Research Article

Upper Bounds for the Euclidean Operator Radius and Applications

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The main aim of the present paper is to establish various sharp upper bounds for the Euclidean operator radius of an n-tuple of bounded linear operators on a Hilbert space. The tools used are provided by several generalizations of Bessel inequality due to Boas-Bellman, Bombieri, and the author. Natural applications for the norm and the numerical radius of bounded linear operators on Hilbert spaces are also given.

1. Introduction

Following Popescu's work [1], we present here some basic properties of the *Euclidean operator radius* of an n-tuple of operators $(T_1, \ldots, T_n)$ that are defined on a Hilbert space $(H; \langle \cdot, \cdot \rangle)$. This radius is defined by

$$w_e(T_1, \ldots, T_n) := \sup_{\|h\|=1} \left(\sum_{i=1}^n |\langle T_i h, h \rangle|^2 \right)^{1/2}. \tag{1.1}$$

We can also consider the following norm and spectral radius on $B(H)^{(n)} := B(H) \times \cdots \times B(H)$, by setting [1]

$$\big\| (T_1, \ldots, T_n) \big\|_e := \sup_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n} \big\| \lambda_1 T_1 + \cdots + \lambda_n T_n \big\|,$$

$$r_e(T_1, \ldots, T_n) = \sup_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n} r(\lambda_1 T_1 + \cdots + \lambda_n T_n), \tag{1.2}$$

where $r(T)$ denotes the usual spectral radius of an operator $T \in B(H)$ and B_n is the closed unit ball in $\mathbb{C}^n$.

Notice that $\|\cdot\|_e$ is a norm on $B(H)^{(n)}$:

$$\|(T_1, \ldots, T_n)\|_e = \|(T_1^*, \ldots, T_n^*)\|_e, \qquad r_e(T_1, \ldots, T_n) = r_e(T_1^*, \ldots, T_n^*). \qquad (1.3)$$

Now, if we denote by $\|[T_1, \ldots, T_n]\|$ the square root of the norm $\|\sum_{i=1}^n T_i T_i^*\|$, that is,

$$\|[T_1, \ldots, T_n]\| := \left\| \sum_{i=1}^n T_i T_i^* \right\|^{1/2}, \qquad (1.4)$$

then we can present the following result due to Popescu [1] concerning some sharp inequalities between the norms $\|[T_1, \ldots, T_n]\|$ and $\|(T_1, \ldots, T_n)\|_e$.

Theorem 1.1 (see [1]). *If $(T_1, \ldots, T_n) \in B(H)^{(n)}$, then*

$$\frac{1}{\sqrt{n}} \|[T_1, \ldots, T_n]\| \leq \|(T_1, \ldots, T_n)\|_e \leq \|[T_1, \ldots, T_n]\|, \qquad (1.5)$$

where the constants $1/\sqrt{n}$ and 1 are best possible in (1.5).

Following [1], we list here some of the basic properties of the Euclidean operator radius of an n-tuple of operators $(T_1, \ldots, T_n) \in B(H)^{(n)}$.

(i) $w_e(T_1, \ldots, T_n) = 0$ if and only if $T_1 = \cdots = T_n = 0$;

(ii) $w_e(\lambda T_1, \ldots, \lambda T_n) = |\lambda| w_e(T_1, \ldots, T_n)$ for any $\lambda \in \mathbb{C}$;

(iii) $w_e(T_1 + T_1', \ldots, T_n + T_n') \leq w_e(T_1, \ldots, T_n) + w_e(T_1', \ldots, T_n')$;

(iv) $w_e(U^* T_1 U, \ldots, U^* T_n U) = w_e(T_1, \ldots, T_n)$ for any unitary operator $U : K \to H$;

(v) $w_e(X^* T_1 X, \ldots, X^* T_n X) \leq \|X\|^2 w_e(T_1, \ldots, T_n)$ for any operator $X : K \to H$;

(vi) $(1/2)\|(T_1, \ldots, T_n)\|_e \leq w_e(T_1, \ldots, T_n) \leq \|(T_1, \ldots, T_n)\|_e$;

(vii) $r_e(T_1, \ldots, T_n) \leq w_e(T_1, \ldots, T_n)$;

(viii) $w_e(I_\varepsilon \otimes T_1, \ldots, I_\varepsilon \otimes T_n) = w_e(T_1, \ldots, T_n)$ for any separable Hilbert space ε;

(ix) w_e is a continuous map in the norm topology;

(x) $w_e(T_1, \ldots, T_n) = \sup_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n} w(\lambda_1 T_1 + \cdots + \lambda_n T_n)$;

(xi) $(1/2\sqrt{n})\|[T_1, \ldots, T_n]\| \leq w_e(T_1, \ldots, T_n) \leq \|[T_1, \ldots, T_n]\|$ and the inequalities are sharp.

Due to the fact that the particular cases $n = 2$ and $n = 1$ are related to some classical and new results of interest which naturally motivate the research, we recall here some facts of significance for our further considerations.

For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well known that $w(\cdot)$ defines a norm on $B(H)$, and for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \le w(A) \le \|A\|. \tag{1.6}$$

For other results concerning the numerical range and radius of bounded linear operators on a Hilbert space, see [2, 3].

In [4], Kittaneh has improved (1.6) in the following manner:

$$\frac{1}{4}\|A^*A + AA^*\| \le w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|, \tag{1.7}$$

with the constants $1/4$ and $1/2$ as best possible.

Let (C, D) be a pair of bounded linear operators on H, the *Euclidean operator radius* is

$$w_e(C, D) := \sup_{\|x\|=1} \left(|\langle Cx, x\rangle|^2 + |\langle Dx, x\rangle|^2\right)^{1/2} \tag{1.8}$$

and, as pointed out in [1], $w_e : B^2(H) \to [0, \infty)$ is a norm and the following inequality holds:

$$\frac{\sqrt{2}}{4}\|C^*C + D^*D\|^{1/2} \le w_e(C, D) \le \|C^*C + D^*D\|^{1/2}, \tag{1.9}$$

where the constants $\sqrt{2}/4$ and 1 are best possible in (1.9).

We observe that, if C and D are self-adjoint operators, then (1.9) becomes

$$\frac{\sqrt{2}}{4}\|C^2 + D^2\|^{1/2} \le w_e(C, D) \le \|C^2 + D^2\|^{1/2}. \tag{1.10}$$

We observe also that if $A \in B(H)$ and $A = B + iC$ is the *Cartesian decomposition* of A, then

$$\begin{aligned}
w_e^2(B, C) &= \sup_{\|x\|=1}\left[|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right] \\
&= \sup_{\|x\|=1}|\langle Ax, x\rangle|^2 \\
&= w^2(A).
\end{aligned} \tag{1.11}$$

By the inequality (1.10) and since (see [4])

$$A^*A + AA^* = 2(B^2 + C^2), \tag{1.12}$$

then we have

$$\frac{1}{16}\left\|A^*A + AA^*\right\| \le w^2(A) \le \frac{1}{2}\left\|A^*A + AA^*\right\|. \tag{1.13}$$

We remark that the lower bound for $w^2(A)$ in (1.13) provided by Popescu's inequality (1.9) is not as good as the first inequality of Kittaneh from (1.7). However, the upper bounds for $w^2(A)$ are the same and have been proved using different arguments.

In order to get a natural generalization of Kittaneh's result for the Euclidean operator radius of two operators, we have obtained in [5] the following result.

Theorem 1.2. *Let* $B, C : H \rightarrow H$ *be two bounded linear operators on the Hilbert space* $(H; \langle \cdot, \cdot \rangle)$. *Then*

$$\frac{\sqrt{2}}{2}\left[w\left(B^2 + C^2\right)\right]^{1/2} \le w_e(B, C)\left(\le \left\|B^*B + C^*C\right\|^{1/2}\right). \tag{1.14}$$

The constant $\sqrt{2}/2$ *is best possible in the sense that it cannot be replaced by a larger constant.*

Corollary 1.3. *For any two self-adjoint bounded linear operators* B, C *on* H, *one has*

$$\frac{\sqrt{2}}{2}\left\|B^2 + C^2\right\|^{1/2} \le w_e(B, C)\left(\le \left\|B^2 + C^2\right\|^{1/2}\right). \tag{1.15}$$

The constant $\sqrt{2}/2$ *is sharp in* (1.15).

Remark 1.4. The inequality (1.15) is better than the first inequality in (1.10) which follows from Popescu's first inequality in (1.9). It also provides, for the case that B, C are the self-adjoint operators in the Cartesian decomposition of A, exactly the lower bound obtained by Kittaneh in (1.7) for the numerical radius $w(A)$.

For other inequalities involving the Euclidean operator radius of two operators and their applications for one operator, see the recent paper [5], where further references are given.

Motivated by the useful applications of the Euclidean operator radius concept in multivariable operator theory outlined in [1], we establish in this paper various new sharp upper bounds for the general case $n \ge 2$. The tools used are provided by several generalizations of Bessel inequality due to Boas-Bellman, Bombieri, and the author. Also several reverses of the Cauchy-Bunyakovsky-Schwarz inequalities are employed. The case $n = 2$, which is of special interest since it generates for the Cartesian decomposition of a bounded linear operator various interesting results for the norm and the usual numerical radius, is carefully analyzed.

2. Upper bounds via the Boas-Bellman-type inequalities

The following inequality that naturally generalizes Bessel's inequality for the case of nonorthonormal vectors $y_1, \ldots, y_n$ in an inner product space is known in the literature as the *Boas-Bellman inequality* (see [6, 7], or [8, chapter 4]):

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \left[\max_{1 \leq i \leq n} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle|^2 \right)^{1/2} \right], \tag{2.1}$$

for any $x \in H$.

   Obviously, if $\{y_1, \ldots, y_n\}$ is an orthonormal family, then (2.1) becomes the classical *Bessel's inequality*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2, \quad x \in H. \tag{2.2}$$

   The following result provides a natural upper bound for the Euclidean operator radius of n bounded linear operators.

Theorem 2.1. *If* $(T_1, \ldots, T_n) \in B(H)^{(n)}$, *then*

$$w_e(T_1, \ldots, T_n) \leq \left[\max_{1 \leq i \leq n} \|T_i\|^2 + \left\{ \sum_{1 \leq i \neq j \leq n} w^2(T_j^* T_i) \right\}^{1/2} \right]^{1/2}. \tag{2.3}$$

Proof. Utilizing the Boas-Bellman inequality for $x = h$, $\|h\| = 1$ and $y_i = T_i h$, $i = 1, \ldots, n$, we have

$$\sum_{i=1}^{n} |\langle T_i h, h \rangle|^2 \leq \max_{1 \leq i \leq n} \|T_i h\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle T_j^* T_i h, h \rangle|^2 \right)^{1/2}. \tag{2.4}$$

Taking the supremum over $\|h\| = 1$ and observing that

$$\sup_{\|h\|=1} \left[\max_{1 \leq i \leq n} \|T_i h\|^2 \right] = \max_{1 \leq i \leq n} \|T_i\|^2,$$

$$\sup_{\|h\|=1} \left(\sum_{1 \leq i \neq j \leq n} |\langle T_j^* T_i h, h \rangle|^2 \right)^{1/2} \leq \left[\sum_{1 \leq i \neq j \leq n} \sup_{\|h\|=1} |\langle T_j^* T_i h, h \rangle|^2 \right]^{1/2} \tag{2.5}$$

$$= \left[\sum_{1 \leq i \neq j \leq n} w^2(T_j^* T_i) \right]^{1/2},$$

then by (2.4) we deduce the desired inequality (2.3). $\qquad \square$

Remark 2.2. If $(T_1, \ldots, T_n) \in B(H)^{(n)}$ is such that $T_j^* T_i = 0$ for $i, j \in \{1, \ldots, n\}$, then from (2.3), we have the inequality:

$$w_e(T_1, \ldots, T_n) \leq \max_{1 \leq i \leq n} \|T_i\|. \tag{2.6}$$

We observe that a sufficient condition for $T_j^* T_i = 0$, with $i \neq j$, $i, j \in \{1, \ldots, n\}$ to hold, is that $\text{Range}(T_i) \perp \text{Range}(T_j)$ for $i, j \in \{1, \ldots, n\}$, with $i \neq j$.

Remark 2.3. If we apply the above result for two bounded linear operators on $H, B, C : H \to H$, then we get the simple inequality

$$w_e^2(B, C) \leq \max\left\{\|B\|^2, \|C\|^2\right\} + \sqrt{2} w(B^* C). \tag{2.7}$$

Remark 2.4. If $A : H \to H$ is a bounded linear operator on the Hilbert space H and if we denote by

$$B := \frac{A + A^*}{2}, \qquad C := \frac{A - A^*}{2i} \tag{2.8}$$

its Cartesian decomposition, then

$$w_e^2(B, C) = w^2(A),$$

$$w(B^* C) = w(C^* B) = \frac{1}{4} w\left[(A^* - A)(A + A^*)\right], \tag{2.9}$$

and from (2.7), we get the inequality

$$w^2(A) \leq \frac{1}{4}\left\{\max\left\{\|A + A^*\|^2, \|A - A^*\|^2\right\} + \sqrt{2} w\left[(A^* - A)(A + A^*)\right]\right\}. \tag{2.10}$$

In [9], the author has established the following Boas-Bellman type inequality for the vectors $x, y_1, \ldots, y_n$ in the real or complex inner product space $(H, \langle \cdot, \cdot \rangle)$:

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \left\{\max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle|\right\}. \tag{2.11}$$

For orthonormal vectors, (2.11) reduces to Bessel's inequality as well. It has also been shown in [9] that the Boas-Bellman inequality (2.1) and the inequality (2.11) cannot be compared in general, meaning that in some instances the right-hand side of (2.1) is smaller than that of (2.11) and vice versa.

Now, utilizing the inequality (2.11) and making use of the same argument from the proof of Theorem 2.1, we can state the following result as well.

Theorem 2.5. *If* $(T_1, \ldots, T_n) \in B(H)^{(n)}$, *then*

$$w_e(T_1, \ldots, T_n) \leq \left[\max_{1 \leq i \leq n} \|T_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} w(T_j^* T_i) \right]^{1/2}. \tag{2.12}$$

If in (2.12) one assumes that $T_j^* T_i = 0$ for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$, then one gets the result from (2.6).

Remark 2.6. We observe that, for $n = 2$, we get from (2.12) a better result than (2.7), namely,

$$w_e^2(B, C) \leq \max \left\{ \|B\|^2, \|C\|^2 \right\} + w(B^* C), \tag{2.13}$$

where B, C are arbitrary linear bounded operators on H. The inequality (2.13) is sharp. This follows from the fact that for $B = C = A \in B(H)$, A a normal operator, we have

$$w_e^2(A, A) = 2w^2(A) = 2\|A\|^2,$$
$$w(A^* A) = \|A\|^2, \tag{2.14}$$

and we obtain in (2.13) the same quantity in both sides. The inequality (2.13) has been obtained in [5, (12.23)] on utilizing a different argument.

Also, for the operator $A : H \to H$, we can obtain from (2.13) the following inequality:

$$w^2(A) \leq \frac{1}{4} \left\{ \max \left\{ \|A + A^*\|^2, \|A - A^*\|^2 \right\} + w \left[(A^* - A)(A + A^*) \right] \right\}, \tag{2.15}$$

which is better than (2.10). The constant $1/4$ in (2.15) is sharp. The case of equality in (2.15) follows, for instance, if A is assumed to be self-adjoint.

Remark 2.7. If in (2.13) we choose $C = A$, $B = A^*$, $A \in B(H)$, and take into account that

$$w_e^2(A^*, A) = 2w^2(A), \tag{2.16}$$

then we get the inequality

$$w^2(A) \leq \frac{1}{2} \left[\|A\|^2 + w(A^2) \right] \left(\leq \|A\|^2 \right), \tag{2.17}$$

for any $A \in B(H)$. The constant $1/2$ is sharp.

Note that this inequality has been obtained in [10] by the use of a different argument based on the Buzano inequality [11].

A different approach is incorporated in the following result.