Stats Review

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Counting:

(a) Counting n tasks(m_i) in order:

$$\text{Mult Principle: } |A| = \prod_{i=0}^{n} m_i \tag{1}$$

(b) The number of ways to do n non-overlapping tasks:

$$\text{Add Principle: } |A| = \sum_{i=0}^{n} m_i \tag{2}$$

(c) Choose k ordered elements of $A, |A| = n$:

$$\text{Permutations: } P(n,k) = \frac{n!}{(n-k)!} \tag{3}$$

(d) The number of unique non-ordered lists:

$$\text{Combinations: } C(n,k) = \binom{n}{k} = \frac{P(n,k)}{P(k,k)} = \frac{n!}{(n-k)!k!} \tag{4}$$

(e) Size of the set(X) of all possible events in set A, with size n:

$$\text{Counting events: } |X| = 2^n \tag{5}$$

(f) Binomial Coefficient

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{6}$$

Probability Function:

(a) $\mathbb{P}(A) : Event \to [0,1]$

(b) $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$, where $\emptyset$ is the empty set, and Ω is the problem space

(c) $\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A|B)$, Condition probability

(d) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

(e) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, Reason: $A \cap A^c = \emptyset \wedge \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) \Rightarrow \mathbb{P}(A) + \mathbb{P}(A^c) = 1$

(f) Given: A has mutually disjoint, $\mathbb{P}(\cup_{i=0}^{n} A_i) = \sum_{i=0}^{n} \mathbb{P}(A_i)$

(g) $\mathbb{P}(A) = \sum_{i=0}^{n} \mathbb{P}(t_i)$, assuming $t_i \cap t_j = \emptyset$

(h) $A \subseteq B \to \mathbb{P}(A) \leq \mathbb{P}(B)$

(i) Independent iff: $\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(B)$, alternatively written $\mathbb{P}(A|B) = \mathbb{P}(A)$ for $\mathbb{P}(B) \neq 0$

Random Variables:

(a) Def:

$$\text{Random Variable}(X)\text{: } X : \Omega \to \mathbb{R} \tag{7}$$

(b)

$$\text{Prob mass function: } f(x) : \mathbb{P}(X = a) : X \to [0, 1] \tag{8}$$

(c)

$$\text{Prob density function: } f(x) : \tag{9}$$

(d) Cumdist Properties (continuous):

(a) Def:

$$\text{(Countable): } F(x) \equiv \mathbb{P}(X \leq a) = \sum_{i=0}^{a} \mathbb{P}(X_i), a \in \Omega \tag{10}$$

$$\text{(Continuous): } F(x) \equiv \mathbb{P}(X \leq a) = \int_{-\infty}^{a} \mathbb{P}(X_i) dx \tag{11}$$

(b) $0 \leq F(x) \leq 1$

(c) $\lim_{x \to -\infty} F(x) = 0 = \mathbb{P}(\emptyset) \wedge \lim_{x \to \infty} F(x) = 1 = \mathbb{P}(\Omega)$

(d) $\forall (x, y) x < y \to F(x) \leq F(y)$

(e) Cumdist/Prob Density

$$\text{Cumdist/Prob density: } F(x) \equiv \int_{-\infty}^{\infty} f(x) dx \tag{12}$$

$$\text{Prob density/Cumdist: } f(x) \equiv \frac{d}{dx} F(x) \tag{13}$$

(f) Expected(μ)

$$\text{(Countable): } \mathbb{E}(X) \equiv \sum_{i=0}^{n} X_i \mathbb{P}(X_i) \tag{14}$$

$$\text{(Continuous): } \mathbb{E}(X) \equiv \int_{-\infty}^{\infty} X_i \mathbb{P}(X_i) dx \tag{15}$$

Moments:

(a) Definition

$$nth \text{ Moment: } \mu_n \equiv \mathbb{E}[(X-b)^n] \tag{16}$$

$n=0$	Constant
$n=1$	Expected Value (center)
$n=2$	Measure of Dispersion
$n=3$	Measure of asymmetry
$n=4$	Measure of peakedness

(b) Types of Moments

$$\text{Raw Moments: } \mu'_n \equiv \mathbb{E}[(x-0)^n] \tag{17}$$

$$\text{Central Moments: } \mu_n \equiv \mathbb{E}[(x-\mu)^n] \tag{18}$$

(c) Getting Moments

i. Use def of moment (i.e.) $mu_n \equiv \mathbb{E}[(X-b)^n$

ii. $[PGF]$ Probability Generating Function (Descrete only)

iii. $[MGF]$ Moment Generating Function (May not exist)

A. Def

$$M_x(t) \equiv \mathbb{E}(e^{tx}) \tag{19}$$

B. Raw moment from MGF

$$\frac{\partial^n}{\partial t^n} M_x(t)|_{t=0} = \mathbb{E}(X^n) \tag{20}$$

iv. Characteristic equation

(d) Generating Central Moments from Raw Moments

$$\mu_n = \Sigma_{j=0}^{b} \binom{n}{k} (-1)^{n-j} \mu'_j \mu^{n-j} \tag{21}$$

(e) Important Moments

i. Mean$(\mu) \equiv \mu'_1 = \mathbb{E}(X)$

ii. Variance$(\sigma^2) \equiv \mu_2 = \mathbb{E}((X-\mu)^2) = \mathbb{E}(X^2) - \mu^2$

iii. Skew$(\gamma_2) \equiv \frac{\mu_3}{\sigma^3} = \frac{\mathbb{E}((X-\mu)^3)}{\sigma^3}$

iv. Kurtosis$(\gamma_3) \equiv \frac{\mu_4}{\sigma^4} - 3$

Joint Probability:

(a) $f(x,y)$ is the joint probability density function of the sets X and Y

(b) Marginal Density: $f(x) = \int_{-\infty}^{\infty} f(x,y)dy$

(c) X and Y are Independent iff $f(x,y) = f(x)f(y)$

(d) $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dxdy$

(e) $f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$

(f)

$$\text{Correlation}(r) = \frac{\mathbb{E}[(X-\mu_x)(Y-\mu_y)]}{\sigma_x \sigma_y} \tag{22}$$

Limit Theorems:

(a) Central Limit Theorem

  (a) Assumptions

    i. All $x \in X$ are independent for a random sample

    ii. All X have the same distribution, μ and σ

  (b) CLT approximated moments

    i. Sum of Averages $= n\mu$

    ii. Sum of Variances $= n\sigma^2$

    iii. $\bar{\sigma}^2 = \frac{\sigma^2}{n}$

    iv. $\mathbb{E}(\bar{X}) = \mu$

  (c) Normal Approximation to binomial

$$\mathbb{P}(a \leq \mathbb{S} \leq b) \approx \mathbb{P}(A - \frac{1}{2} \leq X \leq b + \frac{1}{2}),\ mp > s, m(1-p) > s$$

  (d) Worst Case Approximation:

$$\mathbb{P}(|\bar{x} - p| \leq \epsilon) \equiv \mathbb{P}(|\mathbb{Z}| \leq 2\epsilon\sqrt{n})$$

(b) Markov's inequality: applies to any non-negative random variable

$$\mathbb{P}(X \geq a) \leq \frac{\mu}{a}, a > 0 \tag{23}$$

$$\mathbb{P}(X \geq k\mu) \leq \frac{1}{k} \tag{24}$$

(c) Chebyshev's inequality: upper bound on dist

$$\mathbb{P}(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}, c > 0 \tag{25}$$

$$\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \tag{26}$$

(d) Law of large numbers

$$s \tag{27}$$

Common Random Variables:

(a) Bernoulli trials: Independent repeated trials of an experiment with two outcomes only

(b) Poisson Process: a stochastic process in which events occur continuously and independently of one another

Distributions:

(a) $S =$ number of successes, $S : \mathbb{N} \to \mathbb{N}$

(b) $W =$ waiting time, $W : \mathbb{N} \to \mathbb{N}$

(c) $\mathbb{L} =$ life time, $\mathbb{L} : \mathbb{R} \to \mathbb{R}$

(d) $p =$ prob of success

(e) Binomial Dist. (countable)

$$|\Omega| = 2^n$$
$$\mathbb{P}(S = k) = \binom{n}{k} p^k (1-p)^{m-k}$$
$$\mu = np$$
$$\sigma^2 = np(1-p)$$

(f) K-nomial(multinomial) Dist. (countable)

$$|\Omega| = K^n$$
$$\mathbb{P}(S = k) = \prod_{i=0}^{k} \binom{n - m_i}{m_i} p_i^{m_i}$$

(g) Negative Binomial Dist. (countable)

$r =$ number of failures until the experiment is stopped

$$f(x) = \binom{k + r - 1}{k} (1-p)^r p^k$$
$$mgf = \frac{1-p}{1-pe^t}^{r}$$

(h) Geometric Dist. (countable)

$$\mathbb{P}(W = k) = p(1-p)^{(k-1)}$$
$$\mathbb{P}(W \leq K) = 1 - (1-p)^k$$
$$\mu = \frac{1}{p}$$
$$\sigma^2 = \frac{1-p}{p^2}$$

(i) Poisson Dist. (countable)

$$\begin{aligned}
&\Lambda = t\lambda, \lambda = \text{ Longterm average rate} \\
&\mathbb{P}(S = k) = e^{-\Lambda}\frac{\Lambda^k}{k!} \\
&\mu = \Lambda \\
&\sigma^2 = \Lambda
\end{aligned}$$

(j) Uniform Dist (continuous):

$$p = \frac{1}{b-a} \tag{28}$$
$$f(x) = \begin{cases} p & : x \in [a,b] \\ 0 & : x \notin [a,b] \end{cases}$$

(k) Linear Dist (continuous):

$$C(b-a) + D(b-a)^2 = 1 \tag{29}$$
$$f(x) = \begin{cases} C + Dx & : x \in [a,b] \\ 0 & : x \notin [a,b] \end{cases}$$

(l) Exponential Dist. (continuous)

$$\begin{aligned}
&\mathbb{P}(\mathbb{L} > t) = \mathbb{P}(\mathbb{S} = 0) = \frac{\Lambda^0}{0!}e^{-\Lambda}, t > 0 \\
&F(t) = \mathbb{P}(\mathbb{L} \leq t) = 1 - \mathbb{P}(\mathbb{L} > t) = 1 - e^{-\Lambda} \\
\\
&f(t) = \frac{d}{dt}F(t) = f(x) = \begin{cases} \Lambda e^{-\Lambda} & : x \geq 0 \\ 0 & : x < 0 \end{cases} \\
&\mu = \int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} t\lambda e^{-\lambda x}dx = \frac{1}{\lambda} \\
&\sigma^2 = \frac{1}{\lambda^2}
\end{aligned}$$

Non-Poisson trials (aging): $\mathbb{P}(\mathbb{L} > t^* | \mathbb{L} > t_0) = e^{-\int_{t_0}^{t^*} \lambda(x)dx}$

(m) Standard Normal Dist. (continuous)

$$\begin{aligned}
&\phi = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \\
&\Phi(\mathbb{Z}) = \mathbb{P}(\mathbb{Z} \leq z) = \int_{-\infty}^{z} \phi(x)dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&\mu = 0, \sigma^2 = 0 \\
&X = \sigma\mathbb{Z} + \mu \Leftrightarrow \mathbb{Z} = \frac{X - \mu}{\sigma}
\end{aligned}$$