Formal Proof of Nonconvexity and Discontinuity in Ride-Sharing Optimization

Problem Setup

Let:

- $D = \{d_1, d_2, \ldots, d_{|D|}\}$ be the set of drivers,
- $R = \{r_1, r_2, \ldots, r_{|R|}\}$ be the set of riders.

Define the binary decision variable:

$$I_{d_i,r_j} \in \{0,1\}, \quad \forall d_i \in D,\ \forall r_j \in R,$$

with the interpretation:

- $I_{d_i,r_j} = 1$: rider r_j is assigned to driver d_i,
- $I_{d_i,r_j} = 0$: otherwise.

Let the full vector of decision variables be:

$$\mathbf{I} = \left(I_{d_1,r_1}, I_{d_1,r_2}, \ldots, I_{d_{|D|},r_{|R|}}\right) \in \{0,1\}^n, \quad \text{where } n = |D| \cdot |R|.$$

The objective is to maximize a linear function:

$$\max \sum_{i=1}^{|D|} \sum_{j=1}^{|R|} I_{d_i,r_j}.$$

Subject to Constraints

1. **Unique Assignment Constraint:**

$$\sum_{i \in D} I_{d_i,r_j} \leq 1, \quad \forall j \in \{1, 2, \ldots, |R|\} \tag{1}$$

2. **Driver Capacity Constraint:**

$$\sum_{j \in R} I_{d_i,r_j} \leq n_i, \quad \forall i \in \{1, 2, \ldots, |D|\} \tag{2}$$

3. **Path Deviation Constraint:**

$$\Delta_i \leq t_i \cdot |P_i|, \quad \forall i \in \{1, 2, \ldots, |D|\} \tag{3}$$

Convex Optimization Problem (Standard Form)

This definition is from *Convex Optimization*, pages 136–137.

Definition:

A convex optimization problem is one of the form:

$$\begin{aligned} \text{Minimize} \quad & f_0(x) \\ \text{Subject to} \quad & f_i(x) \le 0, \quad i = 1, \ldots, m, \\ & a_i^T x = b_i, \quad i = 1, \ldots, p, \end{aligned}$$

where:

- $f_0(x)$ is the **objective function** (to be minimized),
- $f_i(x)$ are **inequality constraint functions**,
- $a_i^T x = b_i$ are **equality constraints**.

Convexity Requirements

To qualify as a convex optimization problem, the following must hold:

1. $f_0(x)$ is a **convex function** over a **convex feasible set**,
2. Each inequality constraint function $f_i(x)$ is **convex**,
3. Each equality constraint $a_i^T x = b_i$ is **affine** (i.e., linear).

Convex function: $f(x_1, x_2, \ldots, x_n)$ is convex if, for each pair of points on the graph of f, the line segment joining these two points lies entirely above or on the graph of f.

Convex set: A convex set is a collection of points such that, for each pair of points in the collection, the entire line segment joining these two points is also in the collection.

Both definitions are from *Introduction to Operations Research* by Frederick S. Hillier and Gerald J. Lieberman, pages 995 and 997.

2. Proof of Nonconvexity (by Counterexample)

We demonstrate nonconvexity by constructing a specific counterexample.

2.1 Simplified Problem

Consider a minimal problem with:

- One driver d_1, i.e., $|D| = 1$,
- Two riders r_1 and r_2, i.e., $|R| = 2$,
- $x \equiv I_{d_i,r_j}$ and objective

$$f_0(x) = f_0(I_{d_i,r_j}) = -\sum_{i=1}^{|D|} \sum_{j=1}^{|R|} I_{d_i,r_j},$$

- All constraints are trivially satisfied (e.g., unlimited capacity).

The decision variables reduce to:

$$I_{d_1,r_1}, \quad I_{d_1,r_2} \in \{0,1\}.$$

Convexity Requirement

Condition 1: $f_0(I_{d_i,r_j})$ is a **convex function** over a **convex feasible set**.

First, we prove that the feasible set is convex, and then we prove $f_0(I_{d_i,r_j})$ is a convex function. So the feasible region is:

$$\mathcal{C} = \{(0,0),(1,0),(0,1),(1,1)\} \subseteq \{0,1\}^2.$$

According to the definition of a convex set, let:

$$x = (1,0) \in \mathcal{C}, \quad y = (0,1) \in \mathcal{C}.$$

Take a convex combination with $\theta = 0.5$. When $\theta = 0$, you get y; when $\theta = 1$, you get x; and when $0 < \theta < 1$, you get points strictly between x and y:

$$z = \theta x + (1-\theta)y = 0.5 \cdot (1,0) + 0.5 \cdot (0,1) = (0.5, 0.5).$$

Contradiction

Clearly,

$$z = (0.5, 0.5) \notin \mathcal{C} = \{0,1\}^2.$$

Therefore, the set $\mathcal{C}$ is **not** a convex set. Hence, Ride-Sharing Optimization is **not a convex problem**. We will not check for other requirements.

Note:

1. The feasible sets for discrete optimization problems can be thought of as exhibiting an extreme form of nonconvexity, as a convex combination of two feasible points is in general not feasible.(Nocedal and Wright, *Numerical Optimization*, 2nd ed., Springer, 1999, p. 5).
2. Both ILP and BILP are fundamental discrete optimization models because many real-world problems can be modeled using integer or binary decisions.