Homework - 1

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Exercise 1

Assuming we have a generic 2x2 matrix as below.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

To find the eigenvalues of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we solve the characteristic equation obtained from the determinant of $A - \lambda I$, where I is the identity matrix:

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (\lambda^2 - (a + d)\lambda + (ad - bc))$$

So, finding the eigenvalues is equivalent to solving the following equation: $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$.

Once we have the eigenvalues λ_1, λ_2, we can find the corresponding eigenvectors x_1, x_n by solving the system of linear equations $(A - \lambda_i I)x_i = 0$ for each i:

$$(A - \lambda_i I)x_i = \begin{bmatrix} a - \lambda_i & b \\ c & d - \lambda_i \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If c is not zero, and the eigenvalues are λ_1 and λ_2, the corresponding eigenvectors can be expressed as:

For λ_1:

$$x_1 = \begin{bmatrix} \lambda_1 - d \\ c \end{bmatrix}$$

For λ_2:

$$x_2 = \begin{bmatrix} \lambda_2 - d \\ c \end{bmatrix}$$

If b is not zero, and the eigenvalues are λ_1 and λ_2, the corresponding eigenvectors can be expressed as:

For λ_1:

$$x_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix}$$

For λ_2:

$$x_2 = \begin{bmatrix} b \\ \lambda_2 - a \end{bmatrix}$$

If both b and c are zero, and the eigenvalues are λ_1 and λ_2, the corresponding eigenvectors are:

For λ_1:

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For λ_2:

$$x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Finally, the formula for an inverse matrix is as follows:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Applying the formulas above.

- For $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

Since b is not zero so $X = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$X^{-1} = \frac{1}{(1)(1) - (1)(0)} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

- For $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$, $\lambda^2 - 4\lambda = 0$

$$\lambda_1 = 0$$
$$\lambda_2 = 4$$

Since b is not zero so $X = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$$X^{-1} = \frac{1}{(1)(3) - (-1)(1)} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

$$A^3 = X\Lambda^3X^{-1}$$

$$A^{-1} = X\Lambda^{-1}X^{-1}$$

, because

$$A^{-1}A = X\Lambda^{-1}X^{-1}X\Lambda X^{-1} = X\Lambda^{-1}I\Lambda X^{-1} = XIX^{-1} = I$$

Exercise 3

We can replace the fact that $A = X\Lambda X^{-1}$ and $I = XX^{-1}$ into $A + 2I$, we will have:

$$A + 2I = X\Lambda X^{-1} + 2XX^{-1} = X(\Lambda + 2I)X^{-1}$$

So, the eigenvalue matrix is $\Lambda + 2I$, the eigenvector matrix is X.

Exercise 4

(a) False. Even if all eigenvectors of A are linearly independent, some still can correspond to eigenvalue $\lambda = 0$. Because the determinant of the matrix A is the product of its eigenvalues, we have $\det(A) = 0$, which implies that A cannot be invertible.
(b) True. Because we have n independent eigenvectors that can form a basis $\mathbb{R}^n$ for matrix $A \in \mathbb{R}^{n\times n}$, it follows that A is diagonalizable.
(c) True. Because all the columns of X are independent, X is a full-rank matrix, which implies that X can be inverted.
(d) False. Not enough to draw a conclusion, because there are invertible matrices can not be diagonalize.

Exercise 6

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$$

$\det(\lambda I - A) = 0$
$\Rightarrow \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} \lambda - 4 & 0 \\ 0 & \lambda - 2 \end{bmatrix} = 0$. So there are two roots: $\lambda = 2$ or $\lambda = 4$.

Find eigenvectors for $\lambda = 4$. Find all vectors $X \neq 0$ such that $AX = 4X$.
We have: $(4I - A)X = 0 \Rightarrow X = \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0 \Rightarrow x = \begin{bmatrix} 2i \\ i \end{bmatrix}$

Find eigenvectors for $\lambda = 2$. Find all vectors $X \neq 0$ such that $AX = 2X$.
We have: $(2I - A)X = 0 \Rightarrow X = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0 \Rightarrow x = \begin{bmatrix} i \\ 0 \end{bmatrix}$

$\Rightarrow S = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Having that $A = X \cap X^{-1} = X^{-1} \cap (X^{-1})^{-1}$.
We can conclude that all the matrices are diagonal. A^{-1} are the inverse matrices X^{-1} of matrix X.

Exercise 11

(a) True. $\det(A) = \lambda_1\lambda_2\lambda_3 = 2 \times 2 \times 5 \neq 0 \Rightarrow A$ can be invertible.
(b) False. Because we have 2 eigenvalues with the same value, we lack independent eigenvectors to form a basis $\mathbb{R}^n$.
(c) True.

Exercise 12

(a) False. Multiples of $(1, 4)$ eigenvector could respond to a nonzero eigenvalue $\Rightarrow \det(A) \neq 0 \Rightarrow A$ is invertible.
(b) True. If not, we would have distinct (independent) eigenvectors.
(c) True, since there are not enough independent eigenvectors $\Rightarrow$ eigenvector matrix X is not invertible $\Rightarrow A$ is not diagonalizable.

Exercise 13

With $A = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$, we need to find x and y to fill in the matrix $A = \begin{bmatrix} 8 & x \\ y & 2 \end{bmatrix}$.

Because we know that A has a repeated eigenvalue $\lambda = 5$, we can calculate $\det(A - \lambda I) = (\lambda - 5)^2$. This yields the equation $(8-\lambda) \times (2-\lambda) - x \times y = (\lambda - 5)^2$.
$\rightleftharpoons x \times y = -9 \rightleftharpoons (x = -3$ and $y = 3) \vee (x = 3$ and $y = -3) \vee (x = 9$ and $y = 0) \vee (x = -9$ and $y = 1)$
$\Rightarrow A = \begin{bmatrix} 8 & -3 \\ 3 & 2 \end{bmatrix}$ or $\begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ or $\begin{bmatrix} 8 & 9 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 8 & -9 \\ 1 & 2 \end{bmatrix}$.
With $A = \begin{bmatrix} 8 & -3 \\ 3 & 2 \end{bmatrix}$, we find the corresponding eigenvectors for $\lambda = 5$.
$\Rightarrow \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\Rightarrow k_1 = k_2$, $x = \begin{bmatrix} c \\ c \end{bmatrix}$ with $c \in \mathbb{R}$
Similarly for other matrices we can have the following result:
$A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix} \Rightarrow k_1 = k_2$, $x = \begin{bmatrix} c \\ c \end{bmatrix}$ with $c \in \mathbb{R}$
$A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix} \Rightarrow k_1 = k_2$, $x = \begin{bmatrix} c \\ c \end{bmatrix}$ with $c \in \mathbb{R}$