Diff Eq Notes

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Contents

1 Initial Definitions

- Definition:
  - DE is an equation that describes the properties of an unkown
- Ordinary DE:
  - describes functions of 1 variable
- Partial DE:
  - describes multivariable functions
- Notation:
  - independent variable: y
  - dependent variable: t

2 Operator Notation

Definition: $\frac{d^n}{dt^n} = D^n \to f^{(n)} = D^n(f)$

3 Linear Diff Equations

Definition: For an operator,L, the DE: L(y) = 0 is linear iff:

- L(y_1+y_1) = L(y_1)+L(y_2)
- L(cy) = cL(y)

4 Stability

4.1 Definition

1. Stability ≡ a system in which long term behvaior does not depend within some variation on initial conditions
2. For Linear D.E. y is Stable iff:
$$\lim_{t\to\infty} y_h = 0 \pm \epsilon \tag{1}$$
3. For S.O.L.E
   (a) Stable iff either of the equivalent statments is true:
      i. $Re(r) < 0$
      ii. $a, b, c > 0$ or $a, b, c < 0$

5 Initial Value Problems

$$IVP = \begin{cases} DE \\ y = C_1 \\ \vdots \\ y^{(n)} = C_n \end{cases} \tag{2}$$

6 Seperable DE

6.1 Definition

6.1.1 Form

$$\frac{dy}{dt} = f(y)g(t) \tag{3}$$

6.1.2 Solution

1. Let DE be $\frac{dy}{dt} = f(y)g(t)$
2. Seperating terms: $\frac{dt}{f(y)}\frac{dy}{dt} = \frac{dt}{f(y)} f(y)g(t) \implies \frac{dy}{f(y)} = g(t)dt$
   (a) If $f(y)^{-1}$ is not defined from some t, solve outside of that range and consider those t separately
3. Integrate both sides: $\int \frac{dy}{f(y)} = \int g(t)dt$
4. If possible solve for $y(t)$

7 Homogenous Equations

7.1 Form

$$\frac{dy}{dt} = f(\frac{y}{t}) \tag{4}$$

7.2 Technique for Solving:

1. Let $v = \frac{y}{t} \implies y = vt$
2. Change of variable: $y' = v't + v \implies v't + v = f(v)$
3. Moving v to left side: $t\frac{dv}{dt} = f(v) - v = g(v)$
4. Seperating: $\frac{dv}{g(v)} = \frac{dt}{t}$
5. Solve using Seperable D.E. techniques

8 Exact Equations

8.1 Form

$$\frac{\partial}{\partial t}\Psi(f(t), y(t)) = \frac{\partial \Psi}{\partial f}\frac{df}{dt} + \frac{\partial \Psi}{\partial y}\frac{dy}{dt} \tag{5}$$

8.2 Technique for Solving:

1. Suppose DE is of the form: $M(x, y) + N(x, y)y_x = 0$
2. If $\mathrm{M}_y = \mathrm{N}_x$, then DE is an Exact Eq, solve for $\Psi(x, y) = \Psi(f(x), y(x))$

9 First Order Linear Eq

9.1 Form

$$y' + p(t)y = f(t) \tag{6}$$

9.2 Solving using Integration Factors

1. Let μ be a mult factor s.t. $\mu y' + \mu' y = g(t) \implies [\mu(t)y(t)]' = g(t)$
2. Thus $\mu' = \mu p(t) \implies \frac{d\mu}{\mu} = p(t)dt \implies \mu = e^{\int p(t)dt}$
3. Therfore

$$y(t)e^{\int p(t)dt} = \int g(t)dt \tag{7}$$

$\square$

9.3 Bernoulli's equations

9.3.1 Form

$$y' + p(t)y = q(t)y^n, n \in \mathbb{Z} \tag{8}$$

9.3.2 Solution

1. Let $v = y^{2-n} \implies v' = (1-n)y^{-n}y'$
2. Thus $y' = \frac{v'}{1-n}$ and $y = y^n v$
3. Subsituting in Bernoulli equation: $\frac{v'}{1-n}y^n + p(t)y^n v = q(t)y^n$
4. Moving into standard form:
$$v' + (1-n)p(t)v = (1-n)q(t) \tag{9}$$
5. Solve using Integration Factors $\square$

9.4 Picard Iteration

9.4.1 Integral Equations

Suppose f is continous, then a function $y = \Phi(t)$ solves the IVP iff $y = \Phi(t)$ solves the corresponding integral equation:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{10}$$

9.4.2 Idea

1. Let $f(t) = \frac{dy}{dt}$
2. Construct a sequence of functions $\{g_n(t) : n \geq 0, n \in \mathbb{Z}\}$ that converges to soln:
   (a) $y_0(t) = y_0$
   (b) $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(t))ds$

9.5 Lipsichitz Condition

1. For $f(t, y) \in \mathbb{R}$, f is Lipsichitz iff $\exists L \in \mathbb{R}$:
$$|f(t_1, y_1) - f(t_2, y_2)| \leq L \cdot |(y_1 - y_2)| \tag{11}$$
2. If $\Delta y \neq 0$ then this can be thought of as:
$$\left|\frac{\Delta f}{\Delta y}\right| \leq L \tag{12}$$
3. Lemma: if f_y is bounded then f is Lipsichitz

9.6 Uniform Convergence (U.C.)

9.6.1 Definition:

A sequence of functions $\{f_n(t) : n \geq 0; n \in \mathbb{Z}\}$ defined on the inverval I uniformially converges to $f(t)$ iff $\forall t > 0, \exists N \in \mathbb{Z}$ s.t. $|f_n(t) - f(t)| < \epsilon$ everywhere on I $\forall n > N$

9.6.2

Theorem: Given $_n(t)$ is continuous on I, if $\lim_{n\to\infty} f_n(t) \to f(t)$ with U.C, then:

1. f is continuous
2. If f_n is differtiable, then f is differtiable and f_n' U.C. to f'
3. The limit is communitive with respect to integration

$$\lim_{n\to\infty}\int_I f_n(t)dt = \int_I \lim_{n\to\infty} f_n(t)dt \tag{13}$$

9.6.3 Weirstress M Test

Theorem:

- If $\forall n \in I, |f_n(t)| \leq M_n$ and if $\sum_{n=0}^{\infty} M_n < L$ for some $L \in \mathbb{R}$,
- Then $\sum_{n=0}^{\infty} f_n(t)$ Converges Uniformially on I

9.7 Existence Theorem

1. Claim:

   (a) If:

      i. $f(y)$ is continous

      ii. f is Lipsichitz w.r.t. $y \in R \equiv \{t, y) : |t - t_0| \leq T$ and $|y - y_0| \leq k\}$

      iii. $\sum_{k=1}^{\infty}[y_k(t) - y_{k-1}(t)]$ converges uniformially

   (b) Then: $\exists$ a solution to the IVP on the interval $|t - t_0| \leq T_1 = min(T, \frac{k}{m})$ where $|f(t, y)| \leq M \in R$

2. Proof:

   (a) Converting the IVP to an I.E.: $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$

   (b) Note theat: $|y_k(t) - y_{k-1}(t)| \leq \frac{M}{L}\frac{L^n(t-t_0)^n}{n!} \leq \frac{M}{L}\frac{L^nT_1^n}{n!}$

   (c) Define: $M_n \equiv \sum_{k=1}^{\infty} \frac{M}{L}\frac{(LT_1)^n}{n!} = \frac{M}{L}(e^{LT_1} - 1)$

   (d) Apply the Weirstress M Test, because $\frac{M}{L}\frac{(LT_1)^n}{n!}$ converges, then $\sum_{k=1}^{\infty}[y_k - y_{k-1}]$ converges

   (e) Thus the series $\{y_n : n \geq 1\}$ converges uniformially on the interval.

   (f) Therefore $\exists$ a solution to the IVP $\square$

9.8 Uniqueness Theorm

1. Claim:

   (a) If $\Phi(t)$ and $\Psi(t)$ are solutions of $y' \equiv f(y, t) \in R$ and if f is Lipseitz w.r.t. $y \in R$

   (b) Then $|\Phi(t) - Psi(t)| \leq e^{L|t-t_0|}|\Phi(t_0) - \Psi(t_0)| = 0$

      i. Because they solve the same I.V.P. $|\Phi(t_0) - \Psi(t_0)| = 0$

   (c) Equivalently: Then $\Psi(t) = \Psi(t)$

2. Proof:

   (a) $E \equiv |\Phi(t) - \Psi(t)|^2$

i. Note that $E \geq 0$

(b) $\frac{d}{dt}E = 2(\Phi(t) - \Psi(t))(\Phi'(t) - \Psi'(t))$

(c) $E' \overset{DE}{=} 2(\Phi(t) - \Psi(t))(f(t,\Phi) - f(t,\Psi))$

(d) $E' \overset{Lip}{\leq} 2\left|\Phi(t) - \Psi(t)\right| L \left|\Phi(t) - \Psi(t)\right|$

(e) Thus $E' \leq 2LE \implies E' - 2LE \leq 0 \implies (E(t)e^{-2Lt})' \leq 0$

i. Note that E' is stricly decreasing

(f) Therefore: $e^{-t}E(t) \leq e^{2Lt_0}E(t_0) \implies E(t) \leq e^{2L(t-t_0)}E(t_0)$

(g) Substituting: $\left|\Phi(t) - \Psi(t)\right|^2 \leq e^{2L(t-t_0)}\left|\Phi(t_0) - \Psi(t_0)\right|^2$

(h) Because of absolute value: $\left|\Phi(t) - \Psi(t)\right| \leq e^{2L(t-t_0)}\left|\Phi(t_0) - \Psi(t_0)\right|$

(i) Because they solve the same I.V.P. $\left|\Phi(t_0) - \Psi(t_0)\right| = 0$

(j) Thus $\Phi(t) = \Psi(t)\square$

10 Autonomous Equations

10.1 Form

$$y' = f(y) \tag{14}$$

10.2 Anaylsis

10.2.1 Differentiation Fields

11 Second Order Linear Eq

11.1 The Wronskian:

$$W(f,g)(t) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}, \qquad x \in I \tag{15}$$

11.2 Existence Theorem

1. Claim: For all D.E. there exists a $y(t)$ that satisfies it locally on some interval

2. Proof:

(a) Let $y' = v \to v' = y''$ 2) Therefore $v' = -py' - qy = -pv - qy$, by plugging into the DE

(b) In matrix form:

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} \tag{16}$$

(c) Note that this is a linear first order matrix system which there is an existence therom for

11.3 Uniqueness

11.4 Second Order Linear Homogenous Diff Eq (S.O.L.H.D.E)

11.4.1 Form

$$a(x)\frac{d^2y}{dt^2} + b(x)\frac{dy}{dt} + c(x)y = 0 \tag{17}$$

11.4.2 Theorm: The general solution to S.O.L.H.E

Claim: The general soln of eq1$\equiv [y'' + p(t)y' + g(t)y = 0]$ is:

$$y_h = c_1 y_1 + c_2 y_2 \tag{18}$$

11.4.3 Proof:

- Q1:

  Given y_1 and y_2 are solutions, why is $c_1y_1 + c_2y_2$ a solution

  1. $Eq1 = D^2(y) + p(t)D(y) + q(t)y = 0$
  2. $Eq1 = [D^2 + p(t)D + q(t)]y = 0$
  3. Let $L = [D^2 + p(t)D + q(t)] \to eq1 \equiv L(y) = 0$
  4. Notice the L is a linear operator and thus obeys the superposition principle
  5. Thus $y = c_1y_1 + c_2y_2$ is a solution $\square$

- Q2:

  Given 2 indepent solutions y_1 and y_2 for the DE, $\forall$ IVP and its unique solution y, $\exists(c_1, c_2) \in \mathbb{C}^2$ s.t. $y = c_1y_1 + c_2y_2 \equiv \vec{y} \cdot \vec{c}$

- Q3:

11.4.4 Generating Second Solution

1. Claim: if $y_1 \neq 0$ be a solution to the D.E. then,

$$y_2 = Cy_1 \int \frac{e^{-\int pdt}}{y_1^2} \tag{19}$$

   and y_2 = solution independent of y_1

2. Proof:

   (a) Consider $(\frac{y_2}{y_1})' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W(y_1,y_2)}{y_1^2}$

   (b) Given that $W' + p(t)W = 0 \implies W(t) = ce^{-\int p(t)dt}$

   (c) Thus: $\int(\frac{y_2}{y_1})'dt = C\int \frac{e^{-\int pdt}}{y_1^2}$

   (d) Solving: $y_2 = Cy_1 \int \frac{e^{-\int pdt}}{y_1^2} \square$

11.5 Second Order Linear Inhomogenious Diff Eq (S.O.L.I.D.E)

11.5.1 Form

$$y^{(n)}(t) + p(t)y'(t) + q(t)y = f(t) \tag{20}$$

11.5.2 General Solution

1. <u>Claim</u>: The general soln of $y^{(n)}(t) + p(t)y'(t) + q(t)y = f(t)$ is:

$$y = y_h + k(t) \tag{21}$$

   (a) $y_h = c_1 y_1 + c_2 y_2$ is the solution to the homogenous equation i.e. $f(t) = 0$
   (b) Functional Offset ($k(t)$): variation or 'offset' from the homogenous equation

2. <u>Proof</u>:

   (a) Sub-Claim: $y_h + k$ is a solution
      i. Using Operator notation: $D^2 y + pDy + qy = f \implies [D^2 + pD + q](y) = f$
      ii. Let $L^2 + pD + q \implies L(y) = f$
         A. Note that L is linear
      iii. $L(y_h + k) = L(y_h + k) = L(y_h) + L(y_p)$
      iv. $L(y_h) = 0, L(y_p) = f \implies L(y_h + k) = f + 0 = f\square$
   (b) Sub-Claim: $\forall y_i$, if y_i is a solution to the S.O.L.I.D.E, then $y_i = y_h + k$
      i. $[L(y_i) = f$ and $L(k) = f]$ \$ $\implies$ $L(y_i$-k$)$ = f-f=0\$
      ii. By Existence of S.O.L.H.D.E, $L(y_i - k) = 0 \implies$ \$y$_i$-k=y$_h$\$
      iii. Thus $y_i = k + y_h\square$

11.5.3 Exponential Shift Law

$$P(D)[e^{\alpha u(t)}] = p(D + \alpha)u(t)[e^{\alpha t}] \tag{22}$$

11.5.4 Expontial-Polynomial Functional offesets

- Form

$$ay'' + by' + cy = e^{\alpha t} g(t); \alpha \in \mathbb{C} \tag{23}$$

- Characteristic Polynomial

$$p(r) = aD^2 + bD + c \tag{24}$$

  Note that the DE in Operator notation is: $[aD^2 + bD + c]$

- Finding Particular Solution for S.O.L.E

  <u>Theorem</u>:

  - Let k be s.t. $(r - \alpha)^k$ are roots of $p(\alpha)$
  - Then

$$y_p = \frac{t^k e^{\alpha t}}{p^{(k)}(\alpha)} \tag{25}$$

- Method of Undetermined Coefficents

  Idea: if f(t) is a comprised of strict multiplications (no division) sinusoidal,exponetials, and polynomials then the solution of the S.O.L.E with const coefficents is in terms of of the same types you began with.

  Cases:

  if $f(t) = e^{\alpha t}$ (polynomial of $deg(k+m)$), then guess $y_p = e^{\alpha t}\Sigma_{j=0}^{k} C_j t^j$

11.5.5 Lagrange Variation of Parameters

- Equation:

$$y_p = \int \frac{y_1 f(x)}{W(y_1, y_2)} dt + \int \frac{y_2 f(x)}{W(y_1, y_2)} dt \tag{26}$$

- Derivation:

  See General Derivation

11.6 Foulrier Transform

11.7 Strum Comparison Theorem

Theorem:

1. If:
   (a) $u'' + q_1(t)u = 0$ and $v'' + q_2 v = 0$
   (b) $q_1 > q_2$
2. Then:
   (a) u vanishes as some point between 2 zeros of v

12 General Linear Diff Eq and Variation of Parameters

12.1 Form

$$y^{(n)}(x) + \sum_{k=0}^{n} a_k(x) y^{(k)}(x) = f(x) \tag{27}$$

12.2 Equation:

$$\sum_{k=0}^{n} [y_k(x) \int \frac{W_k(x)}{W(X)} dx] \tag{28}$$

$W(x) \equiv$ Wronskian determinant of the fundamental system and $W_i(x) \equiv$ the Wronskian determinant of the fundamental system with the $i-th$ column replaced by $(0, 0, \ldots, f(x))$

12.2.1 Derivation:

12.3 Theorem for L.D.E

If $u(t) + iv(t)$ is a solution to the D.E. then $u(t) \wedge v(t)$ are solutions