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Stability Radius of Linear Dynamic Equations with Constant Coefficients on Time Scales

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Abstract. This paper considers the exponential stability and stability radius of time-invarying dynamic equations with respect to linear dynamic perturbations on time scales. A formula for the stability radius is given.

Keywords and phrases : time scales, exponential function, linear dynamic equation, exponentially stable, stability radius

1. Introduction

In the last decade, there have been extensive works on studying of robustness measures, where one of the most powerful ideas is the concept of the stability radii, introduced by Hinrichsen and Pritchard [1]. The stability radius is defined as the smallest (in norm) complex or real perturbations destabilizing the system. In [2], if $x^{'} = Ax$ is the nominal system they assume that the perturbed system can be represented in the form

$$x^{'} = (A + BDC)x, \tag{1}$$

where D is an unknown disturbance matrix and B, C are known scaling matrices defining the "structure" of the perturbation. The complex stability radius is given by

$$\left[\max_{t\in i} \|C(tI - A)^{-1}B\|\right]^{-1}. \tag{2}$$

If the nominal system is the difference equation $x_{n+1} = Ax_n$ in [3] they assume that the perturbed system can be represented in the form

$$x_{n+1} = (A + BDC)x_n. \tag{3}$$

Then, the complex stability radius is given by

$$\left[\max_{\omega\in\ :|\omega|=1} \|C(\omega I - A)^{-1}B\|\right]^{-1}. \tag{4}$$

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Earlier results for time-varying systems can be found, e.g., in [4, 5]. The most successful attempt for finding a formula of the stability radius was an elegant result given by Jacob [5]. Using this result, the notion and formula of the stability radius were extended to linear time-invariant differential-algebraic systems [6, 7]; and to linear time-varying differential and difference-algebraic systems [8, 9].

On the other hand, the theory of the analysis on time scales, which has been received a lot of attention, was introduced by Stefan Hilger in his Ph.D thesis in 1988 (supervised by Bernd Aulbach) [10] in order to unify the continuous and discrete analyses. By using the notation of the analysis on time scale, the equations (1) and (3) can be rewritten under the unified form

$$x^{\Delta} = (A + BDC)x, \tag{5}$$

where Δ is the differentiable operator on a time scale $\mathbb{T}$ (see the notions in the section).

Naturally, the question arises whether, by using the theory of analysis on time scale, we can express the formulas (2) and (4) in a unified form. The purpose of this paper is to answer this question.

The difficulty we are faced when dealing with this problem is that although A, B, C are constant matrices but the structure of a time scale is, perhaps, rather complicated and the system (5) in fact is an time-varying system. Moreover, so far there exist some concepts of the exponential stability which have not got a unification of point of view. In [11], author used the classical exponent function to define the asymptotical stability meanwhile the exponent function on time scale has been used in [12]. The first obtained result of this paper is to show that two these definitions are equivalent. To establish a unification formula for computing stability radii of the system (1) and (3) which is at the same time an extention to (5), we follow the way in [12] to define the so-called domain of the exponential stability of a time scale. By the definition of this domain, the problem of stability radius for the equation (5) deduces to one similar to the autonomous case where we know how to solve it as in [13].

This paper is organized as follows. In the section , we summarize some preliminary results on time scales. Section 3 gives a definition of the stability domain for a time scale and find out some its properties. The last section deals with the formula of the stability radius for (5).

2. Preliminaries

A time scale is a nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$. The most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that inherits from the standard topology of the real numbers. We define the *forward jump operator* and the *backward jump operator* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (supplemented by $\inf \emptyset = \sup \mathbb{T}$) and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (supplemented by $\sup \emptyset = \inf \mathbb{T}$). The *graininess* $\mu : \mathbb{T} \to \mathbb{R}^{+} \cup \{0\}$ is given by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$, and *isolated* if t is right-scattered and left-scattered. For every $a, b \in \mathbb{T}$, by $[a, b]$, we mean the set $\{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. For our purpose, we will assume that the time scale $\mathbb{T}$ is unbounded above, i.e., $\sup \mathbb{T} = \infty$. Let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* (or simply: *differentiable*) at $t \in \mathbb{T}$ provided there exists a number, namely $f^{\Delta}(t)$, such that for all $\epsilon > 0$ there is a neighborhood V around t with $|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leqslant \epsilon|\sigma(t) - s|$ for all $s \in$V. If f is differentiable for every $t \in \mathbb{T}$, then f is said to be *differentiable on* $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$ then delta derivative is $f'(t)$ from continuous calculus; if $\mathbb{T} = \mathbb{Z}$ then the delta derivative is the forward difference, Δf, from discrete calculus. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function f defined on $\mathbb{T}$ is *rd-continuous* if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. The set of all rd-continuous function from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. A function f from $\mathbb{T}$ to $\mathbb{R}$ is *regressive* (resp. *positively regressive*) if $1 + \mu(t)f(t) \neq 0$ (resp. $1 + \mu(t)f(t) > 0$) for every $t \in \mathbb{T}$. We denote $\mathcal{R}$ (resp. $\mathcal{R}^+$) the set of regressive functions (resp. positively regressive) from $\mathbb{T}$ to $\mathbb{R}$. The space of rd-continuous, regressive functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ and, $C_{rd}\mathcal{R}^+(\mathbb{T}, \mathbb{R}) := \{f \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}) : 1 + \mu(t)f(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}\}$. The *circle addition* $\oplus$ is defined by $(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)$. For $p \in \mathcal{R}$, the inverse element is given by $(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}$ and if we define *circle subtraction* $\ominus$ by $(p \ominus q)(t) = (p \oplus (\ominus q))(t)$ then $(p \ominus q)(t) = \frac{p(t)-q(t)}{1+\mu(t)q(t)}$.

Let $s \in \mathbb{T}$ and let $(A(t))_{t \geqslant s}$ be a $d \times d$ rd-continuous function. The initial value problem

$$x^\Delta = A(t)x, x(s) = x_0 \tag{6}$$

has a unique solution $x(t, s)$ defined on $t \geqslant s$. For any $s \in \mathbb{T}$, the unique matrix-valued solution, namely $\Phi_A(t, s)$, of the initial value problem $X^\Delta = A(t)X, X(s) = I$, is called the Cauchy operator of (6). It is seen that $\Phi_A(t, s) = \Phi_A(t, \tau)\Phi_A(\tau, s)$ for all $t \geqslant \tau \geqslant s$.

When $d = 1$, for any rd-continuous function $q(\cdot)$, the solution of the dynamic equation $x^\Delta = q(t)x$, with the initial condition $x(s) = 1$ defined a so-called exponential function (defined on the time scale $\mathbb{T}$ if $q(\cdot)$ is regressive; defined only $t \geqslant s$ if $q(\cdot)$ is non-regressive). We denote this exponential function by $e_q(t, s)$. We list some necessary properties that we will use later.

Theorem 2.1. *Assume $p, q : \mathbb{T} \to \mathbb{R}$ are rd-continuous, then the followings hold*

i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$,
ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,
iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$,
iv) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$,
v) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$ *if q is regressive,*
vi) If $p \in \mathcal{R}^+$ *then* $e_p(t, s) > 0$ *for all* $t, s \in \mathbb{T}$,
vii) $\int_a^b p(s)e_p(c, \sigma(s))\Delta s = e_p(c, a) - e_p(c, b)$ *for all* $a, b, c \in \mathbb{T}$,
viii) If $p \in \mathcal{R}^+$ *and* $p(t) \leqslant q(t)$ *for all* $t \geqslant s$ *then* $e_p(t, s) \leqslant e_q(t, s)$ *for all* $t \geqslant s$.

Proof. See [14], [15] and [16].

The following relation is called the constant variation formula.

Theorem 2.2. [See [17], Definition in 5.2 and Theorem 6.4] *If the right-hand side of two equations $x^\Delta = A(t)x$ and $x^\Delta = A(t)x + f(t, x)$ is rd-continuous, then the solution of the initial value problem $x^\Delta = A(t)x + f(t, x), x(t_0) = x_0$ is given by*

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))f(s, x(s))\Delta s, \quad t > t_0.$$

Lemma 2.3. [Gronwall's Inequality]. *Let* $u, a, b \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$, $b(t) \geqslant 0$ *for all* $t \in \mathbb{T}$. *The inequality*

$$u(t) \leqslant a(t) + \int_{t_0}^{t} b(s)u(s)\,\Delta s \ \text{ for all } t \geqslant t_0$$

implies

$$u(t) \leqslant a(t) + \int_{t_0}^{t} a(s)b(s)e_b(t, \sigma(s))\,\Delta s \ \text{ for all } t \geqslant t_0.$$

Corollary 2.4.

1. If $u \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$, $b(t) \equiv L \geqslant 0$ and $u(t) \leqslant a(t) + L\int_{t_0}^{t} u(s)\,\Delta s$ for all $t \geqslant t_0$ implies $u(t) \leqslant a(t) + L\int_{t_0}^{t} e_L(t, \sigma(s))a(s)\,\Delta s$ for all $t \geqslant t_0$.
2. If $u, b \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$, $b(t) \geqslant 0$ for all $t \in \mathbb{T}$ and $u(t) \leqslant u_0 + \int_{t_0}^{t} b(s)u(s)\,\Delta s$ for all $t \geqslant t_0$ then $u(t) \leqslant u_0 e_b(t, t_0)$ for all $t \geqslant t_0$.

To prove the Gronwall's inequality and corollaries, we can find in [14]. For more information on the analysis on time scales, we can refer to [17, 18, 19, 20] .

3. Exponential Stability of Dynamic Equations on Time Scales

Denote $\mathbb{T}^+ = [t_0, \infty) \cap \mathbb{T}$. We consider the dynamic equation on the time scale $\mathbb{T}$

$$x^{\Delta} = f(t, x), \tag{7}$$

where $f : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$ to be a continuous function and $f(t, 0) = 0$.

For the existence, uniqueness and extendibility of solution of initial value problem (7) we can refer to [15]. On exponential stability of dynamic equations on time scales, we often use one of two following definitions.

Let $x(t) = x(t, \tau, x_0)$ be a solution of (7) with the initial condition $x(\tau) = x_0, \tau \geqslant t_0$, where $x_0 \in \mathbb{R}^d$.

Definition 3.1. [See S. Hilger [10, 17], J. J. DaCunha [11], ...] The solution $x \equiv 0$ of the dynamic equation (7) is said to be exponentially stable if there exists a positive constant α with $-\alpha \in \mathcal{R}^+$ such that for every $\tau \in \mathbb{T}^+$ there exists a $N = N(\tau) \geqslant 1$, the solution of (7) with the initial condition $x(\tau) = x_0$ satisfies $\|x(t; \tau, x_0)\| \leqslant N\|x_0\|e_{-\alpha}(t, \tau)$, for all $t \geqslant \tau, t \in \mathbb{T}^+$.

Definition 3.2. [See C. Pötzsche, S. Siegmund, F. Wirth [12],...] The solution $x \equiv 0$ of (7) is called exponentially stable if there exists a constant $\alpha > 0$ such that for every $\tau \in \mathbb{T}^+$ there exists a $N = N(\tau) \geqslant 1$, the solution of (7) with the initial condition $x(\tau) = x_0$ satisfies $\|x(t; \tau, x_0)\| \leqslant N\|x_0\|e^{-\alpha(t-\tau)}$, for all $t \geqslant \tau, t \in \mathbb{T}^+$.

If the constant N can be chosen independent from $\tau \in \mathbb{T}^+$ then the solution $x \equiv 0$ of (7) is called uniformly exponentially stable.

Note that when applying Definition , the condition $-\alpha \in \mathcal{R}^+$ is equivalent to $\mu(t) \leqslant \frac{1}{\alpha}$. This means that we are working on time scales with bounded graininess.

Beside these definitions, we can find other exponentially stable definitions in [21] and [22].

Theorem 3.3. *Two definitions and are equivalent on time scales with bounded graininess.*

Proof. If $-\alpha \in \mathcal{R}^+, t \geqslant \tau$ then $e_{-\alpha}(t,\tau) = \exp\left\{\int\limits_{\tau}^{t} \lim\limits_{u \searrow \mu(s)} \frac{\ln|1-\alpha u|}{u} \Delta s\right\}$ where

$$\lim_{u \searrow \mu(s)} \frac{\ln|1-\alpha u|}{u} = \begin{cases} -\alpha & \text{if } \mu(s) = 0, \\ \frac{\ln(1-\alpha\mu(s))}{\mu(s)} & \text{if } \mu(s) > 0. \end{cases}$$

So

$$\lim_{u \searrow \mu(s)} \frac{\ln|1-\alpha u|}{u} \leqslant -\alpha, \text{ for all } s \in \mathbb{T}.$$

Therefore, $e_{-\alpha}(t,\tau) \leqslant e^{-\alpha(t-\tau)}$ for all $\alpha > 0, -\alpha \in \mathcal{R}^+$ and $t \geqslant \tau$. Hence, the stability due to Definition implies the one due to Definition .

Conversely, with $\alpha > 0$ we put

$$\bar{\alpha}(t) = \lim_{s \searrow \mu(t)} \frac{e^{-\alpha s} - 1}{s} = \begin{cases} -\alpha & \text{if } \mu(t) = 0, \\ \frac{e^{-\alpha\mu(t)}-1}{\mu(t)} & \text{if } \mu(t) > 0. \end{cases}$$

It is obvious that $\bar{\alpha}(\cdot) \in \mathcal{R}^+$ and $e_{\bar{\alpha}(\cdot)}(t,\tau) = e^{-\alpha(t-\tau)}$. Let $M := \sup_{t\in\ +}\mu(t)$. If $M = 0$, i.e., $\mu(t) = 0$ for all $t \in \mathbb{T}$, then $\bar{\alpha}(t) \equiv -\alpha$. When $M > 0$ we consider the function $y = \frac{e^{-\alpha u}-1}{u}$ with $0 < u \leqslant M$. It is easy to see that this function is increasing. In both two cases we have $\bar{\alpha}(t) \leqslant \beta := \lim\limits_{s \searrow M} \frac{e^{-\alpha s}-1}{s}$ for all $t \in \mathbb{T}^+$.

Therefore, $e_{\bar{\alpha}(\cdot)}(t,\tau) = e^{-\alpha(t-\tau)} \leqslant e_{\beta}(t,\tau)$, for all $t \geqslant \tau$. By noting that $-\beta > 0$ and $\beta \in \mathcal{R}^+$ we conclude that Definition implies Definition . The proof is complete.

By virtue of Theorem , in this paper we shall use only Definition to consider the exponential stability.

We now consider the condition of exponential stability for linear time-invariant equations

$$x^{\Delta} = Ax, \tag{8}$$

where $A \in \mathbb{K}^{d\times d}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We denote $\sigma(A) = \{\lambda \in \mathbb{C}, \lambda \text{ is an eigenvalue of } A\}$.

Theorem 3.4. *The trivial solution $x \equiv 0$ of the equation (8) is uniformly exponentially stable if and only if for every $\lambda \in \sigma(A)$, the scalar equation $x^{\Delta} = \lambda x$ is uniformly exponentially stable.*

Proof.

" $\Longrightarrow$ " Assume that the trivial solution $x \equiv 0$ of the equation (8) is uniformly exponentially stable and $\lambda \in \sigma(A)$ with its corresponding eigenvector $v \in \mathbb{C}^d \setminus \{0\}$. It is easy to see that $e_{\lambda}(t,\tau)v$ is a solution of the equation (8). Therefore, there are $N \geqslant 1$ and $\alpha > 0, -\alpha \in \mathcal{R}^+$ such that $|e_{\lambda}(t,\tau)v| \leqslant Ne_{-\alpha}(t,\tau)\|v\|, t \geqslant \tau$. Hence, $|e_{\lambda}(t,\tau)| \leqslant Ne_{-\alpha}(t,\tau), t \geqslant \tau$.

" $\Longleftarrow$ " Let $(\Phi_A(t,\tau))_{t\ \tau}$ be the Cauchy operator of the equation (8). We consider the Jordan form of the matrix A

$$S^{-1}AS = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_n \end{pmatrix},$$

where $J_i \in \mathbb{C}^{d_i \times d_i}$ is a Jordan block

$$J_i := \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ & \lambda_i & 1 & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda_i \end{pmatrix},$$

and $\lambda_i \in \sigma(A), d_1 + d_2 + ... + d_n = d, 1 \leqslant i \leqslant n \leqslant d.$

Since

$$\Phi_A(t,\tau) = S \begin{pmatrix} \Phi_{J_1}(t,\tau) & & \\ & \ddots & \\ & & \Phi_{J_n}(t,\tau) \end{pmatrix} S^{-1},$$

it suffices to prove the reverse relation with

$$A = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ & \lambda & 1 & \cdots & 0 \\ & & \ddots & & \vdots \\ 0 & & & & \lambda \end{pmatrix},$$

where the equation $x^\Delta = \lambda x$ is uniformly exponentially stable. Let $x = (xx_1, x_2, ..., x_d)$. The equation $x^\Delta = Ax$ can be rewritten as follows

$$\begin{cases} x_1^\Delta = \lambda x_1 + x_2 \\ x_2^\Delta = \lambda x_2 + x_3 \\ \cdots\cdots\cdots\cdots \\ x_d^\Delta = \lambda x_d, \end{cases} \tag{9}$$

with the initial conditions $x_k(\tau) = x_k^0, k = 1, \cdots, d$. The assumption that the equation $x^\Delta = \lambda x$ is uniformly exponentially stable implies $|e_\lambda(t,\tau)| \leqslant N e_{-\alpha}(t,\tau)$, with $N, \alpha > 0, -\alpha \in \mathcal{R}^+$ and $t \geqslant \tau$. The last equation of (9) gives $x_d = e_\lambda(t,\tau)x_d^0$. So

$$|x_d(t)| = |e_\lambda(t,\tau)x_d^0| \leqslant N|x_d^0|e_{-\alpha}(t,\tau) \leqslant N\|x_0\|e_{-\alpha}(t,\tau), \quad \text{for all } t \geqslant \tau.$$

By the constant variation formula, we have the representation,

$$x_{d-1}(t) = e_\lambda(t,\tau)x_{d-1}^0 + \int_\tau^t e_\lambda(t,\sigma(s))e_\lambda(s,\tau)x_d^0 \Delta s$$

Therefore,

$$\begin{aligned} |x_{d-1}(t)| &\leqslant N e_{-\alpha}(t,\tau)|x_{d-1}^0| + \int_\tau^t N^2 e_{-\alpha}(t,\sigma(s))e_{-\alpha}(s,\tau)|x_d^0|\Delta s \\ &\leqslant N|x_{d-1}^0|e_{-\alpha}(t,\tau) + N^2|x_d^0| \int_\tau^t e_{-\frac{2\alpha}{3}}(t,\sigma(s))e_{-\frac{2\alpha}{3}}(s,\tau)\Delta s \\ &= N|x_{d-1}^0|e_{-\alpha}(t,\tau) + N^2|x_d^0| \int_\tau^t \frac{1}{(1-\frac{2\alpha}{3}\mu(s))} e_{-\frac{2\alpha}{3}}(t,s)e_{-\frac{2\alpha}{3}}(s,\tau)\Delta s \\ &\leqslant N|x_{d-1}^0|e_{-\frac{\alpha}{3}}(t,\tau) + N^2|x_d^0|e_{-\frac{2\alpha}{3}}(t,\tau) \int_\tau^t \frac{\Delta s}{1-\frac{2\alpha}{3}\mu(s)}. \end{aligned}$$

Since $-\alpha \in \mathcal{R}^+$, we have $1-\alpha\mu(s) > 0$ which is equivalent to $1-\frac{2\alpha}{3}\mu(s) > \frac{1}{3}$ for all $s \in \mathbb{T}$. Hence,

$$|x_{d-1}(t)| \leqslant N|x^0_{d-1}|e_{-\frac{\alpha}{3}}(t,\tau) + 3N^2|x^0_d|(t-\tau)e_{-\frac{2\alpha}{3}}(t,\tau).$$

Further, from the relation $(-\frac{\alpha}{3})\oplus(-\frac{\alpha}{3})(t) = -\frac{2\alpha}{3}+(-\frac{2\alpha}{3})^2\mu(t) \geqslant -\frac{2\alpha}{3}$, it follows that $e_{-\frac{\alpha}{3}}(t,\tau).e_{-\frac{\alpha}{3}}(t,\tau)$ $= e_{(-\frac{\alpha}{3})\oplus(-\frac{\alpha}{3})}(t,\tau) \geqslant e_{-\frac{2\alpha}{3}}(t,\tau)$. On the other hand, $e_{-\frac{\alpha}{3}}(t,\tau) \leqslant \exp(-\frac{\alpha(t-\tau)}{3})$ for any $t > \tau$. Therefore, $(t-\tau)e_{-\frac{\alpha}{3}}(t,\tau) \leqslant (t-\tau)\exp(-\frac{\alpha(t-\tau)}{3}) \leqslant \frac{3\exp(-1)}{\alpha}$. Thus,

$$|x_{d-1}(t)| \leqslant K^1\|x_0\|e_{-\frac{\alpha}{3}}(t,\tau),$$

where $K^1 = N + \frac{3N^2\exp(-1)}{\alpha}$.

Continuing this way, we can find $K > 0$ and $\beta > 0$ with $\beta \in \mathcal{R}^+$ such that

$$\|x\| \leqslant K\|x_0\|e_{-\beta}(t,\tau), \quad \text{for all } t \geqslant \tau.$$

The theorem is proved.

Remark 3.5. It is easy to give an example where on the time scale $\mathbb{T}$, the scalar dynamic equation $x^\Delta = \lambda x$ is exponentially stable but it is not exponentially uniformly stale. Indeed, denote $((a,b)) = \{n \in \mathbb{N} : a < n < b\}$. Consider the time scale

$$\mathbb{T} = \bigcup_n [2^{2n}, 2^{2n+1}] \bigcup_n ((2^{2n+1}, 2^{2n+2})).$$

Let $\lambda = -2$ and $\tau \in \mathbb{T}$, says $2^m \leqslant \tau < 2^{m+1}$. We can choose $\alpha = -1$ and $N = 2^{m+1}$ to obtain $|e_\lambda(t,\tau)| \leqslant Ne_{-1}(t,\tau)$. However, we can not choose N to be independent from τ.

4. The domain of exponential stability of a time scale

We denote

$$S = \{\lambda \in \mathbb{C}, \text{ the scalar equation } x^\Delta = \lambda x \text{ is uniformly exponentially stable}\}.$$

The set S is called the domain of exponential stability of the time scale $\mathbb{T}$. By the definition, if $\lambda \in S$, there exist $\alpha > 0, -\alpha \in \mathcal{R}^+$ and $N \geqslant 1$ such that $|e_\lambda(t,\tau)| \leqslant Ne_{-\alpha}(t,\tau)$ for all $t \geqslant \tau$.

Theorem 4.1. *S is an open set in $\mathbb{C}$.*

Proof.

Let $\lambda \in S$. There are $\alpha > 0, -\alpha \in \mathcal{R}^+$ and $N \geqslant 1$ such that $|e_\lambda(t,\tau)| \leqslant Ne_{-\alpha}(t,\tau)$ for all $t \geqslant \tau$ and assume that $\mu \in \mathbb{C}, |\mu - \lambda| < \epsilon$, where $0 < \epsilon < \frac{\alpha}{N}$. We consider the equation $x^\Delta = \mu x = \lambda x + (\mu-\lambda)x$ with the initial condition $x(\tau) = x_0$.

By the formula of constant variation, we obtain

$$x(t) = e_\lambda(t,\tau)x_0 + \int_\tau^t e_\lambda(t,\sigma(s))(\mu-\lambda)x(s)\Delta s.$$

This implies

$$\begin{aligned}|x(t)| &\leqslant N|x_0|e_{-\alpha}(t,\tau) + \int_\tau^t N\epsilon e_{-\alpha}(t,\sigma(s))|x(s)|\Delta s\\ &= N|x_0|e_{-\alpha}(t,\tau) + \int_\tau^t \frac{N\epsilon}{1-\alpha\mu(s)}e_{-\alpha}(t,s)|x(s)|\Delta s,\end{aligned}$$