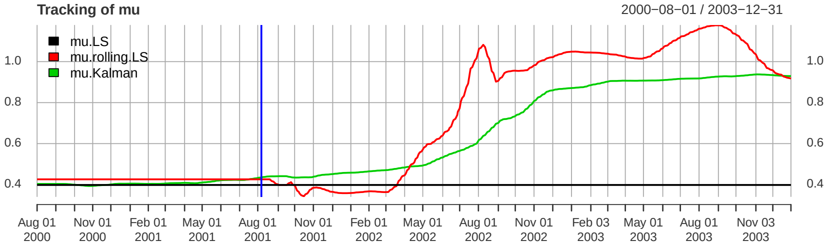
Kalman for pairs trading

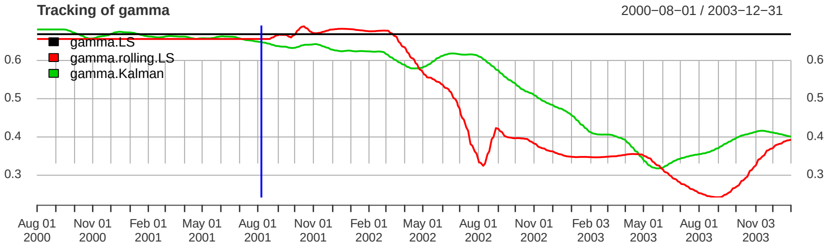
- Log-prices of ETFs EWH and EWZ:



Kalman for pairs trading

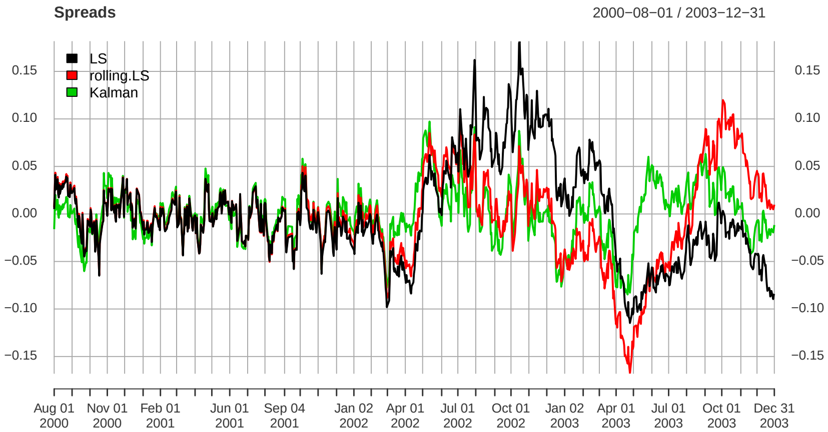
- Tracking of μ and γ by LS, rolling LS, and Kalman:





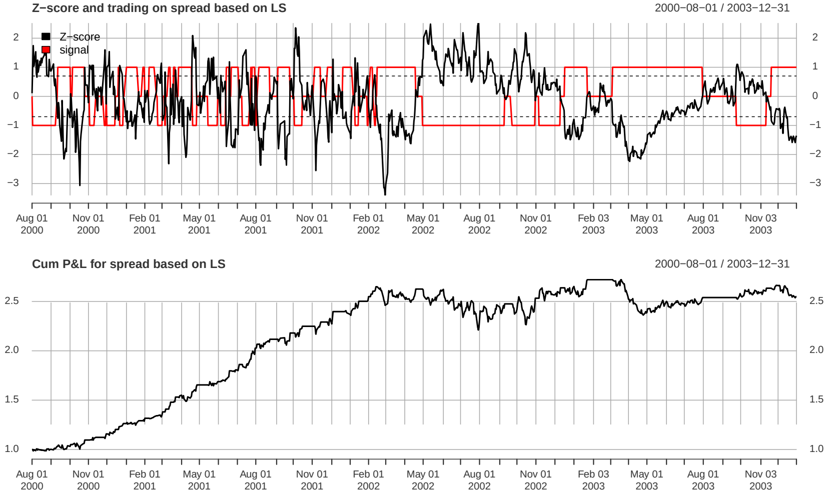
Kalman for pairs trading

- Spreads achieved by LS, rolling LS, and Kalman:



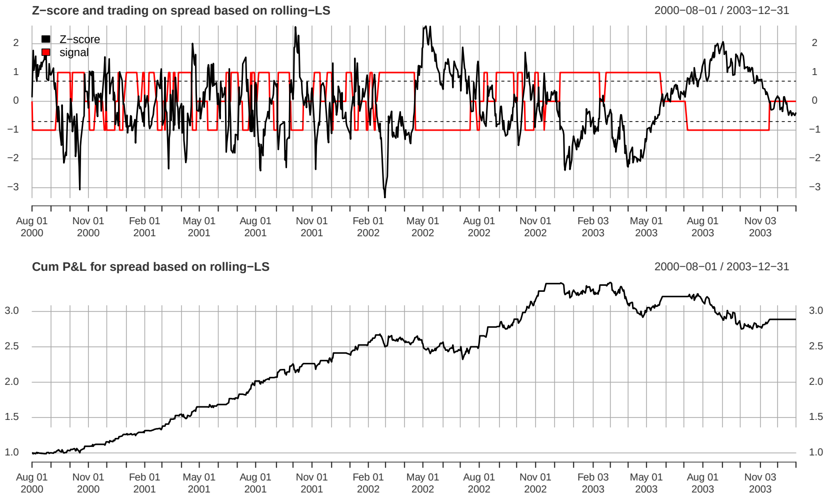
Kalman for pairs trading

- Trading of spread from LS:



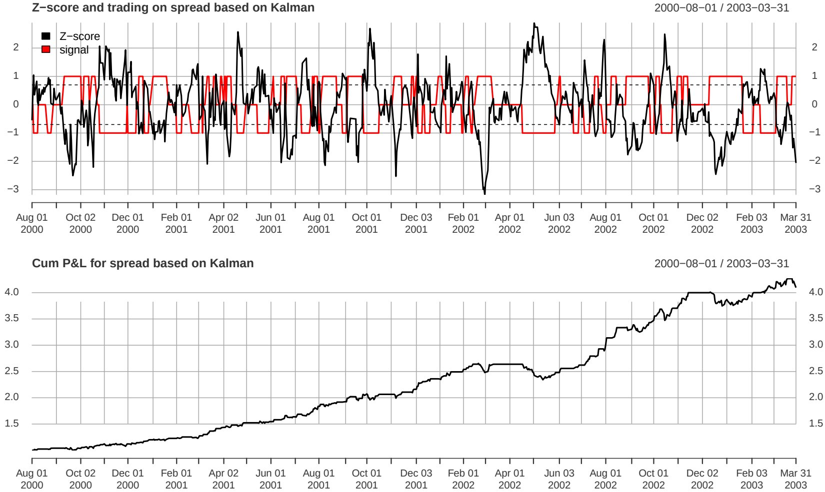
Kalman for pairs trading

- Trading of spread from rolling LS:



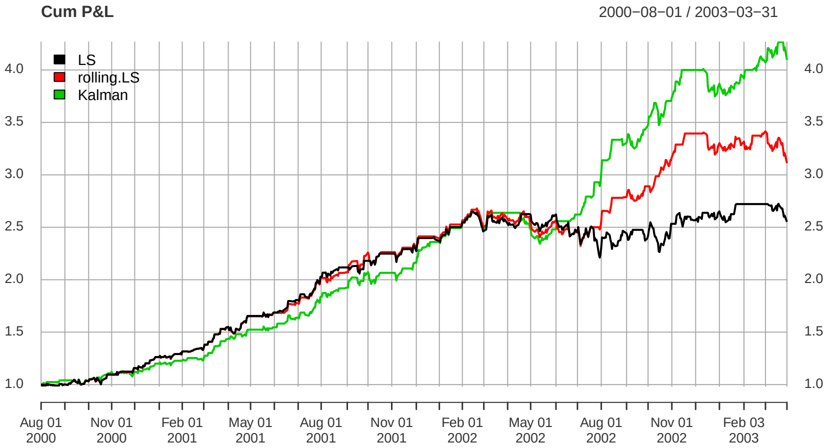
Kalman for pairs trading

- Trading of spread from Kalman:



Kalman for pairs trading

- Wealth comparison:



Kalman filter in finance

- The Kalman filter can and has been used in many aspects of financial time-series modeling as one could expect.[10]
- Examples of univariate time series: rate of inflation, national income, level of unemployment, etc.
- Typical models include: local model, trend-cycle decompositions, seasonality, etc.
- Examples of multivariate time series: inflation and national income.
- Multiple time series allows for more sophisticated models including common factors, cointegration, etc.
- Also data irregularities can be easily handled, e.g., missing observations, outliers, mixed frequencies.
- Plenty of applications for nonlinear and non-Gaussian models as well, e.g., GARCH modeling and stochastic volatility modeling.

[10] A. Harvey and S. J. Koopman, "Unobserved components models in economics and finance: The role of the Kalman filter in time series econometrics," *IEEE Control Systems Magazine*, vol. 29, no. 6, pp. 71–81, 2009.

Outline

From pairs trading to statistical arbitrage

- Pairs trading focuses on finding cointegration between two stocks.
- A more general idea is to extend this statistical arbitrage from two stocks to more stocks.

- The idea is still based on cointegration:

  Try to construct a linear combination of the log-prices of multiple (more than two) stocks such that it is a ***cointegrated mean-reversion*** *process.*

- In the case of two assets, the spread is $z_t = y_{1t} - \gamma y_{2t}$, which can be understood as a portfolio with weights: $\mathbf{w} = \begin{bmatrix} 1 \\ -\gamma \end{bmatrix}$.
- In the general case of many assets, one has to properly design the portfolio $\mathbf{w}$.

Outline

VECM

- Denote the log-prices of multiple stocks as $\mathbf{y}_t$ and the log-returns as $\mathbf{r}_t = \Delta\mathbf{y}_t = \mathbf{y}_t - \mathbf{y}_{t-1}$.
- Most of the multivariate time-series models attempt to model the log-returns $\mathbf{r}_t$ (because the log-prices are nonstationary whereas the log-returns are weakly stationary, at least over some time horizon).
- However, it turns out that differencing the log-prices may destroy part of the structure.
- The VECM[11] tries to fix that issue by including an additional term in the model:

$$\mathbf{r}_t = \phi_0 + \boldsymbol{\Pi}\mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \tilde{\boldsymbol{\Phi}}_i \mathbf{r}_{t-i} + \mathbf{w}_t,$$

where the term $\boldsymbol{\Pi}\mathbf{y}_{t-1}$ is called error correction term.

[11] R. F. Engle and C. W. J. Granger, "Co-integration and error correction: Representation, estimation, and testing," *Econometrica: Journal of the Econometric Society*, pp. 251–276, 1987.

VECM - Matrix $\mathbf{\Pi}$

- The matrix $\mathbf{\Pi}$ is of extreme importance.
- Notice that from the model $\mathbf{r}_t = \phi_0 + \mathbf{\Pi}\mathbf{y}_{t-1} + \sum_{i=1}^{p-1} \tilde{\mathbf{\Phi}}_i \mathbf{r}_{t-i} + \mathbf{w}_t$ one can conclude that $\mathbf{\Pi}\mathbf{y}_t$ must be stationary even though $\mathbf{y}_t$ is not!!!
- If that happens, it is said that $\mathbf{y}_t$ is cointegrated.
- There are three possibilities for $\mathbf{\Pi}$:
    - $\text{rank}(\mathbf{\Pi}) = 0$: This implies $\mathbf{\Pi} = \mathbf{0}$, thus $\mathbf{y}_t$ is not cointegrated (so no mystery here) and the VECM reduces to a VAR model on the log-returns.
    - $\text{rank}(\mathbf{\Pi}) = N$: This implies $\mathbf{\Pi}$ is invertible and thus $\mathbf{y}_t$ must be stationary already.
    - $0 < \text{rank}(\mathbf{\Pi}) < N$: This is the interestinc case and $\mathbf{\Pi}$ can be decomposed as $\mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}^T$ with $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^{N \times r}$ with full column rank. This means that $\mathbf{y}_t$ has r linearly independent cointegrated components, i.e., $\boldsymbol{\beta}^T\mathbf{y}_t$, each of which can be used for pairs trading.

Outline

Statistical arbitrage based on factor models

- Suppose the stock i is cointegrated with some tradable factors:

$$y_{it} = \boldsymbol{\pi}_i^T \mathbf{y}_t^f + w_{it}$$

  where

  - y_{it} is the log-price of the stock i,
  - $\mathbf{y}_t^f$ is the log-price of the tradable factors,
  - $\boldsymbol{\pi}_i$ is the vector of loading coefficients
  - w_{it} is a stationary mean-reversion process.

- It can also be written in a factor model form:

$$r_{it} = \boldsymbol{\pi}_i^T \mathbf{f}_t + \varepsilon_{it}$$

  where

  - $r_{it} = y_{it} - y_{i,t-1}$ is the log-return of stock i,
  - $\mathbf{f}_t = \mathbf{y}_t^f - \mathbf{y}_{t-1}^f$ is the log-returns of the tradable factors, and
  - $\varepsilon_{it} = w_{it} - w_{i,t-1}$ is the specific noise.

Statistical arbitrage based on factor models

- Recall the factor model form expression

$$r_{it} = \boldsymbol{\pi}_i^T \mathbf{f}_t + \varepsilon_{it}$$

- The idea now is to first properly select some tradable factors $\mathbf{f}_t$ and then test whether the cumulative summation of the resulted specific noise ε_{it}, i.e., $w_{it} = \sum_{j=0}^{t} \varepsilon_{ij}$, is stationary or not.
- If positive, then one can define a spread to be

$$\begin{aligned} z_{it} = w_{it} &= \sum_{j=0}^{t} \left(r_{ij} - \boldsymbol{\pi}_i^T \mathbf{f}_j \right) = \begin{bmatrix} 1 & -\boldsymbol{\pi}_i^T \end{bmatrix} \left(\sum_{j=0}^{t} \begin{bmatrix} r_{ij} \\ \mathbf{f}_j \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -\boldsymbol{\pi}_i^T \end{bmatrix} \begin{bmatrix} y_{it} \\ \mathbf{y}_t^f \end{bmatrix} \end{aligned}$$

Statistical arbitrage based on factor models

- Some tradable examples[12] of $\mathbf{f}_t$ are the log-returns of
  - (explicit factors) the sector ETFs and/or
  - (hidden factors) several largest eigen-portfolios[13]
- Again, for each constructed cointegration component, one can study the spread and find the optimal trading thresholds as before.

[12] M. Avellaneda and J.-H. Lee, "Statistical arbitrage in the US equities market," *Quantitative Finance*, vol. 10, no. 7, pp. 761–782, 2010.

[13] A eigen-portfolio is a portfolio whose weight is a eigenvector of the covariance matrix of the stock returns.

Outline

Mean-reverting portfolio (MRP)

- In the case of two assets, the spread is $z_t = y_{1t} - \gamma y_{2t}$, which can be understood as a portfolio with weights: $\mathbf{w} = \begin{bmatrix} 1 \\ -\gamma \end{bmatrix}$.
- In the general case of many assets, one has to properly design the portfolio $\mathbf{w}$.
- One interesting approach is based on a VECM modeling of the universe of stocks:
  - From the parameter β contained in the low-rank matrix $\mathbf{\Pi} = \alpha\beta^T$ one can simply use any column of β (even all of them)
  - Even better, β defines a cointegration subspace and we can then optimize the portfolio within that cointegration subspace.[14]

[14] Z. Zhao and D. P. Palomar, "Mean-reverting portfolio with budget constraint," *IEEE Trans. Signal Process.*, vol. 66, no. 9, pp. 2342–2357, 2018.

Mean-reverting portfolio (MRP)

- Consider the log-prices $\mathbf{y}_t$ and use β to extract several spreads $\mathbf{s}_t = \beta^T \mathbf{y}_t$.
- Let's now use a portfolio $\mathbf{w}$ to extract the best mean-reverting spread from $\mathbf{s}_t$ as $z_t = \mathbf{w}^T \mathbf{s}_t$.
- To design the the portfolio $\mathbf{w}$ we have two main objectives (recall that total profit equals: profit of each trade $\times$ number of trades):
  - we want large variance (profit of each trade): $\mathbf{w}^T \mathbf{M}_0 \mathbf{w}$, where $\mathbf{M}_i = E\left[(\mathbf{s}_t - E[\mathbf{s}_t])(\mathbf{s}_{t+i} - E[\mathbf{s}_{t+i}])^T\right]$
  - we want strong mean reversion (number of trades): many proxies exist like the Portmanteau statistics or crossing statistics.

Mean-reverting portfolio (MRP)

- For example, if we use the Portmanteau statistics as a proxy for the mean reversion, the problem formulation becomes:

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{minimize}} & \sum_{i=1}^{p}\left(\frac{\mathbf{w}^T\mathbf{M}_i\mathbf{w}}{\mathbf{w}^T\mathbf{M}_0\mathbf{w}}\right)^2\\ \text{subject to} & \mathbf{w}^T\mathbf{M}_0\mathbf{w}=\nu\\ & \mathbf{w}\in\mathcal{W}.\end{array}$$

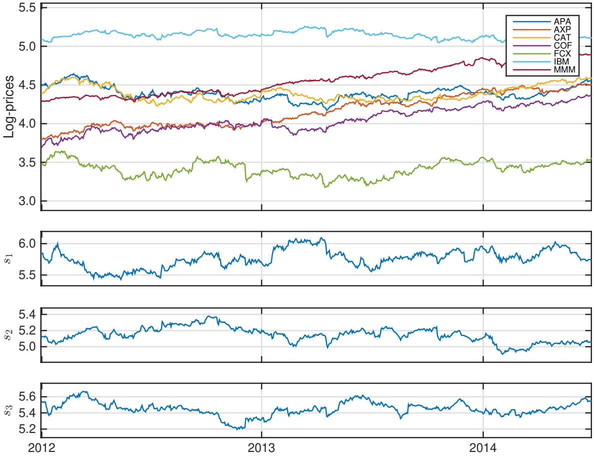
- Using other proxies, the formulation can be expressed more generally as[15]

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^T\mathbf{H}\mathbf{w}+\lambda\sum_{i=1}^{p}\left(\mathbf{w}^T\mathbf{M}_i\mathbf{w}\right)^2\\ \text{subject to} & \mathbf{w}^T\mathbf{M}_0\mathbf{w}=\nu\\ & \mathbf{w}\in\mathcal{W}.\end{array}$$

[15] Z. Zhao and D. P. Palomar, "Mean-reverting portfolio with budget constraint," *IEEE Trans. Signal Process.*, vol. 66, no. 9, pp. 2342–2357, 2018.

MRP in practice

- Observe several stock log-prices and the spreads obtained from β:



MRP in practice

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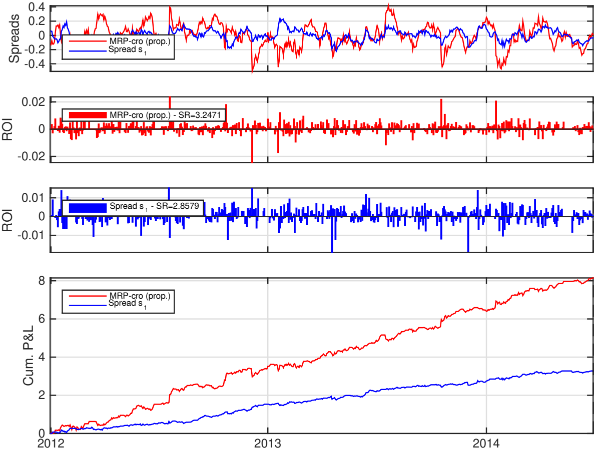

Outline

Summary

- First of all, we have discovered the concept of cointegration.
- We have learned the basic idea of pairs trading for cointegrated assets:
  - searching for a cointegrated spread is the first step
  - making sure that the chosen spread remains cointegrated is key (cointegrated tests)
  - obtaining the cointegration ratio γ and the entering and exiting thresholds are important details.
- We have learned of the use of Kalman (initially developed for tracking missiles) filtering for improved pairs trading.
- We have briefly explored the extension of pairs trading (for two stocks) to statistical arbitrage (for more than two stocks):
  - VECM modeling is an important multivariate time-series modeling tool
  - sophisticated portfolio designs on the cointegration subspace are possible.

Thanks

For more information visit:

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