Assignment 2 - Bacteria Growth Model

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CS166 - Modeling & Analysis of Complex Systems

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Contents

1 Executive Summary

In this report, we explore the dynamics of bacteria and food populations in a system where they interact with each other. A discrete-time model is implemented to simulate how the populations of two observables change over time. The report demonstrates the empirical theoretical analysis of the impact of different values of independent parameters on food and bacteria populations.

2 Simulation Implementation

2.1 Model Setup

The model consists of 5 independent variables, 2 dependent variables and 6 fixed parameters, 4 of which are the initial number of cells where food and bacteria are scattered and the initial amount of food and bacteria to be added to those cells. We treat the values of all these components as continuous. The table below summarizes all of these components.

Variables	Type	Range	Description
g_f	Independent	$[0,\infty]$	Growth rate of food
g_b	Independent	[0,1]	Growth rate of bacteria
d_f	Independent	[0,1]	Diffusion rate of food
d_b	Independent	[0,1]	Diffusion rate of bacteria
c_b	Independent	$[0, \infty]$	Consumption rate of bacteria
p_f	Constant	0.01	Probability of adding one unit of food to a cell
k_f	Constant	100	Maximum food per cell
E_f	Dependent	$[0, \infty]$	Average population of food
E_b	Dependent	$[0, \infty]$	Average population of bacteria

Table 1: **Summary of Model Variables**

2.2 Model Rules

The model involves 7 steps which is taken in a sequential order:

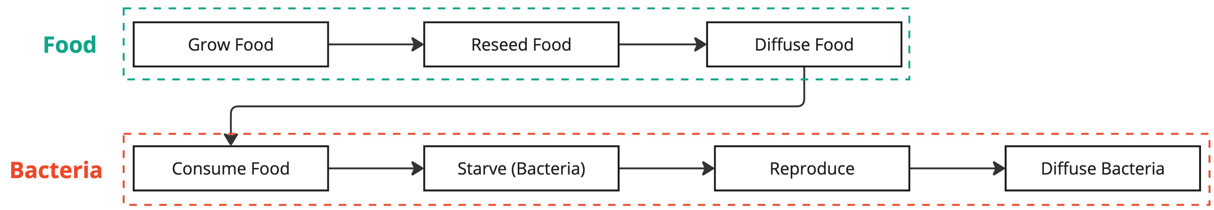

Figure 1: **Update Rule of Bacteria Growth Model**

We start by randomly scattering the food and bacteria in a certain number of cells. This is done by specifying the density of cells where food and bacteria are present and the units of food and bacteria to place on those cells.

After the initialization step, the model grows the food following a logistic growth model where the food population is capped by a carrying capacity of 100 units.

$$f_{t+1} = f_t(1 + g_f(1 - \frac{f_t}{k_f}))$$

Then, the model adds one unit of food to each cell with a probability of 0.01. This is implemented by randomly sample a number between 0 and 1 from a uniform distribution, which is applied to all cells. In each cell, if that number if < 0.01, we will add 1 unit of food to that cell.

Next, a quarter of the food in a single cell diffuses to its four neighboring cells (Von Neumann neighborhood) where each neighbor gets $\frac{1}{4}d_f f$ units of food added and the current cell got subtracted $d_f f$.

Next, the bacteria will consume the food with the rate of c_b. At this step, not all bacteria will get to eat, the fraction of bacteria that can survive is determined by:

$$\min(1, \frac{f_t}{c_b b})$$

If the available food is less than the required food, only a fraction of bacteria survive. The bacteria that do not get to eat will die in the starvation step.

Next, the bacteria that survive grows to $b_t(1 + g_b)$. Finally, the bacteria diffuses following the same rule as the food.

3 Empirical Analysis

3.1 Experimental Results

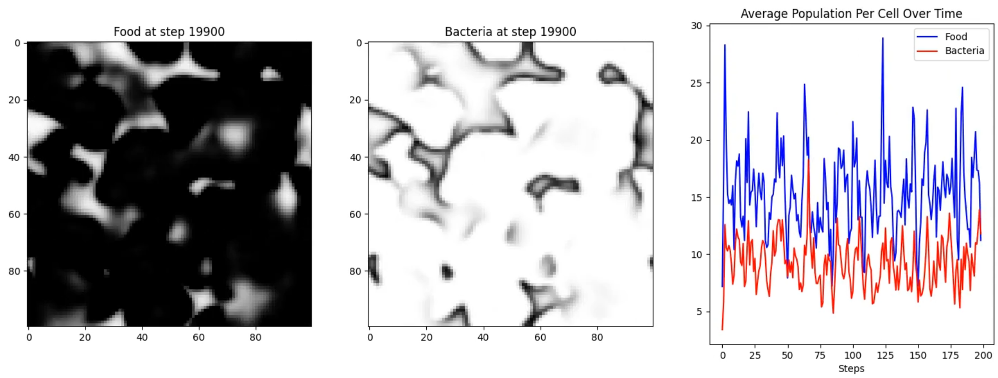

Figure 2: **Patterns & Average Populations of Food & Bacteria in a Balanced System**. In this parameter setting, I set all the independent variables to be equal 0.5 (e.g., food and bacteria grows and diffuses at the same rate and bacteria consumes food at the same rate as they grow). The system is initialized by scattering food in 50 cells, each with 100 units and bacteria in 50 cells, each with 3 bacteria. After 20000 steps, the system seems to reach equilibrium where the amount of food and bacteria fluctuates within a fixed range - between 5 and 11 for bacteria and 10 and 30 for food. There is a noticeable overlap between the food population and the bacteria population. The observed fluctuations and spikes in the plot can be attributed to the balanced rates of growth and consumption. When bacteria consume food rapidly, they deplete the available resources, leading to a temporary decline in the bacterial population as food becomes scarce. Conversely, as the bacterial population decreases, the food supply has a chance to replenish, allowing the bacteria to grow again. This cyclical pattern results in the recurring spikes and drops seen in the data.

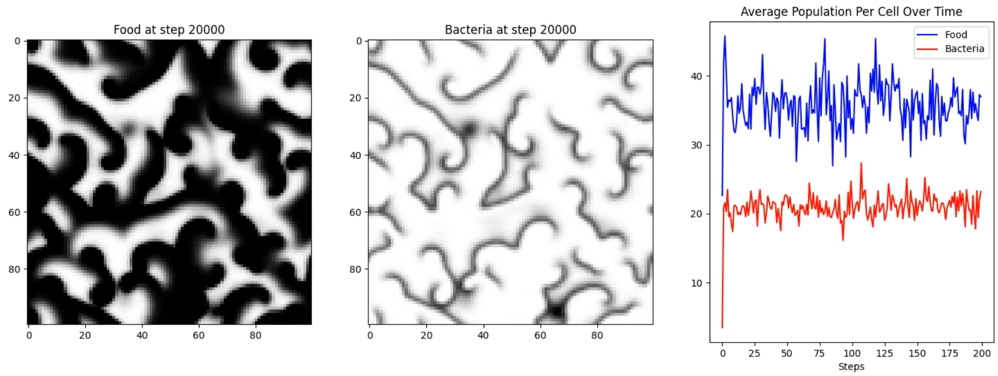

Figure 3: **Patterns & Average Populations of Food & Bacteria When Food Grows and Diffuses Faster Than Bacteria**. The system forms a semi-spiral pattern and the food population equilibrium is systematically higher than that of bacteria over time (around 40 compared to 20, respectively). This pattern makes sense because when food grows and diffuses faster, the bacteria is in a resource-abundant environment. As a result, they will not eat up all the food and grow to the point where food is scarce. The food diffuses faster also ensure that the bacteria in the neighboring cells are provided with enough resources.

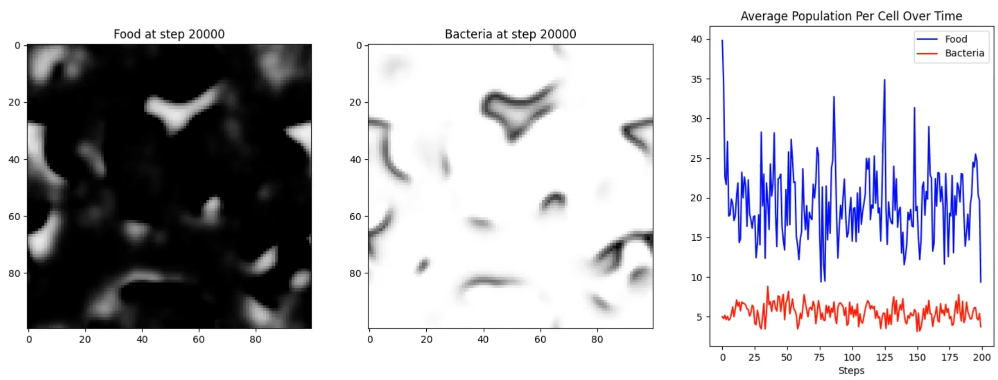

Figure 4: **Patterns & Average Populations of Food & Bacteria When Bacteria Diffuses Faster Than Food**. In this system where the growth rates of food and bacteria are equal, with the same initialization, the average population of bacteria is significantly lower. The equilibrium for food fluctuates within the range of 15 and 25 and that for bacteria slight fluctuates between near 0 and 5. This is because when bacteria diffuse to a cell with not enough food to feed all of them, they will die at a faster rate.

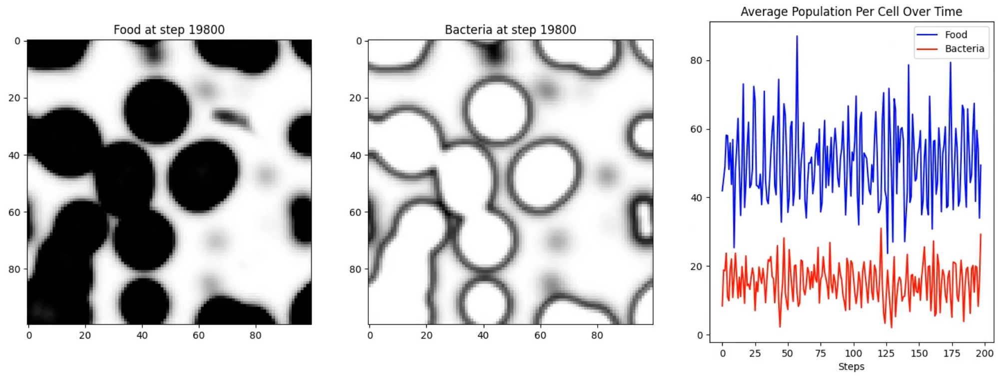

Figure 5: **Patterns & Average Populations of Food & Bacteria When Bacteria Diffuses Faster Than Food But Food Grows Faster Than Bacteria**. In this system where the growth rate of food is greater than that of bacteria, with the same initialization, the average population of bacteria is also significantly lower than food. We can see the oscillatory patterns where food's faster growth rate ensures continuous replenishment and bacteria's faster diffusion allows it to access food resources efficiently.

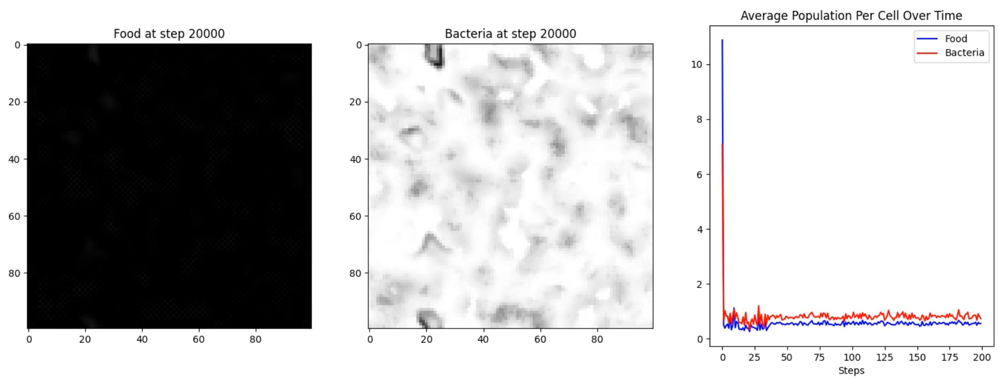

Figure 6: **Patterns & Average Populations of Food & Bacteria When Food Diffuses Faster Than Bacteria But Grows Slower Than Bacteria**. In this system with the same initialization, it is clear that there is not enough food to feed bacteria and the bacteria will eat up all the food rather quickly and then shrinks towards zero. This is an example of a resource-limited environment.

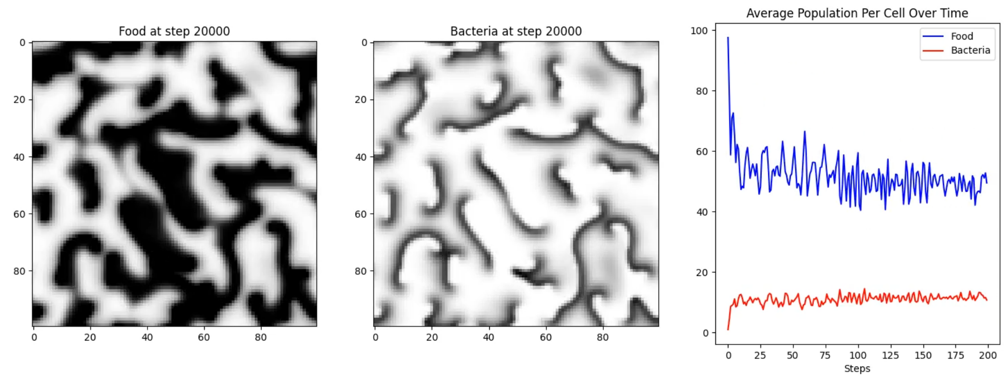

Figure 7: **Patterns & Average Populations of Food & Bacteria When One Bacterium in The Middle & Food Everywhere**. In this system where the growth rates of food and bacteria are equal, with the same initialization, the average population of bacteria is significantly lower and stabilizes at around 18. Food population stabilizes at around 50. This behavior is reasonable given that the population of bacteria is very small compared to the abundance of food and it grows at the equal rate. Therefore, it will not exceed the resources available.

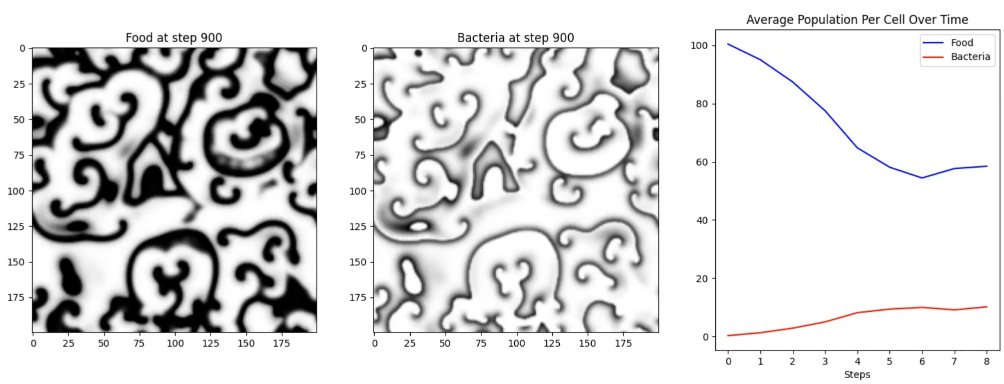

Figure 8: **Another Example of Spiral Patterns Form In The Special Setting**. In a larger grid (200x200), we observe this beautiful spiral pattern when we put one bacterium in the middle and food everywhere.

3.2 Long-term Behavior

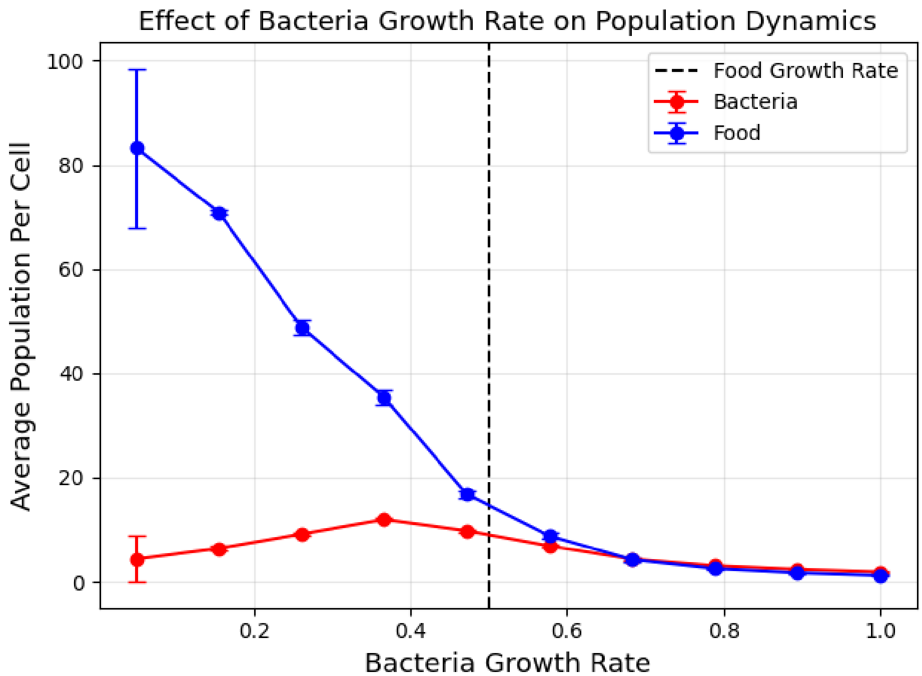

Figure 9: **The Impact of Bacteria Growth Rate On Average Populations of Food & Bacteria After 10 Trials (1000 Steps Each)**. It is clear that when the bacteria growth rate is smaller than the food growth rate, food population is consistently higher than that of bacteria. However, when the growth rate exceeds food growth rate, both of the populations diminish to near 0. This happens when the resources cannot handle the growing population. The 95% confidence interval of the average population corresponding to each bacteria growth rate value is also very narrow, except for the smallest rate of 0.05, which means there's not much variation in the long-term behavior of observable.

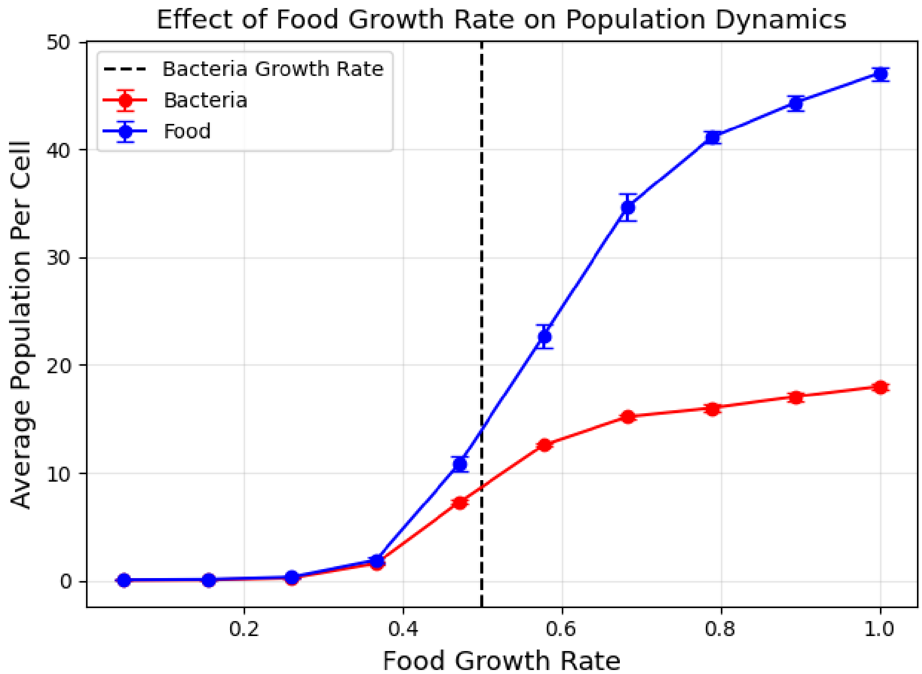

Figure 10: **The Impact of Food Growth Rate On Average Populations of Food & Bacteria After 10 Trials (1000 Steps Each)**. When food grows faster than bacteria, we can see that both populations grow. This is a realistic behavior because now that the resources are abundant and the population does not consume or grow more than that, the amount of bacteria should also increase. Similar to figure 9, the 95% confidence intervals of the populations with respect to different food growth rate values are very narrow, which signals the certainty in the average values of the observables.

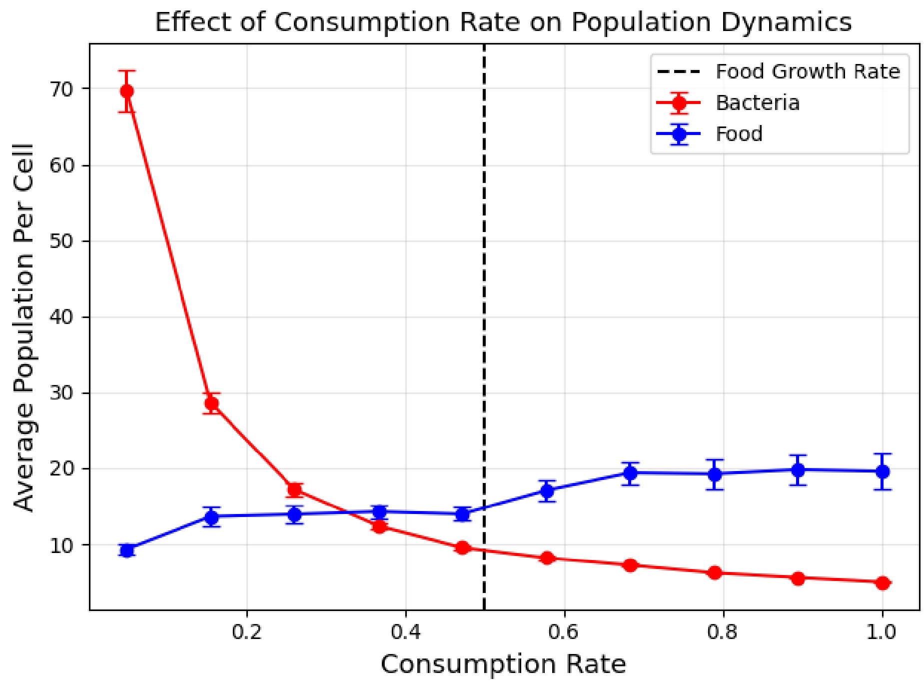

Figure 11: **The Impact Of Consumption Rate On Averge Food & Bacteria Populations After 10 Trials (1000 Steps Each)**. This plot shows an interesting behavior of the system when the consumption rate is smaller than the food growth rate and bacteria growth rate. In this system, I currently set the growth rates of food and bacteria to be equal. At some point (0.3), the population lines intersect where the bacteria continues to decrease and the food population stabilizes. This might be because when the consumption rate surpasses the growth rate of food, the bacteria starve quickly because there is not enough food to eat and as the bacteria dies, only a small amount of them consumes the available food, which is why the food population stagnates (not increasing nor decreasing). The narrow 95% confidence intervals reflect the certainty in the average values.

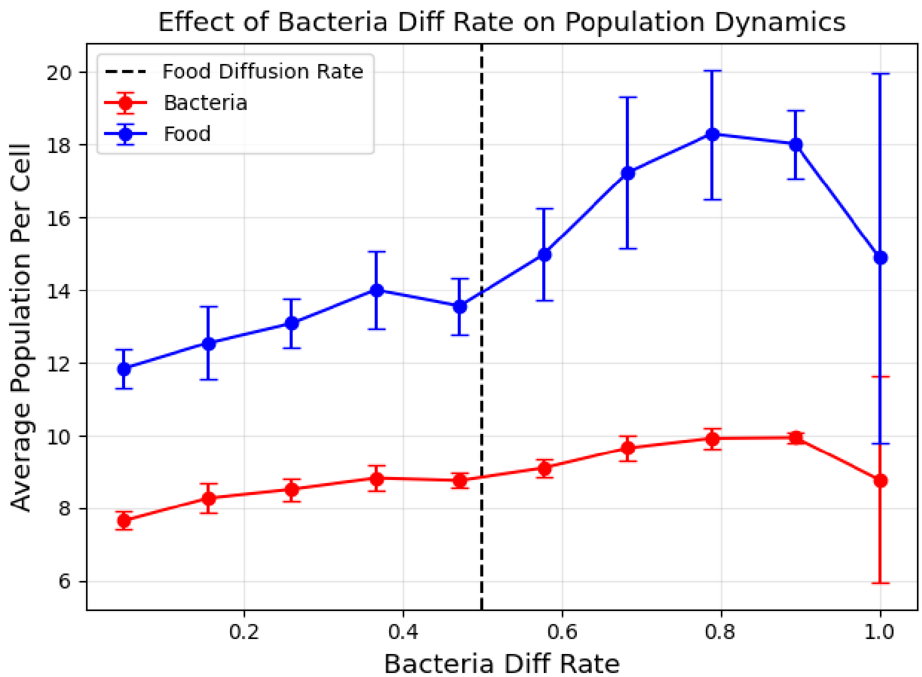

Figure 12: **The Impact Of Bacteria Diffusion Rate On Average Bacteria & Food Populations After 10 Trials (1000 Steps Each)**. It is clear that the food population is consistently larger than the bacteria population (12-18 compared to 8-10). We observe wide 95% confidence intervals of average amount of food corresponding to different bacteria diffusion rates, particularly when the diffusion rate of bacteria reaches 1. This behavior makes sense because the availability of food becomes unstable as bacteria diffuse really fast and consume food in the neighboring cells.

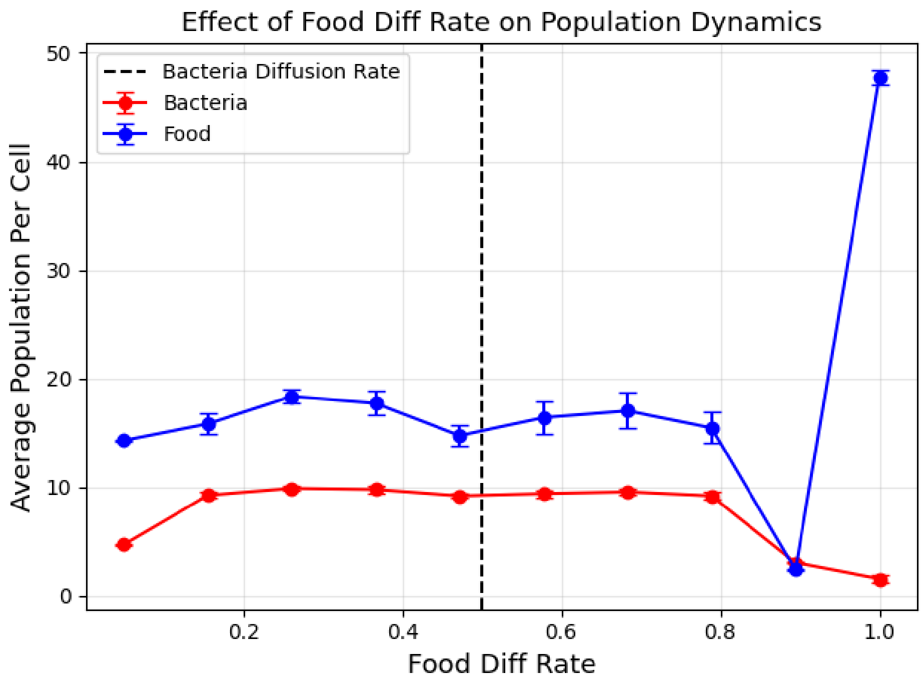

Figure 13: **The Impact Of Food Diffusion Rate on Food & Bacteria Populations After 10 Trials (1000 Steps Each)**. We observe an interesting behavior when the food diffusion rate is at 0.9 and the food population suddenly skyrocketed. This is because when the food diffuses faster but its growth rate is the same as that of the bacteria, it spreads quickly throughout the environment before bacteria can consume it locally. This creates a mismatch between bacterial populations and their food source in many cells - bacteria remain concentrated in certain areas while their food diffuses away too quickly to sustain them. Once the bacteria population is drastically reduced, the food population grows according to its intrinsic growth rate with minimal consumption pressure, explaining the dramatic increase at a diffusion rate of 1.0.

4 Theoretical Analysis

4.1 Food Growth Rate & Food Population

At equilibrium, the amount of food of the previous step is equal to the next step, so we have:

$$1 = 1 + g_f(1 - f_{eq}/k_f) \implies 1 - \frac{f_{eq}}{k_f} = 0 \ (g_f \neq 0)$$

This means that food level reaches equilibrium when it is equal to the carrying capacity in the absence of consumption and diffusion. However, this is not realistic because the food level is dependent on other parameters as well. To account for those factors which balance the food growth, we have:

$$\Delta f_{\text{growth}} = f_{\text{eq}} g_f (1 - \frac{f_{eq}}{k_f})$$

$$\Delta f_{\text{consumption}} = c_b b$$

Thus, the condition for equilibrium is when the food growth equals to the food consumption:

$$f_{\text{eq}} g_f (1 - \frac{f_{eq}}{k_f}) = c_b b$$

$$f_{\text{eq}}^2 - k_f f_{\text{eq}} + \frac{c_b b k_f}{g_f} = 0$$

Then, we can solve for f_{eq} using the quadratic formula:

$$f_{\text{eq}} = \frac{k_f \pm \sqrt{k_f^2 - 4\frac{c_b b k_f}{g_f}}}{2}$$

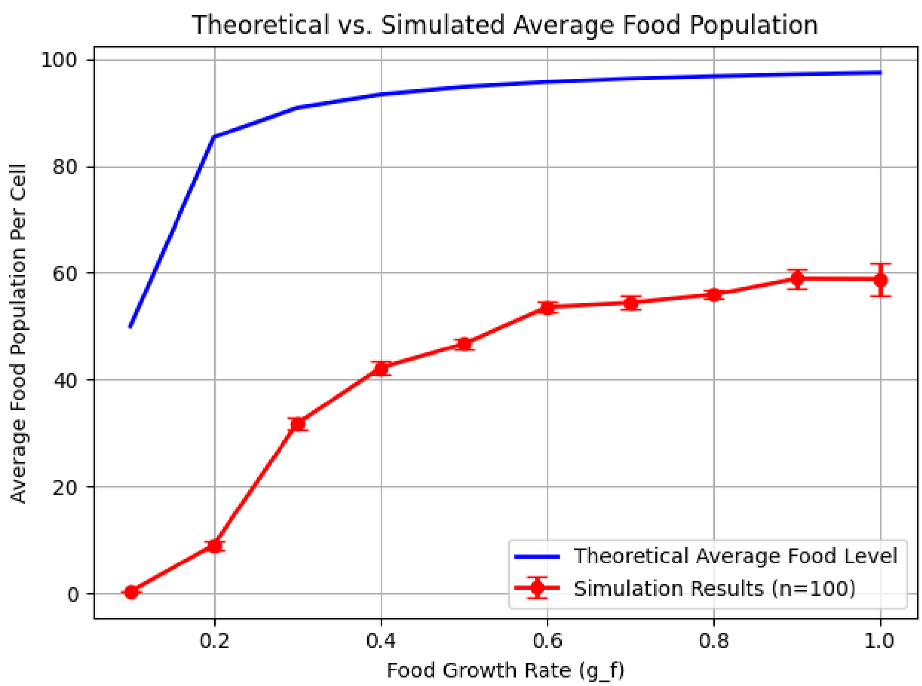

Figure 14: **Theoretical vs. Simulated Average Food Population**. It is clear that despite matching qualitatively, there is a huge gap between the theoretical and the simulated results. This might be because we do not take into account the effect of diffusion where the bacteria can eat up the food in the neighboring cells rather quickly, making the food population lower than expected.

4.2 Bacteria Growth Rate & Bacteria Population

Based on the model, there is an inverse relationship between the consumption rate (c_b) and the maximum sustainable bacteria population. Each cell requires $c_b b$ amount of food to feed all bacteria and the food grows as:

$$f_{t+1} = f^* = f_t(1 + g_f(1 - \frac{f_t}{k_f}))$$

At equilibrium, for the whole grid, the total food growth across all cells equal the total food consumed by all bacteria:

$$\text{Food growth} \times (\text{total number of cells}) = c_b \times \text{Total bacteria}$$

Assuming that the food population is at a stable level where it reaches 50% of its carrying capacity. This is because the change in the food amount is $\Delta f = f g_f(1 - \frac{f}{k_f})$ and we find the maximum growth rate by setting $d(\Delta f)/d_f = 0$. Applying the product rule, we have:

$$\frac{d(\Delta f)}{d_f} = g_f(1 - \frac{f}{k_f}) + f g_f(-\frac{1}{k_f})$$

$$\frac{d(\Delta f)}{d_f} = g_f(1 - \frac{2f}{k_f})$$

Then, if we set the derivative to be equal to 0 and we solve for f in the case where $g_f \neq 0$, we'll get $f = \frac{k_f}{2}$. Using second derivative check where it is negative, we verify that this is a maximum. This means the maximum growth rate occurs at around 50% of the carrying capacity.

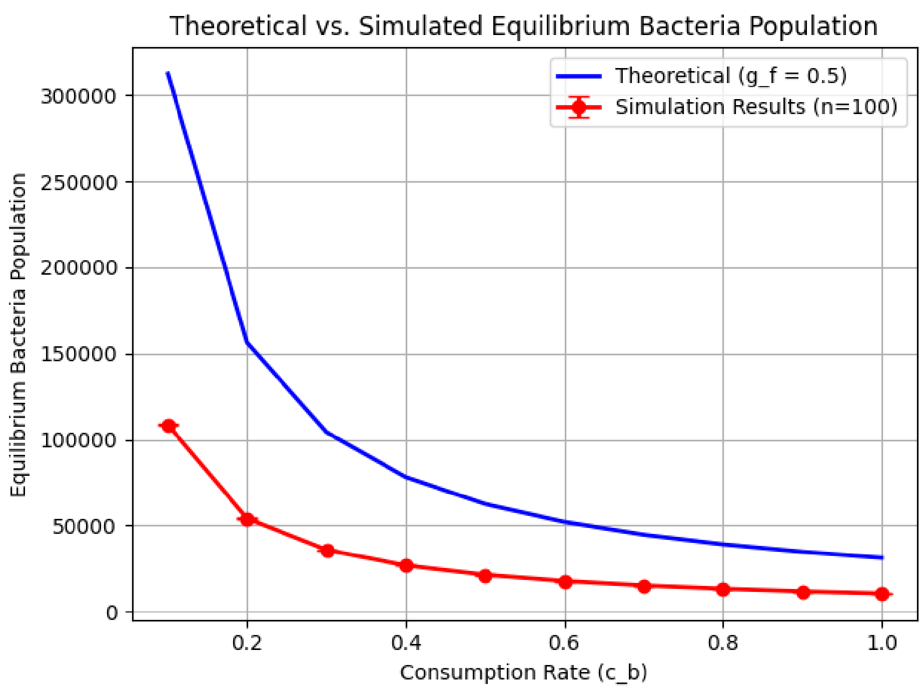

Figure 15: Theoretical vs. Simulated Equilibrium Bacteria Population (Grid Size = 20, Number of Trials = 1000). The simulated result (with 95% confidence interval) is slightly slower than the theoretical one but in general, it is approximately similar to the theoretical result. They also share the same qualitative trend of decreasing as the consumption rate increases.

4.3 Discrete-Time Model - Bacteria & Food Population Over Time

A system reaches an equilibrium state when the next state is approximately equal to the previous state. Because the system is updated discretely, we will use difference equations to model the relationship of the previous state and the next state of food and bacteria populations. $x_{t+1} = F(x_t)$. The system we are trying to model is a first-order autonomous linear system with a linear combinations of state variables, the variable time t is not explicitly included and we assume that the current state is only dependent on the immediate previous state. Here, we are working with a coupled system of difference equations where both variables interact with one another:

$$\begin{cases} f_{t+1} = F(f_t, b_t) \\ b_{t+1} = G(f_t, b_t) \end{cases}$$

The system reaches equilibrium when:

$$f_{t+1} = f_t = f^* \ (1)$$

$$b_{t+1} = b_t = b^* \ (2)$$

We can derive the equilibrium conditions for food and bacteria:

$$f^* = f^*(1 + g_f(1 - \frac{f^*}{k_f})) - c_b b^* + r$$

$$b^* = b^*(1 + g_b)\min(1, \frac{f^*}{c_b b^*})$$

Now, we can solve for equilibrium in two different scenarios - when food is limited and when food is abundant. Firstly, when food is limited ($f^* \leq c_b b^*$), we have:

$$b^* = b^*(1 + g_b)(\frac{f^*}{c_b b^*}) \implies 1 = 1(1 + g_b)(\frac{f^*}{c_b b^*}) \implies f^* = \frac{c_b b^*}{1 + g_b}$$

Substitute this into the food equilibrium equation, we have:

$$\frac{c_b b^*}{1 + g_b} = \frac{c_b b^*}{1 + g_b}(1 + g_f(1 - \frac{\frac{c_b b^*}{1+g_b}}{k_f})) + r - c_b b^*$$

$$\frac{c_b b^*}{1 + g_b} = \frac{c_b b^*}{1 + g_b}(1 + g_f - \frac{g_f c_b b^*}{(1 + g_b)k_f}) + r - c_b b^*$$

$$b^*(g_f - (1 + g_b)) - \frac{g_f c_b}{(1 + g_b)k_f}(b^*)^2 + \frac{r(1 + g_b)}{c_b} = 0$$

$$b^* = \frac{(1 + g_b - g_f) \pm \sqrt{(g_f - (1 + g_b))^2 + \frac{4 g_f r}{k_f}}}{\frac{2 g_f c_b}{(1+g_b)k_f}}$$

4.4 Stability Analysis

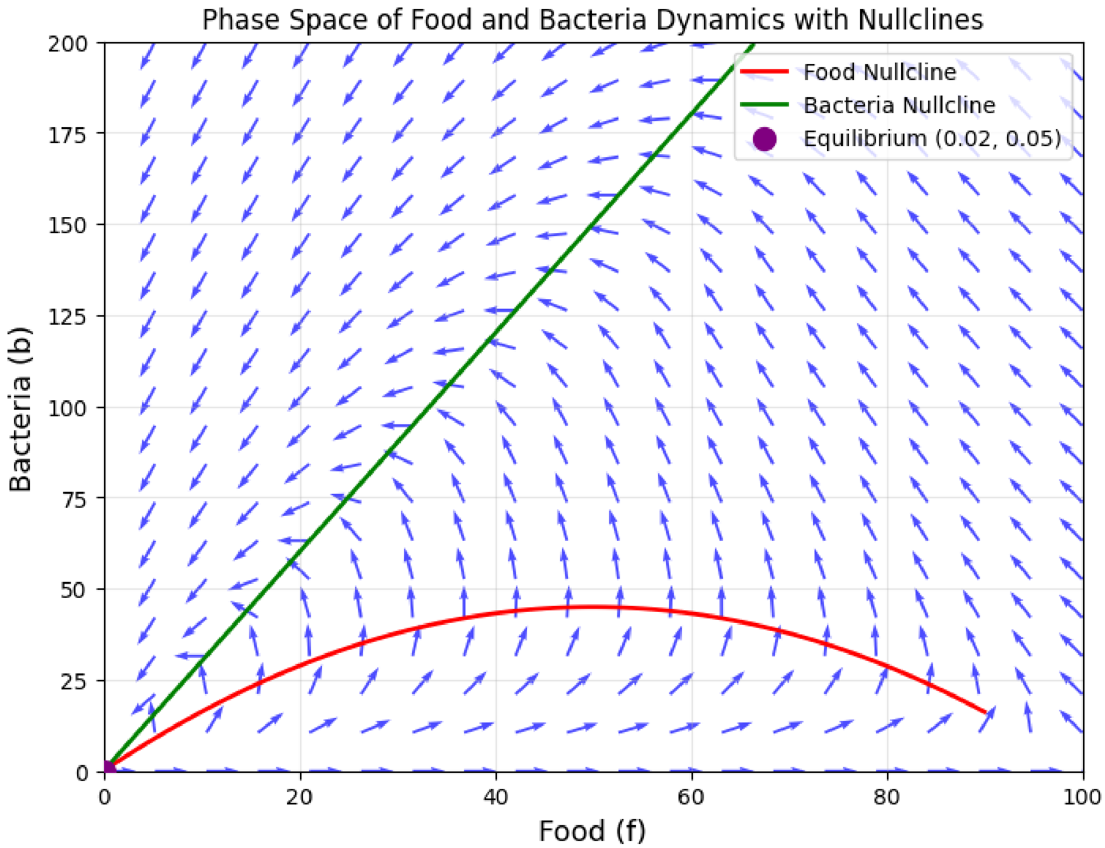

Figure 16: **Phase Space of Food & Bacteria Populations**.This phase space diagram illustrates the dynamic relationship between food and bacteria populations in a predator-prey system. The horizontal axis represents food concentration (f), while the vertical axis shows bacteria population (b).

The red curve represents the food nullcline, where the food population remains constant ($df/dt = 0$). Below this curve, food population increases because growth exceeds consumption by bacteria; above it, food population decreases as bacterial consumption outpaces regeneration. The parabolic shape of this nullcline indicates that food growth is likely density-dependent, with maximum growth rate at intermediate food densities.

The green line represents the bacteria nullcline, where bacteria population remains constant ($db/dt = 0$). To the left of this line, bacteria population decreases due to insufficient food availability; to the right, bacteria increases as food is abundant enough to support population growth. The linear shape suggests that bacteria growth is proportional to food availability.

The purple dot at (0.02, 0.05) marks the equilibrium point where both nullclines intersect. At this point, both food and bacteria populations remain stable if undisturbed. These values are small but positive, indicating a stable coexistence state where both populations persist at low levels.

The blue vectors show the direction and magnitude of population changes throughout the phase space. Their pattern reveals a counterclockwise rotational flow around the equilibrium, indicating oscillatory dynamics typical of predator-prey systems. When disturbed from equilibrium, the system will typically follow a cyclic trajectory, with alternating periods of food abundance followed by bacteria growth, then food scarcity followed by bacteria decline.

5 Conclusion & Limitations

In this report, we explored the bacteria growth model with different parameter settings and how the dynamics of the population change dependent on the interaction between the inputs and observables. We have observed patterns of populations of food and bacteria when the system is resource-abundant and scarce. The biggest limitation of this report is in the theoretical analysis where we did not take into account how the diffusion rates affect the food and bacteria average populations. It can be further improved by 1) investigating the populations on the cell level and how the diffusion rates have significant or negligible impact on the system as a whole and 2) run the empirical analysis for more trials than just 10.

6 AI Statement

I used Claude and DeepSeek mostly to help me with debugging the code when I tried to run and plot theoretical and simulated results. I also asked how to optimized the code because it took so long to run the simulation (3 hours for 10 trials, 1000 steps). Then I used numba and @njit to optimize the performance of the code simply by adding it as a decorator. All the logic of the class and writeup is implemented by myself.

7 Appendix

All code can be found here: `https://colab.research.google.com/drive/19oHLc3llHxDfIz002kR5riyo`
`scrollTo=B7W1MPc7uMZ5`