Homework - 1

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1 Exercise 1

(a) Factor these two matrices into $A = XAX^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

We have that: $\det(\lambda I - A) = 0$
$= \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3$
So there are 2 roots: $\lambda = 1$ or $\lambda = 3$.
For these numbers, the matrix becomes singular (non-determinant). We have
$(A - \lambda \mathbf{I})x_1 = 0$ is $Ax_1 = x_1 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$(A - \lambda \mathbf{I})x_2 = 0$ is $Ax_2 = 3x_2 \Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
We have x_1 and x_2 as eigenvectors $\Rightarrow$ matrix X $= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

To calculate $A = X\Lambda X^{-1}$, we first need to compute X^{-1}, the inverse of matrix X. In this case, X is invertible, so we can calculate X^{-1}.
Let's calculate X^{-1}:

$$X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Now, we can substitute the values into the equation $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Now we can calculate $A^3 = X\Lambda^3 X^{-1} =$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 1 & 27 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$$

Similarly, we can calculate the diagonalized matrix for:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

there are 2 eigenvalues: $\lambda = 0$ or $\lambda = 4$.
The diagonalize matrix is:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/4 & -1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

2 Exercise 3

We can replace the fact that $A = X\Lambda X^{-1}$ and $I = XX^{-1}$ into $A + 2I$, we will have:

$$A + 2I = X\Lambda X^{-1} + 2XX^{-1} = X(\Lambda + 2I)X^{-1}$$

So, the eigenvalue matrix is $\Lambda + 2I$, the eigenvector matrix is X.

3 Exercise 4

(a) False. Even if all eigenvectors of A are linearly independent, some still can correspond to eigenvalue $\lambda = 0$. Because the determinant of the matrix A is the product of its eigenvalues, we have $\det(A) = 0$, which implies that A cannot be invertible.
(b) True. Because we have n independent eigenvectors that can form a basis $\mathbb{R}^n$ for matrix $A \in \mathbb{R}^{n\times n}$, it follows that A is diagonalizable.
(c) True. Because all the columns of X are independent, X is a full-rank matrix, which implies that X can be inverted.
(d) False. Not enough to draw a conclusion, because there are invertible matrices can not be diagonalize.

4 Exercise 6

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$$

$\det(\lambda I - A) = 0$
$\Rightarrow \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} \lambda - 4 & 0 \\ 0 & \lambda - 2 \end{bmatrix} = 0$. So there are two roots: $\lambda = 2$ or $\lambda = 4$.

Find eigenvectors for $\lambda = 4$. Find all vectors $X \neq 0$ such that $AX = 4X$.
We have: $(4I - A)X = 0 \Rightarrow X = \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0 \Rightarrow x = \begin{bmatrix} 2i \\ i \end{bmatrix}$

Find eigenvectors for $\lambda = 2$. Find all vectors $X \neq 0$ such that $AX = 2X$.
We have: $(2I - A)X = 0 \Rightarrow X = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0 \Rightarrow x = \begin{bmatrix} i \\ 0 \end{bmatrix}$

$\Rightarrow S = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Having that $A = X \cap X^{-1} = X^{-1} \cap (X^{-1})^{-1}$.
We can conclude that all the matrices are diagonal. A^{-1} are the inverse matrices X^{-1} of matrix X.

5 Exercise 12

(a) False. Multiples of $(1, 4)$ eigenvector could respond to a nonzero eigenvalue $\Rightarrow \det(A) \neq 0 \Rightarrow A$ is invertible.
(b) True. If not, we would have distinct (independent) eigenvectors.
(c) True, since there are not enough independent eigenvectors $\Rightarrow$ eigenvector matrix X is not invertible $\Rightarrow A$ is not diagonalizable.