# Week 2

## 1 Independent random variables

Definition: Two random variables $X_1, X_2$ are said to be independent if,

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2) \quad \forall\, x_1 \in \mathcal{X}_1 \,\&\, x_2 \in \mathcal{X}_2$$

$\boxed{P(X_1 = x_1, X_2 = x_2) \text{ means that } X_1 = x_1 \textbf{ AND } X_2 = x_2 \text{ occur.}}$

### 1.1 Claim

$$\sum_{x_2 \in \mathcal{X}_2} P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)$$

**Proof**: Suppose $A$ & $B$ are disjoint/mutually exclusive. Then,

$$P(A \cap B) = 0 \quad \& \quad P(A \cup B) = P(A) + P(B)$$

Now the events $(X_1 = x_1) \cap (X_2 = x_2)$ are disjoint for different values of $x_2 \in \mathcal{X}_2$.(if $x_2 \neq x_2' \;\forall\, x_2 \in \mathcal{X}_2$)

Thus,

$$\begin{aligned}
\sum_{x_2 \in \mathcal{X}_2} P(X_1 = x_1, X_2 = x_2) &= \sum_{x_2 \in \mathcal{X}_2} P\big((X_1 = x_1) \cap (X_2 = x_2)\big) \\
&= P\left(\bigcup_{x_2 \in \mathcal{X}_2} (X_1 = x_1) \cap (X_2 = x_2)\right)
\end{aligned}$$

Now,

$$\begin{aligned}
\bigcup_{x_2 \in \mathcal{X}_2} [(X_1 = x_1) \cap (X_2 = x_2)] &= (X_1 = x_1) \bigcap \left[\bigcup_{x_2 \in \mathcal{X}_2} (X_2 = x_2)\right] \\
&= (X_1 = x_1) \bigcap [x_2 \in \mathcal{X}_2]
\end{aligned}$$

As $[x_2 \in \mathcal{X}_2]$ forms the entire sample space,

$$\begin{aligned}
P\left(\bigcup_{x_2 \in \mathcal{X}_2} (X_1 = x_1) \cap (X_2 = x_2)\right) &= P\left((X_1 = x_1) \bigcap [x_2 \in \mathcal{X}_2]\right) \\
&= P\left((X_1 = x_1) \bigcap \Omega\right) \\
&= P(X_1 = x_1)
\end{aligned}$$

# 2 Entropy

Suppose $X_1 \in \mathcal{X}_1$ & $X_2 \in \mathcal{X}_2$ are **independant** random variables. Then,

$$H(X_1,\ X_2) = H(X_1) + H(X_2)$$

Proof:

$$\sum_{\substack{x_1 \in \mathcal{X}_1 \\ x_2 \in \mathcal{X}_2}} P(X_1 = x_1, X_2 = x_2) \log\left(\frac{1}{P(X_1 = x_1, X_2 = x_2)}\right)$$

$$= \sum_{x_1 \in \mathcal{X}_1} \sum_{x_2 \in \mathcal{X}_2} P(X_1 = x_1, X_2 = x_2) \left[\log \frac{1}{P(X_1 = x_1)} + \log \frac{1}{P(X_2 = x_2)}\right]$$

$$= \sum_{x_1 \in \mathcal{X}_1} \log \frac{1}{P(X_1 = x_1)} \left(\sum_{x_2 \in \mathcal{X}_2} P(X_1 = x_1, X_2 = x_2)\right)$$

$$+ \sum_{x_2 \in \mathcal{X}_2} \log \frac{1}{P(X_2 = x_2)} \left(\sum_{x_1 \in \mathcal{X}_1} P(X_1 = x_1, X_2 = x_2)\right)$$

From the previous claim,

$$\begin{aligned} H(X_1, X_2) = &\sum_{x_1 \in \mathcal{X}_1} P(X_1 = x_1) \log \frac{1}{P(X_1 = x_1)} \\ &+ \sum_{x_2 \in \mathcal{X}_2} P(X_2 = x_2) \log \frac{1}{P(X_2 = x_2)} \\ &= H(X_1) + H(X_2) \end{aligned}$$

## 2.1 Conditional probability distribution

What if $X_1$ & $X_2$ are not independent? Then we would use conditional probability distribution. i.e.

$$P(X_2 = x_2 / X_1 = x_1) := \frac{P(X_2 = x_2, X_1 = x_1)}{P(X_1 = x_1)}, \quad P(X_1 = x_1) \neq 0$$

This definition for conditional probability satisfies the probability axioms and hence it is a valid probability measure.

## 2.2 Conditional entropy

Definition:

$$H(X_2/X_1) := \sum_{x_1 \in \mathcal{X}_1} P(X_1 = x_1) H(X_2/X_1 = x_1)$$

where,

$$H(X_2/X_1 = x_1) := \sum_{x_2 \in \mathcal{X}_2} P(X_2 = x_2/X_1 = x_1) \log \frac{1}{P(X_2 = x_2/X_1 = x_1)}$$

### 2.2.1 Support of a function

When $P(X = x) = 0$, it is not considered in calculating the entropy.

$$H(X) = -\sum_{\{x \in \mathcal{X}: P(X=x) \neq 0\}} P(X = x) log(P(X = x))$$

Suppose that f : X → R is a real-valued function whose domain is an arbitrary set X. The set-theoretic support of f, written supp(f), is the set of points in X where f is non-zero:

$$supp(X) = \{x \in X : f(x) \neq 0\}$$

Note: $P(X = x)$ can be denoted as $P_X(x)$ or $P(x)$.

## 2.3 Important Results

$$H(X) = -\sum_{x \in supp(P)} P(x) log(P(x))$$

$$H(X/Y) = \sum_{y \in supp(P_Y)} P_Y(y) H(X/Y = y)$$

$$H(X/Y = y) = -\sum_{x \in supp(P_{X/Y})} P_{X/Y}(x/y) log(P_{X/Y}(x/y))$$

## 2.4 Chain Rule

$$H(X, Y) = H(X) + H(Y/X) = H(Y) + H(X/Y)$$

Proof:

$$H(X) + H(Y/X) = -\sum_{x \in supp(P)} P(x) log(P(x)) + \sum_{x \in supp(P_X)} P(x) \left( \sum_{y \in supp(P_Y)} P(y/x) log \left( \frac{1}{P(y/x)} \right) \right)$$

$$= -\sum_{x \in supp(P_X)} \left( \sum_{y \in supp(P_Y)} P(x, y) \right) log(P(x)) + \sum_{x \in supp(P_X)} \left( \sum_{y \in supp(P_Y)} P(x, y) \right) log \left( \frac{1}{P(y/x)} \right)$$

$$= \sum_{x \in supp(P_X)} \sum_{y \in supp(P_Y)} P(x, y) \left( \log \frac{1}{P(x)P(y/x)} \right)$$

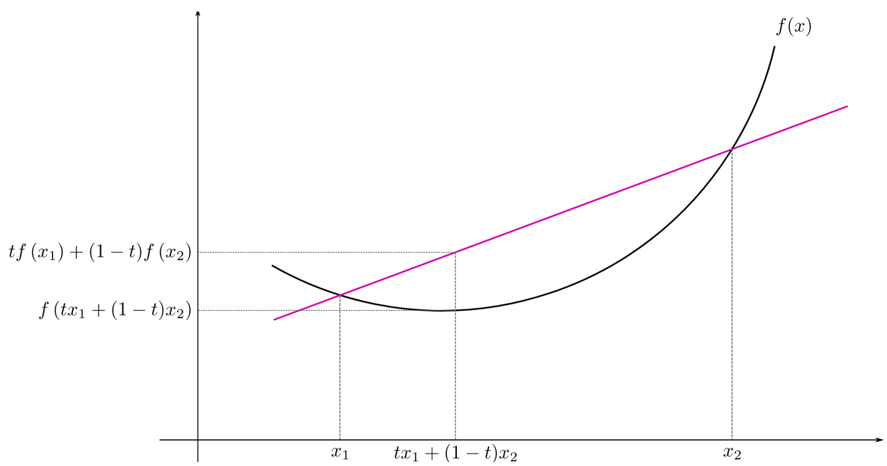
$$= -\sum_x \sum_y P(x, y) \log P(x, y) = H(x, y)$$

## 2.5 Upper and lower bound of entropy

$$0 \leq H(X) \leq |log\mathcal{X}|$$

Proof:
$H(X) = \sum_x P(x)log\frac{1}{P(x)} \rightarrow$ always positive as $P(x) > 0\ \forall x \in supp(P_X)$.
$H(X) = 0$ when $|supp(P_X)| = 1$.

### 2.5.1 Jensen's inequality (Concave/convex functions)




$x_1 < x_3 < x_2$ such that $x_3 = tx_1 + (1-t)x_2$

If $f(x_3) \geq tf(x_1) + (1-t)f(x_2) \rightarrow$ Concave function.

If $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \rightarrow$ Convex function.

Hence, the above graph is a convex function.

We shall use the concavity property of the log function to obtain the proof for $H(X) \leq |log\mathcal{X}|$.

Proof:

$$H(X) = \sum_{x \in supp(P)} P(x)log\frac{1}{P(x)}$$

Let us assume $\lambda_x$ is another notation for $P(x)$.

$$H(X) = \sum_{x \in supp(P)} \lambda_x log\frac{1}{P(x)}$$

This is a convex combination of $log\frac{1}{P(x)} : x \in supp(P_X)$ as $\sum_x P(x) = 1$.

$$H(X) \leq \log\left(\sum_{x \in supp(P_X)} \lambda_x \frac{1}{P(x)}\right) \leq \log|supp(P_x)|$$

$$\Rightarrow H(X) \leq \log|\mathcal{X}|$$

### 2.5.2 When is $H(X)$ is $log|\mathcal{X}|$?

When Jensen's inequality is satisfied with equality, i.e. if $f(x)$ is a straight line, but the $\log x$ curve is strictly concave.

Suppose $f(x)$ is strictly concave(or convex) & $\lambda_1, \lambda_2 \neq 0$ & $\lambda_1 + \lambda_2 = 1$.

If $f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2)$, then $x_1 = x_2$.

More generally, if $\lambda_i \neq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and Jensen's inequality holds with equality, then $x_1 = x_2 \cdots x_n$.

Applying this to $H(X) \leq log|\mathcal{X}|$, from the previous proof,

$$H(X) = \sum_{x \in supp(P)} \lambda_x log \frac{1}{P(x)} \leq \log |supp(P_x)|$$

Suppose equality holds, then by the above claim we must have:

$$\frac{1}{P(X)} = const \quad \forall x \in supp(P_x)$$

$$\Rightarrow const.\ c = P(x) = \frac{1}{|supp(P_x)|}$$

If $|supp(P_x)| = |\mathcal{X}|$, $H(X) = \log |supp(P_x)|$. Then $P(X) = \frac{1}{|\mathcal{X}|}$ $\forall x \in \mathcal{X}$.

We have just proved that $H(X) = \log_2 |\mathcal{X}|$, this can be true only when $P_x$ is uniform. Thus:

Lemma: $H(X) = \log_2 |\mathcal{X}|$ iff $P_x$ is uniform.

# 3 Relative Entropy/ Information Divergence/ Kullback-Leibler Divergence

Suppose there is a random variable $X$ for which we have two probability distributions $p_x$ & $q_x$.

Then the RE or ID or KL is defined as:

$$D(p_X||q_Y) := \sum_{x \in supp(p_x)} p(x) \log \frac{p(x)}{q(x)}$$

($D(p_X||q_Y)$ is a 'kind of' a distance measure between distributions p & q.)

## 3.1 Lower limit of relative entropy

$$D(p||q) \geq 0$$

Proof:

$$D(p||q) = -\sum_{x \in supp(p_x)} p(x) \log \frac{q(x)}{p(x)}$$

$$\geq -\log \left( \sum_{x \in supp(p_x)} p(x) \frac{q(x)}{p(x)} \right)$$

$$\geq -\log\left(\sum_{x\in supp(p_x)} q(x)\right)$$

$$\geq 0 \quad as \quad \sum q(x) \leq 1$$

### 3.2 When is RE equal to zero?

By applying Jensen's inequality for equality condition.

$$\frac{p(x)}{q(x)} = const.\ c \qquad \forall x \in supp(p_x)$$

$$\Rightarrow p(x) = cq(x) \quad \&$$

$$\sum_{x\in supp(p_x)} p(x) = \sum_{x\in supp(p_x)} cq(x) \forall x \in supp(p_x)$$

These togethermean that $c = 1$.

$$\Rightarrow p(x) = q(x) \forall x \in supp(p_x)$$

$$\Rightarrow D(p||q) \quad iff \quad p_x = q_x$$

### 3.3 Upper and lower limits of conditional entropy

$$H(X/Y) = \sum_{y\in supp(P_Y)} P_Y(y) \sum_{x\in supp(P_{X/Y})} P_{X/Y}(x/y)\log\frac{1}{P_{X/Y}(x/y)}$$

$$0 \leq H(X/Y) \leq H(X)$$

Proof:
$H(X/Y) \geq 0$ is true because $H(X/Y = y) \geq 0$ & $P(y) \geq 0$.

$$H(X) - H(X/Y) = \sum_{x\in supp(P_x)} P(x)\log\frac{1}{P(x)} - \sum_{x,y} P(x,y)\log\frac{1}{P(x/y)}$$

$$= \sum_{x,y} P(x,y)\log\frac{1}{P(x)} - \sum_{x,y} P(x,y)\log\frac{1}{P(x/y)}$$

$$= \sum_{x,y} P(x,y)\log\left(\frac{P(x/y)}{P(x)}\right) = \sum_{x,y} P(x,y)\log\left(\frac{P(x,y)}{P(x)P(y)}\right)$$

As both $P(x,y)$ & $P(x)P(y)$ are valid joint distribution of $x, y$. Hence,

$$H(X) - H(X/Y) = D(P(x,y)||P(x)P(y))$$

$$\geq 0 \qquad (as\ D(p||q) \geq 0)$$

$$\Rightarrow H(X/Y) \leq H(X)$$