Diff Eq Notes

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Contents

1 Initial Definitions

- Definition:
  - DE is an equation that describes the properties of an unkown
- Ordinary DE:
  - describes functions of 1 variable
- Partial DE:
  - describes multivariable functions
- Notation:
  - independent variable: y
  - dependent variable: t

2 Operator Notation

Definition: $\frac{d^n}{dt^n} = D^n \to f^{(n)} = D^n(f)$

3 Stability

3.1 Definition

1. Stability $\equiv$ a system in which long term behvaior does not depend within some variation on initial conditions
2. For Linear D.E. y is Stable iff:
$$\lim_{t\to\infty} y_h = 0 \pm \epsilon \tag{1}$$
3. For S.O.L.E
   (a) Stable iff either of the equivalent statments is true:
      i. $Re(r) < 0$
      ii. $a, b, c > 0$ or $a, b, c < 0$

4 Initial Value Problems

$$IVP = \begin{cases} DE \\ y = C_1 \\ \vdots \\ y^{(n)} = C_n \end{cases} \tag{2}$$

5 Seperable DE

5.1 Definition

5.1.1 Form

$$\frac{dy}{dt} = f(y)g(t) \tag{3}$$

5.1.2 Solution

1. Let DE be $\frac{dy}{dt} = f(y)g(t)$
2. Seperating terms: $\frac{dt}{f(y)}\frac{dy}{dt} = \frac{dt}{f(y)}f(y)g(t) \implies \frac{dy}{f(y)} = g(t)dt$
   (a) If $f(y)^{-1}$ is not defined from some t, solve outside of that range and consider those t separately
3. Integrate both sides: $\int \frac{dy}{f(y)} = \int g(t)dt$
4. If possible solve for $y(t)$

6 Homogenous Equations

6.1 Form

$$\frac{dy}{dt} = f(\frac{y}{t}) \tag{4}$$

6.2 Technique for Solving:

1. Let $v = \frac{y}{t} \implies y = vt$
2. Change of variable: $y' = v't + v \implies v't + v = f(v)$
3. Moving v to left side: $t\frac{dv}{dt} = f(v) - v = g(v)$
4. Seperating: $\frac{dv}{g(v)} = \frac{dt}{t}$
5. Solve using Seperable D.E. techniques

7 Exact Equations

7.1 Form

$$\frac{\partial}{\partial t}\Psi(f(t), y(t)) = \frac{\partial \Psi}{\partial f}\frac{df}{dt} + \frac{\partial \Psi}{\partial y}\frac{dy}{dt} \tag{5}$$

7.2 Technique for Solving:

1. Suppose DE is of the form: $M(x, y) + N(x, y)y_x = 0$
2. If $\mathrm{M}_y = \mathrm{N}_x$, then DE is an Exact Eq, solve for $\Psi(x, y) = \Psi(f(x), y(x))$

8 Autonomous Equations

8.1 Form

$$y' = f(y) \tag{6}$$

8.2 Anaylsis

8.2.1 Differentiation Fields

9 Linear Diff Equations

Definition: For an operator,L, the DE: L(y) = 0 is linear iff:

- $\mathrm{L}(\mathrm{y}_1+\mathrm{y}_1) = \mathrm{L}(\mathrm{y}_1)+\mathrm{L}(\mathrm{y}_2)$
- L(cy) = cL(y)

9.1 First Order Linear Eq

9.1.1 Form

$$y' + p(t)y = f(t) \tag{7}$$

9.1.2 Solving using Integration Factors

1. Let μ be a mult factor s.t. $\mu y' + \mu' y = g(t) \implies [\mu(t)y(t)]' = g(t)$
2. Thus $\mu' = \mu p(t) \implies \frac{d\mu}{\mu} = p(t)dt \implies \mu = e^{\int p(t)dt}$
3. Therfore

$$y(t)e^{\int p(t)dt} = \int g(t)dt \tag{8}$$

$\square$

9.1.3 Bernoulli's equations

- Form

$$y' + p(t)y = q(t)y^n, n \in \mathbb{Z} \tag{9}$$

- Solution

  1. Let $v = y^{2-n} \implies v' = (1-n)y^{-n}y'$
  2. Thus $y' = \frac{v'}{1-n}$ and $y = y^n v$
  3. Subsituting in Bernoulli equation: $\frac{v'}{1-n}y^n + p(t)y^n v = q(t)y^n$
  4. Moving into standard form:

$$v' + (1-n)p(t)v = (1-n)q(t) \tag{10}$$

  5. Solve using Integration Factors $\square$

9.1.4 Picard Iteration

- Integral Equations

  Suppose f is continous, then a function $y = \Phi(t)$ solves the IVP iff $y = \Phi(t)$ solves the corresponding integral equation:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{11}$$

- Idea

  1. Let $f(t) = \frac{dy}{dt}$
  2. Construct a sequence of functions $\{g_n(t) : n \geq 0, n \in \mathbb{Z}\}$ that converges to soln:
     (a) $y_0(t) = y_0$
     (b) $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(t))ds$

9.1.5 Lipsichitz Condition

1. For $f(t, y) \in \mathbb{R}$, f is Lipsichitz iff $\exists L \in \mathbb{R}$:

$$|f(t_1, y_1) - f(t_2, y_2)| \leq L \cdot |(y_1 - y_2)| \tag{12}$$

2. If $\Delta y \neq 0$ then this can be thought of as:

$$\left|\frac{\Delta f}{\Delta y}\right| \leq L \tag{13}$$

3. Lemma: if f_y is bounded then f is Lipsichitz

9.1.6 Uniform Convergence (U.C.)

- Definition:

  A sequence of functions $\{f_n(t) : n \geq 0; n \in \mathbb{Z}\}$ defined on the inverval I uniformially converges to $f(t)$ iff $\forall t > 0, \exists N \in \mathbb{Z}$ s.t. $|f_n(t) - f(t)| < \epsilon$ everywhere on I $\forall n > N$

- Theorem: Given $_n(t)$ is continuous on I, if $\lim_{n\to\infty} f_n(t) \to f(t)$ with U.C, then:

  1. f is continuous
  2. If f_n is differtiable, then f is differtiable and f_n' U.C. to f'
  3. The limit is communitive with respect to integration

$$\lim_{n\to\infty} \int_I f_n(t)dt = \int_I \lim_{n\to\infty} f_n(t)dt \tag{14}$$

- Weirstress M Test

  Theorem:

  - If $\forall n \in I, |f_n(t)| \leq M_n$ and if $\sum_{n=0}^{\infty} M_n < L$ for some $L \in \mathbb{R}$,
  - Then $\sum_{n=0}^{\infty} f_n(t)$ Converges Uniformially on I

9.1.7 Existence Theorem

1. Claim:

   (a) If:

   i. $f(y)$ is continous

   ii. f is Lipsichitz w.r.t. $y \in R \equiv \{t, y) : |t - t_0| \leq T$ and $|y - y_0| \leq k\}$

   iii. $\sum_{k=1}^{\infty}[y_k(t) - y_{k-1}(t)]$ converges uniformially

   (b) Then: $\exists$ a solution to the IVP on the interval $|t - t_0| \leq T_1 = min(T, \frac{k}{m})$ where $|f(t, y)| \leq M \in R$

2. Proof:

   (a) Converting the IVP to an I.E.: $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$

   (b) Note theat: $|y_k(t) - y_{k-1}(t)| \leq \frac{M}{L}\frac{L^n(t-t_0)^n}{n!} \leq \frac{M}{L}\frac{L^nT_1^n}{n!}$

   (c) Define: $M_n \equiv \sum_{k=1}^{\infty} \frac{M}{L}\frac{(LT_1)^n}{n!} = \frac{M}{L}(e^{LT_1} - 1)$

   (d) Apply the Weirstress M Test, because $\frac{M}{L}\frac{(LT_1)^n}{n!}$ converges, then $\sum_{k=1}^{\infty}[y_k - y_{k-1}]$ converges

   (e) Thus the series $\{y_n : n \geq 1\}$ converges uniformially on the interval.

   (f) Therefore $\exists$ a solution to the IVP $\square$

9.1.8 Uniqueness Theorm

1. Claim:

   (a) If $\Phi(t)$ and $\Psi(t)$ are solutions of $y' \equiv f(y, t) \in R$ and if f is Lipseitz w.r.t. $y \in R$

   (b) Then $|\Phi(t) - Psi(t)| \leq e^{L|t-t_0|}|\Phi(t_0) - \Psi(t_0)| = 0$

   i. Because they solve the same I.V.P. $|\Phi(t_0) - \Psi(t_0)| = 0$

   (c) Equivalently: Then $\Psi(t) = \Psi(t)$

2. Proof:

   (a) $E \equiv |\Phi(t) - \Psi(t)|^2$

   i. Note that $E \geq 0$

   (b) $\frac{d}{dt}E = 2(\Phi(t) - \Psi(t))(\Phi'(t) - \Psi'(t))$

   (c) $E' \overset{DE}{=} 2(\Phi(t) - \Psi(t))(f(t, \Phi) - f(t, \Psi))$

   (d) $E' \overset{Lip}{\leq} 2|\Phi(t) - \Psi(t)| L |\Phi(t) - \Psi(t)|$

   (e) Thus $E' \leq 2LE \implies E' - 2LE \leq 0 \implies (E(t)e^{-2Lt})' \leq 0$

   i. Note that E' is stricly decreasing

   (f) Therefore: $e^{-t}E(t) \leq e^{2Lt_0}E(t_0) \implies E(t) \leq e^{2L(t-t_0)}E(t_0)$

   (g) Substituting: $|\Phi(t) - \Psi(t)|^2 \leq e^{2L(t-t_0)}|\Phi(t_0) - \Psi(t_0)|^2$

   (h) Because of absolute value: $|\Phi(t) - \Psi(t)| \leq e^{2L(t-t_0)}|\Phi(t_0) - \Psi(t_0)|$

   (i) Because they solve the same I.V.P. $|\Phi(t_0) - \Psi(t_0)| = 0$

   (j) Thus $\Phi(t) = \Psi(t)\square$

9.2 Second Order Linear Eq

9.2.1 The Wronskian:

$$W(f,g)(t) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}, \qquad x \in I \tag{15}$$

9.2.2 Existence Theorem

1. Claim: For all D.E. there exists a $y(t)$ that satisfies it locally on some interval
2. Proof:
   (a) Let $y' = v \to v' = y''$ 2) Therefore $v' = -py' - qy = -pv - qy$, by plugging into the DE
   (b) In matrix form:
   $$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} \tag{16}$$
   (c) Note that this is a linear first order matrix system which there is an existence therom for

9.2.3 Uniqueness

9.2.4 Second Order Linear Homogenous Diff Eq (S.O.L.H.D.E)

- Form

$$a(x)\frac{d^2y}{dt^2} + b(x)\frac{dy}{dt} + c(x)y = 0 \tag{17}$$

- Theorm: The general solution to S.O.L.H.E

  Claim: The general soln of eq1$\equiv [y'' + p(t)y' + g(t)y = 0]$ is:

$$y_h = c_1y_1 + c_2y_2 \tag{18}$$

- Proof:
  - Q1:

    Given y$_1$ and y$_2$ are solutions, why is $c_1y_1 + c_2y_2$ a solution
    1. $Eq1 = D^2(y) + p(t)D(y) + q(t)y = 0$
    2. $Eq1 = [D^2 + p(t)D + q(t)]y = 0$
    3. Let $L = [D^2 + p(t)D + q(t)] \to eq1 \equiv L(y) = 0$
    4. Notice the L is a linear operator and thus obeys the superposition principle
    5. Thus $y = c_1y_1 + c_2y_2$ is a solution $\square$
  - Q2:

    Given 2 indepent solutions y$_1$ and y$_2$ for the DE, $\forall$ IVP and its unique solution y, $\exists(c_1, c_2) \in \mathbb{C}^2$ s.t. $y = c_1y_1 + c_2y_2 \equiv \vec{y} \cdot \vec{c}$

– Q3:

  * Abel's Identity

    1. If u,v solve the D.E. then $W' + p(t)W = 0 \to ce^{-\int p(t)dt}$
    2. Alternatively: $W' + p(t)W = 0 \implies W' = 0 \implies W = C$

  * Finding the general solution

    Goal: The general soln is of the form $y = \vec{y} \cdot \vec{c}$
    1. Recall the matrix form of the D.E. from the Existence theorem proof.
    2. Also Recall that that the equation was only solvable if $W(y_1, y_2)(t_0) \neq 0$
    3. Observe that $W' = (uv' + uv'') - (u'V + u'v') = -pW$
    4. Lemma: if u,v are linearly dependent, then $W(u, v) = 0$ on I

- Generating Second Solution

1. Claim: if $y_1 \neq 0$ be a solution to the D.E. then,

$$y_2 = Cy_1 \int \frac{e^{-\int pdt}}{y_1^2} \tag{19}$$

   and $\mathrm{y_2}$ = solution independent of $\mathrm{y_1}$

2. Proof:
   (a) Consider $(\frac{y_2}{y_1})' = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}$
   (b) Given that $W' + p(t)W = 0 \implies W(t) = ce^{-\int p(t)dt}$
   (c) Thus: $\int (\frac{y_2}{y_1})' dt = C \int \frac{e^{-\int pdt}}{y_1^2}$
   (d) Solving: $y_2 = Cy_1 \int \frac{e^{-\int pdt}}{y_1^2} \square$

9.2.5 Second Order Linear Inhomogenious Diff Eq (S.O.L.I.D.E)

- Form

$$y^{(n)}(t) + p(t)y'(t) + q(t)y = f(t) \tag{20}$$

- General Solution

1. Claim: The general soln of $y^{(n)}(t) + p(t)y'(t) + q(t)y = f(t)$ is:

$$y = y_h + k(t) \tag{21}$$

   (a) $y_h = c_1 y_1 + c_2 y_2$ is the solution to the homogenous equation i.e. $f(t) = 0$
   (b) Functional Offset ($k(t)$): variation or 'offset' from the homogenous equation

2. Proof:
   (a) Sub-Claim: $y_h + k$ is a solution
      i. Using Operator notation: $D^2 y + pDy + qy = f \implies [D^2 + pD + q](y) = f$
      ii. Let $L^2 + pD + q \implies L(y) = f$

A. Note that L is linear

iii. $L(y_h + k) = L(y_h + k) = L(y_h) + L(y_p)$

iv. $L(y_h) = 0, L(y_p) = f \implies L(y_h + k) = f + 0 = f\square$

(b) Sub-Claim: $\forall y_i$, if y_i is a solution to the S.O.L.I.D.E, then $y_i = y_h + k$

i. $[L(y_i) = f$ and $L(k) = f] \implies L(y_i - k) = f - f = 0$

ii. By Existence of S.O.L.H.D.E, $L(y_i - k) = 0 \implies$ \$y$_i$-k=y$_h$\$

iii. Thus $y_i = k + y_h\square$

- Exponential Shift Law

$$P(D)[e^{\alpha u(t)}] = p(D+\alpha)u(t)[e^{\alpha t}] \tag{22}$$

- Expontial-Polynomial Functional offesets

  - Form

$$ay'' + by' + cy = e^{\alpha t}g(t); \alpha \in \mathbb{C} \tag{23}$$

  - Characteristic Polynomial

$$p(r) = aD^2 + bD + c \tag{24}$$

Note that the DE in Operator notation is: $[aD^2 + bD + c]$

  - Finding Particular Solution for S.O.L.E

<u>Theorem</u>:

    * Let k be s.t. $(r-\alpha)^k$ are roots of $p(\alpha)$
    * Then

$$y_p = \frac{t^k e^{\alpha t}}{p^{(k)}(\alpha)} \tag{25}$$

  - Method of Undetermined Coefficents

<u>Idea</u>: if f(t) is a comprised of strict multiplications (no division) sinusoidal,exponetials, and polynomials then the solution of the S.O.L.E with const coefficents is in terms of of the same types you began with.

<u>Cases</u>:

if $f(t) = e^{\alpha t}$ (polynomial of $deg(k+m)$), then guess $y_p = e^{\alpha t}\Sigma_{j=0}^{k} C_j t^j$

- Annihilator Method

1. Given: $(D^2 + aD + b)y = e^{\alpha t}f(t)$, where $f(t)$ is a polynomial
2. Attempt to multiply both sides by a Differential Operator (G) s.t.

$$G(e^{\alpha t}f(t)) = 0 \implies G(D^2 + aD + b)y = 0 \tag{26}$$

3. Notice that this is now a homogenous equation. Solve by finding the roots of the characteristic equation and applying the method of undetermined coefficents

- Lagrange Variation of Parameters

  - Equation:

$$y_p = \int \frac{y_1 f(x)}{W(y_1, y_2)} dt + \int \frac{y_2 f(x)}{W(y_1, y_2)} dt \tag{27}$$

  - Derivation:

    See General Derivation

9.2.6 Foulrier Transform

9.2.7 Strum Comparison Theorem

Theorem:

1. If:
   (a) $u'' + q_1(t)u = 0$ and $v'' + q_2 v = 0$
   (b) $q_1 > q_2$
2. Then:
   (a) u vanishes as some point between 2 zeros of v

9.3 General Linear Diff Eq and Variation of Parameters

9.3.1 Form

$$y^{(n)}(x) + \sum_{k=0}^{n} a_k(x) y^{(k)}(x) = f(x) \tag{28}$$

9.3.2 Equation:

$$\sum_{k=0}^{n} [y_k(x) \int \frac{W_k(x)}{W(X)} dx] \tag{29}$$

$W(x) \equiv$ Wronskian determinant of the fundamental system and $W_i(x) \equiv$ the Wronskian determinant of the fundamental system with the $i-th$ column replaced by $(0, 0, \ldots, f(x))$

- Derivation:

9.3.3 Theorem for L.D.E

If $u(t) + iv(t)$ is a solution to the D.E. then $u(t) \wedge v(t)$ are solutions

10 First Order Systems of Linear Diff Equations with Const Coefficents

10.1 Form

$$\vec{r}' = A\vec{r} \tag{30}$$

10.1.1 Example 1st Order

$$\vec{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{31}$$

10.2 Solution

1. Guess: $\vec{r} = \vec{a}e^{\lambda t}$, $\vec{a} \in D$
2. Thus $\vec{r}' = \vec{a}e^{\lambda t} \implies \lambda\vec{a} = A\vec{a}$
3. Factoring we get $(A - \lambda I)\vec{a} = \vec{0}$
4. <u>Characteristic Polynomial</u> $\equiv |A - \lambda I|$
   (a) For the 2-d version we get
   $$|A - \lambda I| = \lambda^2 + tr(A) + |A| \tag{32}$$
5. Solve for Eigen Value and Corresponding Eigen Vectors
   (a) <u>Complete e-value</u>: K-Repeated Eigen value produces K linearly independent eigen vectors
   (b) <u>Incomplete e-value</u>: Does not produce enough e-values. Solution: assume solution is:
   $$\vec{r} = u(t)\vec{a}e^{\lambda t} + \vec{c} \tag{33}$$
      i. Typically $u(t)$ is a polynomial as in the D-1 case

10.3 Classification

Let $p \equiv tr(A)$ and $q \equiv |A|$

10.3.1 Theorem: Change of coordinates

1. <u>Claim</u>: $\vec{x}' = A\vec{x} \equiv \vec{y}' = B\vec{y} \iff |A - I| = |B - I|$ and $(tr[A])^2 - 4|A| \neq 0$
   1. $(tr[A])^2 - 4|A| \neq 0$ excludes repeated roots
   2. <u>Alternate version</u>: $\exists$ non-singular matrix U s.t. $\vec{y} = U\vec{x}$ and $B = UAU^{-1}$
      (a) This is a change of cordinates
   3. <u>Light Proof</u>: Suppose $\vec{x}' = A\vec{x} \equiv \vec{y}' = B\vec{y}'$
      (a) By the claim $|B - \lambda I| = |UAU^{-1} - \lambda I|$
      (b) Note that $\lambda I = \lambda UU^{-1} = U\lambda IU^{-1} \implies |B - \lambda I| = |UAU^{-1} - U(\lambda I)U^{-1}|$
      (c) Factoring $|U(A - \lambda I)U^{-1}| = |U||A - \lambda I||U^{-1}$
      (d) Using the communitive property of multiplication: $|U||U^{-1}||A - \lambda I| = |A - \lambda I|$
      (e) Thus: $|B - \lambda I| = |A - \lambda I|$